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The blow-up formula for the instanton part of Vafa–Witten invariants on projective surfaces

Nikolas Kuhn, Oliver Leigh and Yuuji Tanaka

Abstract

We prove a blow-up formula for the generating series of virtual χ_y-genera for moduli spaces of sheaves on projective surfaces, which is related to a conjectured formula of Göttsche for topological χ_y-genera. Our formula is a refinement of one by Vafa–Witten relating to S-duality. We prove the formula simultaneously in the setting of Gieseker stable sheaves on polarised surfaces and also in the setting of framed sheaves on $\mathbb{P}^2$. The proof is based on the blow-up algorithm of Nakajima–Yoshioka for framed sheaves on $\mathbb{P}^2$, which has recently been extended to the setting of Gieseker H-stable sheaves on H-polarised surfaces by Kuhn–Tanaka.

1. Introduction

Hilbert schemes of points on a projective surface have been extensively studied, and while the picture is not complete, their geometry is well understood. In particular, there are many deep results concerning their enumerative invariants (for example [Göt90, Che96, EGL01, MOP19]). For their higher-rank analogues, the moduli spaces of (semi-)stable sheaves on a surface, the situation is more complicated. Results for topological invariants for moduli spaces in ranks 2 and 3 on some surfaces have been obtained for example in [LQ99, Yos96, MM18, Koo15], but, in general, one does not expect any reasonable formulas for the topological invariants.

More recently, there has been an interest in studying *virtual* analogues of the topological invariants, which are defined using the perfect obstruction theory coming from the moduli problem. This interest is partially motivated by the S-duality conjecture [VW94] concerning Vafa–Witten invariants and by the papers [TT17, TT20], where Tanaka–Thomas proposed a mathematical definition of these invariants as the sum of two parts – one of which is given by the signed virtual Euler characteristics of moduli spaces of stable sheaves. The virtual invariants have the benefit of being invariant under deformations of the surface that preserve the polarisation, and – strikingly – they are in many cases predicted to depend only on finitely many basic invariants of the

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underlying surface. For example, explicit formulas up to rank 5 have been conjectured for the virtual Euler characteristics in [GKL25] (see also [GK18, GK20, GK19a, GKW21], and [GK19b] for an excellent survey).

In this paper, we prove the first completely general result for the virtual χ_y-genus and Euler characteristic of the higher-rank moduli spaces: Namely, we prove a blow-up formula for the virtual χ_y-genera of the moduli spaces, and thereby for virtual Euler characteristics. This is a first and important step towards establishing the existing conjectures, and provides further evidence for them. Similar formulas for the topological invariants had been obtained in [LQ99] for the virtual Hodge polynomial in rank 2, and in [Göt99, Proposition 3.1] for virtual Poincaré polynomials of arbitrary rank moduli spaces. Our result also confirms part of a conjecture of Göttsche [Göt99, Remark 3.2] (stated more generally for virtual Hodge polynomials) when the moduli spaces are unobstructed, so their virtual and topological χ_y-genera agree.

In the course of our proof, we prove a blow-up formula for an equivariant analogue of the χ_y-genus defined on moduli spaces of framed sheaves on $\mathbb{P}^2$ and on its blow-up. These moduli spaces of framed sheaves and their invariants are of independent interest in physics through in Nekrasov's conjecture and related topics (see for example [Nek03, NY05a, NO06, BE06, GNY08, GL10, GNY11]).

For the rest of the paper, we fix a positive integer $r > 0$, which will denote the rank of the sheaves being considered. As mentioned above, we will prove blow-up formulas for two different kinds of moduli spaces. These two situations are as follows.

Situation A (Gieseker stable sheaves, [Moc09, KT21]). In this situation, X is a smooth projective (connected) surface with a fixed polarisation H, and $p\colon \widehat{X} \to X$ is the blow-up of X at a chosen reduced point $\mathrm{pt} \in X$ with exceptional divisor C. Moreover, L_X is a fixed line bundle on X with $c_1 := c_1(L_X)$ such that the intersection number (L_X, H) is coprime to r.

We consider classes $c := r + c_1 + \mathrm{ch}_2 \in H^*(X, \mathbb{Q})$ for various $\mathrm{ch}_2 \in H^4(X, \mathbb{Q})$. On the blow-up $\widehat{X}$, we will write $\widehat{c}_1 := c_1 - k[C]$ for various $k \in \mathbb{Z}$ and consider classes $\widehat{c} := r + \widehat{c}_1 + \mathrm{ch}_2$ for various $\mathrm{ch}_2 \in H^*(X, \mathbb{Q})$. In this situation, we consider the following moduli stacks:

(i) $M(c)$, the moduli stack of oriented Gieseker H-stable sheaves on X with Chern character c (here an oriented sheaf is a pair $(E, \psi\colon \det E \to L_X)$, where E is a Gieseker H-stable sheaf and ψ is an isomorphism),

(ii) $\widehat{M}(\widehat{c})$, the moduli stack of oriented p^*H slope-stable sheaves on $\widehat{X}$ with Chern character $\widehat{c}$[1] (in this case, we define an orientation to be an isomorphism $\det E \xrightarrow{\sim} p^*L_X \otimes \mathcal{O}_{\widehat{X}}(-kC)$ if $\widehat{c}_1 = p^*c_1 - k[C]$).

Situation B (Framed sheaves on $\mathbb{P}^2$, [NY05a, NY05b]). In this situation, we let $X := \mathbb{P}^2$ and consider the blow-up $p\colon \widehat{X} = \widehat{\mathbb{P}}^2 \to X$ at $\mathrm{pt} = [1 : 0 : 0]$ with exceptional divisor C. We set $L_X = \mathcal{O}_{\mathbb{P}^2}$ and $c_1 = 0$. We consider classes on X of the form $c := r + 0 + \mathrm{ch}_2$ for various $\mathrm{ch}_2 \in H^4(\mathbb{P}^2, \mathbb{Q})$. For classes on the blow-up, we consider $\widehat{c}_1 := -k[C]$ for some $k \in \mathbb{Z}$ and $\widehat{c} := r + \widehat{c}_1 + \mathrm{ch}_2$ for various $\mathrm{ch}_2 \in H^4(\widehat{\mathbb{P}}^2, \mathbb{Q})$. In this situation, we consider the following moduli spaces:

[1]Here, slope stability for the nef line bundle p^*H is the same as Gieseker stability for the polarisation $p^*H - \varepsilon[C]$ for any small enough $\varepsilon \in \mathbb{Q}_{>0}$ depending on $\widehat{c}$ (this follows for example from [Kos21, Proposition 4.3]). Since the choice of ε may depend on ch_2, working with slope stability is slightly more convenient when considering generating series.

(i) $M(c)$, the moduli space of *framed sheaves on* $\mathbb{P}^2$ with Chern character c, that is, pairs (E, φ), where E is a coherent sheaf on X which is locally free in a neighbourhood of the line $\ell_\infty := \{[0 : z_1 : z_2]\}$ and where φ is an isomorphism $E|_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}^{\oplus r}$;

(ii) $\widehat{M}(\widehat{c})$, the moduli space of *framed sheaves on* $\widehat{X}$ with Chern character $\widehat{c}$, where the framing is with respect to the preimage $p^{-1}\ell_\infty$ of ℓ_∞ under p, which we denote again by ℓ_∞.

Remark 1.1. In Situation A, we use moduli stacks of oriented sheaves instead of the usual moduli spaces, since the stacks always have universal sheaves, which simplifies some of the presentation. The moduli stacks are degree $1/r$ gerbes over the coarse moduli spaces, and all the invariants that we consider, differ only by a factor of $1/r$.

We will now address how one can define algebraic invariants in Situations A and B. In Situation A, this uses that the moduli stacks are proper and carry a perfect obstruction theory, while in Situation B, one uses the fact that the moduli spaces carry a natural torus action with isolated torus-fixed points.

In what follows, we let M denote either $M(c)$ or $\widehat{M}(\widehat{c})$ in either situation, we let $\mathcal{E}$ denote the universal sheaf on $M \times X$ or $M \times \widehat{X}$, and we let π denote the projection morphism $\pi\colon M \times X \to M$ or $\pi\colon M \times \widehat{X} \to M$. For $M = M(c)$, take $\widehat{c}_1 := c_1$.

(a) *Situation* A: The moduli space M is a proper Deligne–Mumford stack with a canonical perfect obstruction theory of virtual dimension $-2r\operatorname{ch}_2 + \widehat{c}_1^2 - (r^2-1)\chi(X, \mathcal{O}_X)$. The K-theoretic class of the virtual tangent bundle is

$$\mathrm{T}_M^{\mathrm{vir}} = \mathcal{O}_M^{\oplus \chi(X,\mathcal{O}_X)} - R\pi_* R\,\mathcal{H}om(\mathcal{E}, \mathcal{E})\,.$$

(b) *Situation* B: The moduli space M is representable by a smooth quasi-projective variety with dimension given by $-2r\operatorname{ch}_2 + \widehat{c}_1^2$. There is a canonical action of $\widetilde{T} := (\mathbb{C}^*)^{2+r}$ on M whose fixed locus is a set of isolated points. This action extends in a canonical way to $M \times X$ (respectively, $M \times \widehat{X}$), so $\mathcal{E}$ has a canonical structure of equivariant sheaf. The $\widetilde{T}$-equivariant class of the tangent bundle in K-theory is

$$\mathrm{T}_M^{\widetilde{T}} = -R\pi_* R\,\mathcal{H}om(\mathcal{E}, \mathcal{E}(-\ell_\infty))\,.$$

Here are references for these properties. For Situation A, see either [Moc09, § 5.6] or [KT21, Corollary 3.26 and Proposition 5.12], and for Situation B, see [NY05a, §§ 2–3] and [NY11b, § 5]. Using these properties, we now define a notion of "integration" for Chow-cohomology classes on moduli spaces in each situation. We will let F denote a field of characteristic zero, and we write A^* for Chow-cohomology rings with coefficients in F.

Notation 1.2. Let M be a moduli space from either Situation A or Situation B. Define the following integral notation:

(a) *Situation* A: Using the perfect obstruction theory, we get a virtual fundamental class $[M]^{\mathrm{vir}}$. Using this, we define, for $\alpha \in A^*(M)$,

$$\int_M \alpha := \int_{[M]^{\mathrm{vir}}} \alpha := \deg\bigl(\alpha \cap [M]^{\mathrm{vir}}\bigr)\,,$$

where for a Chow class $\gamma \in A_*(M)$, we let $\deg \gamma$ be the number obtained by pushforward along the structure morphism $M \to \operatorname{Spec} \mathbb{C}$.

(b) *Situation* B: In this situation, the moduli space is non-proper but has a proper $\widetilde{T}$-fixed locus. We define integration via the Atiyah–Bott localisation formula. Hence for a $\widetilde{T}$-equivariant

class $\alpha \in A^*_{\widetilde{T}}(M)$,

$$\int_M \alpha := \sum_{x \in M^{\widetilde{T}}} \frac{\alpha|_x}{\mathrm{e}\big(\mathrm{T}^{\widetilde{T}}_M|_x\big)}\,.$$

Moreover, in light of the unified notations of Situations A and B, we suppress the equivariant notation when considering Situation B unless it is necessary. Hence, from now on, in Situation B, we write $A^*(M)$ instead of $A^*_{\widetilde{T}}(M)$ and T_M instead of $\mathrm{T}^{\widetilde{T}}_M$, and we always consider $\mathcal{E}$ as an equivariant sheaf.

Remark 1.3. If we consider the moduli spaces in Situation B as having the trivial perfect obstruction theory, then we have $\mathrm{T}^{\mathrm{vir}}_M = \mathrm{T}_M$, and it makes sense to use the word *virtual* when discussing the enumerative invariants of both situations.

