Algebraic Geometry 13 (6) (2026) 747–795
doi:10.14231/AG-2026-022

Quivers, flow trees, and log curves

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Abstract

Donaldson–Thomas (DT) invariants of a quiver with potential can be expressed in terms of simpler attractor DT invariants by a universal formula. The coefficients in this formula are calculated combinatorially using attractor flow trees. In this paper, we prove that these coefficients are genus 0 log Gromov–Witten invariants of d-dimensional toric varieties, where d is the number of vertices of the quiver. This result follows from a log-tropical correspondence theorem that relates $(d-2)$-dimensional families of tropical curves obtained as universal deformations of attractor flow trees, and rational log curves in toric varieties.

1. Introduction

This paper relates two seemingly different ways to define enumerative invariants given a quiver with potential: quiver Donaldson–Thomas (DT) invariants, defined using moduli spaces of quiver representations, and log Gromov–Witten invariants of toric varieties, defined as counts of rational log curves satisfying constraints specified by the combinatorics of the quiver.

To relate quiver DT invariants to log Gromov–Witten invariants, we use the combinatorial calculation of quiver DT invariants in terms of attractor flow trees following our previous work [AB22]. We explain how to interpret such trees as tropical curves and then prove a new correspondence theorem between counts of tropical curves and log Gromov–Witten invariants of toric varieties. Crucially, our log-tropical correspondence theorem involves non-rigid tropical curves moving in non-trivial families, unlike previous correspondence theorems in the literature where only rigid tropical curves are usually considered.

After a brief description of DT quiver invariants in Section 1.1 and of log Gromov–Witten invariants in Section 1.2, we provide a more detailed overview of the log-tropical correspondence in Section 1.3 and of the quiver DT-log Gromov–Witten correspondence in Section 1.4.

Received 5 April 2023, accepted in final form 26 March 2025.
2020 Mathematics Subject Classification 14N35, 14T15.
Keywords: Donaldson–Thomas invariants, quivers, log Gromov–Witten invariants, tropical curves.
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Hülya Argüz was supported by the NSF grant DMS-2302116, and Pierrick Bousseau was supported by the NSF grant DMS-2302117.

1.1 DT quiver invariants

A quiver with potential (Q,W) is given by a finite oriented graph Q and a finite formal linear combination W of oriented cycles in Q. We denote by Q_0 the set of vertices of Q, and by Q_1 the set of edges of Q. For every dimension vector $\gamma \in N := \mathbb{Z}^{Q_0}$ and stability parameter

$$\theta \in \left(\gamma^\perp\right)_{\mathbb{R}} \subset M_{\mathbb{R}} := \mathrm{Hom}(N,\mathbb{R})\,,$$

where $\left(\gamma^\perp\right)_{\mathbb{R}} := \{\theta \in M_{\mathbb{R}} \mid \theta(\gamma) = 0\}$, the theory of King's stability for quiver representations [Kin94] defines a quasiprojective variety M_γ^θ over $\mathbb{C}$ parametrizing S-equivalence classes of θ-semistable representations of Q of dimension γ. Moreover, the trace of the potential naturally defines a regular function $\mathrm{Tr}(W)_\gamma^\theta \colon M_\gamma^\theta \to \mathbb{C}$.

A stability parameter $\theta \in \left(\gamma^\perp\right)_{\mathbb{R}}$ is called generic if $\theta \in \left(\gamma'^\perp\right)_{\mathbb{R}}$ for $\gamma' \in N$ implies that γ' is collinear with γ. For a generic stability parameter θ, the *DT invariant* $\Omega_\gamma^{+,\theta}$ is an integer that is a virtual count of the critical points of $\mathrm{Tr}(W)_\gamma^\theta$. We have $\Omega_\gamma^{+,\theta} = 0$ if the θ-stable locus in M_γ^θ is empty, and else, $\Omega_\gamma^{+,\theta}$ is the Euler characteristic of M_γ^θ valued in the perverse sheaf obtained by applying the vanishing-cycle functor for $\mathrm{Tr}(W)_\gamma^\theta$ to the intersection cohomology sheaf IC normalized to be the constant sheaf in degree 0 when M_γ^θ is smooth [DM15, DM20, MR19]:

$$\Omega_\gamma^{+,\theta} = e\big(M_\gamma^\theta, \phi_{\mathrm{Tr}(W)_\gamma^\theta}(\mathrm{IC})\big) \in \mathbb{Z}\,. \tag{1.1}$$

