

Real enumerative invariants relative to the toric boundary and their refinement

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ABSTRACT

We introduce new invariants of a class of toric surfaces (including the projective plane) that arise from appropriate enumeration of real curves of genus 1 and 2. These invariants admit a refinement similar to the one introduced by Grigory Mikhalkin in the rational case.

1. Introduction

Refined enumerative geometry, initiated in [BG16, GS14], became one of the central topics in enumerative geometry with important links to closed and open Gromov–Witten invariants and to Donaldson–Thomas invariants. In a big part of known examples, refined invariants appear as one-parameter deformations of complex enumerative invariants (see, for example, [BG16, FS15, GS19, GS14]). In his groundbreaking paper [Mik17], G. Mikhalkin has proposed a refined invariant provided by enumeration of real rational curves and has related this invariant to the refined tropical invariants of F. Block and L. Göttsche [BG16]. Namely, he introduced an integer-valued *quantum index* for real algebraic curves in toric surfaces. The quantum index is defined in [Mik17] for real curves satisfying the following assumptions: The curve has to intersect toric divisors only at real points and to be irreducible and *separating*; the latter condition means that, in the complex point set of the normalization of the curve, the complement of the real part is disconnected, that is, formed by two halves exchanged by complex conjugation (in fact, the quantum index is associated with one half of a separating real curve, while the other half has the opposite quantum index; for detailed definitions, see Section 3.2). Mikhalkin [Mik17] showed that, for an appropriate kind of constraints, a Welschinger-type enumeration of real rational curves (cf. [Wel05]) in a given divisor class and with a given quantum index produces an invariant that can be directly related to the numerator of a Block–Göttsche refined tropical invariant (represented as a fraction with the standard denominator).

The main goal of the present paper is to introduce refined invariants of Mikhalkin’s type in

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the case of curves of genus 1 and 2. We follow the ideas of [Shu15] and choose constraints so that every counted real curve of genus $g = 1$ or 2 appears to be a *maximal* one (that is, it has $g + 1$ global real branches) and hence is separating. More precisely, given a toric surface with the tautological real structure and a very ample divisor class, we fix maximally many real points in a generic Menelaus position (see Section 2.2) on the toric boundary of the positive quadrant, where genus g curves from the given linear system must be tangent to toric divisors with prescribed even intersection multiplicities, and we fix g more generic real points inside different non-positive quadrants as extra constraints (through which the curves under consideration should pass). There are finitely many real curves of genus g matching the constraints, and all these curves are separating. Their halves have a quantum index, and we equip each curve with a certain Welschinger-type sign, generalizing the sign used by Mikhalkin in the rational case. This gives rise to a signed enumeration of halves of real curves of genus $g = 1$ and 2 that match given constraints, belong to a given linear system, and have a prescribed quantum index κ (we also treat the case $g = 0$, slightly generalizing Mikhalkin’s enumeration to the situation with arbitrary even intersection multiplicities).

We prove that, for some toric surfaces (including the projective plane; in the case of genus 2, in order to avoid cumbersome technicalities, this is the only toric surface considered in the paper), the result $W_g^\kappa(\Delta, \hat{\mathbf{w}})$ of such an enumeration, where the sequence Δ encodes the orders of tangencies and the sequence $\hat{\mathbf{w}}$ encodes the position of the constraints, does not depend on the choice of a (generic) sequence $\hat{\mathbf{w}}$. Precise statements can be found in Section 3.2 (rational case, Theorem 3.2), Section 3.3 (elliptic case, Theorem 3.6), and Section 3.4 (genus 2 case, Theorem 3.8). The obtained invariants are said to be *refined*. In particular, we get new real enumerative invariants (without prescribing values for quantum index) in genus 1 and 2. Tropical counterparts of the above refined invariants, their computation, and comparison with the Block–Göttsche refined tropical invariants are presented in a forthcoming paper by the authors.

The proofs of invariance are decomposed into four steps (these steps are common for all genera $g = 0, 1$, and 2):

- (1) identifying *walls*, that is, loci of codimension 1 in the space of constraints, corresponding to degenerations in the set of counted curves;
- (2) analyzing (local) deformations of germs of degenerate curves at their singular points;
- (3) establishing *transversality conditions* that ensure the existence of 1-dimensional families in the space of constraints transversally crossing the walls and realizing prescribed local deformations of degenerate counted curves;
- (4) verifying the persistence of the count in all wall-crossing events.

In addition to the signed enumeration mentioned above, we suggest another choice for a Welschinger-type sign in the considered enumerative problems. This choice also yields an invariant enumeration. The proofs of the invariance for these two signed enumerations are literally the same, so we treat them together. It turns out¹ that the results $W_g^\kappa(\Delta, \hat{\mathbf{w}})$ and $\widetilde{W}_g^\kappa(\Delta, \hat{\mathbf{w}})$ of the two signed enumerations (see Section 3 for the notation) coincide up to a global sign depending only on Δ , g , and κ . This statement is proved in [IS26]. In fact, the two introduced signs coincide (up to the above-mentioned global sign) for individual curves under enumeration.

This paper is organized as follows. Section 2 contains preparatory material. In Section 3, we introduce refined invariants arising in the enumeration of real curves of genus 0, 1, and 2 (the

¹The remainder of this paragraph was revised in proof.

genus 0 version slightly generalizes Mikhalkin's refined rational invariants [Mik17]). Sections 4, 5, and 6 are devoted to the proofs of the invariance statements in genus 0, 1, and 2, respectively.

2. Preparation

2.1 Convex lattice polygons and real toric surfaces

Consider the lattice $\mathbb{Z}^2$ and its ambient plane $\mathbb{R}^2 = \mathbb{Z}^2 \otimes \mathbb{R}$. Let $P \subset \mathbb{R}^2$ be a non-degenerate convex lattice polygon. For a lattice segment $\sigma \subset \mathbb{R}^2$ (respectively, a vector $\mathbf{a} \in \mathbb{Z}^2 \setminus \{(0,0)\}$), denote by $\|\sigma\|_{\mathbb{Z}}$ (respectively, $\|\mathbf{a}\|_{\mathbb{Z}}$) its *lattice length*, that is, the ratio of the Euclidean length and the minimal length of a non-zero parallel integral vector.

Denote by $\text{Tor}(P)$ the complex toric surface associated with P . Let $\text{Tor}(P)^\times \simeq (\mathbb{C}^*)^2$ (respectively, $\text{Tor}(\partial P)$) be the dense orbit (respectively, the union of all toric divisors) of $\text{Tor}(P)$. The toric surface $\text{Tor}(P)$ has the tautological real structure, and the real part $\text{Tor}_{\mathbb{R}}(P)$ of $\text{Tor}(P)$ contains the positive quadrant $\text{Tor}_{\mathbb{R}}(P)^+ \simeq (\mathbb{R}_{>0})^2$. The closure $\text{Tor}_{\mathbb{R}}^+(P)$ of $\text{Tor}_{\mathbb{R}}(P)^+$ (with respect to the Euclidean topology) is diffeomorphically taken onto P by the moment map. We pull back the standard orientation and the metric of $P \subset \mathbb{R}^2$ to $\text{Tor}_{\mathbb{R}}^+(P)$ and induce an orientation and a metric on the boundary $\partial \text{Tor}_{\mathbb{R}}^+(P)$ of $\text{Tor}_{\mathbb{R}}^+(P)$. In particular, we get a cyclic order on the sides of P ; we call this order *positive*. Denote by $\mathcal{L}_P$ the tautological line bundle over $\text{Tor}(P)$. The global sections of $\mathcal{L}_P$ are spanned by the monomials z^ω , $z = (z_1, z_2)$, $\omega \in P \cap \mathbb{Z}^2$.

For each edge $\sigma \subset \partial P$, we consider the toric curve $\text{Tor}(\sigma)$, its dense orbit $\text{Tor}(\sigma)^\times$, the real part $\text{Tor}_{\mathbb{R}}(\sigma)$ of $\text{Tor}(\sigma)$, and the positive half-axis $\text{Tor}_{\mathbb{R}}(\sigma)^+ \subset \text{Tor}(\sigma)^\times \cap \text{Tor}_{\mathbb{R}}(\sigma)$. The closure $\text{Tor}_{\mathbb{R}}^+(\sigma)$ of $\text{Tor}_{\mathbb{R}}(\sigma)^+$ coincides with $\text{Tor}(\sigma) \cap \partial \text{Tor}_{\mathbb{R}}^+(P)$.

2.2 Toric degree and Menelaus condition

A multi-set $\Delta \subset \mathbb{Z}^2 \setminus \{(0,0)\}$ is said to be

- *balanced* if the sum of vectors in Δ is equal to 0,
- *non-degenerate* if the vectors of Δ span $\mathbb{R}^2$,
- *even* if $\Delta \subset (2\mathbb{Z})^2$.

A balanced non-degenerate multi-set $\Delta \subset \mathbb{Z}^2 \setminus \{(0,0)\}$ is called a (*toric*) *degree*.

Let $\Delta \subset \mathbb{Z}^2 \setminus \{(0,0)\}$ be a non-degenerate balanced multi-set. For each vector $\mathbf{a} \in \Delta$, denote by $\tilde{\mathbf{a}}$ the vector obtained from $\mathbf{a}$ by the counter-clockwise rotation by $\pi/2$. The vectors $\tilde{\mathbf{a}}$ can be attached to each other so that the next vector starts at the end of the preceding one in order to form a simple broken line bounding a convex lattice polygon whose boundary gets the counter-clockwise orientation. This polygon is denoted by P_Δ ; it is determined by Δ up to translation. For a convex lattice polygon δ , we set $\mathcal{A}(\delta)$ to be the Euclidean area of δ and $\mathcal{I}(\delta)$ to be the number of integer points in the interior of δ . In the case of P_Δ , we write $\mathcal{A}(\Delta)$ and $\mathcal{I}(\Delta)$ for short. The elements of Δ are denoted by

$$\mathbf{a}_i^\sigma, \quad i = 1, \dots, n^\sigma,$$

where σ ranges over the set P_Δ^1 of sides of the polygon P_Δ , and n^σ is the number of vectors in Δ that are outer normals to σ .

From now on, assume that Δ is even. For each $\sigma \in P_\Delta^1$ and each $i = 1, \dots, n^\sigma$, put $2k_i^\sigma = \|\mathbf{a}_i^\sigma\|_{\mathbb{Z}}$.

DEFINITION 2.1. For each $\sigma \in P_\Delta^1$, consider a sequence $\mathbf{z}^\sigma$ of n^σ points in $\text{Tor}(\sigma)^\times$, and denote by $\mathbf{z}$ the double-index sequence (with upper index $\sigma \in P_\Delta^1$ and lower index $i = 1, \dots, n^\sigma$) formed by the sequences $\mathbf{z}^\sigma$. We say that the sequence $\mathbf{z}$ satisfies the Menelaus Δ -condition (cf. [Mik17, Section 5.1]) if there exists a curve $C \in |\mathcal{L}_{P_\Delta}|$ such that C does not contain toric divisors as components and, for each $\sigma \in P_\Delta^1$, the scheme-theoretic intersection of C and $\text{Tor}(\sigma)$ coincides with $\sum_{i=1}^{n^\sigma} 2k_i^\sigma z_i^\sigma$.

LEMMA 2.2. *The sequences $\mathbf{z}$ satisfying the Menelaus Δ -condition form an algebraic hypersurface $\text{Men}(\Delta) \subset \prod_{\sigma \in P_\Delta^1} \text{Tor}(\sigma)^{n^\sigma}$.*

Proof. We present an explicit equation of $\text{Men}(\Delta)$. Consider a linear functional $\lambda: \mathbb{R}^2 \rightarrow \mathbb{R}$ that is injective on $\mathbb{Z}^2$. The maximal and minimal points of λ on P_Δ divide the boundary ∂P_Δ of P_Δ into two broken lines B' and B'' , and the maximal and minimal points of λ on each side σ define its orientation. An automorphism of $\mathbb{Z}^2$, which takes an oriented side σ of P onto the naturally oriented segment $[0, \|\sigma\|_{\mathbb{Z}}]$ of the first axis of $\mathbb{R}^2$, defines an isomorphism of $\text{Tor}(\sigma)^\times$ with $\mathbb{C}^\times$. Denote by $\xi^\sigma = (\xi_i^\sigma)_{i=1}^{n^\sigma}$ the sequence of images in $\mathbb{C}^\times$ of the points of $\mathbf{z}^\sigma$. Then, $\text{Men}(\Delta)$ is given by the equation

$$\prod_{\sigma \subset B'} \prod_{i=1}^{n^\sigma} (\xi_i^\sigma)^{2k_i^\sigma} = \prod_{\sigma \subset B''} \prod_{i=1}^{n^\sigma} (\xi_i^\sigma)^{2k_i^\sigma}. \quad \square$$

The hypersurface $\text{Men}(\Delta)$ is called the *Menelaus hypersurface*. Notice that it is reducible. More precisely, $\text{Men}(\Delta)$ splits into $2k_0$ components, where

$$k_0 = \gcd\{k_i^\sigma \mid i = 1, \dots, n^\sigma, \sigma \subset \partial P_\Delta\}.$$

Denote by $\text{Men}(\Delta)_{\text{red}}$ the component given by the equation

$$\prod_{\sigma \subset B'} \prod_{i=1}^{n^\sigma} (\xi_i^\sigma)^{k_i^\sigma/k_0} = \prod_{\sigma \subset B''} \prod_{i=1}^{n^\sigma} (\xi_i^\sigma)^{k_i^\sigma/k_0}. \quad (2.1)$$

Note that the isomorphism $\text{Tor}(\sigma)^\times \simeq \mathbb{C}^\times$ introduced above takes $\text{Tor}_{\mathbb{R}}(\sigma)^+$ onto $\mathbb{R}_{>0} \subset \mathbb{C}^\times$. For $\rho > 0$, denote by $\text{Men}_{\mathbb{R}}^\rho(\Delta)$ the part of $\text{Men}(\Delta)_{\text{red}}$ specified by the condition that, for each $\sigma \in P_\Delta^1$, all points of the subsequence $\mathbf{w}^\sigma$ belong to $\text{Tor}_{\mathbb{R}}^+(\sigma)$ and lie at distance at least ρ (with respect to the metric fixed in Section 2.1) from the endpoints of $\text{Tor}_{\mathbb{R}}^+(\sigma)$. Here, we always assume that, in each interval $\text{Tor}_{\mathbb{R}}^+(\sigma)$, the complement of ρ -neighborhoods of the endpoints is non-empty.

LEMMA 2.3. *The (metric) closure $\overline{\text{Men}}_{\mathbb{R}}^\rho(\Delta) \subset \prod_{\sigma \subset \partial P} (\text{Tor}_{\mathbb{R}}^+(\sigma))^{n^\sigma}$ is diffeomorphic to a convex polytope.*

Proof. Since $\text{Men}_{\mathbb{R}}^\rho(\Delta)$ is given by equation (2.1) with positive variables ξ_i^σ satisfying restrictions of the form $0 < \text{const}_1 \leq \xi_i^\sigma \leq \text{const}_2 < \infty$, the coordinate-wise logarithm takes $\overline{\text{Men}}_{\mathbb{R}}^\rho(\Delta)$ onto a hyperplane section of a convex polytope. $\square$

2.3 Deformations of plane-curve germs

Here we introduce notation and recall some known facts on deformations of plane-curve germs.

Let $(C, z) \subset (\mathbb{C}^2, z)$ be the germ of a reduced curve C with an isolated singular point z , and let $L \supset \{z\}$ be a smooth curve intersecting (C, z) only at z . The space $\text{Vers}(C, z)$, defined as the germ at 0 of $\mathcal{O}_{C,z}/\mathfrak{m}_{C,z}^s$, $s \gg 0$, where $\mathfrak{m}_{C,z}$ is the maximal ideal in the local ring $\mathcal{O}_{C,z}$, serves

as the base of a versal deformation of the germ (C, z) . Denote by $Q_1, \dots, Q_r$ the irreducible components of (C, z) , that is, the local branches of C centered at z , and consider the following loci in $\text{Vers}(C, z)$:

- (a) The equisingular locus $\text{ES}(C, z) \subset \text{Vers}(C, z)$ parameterizes curve germs with a singular point topologically equivalent to (C, z) . It is smooth (see [DH88, item (ii) on p. 435]), and by [GS84, Theorem 1 and formula (19)], its tangent space $T(\text{ES}(C, z))$ is contained in

$$\left\{ \varphi \in \text{Vers}(C, z) \mid \text{ord } \varphi|_{Q_i} \geq 2\delta(Q_i) + \sum_{j \neq i} (Q_i \cdot Q_j) + \text{mult } Q_i - 1, i = 1, \dots, r \right\},$$

where $(Q_i \cdot Q_j)$ stands for the intersection multiplicity of the local branches Q_i and Q_j , the δ -invariant $\delta(Q_i)$ denotes the maximal number of nodes that may appear in a deformation of the local branch Q_i , the multiplicity of Q_i at z is denoted by $\text{mult } Q_i$, and finally, $\text{ord } \varphi|_{Q_i}$ is the vanishing order of $\varphi(x(t), y(t))$, where $(x(t), y(t))$ is a parameterization of Q_i .

- (b) The equisingular loci $\text{ES}_L(C, z) \subset \text{ES}(C, z)$ and $\text{ES}_L^{\text{fix}}(C, z) \subset \text{ES}_L(C, z)$ are additionally constrained by the requirement to intersect L at a point close to z with multiplicity $(C \cdot L)_z$, or to intersect L at z with multiplicity $(C \cdot L)_z$, respectively. Both loci are smooth, and by [GS84, Theorem 2 and formulas (21), (22)], their tangent spaces satisfy

$$T(\text{ES}_L(C, z)) \subset \left\{ \varphi \in \text{Vers}(C, z) \mid \begin{aligned} \text{ord } \varphi|_{Q_i} &\geq 2\delta(Q_i) + \sum_{j \neq i} (Q_i \cdot Q_j) \\ &\quad + (Q_i \cdot L) - 1, i = 1, \dots, r \end{aligned} \right\},$$

$$T(\text{ES}_L^{\text{fix}}(C, z)) \subset \left\{ \varphi \in \text{Vers}(C, z) \mid \begin{aligned} \text{ord } \varphi|_{Q_i} &\geq 2\delta(Q_i) + \sum_{j \neq i} (Q_i \cdot Q_j) \\ &\quad + \text{mult } Q_i + (Q_i \cdot L) - 1, i = 1, \dots, r \end{aligned} \right\}.$$

- (c) The equigeneric locus $\text{EG}(C, z) \subset \text{Vers}(C, z)$ parameterizes equigeneric (or $\delta = \text{const.}$) deformations. It is smooth if and only if all local branches $Q_1, \dots, Q_r$ are smooth. In any case, its tangent cone is the linear space

$$J^{\text{cond}}(C, z)/\mathfrak{m}_{C, z}^s = \left\{ \varphi \in \text{Vers}(C, z) \mid \text{ord } \varphi|_{Q_i} \geq 2\delta(Q_i) + \sum_{j \neq i} (Q_i \cdot Q_j), i = 1, \dots, r \right\}$$

of codimension $\text{codim } \text{EG}(C, z) = \delta(C, z)$, where

$$J^{\text{cond}}(C, z) = \text{Ann}(\Phi_* \mathcal{O}_{\widehat{C}} / \mathcal{O}_C) \subset \mathcal{O}_{C, z}$$

is the conductor ideal (see details in [DH88, item (iii) on p. 435 and Theorem 4.15]), and $\delta(C, z) = \dim \mathcal{O}_{C, z} / J^{\text{cond}}(C, z)$.

- (d) The equigeneric loci $\text{EG}_L(C, z) \subset \text{EG}(C, z)$ and $\text{EG}_L^{\text{fix}}(C, z) \subset \text{EG}_L(C, z)$ are additionally constrained by the requirement to intersect L at a point close to z with multiplicity $(C \cdot L)_z$, or to intersect L at z with multiplicity $(C \cdot L)_z$, respectively. If (C, z) is irreducible, that is, $r = 1$, then these loci are smooth [Shu17, Lemma 3(1)], and their tangent spaces satisfy

$$T(\text{EG}_L(C, z)) = \{ \varphi \in \text{Vers}(C, z) \mid \text{ord } \varphi|_{(C, z)} \geq 2\delta(C, z) + (C \cdot L)_z - 1 \},$$

$$T(\text{EG}_L^{\text{fix}}(C, z)) = \{ \varphi \in \text{Vers}(C, z) \mid \text{ord } \varphi|_{(C, z)} \geq 2\delta(C, z) + (C \cdot L)_z \}.$$

2.4 Curves on toric surfaces

Given a morphism $\Phi: \widehat{C} \rightarrow \text{Tor}(P_\Delta)$ of a smooth curve $\widehat{C}$ to $\text{Tor}(P_\Delta)$, we denote by

- $\Phi_*(\widehat{C})$ the scheme-theoretic push-forward, that is, the 1-dimensional part of the image, whose components are taken with the corresponding multiplicities;

- $C := \Phi(\widehat{C})$ the reduced model of $\Phi_*(\widehat{C})$, where all components are taken with multiplicity 1; we split Φ into $\Phi: \widehat{C} \rightarrow C \xrightarrow{\text{in}} \text{Tor}(P_\Delta)$;
- $\Phi^*(D)$, where $D \subset \text{Tor}(P_\Delta)$ is a divisor intersecting C in finitely many points, the divisor on $\widehat{C}$ that is the pull-back of the scheme-theoretic intersection $D \cap \Phi_*(\widehat{C})$.

Denote by $\mathcal{M}_{g,m}(\text{Tor}(P_\Delta), \mathcal{L}_{P_\Delta})$ the moduli space of stable maps $\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)$, where $\widehat{C}$ is a smooth connected curve of genus g and $\mathbf{p}$ is a sequence of m distinct points in $\widehat{C}$, such that $\Phi_*(\widehat{C}) \in |\mathcal{L}_{P_\Delta}|$. Correspondingly, we denote by $\overline{\mathcal{M}}_{g,m}(\text{Tor}(P_\Delta), \mathcal{L}_{P_\Delta})$ the Kontsevich compactification of the above moduli space. This compactification is obtained by adding isomorphism classes of stable maps $\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)$, where $\widehat{C}$ is a connected nodal curve of arithmetic genus g and $\mathbf{p}$ is a sequence of m distinct points in $\widehat{C} \setminus \text{Sing}(\widehat{C})$, such that $\Phi_*(\widehat{C}) \in |\mathcal{L}_{P_\Delta}|$ (here $\text{Sing}(\widehat{C})$ stands for the set of singularities of $\widehat{C}$). In what follows, we work with certain subspaces of these moduli spaces specified for genus $g = 0$ in Section 3.2, for genus $g = 1$ in Section 3.3, and for genus $g = 2$ in Section 3.4. For any subset $\mathcal{M} \subset \mathcal{M}_{g,m}(\text{Tor}(P_\Delta), \mathcal{L}_{P_\Delta})$, denote by $\overline{\mathcal{M}}$ the closure of $\mathcal{M}$ in the compactified moduli space $\overline{\mathcal{M}}_{g,m}(\text{Tor}(P_\Delta), \mathcal{L}_{P_\Delta})$.

We denote by $(C_1 \cdot C_2)_z$ the intersection multiplicity of curves $C_1, C_2 \subset \text{Tor}(P_\Delta)$ at a smooth point z of $\text{Tor}(P_\Delta)$. By $C_1 C_2$ we mean the total intersection multiplicity of curves $C_1, C_2 \subset \text{Tor}(P_\Delta)$ provided that all intersection points of these curves are smooth in $\text{Tor}(P_\Delta)$.

Recall that deformations of a morphism $\Phi: \widehat{C} \rightarrow \text{Tor}(P_\Delta)$ of a smooth curve $\widehat{C}$ are encoded by the normal sheaf on $\widehat{C}$ (see, for instance, [GK89]):

$$\mathcal{N}_\Phi := \Phi^* \mathcal{T} \text{Tor}(P_\Delta) / \mathcal{T} \widehat{C}$$

(where $\mathcal{T}$ denotes the tangent bundle). In the case of an immersion, $\mathcal{N}_\Phi$ is a line bundle of degree

$$\deg \mathcal{N}_\Phi = c_1(\text{Tor}(P_\Delta)) c_1(\mathcal{L}_{P_\Delta}) - 2. \quad (2.2)$$

If $\Phi: \widehat{C} \rightarrow C \hookrightarrow \text{Tor}(P_\Delta)$ is birational onto its image C but not necessarily an immersion, we consider deformations realizing specific local deformations of C at its singular points (see the details in Section 2.3).

Let $\Delta = (\mathbf{a}_i^\sigma, i = 1, \dots, n^\sigma, \sigma \in P_\Delta^1)$ be a toric degree as introduced in Section 2.2. Fix a non-negative integer $g \leq \mathcal{I}(\Delta)$ and a non-negative integer $n(\sigma) \leq n^\sigma$ for each edge $\sigma \in P_\Delta^1$ such that $\sum_{\sigma \in P_\Delta^1} n(\sigma) < \sum_{\sigma \in P_\Delta^1} n^\sigma$. Put $n_\partial = \sum_{\sigma \in P_\Delta^1} n(\sigma)$ and $n_{\text{in}} = n - n_\partial$, where

$$n = \sum_{\sigma \in P_\Delta^1} n^\sigma + g - 1.$$

Choose a sequence $\mathbf{w}$ of n distinct points in $\text{Tor}(P_\Delta)$ splitting into two subsequences:

- $\mathbf{w}_\partial$, consisting of n_∂ points in general position on $\text{Tor}(\partial P_\Delta)$ so that, for each side $\sigma \in P_\Delta^1$, the sequence $\mathbf{w}_\partial$ contains $n(\sigma)$ points $\mathbf{w}_i^\sigma$, $1 \leq i \leq n(\sigma)$, lying on $\text{Tor}(\sigma)$;
- $\mathbf{w}_{\text{in}}$, consisting of n_{in} points w_j , $j = 1, \dots, n_{\text{in}}$, in general position in $(\mathbb{C}^\times)^2 \subset \text{Tor}(P_\Delta)$.

Introduce the subset $\mathcal{M}_{g,n}(\Delta, \mathbf{w}) \subset \mathcal{M}_{g,n}(\text{Tor}(P_\Delta), \mathcal{L}_{P_\Delta})$ consisting of the elements

$$[\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)]$$

satisfying the following assumptions:

- (M1) The sequence $\mathbf{p}$ splits into two disjoint subsequences $\mathbf{p}_\partial$ and $\mathbf{p}_{\text{in}}$ containing n_∂ points p_i^σ , $\sigma \in P_\Delta^1$, $1 \leq i \leq n(\sigma)$, and n_{in} points p_j , $j = 1, \dots, n_{\text{in}}$, respectively, such that $\Phi(\mathbf{p}_\partial) = \mathbf{w}_\partial$

(more precisely, for each $\sigma \in P_\Delta^1$, one has $\Phi(p_i^\sigma) = w_i^\sigma$, $i = 1, \dots, n(\sigma)$) and $\Phi(p_j) = w_j$, $j = 1, \dots, n_{\text{in}}$.

(M2) For each $\sigma \in P_\Delta^1$, one has

$$\Phi^*(\text{Tor}(\sigma)) = \sum_{i=1}^{n^\sigma} k_i^\sigma p_i^\sigma \in \text{Div}(\widehat{C}),$$

where $\Phi(p_i^\sigma) = w_i^\sigma$ for all $p_i^\sigma \in \mathbf{p}_\partial$ and $w_i^\sigma \in \mathbf{w}_\partial$, $\sigma \in P_\Delta^1$, $1 \leq i \leq n(\sigma)$, and the p_i^σ for $n(\sigma) < i \leq n^\sigma$ are some points of $\widehat{C}$ not counted in the sequence $\mathbf{p}_\partial$; finally, $\text{Div}(\widehat{C})$ denotes the group of Weil divisors on $\widehat{C}$.

LEMMA 2.4. Suppose that the sequence $\mathbf{w}$ is in general position subject to the location with respect to the toric divisors as indicated above. Then, the space $\mathcal{M}_{g,n}(\Delta, \mathbf{w})$ is finite. Moreover, for each element $[\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)] \in \mathcal{M}_{g,n}(\Delta, \mathbf{w})$, the map Φ takes $\widehat{C}$ birationally onto an immersed curve $C = \Phi(\widehat{C})$ that, for each $\sigma \in P_\Delta^1$, intersects the toric divisor $\text{Tor}(\sigma)$ at n^σ distinct points and C is smooth at each of these intersection points. Furthermore, $\Phi(\mathbf{p}) \cap \text{Sing}(C) = \emptyset$.

Proof. Consider the family

$$\begin{aligned} \mathcal{M}'_{g,n}(\Delta) = \{ & [\Phi: (\widehat{C}, \mathbf{p}_\partial) \rightarrow \text{Tor}(P_\Delta)] \in \mathcal{M}_{g,n}(\text{Tor}(P_\Delta), \mathcal{L}_{P_\Delta}) \text{ such that} \\ & \Phi: \widehat{C} \rightarrow C = \Phi(\widehat{C}) \text{ is birational and} \\ & [\Phi: (\widehat{C}, \mathbf{p}_\partial) \rightarrow \text{Tor}(P_\Delta)] \text{ satisfies condition (M2)} \}. \end{aligned} \quad (2.3)$$

We claim that

$$\dim \mathcal{M}'_{g,n}(\Delta) \leq n. \quad (2.4)$$

Assuming (2.4) and imposing the conditions $\Phi(p_i^\sigma) = w_i^\sigma \in \mathbf{w}_\partial$, $1 \leq i \leq n(\sigma)$, for every edge $\sigma \in P_\Delta^1$, as well as imposing n_{in} extra point constraints $\mathbf{w}_{\text{in}}$ in general position in $\text{Tor}(P_\Delta)^\times$, we obtain a finite set.