Using the above notation, in this article, we will consider invariants of the form

$$\int_M \Theta\big(\mathrm{T}^{\mathrm{vir}}\big)\,,$$

where $\Theta\colon K(M) \to A^*(M)$ is some characteristic class. In particular, we are interested in the case where $\Theta = \Theta_y$ is the multiplicative class defined on line bundles by $\Theta_y(L) = (1 - y\,\mathrm{ch}(L)^{-1}) \cdot \mathrm{td}(L)$, where y is a formal variable. In this case, we set

$$\chi^{\mathrm{vir}}_{-y}(M) := \int_M \Theta_y\big(\mathrm{T}^{\mathrm{vir}}\big) \in \mathbb{Q}[y]\,.$$

This is the *virtual χ_y-genus* of M as defined by Fantechi–Göttsche in [FG10, § 4]. The main result of the article is the following theorem.

Theorem 1.4 (Main theorem). *Consider Situation A or B with $0 \leqslant k < r$. Consider the generating series of virtual χ_y-genera*

$$\mathsf{Z}(q,y) := \sum_{\mathrm{ch}_2} \chi^{\mathrm{vir}}_{-y}(M(r + c_1 + \mathrm{ch}_2))q^{\mathrm{vdim}M(r+c_1+\mathrm{ch}_2)} \quad \textit{and}$$

$$\widehat{\mathsf{Z}}(q,y) := \sum_{\mathrm{ch}_2} \chi^{\mathrm{vir}}_{-y}\big(\widehat{M}(r + \widehat{c}_1 + \mathrm{ch}_2)\big)q^{\mathrm{vdim}\widehat{M}(r+\widehat{c}_1+\mathrm{ch}_2)}\,.$$

Then, we have the identity

$$\widehat{\mathsf{Z}} = \mathsf{Y}_k \cdot \mathsf{Z}\,,$$

where

$$\mathsf{Y}_k = \mathsf{Y}_k(q,y) := \prod_{n>0}\big(1 - \big(q^2y\big)^{rn}\big)^{-r} \sum_{k_1+\cdots+k_r=k} \big(q^2y\big)^{\sum_{i<j}(k_i-k_j)^2/2}\, y^{\sum_{i<j}(k_i-k_j)/2}\,.$$

Remark 1.5. The factor Y_k appearing in Theorem 1.4 is independent of all input data except k and the rank r. This is the same blow-up behaviour observed for the topological invariants studied in [Göt90, Che96, LQ98, LQ99, Göt99].

Remark 1.6. The result in Theorem 1.4 was originally conjectured in Situation A for the topological χ_y-genus by Göttsche in [Göt99, Remark 3.2] (stated more generally for virtual Hodge polynomials). The factor Y_k can be rewritten in Göttsche's notation as follows. Let $A = (a_{ij})_{ij}$ be the $(r-1) \times (r-1)$ matrix with entries $a_{ij} = 1$ for $i \leqslant j$ and $a_{ij} = 0$ otherwise, and let I be the column vector of length $r-1$ with all entries equal to 1. Let η denote the Dedekind eta

function. Then we have

$$\mathsf{Y}_k = \frac{\left(q^{2r}y^r\right)^{r/24}}{\eta\left(q^{2r}y^r\right)^r} \sum_{v\in\mathbb{Z}^{r-1}+(k/r)I} \left(q^{2r}y^r\right)^{v^tAv} y^{v^tAI}\,.$$

In the cases where the moduli problem is unobstructed (that is, if every stable sheaf E on X and $\widehat{X}$ has $\mathrm{Ext}^2(E,E)=0$), the topological and virtual χ_y-genera coincide. Hence, in these cases, Theorem 1.4 proves Göttsche's topological conjecture.

One can also consider the virtual Euler characteristic and the virtual holomorphic Euler characteristic. These are defined, respectively, as

$$e^{\mathrm{vir}}(M) := \int_M \mathrm{e}\left(\mathrm{T}^{\mathrm{vir}}_M\right) \quad \text{and} \quad \chi^{\mathrm{vir}}(M) := \int_M \mathrm{td}\left(\mathrm{T}^{\mathrm{vir}}_M\right);$$

they are the respective specialisations χ^{vir}_{-1} and χ^{vir}_0 of χ^{vir}_y. Using these characterisations, Theorem 1.4 has the following corollary.

Corollary 1.7. *In the setting of Theorem 1.4, using appropriately defined generating series, we have*

$$\widehat{\mathsf{Z}}_{e^{\mathrm{vir}}} = \mathsf{Y}_{k,e^{\mathrm{vir}}}\cdot \mathsf{Z}_{e^{\mathrm{vir}}} \quad \textit{and} \quad \widehat{\mathsf{Z}}_{\chi^{\mathrm{vir}}} = \mathsf{Y}_{k,\chi^{\mathrm{vir}}}\cdot \mathsf{Z}_{\chi^{\mathrm{vir}}}\,,$$

where

$$\mathsf{Y}_{k,e^{\mathrm{vir}}} = \prod_{n>0}\left(1-q^{2rn}\right)^{-r} \sum_{\sum_{i=1}^r k_i = k} q^{\sum_{i<j}(k_i-k_j)^2} \quad \textit{and} \quad \mathsf{Y}_{k,\chi^{\mathrm{vir}}} = \begin{cases} 1 & \textit{if } k=0\,,\\ 0 & \textit{else}\,. \end{cases}$$

Remark 1.8. The case $k=0$ of the blow-up formula for virtual holomorphic Euler characteristics of Corollary 1.7 was previously proved by Nakajima–Yoshioka in [NY11c, Theorem 2.11(i)] for Situation B.

1.1 Outline of the article

In order to prove Theorem 1.4, we use the blow-up algorithm of Nakajima–Yoshioka [NY11c] (extended to Situation A by [KT21]) to obtain a weak universal blow-up formula for invariants of the form being studied. A key aspect of this formula is its universality; namely, it shows that the desired blow-up formula exists and is independent of much of the input data. In particular, the formula will be the same in Situations A and B. We then analyse the equivariant geometry arising in Situation B to determine the exact form of the desired formula. This proceeds via a reduction to rank 1, where we can use the (q,t)-Nekrasov–Okounkov formula [RW18, Theorem 1.3] to conclude.

The article is structured as follows. In Section 2, we recall the blow-up algorithm of Nakajima–Yoshioka [NY11c] and show how it is extended to both situations by [KT21]. In Section 3, we state and prove a weak universal blow-up formula for invariants defined by classes which are multiplicative in the tangent bundle. In Section 4, we prove the main theorem, Theorem 1.4, by determining the coefficients from the weak universal blow-up formula. We do this by examining the equivariant structure of Situation B.

1.2 Basic notation

We fix a positive integer $r>0$ and will always assume the notation described in Situations A and B. In particular, we consider classes of the form $c := r + c_1 + \mathrm{ch}_2 \in A^*(X)$ and $\widehat{c} := r + \widehat{c}_1 + \mathrm{ch}_2 \in A^*(\widehat{X})$ described there. We will also use the integral notation described in Notation 1.2.

Other basic notation includes the following:

- $C_m := \mathcal{O}_C(-m-1) := \mathcal{O}_C \otimes \mathcal{O}_{\widehat{X}}((m+1)C)$ denotes the unique degree $-m-1$ line bundle on the exceptional divisor C. In Situation B, we endow C_m with the equivariant structure obtained from the inclusion into $\mathcal{O}_C$ if $m \geqslant -1$ (respectively, the dual one for $m < -1$).
- $\epsilon_m := \mathrm{ch}(j_* C_m) = [C] - (m+1/2)[\mathrm{pt}]$ is the Chern character of C_m taken as a coherent sheaf on $\widehat{X}$ via the inclusion $j\colon C \hookrightarrow \widehat{X}$.
- For a moduli space M of sheaves on X (or $\widehat{X}$), we denote the universal sheaf on $M \times X$ (or $M \times \widehat{X}$) by $\mathcal{E}$ and let $\pi\colon M \times X \to M$ (respectively, $\pi\colon M \times \widehat{X} \to M$) be the projection. We will denote $\mathrm{T}^{\mathrm{vir}}_M$ simply by $\mathrm{T}^{\mathrm{vir}}$.
- We set $R\mathcal{H}om_\pi(\mathcal{F}, \mathcal{G}) := R\pi_* R\,\mathcal{H}om(\mathcal{F}, \mathcal{G})$.
- For a $\mathbb{C}^*$-representation $\hbar$, we write $\boldsymbol{e}^{\hbar}$ for the associated equivariant line bundle.
- We let F denote the coefficient field for Chow-cohomology groups.
- In Situation B, we take $\widetilde{T} := (\mathbb{C}^*)^{2+r}$. The $\widetilde{T}$-action on M is explained in Definition 4.2.
- We denote by $\mathbf{t}_1, \mathbf{t}_2, \mathbf{e}_1, \dots, \mathbf{e}_r$ the fundamental 1-dimensional $\widetilde{T}$-representations. See Notation 4.3 for more details.
- For a coherent sheaf $\mathcal{F}$, we define the following notation in Situation A:

$$\mathrm{T}^{\mathrm{vir}}_{\mathcal{F}} := -\mathcal{O}_M^{\oplus \chi(X, \mathcal{O}_X)} - R\pi_* R\,\mathcal{H}om(\mathcal{F}, \mathcal{F})$$

and the following in Situation B:

$$\mathrm{T}^{\mathrm{vir}}_{\mathcal{F}} := -R\pi_* R\,\mathcal{H}om(\mathcal{F}, \mathcal{F}(-\ell_\infty))\,.$$

2. Nakajima–Yoshioka blow-up algorithm

In [NY11c], Nakajima and Yoshioka introduced an algorithm to express certain tautological integrals on $\widehat{M}(\widehat{c})$ in terms of similar integrals on $M(c + n[\mathrm{pt}])$ for various n, which applies to Situation B. The results necessary to apply the algorithm in Situation A were established in [KT21]. In this section, we will review the Nakajima–Yoshioka blow-up algorithm in both situations simultaneously and draw some conclusions. First, we recall the concept of m-stable sheaves.

Definition 2.1 (*m-Stable sheaves*, [KT21, Lemma 3.29] and [NY11c, § 1.1]). Let $m \in \mathbb{Z}$. We denote by $\widehat{M}^m(\widehat{c})$ the moduli stack of m-stable sheaves which parametrises

(i) *in Situation A*: sheaves E on $\widehat{X}$ with Chern character $\widehat{c}$ such that the torsion-free quotient of the coherent sheaf p_*E is H-stable on X, and such that $\mathrm{Hom}(\mathcal{O}_C(-m), E) = 0$ and $\mathrm{Hom}(E, \mathcal{O}_C(-m-1)) = 0$;

(ii) *in Situation B*: framed sheaves (E, Φ) on $\widehat{\mathbb{P}}^2$ such that $\mathrm{ch}(E) = \widehat{c}$ and E is torsion-free away from the exceptional divisor C, and such that $\mathrm{Hom}(\mathcal{O}_C(-m), E) = 0$ as well as $\mathrm{Hom}(E, \mathcal{O}_C(-m-1)) = 0$.

Remark 2.2. The discussion preceding Notation 1.2 also holds for $M = \widehat{M}^m(\widehat{c})$. In Situation A, they are proper Deligne–Mumford stacks with a canonical perfect obstruction theory. In Situation B, they are smooth, quasi-projective varieties and carry a canonical $\widetilde{T}$-action with isolated fixed points. In either case, the (virtual) dimension is the same as that of $\widehat{M}(\widehat{c})$. Hence, we can extend the integral notation from Notation 1.2 to $M = \widehat{M}^m(\widehat{c})$.

In order to simplify the exposition, we will only state the results of this section for spaces of m-stable sheaves. This is justified by the following result.

Proposition 2.3. *Suppose that we are in either Situation A or Situation B.*

(i) *If $\widehat{c}$ is of the form p^*c, then there is a natural isomorphism $p_*(-)\colon \widehat{M}^0(\widehat{c}) \xrightarrow{\sim} M(c)$ whose inverse is $p^*(-)\colon M(c) \xrightarrow{\sim} \widehat{M}^0(\widehat{c})$ (see* [KT21, Proposition 3.25 and Corollary 3.27] and [NY11b, Proposition 3.3]).