For example, if γ is primitive and $W = 0$, then $\Omega_\gamma^{+,\theta}$ is simply the topological Euler characteristic of M_γ^θ. For every $\gamma \in N$, we set

$$\kappa(\gamma) := (-1)^{\chi(\gamma,\gamma)} \in \{\pm 1\}\,,$$

where $\chi \colon N \times N \to \mathbb{Z}$ is the Euler form of Q, given by

$$\chi(\gamma,\gamma') = \sum_{i \in Q_0} \gamma_i \gamma_i' - \sum_{(\alpha\colon i \to j) \in Q_1} \gamma_i \gamma_j'\,.$$

It is often convenient to repackage the DT invariants $\Omega_\gamma^{+,\theta}$ into the *rational DT invariants*

$$\overline{\Omega}_\gamma^{+,\theta} := \sum_{\substack{\gamma' \in N \\ \gamma = k\gamma',\, k \in \mathbb{Z}_{\geqslant 1}}} \frac{\kappa(\gamma')^{k-1}}{k^2}\, \Omega_{\gamma'}^{+,\theta} \in \mathbb{Q}\,, \tag{1.2}$$

which can also be defined using the motivic Hall algebra [JS12, KS08, Rei10, Rei11]. In the literature on DT invariants, it is more common to work with DT invariants Ω_γ^θ defined by

$$\Omega_\gamma^\theta := (-1)^{\dim M_\gamma^\theta} e\big(M_\gamma^\theta, \phi_{\mathrm{Tr}(W)_\gamma^\theta}(\mathrm{IC})\big) \in \mathbb{Z}$$

when the θ-stable locus is non-empty, and the rational DT invariants $\overline{\Omega}_\gamma^\theta$ defined by

$$\overline{\Omega}_\gamma^\theta := \sum_{\substack{\gamma' \in N_Q \\ \gamma = k\gamma',\, k \in \mathbb{Z}_{\geqslant 1}}} \frac{1}{k^2} \Omega_{\gamma'}^\theta \in \mathbb{Q}\,.$$

When the θ-stable locus is non-empty, we have $\dim M_\gamma^\theta = 1 - \chi(\gamma,\gamma)$, and so we have

$$\Omega_\gamma^{+,\theta} = -\kappa(\gamma)\Omega_\gamma^\theta \quad \text{and} \quad \overline{\Omega}_\gamma^{+,\theta} = -\kappa(\gamma)\overline{\Omega}_\gamma^\theta\,.$$

In this paper, we work with the DT invariants $\Omega_\gamma^{+,\theta}$ as they are more directly related to log Gromov–Witten invariants.

The DT invariants $\Omega_\gamma^{+,\theta}$ are locally constant functions of the generic stability parameter $\theta \in \gamma^\perp$, and their jumps across the loci of non-generic stability parameters are controlled by the wall-crossing formula of Joyce–Song and Kontsevich–Soibelman [JS12, KS08]. Using the wall-crossing formula, the DT invariants can be computed in terms of the simpler *attractor DT invariants*, which are DT invariants at specific values of the stability parameter. Let $\omega\colon N \times N \to \mathbb{Z}$ be the skew-symmetric form on N given by

$$\omega(\gamma,\gamma') := \sum_{i,j\in Q_0} (a_{ij} - a_{ji})\gamma_i\gamma'_j\,, \tag{1.3}$$

where a_{ij} is the number of arrows in Q from the vertex i to the vertex j. The specific point $\iota_\gamma\omega := \omega(\gamma,-) \in \gamma^\perp \subset M_\mathbb{R}$ is called the *attractor point* for γ; see [AP19, MP25]. In general, the attractor point $\iota_\gamma\omega$ is not generic, and we define the attractor DT invariants $\Omega_\gamma^{+,*}$ by

$$\Omega_\gamma^{+,*} := \Omega_\gamma^{+,\theta_\gamma}\,, \tag{1.4}$$

where θ_γ is a small generic perturbation of $\iota_\gamma\omega$ in $\gamma^\perp$; see [AP19, MP25]. One can check that $\Omega_\gamma^{+,*}$ is independent of the choice of the small perturbation [AP19, MP25].

Through an iterative use of the wall-crossing formula, the rational DT invariants $\overline{\Omega}_\gamma^{+,\theta}$ for generic stability parameters $\theta \in \gamma^\perp$ are expressed in terms of the rational attractor DT invariants $\overline{\Omega}_\gamma^{+,*}$ by a formula of the form

$$\overline{\Omega}_\gamma^{+,\theta} = \sum_{r\geqslant 1} \sum_{\substack{\{\gamma_i\}_{1\leqslant i\leqslant r}\\ \sum_{i=1}^r \gamma_i=\gamma}} \frac{1}{|\mathrm{Aut}(\{\gamma_i\}_i)|} F_r^\theta(\gamma_1,\dots,\gamma_r) \prod_{i=1}^r \overline{\Omega}_{\gamma_i}^{+,*}\,, \tag{1.5}$$