To prove (2.4), take a generic element $\xi = [\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)]$ in an irreducible component of $\mathcal{M}'_{g,n}(\Delta)$. It is a smooth point of that component, and the germ of $\mathcal{M}'_{g,n}(\Delta)$ at ξ realizes an equisingular deformation. We estimate the dimension of the tangent space to this germ. Put $C = \Phi(\widehat{C})$. Taking into account the formulas in Section 2.3, we derive that

$$\dim \mathcal{M}'_{g,n}(\Delta) \leq h^0(C, \mathcal{J}_{Z/C} \otimes_{\mathcal{O}_C} \text{in}^* \mathcal{L}_{P_\Delta}) = h^0(\widehat{C}, \mathcal{O}_{\widehat{C}}(-\mathbf{d}) \otimes_{\mathcal{O}_{\widehat{C}}} \Phi^* \mathcal{L}_{P_\Delta}), \quad (2.5)$$

where

- $Z \subset C$ is the 0-dimensional scheme supported at $\text{Sing}(C) \cup (C \cap \text{Tor}(\partial P_\Delta))$, defined at each point $z \in \text{Sing}(C) \cap \text{Tor}(P_\Delta)^\times$ by the conductor ideal $I^{\text{cond}}(C, z)$, and defined at each point $w \in C \cap \text{Tor}(\sigma)$, $\sigma \in P_\Delta^1$, by the ideal described by the right-hand side of the first formula of item (b) in Section 2.3;
- $\mathcal{J}_{Z/C} \subset \mathcal{O}_C$ is the ideal sheaf of the scheme $Z \subset C$;
- the divisor $\mathbf{d}$ satisfies

$$\begin{aligned} \deg \mathbf{d} = & 2 \sum_{z \in \text{Sing}(C)} \delta(C, z) + \sum_{\sigma \in P_\Delta^1} \sum_{w \in C \cap \text{Tor}(\sigma)} \sum_{Q \subset (C, w)} ((Q \cdot \text{Tor}(\sigma))_w - 1) \\ & + \sum_{z \in \text{Sing}(C) \cap \text{Tor}(P_\Delta)^\times} \sum_{Q \subset (C, z)} (\text{mult } Q - 1). \end{aligned}$$

It follows that

$$\begin{aligned}
 \deg(\mathcal{O}_{\widehat{C}}(-\mathbf{d}) \otimes_{\mathcal{O}_{\widehat{C}}} \Phi^* \mathcal{L}_{P_{\Delta}}) &= C^2 - \deg \mathbf{d} \\
 &= C^2 - (C^2 + CK_{\text{Tor}(P_{\Delta})} + 2 - 2g) + CK_{\text{Tor}(P_{\Delta})} \\
 &\quad + \#\{Q \subset (C, w), w \in C \cap \text{Tor}(\sigma)\} - \sum_{z \in \text{Sing}(C) \cap \text{Tor}(P_{\Delta})^{\times}} \sum_{Q \subset (C, z)} (\text{mult } Q - 1) \\
 &= (2g - 2) + \#\{Q \subset (C, w), w \in C \cap \text{Tor}(\sigma)\} - \sum_{z \in \text{Sing}(C) \cap \text{Tor}(P_{\Delta})^{\times}} \sum_{Q \subset (C, z)} (\text{mult } Q - 1),
 \end{aligned} \tag{2.6}$$

where the symbol $\#$ stands for the number of elements of a given set. Since

$$3 \leq \#\{Q \subset (C, w), w \in C \cap \text{Tor}(\sigma)\} \leq \#\Delta,$$

by (2.6) and by the Riemann–Roch formula, the right-hand side of (2.5) does not exceed

$$\begin{aligned}
 2g - 2 + \#\{Q \subset (C, w), w \in C \cap \text{Tor}(\sigma)\} - g + 1 \\
 = g - 1 + \#\{Q \subset (C, w), w \in C \cap \text{Tor}(\sigma)\} \leq g - 1 + \#\Delta = n.
 \end{aligned}$$

Furthermore, if either $\#\{Q \subset (C, w), w \in C \cap \text{Tor}(\sigma)\} < \#\Delta$ or there is a singular local branch Q of C centered in $\text{Tor}(P_{\Delta})^{\times}$, then we get $\mathcal{M}'_{g,n}(\Delta, \mathbf{w}) = \emptyset$. Thus, we are left with the following case: $\dim \mathcal{M}'_{g,n}(\Delta) = n$, the curve C is immersed in $\text{Tor}(P_{\Delta})^{\times}$, and we have $\#\{Q \subset (C, w), w \in C \cap \text{Tor}(\sigma)\} = \#\Delta$. Now we show that C is smooth along the toric divisors. Suppose that C has a local singular branch Q_0 centered at some point $w \in \text{Tor}(\sigma)$, where $\sigma \in P_{\Delta}^1$. Fixing the position of the point w , we obtain a subfamily $\mathcal{V}_1 \subset \mathcal{M}'_{g,n}(\Delta)$ of dimension at least $n - 1$. On the other hand, computations as above yield that

$$\dim \mathcal{V}_1 \leq h^0(\widehat{C}, \mathcal{O}_{\widehat{C}}(-\mathbf{d}_1) \otimes_{\mathcal{O}_{\widehat{C}}} \Phi^* \mathcal{L}_{P_{\Delta}}),$$

where by the second formula of item (b) in Section 2.3,

$$\deg \mathbf{d}_1 = 2 \sum_{z \in \text{Sing}(C)} \delta(C, z) + \sum_{\sigma \in P_{\Delta}^1} \sum_{w \in C \cap \text{Tor}(\sigma)} \sum_{Q \subset (C, w)} ((Q \cdot \text{Tor}(\sigma))_w - 1) + \text{mult } Q_0.$$

This implies that

$$\deg(\widehat{C}, \mathcal{O}_{\widehat{C}}(-\mathbf{d}_1) \otimes_{\mathcal{O}_{\widehat{C}}} \Phi^* \mathcal{L}_{P_{\Delta}}) = 2g - 2 + \#\Delta - \text{mult } Q_0 \leq 2g - 2 + \#\Delta - 2,$$

and by the Riemann–Roch formula, we obtain

$$\dim \mathcal{V}_1 \leq \#\Delta + g - 1 - 2 = n - 2,$$

which gives a contradiction. Thus, C must be immersed. Now suppose that some point w in $C \cap \text{Tor}(\sigma)$, where $\sigma \in P_{\Delta}^1$, is a center of local branches $Q_1, \dots, Q_r$, $r \geq 2$. Again, fixing the position of the point w , we obtain a subfamily $\mathcal{V}_2 \subset \mathcal{M}'_{g,n}(\Delta)$ of dimension at least $n - 1$. On the other hand, computations as above yield that

$$\dim \mathcal{V}_2 \leq h^0(\widehat{C}, \mathcal{O}_{\widehat{C}}(-\mathbf{d}_2) \otimes_{\mathcal{O}_{\widehat{C}}} \Phi^* \mathcal{L}_{P_{\Delta}}),$$

where

$$\deg \mathbf{d}_2 = 2 \sum_{z \in \text{Sing}(C)} \delta(C, z) + \sum_{\sigma \in P_{\Delta}^1} \sum_{w \in C \cap \text{Tor}(\sigma)} \sum_{Q \subset (C, w)} ((Q \cdot \text{Tor}(\sigma))_w - 1) + r,$$

which implies

$$\deg(\widehat{C}, \mathcal{O}_{\widehat{C}}(-\mathbf{d}_2) \otimes_{\mathcal{O}_{\widehat{C}}} \Phi^* \mathcal{L}_{P_{\Delta}}) = 2g - 2 + \#\Delta - r \leq 2g - 2 + \#\Delta - 2,$$

and, by the Riemann–Roch formula, we obtain

$$\dim \mathcal{V}_2 \leq \#\Delta + g - 1 - 2 = n - 2,$$

which gives a contradiction. Hence, C is smooth along the toric divisors.

It remains to show that $\mathcal{M}_{g,n}(\Delta, \mathbf{w})$ does not contain elements $[\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \mathrm{Tor}(P_\Delta)]$ such that $\Phi: \widehat{C} \rightarrow \mathrm{Tor}(P_\Delta)$ is a multiple covering of the image $C' = \Phi(\widehat{C})$. Following the preceding argument, we assume that $\dim \mathcal{M}_{g,n}(\Delta) = n$ and that a generic element of some component $\mathcal{V}_0 \subset \mathcal{M}_{g,n}(\Delta)$ is represented by a multiple covering $\Phi: \widehat{C} \rightarrow C' \hookrightarrow \mathrm{Tor}(P_\Delta)$, and then come to a contradiction. Denote by $\Phi': \widehat{C}' \rightarrow C'$ the normalization. Notice that the genus g' of $\widehat{C}'$ is at most g , and that the number of local branches of C' centered on the toric divisors is bounded from above by $\#\Delta$. Furthermore, we can assume that the variation of $\Phi: \widehat{C} \rightarrow \mathrm{Tor}(P_\Delta)$ along $\mathcal{V}_0$ induces an equisingular deformation of C' in the linear system $|C'|$. Computations as in the preceding paragraph imply that

$$n = \mathcal{V}_0 \leq \# \left\{ Q \subset (C', w), w \in C' \cap \bigcup_{\sigma} \mathrm{Tor}(\sigma) \right\} + g' - 1,$$

which, in turn, yields that

$$\# \left\{ Q \subset (C', w), w \in C' \cap \bigcup_{\sigma} \mathrm{Tor}(\sigma) \right\} = \#\Delta \quad \text{and} \quad g' = g.$$

Moreover, the argument of the preceding paragraph would imply that C' is immersed and smooth along the toric divisors. That means that $\Phi: \widehat{C} \rightarrow \widehat{C}'$ is ramified at each of the $\#\Delta \geq 3$ points of $\widehat{C}'$ mapped to the toric divisors, and the ramification index at each of them is maximal, but this contradicts the Riemann–Hurwitz formula. $\square$

A map $\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \mathrm{Tor}(P_\Delta)$ such that $[\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \mathrm{Tor}(P_\Delta)] \in \mathcal{M}_{g,n}(\Delta, \mathbf{w})$ is said to be *real* if

- (i) the sequence $\mathbf{w}$ is invariant with respect to the tautological real structure on $\mathrm{Tor}(P_\Delta)$,
- (ii) $(\widehat{C}, \mathbf{p})$ is equipped with a real structure, and
- (iii) $\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \mathrm{Tor}(P_\Delta)$ commutes with the real structures in the source and in the target.

The set of equivalence classes of such real maps $\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \mathrm{Tor}(P_\Delta)$ considered up to equivariant isomorphism is denoted by $\mathcal{M}_{g,n}^{\mathbb{R}}(\Delta, \mathbf{w})$. We say that an element $[\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \mathrm{Tor}(P_\Delta)]$ of $\mathcal{M}_{g,n}^{\mathbb{R}}(\Delta, \mathbf{w})$ is *separating* if the complement in $\widehat{C}$ to the real point set $\mathbb{R}\widehat{C}$ is disconnected, that is, consists of two connected components. The choice of one of these halves induces the so-called *complex orientation* on $\mathbb{R}\widehat{C}$, as well as on $\Phi(\mathbb{R}\widehat{C})$ if Φ is birational onto its image. Denote by $\overrightarrow{\mathcal{M}}_{g,n}^{\mathbb{R}}(\Delta, \mathbf{w})$ the set of separating elements $[\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \mathrm{Tor}(P_\Delta)] \in \mathcal{M}_{g,n}^{\mathbb{R}}(\Delta, \mathbf{w})$ equipped with a choice of one half $\widehat{C}_+$ of $\widehat{C} \setminus \mathbb{R}\widehat{C}$.

Following [Mik17], we assign to each element $\xi = ([\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \mathrm{Tor}(P_\Delta)], \widehat{C}_+) \in \overrightarrow{\mathcal{M}}_{g,n}^{\mathbb{R}}(\Delta, \mathbf{w})$ its *quantum index*

$$\mathrm{QI}(\xi) = \frac{1}{\pi^2} \int_{\widehat{C}_+} \Phi^* \left(\frac{dx_1 \wedge dx_2}{x_1 x_2} \right), \quad x_1 = |z_1|, \quad x_2 = |z_2|, \quad (2.7)$$

with z_1, z_2 coordinates in the torus $(\mathbb{C}^\times)^2 \simeq \mathrm{Tor}(P_\Delta)^\times$ such that the form $dx_1 \wedge dx_2$ agrees with the orientation of $\mathrm{Tor}_{\mathbb{R}}^+(P_\Delta)$ defined in Section 2.1. By [Mik17, Theorem 3.1], if all intersection points of $\Phi(\widehat{C})$ with the toric divisors are real, then $\mathrm{QI}(\xi) \in \frac{1}{2}\mathbb{Z}$ and $|\mathrm{QI}(\xi)| \leq \mathcal{A}(\Delta)$.

3. Real refined enumerative invariants in genus 0, 1, and 2

3.1 Preliminary remarks

Recall that the Welschinger-signed count of real plane curves of a given degree and genus, passing through a generic conjugation-invariant configuration of points in the plane, is non-invariant as long as the genus is positive [IKS03, Theorem 3.1]. However, in the case of real rational surfaces with a disconnected real part, Welschinger invariants survive for certain positive genera g , provided that some conditions on constraints are fulfilled: In particular, each real connected component of the ambient surface contains at most one *oval* (that is, global real branch) of any curve under consideration, and the counted curves have the maximal possible number $g + 1$ of ovals; see [Shu15]. We exploit here a similar approach in order to extend the invariant count of real rational curves [Mik17] to positive genera. Our idea is to consider real curves that are tangent to toric divisors with even intersection multiplicity at each intersection point. In such a case, every quadrant of a real toric surface plays the role of a separate component of the real part of a real surface. The lack of the “bad” wall in the space of constraints indicated in [IKS03, Section 3] is expressed as a cohomology-vanishing condition proved below in Lemmas 4.1, 5.1, and 6.1.

Remark 3.1. The above idea does not work for genus $g \geq 3$: If we fix constraints on the boundary of the positive quadrant (see Sections 3.2, 3.3, and 3.4) and g more extra points in the big real torus, the set of counted curves will necessarily include non-dividing real curves with at most g ovals, which does not allow one to perform Mikhalkin’s quantization.

3.2 Refined rational invariants

In the notation of Section 2.4, assume that

$$g = 0, \quad n_{\text{in}} = 0, \quad n_{\partial} = n = \sum_{\sigma \in P_{\Delta}^1} n^{\sigma}, \quad \text{and} \quad \mathbf{w} = \mathbf{w}_{\partial} \subset \partial \text{Tor}_{\mathbb{R}}^+(P_{\Delta}).$$

In particular, for all $\sigma \in P_{\Delta}^1$ but one, we have $n(\sigma) = n^{\sigma}$, and for the remaining edge σ' , we have $n(\sigma') = n^{\sigma'} - 1$. Note that for a given $\mathbf{w}$, there exists a unique point $w_{n^{\sigma'}}^{\sigma'} \in \partial \text{Tor}_{\mathbb{R}}^+(P_{\Delta})$ such that the sequence $\hat{\mathbf{w}} := \mathbf{w} \cup \{w_{n^{\sigma'}}^{\sigma'}\}$ belongs to the hypersurface $\text{Men}(\Delta)$. We denote by $\mathcal{M}_{0,n}(\Delta, \hat{\mathbf{w}}) \subset \mathcal{M}_{0,n}(\Delta, \mathbf{w})$ the subset formed by the curves passing through $\hat{\mathbf{w}}$.

Denote by $\text{Men}_0(\Delta)$ the subset of $\text{Men}(\Delta)$ formed by the above sequences $\hat{\mathbf{w}}$ such that $\mathcal{M}_{0,n}(\Delta, \hat{\mathbf{w}})$ satisfies the conclusions of Lemma 2.4. The closure

$$\overline{\bigcup_{\hat{\mathbf{w}} \in \text{Men}_0(\Delta)} \mathcal{M}_{0,n}^{\mathbb{R}}(\Delta, \hat{\mathbf{w}})} \subset \overline{\mathcal{M}}_{0,n}^{\mathbb{R}}(\text{Tor}(P_{\Delta}), \mathcal{L}_{P_{\Delta}})$$

is naturally fibered over the space of sequences $\hat{\mathbf{w}} \in \text{Men}(\Delta)$; we denote the fibers by $\overline{\mathcal{M}}_{0,n}^{\mathbb{R}}(\Delta, \hat{\mathbf{w}})$.

Let $\mathcal{M}_{0,n}^{\mathbb{R},+}(\Delta, \hat{\mathbf{w}}) \subset \mathcal{M}_{0,n}^{\mathbb{R}}(\Delta, \hat{\mathbf{w}})$ be the subset formed by the elements $[\Phi: (\mathbb{P}^1, \mathbf{p}) \rightarrow \text{Tor}(P_{\Delta})]$ such that $\Phi(\mathbb{P}_{\mathbb{R}}^1) \subset \text{Tor}_{\mathbb{R}}^+(P_{\Delta})$.² For every $[\Phi: (\mathbb{P}^1, \mathbf{p}) \rightarrow \text{Tor}(P_{\Delta})] \in \mathcal{M}_{0,n}^{\mathbb{R},+}(\Delta, \hat{\mathbf{w}})$, the curve $C = \Phi(\mathbb{P}^1)$ intersects each toric divisor $\text{Tor}(\sigma)$ in points w_i^{σ} , $1 \leq i \leq n^{\sigma}$. Hence, each element ξ

²Note that $\mathcal{M}_{0,n}^{\mathbb{R},+}(\Delta, \hat{\mathbf{w}}) \subsetneq \mathcal{M}_{0,n}^{\mathbb{R}}(\Delta, \hat{\mathbf{w}})$ when there exists a connected component of $\text{Tor}_{\mathbb{R}}(P_{\Delta})^{\times} \setminus \text{Tor}_{\mathbb{R}}(P_{\Delta})^{+}$ such that the boundary of the component coincides with $\partial \text{Tor}_{\mathbb{R}}^+(P_{\Delta})$.

of the set

$$\begin{aligned} \vec{\mathcal{M}}_{0,n}^{\mathbb{R},+}(\Delta, \widehat{\mathbf{w}}) = \{ ([\Phi: (\mathbb{P}^1, \mathbf{p}) \rightarrow \mathrm{Tor}(P_\Delta)], \mathbb{P}_+^1) \in \vec{\mathcal{M}}_{0,n}^{\mathbb{R}}(\Delta, \widehat{\mathbf{w}}) \text{ such that} \\ [\Phi: (\mathbb{P}^1, \mathbf{p}) \rightarrow \mathrm{Tor}(P_\Delta)] \in \mathcal{M}_{0,n}^{\mathbb{R},+}(\Delta, \widehat{\mathbf{w}}) \} \end{aligned}$$

possesses a quantum index $\mathrm{QI}(\xi) \in \frac{1}{2}\mathbb{Z}$.

Furthermore, if $\widehat{\mathbf{w}} \in \mathrm{Men}_0(\Delta)$, then there are well-defined *Welschinger signs*

$$W_0(\xi) = (-1)^{e_+(C)} \prod_{\substack{\sigma \in P_\Delta^1, 1 \leq i \leq n^\sigma \\ k_i^\sigma \equiv 0 \pmod{2}}} \mathrm{sign}(\xi, w_i^\sigma), \quad (3.1)$$

$$\widetilde{W}_0(\xi) = (-1)^{e(C)} \prod_{\sigma \in P_\Delta^1, 1 \leq i \leq n^\sigma} \mathrm{sign}(\xi, w_i^\sigma), \quad (3.2)$$

where $e_+(C)$, respectively $e(C)$, is the number of elliptic nodes (that is, real intersection points of a pair of smooth transversal complex-conjugate local branches) of $C = \Phi(\mathbb{P}^1)$ that arise from a real nodal equigeneric deformation of all singular points of C in the positive quadrant $\mathrm{Tor}_\mathbb{R}^+(P_\Delta)$, respectively, in the torus $\mathrm{Tor}_\mathbb{R}(P_\Delta)^\times$ (cf. [IKS18, Section 1.1]), and $\mathrm{sign}(\xi, w_i^\sigma)$ equals 1 or -1 according to whether the complex orientation of $\mathbb{R}C = \Phi(\mathbb{P}_\mathbb{R}^1)$ at w_i^σ agrees or not with the fixed orientation of $\partial \mathrm{Tor}_\mathbb{R}^+(P_\Delta)$.

For each $\kappa \in \frac{1}{2}\mathbb{Z}$ such that $|\kappa| \leq \mathcal{A}(\Delta)$, put

$$W_0^\kappa(\Delta, \widehat{\mathbf{w}}) = \sum_{\substack{\xi \in \vec{\mathcal{M}}_{0,n}^{\mathbb{R},+}(\Delta, \widehat{\mathbf{w}}) \\ \mathrm{QI}(\xi) = \kappa}} W_0(\xi), \quad \widetilde{W}_0^\kappa(\Delta, \widehat{\mathbf{w}}) = \sum_{\substack{\xi \in \vec{\mathcal{M}}_{0,n}^{\mathbb{R},+}(\Delta, \widehat{\mathbf{w}}) \\ \mathrm{QI}(\xi) = \kappa}} \widetilde{W}_0(\xi).$$

THEOREM 3.2. *For each $\kappa \in \frac{1}{2}\mathbb{Z}$ such that $|\kappa| \leq \mathcal{A}(\Delta)$, the values $W_0^\kappa(\Delta, \widehat{\mathbf{w}})$ and $\widetilde{W}_0^\kappa(\Delta, \widehat{\mathbf{w}})$ do not depend on the choice of a sequence $\widehat{\mathbf{w}} \in \mathrm{Men}_0(\Delta)$.*

We prove this theorem in Section 4. Since the values $W_0^\kappa(\Delta, \widehat{\mathbf{w}})$ and $\widetilde{W}_0^\kappa(\Delta, \widehat{\mathbf{w}})$ do not depend on the choice of a sequence $\widehat{\mathbf{w}} \in \mathrm{Men}_0(\Delta)$, we use the notation $W_0^\kappa(\Delta)$ and $\widetilde{W}_0^\kappa(\Delta)$, respectively, for these values.

Remark 3.3. (1) Theorem 3.2 slightly generalizes the statement of [Mik17, Theorem 5], because we allow tangencies of any even intersection multiplicity.

(2) One can show [Mik17, IS26] that the invariants $W_0^\kappa(\Delta)$ and $\widetilde{W}_0^\kappa(\Delta)$ vanish for all $\kappa \notin 2\mathbb{Z}$ and for all $\kappa \in 2\mathbb{Z}$ such that $\kappa \not\equiv \mathcal{A}(\Delta) \pmod{4}$. In general, the invariants $W_0^\kappa(\Delta)$ can be computed via the formula in [Blo25, Theorem 3.4].

3.3 Refined elliptic invariants

In the notation of Section 2.4, assume that

$$g = 1, \quad n_{\mathrm{in}} = 1, \quad n_\partial = \sum_{\sigma \in P_\Delta^1} n^\sigma - 1, \quad \mathbf{w}_\partial \subset \partial \mathrm{Tor}_\mathbb{R}^+(P_\Delta), \quad \mathbf{w}_{\mathrm{in}} = \{w_0\} \subset \mathcal{Q},$$

where $\mathcal{Q}$ is an open quadrant in $\mathrm{Tor}_\mathbb{R}(P_\Delta)^\times \setminus \mathrm{Tor}_\mathbb{R}^+(P_\Delta)$. As in the preceding section, for all $\sigma \in P_\Delta^1$ but one, we have $n(\sigma) = n^\sigma$, and for the remaining edge σ' , we have $n(\sigma') = n^{\sigma'} - 1$. As in Section 3.2, we define the completion $\widehat{\mathbf{w}}_\partial := \mathbf{w}_\partial \cup \{w_{n^{\sigma'}}^{\sigma'}\} \in \mathrm{Men}(\Delta)$, where $w_{n^{\sigma'}}^{\sigma'} \in \partial \mathrm{Tor}_\mathbb{R}^+(P_\Delta)$, and put $\widehat{\mathbf{w}} = \widehat{\mathbf{w}}_\partial \cup \mathbf{w}_{\mathrm{in}}$. We denote by $\mathcal{M}_{1,n}(\Delta, \widehat{\mathbf{w}}) \subset \mathcal{M}_{1,n}(\Delta, \mathbf{w})$ the subset formed by the curves passing through $\widehat{\mathbf{w}}$.

Denote by $\text{Men}_1(\Delta)$ the set of sequences $\widehat{\mathbf{w}} \in \text{Men}(\Delta) \times \mathcal{Q}$ such that the set $\mathcal{M}_{1,n}(\Delta, \widehat{\mathbf{w}})$ satisfies the conclusions of Lemma 2.4. The closure

$$\overline{\bigcup_{\widehat{\mathbf{w}} \in \text{Men}_1(\Delta)} \mathcal{M}_{1,n}^{\mathbb{R}}(\Delta, \widehat{\mathbf{w}})} \subset \overline{\mathcal{M}}_{1,n}^{\mathbb{R}}(\text{Tor}(P_\Delta), \mathcal{L}_{P_\Delta})$$

is naturally fibered over the space of sequences $\widehat{\mathbf{w}} \in \text{Men}(\Delta) \times \mathcal{Q}$; we denote the fibers by $\overline{\mathcal{M}}_{1,n}^{\mathbb{R}}(\Delta, \widehat{\mathbf{w}})$.

Consider the subset $\mathcal{M}_{1,n}^{\mathbb{R},+}(\Delta, \widehat{\mathbf{w}}) \subset \mathcal{M}_{1,n}^{\mathbb{R}}(\Delta, \widehat{\mathbf{w}})$ formed by the elements

$$[\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)], \quad \text{where } \widehat{C} \text{ is a smooth elliptic curve,}$$

defining real curves $C = \Phi(\widehat{C}) \subset \text{Tor}(P_\Delta)$ that have a 1-dimensional real branch in $\text{Tor}_{\mathbb{R}}^+(P_\Delta)$ and that intersect the toric divisor $\text{Tor}(\sigma)$ in w_i^σ , $1 \leq i \leq n^\sigma$, for all $\sigma \in P_\Delta^1$. It follows from Lemma 2.4 that for $\widehat{\mathbf{w}} \in \text{Men}_1(\Delta)$, the curve C has two 1-dimensional branches, one in $\text{Tor}_{\mathbb{R}}^+(P_\Delta)$ and the other in the closed quadrant $\overline{\mathcal{Q}}$ containing the point w_0 . This defines a unique real structure on $\widehat{C}$ whose fixed-point set $\mathbb{R}\widehat{C}$ consists of two ovals, making $\Phi: \widehat{C} \rightarrow C$ a separating real curve. The choice of a component of $\widehat{C} \setminus \mathbb{R}\widehat{C}$ defines a complex orientation on the 1-dimensional branches of $\mathbb{R}C$.

We impose a further restriction on the choice of the quadrant $\mathcal{Q} \ni w_0$ dictated by our wish to avoid specific wall-crossing events that may occur in the proof of the invariance of the count to be specified below: first, the escape of a real 1-dimensional branch of counted curves out of the positive quadrant and, second, degenerations in which the union of the toric divisors splits off.

DEFINITION 3.4. We say that the quadrant $\mathcal{Q} \subset \text{Tor}_{\mathbb{R}}(P_\Delta)^\times$ satisfies the admissible quadrant condition (AQC) if the closure $\overline{\mathcal{Q}}$ of $\mathcal{Q}$ shares at most one side with $\partial \text{Tor}_{\mathbb{R}}^+(P_\Delta)$.

Example 3.5. Let d, d_1 , and d_2 be positive integers. Condition (AQC) imposes no restriction on the choice of $\mathcal{Q}$ in the case of the triangle $\text{conv}\{(0,0), (d,0), (0,d)\}$ (the convex hull of the points $(0,0), (d,0), (0,d)$), which has the projective plane $\mathbb{P}^2$ as associated toric surface, while for the rectangle $\text{conv}\{(0,0), (d_1,0), (d_1,d_2), (0,d_2)\}$, which has $\mathbb{P}^1 \times \mathbb{P}^1$ as associated toric surface, the condition authorizes only the quadrant $\mathcal{Q}$ where both coordinates are negative.