(ii) *For each c, there is an $m' \geqslant 0$ such that $\widehat{M}^m(\widehat{c}) = \widehat{M}(\widehat{c})$ for all $m \geqslant m'$ (see* [NY11b, Proposition 3.37] and [NY11a, Proposition 7.1]).

(iii) *In Situation A, the isomorphisms in points (i) and (ii) preserve the perfect obstruction theories. In Situation B, the isomorphisms are compatible with the $\widetilde{T}$-equivariant structure.*

The Nakajima–Yoshioka blow-up algorithm will be applied to integrals with the type of integrand classes described in Notation 2.4.

Notation 2.4. Suppose that we are in either Situation A or Situation B, and let M be $\widehat{M}^m(\widehat{c})$ with $\pi\colon \widehat{X} \times M \to M$ as the projection. We use the symbol Φ to denote a rule that associates a class $\Phi(\mathcal{F}) \in A^*(M)$ with each coherent sheaf $\mathcal{F}$ on $\widehat{X} \times M$. More precisely, we require that $\Phi(\mathcal{F})$ is given as a power series in classes of the form $\pi_*(c_{i_1}(\mathcal{F}) \cdots c_{i_N}(\mathcal{F}) \cdot \alpha)$ for some list of integers $(i_1, \ldots, i_N)$ and some Chow class $\alpha \in A^*(\widehat{X})$.

We also use this notation in the more general setting where M' is a smooth projective scheme over M that either carries an obstruction theory compatible with that on M (in Situation A) or a $\widetilde{T}$-action for which the map $M' \to M$ is equivariant (in Situation B).

Example 2.5. In both Situation A and B, any polynomial in Chern classes of $\mathrm{T}^{\mathrm{vir}}$, $R\mathcal{H}om_\pi(\mathcal{E}, C_m)$ or $R\mathcal{H}om_\pi(C_m, \mathcal{E})$ will satisfy the conditions of Notation 2.4 for $\Phi(\mathcal{E})$. This follows from the virtual and equivariant versions of the Grothendieck–Riemann–Roch theorem and the characterisation of the respective tangent bundles given by Remark 2.2 and the discussion preceding Notation 1.2.

In what follows, it will be useful to better understand the K-theory classes mentioned in Example 2.5.

Lemma 2.6. *Suppose that we are in Situation A or B, and consider the moduli space $M = \widehat{M}^{m_0}(\widehat{c})$ with universal sheaf $\mathcal{E}$ and projection $\pi\colon \widehat{X} \times M \to M$. Then we have the following:*

(i) *For any $m \in \mathbb{Z}$, the respective K-theoretic ranks of $R\mathcal{H}om_\pi(\mathcal{E}, C_m)[1]$ and $R\mathcal{H}om_\pi(C_m, \mathcal{E})[1]$ are given by*

$$rm + (\widehat{c}_1, [C]) \quad \text{and} \quad r(m+1) + (\widehat{c}_1, [C])\,.$$

(ii) *For $m = m_0$, we have the vector bundles*

$$R\mathcal{H}om_\pi(\mathcal{E}, C_m)[1] = \mathrm{Ext}^1_\pi(\mathcal{E}, C_m) \quad \text{and} \quad R\mathcal{H}om_\pi(C_{m-1}, \mathcal{E})[1] = \mathrm{Ext}^1_\pi(C_{m-1}, \mathcal{E})\,.$$

(iii) *Suppose that $\widehat{c}$ satisfies $(\widehat{c}_1, [C]) = 0$, and consider the space $\widehat{M}^0(\widehat{c})$. Then for $m \in \mathbb{Z}$, we have the following identities of K-theory classes:*

$$R\mathcal{H}om_\pi(C_m, \mathcal{E}) = -\mathbf{t}_1\mathbf{t}_2 R\Gamma(C_{m+1})^\vee \otimes R\mathcal{H}om_\pi(C_0, \mathcal{E}) \quad \text{and}$$
$$R\mathcal{H}om_\pi(\mathcal{E}, C_m) = -R\Gamma(C_m) \otimes R\mathcal{H}om_\pi(C_0, \mathcal{E})^\vee\,.$$

In Situation B, this uses the equivariant structure defined in Section 1.2, and the $R\Gamma(C_m)$ are $\widetilde{T}$-representations explicitly given by (2.1). In Situation A, the identity holds if one replaces both $\mathbf{t}_1$ and $\mathbf{t}_2$ with the trivial line bundle.

(iv) *Let m_1, m_2 be integers. The object $R\operatorname{Hom}_{\widehat{X}}(C_{m_1}, C_{m_2})$ has K-theoretic rank equal to -1. In Situation B, the K-theory class of $R\operatorname{Hom}_{\widehat{X}}(C_{m_1}, C_{m_2})$ equals a sum of terms of the form $\pm \mathbf{t}_1^i \mathbf{t}_2^j$.*

Proof. Point (i) follows from a Grothendieck–Riemann–Roch computation. Point (ii) follows from the definition of m-stability as explained prior to Lemma 3.22 in [NY11b]. For point (iii), using Bott's formula for pushforwards in equivariant K-theory, we have, for any $E \in K^{T^2}(\widehat{\mathbb{P}}^2)$,

$$R\Gamma(E) = \sum_F \frac{\iota_F^*(E)}{\Lambda_{-1} \mathrm{T}_F^* \widehat{\mathbb{P}}^2},$$

where the sum ranges over the torus-fixed points of $\widehat{\mathbb{P}}^2$. Using this, one can compute

$$\begin{aligned} R\Gamma(\mathcal{O}_C(m)) &= \frac{\mathbf{t}_1^{-m} - \mathbf{t}_1^{-m-1}}{1 - \mathbf{t}_1^{-1} - \mathbf{t}_1\mathbf{t}_2^{-1} + \mathbf{t}_2^{-1}} + \frac{\mathbf{t}_2^{-m} - \mathbf{t}_2^{-m-1}}{1 - \mathbf{t}_2^{-1} - \mathbf{t}_2\mathbf{t}_1^{-1} + \mathbf{t}_1^{-1}} \\ &= \begin{cases} -\mathbf{t}_1\mathbf{t}_2 \left(\displaystyle\sum_{i=0}^{-m-2} \mathbf{t}_1^i \mathbf{t}_2^{-m-2-i} \right) & \text{if } m < 0\,, \\ (\mathbf{t}_1\mathbf{t}_2)^{-m} \left(\displaystyle\sum_{i=0}^{m} \mathbf{t}_1^i \mathbf{t}_2^{m-i} \right) & \text{if } m \geqslant 0\,. \end{cases} \end{aligned} \tag{2.1}$$

By the equivariant localisation formula in K-theory, we also have $Rp_*(\mathcal{O}_C(m)) = R\Gamma(\mathcal{O}_C(m))\mathcal{O}_{\mathrm{pt}}$. These facts also hold non-equivariantly on any surface if one imposes $\mathbf{t}_1 = \mathbf{t}_2 = \mathbb{C}$. Now, point (iii) follows from a calculation using equivariant Serre duality and the observation that, on $\widehat{M}^0(p^*c)$, we have the adjunction formula $R\mathcal{H}om_\pi(p^*\mathcal{F}, C_m) = R\mathcal{H}om_\pi(\mathcal{F}, p_*C_m)$, where $\mathcal{F}$ is the universal sheaf over $\mathbb{P}^2 \times M_{\mathbb{P}^2}(c)$. Point (iv) follows similarly, using that K-theoretically, we have $C_m = \mathcal{O}_{\widehat{X}}((m+1)C) - \mathcal{O}_{\widehat{X}}(mC)$. □

The Nakajima–Yoshioka blow-up algorithm allows us to express intersections of the form $\int_{\widehat{M}^m(\widehat{c})} \Phi(\mathcal{E})$ in terms of integrals over $M(c')$ for various c'. Indeed, it gives a consistent way to compute classes $\Phi_n(\mathcal{E})$ such that

$$\int_{\widehat{M}^m(\widehat{c})} \Phi(\mathcal{E}) = \sum_{n \geqslant 0} \int_{M(c+n\mathrm{pt})} \Phi_n(\mathcal{E})\,.$$

For special choices of Φ, we can say much more about the classes Φ_n (see Theorem 3.2). We now state the key components of the algorithm.

Theorem 2.7 (Wall-crossing formula, [KT21, Theorem 1.1] and [NY11c, Theorem 1.5]). *Suppose that we are in Situation A or B, and let $\Phi(\mathcal{E})$ be as in Notation 2.4. Then we have the following identity:*

$$\begin{aligned} &\int_{\widehat{M}^{m+1}(\widehat{c})} \Phi(\mathcal{E}) - \int_{\widehat{M}^m(\widehat{c})} \Phi(\mathcal{E}) \\ &\qquad = \sum_{j>0} \frac{1}{j!} \int_{\widehat{M}^m(\widehat{c} - j\epsilon_m)} \operatorname*{Res}_{h_j=0} \cdots \operatorname*{Res}_{h_1=0} \Phi\left(\mathcal{E} \oplus \bigoplus_{i=1}^{j} C_m \otimes \boldsymbol{e}^{-h_i} \right) \Psi_m^j(\mathcal{E})\,, \end{aligned}$$

where $C_m := \mathcal{O}_C(-m-1)$, $\epsilon_m := \operatorname{ch}(C_m)$ and where

$$\Psi_m^j(\mathcal{E}) := \frac{\prod_{1 \leqslant i_1 \neq i_2 \leqslant j} (h_{i_1} - h_{i_2})}{\prod_{i=1}^j \mathrm{e}(-R\mathcal{H}om_\pi(\mathcal{E}, C_m)\boldsymbol{e}^{-h_i})\, \mathrm{e}(-R\mathcal{H}om_\pi(C_m, \mathcal{E})\boldsymbol{e}^{h_i})}.$$

Remark 2.8. (i) In Theorem 2.7, the notation for taking the residue at $h_i = 0$ means *taking the coefficient of* h_i^{-1} *after expanding at* $h_i = 0$.
(ii) We have written the right-hand side of Theorem 2.7 as an infinite sum. However, from Remark 2.2 and the discussion preceding Notation 1.2, we know that the (virtual) dimension of $\widehat{M}^m(\widehat{c} - j\epsilon_m)$ decreases strictly as $j \gg 0$ increases. Hence, for any choice of $\widehat{c}$, the right-hand side is a finite sum.

Remark 2.9. In the version of Theorem 2.7 stated in [NY11c], only the action of a certain subtorus of $\widetilde{T} = (\mathbb{C}^*)^{r+2}$ is considered. In correspondence with H. Nakajima, it was explained to us that this restriction is inessential and that the results still hold when one considers the whole $\widetilde{T}$-action.

Lemma 2.10 (Twisting by $\mathcal{O}(C)$, [KT21, Remark 3.31] and [NY11c, § 1.5.4]). *Suppose that we are in Situation A or B. For each* $n \in \mathbb{Z}$, *there is a natural isomorphism*

$$\widehat{M}^m(\widehat{c}) \xrightarrow{\sim} \widehat{M}^{m+n}\big(\widehat{c} \cdot \mathrm{ch}(\mathcal{O}_{\widehat{X}}(nC))\big)$$

defined by sending a family of m*-stable sheaves* $\mathcal{F}$ *to its twist* $\mathcal{F}(nC)$. *In Situation A, this isomorphism preserves the perfect obstruction theory, and in Situation B, it is compatible with the* $\widetilde{T}$*-equivariant structure.*

Remark 2.11. Using the notation of Lemma 2.10, suppose $\widetilde{c} = \widehat{c} \cdot \mathrm{ch}(\mathcal{O}_{\widehat{X}}(nC))$. Then we have $(\widetilde{c}_1, [C]) = (\widehat{c}_1, [C]) - rn$.