where the second sum is over the multisets $\{\gamma_i\}_{1\leqslant i\leqslant r}$ with $\gamma_i \in N$ and $\sum_{i=1}^r \gamma_i = \gamma$; see [AP19, KS14]. Here, the denominator $|\mathrm{Aut}(\{\gamma_i\}_i)|$ is the order of the symmetry group of $\{\gamma_i\}$: If $m_{\gamma'}$ is the number of times that $\gamma' \in N$ appears in $\{\gamma_i\}_i$, then $|\mathrm{Aut}(\{\gamma_i\}_i)| = \prod_{\gamma'\in N}(m_{\gamma'}!)$. The coefficients $F_r^\theta(\gamma_1,\dots,\gamma_r)$ are integers and are universal in the sense that they depend on (Q,W) only through the skew-symmetric form ω on N. The dependence on the potential W is entirely contained in the attractor DT invariants $\overline{\Omega}_{\gamma_i}^{+,*}$. Moreover, see [KS14, § 3.2], the coefficients $F_r^\theta(\gamma_1,\dots,\gamma_r)$ have a natural decomposition

$$F_r^\theta(\gamma_1,\dots,\gamma_r) = \sum_h F_{r,h}^\theta(\gamma_1,\dots,\gamma_r) \tag{1.6}$$

indexed by *attractor flow trees* $h\colon T \to M_\mathbb{R}$, which are rooted balanced trees in $M_\mathbb{R}$ with root at θ and with r leaves decorated by $\gamma_1,\dots,\gamma_r$; see Definition 5.2 for details. The contribution $F_{r,h}^\theta(\gamma_1,\dots,\gamma_r)$ of $h\colon T \to M_\mathbb{R}$ is obtained by applying the wall-crossing formula at each vertex of T.

In our previous paper [AB22], we proved an explicit formula, called the *flow tree formula* and conjectured by Alexandrov–Pioline [AP19], which computes the coefficients $F_{r,h}^\theta(\gamma_1,\dots,\gamma_r)$ in (1.6) in terms of a sum over trivalent perturbed attractor flow trees obtained as small perturbations of $h\colon T \to M_\mathbb{R}$. A different combinatorial formula for $F_r^\theta(\gamma_1,\dots,\gamma_r)$ was proved by Mozgovoy in [Moz22]. The main goal of the present paper is to give a geometric interpretation of these coefficients $F_{r,h}^\theta(\gamma_1,\dots,\gamma_r)$ in terms of genus 0 log Gromov–Witten invariants of toric varieties.

1.2 Log Gromov–Witten invariants

Log Gromov–Witten theory, developed by Abramovich–Chen and Gross–Siebert [AC14, GS13], provides a framework to enumerate algebraic curves in an algebraic variety X having pre-

scribed tangency conditions along a divisor $D \subset X$ that is allowed to have particular types of singularities—typically, we consider normal crossing divisors. Generalizing the notion of a stable map $f\colon C \to X$ to the log setting amounts to endowing all spaces with (fine, saturated) log structures and lifting all morphisms to morphisms of log spaces; see Section 3.1 for a brief review of log schemes and stable log maps.

The additional data of a log structure on a variety X encodes combinatorial information that is used to define a cone complex associated with X, referred to as the *tropicalization* of X. This makes it feasible to count log curves in terms of their tropical counterparts. In [NS06], Nishinou–Siebert used toric degenerations and log deformation theory to prove a correspondence theorem between counts of genus 0 complex curves in d-dimensional toric varieties and counts of tropical curves in $\mathbb{R}^d$. More precisely, the main result of [NS06] provides a correspondence between counts of stable log maps and counts of rigid tropical trees. To obtain such a correspondence, one considers log curves generically contained in the big torus orbit of the toric variety and with generically trivial log structure.

In the present paper, given a quiver with d vertices, we prove a correspondence theorem between counts of log curves in d-dimensional toric varieties and tropical counts associated with $(d-2)$-dimensional families of tropical curves in $\mathbb{R}^d$ obtained as universal deformations of the attractor flow trees. A key and major difference with the setup of [NS06] is that for $d \geqslant 3$, the relevant log curves are entirely contained in the toric boundary divisor, have generically non-trivial log structures, and correspond to non-rigid tropical curves moving in non-trivial $(d-2)$-dimensional families.