Denote by $\overrightarrow{\mathcal{M}}_{1,n}^{\mathbb{R},+}(\Delta, \widehat{\mathbf{w}})$ the subset of $\overrightarrow{\mathcal{M}}_{1,n}^{\mathbb{R}}(\Delta, \widehat{\mathbf{w}})$ formed by the elements

$$([\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)], \widehat{C}_+)$$

such that $[\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)] \in \mathcal{M}_{1,n}^{\mathbb{R},+}(\Delta, \widehat{\mathbf{w}})$ and $\widehat{C}_+$ is one half of $\widehat{C} \setminus \mathbb{R}\widehat{C}$. According to [Mik17, Theorem 1], each element ξ of $\overrightarrow{\mathcal{M}}_{1,n}^{\mathbb{R},+}(\Delta, \widehat{\mathbf{w}})$ has quantum index $\text{QI}(\xi) \in \frac{1}{2}\mathbb{Z}$ with $|\text{QI}(\xi)| \leq \mathcal{A}(\Delta)$ (defined by formula (2.7) with integration domain $\widehat{C}_+$), and for $\mathbf{w}$ such that we have $\widehat{\mathbf{w}} \in \text{Men}_1(\Delta)$, according to the argument in [IKS18, Section 1.1], there is a well-defined *Welschinger sign*

$$W_1(\xi) = (-1)^{e_+(C)} \cdot (-1)^{h(C, \mathcal{Q})} \cdot \prod_{\substack{\sigma \in P_\Delta^1, 1 \leq i \leq n^\sigma \\ k_i^\sigma \equiv 0 \pmod{2}}} \text{sign}(\xi, w_i^\sigma), \quad (3.3)$$

where $h(C, \mathcal{Q})$ is the number of hyperbolic nodes of $C = \Phi(\widehat{C})$ in the quadrant $\mathcal{Q}$ that arise in any real nodal equigeneric deformation of all singularities of C in $\mathcal{Q}$ (here $e_0(C)$ and $\text{sign}(\xi, w_i^\sigma)$ are straightforward analogs of the corresponding ingredients in formula (3.1)).

For each $\kappa \in \frac{1}{2}\mathbb{Z}$ such that $|\kappa| \leq \mathcal{A}(\Delta)$, put

$$W_1^\kappa(\Delta, \widehat{\mathbf{w}}) = \sum_{\substack{\xi \in \overrightarrow{\mathcal{M}}_{1,n}^{\mathbb{R},+}(\Delta, \widehat{\mathbf{w}}) \\ \text{QI}(\xi) = \kappa}} W_1(\xi), \quad \widetilde{W}_1^\kappa(\Delta, \widehat{\mathbf{w}}) = \sum_{\substack{\xi \in \overrightarrow{\mathcal{M}}_{1,n}^{\mathbb{R},+}(\Delta, \widehat{\mathbf{w}}) \\ \text{QI}(\xi) = \kappa}} \widetilde{W}_1(\xi),$$

where (cf. formula (3.2))

$$\widetilde{W}_1(\xi) = (-1)^{e(C)} \prod_{\sigma \in P_\Delta^1, 1 \leq i \leq n^\sigma} \text{sign}(\xi, w_i^\sigma).$$

THEOREM 3.6. *Suppose that $n \geq 4$. Then, the values $W_1^\kappa(\Delta, \widehat{\mathbf{w}})$ and $\widetilde{W}_1^\kappa(\Delta, \widehat{\mathbf{w}})$ do not depend on the choice of a sequence $\widehat{\mathbf{w}} \in \text{Men}_1(\Delta) \times \mathcal{Q}$ subject to the following conditions: $\widehat{\mathbf{w}}_\partial \subset \partial \text{Tor}_\mathbb{R}^+(P_\Delta)$ and $w_0 \in \mathcal{Q}$, where $\mathcal{Q}$ is a fixed non-positive quadrant satisfying (AQC).*

We prove this theorem in Section 5. Since the values $W_1^\kappa(\Delta, \widehat{\mathbf{w}})$ and $\widetilde{W}_1^\kappa(\Delta, \widehat{\mathbf{w}})$ do not depend on the choice of a sequence $\widehat{\mathbf{w}} \in \text{Men}_1(\Delta) \times \mathcal{Q}$, we use the notation $W_1^\kappa(\Delta, \mathcal{Q})$ and $\widetilde{W}_1^\kappa(\Delta, \mathcal{Q})$, respectively, for these values.

Remark 3.7. (1) We expect that the statement of Theorem 3.6 holds for the remaining value $n = 3$ too. This value corresponds to $\#\Delta = 3$ and P_Δ a lattice triangle, while the counted elliptic curves in $\text{Tor}(P_\Delta)$ intersect each toric divisor at one point, where they are unibranch.

(2) One can show [IS26] that the invariants $W_1^\kappa(\Delta, \mathcal{Q})$ and $\widetilde{W}_1^\kappa(\Delta, \mathcal{Q})$ vanish for all $\kappa \notin 2\mathbb{Z}$ and for all $\kappa \in 2\mathbb{Z}$ such that $\kappa \not\equiv \mathcal{A}(\Delta) \pmod{4}$.

3.4 Refined invariants of genus 2

For genus $g = 2$, we restrict our attention to the case $\text{Tor}(P_\Delta) \simeq \mathbb{P}^2$. That is, Δ consists of positive multiples of the vectors $(1, 1), (-1, 0), (0, -1)$ and defines plane curves of an even degree $2d$. The boundary of the positive quadrant $(\mathbb{P}_\mathbb{R}^2)^+$ consists of the three segments D_s^+ , $s = 0, 1, 2$, where $D_s^+ = D_s \cap \text{Cl}(\mathbb{P}_\mathbb{R}^2)^+$, $D_s = \{x_s = 0\} \subset \mathbb{P}^2$, corresponding to the sides

$$[(0, d), (d, 0)], [(0, 0), (0, d)], [(0, 0), (d, 0)] \in P_\Delta^1,$$

respectively. Introduce the completions $\widehat{\mathbf{w}}_\partial = \mathbf{w}_\partial \cup \{w_{n_0}^{(0)}\} \in \text{Men}(\Delta)$ and $\widehat{\mathbf{w}} = \widehat{\mathbf{w}}_\partial \cup \mathbf{w}_{\text{in}}$, where $\mathbf{w}_{\text{in}} = \{w_1, w_2\}$,

$$w_1 \in (\mathbb{P}_\mathbb{R}^2)^{(-,+)} = \left\{ (x_0 : x_1 : x_2) \in \mathbb{P}_\mathbb{R}^2 \mid \frac{x_1}{x_0} < 0, \frac{x_2}{x_0} > 0 \right\},$$

$$w_2 \in (\mathbb{P}_\mathbb{R}^2)^{(+,-)} = \left\{ (x_0 : x_1 : x_2) \in \mathbb{P}_\mathbb{R}^2 \mid \frac{x_1}{x_0} > 0, \frac{x_2}{x_0} < 0 \right\}.$$

Denote by $\text{Men}_2(\Delta)$ the set of sequences $\widehat{\mathbf{w}} \in \text{Men}(\Delta) \times (\mathbb{P}_\mathbb{R}^2)^{(-,+)} \times (\mathbb{P}_\mathbb{R}^2)^{(+,-)}$ such that the set $\mathcal{M}_{2,n}(\Delta, \widehat{\mathbf{w}})$ satisfies the conclusions of Lemma 2.4. The closure

$$\overline{\bigcup_{\widehat{\mathbf{w}} \in \text{Men}_2(\Delta)} \mathcal{M}_{2,n}^\mathbb{R}(\Delta, \widehat{\mathbf{w}})} \subset \overline{\mathcal{M}}_{2,n}^\mathbb{R}(\text{Tor}(P_\Delta), \mathcal{L}_{P_\Delta})$$

is naturally fibered over the space of sequences $\widehat{\mathbf{w}} \in \text{Men}(\Delta) \times (\mathbb{P}_\mathbb{R}^2)^{(-,+)} \times (\mathbb{P}_\mathbb{R}^2)^{(+,-)}$; we denote the fibers by $\overline{\mathcal{M}}_{2,n}^\mathbb{R}(\Delta, \widehat{\mathbf{w}})$.

Denote by $\mathcal{M}_{2,n}^\mathbb{R}(\Delta, \widehat{\mathbf{w}})$ the space of equivariant isomorphism classes of real stable maps of curves of genus 2. Due to the choice of the points of $\widehat{\mathbf{w}}$, each element is of the form $\xi =$

$[\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \mathbb{P}^2] \in \mathcal{M}_{2,n}^{\mathbb{R}}(\Delta, \widehat{\mathbf{w}})$, where $\widehat{C}$ stands for a real Riemann surface of genus 2 and ξ defines an M -curve; that is, $\mathbb{R}\widehat{C}$ consists of three ovals that are mapped to the quadrants $\text{Cl}(\mathbb{P}_{\mathbb{R}}^2)^+$, $\text{Cl}(\mathbb{P}_{\mathbb{R}}^2)^{(-,+)}$, $\text{Cl}(\mathbb{P}_{\mathbb{R}}^2)^{(+,-)}$, respectively. The complement $\widehat{C} \setminus \mathbb{R}\widehat{C}$ consists of two components, and the choice of one of them, $\widehat{C}^+$, defines complex orientations on $\mathbb{R}\widehat{C}$ and on $\Phi(\mathbb{R}\widehat{C})$. Denote by $\vec{\mathcal{M}}_{2,n}^{\mathbb{R}}(\Delta, \widehat{\mathbf{w}})$ the set of elements of $\mathcal{M}_{2,n}^{\mathbb{R}}(\Delta, \mathbf{w})$ equipped with a complex orientation. Similarly to formula (3.3), for each element $\xi = [\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \mathbb{P}^2] \in \vec{\mathcal{M}}_{2,n}^{\mathbb{R}}(\Delta, \widehat{\mathbf{w}})$, we define

$$W_2(\xi) = (-1)^{e_+(C)} \cdot (-1)^{h_-(C)} \cdot \prod_{\sigma \in P_{\Delta}^1} \prod_{k_i^{\sigma} \equiv 0 \pmod{2}} \text{sign}(\xi, w_i^{\sigma}),$$

where $C = \Phi(\widehat{C})$ and $h_-(C)$ is the number of hyperbolic nodes that appear in real local equi-generic nodal deformations of singular points of C in $(\mathbb{P}_{\mathbb{R}}^2)^{(-,+)} \cup (\mathbb{P}_{\mathbb{R}}^2)^{(+,-)}$. Put

$$W_2^{\kappa}(\Delta, \widehat{\mathbf{w}}) = \sum_{\substack{\xi \in \vec{\mathcal{M}}_{2,n}^{\mathbb{R}}(\Delta, \widehat{\mathbf{w}}) \\ \text{QI}(\xi) = \kappa}} W_2(\xi), \quad \widetilde{W}_2^{\kappa}(\Delta, \widehat{\mathbf{w}}) = \sum_{\substack{\xi \in \vec{\mathcal{M}}_{2,n}^{\mathbb{R}}(\Delta, \widehat{\mathbf{w}}) \\ \text{QI}(\xi) = \kappa}} \widetilde{W}_2(\xi),$$

where (cf. formula (3.2))

$$\widetilde{W}_2(\xi) = (-1)^{e(C)} \prod_{\sigma \in P_{\Delta}^1, 1 \leq i \leq n^{\sigma}} \text{sign}(\xi, w_i^{\sigma}).$$

THEOREM 3.8. *Suppose that $n \geq 5$. For any toric degree Δ as above and any $\kappa \in \frac{1}{2}\mathbb{Z}$ such that $|\kappa| \leq 2d^2$, the values $W_2^{\kappa}(\Delta, \widehat{\mathbf{w}})$ and $\widetilde{W}_2^{\kappa}(\Delta, \widehat{\mathbf{w}})$ do not depend on the choice of a sequence $\widehat{\mathbf{w}} \in \text{Men}_2(\Delta) \times (\mathbb{P}_{\mathbb{R}}^2)^{(-,+)} \times (\mathbb{P}_{\mathbb{R}}^2)^{(+,-)}$.*

We prove this theorem in Section 6. Since the values $W_2^{\kappa}(\Delta, \widehat{\mathbf{w}})$ and $\widetilde{W}_2^{\kappa}(\Delta, \widehat{\mathbf{w}})$ do not depend on the choice of a sequence $\widehat{\mathbf{w}} \in \text{Men}_2(\Delta) \times (\mathbb{P}_{\mathbb{R}}^2)^{(-,+)} \times (\mathbb{P}_{\mathbb{R}}^2)^{(+,-)}$, we use the notation $W_2^{\kappa}(\Delta)$ and $\widetilde{W}_2^{\kappa}(\Delta)$, respectively, for these values.

Remark 3.9. (1) We expect that the statement of Theorem 3.8 holds for the remaining value $n = 4$ too. This value corresponds to $\#\Delta = 3$ and T_{Δ} a lattice triangle, while the counted curves of genus 2 in $\text{Tor}(P_{\Delta})$ intersect each toric divisor at one point, where they are unibranch.

(2) One can show (see [IS26]) that the invariants $W_2^{\kappa}(\Delta)$ and $\widetilde{W}_2^{\kappa}(\Delta)$ vanish for all $\kappa \notin 2\mathbb{Z}$ and for all $\kappa \in 2\mathbb{Z}$ such that $\kappa \not\equiv \mathcal{A}(\Delta) \pmod{4}$.³

4. Invariance in genus 0: Proof of Theorem 3.2

Note that $\text{Men}_0(\Delta)$ is an open dense semialgebraic subset of $\text{Men}_{\mathbb{R}}^0(\Delta)$. We call its connected components the *chambers*. The complement to this subset is the union of finitely many semialgebraic strata of codimension at least 1. We call the strata of codimension 1 the *walls*.

Now we pick two sequences $\widehat{\mathbf{w}}(0)$ and $\widehat{\mathbf{w}}(1)$ satisfying the hypotheses of Theorem 3.2. We can join these sequences by a generic path $\{\widehat{\mathbf{w}}(t)\}_{0 \leq t \leq 1} \subset \text{Men}_{\mathbb{R}}^0(\Delta)$ that avoids the loci of codimension greater than 1. To prove Theorem 3.2, we need to verify the constancy of $W_0^{\kappa}(\Delta, \widehat{\mathbf{w}}(t))$ and $\widetilde{W}_0^{\kappa}(\Delta, \widehat{\mathbf{w}}(t))$ inside any given chamber and in all possible wall-crossing events, for all κ .

³Item (2) was added in proof.

4.1 Invariance inside chambers

Here, we verify that $W_0^\kappa(\Delta, \widehat{\mathbf{w}}(t))$ does not change along intervals $t' < t < t''$ for which $\widehat{\mathbf{w}}(t)$ lies in a given chamber. To this end, we show that the projection of $\mathcal{M}_{0,n}^{\mathbb{R},+}(\Delta, \widehat{\mathbf{w}}(t))_{t' < t < t''}$ to the interval (t', t'') does not have critical points. In such a case, the family $\mathcal{M}_{0,n}^{\mathbb{R},+}(\Delta, \widehat{\mathbf{w}}(t))_{t' < t < t''}$ is the union of intervals trivially covering (t', t'') . Hence it lifts to a trivial family of complex oriented curves, and the quantum index persists along each of the components of that family.

It is convenient to look at the elements $\widehat{\mathbf{w}} \in \text{Men}_\mathbb{R}^\rho(\Delta)$ as follows: Pick some pair (σ_0, i_0) , where $\sigma_0 \in P_\Delta^1$ and $1 \leq i_0 \leq n^{\sigma_0}$, and think of $w_{i_0}^{\sigma_0}$ as a mobile point, whose position is determined by the other points $w_i^\sigma \in \widehat{\mathbf{w}}$ via the Menelaus condition. Put $C = \Phi_*(\mathbb{P}^1)$. Observe that the tangent space at C to the family of curves $C' \in |\mathcal{L}_{P_\Delta}|$ intersecting $\text{Tor}(\sigma)$ at w_i^σ (where $(\sigma, i) \neq (\sigma_0, i_0)$) with multiplicity at least $2k_i^\sigma$ can be identified with the linear system $\{C' \in |\mathcal{L}_{P_\Delta}| \mid (C' \cdot C)_{w_i^\sigma} \geq 2k_i^\sigma\}$. In turn, the tangent space at C to the family of curves $C' \in |\mathcal{L}_{P_\Delta}|$ intersecting $\text{Tor}(\sigma_0)$ at a point close to $w_{i_0}^{\sigma_0}$ with multiplicity at least $2k_{i_0}^{\sigma_0}$ can be identified with the linear system $\{C' \in |\mathcal{L}_{P_\Delta}| \mid (C' \cdot C)_{w_{i_0}^{\sigma_0}} \geq 2k_{i_0}^{\sigma_0} - 1\}$. To ensure the behavior of $\mathcal{M}_{0,n}^{\mathbb{R},+}(\Delta, \widehat{\mathbf{w}}(t))_{t' < t < t''}$ as indicated in the previous paragraph, we have to show the transversality of the intersection of the tangent space at C to the family of rational curves and the tangent spaces to families of curves matching the above tangency conditions. The required transversality is reformulated in cohomology language (and proved) in the following lemma.

LEMMA 4.1. *Let $\xi = [\Phi: (\mathbb{P}^1, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)] \in \mathcal{M}_{0,n}^{\mathbb{R},+}(\Delta, \widehat{\mathbf{w}})$ with $\widehat{\mathbf{w}} \in \text{Men}_0(\Delta)$. Then,*

$$\begin{aligned} H^0\left(\mathbb{P}^1, \mathcal{N}_\Phi\left(-\sum_{\sigma \neq \sigma_0, i \neq i_0} 2k_i^\sigma p_i^\sigma - (2k_{i_0}^{\sigma_0} - 1)p_{i_0}^{\sigma_0}\right)\right) \\ = H^1\left(\mathbb{P}^1, \mathcal{N}_\Phi\left(-\sum_{\sigma \neq \sigma_0, i \neq i_0} 2k_i^\sigma p_i^\sigma - (2k_{i_0}^{\sigma_0} - 1)p_{i_0}^{\sigma_0}\right)\right) = 0. \end{aligned}$$

Proof. We have

$$\deg \mathcal{N}_\Phi\left(-\sum_{\sigma \neq \sigma_0, i \neq i_0} 2k_i^\sigma p_i^\sigma - (2k_{i_0}^{\sigma_0} - 1)p_{i_0}^{\sigma_0}\right) \stackrel{(2.2)}{=} -1 > -2.$$

Thus, the claim of the lemma follows from the Riemann–Roch theorem. $\square$

4.2 Walls in the space of constraints

Here, we characterize elements $\xi \in \overline{\mathcal{M}}_{0,n}^{\mathbb{R},+}(\Delta, \widehat{\mathbf{w}}(t))_{0 \leq t \leq 1}$ such that $\widehat{\mathbf{w}}(t)$ is a generic member of a stratum of dimension $n - 1$ in $\text{Men}_\mathbb{R}^\rho(\Delta) \setminus \text{Men}_0(\Delta)$ (that is, of codimension 1 in $\text{Men}_\mathbb{R}^\rho(\Delta)$). In view of the results of Section 4.1, we can ignore the walls along which $\overline{\mathcal{M}}_{0,n}^{\mathbb{R}}(\Delta, \widehat{\mathbf{w}}) = \mathcal{M}_{0,n}^{\mathbb{R}}(\Delta, \widehat{\mathbf{w}})$. The following lemma classifies the other walls.

LEMMA 4.2. (1) *The following elements $\xi = [\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)]$ cannot occur in the space $\overline{\mathcal{M}}_{0,n}^{\mathbb{R},+}(\Delta, \widehat{\mathbf{w}}(t))_{0 \leq t \leq 1} \setminus \mathcal{M}_{0,n}^{\mathbb{R},+}(\Delta, \widehat{\mathbf{w}}(t))_{0 \leq t \leq 1}$:*

- (1i) $\widehat{C}$ is a reducible connected curve of arithmetic genus 0 with a component mapped onto a toric divisor,
- (1ii) $\widehat{C}$ is a reducible connected curve of arithmetic genus 0 with at least three irreducible components.

(2) *If $\xi = [\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)] \in \overline{\mathcal{M}}_{0,n}(\Delta, \widehat{\mathbf{w}}(t^*))$, where $\widehat{\mathbf{w}}(t^*)$ is a generic element in an $(n - 1)$ -dimensional stratum in $\text{Men}_\mathbb{R}^\rho(\Delta) \setminus \text{Men}_0(\Delta)$, then ξ is of one of the following types:*

- (2i) The element $\widehat{\mathbf{w}}(t^*)$ consists of $n+1$ distinct points, $\widehat{C} \simeq \mathbb{P}^1$, and the map Φ is birational onto its image and satisfies the following: Either it is smooth at $\widehat{\mathbf{w}}(t^*)$ but has singular branches in $\text{Tor}(P_\Delta)^\times$; or C is immersed in $\text{Tor}(P_\Delta)^\times$, and at some points $w_i^\sigma \in \widehat{\mathbf{w}}(t^*)$, it has a singularity of type A_{2m} , $m \geq k_i^\sigma$, whereas C is smooth at the remaining points of $\widehat{\mathbf{w}}(t^*)$.
- (2ii) Two points of the sequence $\widehat{\mathbf{w}}(t^*)$ coincide ($w_i^\sigma(t^*) = w_j^\sigma(t^*)$ for some $\sigma \in P_\Delta^1$ and $i \neq j$), $\widehat{C} \simeq \mathbb{P}^1$, and the map Φ is an immersion such that the point $w_i^\sigma(t^*) = w_j^\sigma(t^*)$ is a center of one or two smooth branches.
- (2iii) The element $\widehat{\mathbf{w}}(t^*)$ consists of $n+1$ distinct points, $\widehat{C} = \widehat{C}_1 \cup \widehat{C}_2$, where $\widehat{C}_1 \simeq \widehat{C}_2 \simeq \mathbb{P}^1$ and $\widehat{C}_1 \cap \widehat{C}_2$ is one point p , each map $\Phi: \widehat{C}_i \rightarrow \text{Tor}(P_\Delta)$, $i = 1, 2$, is either an immersion that is smooth along $\text{Tor}(\partial P_\Delta)$, or a multiple covering of a line intersecting only two toric divisors, while these divisors correspond to opposite parallel sides of P_Δ and the intersection points with these divisors are ramification points of the covering with the maximal ramification index. Furthermore, either $\Phi(p) \in \text{Tor}(P_\Delta)^\times$, and then all intersection points of the curves $C_1 = \Phi(\widehat{C}_1)$, $C_2 = \Phi(\widehat{C}_2)$ are ordinary nodes; or $w = \Phi(p) \in \mathbf{w}(t^*)$, and at least one of the maps $\Phi: \widehat{C}_i \rightarrow \text{Tor}(P_\Delta)$ is birational onto its image, and the curves $C_1 = \Phi(\widehat{C}_1)$ and $C_2 = \Phi(\widehat{C}_2)$ intersect $\text{Tor}(\sigma)$ at w with even multiplicities and do not have common point in $\widehat{\mathbf{w}}(t^*) \setminus \{w\}$.
- (2iv) The polygon P_Δ is a triangle, $n = 3$, two points of the sequence $\widehat{\mathbf{w}}(t^*)$ coincide, the curve $C = \Phi(\mathbb{P}^1)$ is rational and smooth at $\widehat{\mathbf{w}}(t^*)$, and $\Phi: \mathbb{P}^1 \rightarrow C$ is a double covering ramified at two distinct points of $\widehat{\mathbf{w}}(t^*)$.

Proof. (1i) A toric divisor $\text{Tor}(\sigma)$ cannot split off alone, since, otherwise, in the deformation along the path $\{\widehat{\mathbf{w}}(t)\}_{0 \leq t \leq 1}$, the intersection points with the neighboring toric divisors would yield points on these toric divisors at distance less than ρ from the corners of $\text{Tor}_\mathbb{R}^+(P_\Delta)$. For the same reason, we obtain that only all toric divisors together may split off, while their intersection points must smooth out in the deformation. However, this contradicts the rationality of the considered curves.

(1ii) First, note that $\Phi(\widehat{C})$ intersects $\text{Tor}(\partial P_\Delta)$ only at $\widehat{\mathbf{w}}(t^*)$. Furthermore, if the images of two irreducible components $\widehat{C}_1$ and $\widehat{C}_2$ of $\widehat{C}$ contain the same point $w_i^\sigma(t^*)$ that is different from any other point $w_j^\sigma(t^*)$, $j \neq i$, then $w_i^\sigma(t^*) = \Phi(\widehat{C}_1 \cap \widehat{C}_2)$ (cf. Lemma 2.4). Due to the genus restriction, it follows that the sequence $\widehat{\mathbf{w}}(t^*)$ satisfies at least three Menelaus conditions, which cuts off a polytope $\text{Men}_\mathbb{R}^\rho(\Delta)$ of dimension at most $n-2$, contrary to the dimension $n-1$ assumption.

Let ξ satisfy the hypotheses of item (2).

(2i) Let $\widehat{C} \simeq \mathbb{P}^1$, $\widehat{\mathbf{w}}(t^*)$ consist of $n+1$ distinct points, and let Φ take $\widehat{C}$ birationally onto $C = \Phi(\widehat{C}) \subset \text{Tor}(P_\Delta)$.

First, we consider the minimal value $n = 2$. In such a case, P_Δ is a triangle, and, for each $\widehat{\mathbf{w}} \in \text{Men}_\mathbb{R}^\rho(\Delta)$, there exists a unique curve $C = \Phi(\mathbb{P}^1)$ (where $\xi = [\Phi: (\mathbb{P}^1, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)] \in \mathcal{M}_{0,n}^{\mathbb{R},+}(\Delta, \widehat{\mathbf{w}})$) that is smooth along the toric divisors and is nodal, while the number of elliptic nodes in $\text{Tor}_\mathbb{R}^+(P_\Delta)$ does not depend on the choice of $\widehat{\mathbf{w}} \in \text{Men}_\mathbb{R}^\rho(\Delta)$; see [Shu06, Lemma 3.5].

Suppose that $n \geq 3$. Note that ξ must be unibranch at each point of $\widehat{\mathbf{w}}(t^*)$. Since ξ is a generic element of an $(n-1)$ -dimensional family in $\mathcal{M}_{0,n}(\Delta)$, fixing the position of a point $w_i^\sigma \in \widehat{\mathbf{w}}(t^*)$, we obtain a family of dimension at least $n-2 \geq 1$. Applying [IKS18, inequality (5) in Lemma 2.1],

we get

$$\begin{aligned} -CK_{\text{Tor}(P_\Delta)} &\geq 2 + (-CK_{\text{Tor}(P_\Delta)} - n) + (n - 3) + \sum_B (\text{mult } Q - 1) \\ &= -CK_{\text{Tor}(P_\Delta)} - 1 + \sum_B (\text{mult } Q - 1) \implies \sum_B (\text{mult } Q - 1) \leq 1, \end{aligned} \quad (4.1)$$

where B is the set of all singular branches of ξ in $\text{Tor}(P_\Delta)^\times \cup \{w_i^\sigma\}$. In particular, if C is singular at some point w_i^σ , the singularity is of type A_{2m} . The requirements of item (2i) are fulfilled.

(2ii) Suppose that $\widehat{C} \simeq \mathbb{P}^1$ is birationally taken by Φ onto its image in $\text{Tor}(P_\Delta)$, and some of the points of the sequence $\widehat{\mathbf{w}}(t^*)$ coincide. The latter is possible only if $n \geq 3$. For dimension reasons, the number $\#(\widehat{\mathbf{w}}(t^*))$ of points in $\widehat{\mathbf{w}}(t^*)$ is equal to n (that is, $w_i^\sigma(t^*) = w_j^\sigma(t^*) = w$ for some $\sigma \subset \partial P_\Delta$ and $i \neq j$), and all these $n \geq 3$ points are in general position on $\text{Tor}(\partial P_\Delta)$ subject to the Menelaus relation (2.1). If $C = \Phi(\widehat{C})$ is unibranch at each point of $\widehat{\mathbf{w}}(t^*)$, then, by Lemma 2.4, the curve C is immersed and smooth along the toric divisors. If C is not unibranch at w , then, by Lemma 2.4, it has two local branches at w ; furthermore, C must be unibranch at each point $w_{i'}^{\sigma'}(t^*) \neq w$. So, assume that C has two local branches at w .

If $n - 1 = 2$, then P_Δ is a lattice triangle. Using affine automorphisms of $\mathbb{Z}^2$, we can take $P_\Delta = \text{conv}\{(m_1, 0), (m_1 + k_1 + k_2, 0), (0, m_2)\}$ with m_1, m_2, k_1, k_2 even and write a parameterization of C in the form

$$x = a\tau^{m_2}, \quad y = b\tau^{m_1}(\tau - 1)^{k_1}(\tau - \theta)^{k_2}, \quad a, b, \theta \in \mathbb{R}^*.$$

Since $x(1) = x(\theta)$, we obtain $\theta = -1$. Note that $k_1 \neq k_2$, since otherwise, x and y would be functions of τ^2 , that is, $\Phi: \widehat{C} \rightarrow C$ would be a double covering against the initial assumption. The relations

$$x = a(1 + m_2(\tau - 1) + O((\tau - 1)^2)), \quad x = a(1 - m_2(\tau + 1) + O((\tau + 1)^2))$$

show that the local branches of C at w are smooth. Furthermore, we have

$$x = a\tau^{m_2}, \quad y = b\tau^{m_1} + b(k_2 - k_1)\tau^{m_1+1} + O(\tau^{m_1+2});$$

hence, C is smooth at the two remaining points of $\widehat{\mathbf{w}}$. In addition, C is immersed everywhere, since $dx/d\tau \neq 0$ for $\tau \neq 0$.

Let $n - 1 \geq 3$. We now show that C is immersed. Fixing the position of w and the position of one more point $w' \in \mathbf{w}(t^*) \setminus \{w\}$, we obtain a family of dimension at least $n - 3 \geq 1$; thus, [IKS18, inequality (5) in Lemma 2.1] applies:

$$\begin{aligned} -CK_{\text{Tor}(P_\Delta)} &\geq 2 + (-CK_{\text{Tor}(P_\Delta)} - n + 2) + \sum_B (\text{mult } Q - 1) + (n - 4) \\ &= -CK_{\text{Tor}(P_\Delta)} + \sum_B (\text{mult } Q - 1), \end{aligned}$$

where B is the set of all singular local branches of C in $\text{Tor}(P_\Delta)^\times \cup \{w, w'\}$, and hence C is immersed.