The final ingredient is the following.

Proposition 2.12 (Grassmann-bundle formula). *Suppose that we are in Situation A or B, and that* $\widehat{c}$ *satisfies* $(\widehat{c}_1, [C]) = -k$ *for some* $0 < k < r$. *Let* $\Phi(\mathcal{E})$ *be as in Notation 2.4. Then*

$$\int_{\widehat{M}^1(\widehat{c})} \Phi(\mathcal{E}) = \frac{1}{k!} \int_{\widehat{M}^1(\widehat{c} - k\epsilon_0)} \operatorname*{Res}_{h_k=0} \cdots \operatorname*{Res}_{h_1=0} \Phi\Big(\mathcal{E} \oplus \bigoplus_{i=1}^{k} C_0 \otimes \boldsymbol{e}^{-h_i}\Big) \overline{\Psi}^k(\mathcal{E}),$$

where $C_0 = \mathcal{O}_C(-1)$ *and where*

$$\overline{\Psi}^k(\mathcal{E}) := \frac{\prod_{1 \leqslant i_1 \neq i_2 \leqslant k} (h_{i_1} - h_{i_2})}{\prod_{i=1}^{k} \mathrm{e}(-R\mathcal{H}om_\pi(C_0, \mathcal{E})\boldsymbol{e}^{h_i})}.$$

Proof. This is a reformulation of the results from [KT21, Theorem 1.2 and Proposition 3.36] and [NY11c, Proposition 1.2]. More precisely, by the results in [KT21, NY11c], we have

$$\int_{\widehat{M}^1(\widehat{c})} \Phi(\mathcal{E}) = \int_{\widehat{M}^1(\widehat{c} - k\epsilon_0)} q_* \Phi(q^*\mathcal{E} \oplus C_0 \otimes \mathcal{V}),$$

where $q\colon \mathrm{Gr}\big(k, \mathrm{Ext}^1_\pi(C_0, \mathcal{E})\big) \to \widehat{M}^1(\widehat{c} - k\epsilon_0)$ is the relative Grassmannian of k-planes and $\mathcal{V}$ is the associated universal sub-bundle.

Let r denote the rank of $\mathrm{Ext}^1_\pi(C_0, \mathcal{E})$. By the formula of [DP20, Example 1], the expression $q_*\Phi(q^*\mathcal{E} \oplus C_0 \otimes \mathcal{V})$ is equal to the following expression:

$$\mathrm{Coeff}_{h_1^{r-1} \cdots h_k^{r-k}} \left[\Phi\Big(q^*\mathcal{E} \oplus \bigoplus_{i=1}^{k} C_0 \otimes \boldsymbol{e}^{-h_i}\Big) \prod_{1 \leqslant i_1 < i_2 \leqslant k} (h_{i_1} - h_{i_2}) \prod_{1 \leqslant i \leqslant k} s_{1/h_i}\big(\mathrm{Ext}^1_\pi(C_0, \mathcal{E})\big) \right].$$

Here, s_{1/h_i} denotes the Segre polynomial in the variable $1/h_i$. We have

$$s_{1/h_i}\big(\mathrm{Ext}^1_\pi(C_0, \mathcal{E})\boldsymbol{e}^{h_i}\big) = h_i^r\, \mathrm{e}\big(\mathrm{Ext}^1_\pi(C_0, \mathcal{E})\boldsymbol{e}^{h_i}\big)^{-1},$$

and $\operatorname{Ext}^1_\pi(C_0,\mathcal{E}) = -R\mathcal{H}om_\pi(C_0,\mathcal{E})$, from which we find that $q_*\Phi(q^*\mathcal{E}\oplus C_0\otimes\mathcal{V})$ is

$$\operatorname{Coeff}_{h_1^{-1}\cdots h_k^{-k}}\left[\frac{\Phi\bigl(q^*\mathcal{E}\oplus\bigoplus_{i=1}^k C_0\otimes \boldsymbol{e}^{-h_i}\bigr)\prod_{1\leqslant i_1<i_2\leqslant k}(h_{i_1}-h_{i_2})}{\mathrm{e}(-R\mathcal{H}om_\pi(C_0,\mathcal{E})\boldsymbol{e}^{h_i})}\right]. \tag{2.2}$$

We claim that averaging the expression in (2.2) over all permutations of $1,\dots,k$ recovers the iterated residue term in the statement of the proposition. Indeed, in our case the expansion at $h_1=\cdots=h_k=0$ is independent of the ordering of the h_i, so the iterated residue is simply the coefficient of $h_1^{-1}\cdots h_k^{-1}$. Now the claim follows from the Vandermonde identity

$$\sum_{\sigma\in\Sigma_k}\operatorname{sgn}\sigma\, h_{\sigma 1}^0\cdots h_{\sigma k}^{k-1} = \prod_{1\leqslant i_1<i_2\leqslant k}(h_{i_2}-h_{i_1})\,. \qquad \square$$

ALGORITHM 2.13 (Blow-up algorithm for single terms). Let $0\leqslant\ell<r$ and $n\geqslant 0$ be integers, and set $\widehat{c}:=p^*c-\ell\epsilon_0+n[\mathrm{pt}]$.

- `INPUT:` The input for the algorithm is an integral of the form
$$\int_{\widehat{M}^m(\widehat{c})}\Phi(\mathcal{E})\,.$$
- `OUTPUT:` The output is an identity of the form
$$\int_{\widehat{M}^m(\widehat{c})}\Phi(\mathcal{E}) = \int_{M(c+n[\mathrm{pt}])}\Phi'(\mathcal{E}) + \sum_{i\in I}\int_{M_i}\Phi_i(\mathcal{E})\,,$$
where
  - (i) Φ' is as in Notation 2.4, and is zero if $0<\ell<r$;
  - (ii) I is a finite collection indexing pairs (M_i,Φ_i), where Φ_i is as in Notation 2.4 and M_i is of the form $\widehat{M}^{m_i}(b_i)$ with $b_i=p^*c-\ell_i\epsilon_0+n_i[\mathrm{pt}]$ for integers $0\leqslant\ell_i<r$ and $n_i\in\mathbb{Z}$ such that $\operatorname{vdim}\widehat{M}^{m_i}(b_i)<\operatorname{vdim}\widehat{M}^m(\widehat{c})$.
- `PROCEDURE:` The algorithm proceeds as follows:

  A Use Theorem 2.7 a total of $m-1$ times to obtain the expression
$$\int_{\widehat{M}^m(\widehat{c})}\Phi(\mathcal{E}) = \int_{\widehat{M}^0(\widehat{c})}\Phi(\mathcal{E}) + \sum_{\tilde{m}=0}^{m-1}\sum_{j>0}\int_{\widehat{M}^{\tilde{m}}(\widehat{c}-j\epsilon_0+j\tilde{m}[\mathrm{pt}])}\Phi_{\tilde{m},j}(\mathcal{E})\,.$$
  Here, we also used $\epsilon_m=\epsilon_0-m[\mathrm{pt}]$ from Section 1.2.

  B If $\ell=0$, then we apply Proposition 2.3 to the first term on the right-hand side of Step A. The second term on the right-hand side of Step A becomes the collection indexed by I. We apply Lemma 2.10 for each term if necessary to obtain moduli spaces of the desired form. More precisely, for the term in the sum corresponding to indices $\tilde{m}$ and j, we apply the lemma for the twist by $\mathcal{O}(\lfloor j/r\rfloor C)$. This completes the algorithm in the case $\ell=0$.

  C If $0<\ell<r$, then we twist the first term on the right-hand side of Step A by $\mathcal{O}(C)$ and apply Lemma 2.10. The resulting term has Chern character $\widehat{c}+r\epsilon_0+\ell[\mathrm{pt}]$ and satisfies the conditions of Proposition 2.12 with $k=r-\ell$. Applying Proposition 2.12 strictly decreases the virtual dimension. The term $\int_{\widehat{M}^0(\widehat{c})}\Phi(\mathcal{E})$ from Step A becomes
$$\int_{\widehat{M}^1(p^*c+(n+\ell)[\mathrm{pt}])}\Phi''(\mathcal{E})\,.$$
  This and the remaining terms from Step A become the collection indexed by I (observing

that $\Phi' = 0$ in this case). As detailed in Step B, we apply Lemma 2.10 for various twists to bring the remaining terms in I into the desired form. This completes the algorithm.

Algorithm 2.14 (Nakajima–Yoshioka blow-up algorithm). Let $\ell \geqslant 0$ be an integer and suppose $\widehat{c} := p^*c - \ell\epsilon_0$.

- ▶ `INPUT:` The input for the algorithm is an integral of the form
$$\int_{\widehat{M}^m(\widehat{c})} \Phi(\mathcal{E})\,.$$
- ▶ `OUTPUT:` The output is an identity of the form
$$\int_{\widehat{M}^m(\widehat{c})} \Phi(\mathcal{E}) = \sum_{n\geqslant 0} \int_{M(c+n[\mathrm{pt}])} \Phi'_n(\mathcal{E})\,.$$
- ▶ `PROCEDURE:` The algorithm proceeds as follows:

  A We can assume $0 \leqslant \ell < r$ by applying Lemma 2.10 an appropriate number of times. Apply Algorithm 2.13 to obtain
  $$\int_{\widehat{M}^m(\widehat{c})} \Phi(\mathcal{E}) = \int_{M(c)} \Phi'_0(\mathcal{E}) + \sum_{i\in I} \int_{M_i} \Phi_i(\mathcal{E})\,.$$

  B Apply Algorithm 2.13 to those terms indexed by $i \in I$ for which M_i has maximal virtual dimension, and remove those terms from I. Collect the resulting output terms of type (i) to form Φ'_n for the appropriate n (since the virtual dimensions of the spaces appearing in the algorithm are bounded above by $M(c)$, one automatically has $n \geqslant 0$). Collect the resulting output terms of type (ii) of lower virtual dimension, and add those terms to I. We now use I to denote the indexing set for this new collection.

  C Iterate Step B until I indexes the empty set. This will terminate, because after each step, I is finite, and the maximal virtual dimension of the moduli spaces indexed by I decreases at each step.

3. Weak structure theorem

Here we explain a refined version of Nakajima–Yoshioka's blow-up formalism which holds when integrating a multiplicative class of the tangent bundle.[2] This gives a unified version of Nakajima–Yoshioka [NY11c, Theorem 2.6] and Kuhn–Tanaka [KT21, Theorem 1.5]. In particular, we give a concise presentation of how the two situations are related, which is only implicit in their work. We begin this section by defining some useful notation.

Notation 3.1. Denote by $\mathcal{S}$ the collection of symbols
$$\big\{c_i(R\mathcal{H}om_\pi(C_m, -)), c_i(R\mathcal{H}om_\pi(-, C_m))\big\}_{i\in\mathbb{Z}_{>0},\, m\in\mathbb{Z}}\,,$$
and let $\mathfrak{R} := \mathrm{F}[\![\mathcal{S}]\!][\![\varepsilon_1, \varepsilon_2]\!]$ be the ring of power series in the symbols from $\mathcal{S}$ and in variables $\varepsilon_1, \varepsilon_2$. For an element $P \in \mathfrak{R}$ and for $\mathcal{F}$ a sheaf on $M \times \widehat{X}$, we denote by $P(\mathcal{F})$ the Chow class in $A^*(M)$ obtained by evaluating the symbols of $\mathcal{S}$ at $\mathcal{F}$ and by setting $\varepsilon_i = c_1(\mathbf{t}_i)$ in Situation B, respectively $\varepsilon_i = 0$ in Situation A.

We now state the main result of this section.

[2] Recall that a characteristic class $\Theta\colon K(-) \to A^*(-)$ is multiplicative if it satisfies the identity $\Theta(F + E) = \Theta(F)\cdot\Theta(E)$ for all choices of F and E.