We now describe more precisely our log-tropical correspondence theorem. As in Section 1.1, we fix a quiver Q, with set of vertices Q_0 of cardinality d, and let ω be the skew-symmetric form on N as in (1.3). We write

$$N = \mathbb{Z}^{Q_0} \simeq \mathbb{Z}^d \quad \text{and} \quad M_{\mathbb{R}} = \mathrm{Hom}(N, \mathbb{R}) \simeq \mathbb{R}^d .$$

We also fix a tuple $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_r)$ of elements $\gamma_i \in N$ such that $\iota_{\gamma_i}\omega \neq 0$ for all $1 \leqslant i \leqslant r$ and $\iota_{\gamma}\omega \neq 0$, where $\gamma = \sum_{i=1}^r \gamma_i$. We count curves in a d-dimensional projective toric variety X_Σ defined by a fan in $M_{\mathbb{R}}$ containing the rays $\mathbb{R}_{\geqslant 0}\iota_{\gamma_i}\omega$ and such that the hyperplanes $\left(\gamma_i^{\perp}\right)_{\mathbb{R}}$ in $M_{\mathbb{R}}$ are unions of cones of Σ. We refer to such a fan as a $\boldsymbol{\gamma}$-fan (see Definition 3.5). In particular, for every $1 \leqslant i \leqslant r$, we have a toric divisor D_i of X_Σ corresponding to the ray $\mathbb{R}_{\geqslant 0}\iota_{\gamma_i}\omega$. The toric variety X_Σ has a natural divisorial log structure defined by its toric boundary D_Σ, and elements of N naturally define tangency conditions along D_Σ at marked point for stable log maps to X_Σ; see Section 3.1 for details.

Consider hypersurfaces H_i in D_i given by an equation of the form $z^{\gamma_i/|\gamma_i|} = c_i$ for some constant c_i, where $|\gamma_i|$ is the divisibility of γ_i in N, and let $\mathbf{H} = (H_1, \dots, H_r)$. We denote by

$$\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma},\mathbf{H}}(X_\Sigma)$$

the moduli space of genus 0 stable log maps $f\colon C \to X_\Sigma$ having $r+1$ marked points $x_1, \dots, x_r, x_{\mathrm{out}}$, with tangency condition along D_Σ given by $\iota_{\gamma_i}\omega$ at x_i and $-\iota_{\gamma}\omega$ at x_{out}, and such that $f(x_i) \in H_i$ for all $1 \leqslant i \leqslant r$; see Figure 1. The moduli space $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma},\mathbf{H}}(X_\Sigma)$ is naturally stratified by the combinatorics of the log structure and is log smooth of dimension $d-2$ for general H_i; see Theorem 3.8. If $f\colon C \to X_\Sigma$ is a general point in a codimension k stratum of $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma},\mathbf{H}}(X_\Sigma)$, then the tropicalization of $f\colon C \to X_\Sigma$ is naturally a k-dimensional face of the moduli space $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}(\Sigma)$ of genus 0 tropical curves $h\colon \Gamma \to M_{\mathbb{R}}$ in $M_{\mathbb{R}}$ with $r+1$ legs $L_1, \dots, L_r, L_{\mathrm{out}}$, of weighted directions $\iota_{\gamma_i}\omega$ and $-\iota_{\gamma}\omega$, and such that $h(L_i) \subset A_i^0 := \left(\gamma_i^{\perp}\right)_{\mathbb{R}}$; see Figure 2. In particular, given a

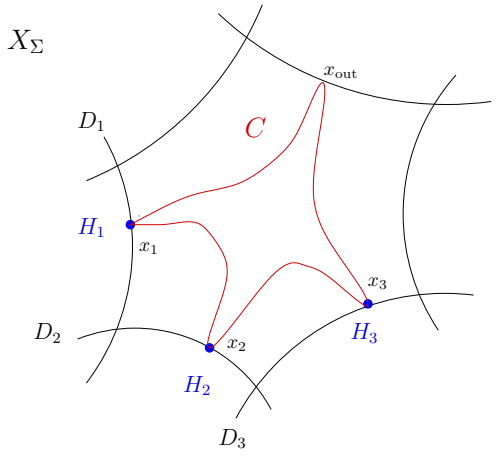


FIGURE 1. A stable log map in $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma},\mathbf{H}}(X_\Sigma)$ for $d=2$ and $r=3$.

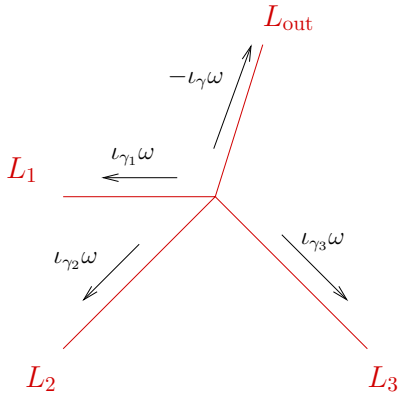


FIGURE 2. A tropical curve in $\mathcal{M}^{\text{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}(\Sigma)$ for $d=2$ and $r=3$.