(2iii) Consider the case of $\widehat{C} = \widehat{C}_1 \cup \widehat{C}_2$, where $\widehat{C}_1 \simeq \widehat{C}_2 \simeq \mathbb{P}^1$ and $\widehat{C}_1 \cap \widehat{C}_2 = \{p\}$ is one point. For dimension reasons, the points of $\widehat{\mathbf{w}}(t^*)$ are in general position subject to exactly two Menelaus conditions (induced by the components of $\widehat{C}$), and they are all distinct. Each of the curves C_1 and C_2 , which pass through at least three points of $\widehat{\mathbf{w}}(t^*)$, is immersed and smooth along the toric divisors and has even intersection multiplicity with $\text{Tor}(\partial P_\Delta)$ at any point, by Lemma 2.4. Indeed, suppose that C_1 or C_2 passes through exactly two points of $\widehat{\mathbf{w}}(t^*)$; then

it is a multiple covering of a line as described in item (2iv). Let $\Phi(p) \in \text{Tor}(P_\Delta)^\times$. Then, by Lemma 2.4, the curves C_1 and C_2 do not share points in $\widehat{\mathbf{w}}(t^*)$, and we claim that all their intersection points are ordinary nodes. Due to the genericity assumptions for $\widehat{\mathbf{w}}(t^*)$, we only have to study the case where both C_1 and C_2 are immersed. If we freely move the points of $C_1 \cap \widehat{\mathbf{w}}(t^*)$ so that the corresponding Menelaus condition induced by C_1 still holds and fix the curve C_2 , then the persisting tangency condition of a germ $\Phi: (\widehat{C}_1, q) \rightarrow \text{Tor}(P_\Delta)$ to the curve C_2 would yield that the tangent space to the considered family of curves in the linear system $|C_1|$ would be contained in $H^0(\widehat{C}_1, \mathcal{O}_{\widehat{C}_1}(\mathbf{d}_1))$, where

$$\begin{aligned} \deg \mathbf{d}_1 &= C_1^2 - (C_1^2 + C_1 K_{\text{Tor}(P_\Delta)} + 2) - (-C_1 K_{\text{Tor}(P_\Delta)} - \#(\mathbf{w}(t^*) \cap C_1)) - 1 \\ &= \#(\widehat{\mathbf{w}}(t^*) \cap C_1) - 3 > -2, \end{aligned}$$

and hence, by the Riemann–Roch theorem,

$$h^0(\widehat{C}_1, \mathcal{O}_{\widehat{C}_1}(\mathbf{d}_1)) = \#(\widehat{\mathbf{w}}(t^*) \cap C_1) - 3 + 1 = \#(\widehat{\mathbf{w}}(t^*) \cap C_1) - 2 < \#(\widehat{\mathbf{w}}(t^*) \cap C_1) - 1,$$

which gives a contradiction. If $\Phi(p) = w_i^\sigma(t^*) \in \widehat{\mathbf{w}}(t^*)$, then the intersection multiplicities of the germs $\Phi(\widehat{C}_1, p)$ and $\Phi(\widehat{C}_2, p)$ with $\text{Tor}(\sigma)$ are even. Indeed, otherwise, C_1 and C_2 would have at least one additional local branch centered on some $\text{Tor}(\sigma')$ for $\sigma' \neq \sigma$ and intersecting $\text{Tor}(\sigma')$ with an odd multiplicity, which contradicts our assumptions.

(2iv) Suppose that $\Phi: \mathbb{P}^1 \rightarrow \text{Tor}(P_\Delta)$ is an s -multiple covering onto its image, with $s \geq 2$. For a generic $t \in [0, 1]$, the set $\widehat{\mathbf{w}}(t)$ consists of $n + 1$ distinct points; hence the sequence $\widehat{\mathbf{w}}(t^*)$ contains at least n distinct points. If $\widehat{\mathbf{w}}(t^*)$ consists of $n + 1$ distinct points, then at each point of $\widehat{\mathbf{w}}(t^*)$, we have ramification index s . Thus, by the Riemann–Hurwitz formula, we get

$$2 \leq 2s - (s - 1)(n + 1), \quad (4.2)$$

which gives $n \leq 1$; the latter inequality contradicts the fact that $n + 1$ is bounded from below by the number of sides of P_Δ . If the sequence $\widehat{\mathbf{w}}(t^*)$ contains only n distinct points, we have $n \geq 3$. Note that Φ has an irreducible preimage in at least $n - 1$ points of $\widehat{\mathbf{w}}(t^*)$ with ramification index s , and hence $n = 3$ (cf. (4.2)). It also follows that there are no other ramifications and that $s = 2$, since the remaining point of $\widehat{\mathbf{w}}(t^*)$, where Φ is not ramified, lifts to at most two points in $\mathbb{P}^1$. Thus, we are left with the case described in item (2iv) of the lemma. $\square$

4.3 Wall-crossing

We complete the proof of Theorem 3.2 with the following lemma.

LEMMA 4.3. *Let $\{\widehat{\mathbf{w}}(t)\}_{0 \leq t \leq 1}$ be a generic path in $\text{Men}_{\mathbb{R}}^o(\Delta)$, and let $t^* \in (0, 1)$ be such that $\overline{\mathcal{M}}_{0,n}^{\mathbb{R}}(\Delta, \widehat{\mathbf{w}}(t^*))$ contains an element ξ as described in one of the items of Lemma 4.2(2). Then, for each $\kappa \in \frac{1}{2}\mathbb{Z}$ such that $|\kappa| \leq \mathcal{A}(\Delta)$, the numbers $W_0^\kappa(t) = W_0^\kappa(\Delta, \widehat{\mathbf{w}}(t))$ and $\widetilde{W}_0^\kappa(t) = \widetilde{W}_0^\kappa(\Delta, \widehat{\mathbf{w}}(t))$ do not change as t varies in a neighborhood of t^* .*

Proof. We always can assume that in a neighborhood of t^* , the path $\{\widehat{\mathbf{w}}(t)\}_{0 \leq t \leq 1}$ is defined by fixing the position of some $n - 1$ points of $\widehat{\mathbf{w}}(t^*)$, while the other two points remain mobile (the choice of the two mobile points may depend on the considered degeneration). We also notice that, in the degenerations as in Lemma 4.2(2i, 2ii), the source curve and its real structure remain fixed, which implies that the quantum index is constant in these wall-crossings. Except for the case of Lemma 4.2(2iii) describing reducible degenerations, we work with families of curves that are trivially covered by families of complex oriented curves, so the quantum index persists along each component of the family of oriented curves.

The proof is separated into parts (1)–(6) in accordance with the classification of walls in Lemma 4.2(2) and specific degenerations occurring on these walls.

(1) Suppose that $\xi \in \overline{\mathcal{M}}_{0,n}^{\mathbb{R},+}(\Delta, \widehat{\mathbf{w}}(t^*))$ is as in Lemma 4.2(2i) and ξ has a singular local branch centered in $\text{Tor}(P_\Delta)^\times$. Then the constancy of $W_0^\kappa(t)$, $|t - t^*| < \varepsilon$, follows from [IKS17, Lemmas 13 and 15], provided that we establish the following transversality condition. Choose a sufficiently large integer s . For each point $z \in \text{Sing}(C) \cap \text{Tor}(P_\Delta)^\times$, we set $I_z = I^{\text{cond}}(C, z)/\mathfrak{m}_{C,z}^s \subset \mathcal{O}_{C,z}/\mathfrak{m}_z^s$; this is the quotient of the conductor ideal by the power of the maximal ideal, which can be viewed as the tangent cone to the stratum parameterizing equigeneric deformations (see [DH88, Theorem 4.15]). The condition we need reads

$$h^1(C, \mathcal{J}_{Z/C} \otimes_{\mathcal{O}_{\widehat{C}}} \text{in}^* \mathcal{L}_{P_\Delta}) = 0,$$

where $\mathcal{J}_{Z/C}$ is the ideal sheaf of the 0-dimensional scheme $Z \subset C$ defined at each point $z \in \text{Sing}(C)$ by the conductor ideal $I^{\text{cond}}(C, z)$ and at each point $w = w_i^\sigma \in \widehat{\mathbf{w}}(t^*)$ by the ideal

$$I_w = \{\varphi \in \mathcal{O}_{C,w} \mid (\varphi \cdot C)_w \geq 2k_i^\sigma - 1\} \subset \mathcal{O}_{C,w}.$$

The above h^1 -vanishing lifts to $\mathbb{P}^1$ in the form

$$h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\mathbf{d})) = 0, \quad (4.3)$$

where

$$\begin{aligned} \deg \mathbf{d} &= C^2 - (C^2 + CK_{\text{Tor}(P_\Delta)} + 2) - (CK_{\text{Tor}(P_\Delta)} - n - 1) \\ &= n - 1 > -2 = 2g - 2, \end{aligned} \quad (4.4)$$

and hence the required h^1 -vanishing comes from the Riemann–Roch theorem.

(2) Suppose that $\xi \in \overline{\mathcal{M}}_{0,n}^{\mathbb{R},+}(\Delta, \widehat{\mathbf{w}}(t^*))$ is as in Lemma 4.2(2i) with singularities of type A_{2m} at some points $w_i^\sigma \in \widehat{\mathbf{w}}(t^*)$, $m \geq k_i^\sigma$.

The germ at ξ of $\mathcal{M}_{0,n}(\Delta)$ (defined by (2.3)) can be identified with the germ at 0 of $H^0(C, \text{in}^* \mathcal{L}_{P_\Delta})$, which in turn can be naturally embedded into the product

$$\prod_{\sigma,i} \text{Vers}(C, w_i^\sigma) \times \prod_{z \in \text{Sing}(C) \cap \text{Tor}(P_\Delta)^\times} \text{Vers}(C, z).$$

We claim that the image of the germ at 0 of $H^0(C, \text{in}^* \mathcal{L}_{P_\Delta})$ in this product intersects transversally with $\prod_{\sigma,i} \text{EG}_{\text{Tor}(\sigma)}(C, w_i^\sigma) \times \prod_{z \in \text{Sing}(C) \cap \text{Tor}(P_\Delta)^\times} \text{EG}(C, z)$. Hence, this intersection $\mathcal{V}$ is smooth of the expected dimension n since the latter product is smooth (see Section 2.3, items (c) and (d)). The required transversality amounts to the validation of the relation

$$h^1(C, \mathcal{J}_{Z_1/C} \otimes_{\mathcal{O}_C} \text{in}^* \mathcal{L}_{P_\Delta}) = 0, \quad (4.5)$$

where the 0-dimensional scheme $Z_1 \subset C$ is supported at $\widehat{\mathbf{w}}(t^*) \cup \text{Sing}(C)$ and defined by the ideals $I^{\text{cond}}(C, z)$ at all $z \in \text{Sing}(C) \cap \text{Tor}(P_\Delta)^\times$ and by the ideals

$$\{\varphi \in \mathcal{O}_{C,w_i^\sigma} \mid \text{ord } \varphi|_{C,w_i^\sigma} \geq 2k_i^\sigma - 1 + 2\delta(C, w_i^\sigma)\} \quad \text{for all } w_i^\sigma \in \widehat{\mathbf{w}}(t^*)$$

(cf. Section 2.3, items (c) and (d)). Relation (4.5) is equivalent to

$$h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\mathbf{d}_1)) = 0,$$

where $\deg \mathbf{d}_1$ is given by (4.4), and hence it holds by the Riemann–Roch theorem.

Note that ξ is chosen as a generic member of the wall; that is, it represents a smooth point of the wall. Let us show that the germ $\mathcal{V}_1$ of the wall at ξ has codimension 1 in $\mathcal{V}$. Pick a point $w_i^\sigma \in \widehat{\mathbf{w}}(t^*)$, where $\Phi(\widehat{C})$ has a singularity of type A_{2m} with $m \geq k_i^\sigma$, and consider

the intersection $\mathcal{V}_2$ of the image of the germ at 0 of $H^0(C, \text{in}^* \mathcal{L}_{P_\Delta})$ in $\prod_{\sigma', i'} \text{Vers}(C, w_{i'}^{\sigma'}) \times \prod_{z \in \text{Sing}(C) \cap \text{Tor}(P_\Delta)^\times} \text{Vers}(C, z)$ with

$$\prod_{(\sigma', i') \neq (\sigma, i)} \text{ES}_{\text{Tor}(\sigma')}(C, w_{i'}^{\sigma'}) \times \text{ES}_{\text{Tor}(\sigma)}^{\text{fix}}(C, w_i^\sigma) \times \prod_{z \in \text{Sing}(C) \cap \text{Tor}(P_\Delta)^\times} \text{EG}(C, z). \quad (4.6)$$

It has dimension $\dim \mathcal{V}_2 \geq \dim \mathcal{V}_1 - 1$. We claim that $\dim \mathcal{V}_2 \leq \dim \mathcal{V} - 2 = n - 2$. Indeed,

$$h^1(C, \mathcal{J}_{Z_2/C} \otimes_{\mathcal{O}_C} \text{in}^* \mathcal{L}_{P_\Delta}) = 0, \quad (4.7)$$

where the 0-dimensional scheme Z_2 supported at $\widehat{\mathbf{w}}(t^*) \cup \text{Sing}(C)$ coincides with Z_1 (appearing in (4.5)) along $(\widehat{\mathbf{w}}(t^*) \setminus \{w_i^\sigma\}) \cup \text{Sing}(C)$ and is defined at w_i^σ by the ideal

$$\{\varphi \in \mathcal{O}_{C, w_i^\sigma} \mid \text{ord } \varphi|_{C, w_i^\sigma} \geq 2k_i^\sigma + 1 + 2\delta(C, w_i^\sigma)\}$$

(cf. Section 2.3, item (b)). To see this, as in the preceding paragraph, we rewrite (4.7) in the form

$$h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\mathbf{d}_2)) = 0,$$

where

$$\deg \mathbf{d}_2 = C^2 - (C^2 + CK_{\text{Tor}(P_\Delta)} + 2) - (CK_{\text{Tor}(P_\Delta)} - n) - 1 = n - 3 > -2.$$

Thus, by the Riemann–Roch theorem, we have

$$\dim \mathcal{V}_2 \leq h^0(C, \mathcal{J}_{Z_2/C} \otimes_{\mathcal{O}_C} \text{in}^* \mathcal{L}_{P_\Delta}) = (n - 3) + 1 = n - 2,$$

as required. Furthermore, relation (4.7) yields the existence of the germ of a smooth path through 0 in $\mathcal{V}$ transversally intersecting $\mathcal{V}_1$ and regularly projecting onto a smooth path through $\widehat{\mathbf{w}}(t^*)$ in $\text{Men}(\Delta)$.

Having a fragment of the path $\{\widehat{\mathbf{w}}(t)\}_{|t-t^*|<\varepsilon}$ whose existence we verified above, it remains to show that the value of $W_0^\kappa(\Delta, \widehat{\mathbf{w}}(t))$ is the same for $t < t^*$ and for $t > t^*$. Given a point $w_i^\sigma \in \widehat{\mathbf{w}}(t^*)$ where C has a singularity of type A_{2m} , $m \geq k_i^\sigma$, take local coordinates x, y in a neighborhood $U \subset \text{Tor}(P_\Delta)$ of w_i^σ so that $\text{Tor}(\sigma) = \{y = 0\}$. Suppose that we have $C \cap U = \{f(x, y) = y^2 - 2yx^k + x^{2k} + \text{h.o.t.} = 0\}$, where $k = k_i^\sigma$ and h.o.t. denotes terms above the Newton diagram. In particular, the real part $\mathbb{R}C \cap U$ lies in $\{y \geq 0\}$. Without loss of generality, we can assume that C deforms along the path $\{\widehat{\mathbf{w}}(t)\}_{|t-t^*|<\varepsilon}$ into nodal curves $C^{(t)} \cap U$, $0 < |t - t^*| < \varepsilon$. We show that all hyperbolic nodes live in the domain $\{x > 0, y > 0\}$, and the elliptic nodes live in the domain $\{y > 0\}$ for an even k and in the domain $\{y < 0\}$ for an odd k . Then, the constancy of $W_0^\kappa(\Delta, \widehat{\mathbf{w}}(t))$, $0 < |t - t^*| < \varepsilon$, follows from the fact that the total number of real nodes of $C^{(t)} \cap U$ equals m modulo 2 and that the orientation of $C^{(t)} \cap U$ at w_i^σ correlates with the number of the hyperbolic nodes; see Figure 1(a,b) for an even k and Figure 1(c,d) for an odd k .

Since the deformation $\{C^{(t)}\}_{|t-t^*|<\varepsilon}$ is equigeneric, we can describe it *via* the deformation of local parameterizations; see [Tei80, Theorem 1 on p. 73] and [CL06, Theorems 4.1 and 4.2]. Thus, $C \cap U = C^{(t^*)} \cap U$ admits a (real) parametrization

$$x = \tau^2 + a_{2m+1}\tau^{2m+1} + \sum_{2 \leq r \leq m} a_{2r}\tau^{2r} + \sum_{r \geq 2m-2k+2} a_r\tau^r, \quad y = \tau^{2k},$$

where $\tau \in (\mathbb{C}, 0)$ and $a_{2m-2k+1} \neq 0$, and the curves $C^{(t)} \cap U$ are parameterized by

$$x = \tau^2 + a_{2m-2k+1}\tau^{2m-2k+1} + \sum_{2 \leq r \leq m} a_{2r}\tau^{2r} + \sum_{r \geq 2m-2k+2} a_r\tau^r + \sum_{r \geq 1} b_r\tau^r, \quad y = \tau^{2k},$$

where the $b_r = b_r(t)$ are analytic functions defined over $\mathbb{R}$ and vanishing at $t = t^*$.

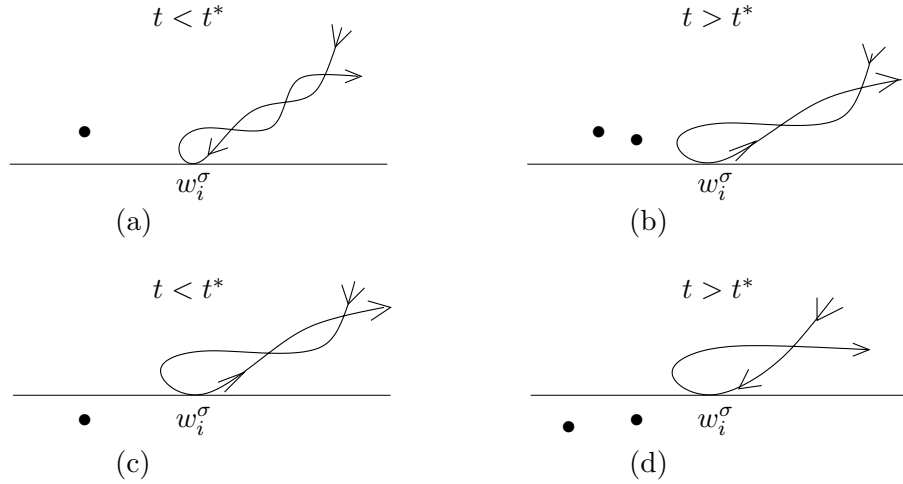


FIGURE 1. Proof of Lemma 4.3, part (2)

The curves $C^{(t)}$, $t \neq t^*$, satisfy the conditions of Lemma 2.4; that is, we have $\alpha_1(t) > 0$ for $0 < |t - t^*| < \varepsilon$. Since $y = \tau^{2k} > 0$ as $\tau \in \mathbb{R}^*$, all hyperbolic nodes are located in the domain $\{y > 0\}$. Furthermore, a hyperbolic node corresponds to values $\tau_1, \tau_2 \in \mathbb{R}$, $\tau_2 = -\tau_1$, such that $x(\tau_1) = x(\tau_2)$, which yields $x(\tau_1) = \tau_1^2 + O(\tau_1^4) > 0$. Consider an elliptic node of $C^{(t)}$, $t \neq t^*$. It corresponds to a $\tau \in (\mathbb{C}, 0) \setminus (\mathbb{R}, 0)$ such that $x(\tau), y(\tau) \in \mathbb{R}$. Let k be odd. Suppose that $y(\tau) > 0 = \tau^{2k} > 0$, that is, $\tau = \theta \zeta_{2k}$, where $\theta > 0$ and $(\zeta_{2k})^{2k} = 1$ with $\zeta_{2k} \neq \pm 1$. Then,

$$x(\tau) = \alpha_1 \theta \zeta_{2k} + (1 + \alpha_2) \theta^2 (\zeta_{2k})^2 + O(\theta^3) \notin \mathbb{R},$$

since (recall that k is odd)

$$(1 + \alpha_2) \theta^2 (\zeta_{2k})^2 \notin \mathbb{R} \quad \text{and} \quad \frac{\alpha_1 \theta \zeta_{2k}}{(1 + \alpha_2) \theta^2 (\zeta_{2k})^2} \notin \mathbb{R}.$$

Hence, all elliptic nodes live in the domain $\{y < 0\}$. Let k be even. Suppose that $y(\tau) = \tau^{2k} < 0$, that is, $\tau = \theta \eta \zeta_{2k}$, where $\theta > 0$, $(\zeta_{2k})^{2k} = 1$, $\eta = \exp(\pi \sqrt{-1}/2k)$. Thus,

$$x(\tau) = \alpha_1 \theta \eta \zeta_{2k} + (1 + \alpha_2) \theta^2 \eta^2 (\zeta_{2k})^2 + O(\theta^3),$$

and again we have

$$(1 + \alpha_2) \theta^2 (\eta^2 \zeta_{2k})^2 \notin \mathbb{R} \quad \text{and} \quad \frac{\alpha_1 \theta \zeta_{2k}}{(1 + \alpha_2) \theta^2 \eta^2 (\zeta_{2k})^2} \notin \mathbb{R}.$$

Hence, all elliptic nodes live in the domain $\{y > 0\}$.

(3) Suppose that ξ is as in Lemma 4.2(2ii). Here we choose a path $\{\widehat{w}(t)\}_{0 \leq t \leq 1}$ defined in a neighborhood of t^* such that the point $w_i^\sigma(t)$ is fixed and the point $w_j(t^*)$ is mobile (together with some other point of $\widehat{w}(t)$). Since C is immersed, we extract the required local constancy of $W_0^\kappa(t)$ and $\widetilde{W}_0^\kappa(t)$ from the transversality relation in the form of (4.3) that holds due to

$$\deg \mathbf{d} = C^2 - (C^2 + CK_{\text{Tor}(P_\Delta)} + 2) - (-CK_{\text{Tor}(P_\Delta)} - 2) = 0 > -2, \quad (4.8)$$

and from the smoothness of the two following strata in $\mathcal{O}_{C,w}/\mathfrak{m}_w^s$:

- one stratum parameterizes deformations of a smooth branch intersecting $\text{Tor}(\sigma)$ at w with

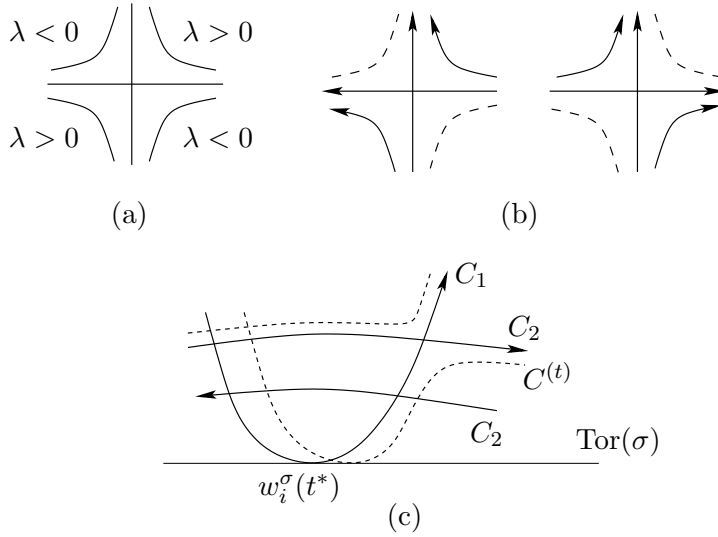


FIGURE 2. Proof of Lemma 4.3, part (4)

multiplicity $2(k_i^\sigma + k_j^\sigma)$ into a smooth branch intersecting $\text{Tor}(\sigma)$ at w with multiplicity $2k_i^\sigma$ and in a nearby point with multiplicity $2k_j^\sigma$;

- the other stratum parameterizes deformations of a couple of smooth branches intersecting $\text{Tor}(\sigma)$ at w with multiplicities $2k_i^\sigma$ and $2k_j^\sigma$, respectively, into a couple of smooth branches, one intersecting $\text{Tor}(\sigma)$ at w with multiplicity $2k_i^\sigma$, and the other intersecting $\text{Tor}(\sigma)$ in a nearby point with multiplicity $2k_j^\sigma$.

Both smoothness statements are straightforward.

(4) Suppose that ξ is as in Lemma 4.2(2iii) and $\Phi(p) \in \text{Tor}_\mathbb{R}(P_\Delta)_+^\times$. We closely follow the proof of [Mik17, Theorem 5], providing here details for the reader's convenience. Consider the path $\{\hat{w}(t)\}_{0 \leq t \leq 1}$ locally obtained from $\hat{w}(t^*)$ by moving one point of $\hat{w}(t^*) \cap C_1$ and one point of $\hat{w}(t^*) \cap C_2$, while fixing the position of the other points of $\hat{w}(t^*)$. Then, we obtain a smooth 1-dimensional germ in $\overline{\mathcal{M}}_{0,n}(\Delta)$, which locally induces a deformation of the hyperbolic node $\Phi(\hat{C}, p)$ equivalent, in suitable local coordinates in a neighborhood of $z = \Phi(p)$, to $uv = \lambda(t)$ with λ a smooth function in a neighborhood of t^* such that $\lambda(t^*) = 0$ and $\lambda'(t^*) > 0$ (see Figure 2(a)). The smoothness and the local deformation claims are straightforward from the (standard) considerations in the proof of [IKS17, Lemma 11(2)], which reduce both claims to the following h^0 -vanishing (cf. [IKS17, formula (16) and computations in the first paragraph on p. 251]):

$$h^0(\hat{C}_r, \mathcal{O}_{\hat{C}_r}(\mathbf{d}_r)) = 0, \quad r = 1, 2, \quad (4.9)$$

where

$$\deg \mathbf{d}_r = C_r^2 - (C_r^2 + C_r K_{\text{Tor}(P_\Delta)} + 2) - (C_r K_{\text{Tor}(P_\Delta)} - 1) = -1 > -2,$$

and hence (4.9) follows.

Now we analyze the change of the refined invariants, closely following the argument from [Mik17, Section 6.4]. To this end, we pass to the consideration of complex oriented curves. Denote by $\hat{C}_1^\pm$ and $\hat{C}_2^\pm$ the connected components of $\hat{C}_1 \setminus \mathbb{R}\hat{C}_1$ and $\hat{C}_2 \setminus \mathbb{R}\hat{C}_2$, respectively. When t varies around t^* , the curve $\hat{C}_1 \cup \hat{C}_2$ turns into $\mathbb{P}^1$, and we encounter two types of deformations: Either $\hat{C}_1^+$, $\hat{C}_1^-$ glue up with $\hat{C}_2^+$, $\hat{C}_2^-$, respectively; or $\hat{C}_1^+$, $\hat{C}_1^-$ glue up with $\hat{C}_2^-$, $\hat{C}_2^+$, respectively.

The type of the deformation agrees with the local complex orientation of the central curve as shown in Figure 2(b). This means that if for $t < t^*$ one encounters a deformation of the first type, then for $t > t^*$ it is of the second type, and vice versa. Put (cf. (3.1) and (3.2))

$$\varepsilon_j^\pm = \prod_{\substack{\sigma \in P_\Delta^1, 1 \leq i \leq n^\sigma \\ w_i^\sigma \in C_j, k_i^\sigma \equiv 0 \pmod{2}}} \text{sign}(\widehat{C}_j^\pm, w_i^\sigma), \quad \widetilde{\varepsilon}_j^\pm = \prod_{\sigma \in P_\Delta^1, 1 \leq i \leq n^\sigma} \text{sign}(\widehat{C}_j^\pm, w_i^\sigma), \quad j = 1, 2.$$

Depending on the point of $C_1 \cap C_2$ chosen for the smoothing, the triple $(\kappa, \varepsilon, \widetilde{\varepsilon})$, where κ is the quantum index and $\varepsilon, \widetilde{\varepsilon}$ denote the second product in (3.1), (3.2), respectively, changes as follows (see Figure 2(c)): Either the real tangent vectors to C_1, C_2 at the smoothed-out intersection points form a negative pair and

$$\begin{cases} (\kappa_1 + \kappa_2, \varepsilon_1^+ \varepsilon_2^+, \widetilde{\varepsilon}_1^+ \widetilde{\varepsilon}_2^+) , \\ (-\kappa_1 - \kappa_2, \varepsilon_1^- \varepsilon_2^-, \widetilde{\varepsilon}_1^- \widetilde{\varepsilon}_2^-) , \end{cases} \implies \begin{cases} (\kappa_1 - \kappa_2, \varepsilon_1^+ \varepsilon_2^-, \widetilde{\varepsilon}_1^+ \widetilde{\varepsilon}_2^-) , \\ (-\kappa_1 + \kappa_2, \varepsilon_1^- \varepsilon_2^+, \widetilde{\varepsilon}_1^- \widetilde{\varepsilon}_2^+) ; \end{cases}$$

or the real tangent vectors to C_1, C_2 form a positive pair, and

$$\begin{cases} (\kappa_1 - \kappa_2, \varepsilon_1^+ \varepsilon_2^-, \widetilde{\varepsilon}_1^+ \widetilde{\varepsilon}_2^-) , \\ (-\kappa_1 + \kappa_2, \varepsilon_1^- \varepsilon_2^+, \widetilde{\varepsilon}_1^- \widetilde{\varepsilon}_2^+) \end{cases} \implies \begin{cases} (\kappa_1 + \kappa_2, \varepsilon_1^+ \varepsilon_2^+, \widetilde{\varepsilon}_1^+ \widetilde{\varepsilon}_2^+) , \\ (-\kappa_1 - \kappa_2, \varepsilon_1^- \varepsilon_2^-, \widetilde{\varepsilon}_1^- \widetilde{\varepsilon}_2^-) . \end{cases}$$

For a simple geometric reason, the number of intersection points of the first kind equals the number of intersection points of the second type. Hence, summing up over smoothings for all real intersection points of C_1, C_2 , we derive the constancy of the numbers $W_0^\kappa(t)$ and $\widetilde{W}_0^\kappa(t)$ along the path $\{\widehat{w}(t)\}_{|t-t^*|<\zeta}$ ($0 < \zeta \ll 1$) in the case where $\Phi: \widehat{C} \rightarrow \text{Tor}(P_\Delta)$ is a birational immersion.