Theorem 3.2 (Weak structure theorem). *Let $k \geqslant 0$ be an integer, Θ a multiplicative class and $\nu_1, \dots, \nu_r$ formal variables. There exists a unique collection of power series $\{\Omega_n\}_{n\geqslant 0}$ in $\mathrm{F}[\![\varepsilon_1, \varepsilon_2, \nu_1, \dots, \nu_r]\!]$ which satisfies the following condition:*

Whenever we are in Situation A or B with $\widehat{c}$ satisfying $(\widehat{c}_1, [C]) = k$, we have

$$\int_{\widehat{M}(\widehat{c})} \Theta\big(\mathrm{T}^{\mathrm{vir}}\big) = \sum_{n\geqslant 0} \int_{M(c+n[\mathrm{pt}])} \Theta\big(\mathrm{T}^{\mathrm{vir}}\big)\Omega_n(\mathcal{E})\,,$$

where $\Omega_n(\mathcal{E})$ denotes the Chow-cohomology class obtained by replacing the variable ν_i by the class $c_i(R\mathcal{H}om_\pi(\mathcal{O}_{\mathrm{pt}}, \mathcal{E}))$.

In fact, the collection $\{\Omega_n\}_{n\geqslant 0}$ is uniquely determined if one only assumes this condition to hold in Situation B.

Remark 3.3. More generally, the existence in Theorem 3.2 holds (with the same proof) if instead of just a multiplicative class of the tangent bundle, one considers a product $\Theta(T^{\mathrm{vir}})\Phi(\mathcal{E})$, where Θ is multiplicative and Φ is as in Notation 2.4 and is also multiplicative in its input, that is, $\Phi(\mathcal{F}_1 \oplus \mathcal{F}_2) = \Phi(\mathcal{F}_1)\cdot\Phi(\mathcal{F}_2)$. Here, Φ may only be defined for a particular choice of X or for a particular situation. The uniqueness statement holds (with the same proof) whenever Φ is defined in a way that makes sense in Situation B.

The idea of the proof is to follow a refined version of the Nakajima–Yoshioka blow-up algorithm (Algorithm 2.14) which keeps track of extra structure related to the coefficients Φ_i. We first collect some consequences from Theorem 2.7 and Proposition 2.12 when applied to multiplicative classes.

Lemma 3.4 (Series version of Nakajima–Yoshioka algorithm components). *For $P \in \mathfrak{R}$, the components of Algorithm 2.14 define canonical series in $\mathfrak{R}$ in the following ways:*

(i) Wall-crossing formula (Theorem 2.7): *For integers $k, m \geqslant 0$, there exists a canonical collection of power series $\{Q_j\}_{j\geqslant 1}$ in $\mathfrak{R}$ which satisfies the following condition:*

Whenever we are in Situation A or B with $\widehat{c}$ satisfying $(\widehat{c}_1, [C]) = k$, we have

$$\int_{\widehat{M}^{m+1}(\widehat{c})} \Theta\big(\mathrm{T}^{\mathrm{vir}}\big)P(\mathcal{E}) - \int_{\widehat{M}^{m}(\widehat{c})} \Theta\big(\mathrm{T}^{\mathrm{vir}}\big)P(\mathcal{E}) = \sum_{j\geqslant 1} \int_{\widehat{M}^{m}(\widehat{c}-j\epsilon_m)} \Theta\big(\mathrm{T}^{\mathrm{vir}}\big)Q_j(\mathcal{E})\,.$$

(ii) Twisting by $\mathcal{O}(C)$ (Lemma 2.10): *There exists a canonical power series $P^\dagger \in \mathfrak{R}$ satisfying $P^\dagger(\mathcal{E}) = P(\mathcal{E}(C))$ whenever we are in Situation A or B with arbitrary $\widehat{c}$.*

(iii) Gr-bundle formula (Proposition 2.12): *For each integer $0 < k < r$, there exists a canonical power series Q in $\mathfrak{R}$ which satisfies the following condition:*

Whenever we are in Situation A or B with $\widehat{c}$ satisfying $(\widehat{c}_1, [C]) = -k$, we have

$$\int_{\widehat{M}^{1}(\widehat{c})} \Theta\big(\mathrm{T}^{\mathrm{vir}}\big)P(\mathcal{E}) = \int_{\widehat{M}^{1}(\widehat{c}-k\epsilon_0)} \Theta\big(\mathrm{T}^{\mathrm{vir}}\big)Q(\mathcal{E})\,.$$

Proof. For part (ii), one obtains $P^\dagger$ by replacing every occurrence of the functors $R\mathcal{H}om_\pi(C_m, -)$ and $R\mathcal{H}om_\pi(-, C_m)$ in P by $R\mathcal{H}om_\pi(C_{m-1}, -)$ and $R\mathcal{H}om_\pi(-, C_{m-1})$, respectively. For statements (i) and (iii), we apply Theorem 2.7 and Proposition 2.12, respectively, to the class $\Phi(\mathcal{F}) := \Theta\big(\mathrm{T}^{\mathrm{vir}}_{\mathcal{F}}\big)P(\mathcal{F})$ (where $\mathrm{T}^{\mathrm{vir}}_{\mathcal{F}}$ is defined in Section 1.2). The description of the virtual tangent bundle from Remark 2.2 and the discussion preceding Notation 1.2 show that $\Phi\big(\mathcal{E} \oplus \bigoplus_{i=1}^{j} C_m e^{-h_i}\big)$ can

be written as

$$\Theta\big(\mathrm{T}^{\mathrm{vir}}\big)P\Big(\mathcal{E}\oplus\bigoplus_{i=1}^{j}C_m\boldsymbol{e}^{-h_i}\Big)\prod_{1\leqslant i_1,i_2\leqslant j}\Theta\big(R\mathcal{H}om_\pi(C_m,C_m)\boldsymbol{e}^{h_{i_2}-h_{i_1}}\big)$$
$$\times\prod_{i=1}^{j}\Theta\big(R\mathcal{H}om_\pi(\mathcal{E},C_m)\boldsymbol{e}^{-h_i}\big)\Theta\big(R\mathcal{H}om_\pi(C_m,\mathcal{E})\boldsymbol{e}^{h_i}\big)\,.$$

We can make use of Lemma 2.6(iv) and the basic properties of Chern classes to rewrite this expression canonically as an expression in variables h_i, elements of $\mathcal{S}$ and in ε_1, ε_2. It follows that the expressions appearing on the right-hand side of the equations in Theorem 2.7 and Proposition 2.12 are of the desired form. This implies parts (i) and (iii). □

We can now state a refined version of Algorithm 2.13.

Algorithm 3.5 (Refined blow-up algorithm for single terms). Let $n,\ell\geqslant 0$ be integers, and set $\widehat{c}:=p^*c-\ell\epsilon_0+n[\mathrm{pt}]$.

- `INPUT:` The input for the refined algorithm is an expression of the form
$$\int_{\widehat{M}^m(\widehat{c})}\Theta\big(\mathrm{T}^{\mathrm{vir}}\big)P(\mathcal{E})\,,$$
where $P\in\mathfrak{R}$.
- `OUTPUT:` The output is an identity of the form
$$\int_{\widehat{M}^m(\widehat{c})}\Theta\big(\mathrm{T}^{\mathrm{vir}}\big)P(\mathcal{E})=\int_{M(c+n[\mathrm{pt}])}\Theta\big(\mathrm{T}^{\mathrm{vir}}\big)\Omega(\mathcal{E})+\sum_{i\in I}\int_{M_i}\Theta\big(\mathrm{T}^{\mathrm{vir}}\big)P_i(\mathcal{E})\,,$$
where
  (i) $\Omega(\mathcal{E})$ is a power series in the symbols $c_i(R\mathcal{H}om_\pi(\mathcal{O}_{\mathrm{pt}},\mathcal{E}))$, and is zero if $0<\ell<r$;
  (ii) I is a finite collection indexing tuples (m_i,ℓ_i,d_i,P_i), where $m_i,d_i,\ell_i\geqslant 0$, and where $P_i\in\mathfrak{R}$. Moreover, for $M_i:=\widehat{M}^{m_i}(b_i)$ with $b_i=p^*c-\ell_i\epsilon_0+(n+d_i)[\mathrm{pt}]$, they satisfy $\mathrm{vdim}\widehat{M}^{m_i}(b_i)<\mathrm{vdim}\widehat{M}^{m}(\widehat{c})$.
- `PROCEDURE:` The algorithm proceeds as follows:
  A Follow Algorithm 2.13 with the wall-crossing component (Theorem 2.7), the twisting component (Lemma 2.10) and the Gr-bundle component (Proposition 2.12) replaced by their counterparts from Lemma 3.4.
  B If $0<\ell<r$, then the algorithm is complete. If $\ell=0$, then consider the term
$$\int_{M(c+n[\mathrm{pt}])}\Phi(\mathcal{E})=\int_{\widehat{M}^0(\widehat{c})}\Theta\big(\mathrm{T}^{\mathrm{vir}}\big)P(\mathcal{E})$$
arising from integrands of the form described in item (i) of the `OUTPUT` part Algorithm 2.13. Apply (to this term) the following procedure:
    (a) Use the formulas from Lemma 2.6(iii) to rewrite all Chern classes $c_i(R\mathcal{H}om_\pi(\mathcal{E},C_m))$ and $c_i(R\mathcal{H}om_\pi(C_m,\mathcal{E}))$ appearing in P in terms of the classes $c_i(-R\mathcal{H}om_\pi(C_0,\mathcal{E}))$ for $i\geqslant 1$ and $\varepsilon_a=c_1(\mathbf{t}_a)$ for $a=1,2$ (the sign introduced here is important to obtain a vector bundle for Step (c)). Denote the resulting power series by P'.
    (b) Use the wall-crossing formula from Lemma 3.4 to get
$$\int_{M(c+n[\mathrm{pt}])}\Phi(\mathcal{E})=\int_{\widehat{M}^1(\widehat{c})}\Theta\big(\mathrm{T}^{\mathrm{vir}}\big)P'(\mathcal{E})-\sum_{j\geqslant 1}\int_{\widehat{M}^0(\widehat{c}-j\epsilon_0)}\Theta\big(\mathrm{T}^{\mathrm{vir}}\big)Q_j(\mathcal{E})$$

for some $Q_j \in \mathfrak{R}$.

(c) By Lemma 2.6(ii), we have that $-R\mathcal{H}om_\pi(C_0, \mathcal{E})$ is the K-theory class of a rank r vector bundle on $\widehat{M}^1(\widehat{c})$. Therefore, set all Chern classes of degree higher than r equal to zero in P' to get a power series $\Omega \in \mathbb{Z}[\![\varepsilon_1, \varepsilon_2, \nu_1, \dots, \nu_r]\!]$ such that

$$\int_{\widehat{M}^1(\widehat{c})} \Theta\big(\mathrm{T}^{\mathrm{vir}}\big) P'(\mathcal{E}) = \int_{\widehat{M}^1(\widehat{c})} \Theta\big(\mathrm{T}^{\mathrm{vir}}\big) \Omega(\mathcal{E})\,.$$

(d) Use the wall-crossing formula from Lemma 3.4 on the term from Step (c) to get the identity

$$\int_{M(c+n[\mathrm{pt}])} \Phi(\mathcal{E}) = \int_{\widehat{M}^0(\widehat{c})} \Theta\big(\mathrm{T}^{\mathrm{vir}}\big) \Omega(\mathcal{E}) + \sum_{j \geqslant 1} \int_{\widehat{M}^0(\widehat{c} - j\epsilon_0)} \Theta\big(\mathrm{T}^{\mathrm{vir}}\big) \big(Q'_j(\mathcal{E}) - Q_j(\mathcal{E})\big)$$

for canonical power series Q'_j.

(e) Apply Proposition 2.3 to the first term on the right-hand side of Step (d). The second term on the right-hand side of Step (d) is absorbed into the collection indexed by I. This completes the algorithm in the case $\ell = 0$.