$(d-2)$-dimensional face ρ of $\mathcal{M}^{\text{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}(\Sigma)$, one obtains a log Gromov–Witten invariant $N^{\text{toric}}_\rho(X_\Sigma)$ by counting the 0-dimensional strata of $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma},\mathbf{H}}(X_\Sigma)$ with tropicalization ρ.

1.3 Log-tropical correspondence

To compute the log Gromov–Witten invariant $N^{\text{toric}}_\rho(X_\Sigma)$ tropically, one considers general affine perturbations A_i of the linear hyperplanes $A_i^0 = (\gamma_i^\perp)_\mathbb{R}$ and the corresponding moduli space $\mathcal{M}^{\text{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ of tropical curves in $M_\mathbb{R}$ with $h(L_i)\subset A_i$ for all $1\leqslant i\leqslant r$. There is a finite set S_ρ of $(d-2)$-dimensional faces σ of $\mathcal{M}^{\text{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ that in the limit $\mathbf{A}\to\mathbf{A}^0$ recovers ρ. While a general tropical curve in the face ρ might have vertices of arbitrary valency, general tropical curves in the faces σ are trivalent; see Lemma 2.12 and Figure 3.

In particular, one can define the tropical multiplicity N^{trop}_σ of σ either as a lattice index (see Definition 2.20) or as a product over the vertices of local multiplicities determined by the skew-symmetric form ω (see Lemma 2.24). Moreover, if $\dim\mathfrak{d}^\rho_{L_{\text{out}}} = d-1$, where $\mathfrak{d}^\rho_{L_{\text{out}}}\subset M_\mathbb{R}$ is the union of the loci $h(L_{\text{out}})$ for $h\in\rho$, we define a tropical coefficient $k_\rho\in\mathbb{Z}_{\geqslant 1}$ as an explicit lattice index; see (2.10). We also define a similar coefficient k_σ for $\sigma\in S_\rho$.

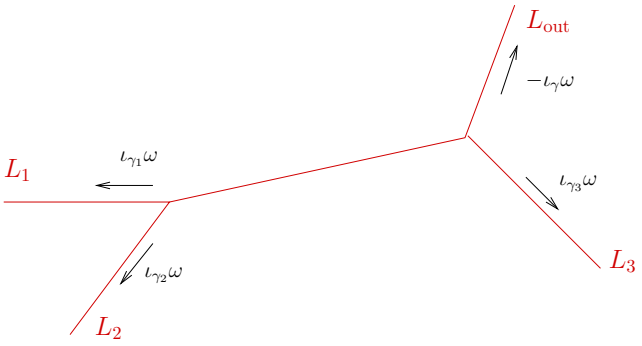


FIGURE 3. A tropical curve in $\mathcal{M}^{\text{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}(\Sigma)$ for $d = 2$ and $r = 3$.

We can now state the log-tropical correspondence theorem for the log Gromov–Witten invariants $N^{\text{toric}}_\rho(X_\Sigma)$.

THEOREM A. *Fix a skew-symmetric form ω on N and a tuple $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_r)$ of elements $\gamma_i \in N$ such that $\iota_{\gamma_i}\omega \neq 0$ and $\iota_\gamma\omega \neq 0$, where $\gamma = \sum_{i=1}^r \gamma_i$. Also fix a $\boldsymbol{\gamma}$-fan Σ. Then, for every general tuple $\mathbf{A} = (A_1, \dots, A_r)$ of affine hyperplane in $M_\mathbb{R}$ parallel to the linear hyperplanes $\left(\gamma_i^\perp\right)_\mathbb{R}$, and for every $(d-2)$-dimensional face ρ of $\mathcal{M}^{\text{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}(\Sigma)$ such that $\dim \mathfrak{d}^\rho_{L_{\text{out}}} = d-1$, the log Gromov–Witten invariant $N^{\text{toric}}_\rho(X_\Sigma)$ of the toric variety X_Σ satisfies*

$$k_\rho N^{\text{toric}}_\rho(X_\Sigma) = \sum_{\sigma \in S_\rho} k_\sigma N^{\text{trop}}_\sigma . \tag{1.7}$$

We prove Theorem A using a toric degeneration of X_Σ and the recent theory of punctured log maps of Abramovich–Chen–Gross–Siebert [ACGS25] to construct stable log maps to the special fiber by gluing.

1.4 Quiver DT-log Gromov–Witten correspondence

Combining the flow tree formula of [AB22] with the log-tropical correspondence given by Theorem A, we obtain a geometric interpretation of the coefficients $F^\theta_{r,h}(\gamma_1, \dots, \gamma_r)$ in terms of genus 0 log Gromov–Witten invariants of toric varieties.