The case of multiple coverings can be treated in the same manner. We leave the details to the reader.

(5) Suppose that ξ is as in Lemma 4.2(2iii), and set $\Phi(p) = w_i^\sigma(t^*) =: w$. Observe that $(\Phi|_{\widehat{C}_r})^*(w) = 2l_r p$, $r = 1, 2$, where $l_1 + l_2 = k_i^\sigma$.

(5a) Suppose that $\Phi|_{\widehat{C}_1}$ and $\Phi|_{\widehat{C}_2}$ are both immersions and that $l_1 \leq l_2$.

Define the path $\{\widehat{w}(t)\}_{|t-t^*|<\zeta}$, $0 < \zeta \ll 1$, by picking one mobile point in $C_1 \cap \widehat{w}(t^*) \setminus \{w\}$ and the other in $C_2 \cap \widehat{w}(t^*) \setminus \{w\}$. This defines a 1-dimensional subvariety of $\overline{\mathcal{M}}_{0,n}(\Delta)$. Let $\mathcal{V} = \{[\Phi_t: (\widehat{C}^{(t)}, p_t) \rightarrow \text{Tor}(P_\Delta)]\}_{|t-t^*|<\zeta}$ be one of the irreducible germs of that variety at ξ , where $\widehat{C}^{(t)} \simeq \mathbb{P}^1$ as $t \neq 0$. The 1-dimensional family of curves $C^{(t)} = \Phi_t(\widehat{C}^{(t)}) \in |\mathcal{L}_{P_\Delta}|$, $|t - t^*| < \zeta$, has a tangent cone at $C_1 \cup C_2$ spanned by $C_1 \cup C_2$ and some curve $C^* \in |\mathcal{L}_{P_\Delta}| \setminus \{C_1 \cup C_2\}$.

We now describe the local behavior of C^* at the point w . Choose local conjugation-invariant coordinates x, y in a neighborhood of w so that $w = (0, 0)$, $\text{Tor}(\sigma) = \{y = 0\}$, $\text{Tor}_{\mathbb{R}}^+(P_\Delta) = \{y \geq 0\}$, and $C_s = \{y = \eta_s x^{l_s} + \text{h.o.t.}\}$ with $\eta_s > 0$, $s = 1, 2$, where without loss of generality we assume $\eta_1 \neq \eta_2$, and h.o.t. denotes terms above the Newton diagram. Thus, the equation of $C_1 \cup C_2$ has Newton diagram Γ at w as shown in Figures 3(a,b) by fat lines. We claim that under Γ , the curve C^* has the monomial $x^{2l_1-1}y$ with a non-zero coefficient and no other monomials. Since C^* can be taken close to $C_1 \cup C_2$, its Newton diagram either coincides with or lies below Γ . Coincidence is not possible for the following reason. The curve C^* passes through $C_1 \cap C_2 \cap \text{Tor}(P_\Delta)^\times$ (which consists of only nodes by Lemma 4.2(2iii)) and, if C^* has Newton diagram Γ at w , intersects C_1 at w with multiplicity at least $2l_1 = (C_1 \cdot C_2)_w$. Hence, by Noether's fundamental theorem, C^* lies in the subspace of $|\mathcal{L}_{P_\Delta}|$ spanned by the curves of type $C_1 \cup C'_2$ with $C'_2 \in |C_2|$ and of type $C'_1 \cup C_2$ with $C'_1 \in |C_1|$. However, the Bézout-type restriction dictates that $C'_1 = C_1$ and $C'_2 = C_2$: For example, at each point $z \in \text{Sing}(C_1)$, the curve C'_1 must induce in $\mathcal{O}_{C_1,z}$ an

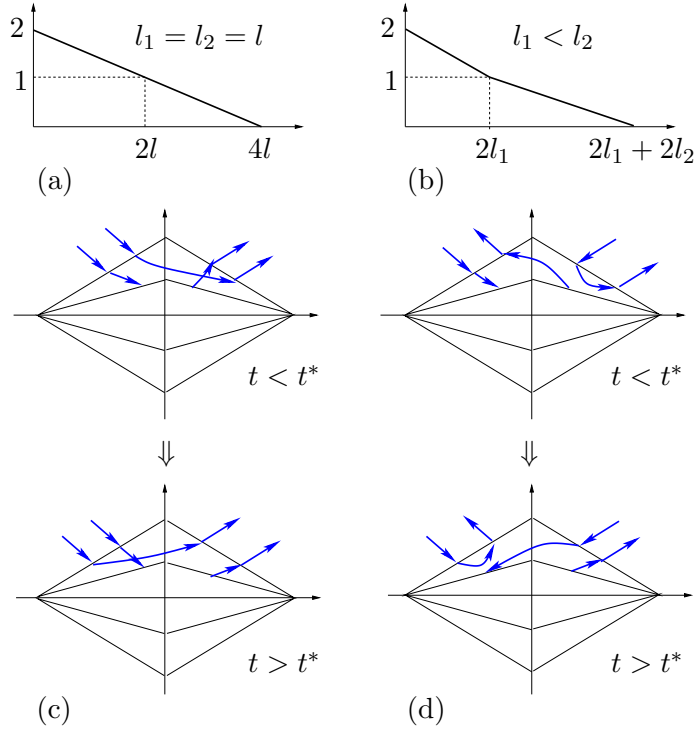


FIGURE 3. Proof of Lemma 4.3, part (5), I

element of the conductor ideal (cf. [DH88, Theorem 4.15]), and hence $(C'_1 \cdot C_1)_z \geq 2\delta(C_1, z)$ at each fixed point $w_{i'}^{\sigma'}(t^*) \in \widehat{w}(t^*) \cap C_1$ (including w), we have $(C'_1 \cdot C_1)_{w_{i'}^{\sigma'}(t^*)} \geq 2k_{i'}^{\sigma'}$, and finally, at the mobile point $w_{i'}^{\sigma'}(t^*) \in \widehat{w}(t^*) \cap C_1$, we have $(C'_1 \cdot C_1)_{w_{i'}^{\sigma'}(t^*)} \geq 2k_{i'}^{\sigma'} - 1$, which altogether amounts to

$$\sum_{z \in \text{Sing}(C_1)} \delta(C_1, z) - C_1 K_{\text{Tor}(P_\Delta)} - 1 = C_1^2 + 1.$$

Furthermore, a monomial in an equation of C^* below Γ cannot be x^s , $0 \leq s < 2(l_1 + l_2)$, since $(C^* \cdot \text{Tor}(\sigma))_w \geq 2(l_1 + l_2)$, and it cannot be $x^s y$ with $0 \leq s \leq 2l_1 - 2$. For the latter claim, we observe that each curve $C^{(t)}$, $t \neq 0$, has $2l_1 - 1$ nodes in a neighborhood of w and satisfies $(C^{(t)} \cdot \text{Tor}(\sigma))_w = 2(l_1 + l_2)$. Thus, if the tangent line at $C^{(t)}$ to the considered family of curves is spanned by $C^{(t)}$ and $(C^*)^{(t)}$, then $C^{(t)}$ and $(C^*)^{(t)}$ intersect in a neighborhood of w with multiplicity at least $2(2l_1 - 1) + 2(l_1 + l_2) = 4l_1 + 2l_2 - 2$. Hence, C^* and $C_1 \cup C_2$ intersect at w with multiplicity at least $4l_1 + 2l_2 - 2$, which leaves the only possibility of the extra monomial $x^{2l_1-1}y$.

If $l_1 = l_2 = l$, we can write down an equation of $C^{(t)}$ in a neighborhood of w in the form

$$y^2(1 + O(t_1^{>0})) - (\eta_1 + \eta_2 + O(t_1^{>0}))x^{2l}y + x^{4l}(\eta_1\eta_2 + O(t_1^{>0})) + \text{h.o.t.} + \sum_{r=0}^{2l-1} a_{r1}(t_1)x^r y = 0,$$

where h.o.t. denotes the terms above the Newton diagram, and

$$t_1 = t - t^*, \quad a_{r1}(0) = 0 \quad \text{for all } r = 0, \dots, 2l - 1.$$

Consider the tropical limit at $t_1 \rightarrow 0$. It includes a subdivision of the triangle

$$T = \text{conv}\{(0, 1), (4l, 0), (0, 2)\}$$

and certain limit curves in the toric surfaces associated with the pieces of the subdivision. Since locally $C^{(t)}$ is an immersed cylinder, we obtain that the union of the limit curves is a curve of arithmetic genus 0, which finally allows only the following tropical limit: The triangle T is the unique piece of the subdivision, and

$$a_{r1}(t_1) = t_1^{\lambda(2l-r)} (a_{r1}^0 + O(t_1^{>0})) , \quad r = 0, \dots, 2l-1 ,$$

where $a_{2l-1,1}^0 \neq 0$ in view of the above conclusion on the Newton diagram of C^* , and in addition, the limit curve with Newton triangle T given by

$$y^2 - (\eta_1 + \eta_2)x^{2l}y + \eta_1\eta_2x^{4l} + \sum_{r=0}^{2l-1} a_{r1}^0 x^r y = 0 \quad (4.10)$$

is rational. It also follows that $\mathcal{V}$ is smooth, regularly parameterized by t_1 (that is, $\lambda = 1$), and its real part submersively projects onto the path $\{\widehat{\mathbf{w}}(t)\}_{|t-t^*|<\zeta}$. By the patchworking theorem [Shu98, Theorems 3.1 and 4.2], we can uniquely restore $\mathcal{V}$ as long as we compute the coefficients a_{r1}^0 , $0 \leq r \leq 2l-2$ (here $a_{2l-1,1}^0$ is determined by C^*). The rationality of the curve (4.10) can be expressed as follows. Write equation (4.10) in the form $y^2 - 2P(x)y + \eta_1\eta_2x^{4l} = 0$; then resolve with respect to y :

$$y = P(x) \pm \sqrt{P(x)^2 - \eta_1\eta_2x^{4l}} = P(x) \pm x^{2l}\sqrt{Q(1/x)^2 - \eta_1\eta_2}, \quad \deg Q = 2l.$$

The rationality means that the expression under the radical has $2l-1$ double roots (corresponding to the nodes of the curve). This means that Q is a modified Chebyshev polynomial:

$$Q(u) = \lambda_1 \text{Cheb}_{2l}(u + \lambda_2) + \lambda_3, \quad \text{Cheb}_{2l}(u) = \cos(2l \cdot \arccos u),$$

where $\lambda_1, \lambda_2, \lambda_3$ can be computed out of η_1, η_2 , and $a_{2l-1,1}^0$. As noticed in [Shu06, proof of Proposition 6.1], there are exactly two real solutions. One corresponds to a curve with the real part, shown in Figure 3(c) by lines inside the four Newton triangles designating four real quadrants, and this curve has one hyperbolic node and $2l-2$ non-real nodes, while the other curve, shown in Figure 3(d), has $2l-1$ elliptic nodes. Bringing the complex orientations into play, we see that the former solution appears when the local real branches $(\mathbb{R}C_1, w)$ and $(\mathbb{R}C_2, w)$ are oriented coherently, while the latter solution corresponds to the case when these real branches have opposite orientations; see Figures 3(c,d). That is, on both sides of the path $\{\mathbf{w}(t)\}_{|t-t^*|<\zeta}$, we encounter four elements of $\overrightarrow{\mathcal{M}}_{0,n}^{\mathbb{R},+}(\Delta, \widehat{\mathbf{w}}(t))$ having respective quantum indices

$$\kappa_1 + \kappa_2, \quad -\kappa_1 - \kappa_2, \quad \kappa_1 - \kappa_2, \quad \kappa_2 - \kappa_1.$$

It remains to notice that the Welschinger signs (3.1) and (3.2) of the elements with quantum index $\kappa_1 + \kappa_2$ or $-\kappa_1 - \kappa_2$ are the same on both sides, since the modified limit curves shown in Figure 3(c) for $t > t^*$ and $t < t^*$ are obtained from each other by reflection with respect to the vertical axis. The same holds for the elements with quantum index $\kappa_1 - \kappa_2$ or $\kappa_2 - \kappa_1$ (see Figure 3(d)). This completes the proof of the constancy of the functions $W_0^\kappa(t)$ and $\widetilde{W}_0^\kappa(t)$ in the considered case $l_1 = l_2$.

In the case $l_1 < l_2$, an equation of $C^{(t)}$ in a neighborhood of w takes the form

$$(1 + O(t_1^{>0}))y^2 - (\eta_1 + O(t_1^{>0}))x^{2l_1}y + (\eta_1\eta_2 + O(t_1^{>0}))x^{2(l_1+l_2)} + \sum_{r=0}^{2l-1} a_{r1}(t_1)x^r y + \text{h.o.t.} = 0,$$

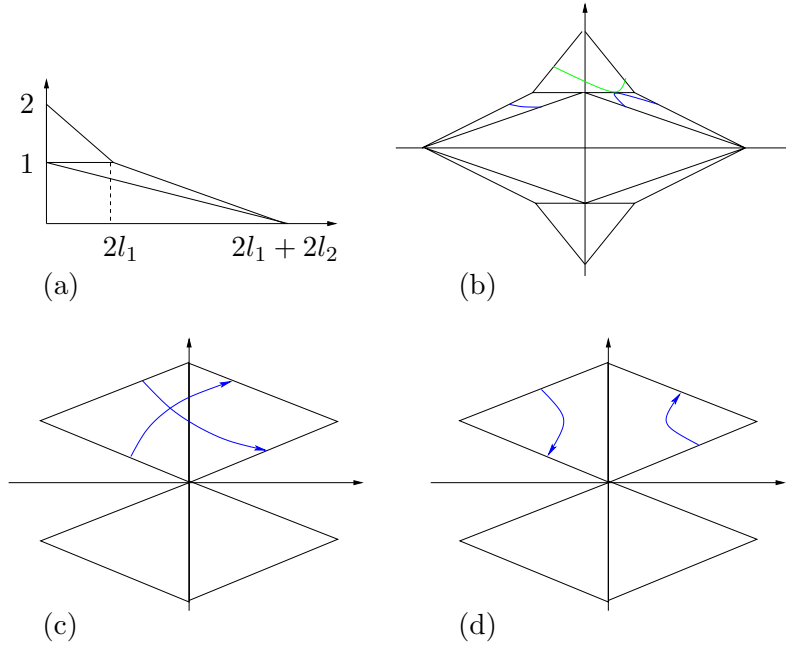


FIGURE 4. Proof of Lemma 4.3, part (5), II

where $t_1 = t - t^*$, $a_{r1}(0) = 0$ for all $r = 0, \dots, 2l - 1$, and h.o.t. stands for the terms above the Newton diagram. In this situation, due to the condition of arithmetic genus 0, the tropical limit is defined uniquely: The area under the Newton diagram is divided into two triangles

$$\text{conv}\{(0, 1), (2l_1, 1), (2(l_1 + l_2), 0)\} \quad \text{and} \quad \text{conv}\{(0, 1), (2l_1, 0), (0, 2)\}$$

(see Figure 4(a)), while the monomials on the segment $[(0, 1), (2l_1, 1)]$ sum up to $\eta_1(x + \lambda t_1)^{2l_1}y$, where $\lambda \neq 0$ is uniquely determined by the coefficient $a_{2l_1-1,1}^0$ coming from C^* . The two limit curves have branches intersecting each other with multiplicity $2l_1$ along the toric divisor $\text{Tor}([(0, 1), (2l_1, 0)])$; see Figure 4(b). The genuine geometry of $C^{(t)}$ can be recovered when we deform that intersection point into $2l_1 - 1$ nodes. The modification describes such a deformation as a replacement of the intersection point by one of the $2l_1$ modified limit curves, among which exactly two are real, and their real parts are shown in Figure 4(c,d) (cf. Figure 3(c,d)): One of them has a hyperbolic node and $2l_1 - 2$ non-real nodes, and the other has $2l_1 - 1$ elliptic nodes. As in the preceding paragraph, the former modified limit curve fits the case when the local real branches $(\mathbb{R}C_1, w)$ and $(\mathbb{R}C_2, w)$ are oriented coherently, which means that on each side of the path $\{\widehat{w}(t)\}_{|t-t^*|<\zeta}$, we have two elements of $\overrightarrow{\mathcal{M}}_{0,n}^{\mathbb{R},+}(\Delta, \widehat{w}(t))$ with quantum indices $\kappa_1 + \kappa_2, -\kappa_1 - \kappa_2$, while all four elements have the same Welschinger sign (3.1) or (3.2) since the constructions for $t > t^*$ and $t < t^*$ are symmetric with respect to the vertical axis. Hence, we obtain the constancy of the functions $W_0^\kappa(t)$ and $\widetilde{W}_0^\kappa(t)$. The same holds in the case of the opposite orientation of the branches $(\mathbb{R}C_1, w)$ and $(\mathbb{R}C_2, w)$ with the use of the second modified limit curve. We also point out that the final deformation including the modification still depends on $t_1 = t - t^*$.

The realizability of the deformations described in this item by variation of the constraints requires an independent variation of the coefficients of the monomials under the Newton diagram Γ while keeping the total δ -invariant outside the point w and keeping the given intersection number $2k_{i'}^{\sigma'}$ with the toric divisor $\text{Tor}(\sigma')$ at a point close to $w_{i'}^{\sigma'}$, for each $(i', \sigma') \neq (i, \sigma)$. This can

be expressed as

$$h^1(C, \mathcal{J}_{Z/C} \otimes_{\mathcal{O}_C} \text{in}^* \mathcal{L}_{P_\Delta}) = 0, \quad (4.11)$$

where $C = C_1 \cup C_2$ and $C_r = \Phi(\widehat{C}_r)$, $r = 1, 2$, and where the 0-dimensional scheme $Z \subset C$ is supported at $\text{Sing}(C) \cup \widehat{\mathbf{w}}(t^*)$ and defined by the ideal $J^{\text{cond}}(C, z)$ at each $z \in \text{Sing}(C) \cap \text{Tor}(P_\Delta)^\times$, by the ideal $\{\varphi \in \mathcal{O}_{C, w_{i'}^{\sigma'}} \mid (\varphi \cdot C)_{w_{i'}^{\sigma'}} \geq 2k_{i'}^{\sigma'} - 1\}$ at each $w_{i'}^{\sigma'} \in \widehat{\mathbf{w}}(t^*) \setminus \{w\}$, and by the ideal generated by the monomials on the Newton diagram Γ and above it at the point w . This h^1 -vanishing lifts to the disjoint union of $\widehat{C}_1$ and $\widehat{C}_2$ in the form

$$h^1(\widehat{C}_r, \mathcal{O}_{\widehat{C}_r}(\mathbf{d}_r)) = 0, \quad r = 1, 2, \quad (4.12)$$

where

$$\begin{aligned} \deg \mathbf{d}_r &= C_r^2 - (C_r^2 + C_r K_{\text{Tor}(P_\Delta)} + 2) - (-C_r K_{\text{Tor}(P_\Delta)} - |C_r \cap \text{Tor}(\partial P_\Delta)| + 1) \\ &= |C_r \cap \text{Tor}(\partial P_\Delta)| - 3 \geq -1 > -2, \quad r = 1, 2, \end{aligned} \quad (4.13)$$

which confirms (4.12).

(5b) The remaining case is as follows: $\Phi: \widehat{C}_1 \rightarrow \text{Tor}(P_\Delta)$ is a $2l_1$ -multiple covering of a line $C_1 \simeq \mathbb{P}^1$ through w , and $\Phi: \widehat{C}_2 \rightarrow \text{Tor}(P_\Delta)$ is a (birational) immersion onto a rational curve C_2 having a smooth branch at w intersecting $\text{Tor}(\sigma)$ with multiplicity $2l_2$. Here, we remove the restriction $l_1 \leq l_2$. We extend the range of the coordinate system x, y to a neighborhood of the line C_1 , assuming that $C_1 = \{x = 0\}$. Then, the Newton diagram of $\Phi_*(\widehat{C})$ in this neighborhood of C_1 is as shown in Figure 5(a) by fat lines, where N is the intersection multiplicity of the curves C_1 and C_2 . Following the recipe used in the preceding cases and taking into account that the curves $C^{(t)}$, $0 < |t - t^*| \ll 1$, are rational, we obtain the following:

- The combinatorial part of the tropical limit consists of the triangle $\text{conv}\{(0, 1), (2l_1, 1), (2(l_1 + l_2), 0)\}$ and the rectangle $\text{conv}\{(0, 1), (0, N), (2l_1, 1), (2l_1, N)\}$ (see Figure 5(b)).
- The limit curve associated with the triangle is a rational curve that intersect the toric divisor $\text{Tor}([(0, 1), (2l_1, 1)])$ at one point, where it is smooth, while the limit curve associated with the rectangle is split into $2l_1$ -multiple line intersecting the two horizontal toric divisors and $N - 1$ distinct lines intersecting the two vertical toric divisors. Here, the position of the $N - 1$ horizontal lines is determined by the intersection points of C_2 with $\text{Tor}([(2l_1, 1), (2l_1, N)])$, and the position of the vertical line is determined by the intersection point of the limit curve from the preceding item with $\text{Tor}([(0, 1), (2l_1, 1)])$.

Note that the $2l_1$ -multiple component of the limit curve associated with the rectangle is, in fact, the image of a $2l_1$ -multiple covering $\Phi: \widehat{C}_1 \rightarrow C_1$, and the point p is mapped to the point $x = \tau$ on the toric divisor $\text{Tor}([(0, 1), (2l_1, 1)])$. The deformation can be written in the form

$$ay(x - \tau)^{2l_1} \prod_{j=1}^{N-1} (y - y_j) + f(x, y) + \text{h.o.t.} = 0,$$

where $\tau = t - t^*$, the non-zero constants $a, y_1, \dots, y_{N-1}$ are determined by C_2 , and $f(x, y)$ is the sum of monomials strictly to the right of the Newton diagram with coefficients determined by C_2 . The higher-order terms (h.o.t.) yield the appearance of $2l_1(N - 1)$ nodes of $C^{(t)}$ ($t \neq t^*$) in a neighborhood of C_1 , a local equigeneric deformation of each singular point of C_2 , the fixed intersection multiplicity $2l_1 + 2l_2$ of $C^{(t)}$ and $\text{Tor}(\sigma)$ at w , and intersection multiplicities of $C^{(t)}$ with $\text{Tor}(\partial P_\Delta) \setminus \{w\}$ prescribed by the condition $C^{(t)} \in \mathcal{M}_{0,n}(\Delta, \mathbf{w}(t))$. Since this deformation does not produce elliptic nodes in a neighborhood of C_1 , the functions $W_0^\kappa(\widehat{\mathbf{w}}(t))$ and $\widetilde{W}_0^\kappa(\widehat{\mathbf{w}}(t))$ remain constant for $t \in (t^* - \varepsilon, t^* + \varepsilon)$.

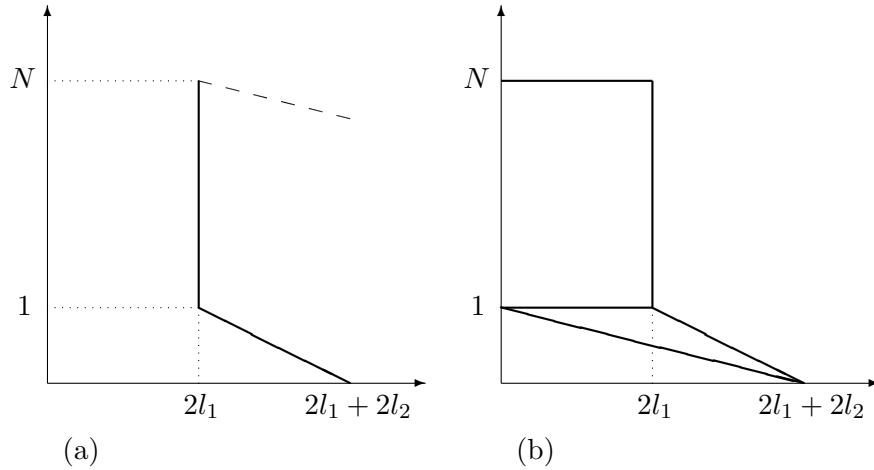


FIGURE 5. Proof of Lemma 4.3, part (5), III

A sufficient condition for the existence of the required deformation can be written as

$$h^1(C, \mathcal{J}_{Z/C} \otimes_{\mathcal{O}_C} \text{in}^* \mathcal{L}_{P_\Delta}) = 0, \quad (4.14)$$

where C is the non-reduced union of a $2l_1$ -multiple component C_1 and of C_2 , and the 0-dimensional scheme $Z \subset C$ is supported at $\text{Sing}(C_2) \cup (C \cap \text{Tor}(\partial P_\Delta)) \cup (C_1 \cap C_2)$ and is defined by the ideal $I^{\text{cond}}(C_2(z))$ at each $z \in \text{Sing}(C_2)$, by the ideal $I_z = \langle x^{2l_1}, y - y_i \rangle$ at each point $(0, y_i) \in C_1 \cap C_2$, $i = 1, \dots, N_1$, by the ideal

$$\{\varphi \in \mathcal{O}_{C_2, w_{i'}^{\sigma'}} \mid (\varphi \cdot C_2)_{w_{i'}^{\sigma'}} \geq 2k_{i'}^{\sigma'} - 1\}$$

at some point $w_{i'}^{\sigma'} \in (C_2 \cap \text{Tor}(\partial P_\Delta)) \setminus \{w\}$, by the ideal

$$\{\varphi \in \mathcal{O}_{C_2, w_{i''}^{\sigma''}} \mid (\varphi \cdot C_2)_{w_{i''}^{\sigma''}} \geq 2k_{i''}^{\sigma''}\}$$

at each point $w_{i''}^{\sigma''} \in C_2 \cap \text{Tor}(\partial P_\Delta) \setminus \{w, w_{i'}^{\sigma'}\}$, and finally, at the point w , by the ideal I_w determined by the vanishing of the coefficients of the monomials x^r for $0 \leq r < 2l_1 + 2l_2$ and $x^r y$ for $0 \leq r < 2l_1$; see Figure 5. We establish (4.14) using the basic Horace method [Hir85] (see also [GLS18, Section 3.4.1]): The vanishing relation (4.14) follows from the series of vanishing relations

$$\begin{cases} h^1(C_1, \mathcal{J}_{(Z:jC_1) \cap C_1/C_1} \otimes_{\mathcal{O}_{C_1}} \text{in}^* \mathcal{L}_{P_\Delta}(-jC_1)) = 0, & j = 0, \dots, 2l_1 - 1, \\ h^1(C_2, \mathcal{J}_{Z:(2l_1 C_1)/C_2} \otimes_{\mathcal{O}_{C_2}} \text{in}^* \mathcal{L}_{P_\Delta}(-2l_1 C_1)) = 0. \end{cases} \quad (4.15)$$

Each of the $2l_1$ relations in the upper line of (4.15) reads

$$h^1(C_1, \mathcal{O}_{C_1}(-w)) = 0,$$

which follows by the Riemann–Roch theorem. The residue scheme $Z : (2l_1 C_1)$ in the lower line of (4.15) coincides with Z at $\text{Sing}(C_2) \cup (C_2 \cap \text{Tor}(\partial P_\Delta)) \setminus \{w\}$ and is defined at w by the ideal $I_w = \langle x^{2l_2}, y \rangle$. Then, it lifts to $\widehat{C}_2 \simeq \mathbb{P}^1$ in the form

$$h^1(\widehat{C}_2, \mathcal{O}_{\widehat{C}_2}(\mathbf{d})) = 0,$$

where

$$\deg \mathbf{d} = C_2^2 - (C_2^2 + C_2 K_{\text{Tor}(P_\Delta)} + 2) - (-C_2 K_{\text{Tor}(P_\Delta)} - 1) = -1 > -2,$$

and hence (4.15) together with (4.14) follow.