Remark 3.6. It follows from the algorithm that the output depends only on Θ, P, ℓ and m, but is independent of c and n, and on whether we are in Situation A or B, in the following sense: the series Ω depends only on Θ, P, ℓ and m. Moreover, there is an infinite collection $\overline{I}$ of tuples (m_i, ℓ_i, d_i, P_i) which depends only on Θ, P, ℓ and m, such that I is exactly the subset of $\overline{I}$ consisting of those tuples for which $\operatorname{vdim} \widehat{M}^{m_i}(p^*c - \ell_i\epsilon_0 + (n + d_i)[\mathrm{pt}])$ is non-negative.

Proof of Theorem 3.2. *Existence of* $\{\Omega_n\}_{n \geqslant 0}$. We have a refined version of Algorithm 2.14 (the Nakajima–Yoshioka blow-up algorithm), obtained by replacing Algorithm 2.13 by the refined version, Algorithm 3.5, and by replacing Lemma 2.10 by its counterpart in Lemma 3.4. Then, this modified algorithm applies to any $\Phi(\mathcal{E}) = \Theta(T^{\mathrm{vir}})P(\mathcal{E})$ with $P \in \mathfrak{R}$ whenever we are in Situation A or B with $\widehat{c}$ such that $(\widehat{c}_1, [C]) = k \geqslant 0$. It shows that there are classes $\Omega_n(\mathcal{E})$ that are power series in $c_i(R\mathcal{H}om_\pi(\mathcal{O}_{\mathrm{pt}}, \mathcal{E}))$ and that satisfy the equation

$$\int_{\widehat{M}(\widehat{c})} \Theta\big(\mathrm{T}^{\mathrm{vir}}\big) = \sum_{n \geqslant 0} \int_{M(c+n[\mathrm{pt}])} \Theta\big(\mathrm{T}^{\mathrm{vir}}\big) \Omega_n(\mathcal{E})\,. \tag{3.1}$$

We observe that the algorithm creates the same collection of formal power series $\{\Omega_n\}_{n \geqslant 0}$ in $\mathrm{F}[\![\varepsilon_1, \varepsilon_2, \nu_1, \dots, \nu_r]\!]$ for any choice of data from Situation A or B with $\widehat{c}$ such that $(\widehat{c}_1, [C]) = k$. This is due to the independence in the single step asserted in Remark 3.6.

Uniqueness of $\{\Omega_n\}_{n \geqslant 0}$. We prove that any collection of power series $\{\Omega_n\}$ satisfying the condition in the statement of Theorem 3.2 must be unique. As mentioned there, the following stronger statement is true:

> The collection $\{\Omega_n\}$ is the unique collection which satisfies the requirement that (3.1) holds *in Situation B* for all $\widehat{c} = r - k\epsilon_0 + n_0[\mathrm{pt}]$ with $n_0 \in \mathbb{Z}_{\geqslant 0}$.

Hence, for the rest of the proof, we will assume that we are in Situation B and that we have some collection $\{\Omega_n\}$ for which (3.1) holds for all $\widehat{c} = r - k\epsilon_0 + n_0[\mathrm{pt}]$ with $n_0 \in \mathbb{Z}_{\geqslant 0}$. Now, for any $n_0 \in \mathbb{Z}_{\geqslant 0}$, equation (3.1) can be rearranged as

$$\int_{M(r-0[\mathrm{pt}])} \Theta\big(\mathrm{T}^{\mathrm{vir}}\big) \Omega_{n_0}(\mathcal{E}) = \int_{\widehat{M}(r-k\epsilon_0-n_0[\mathrm{pt}])} \Theta\big(\mathrm{T}^{\mathrm{vir}}\big) - \sum_{n=0}^{n_0-1} \int_{M(r-(n_0-n)[\mathrm{pt}])} \Theta\big(\mathrm{T}^{\mathrm{vir}}\big) \Omega_n(\mathcal{E})\,. \tag{3.2}$$

As shown, for example, by Nakajima–Yoshioka in [NY05a, Proposition 2.9 and Theorem 2.11], the moduli space $M(r+0[\mathrm{pt}])$ is a point with tangent bundle $\mathrm{T}^{\mathrm{vir}}_{M(r+0[\mathrm{pt}])} = 0$. Now, letting $\mathbf{t}_1, \mathbf{t}_2, \mathbf{e}_1, \dots, \mathbf{e}_r$ be the characters of the (trivial) $\widetilde{T}$-action on $M(r+0[\mathrm{pt}])$, the universal sheaf of $M(r+0[\mathrm{pt}])$ is $\mathcal{E}_0 = \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^2}\mathbf{e}_i$.[3] In particular, one can compute that $R\mathcal{H}om_\pi(\mathcal{O}_{\mathrm{pt}}, \mathcal{E}_0) = \bigoplus_{i=1}^r \mathbf{e}_i$, and we have the following equation in $A^*_{\widetilde{T}}(\mathrm{pt})$:

$$\Omega_{n_0}\left(\bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^2}\mathbf{e}_i\right) = \int_{\widehat{M}(r-k\epsilon_0-n_0[\mathrm{pt}])} \Theta\big(\mathrm{T}^{\mathrm{vir}}\big) - \sum_{n=0}^{n_0-1} \int_{M(r-(n_0-n)[\mathrm{pt}])} \Theta\big(\mathrm{T}^{\mathrm{vir}}\big)\Omega_n(\mathcal{E})\,.$$

We observe that the left-hand side is by definition a power series in the Chern classes $c_i(R\mathcal{H}om_\pi(\mathcal{O}_{\mathrm{pt}}, \mathcal{E})) = \sigma_i(c_1(\mathbf{e}_1), \dots, c_1(\mathbf{e}_r))$ for $1 \leqslant i \leqslant r$, where σ_i denotes the ith elementary symmetric polynomial. These are analytically independent (since the terms $c_1(\mathbf{e}_i)$ are), so equality (3.2) determines Ω_{n_0} inductively from the Ω_n for $n = 0, \dots, n_0 - 1$. (Note that this includes the base case of the induction.) Thus the collection of all the Ω_n is uniquely determined. □

4. Proof of the blow-up formula for the χ_y-genus

In this section, we prove our main theorem, Theorem 1.4. For this, we apply Theorem 3.2 to the class Θ_y giving rise to the χ_y-genus. Then, it is clearly enough to show that each power series Ω_n is constant and equal to the coefficient of q^n in the power series Y_k appearing in Theorem 1.4. To do this, we use the fact that by (3.2), the Ω_n can be uniquely determined from the equivariant theory of framed moduli spaces in Situation B.

Remark 4.1. An easier version of the proof using the total Chern class $\Theta(-) = c(-)$ in place of Θ_y gives a direct proof of the blow-up formula for virtual Euler characteristics.

In the rest of this section, we work in Situation B, and we let $\{\Omega_n\}$ denote the collection of power series obtained from Theorem 3.2 by the choice $\Theta = \Theta_y$. Note that, by definition, these are power series with coefficients in $\mathrm{F} = \mathbb{Q}(y)$. We recall the $\widetilde{T}$-action on the moduli spaces $M(c)$ and $\widehat{M}(\widehat{c})$ from [NY05a].

Definition 4.2 ($\widetilde{T}$-Action, [NY05a, §§ 2–3]). Let M be one of $M(c)$ or $\widehat{M}(\widehat{c})$, and let $(E, \varphi) \in M$ be a framed sheaf. For $(t_1, t_2, e_1, \dots, e_r) \in \widetilde{T} \cong (\mathbb{C}^*)^{2+r}$, we define the following:

(i) In the case of $M = M(c)$, let F_{t_1,t_2} denote the automorphism of $\mathbb{P}^2$ defined via $[z_0 : z_1 : z_2] \mapsto [z_0 : t_1 z_1 : t_2 z_2]$. In the case of $M = \widehat{M}(\widehat{c})$, we denote again by F_{t_1,t_2} the unique lift of this automorphism to $\widehat{\mathbb{P}}^2$.

(ii) In either case, we let $\varphi_{(t_1,t_2,e_1,\dots,e_r)}$ denote the composition

$$\big(F^{-1}_{t_1,t_2}\big)^* E|_{\ell_\infty} \xrightarrow{(F^{-1}_{t_1,t_2})^*\varphi} \big(F^{-1}_{t_1,t_2}\big)^* \mathcal{O}^{\oplus r}_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}^{\oplus r}_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}^{\oplus r}_{\ell_\infty}\,,$$

where the middle arrow comes from the $(\mathbb{C}^*)^2$-action on $\mathbb{P}^2$ (respectively, $\widehat{\mathbb{P}}^2$) and the last arrow is $(x_1, \dots, x_r) \mapsto (e_1 x_1, \dots, e_r x_r)$.

The action of $\widetilde{T}$ on M is defined by

$$(t_1, t_2, e_1, \dots, e_r) \cdot (E, \varphi) = \big(\big(F^{-1}_{t_1,t_2}\big)^* E, \varphi_{(t_1,t_2,e_1,\dots,e_r)}\big)\,.$$

[3] The $\widetilde{T}$-action on the moduli spaces is described further in Definition 4.2.

Notation 4.3 ($\widetilde{T}$-Representations). We denote by $\mathbf{t}_i$ and $\mathbf{e}_a$, the 1-dimensional $\widetilde{T}$-modules given, respectively, by

$$(t_1, t_2, e_1, \dots, e_r) \longmapsto t_i \quad \text{and} \quad (t_1, t_2, e_1, \dots, e_r) \longmapsto e_a\,.$$

We now recall the description of the torus-fixed points of the moduli spaces of framed sheaves on $\mathbb{P}^2$ and $\widehat{\mathbb{P}}^2$ given in [NY05a]. We also recall their description of the equivariant K-theory class given by the restriction of the tangent bundle to each fixed point. The results are as follows.

THEOREM 4.4 ([NY05a, Proposition 2.9 and Theorem 2.11]). *Let $c = r - n[\mathrm{pt}]$ for $n \geqslant 0$ an integer. Then we have the following:*

(i) *The fixed points of $M(c)$ are in bijection with the set of r-tuples of Young diagrams $\boldsymbol{Y} = (Y_1, \dots, Y_r)$ satisfying $\sum_{i=1}^r |Y_i| = n$.*

(ii) *Let $\boldsymbol{Y} = (Y_1, \dots, Y_r)$ correspond to a fixed point of $M(c)$. The equivariant K-theory class of the restricted tangent bundle $\mathrm{T}_{M(c)}|_{\boldsymbol{Y}}$ is given by $\sum_{a,b=1}^r \mathrm{N}_{a,b}^{\boldsymbol{Y}}(\mathbf{t}_1, \mathbf{t}_2)$, where $\mathrm{N}_{a,b}^{\boldsymbol{Y}}(\mathbf{t}_1, \mathbf{t}_2)$ is given by*

$$\mathbf{e}_b \mathbf{e}_a^{-1} \times \Bigg\{ \sum_{s \in Y_a} \big(\mathbf{t}_1^{-l_{Y_b}(s)} \mathbf{t}_2^{a_{Y_a}(s)+1} \big) + \sum_{t \in Y_b} \big(\mathbf{t}_1^{l_{Y_a}(t)+1} \mathbf{t}_2^{-a_{Y_b}(t)} \big) \Bigg\}.$$

Here, $a_Y(i,j) := \lambda_i - j$ and $l_Y(i,j) := \lambda_j^t - i$ are the arm length *and* leg length *of the box (i,j) in Y, where λ is the partition associated with Y with transpose λ^t.*

THEOREM 4.5 ([NY05a, Proposition 3.2 and Theorem 3.4]). *Let $k, n \in \mathbb{Z}$ be integers and $\epsilon_0 = [C] - \frac{1}{2}[\mathrm{pt}]$. If $\widehat{c} = r + k\epsilon_0 - n[\mathrm{pt}]$, then we have the following:*