Let (Q, W) be a quiver with potential with d vertices as in Section 1.1 and ω be the corresponding skew-symmetric form on $N = \mathbb{Z}^{Q_0}$. We fix a dimension vector $\gamma \in N$ such that $\iota_\gamma\omega \neq 0$ and a decomposition $\gamma = \sum_{i=1}^r \gamma_i$ such that $\iota_{\gamma_i}\omega \neq 0$ for all $1 \leqslant i \leqslant r$. We set $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_r)$ and fix a $\boldsymbol{\gamma}$-fan Σ. Then, every attractor flow tree $(h\colon T \to M_\mathbb{R})$ naturally defines a $(d-2)$-dimensional face $\tilde{\rho}_h$ of the moduli space of tropical curves $\mathcal{M}^{\text{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}$. One first turns h into a tropical curve $\overline{h}\colon \overline{T} \to M_\mathbb{R}$ by extending its root to infinity in the direction $-\iota_\gamma\omega$. Then, $\tilde{\rho}_h$ is the smallest face of $\mathcal{M}^{\text{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}$ containing $\overline{h}$, that is, the universal family of deformations of $\overline{h}$ preserving its combinatorial type; see Lemma 5.6. In particular, one can consider as in Section 1.2 the genus 0 log Gromov–Witten invariant $N^{\text{toric}}_{\tilde{\rho}_h}(X_\Sigma)$ of the d-dimensional toric variety X_Σ of fan Σ, and the tropical coefficient $k_{\tilde{\rho}_h}$.

THEOREM B. *Let (Q, W) be a quiver with potential. Fix a dimension vector $\gamma \in N = \mathbb{Z}^{Q_0}$ such that $\iota_\gamma\omega \neq 0$, a decomposition $\gamma = \sum_{i=1}^r \gamma_i$ such that $\iota_{\gamma_i}\omega \neq 0$ for all $1 \leqslant i \leqslant r$, a generic stability parameter $\theta \in \gamma^\perp$, and an attractor flow tree $h\colon T \to M_\mathbb{R}$ with root at θ and r leaves*

decorated by $\gamma_1, \dots, \gamma_r$. *Then, for every* $\boldsymbol{\gamma}$*-fan* Σ, *the coefficient* $F^\theta_r(\gamma_1, \dots, \gamma_r)$ *in* (1.5) *and* (1.6) *satisfies*

$$F^\theta_{r,h}(\gamma_1, \dots, \gamma_r) = \frac{\prod_{i=1}^r |\gamma_i|}{|\gamma|} k_{\widetilde{\rho}_h} N^{\mathrm{toric}}_{\widetilde{\rho}_h}(X_\Sigma)\,.$$

If $\iota_\gamma \omega = 0$, then $\overline{\Omega}^{+,\theta}_\gamma$ does not experience any non-trivial wall-crossing, so $\overline{\Omega}^{+,\theta}_\gamma = \overline{\Omega}^{+,\star}_\gamma$ for every $\theta \in \gamma^\perp$. Moreover, if $r \geqslant 2$ and $\iota_{\gamma_i}\omega = 0$ for some i, then $F^\theta_{r,h}(\gamma_1, \dots, \gamma_r) = 0$. In particular, Theorem B applies to compute $F^\theta_{r,h}(\gamma_1, \dots, \gamma_r)$ in every non-trivial situation.

By the flow tree formula of [AB22], one can compute $F^\theta_{r,h}(\gamma_1, \dots, \gamma_r)$ using generic trivalent perturbations of h. To prove Theorem B, we show that the universal deformations of these perturbed attractor trees are exactly the $(d-2)$-dimensional families of tropical curves σ appearing in the right-hand side of (1.7), and then we use the log-tropical correspondence of Theorem A to rewrite the corresponding tropical counts as the log Gromov–Witten invariant $N^{\mathrm{toric}}_{\widetilde{\rho}_h}(X_\Sigma)$.

1.5 Related works

Another way to construct a geometrical object out of a quiver is based on the theory of cluster algebras [FZ02] and cluster varieties [FG09]. From a quiver Q with d vertices, one can construct the corresponding d-dimensional cluster variety using cluster transformations in a way prescribed by the combinatorics of Q. A key result of Gross–Hacking–Keel–Kontsevich [GHKK18] shows that the algebra of regular functions of a cluster variety admits a canonical basis, which can be constructed using a combinatorial gadget known as the cluster scattering diagram. A version of such a cluster scattering diagram is shown to capture quiver DT invariants in the work of Bridgeland [Bri17]. On the other hand, Gross–Siebert in [GS22] show how to construct a mirror to a given log Calabi–Yau variety from a canonical scattering diagram, defined using counts of log curves. In our previous paper [AB23], we showed how the canonical scattering diagram for cluster varieties relates to the cluster scattering.