(6) Suppose that ξ is as in Lemma 4.2(2iv). Without loss of generality, we can suppose that

$$P = \text{conv}\{(0, 2m), (2p, 0), (2q, 0)\}, \quad 0 \leq p < q, \quad m < q$$

and that Φ is ramified at the points of $\widehat{\mathbf{w}}(t^*)$ on the toric divisors associated with the segments $[(0, 2m), (2p, 0)]$ and $[(0, 2m), (2q, 0)]$, while the two points $w_1^\sigma(t), w_2^\sigma(t) \in \mathbf{w}(t)$ on the toric divisor $\text{Tor}(\sigma)$, where $\sigma = [(2p, 0), (2q, 0)]$, merge to one point as $t \rightarrow t^*$. Then, $C = \Phi(\mathbb{P}^1)$ admits a parametrization

$$x = a\theta^m, \quad y = b\theta^p(\theta - 1)^{q-p}, \quad \theta \in \mathbb{C},$$

and correspondingly Φ is given by

$$x = a\theta^{2m}, \quad y = b\theta^{2p}(\theta^2 - 1)^{q-p}, \quad \theta \in \mathbb{C}.$$

Assuming that the path $\{\widehat{\mathbf{w}}(t)\}_{|t-t^*|<\varepsilon}$ is defined by fixing the point w_1^σ associated with $\theta = 1$ and the point of $\widehat{\mathbf{w}}(t^*)$ on $\text{Tor}([(0, 2m), (2q, 0)])$, we obtain a one-parameter deformation of Φ

$$x = a\theta^{2m}, \quad y = b\theta^{2p}(\theta - 1)^{q-p}(\theta + 1 + \lambda)^{q-p}, \quad |\lambda| \ll 1. \quad (4.16)$$

This deformation is regularly parameterized by the difference of the x -coordinates of $w_2^\sigma(t)$ and $w_1^\sigma(t)$, which is equal to $2ma\lambda + O(\lambda^2)$. It follows from formulas (4.16) that the corresponding element $\xi(t) \in \overline{\mathcal{M}}_{0,n}^{\mathbb{R},+}(\Delta, \widehat{\mathbf{w}}(t))$ has the same orientations at a point on the toric divisor $\text{Tor}([(0, 2m), (2p, 0)])$ for $\lambda < 0$ and for $\lambda > 0$, and similarly for the point on the toric divisor $\text{Tor}([(0, 2m), (2q, 0)])$, while the local branches at $w_1^\sigma(t)$ and $w_2^\sigma(t)$ have opposite orientations with respect to the orientation of $\text{Tor}_{\mathbb{R}}(\sigma)$. Furthermore, $\xi(t)$ has exactly two elliptic nodes in a neighborhood of each elliptic node of C . Thus, the constancy of $W_0^\kappa(t)$ and $\widetilde{W}_0^\kappa(t)$ follows. $\square$

5. Invariance in genus 1: Proof of Theorem 3.6

Take two sequences $\widehat{\mathbf{w}}(0) = (\widehat{\mathbf{w}}_\partial(0), w_0(0))$ and $\widehat{\mathbf{w}}(1) = (\widehat{\mathbf{w}}_\partial(1), w_0(1))$ satisfying the conditions of Theorem 3.6. We may assume that $\widehat{\mathbf{w}}_\partial(0), \widehat{\mathbf{w}}_\partial(1) \in \text{Men}_{\mathbb{R}}^\rho(\Delta)$ for some $\rho > 0$. Then, we join these sequences by a generic path $\{\widehat{\mathbf{w}}_\partial(t)\}_{0 \leq t \leq 1}$ in $\text{Men}_{\mathbb{R}}^\rho(\Delta)$ and the points $w_0(0)$ and $w_0(1)$ by a generic path $\{w_0(t)\}_{0 \leq t \leq 1}$ in $\mathcal{Q}$, and verify the constancy of $W_1^\kappa(\Delta, \widehat{\mathbf{w}}(t))$ and $\widetilde{W}_1^\kappa(\Delta, \widehat{\mathbf{w}}(t))$ in all possible wall-crossing events, for all quantum indices κ .

5.1 Invariance inside chambers

As in the proof of Theorem 3.2, we start with the verification of the constancy of $W_1^\kappa(\Delta, \widehat{\mathbf{w}}(t))$ along intervals $t' < t < t''$ such that $\{\widehat{\mathbf{w}}(t)\}_{t' < t < t''}$ lies in the same chamber of $\text{Men}_1(\Delta)$. To this end, it is enough to show that the projection $\mathcal{M}_{1,n}^{\mathbb{R},+}(\Delta, \widehat{\mathbf{w}}(t))_{t' < t < t''} \rightarrow (t', t'')$ has no critical points (cf. the proof of Theorem 3.2). This requirement amounts to the following statement (cf. Lemma 4.1 and [Shu15, Lemma 2.3]).

LEMMA 5.1. *Let $\xi = [\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)] \in \mathcal{M}_{1,n}^{\mathbb{R},+}(\Delta, \widehat{\mathbf{w}})$, where $\mathbf{w} \in \text{Men}_1(\Delta)$. Then,*

$$H^0\left(\widehat{C}, \mathcal{N}_\Phi\left(-p_0 - \sum_{\sigma \neq \sigma_0, i \neq i_0} 2k_i^\sigma p_i^\sigma - (2k_{i_0}^{\sigma_0} - 1)p_{i_0}^{\sigma_0}\right)\right) = 0. \quad (5.1)$$

Proof. We proceed along the lines of the proof of [Shu15, Lemma 2.3]. Assume that $H^0 \neq 0$ in (5.1). Then, there exists a real curve $C' \in |\mathcal{L}_{P_\Delta}| \setminus \{C\}$, where $C = \Phi_*(\widehat{C})$, which intersects C

- at each point $q \in \text{Sing}(C)$ with multiplicity at least $2\delta(C, q)$,
- at each point w_i^σ , $(\sigma, i) \neq (\sigma_0, i_0)$, with multiplicity at least $2k_i^\sigma$,
- at $w_{i_0}^{\sigma_0}$ with multiplicity at least $2k_{i_0}^{\sigma_0} - 1$,
- and at w_0 .

Since C has a null-homologous immersed circle S in $\overline{\mathcal{Q}}$, there must be an additional intersection point $w' \in C' \cap S$, not mentioned in the above list. However, then

$$\begin{aligned} C'C &\geq 2 \sum_{q \in \text{Sing}(C)} \delta(C, q) + \sum_{\sigma \in P_\Delta^1} \sum_{i=1}^{n^\sigma} 2k_i^\sigma - 1 + 2 \\ &= (C^2 - c_1(\text{Tor}(P_\Delta)[C])) + c_1(\text{Tor}(P_\Delta)[C]) - 1 + 2 = C^2 + 1, \end{aligned}$$

which gives a contradiction. $\square$

5.2 Walls in the space of constraints

The set of sequences $\widehat{w} \in \text{Men}_\mathbb{R}^\rho(\Delta) \times \mathcal{Q}$ satisfying

$$\overline{\mathcal{M}}_{1,n}^\mathbb{R}(\Delta, \widehat{w}) = \mathcal{M}_{1,n}^\mathbb{R}(\Delta, \widehat{w}) \quad \text{and} \quad \widehat{w} \in \text{Men}_1(\Delta)$$

is a dense semialgebraic subset of full dimension $\dim(\text{Men}_\mathbb{R}^\rho(\Delta) \times \mathcal{Q}) = n + 1$. The complement is the union of finitely many semialgebraic strata of codimension at least 1. Since the path $\{(\widehat{w}_\partial(t), w_0(t))\}_{0 \leq t \leq 1}$ is generic, it avoids strata of $\text{Men}_\mathbb{R}^\rho(\Delta) \times \mathcal{Q}$ of codimension at least 2 and intersects strata of codimension 1 only in their generic points. Note that the image $\text{pr}(S) \subset |\mathcal{L}_{P_\Delta}|$ of $\overline{\mathcal{M}}_{1,n}^\mathbb{R}(\Delta, \widehat{w})$ in the linear system $|\mathcal{L}_{P_\Delta}|$, when $\widehat{w}$ ranges over a stratum S of codimension 1, has dimension $n - 1$.

Now, we study the strata S of dimension n in $(\text{Men}_\mathbb{R}^\rho \times \mathcal{Q}) \setminus \text{Men}_1(\Delta)$ in terms of the geometry of elements $\xi \in \overline{\mathcal{M}}_{1,n}^\mathbb{R}(\Delta, \widehat{w}(t)) \setminus \mathcal{M}_{1,n}^\mathbb{R}(\Delta, \widehat{w}(t))$, $0 \leq t \leq 1$. In view of the results of Section 5.1, we can ignore the strata along which $\overline{\mathcal{M}}_{1,n}^\mathbb{R}(\Delta, \widehat{w}) = \mathcal{M}_{1,n}^\mathbb{R}(\Delta, \widehat{w})$. The following lemma classifies other strata.

LEMMA 5.2. (1) *The following elements $\xi = [\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)]$ cannot occur in*

$$\overline{\mathcal{M}}_{1,n}^\mathbb{R}(\Delta, \widehat{w}(t)) \setminus \mathcal{M}_{1,n}^\mathbb{R}(\Delta, \widehat{w}(t)),$$

$0 \leq t \leq 1$, as $\widehat{w} \in S$:

- (1i) $\widehat{C}$ is a reducible curve of arithmetic genus 1 having a component mapped onto a toric divisor;
- (1ii) $\widehat{C}$ is a connected curve of arithmetic genus 1 either having at least three irreducible components, or with two rational irreducible components and no other components.
- (2) If $\xi = [\Phi: (\widehat{C}, \widehat{\mathbf{p}}) \rightarrow \mathbb{P}^2] \in \overline{\mathcal{M}}_{1,n}(\Delta, \widehat{w}(t^*))$ when $\widehat{w}(t^*)$ does not satisfy the conclusions of Lemma 2.4, then ξ is of one of the following types:
 - (2a) either $w_0(t^*)$ is a singular point of the curve $C = \Phi(\widehat{C})$, and the following holds:
 - (2i) $\widehat{w}_\partial(t^*)$ consists of n distinct points, the curve $\widehat{C}$ is smooth elliptic, Φ is an immersion onto a curve C that is smooth along the toric divisors, while $w_0(t^*)$ is a center of at least one real local branch;
 - (2b) or $w_0(t^*)$ is a smooth point of the curve $C = \Phi(\widehat{C})$, and one of the following holds:

- (2ii) $\widehat{w}_\partial(t^*)$ consists of n distinct points, $\widehat{C}$ is a smooth elliptic curve, the map Φ is birational onto its image C but not an immersion; it may have singular local branches centered in $\text{Tor}(P_\Delta)^\times$ as well as singularities of type A_{2m} at some points $w_i^\sigma \in \widehat{w}_\partial(t^*)$, where $m \geq k_i^\sigma$;
- (2iii) $\widehat{w}_\partial(t^*)$ consists of $n - 1$ distinct points, in particular, $w_i^\sigma(t^*) = w_j^\sigma(t^*) =: w$ for some $\sigma \in P_\Delta^1$ and $i \neq j$, the curve $\widehat{C}$ is smooth elliptic, and the map Φ is birational onto its image C ; the curve C is immersed outside $\widehat{w}_\partial(t^*) \setminus \{w\}$ and may have singularities of type A_{2m} at some points $w_{i'}^{\sigma'} \in \widehat{w}_\partial(t^*) \setminus \{w\}$, while at w , the curve C either is smooth or has two smooth branches;
- (2iv) $\widehat{w}_\partial(t^*)$ consists of n distinct points, $\widehat{C} = \widehat{C}_1 \cup \widehat{C}_2$, where $\widehat{C}_1 \simeq \mathbb{P}^1$ and $\widehat{C}_2$ is a smooth elliptic curve, the intersection $\widehat{C}_1 \cap \widehat{C}_2$ consists of one point p ; the map $\Phi: \widehat{C}_1 \rightarrow \text{Tor}(P_\Delta)$ is either an immersion, smooth along $\text{Tor}(\partial P_\Delta)$, or a multiple covering of a line intersecting only two toric divisors, while these divisors correspond to opposite parallel sides of P_Δ and the intersection points with these divisors are ramification points of the covering; the map $\Phi: \widehat{C}_2 \rightarrow \text{Tor}(P_\Delta)$ is an immersion, smooth along $\text{Tor}(\partial P_\Delta)$; the point p either is mapped to $\text{Tor}(P_\Delta)^\times$, and then the curves $C_1 = \Phi(\widehat{C}_1)$ and $C_2 = \Phi(\widehat{C}_2)$ intersect each other only in $\text{Tor}(P_\Delta)^\times$ and each of their intersection points is an ordinary node, or p is mapped to some point $w_i^\sigma(t^*)$, in which case the curves $C_i = \Phi(\widehat{C}_i)$, $i = 1, 2$, intersect $\text{Tor}(\sigma)$ with even multiplicities, and they do not have other common points in $\widehat{w}_\partial(t^*)$;
- (2v) $\widehat{w}_\partial(t^*)$ consists of n distinct points, $\widehat{C}$ is an irreducible rational curve with one node p , the map Φ is an immersion sending p to some point $w_i^\sigma(t^*)$, center of two smooth branches intersecting with multiplicity $\min\{2l_1 + 1, 2l_2 + 1\}$ and intersecting $\text{Tor}(\sigma)$ with odd multiplicities $2l_1 + 1, 2l_2 + 1$, where $l_1 + l_2 + 1 = k_i^\sigma$; furthermore, each of the other points of $\widehat{w}_\partial(t^*)$ is a center of one smooth branch;
- (2vi) $\widehat{C}$ is a smooth elliptic curve, $\widehat{w}_\partial(t^*)$ consists of $n = 4$ points, the curve $C = \Phi(\mathbb{P}^1)$ is rational and smooth at $\widehat{w}_\partial(t^*)$, and $\Phi: \widehat{C} \rightarrow C$ factors through the double covering $\widehat{C} \rightarrow C^\nu \simeq \mathbb{P}^1$ ramified at four points projecting to $\widehat{w}_\partial(t^*)$, where $C^\nu \rightarrow C$ is the normalization.

Proof. (1i) In this case, we must have $\widehat{C} = \widehat{C}' \cup \widehat{C}''$, where $\widehat{C}'$ is a connected curve of arithmetic genus 1 such that $\widehat{C}'$ is mapped onto the union of all toric divisors, and $\widehat{C}''$ is a non-empty (due to $w_0 \in \text{Tor}(P_\Delta)^\times$) union of connected curves of arithmetic genus 0, each one joined with $\widehat{C}'$ in one point (cf. the proof of Lemma 4.2(1i)). Let $\widehat{C}_0''$ be a connected component of $\widehat{C}''$ such that $w_0(t^*) \in \Phi(\widehat{C}_0'')$. Note that all local branches of $\Phi: \widehat{C}_0'' \rightarrow \text{Tor}(P_\Delta)$ centered on toric divisors, except for $\Phi: (\widehat{C}_0'', p) \rightarrow \text{Tor}(P_\Delta)$, where $p = \widehat{C}' \cap \widehat{C}_0''$, are, in fact, centered at some points of $\widehat{w}_\partial(t^*)$. Observe that either $\widehat{C}_0''$ intersects at least three toric divisors of $\text{Tor}(P_\Delta)$, or it intersects two toric divisors corresponding to parallel sides of P_Δ . However, in the both cases, we reach a contradiction with (AQC).

(1ii) In the case of at least three irreducible components, we follow the lines of the proof of Lemma 4.2(1ii). If $\widehat{C}$ contains an elliptic component and at least two rational components, then we similarly obtain at least three independent Menelaus-type conditions for $\widehat{w}_\partial(t^*)$, which bounds from above the dimension of the considered stratum by $(n - 3) + 2 = n - 1 < n$, so we have a contradiction. If all irreducible components of $\widehat{C}$ are rational, then we encounter at least two independent Menelaus conditions on $\widehat{w}_\partial(t^*)$ as well as a condition on the position of the point $w_0(t^*)$ due to the finiteness statement of Lemma 2.4, which altogether bounds from above

the dimension of the considered stratum by $(n - 2) + 1 = n - 1 < n$.

(2a) Let $w_0(t^*)$ be a singular point of C . Then, for dimension reasons, $\widehat{w}_\partial(t^*)$ must be a generic element of $\text{Men}_{\mathbb{R}}^\rho$. This implies, in particular, that $\widehat{C}$ is irreducible and $\widehat{w}_\partial(t^*)$ consists of n distinct points. Furthermore, $\widehat{C}$ cannot be a rational curve with a node. Thus, $\widehat{C}$ is a smooth elliptic curve. Then (see Lemma 2.4), we derive that $\Phi: \widehat{C} \rightarrow \text{Tor}(P_\Delta)$ is an immersion and C is smooth along the toric divisors. Since $w_0(t^*)$ turns into a real smooth point in the deformation along the path $\{(\widehat{w}_\partial(t), w_0(t))\}_{0 \leq t \leq 1}$, it must be a center of at least one real local branch.

(2b) From now on, we can assume that $w_0(t^*)$ is a smooth point of $C = \Phi_*(\widehat{C})$.

(2ii) Suppose that $\widehat{w}_\partial(t^*)$ consists of n distinct points, $\widehat{C}$ is a smooth elliptic curve, and $\Phi: \widehat{C} \rightarrow C = \Phi(\widehat{C})$ is birational but is not an immersion. To verify the conditions of item (2ii) of the lemma, it is enough to consider the case of C having a singular branch at some point $w_i^\sigma \in \widehat{w}_\partial(t^*)$. Fixing the position of the point w_i^σ , we obtain a subfamily $\text{pr}(S^{\text{fix}}(w_i^\sigma)) \subset \text{pr}(S)$ of dimension at least $n - 2 \geq 2$ and then apply [IKS18, inequality (5) in Lemma 2.1] (cf. (4.1)):

$$\begin{aligned} -CK_{\text{Tor}(P_\Delta)} &\geq (-CK_{\text{Tor}(P_\Delta)} - n + 1) + (n - 3) + (\text{mult } Q - 1) \\ &= -CK_{\text{Tor}(P_\Delta)} - 2 + (\text{mult } Q - 1) \implies \text{mult } Q \leq 3, \end{aligned} \quad (5.2)$$

where Q is the local branch of C at w_i^σ . It remains to show that $\text{mult } Q = 2$. Arguing by contradiction, assume that $\text{mult } Q = 3$, that is, Q is topologically equivalent to a singularity $y^3 + x^m$, where $\gcd(3, m) = 1$ and $m \geq 2k_i^\sigma \geq 4$. Without loss of generality, we can assume that the germ of $\text{pr}(S) \subset |\mathcal{L}_{P_\Delta}|$ at C is an equisingular family. According to items (a) and (b) in Section 2.3, we obtain the following upper bound completely analogous to (2.5):

$$\dim \text{pr}(S^{\text{fix}}(w_i^\sigma)) \leq h^0(C, \mathcal{J}_{Z/C} \otimes_{\mathcal{O}_C} \text{in}^* \mathcal{L}_{P_\Delta}) = h^0(\widehat{C}, \mathcal{O}_{\widehat{C}}(-\mathbf{d}) \otimes_{\mathcal{O}_{\widehat{C}}} \Phi^* \mathcal{L}_{P_\Delta}), \quad (5.3)$$

where

$$\begin{aligned} \deg \mathbf{d} &= 2 \sum_{z \in \text{Sing}(C)} \delta(C, z) + (-CK_{\text{Tor}(P_\Delta)} - n + 1) + 2 \\ \implies \deg(\mathcal{O}_{\widehat{C}}(-\mathbf{d}) \otimes_{\mathcal{O}_{\widehat{C}}} \Phi^* \mathcal{L}_{P_\Delta}) &= C^2 - \deg \mathbf{d} = n - 3 \geq 1. \end{aligned}$$

By the Riemann–Roch formula, this yields

$$\dim \text{pr}(S^{\text{fix}}(w_i^\sigma)) \leq n - 3$$

contradicting the lower bound $\dim \text{pr}(S^{\text{fix}}(w_i^\sigma)) \geq n - 2$ pointed out above.

(2iii) Suppose that some of the points of the sequence $\widehat{w}_\partial(t^*)$ coincide. For dimension reasons, we immediately get that $\widehat{w}_\partial(t^*)$ consists of $n - 1$ distinct points, and all these points must be in general position subject to the unique Menelaus relation (2.1). Hence, $\widehat{C}$ must be irreducible. Moreover, $\widehat{C}$ cannot be rational, since otherwise, by Lemma 2.4, one would get that the total dimension of the considered stratum is at most $(n - 2) + 1 = n - 1 < n$, and we would have a contradiction. Thus, $\widehat{C}$ is a smooth elliptic curve. We have $w_i^\sigma(t^*) = w_j^\sigma(t^*) = w$ for some $\sigma \subset \partial P$ and $i \neq j$. If C is unibranch at each point of $\widehat{w}_\partial(t^*)$, then, by Lemma 2.4, the curve C is immersed and smooth along the toric divisors. Otherwise, the curve C has two local branches at w . Moreover, it must be unibranch at each point of $\widehat{w}_\partial(t^*) \setminus \{w\}$.

Let us analyze possible singularities of C . Fixing the position of w and some other point $w' \in \widehat{w}_\partial(t^*) \setminus \{w\}$, we obtain a family $\text{pr}(S^{\text{fix}}(w, w')) \subset \text{pr}(S)$ of dimension at least $n - 3 \geq 1$;

hence, [IKS18, inequality (5) in Lemma 2.1] applies, and similarly to (4.1) yields

$$\begin{aligned} -CK_{\text{Tor}(P_\Delta)} &\geq (-CK_{\text{Tor}(P_\Delta)} - n + 3) + (n - 4) + \sum_B (\text{mult } Q - 1) \\ &= -CK_{\text{Tor}(P_\Delta)} - 1 + \sum_B (\text{mult } Q - 1) \implies \sum_B (\text{mult } Q - 1) \leq 1, \end{aligned} \quad (5.4)$$

where B is the set of all singular local branches of C in $\text{Tor}(P_\Delta)^\times \cup \{w, w'\}$. This means that a singular branch of C (if any) must be of order 2. We show that C is immersed outside $\widehat{w}_\partial(t^*) \setminus \{w\}$ and hence fits the claim of item (2iii).

Let C have a singular local branch Q centered in $\text{Tor}(P_\Delta)^\times$. Note that $\dim \text{pr}(S^{\text{fix}}(w)) \geq n - 2$. Similarly to (5.3), we get

$$\dim \text{pr}(S^{\text{fix}}(w)) \leq h^0(\widehat{C}, \mathcal{O}_{\widehat{C}}(-\mathbf{d}_1) \otimes_{\mathcal{O}_{\widehat{C}}} \Phi^* \mathcal{L}_{P_\Delta}), \quad (5.5)$$

where

$$\begin{aligned} \deg \mathbf{d}_1 &= 2 \sum_{z \in \text{Sing}(C)} \delta(C, z) + (-CK_{\text{Tor}(P_\Delta)} - n + 2) + (\text{mult } Q - 1) \\ \implies \deg(\widehat{C}, \mathcal{O}_{\widehat{C}}(-\mathbf{d}_1) \otimes_{\mathcal{O}_{\widehat{C}}} \Phi^* \mathcal{L}_{P_\Delta}) &= C^2 - \deg \mathbf{d}_1 \\ &= C^2 - (C^2 + CK_{\text{Tor}(P_\Delta)}) - (-CK_{\text{Tor}(P_\Delta)} - n + 2) - (\text{mult } Q - 1) \\ &= n - 3 \geq 1, \end{aligned} \quad (5.6)$$

which yields $\dim \text{pr}(S^{\text{fix}}(w)) \leq n - 3$, and hence gives a contradiction with the lower bound $\dim \text{pr}(S^{\text{fix}}(w)) \geq n - 2$.

Let C have a singular local branch Q centered at w . Then $\dim \text{pr}(S^{\text{fix}}(w)) \geq n - 2$, whereas the computation literally coinciding with that in the preceding paragraph yields the inequality $\dim \text{pr}(S^{\text{fix}}(w)) \leq n - 3$.

(2iv) We are left with the case of $\widehat{w}_\partial(t^*)$ consisting of n distinct points and $\widehat{C}$ either consisting of two components or being a rational curve with a node. Observe that the case of two rational components is not possible, since a generic element of $\mathcal{M}_{0,n}^{\mathbb{R}}(\Delta, \widehat{w}(t))$ will appear after smoothing out two intersection points of the given rational curves, and hence the pairs of rational curves would sweep a locus of codimension 2.

Now suppose that $\widehat{w}_\partial(t^*)$ consists of n distinct points, $\widehat{C} = \widehat{C}_1 \cup \widehat{C}_2$, where $\widehat{C}_1 \simeq \mathbb{P}^1$ and $\widehat{C}_2$ is a smooth elliptic curve, and the intersection $\widehat{C}_1 \cap \widehat{C}_2$ consists of one point p . Then, for dimension reasons, the sequence $\widehat{w}_\partial(t^*)$ is in general position subject to exactly two Menelaus-type relations, while $w_0(t^*)$ is in general position in $\mathcal{Q}$. Thus, by Lemma 2.4, we obtain the immersion and smoothness statements as required. The rest of the argument literally coincides with the corresponding part of the proof of Lemma 4.2(2iii).

(2v) Suppose that $\widehat{C}$ is a rational curve with a node p . Then, this node cannot be mapped to $\text{Tor}(P_\Delta)^\times$, since otherwise one would encounter a real rational curve with two 1-dimensional branches, one in $\text{Tor}_{\mathbb{R}}^+(P_\Delta)$ and the other in $\overline{\mathcal{Q}}$. Hence the node is mapped to $w \in \widehat{w}_\partial(t^*)$, and for the above reason, the intersection multiplicities of the two branches of $C = \Phi(\widehat{C})$ at w are odd. Observe that $\Phi': \mathbb{P}^1 \rightarrow \widehat{C} \rightarrow \text{Tor}(P_\Delta)$ is not a multiple covering of the image. Indeed, in such a case, one would have that P is a triangle, $l_1 = l_2 = l$, and $C = \Phi(\widehat{C})$ is a rational curve intersecting at least two of the toric divisors with odd multiplicity, while the covering is double ramified at the two points of $\widehat{w}_\partial(t^*) \setminus \{w\}$. With an automorphism of $\mathbb{Z}^2$, we can turn the Newton

triangle of C (which is $\frac{1}{2}P_\Delta$) into the triangle

$$\text{conv}\{(0, s), (r, 0), (r + 2l + 1, 0)\}, \quad \gcd(s, r) \equiv 1 \pmod{2},$$

with $w \in \text{Tor}([(r, 0), (r + 2l + 1, 0)])$ and C given by a parametrization (in some affine coordinates x, y)

$$x = a\theta^s, \quad y = b\theta^r(\theta - 1)^{2l+1}, \quad a, b \in \mathbb{R}^\times.$$

Since $x(1) > 0$, we get $a > 0$, and hence $x > 0$ as $0 < \theta \ll 1$. Then $y > 0$ for $0 < \theta \ll 1$, and hence $b < 0$. Thus, if r and s are odd, we obtain $x > 0, y < 0$ as $\theta \rightarrow +\infty$, and $x < 0, y < 0$ as $\theta \rightarrow -\infty$. If r is even and s is odd, we get $x > 0, y < 0$ as $\theta \rightarrow +\infty$, and $x < 0, y > 0$ as $\theta \rightarrow -\infty$. In both cases, it follows that the intersection point with the toric divisor $\text{Tor}([(0, s), (r + 2l + 1, 0)])$ is out of $\partial \text{Tor}_\mathbb{R}^+(P_\Delta)$, which goes against the initial assumption. If r is odd and s is even, we get that the segment of $\mathbb{R}C$ in the quadrant $\{x > 0, y < 0\}$ touches all three toric divisors, which contradicts the condition (AQC). Thus, a multiple covering is excluded. The rest of the claims can be derived in the same way as the statement of Lemma 4.2(2ii) with one exception: We can show that the local branches of C at w intersect each other with multiplicity $\min\{2l_1 + 1, 2l_2 + 1\}$. For $l_1 \neq l_2$, this is immediate. Suppose that $l_1 = l_2 = l$. If $n = 3$, then P is a triangle and, in the above setting, we get the parametrization of C in the form

$$x = a\theta^{2s}, \quad y = b\theta^{2r}(\theta - 1)^{2l+1}(\theta - \theta_0)^{2l+1}, \quad \theta_0 \neq 1, \quad a, b \in \mathbb{R}^\times.$$

Since $x(1) = x(\theta_0)$, we obtain the only real solution $\theta_0 = -1$, and hence $y = b\theta^{2r}(\theta^2 - 1)^{2l-1}$ is a function of θ^2 , as is x , which means that this is a double covering, forbidden above.

(2vi) Now suppose that $\widehat{C}$ is a smooth elliptic curve and $\Phi: \widehat{C} \rightarrow \text{Tor}(P_\Delta)$ is a multiple covering of its image. Since the preimages of at least $n - 2$ points of $\widehat{w}_\partial(t^*)$ in $\widehat{C}$ are irreducible, the map $\Phi: \widehat{C} \rightarrow C \subset \text{Tor}(P_\Delta)$ is ramified, and hence C cannot be elliptic by the Riemann–Hurwitz formula. Thus, C is rational, which for dimension reasons implies that $\widehat{w}_\partial(t^*)$ consists of n points in general position subject to one Menelaus condition. In particular, C is smooth at each point of $\widehat{w}_\partial(t^*)$, and Φ has ramification index s at all points of $\widehat{w}_\partial(t^*)$. By the Riemann–Hurwitz formula,

$$0 \leq 2s - (s - 1)n.$$

Hence, $n \leq 2s/(s - 1)$, which in view of the assumption $n \geq 4$ yields $s = 2$ and $n = 4$ as stated in item (2vi). $\square$

5.3 Wall-crossing

We complete the proof of Theorem 3.6 with the following lemma.