(i) *The fixed points of $\widehat{M}(\widehat{c})$ are in natural bijection with the set of triples $(\boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{k})$, where $\boldsymbol{Y} = (Y_1, \dots, Y_r)$ and $\boldsymbol{Z} = (Z_1, \dots, Z_r)$ are r-tuples of Young diagrams and $\boldsymbol{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$ is such that*

$$\sum k_i = k \quad \text{and} \quad \sum_{j=1}^r (|Y_i| + |Z_i|) + \frac{1}{2r} \sum_{1 \leqslant i < j \leqslant r} (k_i - k_j)^2 = n + \frac{k(r-k)}{2r}\,.$$

(ii) *Let F be the fixed point of $\widehat{M}(\widehat{c})$ given by $(\boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{k})$. The equivariant K-theory class of the restricted tangent bundle $\mathrm{T}_{\widehat{M}(\widehat{c})}|_F$ is given by*

$$\sum_{a,b=1}^r \mathrm{L}_{a,b}(\mathbf{t}_1, \mathbf{t}_2) + \mathbf{t}_1^{k_b - k_a} \mathrm{N}_{a,b}^{\boldsymbol{Y}}(\mathbf{t}_1, \mathbf{t}_2/\mathbf{t}_1) + \mathbf{t}_2^{k_b - k_a} \mathrm{N}_{a,b}^{\boldsymbol{Z}}(\mathbf{t}_1/\mathbf{t}_2, \mathbf{t}_2)\,,$$

where the expressions $N_{a,b}$ are defined as in Theorem 4.4 and

$$\mathrm{L}_{a,b}(\mathbf{t}_1, \mathbf{t}_2) = \mathbf{e}_b \mathbf{e}_a^{-1} \times \begin{cases} \displaystyle\sum_{\substack{i,j \geqslant 0 \\ i+j \leqslant k_a - k_b - 1}} \mathbf{t}_1^{-i} \mathbf{t}_2^{-j} & \text{if } k_a > k_b\,, \\ \displaystyle\sum_{\substack{i,j \geqslant 0 \\ i+j \leqslant k_b - k_a - 2}} \mathbf{t}_1^{i+1} \mathbf{t}_2^{j+1} & \text{if } k_a + 1 < k_b\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Example 4.6. From the dimension formula $c_1^2 - 2r\,\mathrm{ch}_2$, one can see that the spaces $M(r+0)$ and $\widehat{M}(r+0)$ consist of a collection of reduced points, while the spaces $\widehat{M}(r - k\epsilon_0)$ are empty for $k > 0$. From Theorems 4.4 and 4.5, it follows that in fact $M(r+0)$ and $\widehat{M}(r+0)$ consist

of a single reduced point. Moreover, the universal sheaf $\mathcal{E}_0$ on $\mathbb{P}^2 \times M(r) \simeq \mathbb{P}^2$ is isomorphic to a direct sum $\bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^2}\mathbf{e}_i$. It follows that $R\mathcal{H}om_\pi(\mathcal{O}_{\mathrm{pt}}, \mathcal{E}_0) \simeq \bigoplus_{i=1}^r \mathbf{e}_i$.

Proposition 4.7. *The collection $\{\Omega_n\}$ obtained from applying Theorem 3.2 (the weak structure theorem) to $\Theta = \Theta_y$ is a collection of elements from $\mathrm{F}[\![\varepsilon_1, \varepsilon_2]\!]$, where $\mathrm{F} = \mathbb{Q}(y)$.*

Proof. We prove by induction on n that the Ω_n are constant in $\nu_1, \dots, \nu_r$. For this, we use the inductive formula (3.2). For $n_0 = 0$, we get that

$$\Omega_0 = \int_{\widehat{M}(r-k\epsilon_0)} \Theta\big(T^{\mathrm{vir}}\big)\,.$$

By Example 4.6, the right-hand side is equal to zero if $k \neq 0$, and equal to 1 if $k = 0$; therefore, Ω_0 is constant in either case. Now let $n_0 > 0$ be fixed, and suppose that we know that Ω_n lies in $\mathrm{F}[\![\varepsilon_1, \varepsilon_2]\!]$ for each $n < n_0$. Then (3.2) becomes

$$\Omega_{n_0}\left(\bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^2}\mathbf{e}_i\right) = \int_{\widehat{M}(r-n_0[\mathrm{pt}]-k\epsilon_0)} \Theta(\mathrm{T}) - \sum_{n=0}^{n_0-1} \Omega_n \int_{M(r+(n-n_0)[\mathrm{pt}])} \Theta(\mathrm{T})\,. \tag{4.1}$$

By Example 4.6, we know that the left-hand side is a power series in the variables

$$\{c_1(\mathbf{e}_1), \dots, c_1(\mathbf{e}_r), c_1(\mathbf{t}_1), c_1(\mathbf{t}_2)\}$$

which is symmetric in the variables $c_1(\mathbf{e}_i)$. We may equivalently write it as a power series in the variables $d_i := \mathrm{ch}(\mathbf{e}_i) - 1$ and $s_i := \mathrm{ch}(\mathbf{t}_i) - 1$ via the substitutions $c_1(\mathbf{e}_i) = \log(1 + d_i)$ and $c_1(\mathbf{t}_i) = \log(1 + s_i)$, where we use the formal power series expansion of $\log(1 + x)$.

We more closely examine the right-hand side of (4.1). By the definition of integration as a sum over fixed points, we can rewrite each integral appearing on the right-hand side as a sum

$$\sum_G \frac{\Theta(\mathrm{T}|_G)}{\mathrm{e}(\mathrm{T}|_G)}\,,$$

where G ranges over the fixed points of a given moduli space. Note that for a non-trivial $\widetilde{T}$-equivariant line bundle L with $\alpha = c_1(L)$, we have

$$\frac{\Theta_y(L)}{\mathrm{e}(L)} = \frac{1 - ye^{-\alpha}}{1 - e^{-\alpha}} = \frac{1}{\alpha}\,\mathrm{td}(\alpha)(1 - ye^{-\alpha})\,.$$

In particular, this fraction is well defined. Write $e_i := \mathrm{ch}(\mathbf{e}_i)$ and $t_i := \mathrm{ch}(\mathbf{t}_i)$. Using Theorems 4.4 and 4.5, and examining the formulas given there, we find that each integral is a sum of products over terms of the form

$$\frac{1 - y(e_a/e_b)t_1^i t_2^j}{1 - (e_a/e_b)t_1^i t_2^j}\,,$$

where $a, b \in \{1, \dots, r\}$ and where $(i, j) \neq (0, 0)$ if $a = b$. We may rewrite each such term as a fraction of two coprime elements in $\mathrm{F}[e_1, \dots, e_r, t_1, t_2]$ by cancelling the potential negative powers of e_b, t_1 and t_2 appearing in numerator and denominator. After doing this, for each fraction with $a \neq b$, the denominator is now of one of the forms

$$e_b - e_a t_1^i t_2^j\,, \quad e_b t_1^{-i} - e_a t_2^j\,, \quad e_b t_2^{-j} - e_a t_1^i \quad \text{or} \quad e_b t_1^{-i} t_2^{-j} - e_a\,, \tag{4.2}$$

depending on which of the indices i, j are positive. For $a = b$, the resulting denominator is a polynomial in only t_1, t_2. By the Gauss lemma, each polynomial (4.2) is irreducible in $\mathrm{F}[e_1, \dots, e_r, t_1, t_2]$ if $a \neq b$. Moreover, each such polynomial vanishes at $e_1 = \cdots = e_r = t_1 =$

$t_2 = 1$. Changing variables to $d_i := e_i - 1$ and $s_i := t_i - 1$, we find that each expression in (4.2) with $a \neq b$ gives rise to an irreducible polynomial that vanishes at the origin in $\mathrm{F}[d_1, \dots, d_r, s_1, s_2]$.

Multiplying both sides of (4.1) with all the denominators of the form (4.2) appearing in the right-hand side, we obtain an equality of the form

$$P(d_1, \dots, d_r, s_1, s_2)\Omega_{n_0}(d_1, \dots, d_r, s_1, s_2) = Q(d_1, \dots, d_r, s_1, s_2)$$

in the power series ring $\mathrm{F}[\![s_1, s_2, d_1, \dots, d_r]\!]$, where P and Q are polynomials, and where the total degree of Q with respect to the d_i is at most the total degree of P with respect to the d_i. Moreover, we may clear the common irreducible polynomial factors of P and Q and assume that they are relatively prime. We claim that then, P and Q are constant in the d_i. Indeed, P is a product of terms (4.2) and terms not involving the d_i, so each irreducible factor of P that involves the d_i satisfies the assumption of Lemma 4.8 below. Then, by that lemma, each such factor must also divide Q as a polynomial, contradicting our relatively prime assumption. It follows that $\Omega = Q/P$ is also independent of the d_i, which is what we wanted to show. □

Lemma 4.8. *Let $P, Q \in k[x_1, \dots, x_n]$ be elements of a polynomial ring over the field k. Suppose that P is irreducible and vanishes at the origin. Then if P divides Q in the power series ring $k[\![x_1, \dots, x_n]\!]$, it already divides Q in $k[x_1, \dots, x_n]$.*

Proof. Let $I = (x_1, \dots, x_n)$ be the ideal of the origin, and consider the localisation $k[x_1, \dots, x_n]_I$. Since the completion $k[x_1, \dots, x_n]_I \to k[\![x_1, \dots, x_n]\!]$ is faithfully flat, it follows that P divides Q in the power series ring if and only if it does so in the localisation.[4] In particular, there is some equality of the form $PR_1 = QR_2$ in $k[x_1, \dots, x_n]$, where $R_2 \notin I$. Since $P \in I$ by assumption, it cannot divide R_2; therefore, P divides Q by unique factorisation. □

Using Proposition 4.7, one can already deduce Theorem 1.4 for the Gieseker moduli spaces by taking appropriate limits in the ε_i-variables. We want to go further and prove the desired formula in Situation B. This will use the following result, which is nothing but a restatement of the rank 1 case of Theorem 1.4 for the framed moduli spaces.

Proposition 4.9. *Let $\mathsf{W}(t_1, t_2, y, q)$ be the rank 1 (that is, $r = 1$) generating series for equivariant χ_y-genera on the moduli spaces of framed sheaves on $\mathbb{P}^2$. More precisely,*

$$\mathsf{W}(t_1, t_2, y, q) := \sum_Y \prod_{s \in Y} \left(\Theta_y\left(\mathbf{t}_1^{-l_Y(s)}\mathbf{t}_2^{a_Y(s)+1}\right)\Theta_y\left(\mathbf{t}_1^{l_Y(s)+1}\mathbf{t}_2^{-a_Y(s)}\right)\right) q^{|Y|},$$

where the sum ranges over all Young diagrams Y. Then

$$\frac{W(t_1, t_2/t_1, y, q)W(t_1/t_2, t_2, y, q)}{W(t_1, t_2, y, q)} = \prod_{n=1}^{\infty} (1 - (yq)^n)^{-1}.$$

Proof. Using the definition of Θ_y, we find that

$$\mathsf{W}(t_1, t_2, y, q) = \sum_Y \left(\prod_{s \in Y} \frac{1 - y t_1^{l_Y(s)} t_2^{-1-a_Y(s)}}{1 - t_1^{l_Y(s)} t_2^{-1-a_Y(s)}} \frac{1 - y t_1^{-1-l_Y(s)} t_2^{a_Y(s)}}{1 - t_1^{-1-l_Y(s)} t_2^{a_Y(s)}} \right) q^{|Y|}.$$

[4] Note that the divisibility can be restated as the inclusion of ideals $(P) \subseteq (P, Q)$ being an isomorphism. By flatness the ideals play well with base change. By faithful flatness isomorphisms are preserved and reflected under base change.