Building on the results in this paper, in a sequel [AB25], we provide a correspondence between quiver DT invariants, in the situation where attractor DT invariants are trivial, and counts of log curves in cluster varieties, using the comparison between cluster and canonical scattering diagrams. This generalizes the Kronecker–Gromov–Witten correspondence that was shown earlier in the case where the cluster variety is of dimension 2; see [Bou20, GP10, GPS10, RSW12, RW13]. For variants of the Kronecker–Gromov–Witten correspondence in dimension 2, see also [Bou21, RW21]. Such a correspondence between quiver DT invariants and log curve counts in cluster varieties is compatible with the results of the present paper, as cluster varieties can be obtained by non-toric blow-ups of toric varieties, as explained in [GHK15], and therefore admit degenerations into a toric variety and other simpler varieties, as studied in [AG22, GPS10]. In particular, log curve counts in cluster varieties are naturally related to log curve counts in toric varieties considered in this paper.

2. The tropical enumerative problem

Throughout this paper, we work over an algebraically closed field $\Bbbk$ of characteristic 0. We denote by $M \simeq \mathbb{Z}^d$ a free abelian group of finite rank d, and by $M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R} \simeq \mathbb{R}^d$ the associated d-dimensional real vector space. We let $N := \mathrm{Hom}(M, \mathbb{Z})$ denote the dual abelian group.

2.1 Tropical curves

We mainly follow the conventions and notation of [NS06]. In this paper, a *graph* Γ is finite and connected, consisting of vertices, edges connecting pairs of vertices, and legs adjacent to single vertices. Moreover, we assume that legs are marked by distinct labels. We set

$$V(\Gamma) := \text{the set of vertices of } \Gamma\,,$$
$$E(\Gamma) := \text{the set of edges of } \Gamma\,,$$
$$L(\Gamma) := \text{the set of legs of } \Gamma\,.$$

We denote the set of the two vertices adjacent to an edge E by ∂E, and similarly the vertex adjacent to a leg L by ∂L.

Definition 2.1. A *weighted graph* is a graph Γ together with a weight function $w\colon E(\Gamma)\cup L(\Gamma) \to \mathbb{Z}_{\geqslant 1}$ assigning a weight $w(E) \in \mathbb{Z}_{\geqslant 1}$ to every edge or leg E of Γ.

Definition 2.2. A *parametrized tropical curve in* $M_{\mathbb{R}}$ is a weighted graph Γ without divalent vertices, together with a proper continuous map $h\colon \Gamma \to M_{\mathbb{R}}$ satisfying the following conditions:

(i) For every edge or leg E, the restriction $h|_E$ is an embedding with image contained in an affine line with rational slope.

(ii) For every vertex $v \in V(\Gamma)$, the following *balancing condition* holds: Let $E_1, \dots, E_m \in E(\Gamma) \cup L(\Gamma)$ be the edges or legs adjacent to v, and let $\bar{u}_i \in M$ be the primitive integral vector emanating from $h(v)$ in the direction of $h(E_i)$; then

$$\sum_{i=1}^{m} w(E_i)\bar{u}_i = 0\,.$$

An isomorphism of parameterized tropical curves $h\colon \Gamma \to M_{\mathbb{R}}$ and $h_0\colon \Gamma_0 \to M_{\mathbb{R}}$ is a homeomorphism $\Psi\colon \Gamma \to \Gamma_0$ respecting the marking of the legs, the weights of the edges and legs, and such that $h = h_0 \circ \Psi$. A *tropical curve* is an isomorphism class of parameterized tropical curves. The genus of a tropical curve $h\colon \Gamma \to M_{\mathbb{R}}$ is the first Betti number of its domain Γ. A rational tropical curve is a tropical curve of genus 0. In what follows, we focus on rational tropical curves.

Definition 2.3. A *tropical type* for $M_{\mathbb{R}}$ is the data $(\Gamma, \bar{u})$ of a weighted graph Γ and of a map

$$\bar{u}\colon F(\Gamma) \longrightarrow M\,, \quad (v, E) \longmapsto \bar{u}_{v,E}\,,$$

where $F(\Gamma)$ is the set of flags of Γ, that is, of pairs (v, E), where $v \in V(\Gamma)$ is a vertex and $e \in E(\Gamma) \subset L(\Gamma)$ is an edge or leg adjacent to v.

Definition 2.4. The *type* of a tropical curve $h\colon \Gamma \to M_{\mathbb{R}}$ is the tropical type $(\Gamma, \bar{u})$, where for every $(v, E) \in F(\Gamma)$, the image $\bar{u}_{v,E}$ is the primitive integral in M emanating from $h(v)$ in the direction of $h(E)$.