LEMMA 5.3. *Let $\{(\widehat{w}_\partial(t), w_0(t))\}_{0 \leq t \leq 1}$ be a generic path in $\text{Men}_\mathbb{R}^\rho \times \mathcal{Q}$, and let $t^* \in (0, 1)$ be such that $\overline{\mathcal{M}}_{1,n}^\mathbb{R}(\Delta, \widehat{w}(t^*))$ contains an element ξ as described in one of the items of Lemma 5.2(2). Then, for each $\kappa \in \frac{1}{2}\mathbb{Z}$ such that $|\kappa| \leq \mathcal{A}(\Delta)$, the numbers $W_1^\kappa(t) := W_1^\kappa(\Delta, \widehat{w}(t))$ and $\widetilde{W}_1^\kappa(t) := \widetilde{W}_1^\kappa(\Delta, \widehat{w}(t))$ do not change as t varies in a neighborhood of t^* .*

Proof. We can always assume that, in a neighborhood of t^* , the path $\{(\widehat{w}_\partial(t), w_0(t))\}_{0 \leq t \leq 1}$ is defined by fixing the position of $w_0(t^*)$ and some $n - 2$ points of $\widehat{w}_\partial(t^*)$, while the other two points remain mobile. Except for the case of Lemma 5.2(2iv) describing reducible degenerations, we work with families of curves that are trivially covered by families of complex oriented curves so that the quantum index persists along each component of the family of oriented curves.

(1) Suppose that ξ is as in Lemma 5.2(2i). Consider the lifts of $w_0(t^*)$ to $\widehat{C}$ that correspond to real branches of C centered at $w_0(t^*)$. Since C is immersed, we only have to show that each of the lifts induces a germ of a smooth 1-dimensional subfamily in $\mathcal{M}_{1,n}(\text{Tor}(P_\Delta), \mathcal{L}_{P_\Delta})$. This follows from the smoothness statements in [DH88, Proposition 4.17(2)] and in [Shu17, Lemma 3(1)] and [IKS18, Lemma 2.4(2)].

(2) Suppose that ξ is as in Lemma 5.2(2ii). We follow the argument used in steps (1) and (2) of the proof of Lemma 4.3. Since the local treatment of the occurring bifurcations is the same as in the proof of Lemma 4.3, we derive the constancy of $W_1^\kappa(t)$ and $\widetilde{W}_1^\kappa(t)$ along the interval $(t^* - \varepsilon, t^* + \varepsilon)$ from the fact that the image of $H^0(C, \text{in}^* \mathcal{L}_{P_\Delta})$ in the product $\prod_{\sigma,i} \text{Vers}(C, w_i^\sigma) \times \prod_{z \in \text{Sing}(C) \cap \text{Tor}(P_\Delta)} \text{Vers}(C, z)$ transversally intersects with the tangent cone to the locus (4.6), where w_i^σ is a center of a local branch of C of type A_{2m} , $m \geq k_i^\sigma$. This amounts to a relation of the form (4.7), which we rewrite as

$$h^1(\widehat{C}, \mathcal{O}_{\widehat{C}}(\mathbf{d}_1)) = 0, \quad (5.7)$$

where the divisor $\mathbf{d}_1$ satisfies

$$\begin{aligned} \deg \mathbf{d}_1 &= C^2 - (C^2 + CK_{\text{Tor}(P_\Delta)}) - (-CK_{\text{Tor}(P_\Delta)} - n + 1) - (\text{mult}(C, w_i^\sigma) - 1) \\ &= n - 2 > 0 \end{aligned}$$

and $\text{mult}(C, w_i^\sigma)$ stands for the multiplicity of C at w_i^σ . Thus, the required transversality follows.

(3) The treatment of the case where ξ is as in Lemma 5.2(2iii) is the same as that in step (3) of the proof of Lemma 4.3. The required transversality condition now reads as (cf. (4.8))

$$h^1(\widehat{C}, \mathcal{O}_{\widehat{C}}(\mathbf{d})) = 0,$$

where

$$\deg \mathbf{d} = C^2 - (C^2 + CK_{\text{Tor}(P_\Delta)}) - (-CK_{\text{Tor}(P_\Delta)} - 2) = 2 > 0,$$

which immediately yields the above h^1 -vanishing.

(4) Suppose that ξ is as in Lemma 5.2(2iv) and $\Phi(p) \in \text{Tor}(P_\Delta)^\times$. The real part of the elliptic curve $C_2 = \Phi(\widehat{C}_2)$ has two 1-dimensional components, one in $\text{Tor}_\mathbb{R}^+(P_\Delta)$ and the other in $\overline{\mathcal{Q}}$. The real part of the rational curve C_1 lies in either $\text{Tor}_\mathbb{R}^+(P_\Delta)$ or $\overline{\mathcal{Q}}$. Again, the treatment of the wall-crossing is very similar to that in step (4) of the proof of Lemma 4.3. We only comment on the transversality conditions ensuring the realizability of this scenario of the wall-crossing in a suitable variation of the constraints. Thus, we have a combination of two h^1 -vanishing relations: (i) One is for $\widehat{C}_1$, and it coincides with (4.9); (ii) the other reads (cf. (4.9))

$$h^1(\widehat{C}_2, \mathcal{O}_{\widehat{C}_2}(\mathbf{d}_2)) = 0,$$

where

$$\deg \mathbf{d}_2 = C_2^2 - (C_2^2 + CK_{\text{Tor}(P_\Delta)}) - (-CK_{\text{Tor}(P_\Delta)} - 1) = 1 > 0,$$

which yields the desired h^1 -vanishing.

Suppose that ξ is as in Lemma 5.2(2iv) and $\Phi(p) = w_i^\sigma \in \widehat{w}(t^*)$ (to relax notation, below we write w instead of w_i^σ). Again, the local description of the corresponding wall-crossing event can be borrowed from part (5) of the proof of Lemma 4.3, and, correspondingly, the signs $W_1^\kappa(\widehat{w}(t))$ and $\widetilde{W}_1^\kappa(\widehat{w}(t))$ do not change as t transits through t^* . Thus, we should confirm the required transversality conditions.

If $\Phi: \widehat{C}_1 \rightarrow C_1$ and $\Phi: \widehat{C}_2 \rightarrow C_2$ both are immersions, then we need relation (4.11), which reduces to two h^1 -vanishing conditions (4.12) on $\widehat{C}_1 \simeq \mathbb{P}^1$ and on $\widehat{C}_2$, a smooth elliptic curve.

For $\widehat{C}_1$, that h^1 -vanishing holds due to (4.13). For $\widehat{C}_2$, it reads

$$h^1(\widehat{C}_2, \mathcal{O}_{C_2}(\mathbf{d}_2)) = 0, \quad (5.8)$$

where the divisor $\mathbf{d}_2$ satisfies

$$\begin{aligned} \deg \mathbf{d}_2 &= C_2^2 - (C_2^2 + C_2 K_{\text{Tor}(P_\Delta)}) - (-C_2 K_{\text{Tor}(P_\Delta)} - |C_2 \cap \text{Tor}(\partial P_\Delta)| + 1) \\ &= |C_2 \cap \text{Tor}(\partial P_\Delta)| - 1 \geq 2 > 0, \end{aligned}$$

and hence (5.8) holds too.

In the case where $\Phi: \widehat{C}_1 \rightarrow C_1$ is a $2l_1$ -multiple ramified covering and $\Phi: \widehat{C}_2 \rightarrow C_2$ is an immersion, the required transversality condition takes the form of (4.14). Using the Horace method in the same manner as in part (5b) of the proof of Lemma 4.3, we come to the system of h^1 -vanishing relations (4.15). The top-line relations in (4.15) are the same in the current setting and hence hold for the same reason, while the second-line relation turns into

$$h^1(\widehat{C}_2, \mathcal{O}_{\widehat{C}_2}(\mathbf{d})) = 0,$$

where

$$\deg \mathbf{d} = C_2^2 - (C_2^2 + C_2 K_{\text{Tor}(P_\Delta)}) - (-C_2 K_{\text{Tor}(P_\Delta)} - 1) = 1 > 0,$$

and hence it holds too.

(5) Suppose that ξ is as in Lemma 5.2(2v). Since $\widehat{C}$ is rational, both local branches B_1, B_2 of $C = \Phi(\widehat{C})$ at $w = w_i^\sigma(t^*)$ are real, and they intersect $\text{Tor}(\sigma)$ with odd multiplicities $2l_1 + 1$ and $2l_2 + 1$, respectively, where $2l_1 + 2l_2 + 2 = 2k_i^\sigma$ with $l_1 \leq l_2$. Choose two mobile points $w_{i'}^{\sigma'}, w_{i''}^{\sigma''} \in \widehat{\mathbf{w}}_\partial(t^*) \setminus \{w\}$, and fix the other points of $\widehat{\mathbf{w}}(t^*)$. The 1-dimensional stratum $\{\overline{\mathcal{M}}_{1,n}^{\mathbb{R},+}(\Delta, \mathbf{w}(t))\}_{|t-t^*|<\zeta}$, $0 < \zeta \ll 1$, projects to the germ at C of a 1-dimensional variety in $|\mathcal{L}_{P_\Delta}|$; we denote by $\mathcal{V}$ one of its real irreducible components. Let the tangent line to $\mathcal{V}$ at C be spanned by C and $C^* \in |\mathcal{L}_{P_\Delta}| \setminus \{C\}$. Fix conjugation-invariant coordinates x, y in a neighborhood of w so that $w = (0, 0)$ and $\text{Tor}(\sigma) = \{y = 0\}$. Then, in a neighborhood of w , the curve C is given by an equation

$$(y + \eta_1 x^{2l_1+1})(y + \eta_2 x^{2l_2+1}) + \text{h.o.t.} = 0, \quad \eta_1, \eta_2 \in \mathbb{R}^\times,$$

with Newton diagram (see Figure 6(a,b))

$$\Gamma = [(0, 2), (2l_1 + 1, 1)] \cup [(2l_1 + 1, 1), (2l_1 + 2l_2 + 2, 0)]$$

(cf. Figure 4(a,b)). Exactly as in step (5a) of the proof of Lemma 4.3, an equation of C^* has the monomial $x^{2l_1}y$ with a non-zero coefficient and no other monomials below the Newton diagram Γ .

If $l_1 = l_2 = l$, then as in step (5a) of the proof of Lemma 4.3, we obtain that the equation of the germ of $C^{(t)} \in \mathcal{V}$ at $w = (0, 0)$ is as follows:

$$y^2 + P(x)y + \eta_1 \eta_2 x^{4l+2} + \text{h.o.t.} = 0, \quad P(x) = (\eta_1 + \eta_2)x^{2l+1} + \sum_{j=2l}^0 a_j \tau^{2l+1-j} x^j, \quad (5.9)$$

$$\text{where } \tau = t - t^* \quad \text{and} \quad x^{2l+1}P(1/x) = \lambda_1 \text{Cheb}_{2l+1}(\tau x + \lambda_2) + \lambda_3$$

with real numbers $\lambda_1, \lambda_2, \lambda_3$ uniquely determined by η_1, η_2 and by the coefficient of $x^{2l}y$ in the equation of C^* , and where h.o.t. includes higher powers of τ in the present monomials and the sum of monomials above Γ . If $\eta_1 \eta_2 > 0$, the latter formula yields $2l$ elliptic nodes in a neighborhood of w , all of them lying either in the domain $\{y > 0\}$, or in the domain $\{y < 0\}$. Observe that the change of sign of τ yields the symmetry with respect to the origin for the real

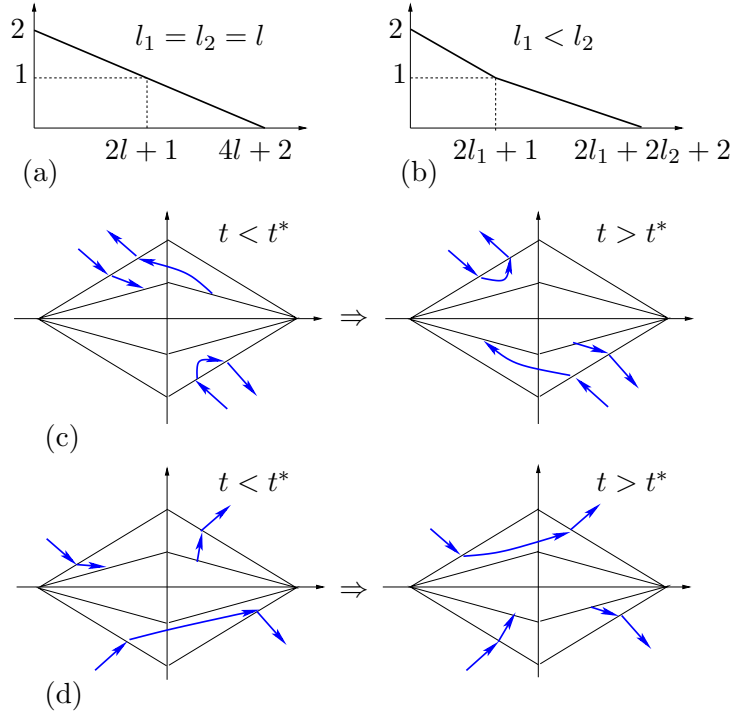


FIGURE 6. Proof of Lemma 5.3, part (5)

part of $C^{(t)}$ (up to h.o.t.), while the orientation of the local branch of $C^{(t)}$ at w persists. Hence the quantum index and the signs $W_1^\kappa(t)$ and $\widetilde{W}_1^\kappa(t)$ remain constant along the path $\{(\widehat{w}(t))\}_{|t-t^*|<\zeta}$. If $\eta_1\eta_2 < 0$, then equation (5.9) implies

$$y = \frac{1}{2} \left(-P(x) \pm \sqrt{P(x)^2 - 4\eta_1\eta_2 x^{4l+2}} \right) + \text{h.o.t.},$$

which means that $C^{(t)}$ has only non-real nodes in a neighborhood of w . Again, the orientation of the local branch of $C^{(t)}$ at w persists (see Figure 6(d)), and we conclude that the signs $W_1^\kappa(t)$ and $\widetilde{W}_1^\kappa(t)$ do not change as t transits through t^* .

Basically, the same local bifurcation occurs when $l_1 < l_2$ (cf. step (5a) of the proof of Lemma 4.3). The family $\mathcal{V}$ appears to be smooth, regularly parameterized by $t_1 = t - t^*$ in the form

$$y^2 + \eta_1(x - \eta_0 t_1)^{2l_1+1}y + \eta_1\eta_2 x^{2l_1+2l_2+2} + \text{h.o.t.} = 0, \quad (5.10)$$

where η_0 is determined by C^* , and h.o.t. includes higher powers of t_1 in the present monomials and the sum of monomials above Γ . We skip the rest of details.

Remark 5.4. In equations (5.9) and (5.10), the sign of the coefficient of $x^{2l_1+2l_2+2}$ remains constant, and the sign of the coefficient of y changes as t_1 changes its sign. Geometrically, this means that, for $t_1 < 0$, the real branch of $C^{(t)}$, which is tangent to $\text{Tor}(\sigma)$ at w , lies in $\text{Tor}_{\mathbb{R}}^+(P_\Delta)$, while, for $t_1 > 0$, this real branch lies in $\mathcal{Q}$, or vice versa (see Figure 6(c,d)).

Finally, we establish the transversality condition sufficient for the realizability of the wall-crossing described above by a variation of constraints. Namely, we prove that representing the local deformation as above, and crossing the wall transversally, we need the following transversality condition: The Zariski tangent space to the germ at C^* of the family constrained by the

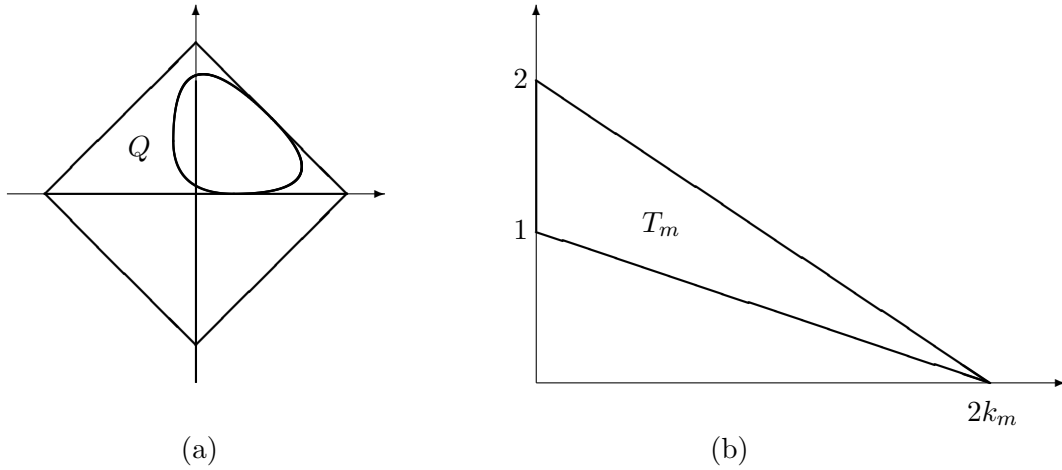


FIGURE 7. Proof of Lemma 4.3, part (6)

given tangency conditions at fixed points of $\widehat{w}_\partial(t^*)$ and at two mobile points of $\widehat{w}_\partial(t^*)$ as well as by fixing the point w_0 has dimension 0. The routine computations, performed in preceding steps, reduce this condition to

$$h^0\left(\mathbb{P}^1, \mathcal{N}_{\Phi'}\left(-\sum_{p_i^\sigma \neq p} 2k_i^\sigma p_i^\sigma + p_{i'}^{\sigma'} + p_{i''}^{\sigma''} - (2l_1 + 1)p' - (2l_2 + 1)p'' - p_0\right)\right) = 0, \quad (5.11)$$

where $\Phi': \mathbb{P}^1 \rightarrow \widehat{C} \xrightarrow{\Phi} \text{Tor}(P_\Delta)$ is Φ combined with the normalization, and p', p'' are the preimages of $p \in \widehat{C}$. This h^1 -vanishing follows from the fact that the degree of the line bundle is

$$(-CK_{\text{Tor}(P_\Delta)} - 2) + (CK_{\text{Tor}(P_\Delta)} - 2k_i^\sigma + 2) - ((2l_1 + 1) + (2l_2 + 1) + 1) = -1.$$

(6) Suppose that ξ is as in Lemma 5.2(2vi). Since the rational curve $C = \Phi(\widehat{C})$ has one global 1-dimensional real branch and contains the point $w_0 \in Q$, it intersects the toric divisor $\text{Tor}(\sigma)$ at two points w_1, w_2 with odd multiplicities k_1, k_2 , respectively, where $\text{Tor}_\mathbb{R}^+(\sigma)$ is the (unique) common side of $\text{Tor}_\mathbb{R}^+(P_\Delta)$ and Q . Correspondingly, P_Δ is a lattice triangle, and each of the two other toric divisors intersects C with an even multiplicity; see Figure 7(a) (the two fragments of $\mathbb{R}C$ shown in bold are doubly covered by the two ovals of $\mathbb{R}\widehat{C}$).

We define the path $\{(\widehat{w}_\partial(t), w_0(t))\}_{|t-t^*|<\varepsilon}$ by fixing the position of w_1, w_2, w_3 , and w_4 and leaving w_0 mobile. Note that w_0 can be chosen in general position on C . Suppose that $\{w_0(t)\}_{|t-t^*|<\varepsilon}$ sweeps the germ of a real line $L \subset Q$ transversally intersecting $\mathbb{R}C$ at $w_0 = w_0(t^*)$. Also note that $(C, (w_0, w_1, w_2, w_3, w_4))$ has two preimages: marked curves $(\widehat{C}, (p'_0, p_1, p_2, p_3, p_4))$ and $(\widehat{C}, (p''_0, p_1, p_2, p_3, p_4))$. It follows from (2.7) that the quantum index of the two latter curves equipped with any complex orientation is 0. The signs (3.2) and (3.3) are preserved too when we reverse the orientation.

The family $\{\overline{\mathcal{M}}_{1,n}^{\mathbb{R},+}(\Delta, \widehat{w}(t))\}_{|t-t^*|<\varepsilon}$ naturally projects to a bunch of real 1-dimensional subvarieties in the linear system $|\mathcal{L}_{P_\Delta}|$. Denote by $\mathcal{V}$ the union of those components of the image that pass through $2C$ and have the same tangent line ($\mathcal{V}$ is determined up to a finite choice). Let this common tangent line Λ be spanned by $2C$ and some $C^* \in |\mathcal{L}_{P_\Delta}| \setminus \{2C\}$. Two cases are possible: Either C^* contains C as a component, or not.

Let $C \not\subset C^*$. Introduce local conjugation-invariant coordinates x, y in a neighborhood of w_0

so that

$$w_0 = (0, 0), \quad L = \{y = 0\}, \quad C = \{x = 0\}, \quad C^* = \left\{ a + \sum_{i+j>0} a_{ij} x^i y^j = 0 \right\},$$

where $a \neq 0$. Let us parameterize Λ as follows:

$$C(\theta) = \left\{ x^2 + \theta \left(a + \sum_{i+j>0} a_{ij} x^i y^j \right) = 0 \right\}, \quad \theta \in (\mathbb{R}, 0).$$

Then, for $a\theta < 0$, the curve $C(\theta)$ intersects L at two points, $x = \sqrt{-a\theta} + o(\sqrt{\theta})$ and $x = -\sqrt{-a\theta} + o(\sqrt{\theta})$. The same holds for any curve $C' \in \mathcal{V}$ projecting to $C(\theta)$; hence, C' contributes to the count of $W_1^0(t)$ and $\widetilde{W}_1^0(t)$ both for $t < t^*$ and for $t > t^*$. This yields the constancy of $W_1^0(t)$ and $\widetilde{W}_1^0(t)$ over the segment $t \in (t^* - \varepsilon, t^* + \varepsilon)$.

Let $C^* = C \cup C'$, $C' \neq C$. In the above coordinates x, y , we have

$$C^* = \left\{ x \left(a + \sum_{i+j>0} a_{ij} x^i y^j \right) = 0 \right\}, \quad a \neq 0.$$

Thus,

$$\begin{aligned} \Lambda &= \left\{ C(\theta) := \left\{ x \left(x + \theta \left(a + \sum_{i+j>0} a_{ij} x^i y^j \right) \right) = 0 \right\} \right\}_{\theta \in (\mathbb{R}, 0)}, \\ C(\theta) \cap L &= \{w_0, w_0(t)\}, \quad w_0(t) = (-a\theta + O(\theta^2), 0). \end{aligned} \quad (5.12)$$

Without loss of generality, we can assume that $\mathcal{V}$ consists of one component. If $\mathcal{V}$ homeomorphically projects onto the germ $(\Lambda, 2C)$, and hence onto the segment $(t^* - \varepsilon, t^* + \varepsilon)$, we establish the constancy of $W_1^0(t)$ and $\widetilde{W}_1^0(t)$ by showing that a curve $C^{(t)}$ makes the same contribution to $W_1^0(t)$ both for $t < t^*$ and for $t > t^*$, and so for $\widetilde{W}_1^0(t)$.

To this end, we analyze the singularities of the curve $C^{(t)}$, $0 < |t - t^*| < \varepsilon$. The curve C has $(C^2 + CK_{\text{Tor}(P_\Delta)} + 2)/2$ nodes. In a neighborhood of each of these nodes, the curve $C^{(t)}$ has four nodes: four hyperbolic nodes in a neighborhood of a hyperbolic node of C , two elliptic and two complex-conjugate nodes in a neighborhood of an elliptic node of C , four non-real nodes in a neighborhood of a non-real node of C . There remain

$$\frac{4C^2 + 2CK_{\text{Tor}(P_\Delta)}}{2} - 4 \cdot \frac{C^2 + CK_{\text{Tor}(P_\Delta)} + 2}{2} = k_1 + k_2 + k_3 + k_4 - 4$$

nodes of $C^{(t)}$. Let us show that they are located in a neighborhood of the points w_1, w_2, w_3, w_4 . Consider the tropical limit of the family of the parameterized curves

$$\{\Phi_t: \widehat{C}^{(t)} \rightarrow C^{(t)} \hookrightarrow \mathfrak{X}_t = \text{Tor}(P_\Delta)\}_{C^{(t)} \in \mathcal{V}, 0 < |t - t^*| < \varepsilon}$$

as $t \rightarrow t^*$ (see details in [GS22, Section 1], where the setting is most suitable for our argument). The limit of the maps $\Phi_t: \widehat{C}^{(t)} \rightarrow \mathfrak{X}_t = \text{Tor}(P_\Delta)$, $t \rightarrow t^*$, is $\Phi_0: \widehat{C}^{(0)} \rightarrow \mathfrak{X}_0$, where $\widehat{C}^{(0)}$ is the union of the elliptic curve $\widehat{C}$ with four marked points p_1, p_2, p_3, p_4 mapped to w_1, w_2, w_3, w_4 , respectively, and of connected curves $\widehat{C}_1, \widehat{C}_2, \widehat{C}_3, \widehat{C}_4$ of arithmetic genus 0 attached to p_1, p_2, p_3, p_4 , respectively. Pick $m \in \{1, 2, 3, 4\}$ and introduce local conjugation-invariant coordinates x_m, y_m in a neighborhood of w_m so that $w_m = (0, 0)$, the toric divisor passing through w_m is given by $\{y = 0\}$, and the germ of C at w_m is given by $\{y - x^{k_m} = 0\}$. In particular, a local equation of $2C$ reads $(y_m - x^{k_m})^2 = 0$. Components of $\widehat{C}_m$ are mapped to toric components of $\mathfrak{X}_0$ that correspond to a certain subdivision of the region under the segment $[(0, 2), (2k_m, 0)]$. In view of the given

data, the only possible subdivision contains just one triangle $T_m = \text{conv}\{(0, 1), (0, 2), (2k_m, 0)\}$ (see Figure 7(b)). The limit curve $C_m \subset \text{Tor}(T_m)$ is rational, quadratically tangent to the toric divisor $\text{Tor}([(0, 2), (2k_m, 0)])$, and it has $k_m - 1$ nodes (cf. [Shu06, Lemma 3.5]). An equation of C_m reads

$$(y_m - x_m^{k_m})^2 + \sum_{i=0}^{k_m-1} a_i^{(m)} x_m^i y_m = 0, \quad a_0^{(m)} \neq 0.$$

Correspondingly, in a neighborhood of w_m , the curves $C^{(t)} \in \mathcal{V}$ are given by

$$(y_m - x_m^{k_m})^2 + \sum_{i=0}^{k_m-1} \tau^{(k_m-i)r_m} a_i^{(m)} x_m^i y_m + \text{h.o.t.} = 0, \quad (5.13)$$

where $\tau = t - t^*$ and h.o.t. denotes terms with higher powers of τ in the preceding monomials as well as monomials above the segment $[(0, 2), (2k_m, 0)]$.

If $m = 1$ or 2 , then k_m is odd. In this case, all exponents of τ in the second summand of (5.13) must be even since otherwise the coefficient of y_m will change sign when τ does, but geometrically this would mean that the local real branch of $C^{(t)}$ at w_m would jump from the quadrant Q to the positive quadrant when t transits through t^* , which would contradict the fact that $C^{(t)}$ must have a global real branch in Q . This observation yields that the distribution among the quadrants of the elliptic nodes in a neighborhood of w_m does not change when τ changes its sign. If $m = 3$ or 4 , then k_m is even. If all exponents of τ in the second summand of (5.13) are even, we obtain the preceding conclusion. If there are odd exponents, then the change of sign of τ geometrically is equivalent to the reflection of the limit curve C_m with respect to the y_m -axis, which again does not affect the distribution of elliptic nodes among quadrants. In all of the cases, we immediately derive the constancy of $W_1^0(t)$ and $\widetilde{W}_1^0(t)$ along the segment $t \in (t^* - \varepsilon, t^* + \varepsilon)$.