We define a function $\widetilde{\mathsf{W}} := \widetilde{\mathsf{W}}(t,q,u,T)$ by renaming variables as follows:[5]

$$t_1 \longmapsto t\,, \quad t_2 \longmapsto q^{-1}\,, \quad y \longmapsto u\,, \quad q \longmapsto T/u\,.$$

Then

$$\widetilde{\mathsf{W}}(t,q,u,T) = \sum_Y \Bigg(\prod_{s\in Y} \frac{\big(1-ut^{l_Y(s)}q^{1+a_Y(s)}\big)\big(1-u^{-1}t^{1+l_Y(s)}q^{a_Y(s)}\big)}{\big(1-t^{l_Y(s)}q^{1+a_Y(s)}\big)\big(1-t^{1+l_Y(s)}q^{a_Y(s)}\big)} \Bigg) T^{|Y|}.$$

The desired formula now becomes

$$\frac{\widetilde{\mathsf{W}}(tq,q,u,T)\widetilde{\mathsf{W}}(t,tq,u,T)}{\widetilde{\mathsf{W}}(t,q,u,T)} = \prod_{n=1}^{\infty}(1-T^n)^{-1}\,. \tag{4.3}$$

By the (q,t)-Nekrasov–Okounkov formula [RW18, Theorem 1.3], we have the product expansion

$$\widetilde{\mathsf{W}}(t,q,u,T) = \prod_{i,j,k\geqslant 1} \frac{\big(1-uq^it^{j-1}T^k\big)\big(1-u^{-1}q^{i-1}t^jT^k\big)}{\big(1-q^{i-1}t^{j-1}T^k\big)\big(1-q^it^jT^k\big)}\,.$$

Expanding each occurrence of $\widetilde{\mathsf{W}}$ according to this formula and keeping track of all the factors proves (4.3). □

Proof of Theorem 1.4. Proposition 4.7 states that each element of the collection $\{\Omega_n\}$ is a power series only in $c_1(\mathbf{t}_1)$ and $c_1(\mathbf{t}_2)$ over F. In particular, each $\Omega_n(\mathcal{E})$ is pulled back from $A^*(\mathrm{pt})$ and does not involve any of the equivariant parameters e_i. Consequently, the main equation from Theorem 3.2 (the weak structure theorem) implies that we have a relation of generating series

$$\sum_n \int_{\widehat{M}(r-k\epsilon_0-n[\mathrm{pt}])} \Theta(\mathrm{T})q^{2rn-k(r+k)} = \Big(\sum_n \Omega_n q^{2rn}\Big)\cdot \sum_n \int_{M(r-n[\mathrm{pt}])} \Theta(\mathrm{T})q^{2rn}\,. \tag{4.4}$$

We wish to show that $\sum_n \Omega_n q^n$ is equal to Y_k as defined in the statement of the main theorem, Theorem 1.4. To do this, we will evaluate the equivariant integrals on both sides of equation (4.4) and specialise the equivariant parameters corresponding to the $\mathbf{e}_i$.

The integrals arising in equation (4.4) are computed by summing over the fixed points of the associated moduli spaces. By Theorems 4.4 and 4.5, the contribution from each fixed point is a product of terms of the form $\theta\big((e_b/e_a)\,t_1^{i_1}t_2^{i_2}\big)$, where

$$\theta(x) := \frac{1-yx^{-1}}{1-x^{-1}}\,,$$

with $e_i = \mathrm{ch}(\mathbf{e}_i)$ and $t_i = \mathrm{ch}(\mathbf{t}_i)$. Now, taking limits in a fixed order, we observe that

$$\lim_{e_r\to 0}\cdots\lim_{e_1\to 0}\theta\big((e_b/e_a)\,t_1^{i_1}t_2^{i_2}\big) = \begin{cases} 1 & \text{if } a<b\,,\\ y & \text{if } a>b\,,\\ \dfrac{1-yt_1^{-i_1}t_2^{-i_2}}{1-t_1^{-i_1}t_2^{-i_2}} = \theta\big(t_1^{i_1}t_2^{i_2}\big) & \text{if } a=b\,. \end{cases} \tag{4.5}$$

We now consider the generating series associated with the moduli spaces $M(c)$. By the definition of the integral notation from Notation 1.2 combined with Theorem 4.4(i), we have

$$\sum_n \int_{M(r-n[\mathrm{pt}])} \Theta(\mathrm{T})q^{2rn} = \sum_{\boldsymbol{Y}} \Theta\big(\mathrm{T}|_{\boldsymbol{Y}}\big)q^{2r\sum_{i=1}^r |Y_i|}\,,$$

[5] Warning: the symbol q plays different roles before and after the change of variables. The usage after the change of variables is in accordance with that from [RW18].

where the sum ranges over r-tuples of Young diagrams $\boldsymbol{Y}=(Y_1,\dots,Y_r)$. Now, by Theorem 4.4(ii), we can compute $\Theta(\mathrm{T}|_{\boldsymbol{Y}})$ to be

$$\prod_{a,b=1}^{r}\Bigg(\prod_{s\in Y_a}\Theta\big(\mathbf{e}_b/\mathbf{e}_a\mathbf{t}_1^{-l_{Y_b}(s)}\mathbf{t}_2^{a_{Y_a}(s)+1}\big)\times\prod_{t\in Y_b}\Theta\big(\mathbf{e}_b/\mathbf{e}_a\mathbf{t}_1^{l_{Y_a}(t)+1}\mathbf{t}_2^{-a_{Y_b}(t)}\big)\Bigg).$$

Using equation (4.5) gives that $\lim_{e_r\to 0}\cdots\lim_{e_1\to 0}\Theta(\mathrm{T}|_{\boldsymbol{Y}})$ is equal to

$$\begin{aligned}&\prod_{1\leqslant b<a\leqslant r}y^{|Y_a|+|Y_b|}\times\prod_{a=1}^{r}\prod_{s\in Y_a}\theta\big(t_1^{-l_{Y_a}(s)}t_2^{a_{Y_a}(s)+1}\big)\theta\big(t_1^{l_{Y_a}(s)+1}t_2^{-a_{Y_a}(s)}\big)\\&\quad=\prod_{a=1}^{r}\Bigg(y^{(r-1)|Y_a|}\prod_{s\in Y_a}\theta\big(t_1^{-l_{Y_a}(s)}t_2^{a_{Y_a}(s)+1}\big)\theta\big(t_1^{l_{Y_a}(s)+1}t_2^{-a_{Y_a}(s)}\big)\Bigg).\end{aligned}$$

We conclude that

$$\begin{aligned}&\lim_{e_r\to 0}\cdots\lim_{e_1\to 0}\sum_n\int_{M(r-n[\mathrm{pt}])}\Theta(T)q^{2rn}\\&\quad=\Bigg(\sum_Y\Bigg(\prod_{s\in Y}\theta\big(t_1^{-l_Y(s)}t_2^{a_Y(s)+1}\big)\theta\big(t_1^{l_{Y_a}(s)+1}t_2^{-a_{Y_a}(s)}\big)\Bigg)\big(q^{2r}y^{r-1}\big)^{|Y|}\Bigg)^r\\&\quad=\big(\mathsf{W}\big(t_1,t_2,y,y^{r-1}q^{2r}\big)\big)^r,\end{aligned}\tag{4.6}$$

where W is as in Proposition 4.9.

We now consider the generating series associated with the moduli spaces $\widehat{M}(\widehat{c})$. By the definition of the integral notation from Notation 1.2 combined with Theorem 4.5(i), we have

$$\begin{aligned}&\sum_n\int_{\widehat{M}(r-n[\mathrm{pt}]-k\epsilon_0)}\Theta(\mathrm{T})q^{2rn-k(r+k)}\\&\quad=\sum_{(\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{k})}\Theta(\mathrm{T}|_{(\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{k})})q^{2r\sum_{i=1}^{r}(|Y_i|+|Z_i|)+\sum_{1\leqslant i<j\leqslant r}(k_i-k_j)^2}.\end{aligned}\tag{4.7}$$

Consider a fixed triple $(\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{k})$. In the notation of Theorem 4.5(ii), we have

$$\begin{aligned}\lim_{e_r\to 0}\cdots\lim_{e_1\to 0}\Theta(\mathrm{L}_{a,b}(\mathbf{t}_1,\mathbf{t}_2))&=\begin{cases}y^{\binom{k_a-k_b+1}{2}} & \text{if } a>b \text{ and } k_a>k_b,\\ y^{\binom{k_b-k_a}{2}} & \text{if } a>b \text{ and } k_a+1<k_b,\\ 1 & \text{else}\end{cases}\\&=\begin{cases}y^{(k_a-k_b)^2/2}y^{(k_a-k_b)/2} & \text{if } a>b,\\ 1 & \text{else}.\end{cases}\end{aligned}$$

In particular, we have the identity

$$\lim_{e_r\to 0}\cdots\lim_{e_1\to 0}\prod_{a,b=1}^{r}\Theta(\mathrm{L}_{a,b}(\mathbf{t}_1,\mathbf{t}_2))=y^{\sum_{a>b}(k_a-k_b)^2/2}y^{\sum_{a>b}(k_a-k_b)/2}.$$

Proceeding as before for $\mathrm{N}^{\boldsymbol{Y}}_{a,b}(-,-)$ gives that $\lim_{e_r\to 0}\cdots\lim_{e_1\to 0}\Theta(\mathrm{T}|_{(\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{k})})$ is equal to

$$\prod_{a=1}^{r}\Big(y^{|Y_a|(r-1)}\prod_{s\in Y_a}\theta\big(t_1^{-l_{Y_a}(s)}(t_2/t_1)^{a_{Y_a}(s)+1}\big)\theta\big(t_1^{l_{Y_a}(s)+1}(t_2/t_1)^{-a_{Y_a}(s)}\big)\Big)$$
$$\times\prod_{a=1}^{r}\Big(y^{|Z_a|(r-1)}\prod_{s\in Z_a}\theta\big((t_1/t_2)^{-l_{Z_a}(s)}t_2^{a_{Z_a}(s)+1}\big)\theta\big((t_1/t_2)^{l_{Z_a}(s)+1}t_2^{-a_{Z_a}(s)}\big)\Big)$$
$$\times\, y^{\sum_{a>b}(k_a-k_b)^2/2}y^{\sum_{a<b}(k_a-k_b)/2}.$$

Now, considering the generating series on the left-hand side of (4.7), we see that we can separate the sums over pairs of r-tuples of Young diagrams into $2r$ sums over Young diagrams after taking limits in the e_i. In conclusion, we find that the limiting series is equal to

$$\lim_{e_r\to 0}\cdots\lim_{e_1\to 0}\sum_{n}\int_{\widehat{M}(r-n[\mathrm{pt}]-k\epsilon_0)}\Theta(\mathrm{T})q^{2rn-k(r+k)}$$
$$=\big(\mathsf{W}\big(t_1/t_2,t_2,y,y^{r-1}q^{2r}\big)\cdot\mathsf{W}\big(t_1,t_2/t_1,y,y^{r-1}q^{2r}\big)\big)^r$$
$$\times\sum_{k_1+\cdots+k_r=k}\big(q^2y\big)^{\sum_{i<j}(k_i-k_j)^2/2}y^{\sum_{i<j}(k_i-k_j)/2}.$$

Comparing this to equation (4.6) and using Proposition 4.9 gives the desired result. □

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Nikolas Kuhn kuhnn@maths.ox.ac.uk
Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany

Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford, OX2 6GG, UK

Oliver Leigh oliverle@math.uio.no
Department of Mathematics, Uppsala University, Box 480, 751 06 Uppsala, Sweden

Department of Mathematics, University of Oslo, Niels Henrik Abels hus, Moltke Moes vei 35, 0851 Oslo, Norway

Yuuji Tanaka ytanaka@bimsa.cn
Beijing Institute of Mathematical Sciences and Applications (BIMSA), No. 544, Hefangkou Village, Huaibei Town, Huairou District, Beijing 101408, China