It remains to show that the case of $\mathcal{V}$ doubly covering one half of the interval $(t^* - \varepsilon, t^* + \varepsilon)$ with ramification at t^* is not possible. We argue by contradiction based on the Bézout theorem. Let $C^{(t')}, C^{(t'')} \in \mathcal{V}$ be two curves with $t' < t^* < t''$ that intersect the line L at the same point $w_0(t') = w_0(t'')$ given by the second formula in (5.12). In a neighborhood of $\text{Sing}(C)$, the curves $C^{(t')}$ and $C^{(t'')}$ have at least

$$8 \cdot \frac{C^2 + CK_{\text{Tor}(P_\Delta)} + 2}{2} = 4C^2 + 4CK_{\text{Tor}(P_\Delta)} + 8 = 4C^2 - \sum_{m=1}^4 (4k_m - 2)$$

intersection points. To estimate the intersection multiplicity of $C^{(t')}$ and $C^{(t'')}$ in a neighborhood of w_m , $m = 1, 2, 3, 4$, we rescale the formula (5.13), bringing it to the following common form for t' and t'' (since $|t' - t^*|$ and $|t'' - t^*|$ are asymptotically equivalent as $t', t'' \rightarrow t^*$):

$$(y_m - x_m^{k_m})^2 + \sum_{i=0}^{k_m-1} (\text{sign } \tau)^{(k_m-i)r_m} a_i^{(m)} x_m^i y_m + O(\tau) = 0.$$

In a neighborhood of w_m , $m = 1, 2, 3, 4$, the curves $C^{(t')}$ and $C^{(t'')}$ intersect with multiplicity at least $4k_m - 2$. Indeed, if the change of sign of τ in (5.13) yields the same limit curve C_m , then $C^{(t')}$ and $C^{(t'')}$ intersect with multiplicity at least $2k_m$ at w_m and intersect with multiplicity at least $2(k_m - 1)$ near the nodes of C_m . If the change of sign of τ induces the reflection of C_m against the y_m -axis, then $C^{(t')}$ and $C^{(t'')}$ intersect in the big torus of $\text{Tor}(T_m)$ with multiplicity $2k_m - 2$ (here we subtract the intersection multiplicity at the common tangency point on the toric divisor $\text{Tor}([(0, 2), (2k_m, 0)])$) and they intersect at w_m with multiplicity at least $2k_m$. Taking into account the extra intersection point $w_0(t') = w_0(t'') \in L$, we arrive at a contradiction with

the Bézout theorem. $\square$

6. Invariance in genus 2: Proof of Theorem 3.8

We closely follow the arguments in the proofs of Theorems 3.2 and 3.6. Take two sequences $\mathbf{w}(0) = (\mathbf{w}_\partial(0), w_1(0), w_2(0))$ and $\mathbf{w}(1) = (\mathbf{w}_\partial(1), w_1(1), w_2(1))$ satisfying the conditions of Theorem 3.6. We may assume that $\widehat{\mathbf{w}}_\partial(0), \widehat{\mathbf{w}}_\partial(1) \in \text{Men}_\mathbb{R}^\rho(\Delta)$ for some $\rho > 0$. Then, we join these sequences by a generic path $\{\widehat{\mathbf{w}}_\partial(t)\}_{0 \leq t \leq 1}$ in $\text{Men}_\mathbb{R}^\rho(\Delta)$, and we join the points $w_i(0), w_i(1)$ by a generic path $\{w_i(t)\}_{0 \leq t \leq 1}$ in the respective open quadrant, where $i = 1, 2$. Then we verify the constancy of $W_2^\kappa(\Delta, \widehat{\mathbf{w}}(t))$ inside the chambers (that is, connected components of $\text{Men}_2(\Delta)$) and in all possible wall-crossing events, for all κ .

6.1 Invariance inside chambers

Let us verify the constancy of $W_2^\kappa(\Delta, \widehat{\mathbf{w}}(t))$ along intervals $t' < t < t''$ such that $\mathbf{w}(t) \in \text{Men}_2(\Delta)$ for all $t' < t < t''$. To this end, it is enough to show that the projection $\mathcal{M}_{2,n}^\mathbb{R}(\Delta, \widehat{\mathbf{w}}(t))_{t' < t < t''} \rightarrow (t', t'')$ has no critical points (cf. the proofs of Theorems 3.2 and 3.6). This requirement is a consequence of the following statement similar to Lemmas 4.1 and 5.1 (cf. [Shu15, Lemma 2.3]).

LEMMA 6.1. *Let $\xi = [\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)] \in \mathcal{M}_{2,n}^\mathbb{R}(\Delta, \widehat{\mathbf{w}})$, where $\widehat{\mathbf{w}} \in \text{Men}_2(\Delta)$. Then,*

$$h^0\left(\widehat{C}, \mathcal{N}_\Phi\left(-p_1 - p_2 - \sum_{s=0}^2 \sum_{i=1}^{n_s} 2k_i^{(s)} p_i^{(s)} + p_{n_0}^{(0)}\right)\right) = 0, \quad (6.1)$$

where $[\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \mathbb{P}^2] \in \mathcal{M}_{2,n}^\mathbb{R}(\mathbb{P}^2, \Delta, \widehat{\mathbf{w}}(t))$.

Proof. Non-vanishing in (6.1) would mean the existence of a real plane curve $C' \neq C = \Phi(\widehat{C})$ of degree $2d$ passing through w_1 and w_2 , intersecting D_s , $s = 0, 1, 2$, at each point $n_i^{(s)}$, $1 \leq i \leq n_s$, where $(s, i) \neq (0, n_0)$, with multiplicity $2k_i^{(s)}$, intersecting D_0 at $w_{n_0}^{(0)}$ with multiplicity $2k_{n_0}^{(0)} - 1$, and intersecting C at each point $z \in \text{Sing}(C)$ with multiplicity at least $\delta(C, z)$. Note that C' must additionally intersect C in an extra point on the real branch in $(\mathbb{P}_\mathbb{R}^2)^{(-,+)}$ and in an extra point on the real branch in $(\mathbb{P}_\mathbb{R}^2)^{(+,-)}$. However, this leads to a contradiction with the Bézout theorem:

$$\begin{aligned} 4d^2 &\geq 4 + \sum_{s=0}^2 \sum_{i=1}^{n_s} 2k_i^{(s)} - 1 + \sum_{z \in \text{Sing}(C)} 2\delta(C, z) \\ &= 4 + (6d - 1) + ((2d - 1)(2d - 2) - 4) = 4d^2 + 1. \end{aligned} \quad \square$$

6.2 Walls in the space of constraints

Let $[\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \mathbb{P}^2] \in \overline{\mathcal{M}}_{2,n}^\mathbb{R}(\Delta, \widehat{\mathbf{w}}(t))_{0 \leq t \leq 1}$. In this section, we assume that $\widehat{C}$ does not contain contractible components, that is, we just contract them (if any), and we call the singularities of $\widehat{C}$ “nodes.” Note that such a contraction does not affect the arithmetic genus, since a generic curve $\widehat{C}$ has three real ovals. We also allow the points of $\widehat{\mathbf{p}}$ to collate. We use the notation of Section 5.2; namely, S denotes a locus in the space of constraints $\text{Men}_\mathbb{R}^\rho(\Delta) \times (\mathbb{P}_\mathbb{R}^2)^{(+,-)} \times (\mathbb{P}_\mathbb{R}^2)^{(-,+)}$, and $\text{pr}(S)$ denotes the image of $\overline{\mathcal{M}}_{2,n}^\mathbb{R}(\Delta, \widehat{\mathbf{w}})$ in the linear system $|\mathcal{L}_{P_\Delta}|$ when $\widehat{\mathbf{w}}$ ranges over S .

In view of the results of Section 6.1, we can ignore the walls along which $\overline{\mathcal{M}}_{2,n}^\mathbb{R}(\Delta, \widehat{\mathbf{w}}) = \mathcal{M}_{2,n}^\mathbb{R}(\Delta, \widehat{\mathbf{w}})$. The following lemma classifies other walls.

LEMMA 6.2. (1) The following elements $\xi = [\Phi: (\widehat{C}, \mathbf{p}) \rightarrow \mathbb{P}^2]$ cannot occur in the space $\overline{\mathcal{M}}_{2,n}^{\mathbb{R}}(\Delta, \widehat{\mathbf{w}}(t))_{0 \leq t \leq 1}$:

- (1i) $\widehat{C}$ is a reducible connected curve of arithmetic genus 2 with a component mapped onto a toric divisor;
- (1ii) $\widehat{C}$ is a connected curve of arithmetic genus 2 with at least two nodes;
- (1iii) $\widehat{C}$ has a component that multiply covers its image.
- (2) If $\xi = [\Phi: (\widehat{C}, \widehat{\mathbf{p}}) \rightarrow \mathbb{P}^2] \in \overline{\mathcal{M}}_{2,n}(\Delta, \widehat{\mathbf{w}}(t^*))$ when $\widehat{\mathbf{w}}(t^*)$ does not satisfy the conclusions of Lemma 2.4, then ξ is of one of the following types:
 - (2a) either exactly one of the points $w_1(t^*), w_2(t^*)$ is a singular point of the curve $C = \Phi(\widehat{C})$, and the following holds:
 - (2i) $\widehat{\mathbf{w}}_{\partial}(t^*)$ consists of n distinct points, the curve $\widehat{C}$ is smooth of genus 2, Φ is an immersion onto a curve C that is smooth along the toric divisors, while $w_i(t^*)$ is a center of at least one real local branch, $i = 1, 2$;
 - (2b) or $w_1(t^*)$ and $w_2(t^*)$ are smooth points of the curve $C = \Phi(\widehat{C})$, and one of the following holds:
 - (2ii) $\widehat{C}$ is a smooth curve of genus 2, $\widehat{\mathbf{w}}_{\partial}(t^*)$ consists of $n - 1$ distinct points, Φ is birational onto its image, but not an immersion; it may have singular local branches centered in $\text{Tor}(P_{\Delta})^{\times}$ as well as singularities of type A_{2m} at some points $w_i^{\sigma} \in \widehat{\mathbf{w}}_{\partial}(t^*)$, where $m \geq k_i^{\sigma}$;
 - (2iii) $\widehat{C}$ is a smooth curve of genus 2, $\widehat{\mathbf{w}}_{\partial}$ contains $n - 2$ distinct points so that $w_i^{(s)}(t^*) = w_j^{(s)}(t^*) =: w$ for some $s \in \{0, 1, 2\}$ and $i \neq j$; the map Φ is birational onto its image C , where the curve C is immersed outside $\widehat{\mathbf{w}}_{\partial}(t^*) \setminus \{w\}$ and may have singularities of type A_{2m} at some points of $\widehat{\mathbf{w}}_{\partial}(t^*) \setminus \{w\}$, while at w , the curve C either is smooth or has two smooth branches;
 - (2iv) $\widehat{C} = \widehat{C}_1 \cup \widehat{C}_2$, where $g(\widehat{C}_1) + g(\widehat{C}_2) = 2$, $\widehat{\mathbf{w}}_{\partial}(t^*)$ consists of $n - 1$ distinct points, and the maps $\Phi: \widehat{C}_i \rightarrow \mathbb{P}^2$, $i = 1, 2$, are immersions; furthermore, the only node p joining $\widehat{C}_1$ and $\widehat{C}_2$ either is mapped to $(\mathbb{R}^{\times})^2 \subset \mathbb{P}_{\mathbb{R}}^2$, or is mapped to some point $w_i^{(s)}(t^*) \in \widehat{\mathbf{w}}(t^*)$; the germ $\Phi(\widehat{C}_j, p)$ intersects D_s with multiplicity $2l_j$, $j = 1, 2$, where $l_1 + l_2 = k_i^{(s)}$, and these two germs intersect each other with multiplicity $\min\{2l_1, 2l_2\}$;
 - (2v) $\widehat{\mathbf{w}}_{\partial}(t^*)$ consists of $n - 1$ distinct points, $\widehat{C}$ is an irreducible elliptic curve with one node p , the map Φ is an immersion that sends p to some point $w_i^{(s)}(t^*)$, center of two smooth branches intersecting D_s with odd multiplicities $2l_1 + 1, 2l_2 + 1$, where $l_1 + l_2 + 1 = k_i^{(s)}$, and intersecting each other with multiplicity $\min\{2l_1 + 1, 2l_2 + 1\}$; furthermore, each of the other points of $\widehat{\mathbf{w}}_{\partial}(t^*)$ is a center of one smooth branch.

Proof. (1i) We can assume that the sequences $\{\mathbf{w}(t)\}_{0 \leq t \leq 1}$ avoid ρ_0 -neighborhoods of the intersection points of the coordinate axes, for some $\rho_0 > 0$. Suppose that $\Phi(\widehat{C})$ contains some coordinate axis. In this case, $\Phi(\widehat{C})$ contains all of them. Indeed, otherwise, we would have $\mathbf{w}(t)$ arbitrarily close to the intersection points of the coordinate axes, contrary to the above assumption. Furthermore, the intersection points of the coordinate axes must lift to nodes of $\widehat{C}$, and they smooth out along the path $\{\mathbf{w}(t)\}_{0 \leq t \leq 1}$. In particular, the union $\widehat{C}_1$ of the components of $\widehat{C}$ mapped onto the coordinate axes is a curve of arithmetic genus at least 1.

The union $\widehat{C}_2$ of the other components of $\widehat{C}$ cannot be mapped onto a plane curve of odd degree. Indeed, otherwise, one would get an odd number of real branches of $\Phi(\widehat{C}_2)$ intersecting D_0

with odd multiplicity yielding a 1-dimensional real arc in the quadrant $\{x_1/x_0 < 0, x_2/x_0 < 0\}$, which is not possible. Thus, $\Phi_*(\widehat{C}_2)$ has even degree. If $\widehat{C}_2$ has arithmetic genus 0, then the real part of $\Phi(\widehat{C}_2)$ contains a connected graph passing through w_1 and w_2 , and then there should be four real local branches of $\Phi(\widehat{C}_2)$ centered on $D_1^+ \cup D_2^+$ (see Section 3.4 for the notation) that intersect $D_1 \cup D_2$ with odd multiplicity. Hence, the centers of these branches must lift to the nodes of $\widehat{C}$, which would yield that the arithmetic genus of $\widehat{C}$ is at least 4, giving a contradiction. Thus, $\widehat{C}_1$ and $\widehat{C}_2$ both have arithmetic genus 1 and are joined by one node. It is easy to see that this node must be mapped by Φ to the center of the unique local branch of $\Phi(\widehat{C}_2)$ on D_0^+ . This local branch must lie in the closure of $(\mathbb{P}_{\mathbb{R}}^2)^+$. Hence the image of one connected component of the real part of $\widehat{C}_2$ must pass through, say, w_1 and contain the real local branch centered on D_0^+ . It follows that $\Phi(\widehat{C}_2)$ has two real local branches centered on D_1^+ and intersecting D_1 with odd multiplicity. Thus, their centers must lift to two nodes of $\widehat{C}$ joining $\widehat{C}_1$ and $\widehat{C}_2$, making the arithmetic genus of $\widehat{C}$ at least 3, which gives a contradiction.

(1ii) We argue by contradiction, assuming that $\widehat{C}$ has $k \geq 2$ nodes and no component of $\widehat{C}$ is mapped onto a coordinate axis.

First, $\widehat{C}$ cannot have three or more components, since this would impose two extra Menelaus conditions on the disposition of $\mathbf{w}_{\partial}$ defining a stratum of codimension 2 in $\mathcal{W}$ that must be avoided by the path $\{\mathbf{w}(t)\}_{0 \leq t \leq 1}$.

Second, $\widehat{C}$ cannot be an irreducible rational curve. Indeed, if $\mathbf{w}_{\partial} \in \mathcal{W}_{\partial}$ is a general position, then $\mathbf{w}(\widehat{C})$ belongs to a finite set; see Lemma 2.4. Hence, if $\mathbf{w}_{\partial} \in \mathcal{W}_{\partial}$ varies in a stratum of codimension 1, then we obtain a 1-dimensional family of rational curves, and this imposes an extra condition on the position of the points w_1, w_2 , altogether defining a stratum of codimension at least 2 in $\mathcal{W}$.

Third, $\widehat{C}$ cannot be the union of a rational and an elliptic component. Then, we obtain an extra Menelaus condition on $\mathbf{w}_{\partial}$, while the parts of $\mathbf{w}_{\partial}$ lying on the image of the rational and the elliptic components are in general position. Thus, $\mathbf{w}_{\partial}$ and one of the points w_1, w_2 determine a finite set of pairs of a rational and an elliptic component of $\widehat{C}$ (see Lemma 2.4), which imposes a condition on the location of the second point in $\{w_1, w_2\}$, hence defining a stratum of codimension at least 2 in $\mathcal{W}$.

(1iii) Suppose that $\widehat{C}$ is irreducible and $\Phi: \widehat{C} \rightarrow C \hookrightarrow \mathbb{P}^2$ is a multiple covering. By the Riemann–Hurwitz formula, the curve C cannot be of geometric genus 2. Also, C cannot be rational since otherwise, the fixed position of $\mathbf{w}_{\partial}$ determines a finite set of rational curves, and hence two conditions are imposed on the position of w_1, w_2 . This defines a stratum of codimension at least 2 in $\mathcal{W}$ avoided by the path $\{\mathbf{w}(t)\}_{0 \leq t \leq 1}$. The only remaining option is an elliptic curve C . As in the case of the rational curve C , we derive that $\mathbf{w}_{\partial}$ varies in an open dense subset of $\mathcal{W}_{\partial}$, while the position of w_1, w_2 is constrained by one condition. It follows that all the points of the sequence $\widehat{\mathbf{w}}_{\partial}$ are distinct, and hence they all must be ramification points of the covering. However, we have $\#\widehat{\mathbf{w}}_{\partial} \geq 3$, which yields a contradiction with the Riemann–Hurwitz formula.

In view of part (1ii), it remains to study the case of two-component $\widehat{C} = \widehat{C}_1 \cup \widehat{C}_2$ such that either $g(\widehat{C}_1) = 2$ and $g(\widehat{C}_2) = 0$, or $g(\widehat{C}_1) = g(\widehat{C}_2) = 1$. It follows that $\mathbf{w}_{\partial}$ is constrained by an extra Menelaus condition, and this is the only condition imposed. Hence, all of the points of $\widehat{\mathbf{w}}_{\partial}$ differ from each other, and the genus of the image of any component must be equal to the genus of that component. The argument of the preceding paragraph leaves the only option of a multiple covering $\Phi: \widehat{C}_2 \rightarrow C_2 \hookrightarrow \mathbb{P}^2$ with rational $\widehat{C}_2$ and C_2 . If m is the degree of the covering, then each point $w_i^{(s)} \in C_2$ is a ramification point of the covering of index $m - 1$, and hence we

get a contradiction with the Riemann–Hurwitz formula:

$$2m - (m - 1) \cdot \#(\widehat{w}_\partial \cap C_2) \geq 2, \quad \#(\widehat{w}_\partial \cap C_2) \geq 3.$$

(2) We mainly follow the arguments in parts (2ii)–(2v) of the proof of Lemma 5.2. Items (2i), (2iv), (2v) of the present lemma can be treated in the same manner as items (2i), (2iv), (2v) in Lemma 5.2. Item (2v) appears to be even much simpler, since multiple coverings are not possible as we established in the preceding part of the proof. Items (2ii) and (2iii) require additional arguments.

(2ii) Suppose that $\widehat{w}_\partial(t^*)$ consists of n distinct points, $\widehat{C}$ is a smooth curve of genus 2, and $\Phi: \widehat{C} \rightarrow C = \Phi(\widehat{C})$ is birational but is not an immersion. To verify the conditions of item (2ii) of the lemma, it is enough to consider the case where C has a singular branch at some point $w_i^\sigma \in \widehat{w}_\partial(t^*)$. Fixing the position of the point w_i^σ , we obtain a subfamily $\text{pr}(S^{\text{fix}}(w_i^\sigma)) \subset \text{pr}(S)$ of dimension at least $n - 2 \geq 3$ and then apply [IKS18, inequality (5) in Lemma 2.1] (cf. (5.2)):

$$\begin{aligned} -CK_{\text{Tor}(P_\Delta)} &\geq -2 + (-CK_{\text{Tor}(P_\Delta)} - n + 2) + (n - 3) + (\text{mult } Q - 1) \\ &= -CK_{\text{Tor}(P_\Delta)} - 3 + (\text{mult } Q - 1) \implies \text{mult } Q \leq 4, \end{aligned}$$

where Q is the local branch of C at w_i^σ . It remains to show that $\text{mult } Q = 2$. Arguing by contradiction, assume that $\text{mult } Q = 3$ or 4. Without loss of generality, we can assume that the germ of $\text{pr}(S) \subset |\mathcal{L}_{P_\Delta}|$ at C is an equisingular family. According to items (a) and (b) in Section 2.3, we obtain the following upper bound completely analogous to (2.5):

$$\dim \text{pr}(S^{\text{fix}}(w_i^\sigma)) \leq h^0(C, \mathcal{J}_{Z/C} \otimes_{\mathcal{O}_C} \text{in}^* \mathcal{L}_{P_\Delta}) = h^0(\widehat{C}, \mathcal{O}_{\widehat{C}}(-\mathbf{d}) \otimes_{\mathcal{O}_{\widehat{C}}} \Phi^* \mathcal{L}_{P_\Delta}),$$

where

$$\begin{aligned} \deg \mathbf{d} &= 2 \sum_{z \in \text{Sing}(C)} \delta(C, z) + (-CK_{\text{Tor}(P_\Delta)} - n + 2) + r, \quad 2 \leq r \leq 3, \\ \implies \deg(\mathcal{O}_{\widehat{C}}(-\mathbf{d}) \otimes_{\mathcal{O}_{\widehat{C}}} \Phi^* \mathcal{L}_{P_\Delta}) &= C^2 - \deg \mathbf{d} = n - r \geq \begin{cases} 3, & r = 2, \\ 2, & r = 3. \end{cases} \end{aligned}$$

In the case $r = 2$, we have $h^1(\mathcal{O}_{\widehat{C}}(-\mathbf{d}) \otimes_{\mathcal{O}_{\widehat{C}}} \Phi^* \mathcal{L}_{P_\Delta}) = 0$ since $2g(\widehat{C}) - 2 = 2$. Thus, by the Riemann–Roch formula, we have

$$\dim \text{pr}(S^{\text{fix}}(w_i^\sigma)) \leq h^0(\mathcal{O}_{\widehat{C}}(-\mathbf{d}) \otimes_{\mathcal{O}_{\widehat{C}}} \Phi^* \mathcal{L}_{P_\Delta}) = n - 3,$$

contradicting the lower bound $\dim \text{pr}(S^{\text{fix}}(w_i^\sigma)) \geq n - 2$ pointed above. In the case $r = 3$, we have

$$h^0(\mathcal{O}_{\widehat{C}}(-\mathbf{d}) \otimes_{\mathcal{O}_{\widehat{C}}} \Phi^* \mathcal{L}_{P_\Delta}) \leq h^0(\mathcal{O}_{\widehat{C}}(-\mathbf{d}') \otimes_{\mathcal{O}_{\widehat{C}}} \Phi^* \mathcal{L}_{P_\Delta}),$$

where $\mathbf{d}' = \mathbf{d} - w$ with w being one of the points of $\mathbf{d}$. Thus,

$$\deg \mathbf{d}' = \deg \mathbf{d} - 1 \implies \deg(\mathcal{O}_{\widehat{C}}(-\mathbf{d}') \otimes_{\mathcal{O}_{\widehat{C}}} \Phi^* \mathcal{L}_{P_\Delta}) = n - r + 1 \geq 3,$$

and we derive the same contradiction.

(2iii) Proceeding along the lines of the proof of part (2iii) of Lemma 5.2, we need to verify suitable analogs of the estimates (5.4), (5.5), and (5.6) in the genus 2 case.

An analog of (5.4) reads

$$\begin{aligned} -CK_{\text{Tor}(P_\Delta)} &\geq -2 + (-CK_{\text{Tor}(P_\Delta)} - n + 4) + (n - 4) + \sum_B (\text{mult } Q - 1) \\ &= -CK_{\text{Tor}(P_\Delta)} - 2 + \sum_B (\text{mult } Q - 1) \implies \sum_B (\text{mult } Q - 1) \leq 2, \end{aligned}$$

where B is the set of all singular local branches of C in $\text{Tor}(P_\Delta)^\times \cup \{w, w'\}$ (recall that w' is any point in $\widehat{\mathbf{w}}_\partial(t^*) \setminus \{w\}$). We have analogs of (5.5) and (5.6):

$$\dim \text{pr}(S^{\text{fix}}(w)) \leq h^0(\widehat{C}, \mathcal{O}_{\widehat{C}}(-\mathbf{d}_1) \otimes_{\mathcal{O}_{\widehat{C}}} \Phi^* \mathcal{L}_{P_\Delta}),$$

where

$$\deg \mathbf{d}_1 = 2 \sum_{z \in \text{Sing}(C)} \delta(C, z) + (-CK_{\text{Tor}(P_\Delta)} - n + 3) + r, \quad r = 1 \text{ or } 2,$$

and hence

$$\begin{aligned} \deg(\widehat{C}, \mathcal{O}_{\widehat{C}}(-\mathbf{d}_1) \otimes_{\mathcal{O}_{\widehat{C}}} \Phi^* \mathcal{L}_{P_\Delta}) &= C^2 - \deg \mathbf{d}_1 \\ &= C^2 - (C^2 + CK_{\text{Tor}(P_\Delta)} - 2) - (-CK_{\text{Tor}(P_\Delta)} - n + 3) - r \\ &= n - 1 - r \geq \begin{cases} 3, & r = 1, \\ 2, & r = 2. \end{cases} \end{aligned}$$

Then, we literally follow the last paragraph of the proof of part (2ii) of the lemma and derive that $\dim \text{pr}(S^{\text{fix}}(w)) \leq n - 3$, contrary to the initial lower bound $\dim \text{pr}(S^{\text{fix}}(w)) \geq n - 2$. This contradiction completes the study of case (2iii) of the lemma. $\square$

6.3 Wall-crossing

In the analysis of wall-crossings, we use arguments similar to the ones presented in Sections 4 and 5. The following lemma completes the proof of Theorem 3.8.

LEMMA 6.3. *The numbers $W_2^\kappa(\Delta, \widehat{\mathbf{w}}(t))$ and $\widetilde{W}_2^\kappa(\Delta, \widehat{\mathbf{w}}(t))$ remain constant in each of the wall-crossing events described in Lemma 6.2(2).*

Proof. All local considerations in the proofs of Theorems 3.2 and 3.6 can literally be transferred to our situation. Thus, we only confirm the global transversality relations, which all take the form of certain cohomology-vanishing conditions.

(1) The wall-crossing described in Lemma 6.2(2ii) coincides with that in Lemma 5.2(2ii). The transversality condition in the treatment of this wall-crossing reads (cf. formula (5.7))

$$H^0(\widehat{C}, \mathcal{O}_{\widehat{C}}(\mathbf{d} - p_1 - p_2)) = 0, \quad \mathbf{d} \in \text{Div}(\widehat{C}), \quad (6.2)$$

where in the notation of the proof of Lemma 5.3(2),

$$\begin{aligned} \deg \mathbf{d} &= C^2 - \sum_{z \in \text{Sing}(C)} 2\delta(C, z) - \sum_{w \in \widehat{\mathbf{w}}_\partial(t^*)} \dim \mathcal{O}_{C,w}/I_w \\ &= 4d^2 - (4d^2 - 6d - 2) - (6d - 1) = 3 > (2g - 2)|_{g=2} = 2. \end{aligned}$$

Thus, the Riemann–Roch formula yields $h^0(\widehat{C}, \mathcal{O}_{\widehat{C}}(\mathbf{d})) = 2$, and hence (6.2) holds due to the general position of w_1, w_2 on $C = \Phi(\widehat{C})$.

(2) The wall-crossing described in Lemma 6.2(2iii) coincides with that in the proof of Lemma 5.2(2iii). The required transversality condition coincides with relation (6.2) confirmed in the preceding paragraph.

(3) The wall-crossing described in Lemma 6.2(2iv) coincides with that in Lemma 5.2(2iv), the subcase where the $\Phi: \widehat{C}_i \rightarrow \text{Tor}(\Delta)$, $i = 1, 2$, are birational immersions. The local study is a word-for-word copy of part (5a) of the proof of Lemma 4.3 (restricted to the case of birational immersions). Similarly to part (5b) of the proof of Lemma 4.3 and part (4) of the proof of

Lemma 5.3, the global transversality condition amounts to

$$h^0(\widehat{C}_1, \mathcal{O}_{\widehat{C}_1}(\mathbf{d}_1 - \{p_1, p_2\} \cap \widehat{C}_1)) = h^0(\widehat{C}_2, \mathcal{O}_{\widehat{C}_2}(\mathbf{d}_1 - \{p_1, p_2\} \cap \widehat{C}_2)) = 0,$$

where

$$\deg \mathbf{d}_1 = 2g(\widehat{C}_1) - 1, \quad \deg \mathbf{d}_2 = 2g(\widehat{C}_2) - 1.$$

Thus, the required h^1 -vanishing holds.

(4) The last wall-crossing to consider is associated with the wall described in Lemma 6.2(2v). That wall is very similar to the wall appearing in Lemma 5.2(2v). The local study of the wall-crossing is a word-for-word copy of the study performed in part (5) of the proof of Lemma 5.3. Thus, we only address the global transversality condition analogous to that in formula (5.11):

$$h^0(\widehat{C}, \mathcal{N}_{\Phi'}(\mathbf{d})) = 0, \quad \mathbf{d} = - \sum_{p_i^{(s)} \neq p} 2k_i^{(s)} p_i^{(s)} + p_{i'}^{(s')} + p_{i''}^{(s'')} - (2l_1 + 1)p' - (2l_2 + 1)p'' - p_1 - p_2,$$

where $\Phi': \widehat{C} \rightarrow \widehat{C} \xrightarrow{\Phi} \mathbb{P}^2$ is a composition of Φ with the normalization, $\mathcal{N}_{\Phi'}$ is the normal bundle, $p_{i'}^{(s')}, p_{i''}^{(s'')}$ are mobile points in $\widehat{\mathbf{w}}_{\partial}$ (while the other points are fixed), and p', p'' are the preimages of the node p in $\widehat{C}$. It is easy to see that $\deg \mathcal{N}_{\Phi'}(\mathbf{d}) = 0$. However, the points w_1, w_2 are in general position on $C = \Phi(\widehat{C})$, and hence p_1, p_2 are in general position on $\widehat{C}$, which by the Riemann–Roch formula implies the desired h^0 -vanishing. $\square$

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