The moduli space of tropical curves of a given type $\tau = (\Gamma, \bar{u})$ is naturally the interior of a polyhedral cone. For example, when Γ is of genus 0, this moduli space is isomorphic to $M_{\mathbb{R}} \times \mathbb{R}_{>0}^{|E(\Gamma)|}$, where $M_{\mathbb{R}}$ parametrizes the position of a chosen vertex of Γ and $\mathbb{R}_{>0}^{|E(\Gamma)|}$ parametrizes the affine lengths of the images $h(E)$ of the edges $E \in E(\Gamma)$.

Definition 2.5. The *degree* of a tropical type $(\Gamma, \bar{u})$ is the tuple

$$\Delta = (w(L)\bar{u}_{v,L})_{L \in L(\Gamma)} \in M^{L(\Gamma)}$$

of weighted directions of the legs, where v is the unique vertex of Γ adjacent to the leg L. The degree of a tropical curve is the degree of its type.

By [NS06, Proposition 2.1], there are finitely many tropical types of given genus and degree that are realized by tropical curves. In particular, the moduli space $\mathcal{M}^{\mathrm{trop}}(\Delta)$ of rational tropical curves of given degree Δ is naturally a finite cone complex, obtained by gluing together the cones obtained as the closures of the spaces of tropical curves of given types (when the length of an edge goes to 0, a vertex of higher valency is produced).

2.2 Marked tropical curves

Let $\omega \in \bigwedge^2 M$ be a skew-symmetric form on N. For every $n \in N$, we denote by $\iota_n\omega \in M$ the contraction of ω by n, that is, the linear form $\omega(n,-)$ on N. Let $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_r)$ be an r-tuple of elements $\gamma_i \in N$ such that $\iota_{\gamma_i}\omega \neq 0$ for all $1 \leqslant i \leqslant r$ and $\iota_\gamma\omega \neq 0$, where $\gamma := \sum_{i=1}^r \gamma_i$.

Definition 2.6. An $(\omega, \boldsymbol{\gamma})$-marked tropical curve in $M_{\mathbb{R}}$ is a rational tropical curve $h\colon \Gamma \to M_{\mathbb{R}}$ of degree

$$\Delta_{\omega,\boldsymbol{\gamma}} := (\iota_{\gamma_1}\omega, \dots, \iota_{\gamma_r}\omega, -\iota_\gamma\omega)\,.$$

In other words, Γ is a genus 0 graph, with $r+1$ legs $L_1, \dots, L_r, L_{\mathrm{out}}$, such that $w(L_i)\bar{u}_{v_i,L_i} = -\iota_{\gamma_i}\omega$ for all $1 \leqslant i \leqslant r$ and $w(L_{\mathrm{out}})\bar{u}_{v_{\mathrm{out}},L_{\mathrm{out}}} = -\iota_\gamma\omega$, where v_i (respectively, v_{out}) is the unique vertex of Γ adjacent to L_i (respectively, L_{out}).

Given an $(\omega, \boldsymbol{\gamma})$-marked tropical curve $h\colon \Gamma \to M_{\mathbb{R}}$, we usually view Γ as a tree with root leg L_{out} and leaves L_i for $1 \leqslant i \leqslant r$. Correspondingly, we say that an edge E is a child (respectively, a parent) of a vertex v if E is adjacent to v and is not (respectively, is) contained in the unique path in Γ between v and the root. We similarly define descendants and ancestors. For every edge or leg E of Γ, we set

$$u_E := w(E)\bar{u}_{v,E}\,, \tag{2.1}$$

where v is the unique vertex such that E is a parent of v. In other words, u_E is the weighted direction of E following the flow on Γ starting at the leaves and ending at the root.

Definition 2.7. Let $h\colon \Gamma \to M_{\mathbb{R}}$ be an $(\omega, \boldsymbol{\gamma})$-marked tropical curve in $M_{\mathbb{R}}$. Then for every edge or leg E of Γ, the *class of* E is the element $\gamma_E \in N$ defined by

$$\gamma_E = \sum_i \gamma_i\,,$$

where the sum is over the indices $1 \leqslant i \leqslant r$ such that the leg L_i is a descendant of E.

Lemma 2.8. *Let $h\colon \Gamma \to M_{\mathbb{R}}$ be an $(\omega, \boldsymbol{\gamma})$-marked tropical curve in $M_{\mathbb{R}}$. Then for every edge or leg E of Γ, we have $u_E = -\iota_{\gamma_E}\omega$. In particular, the class γ_E is non-zero.*

Proof. The equality $u_E = -\iota_{\gamma_E}\omega$ is true for leaves by the definition of a $\boldsymbol{\gamma}$-marked tropical curve (Definition 2.6) and then follows for the other edges and the root leg by the balancing condition in Definition 2.2(ii). This implies that $\gamma_E \neq 0$ because $u_E \neq 0$ by the definition of a tropical curve in $M_{\mathbb{R}}$ (see (2.1) and Definition 2.2(i)). □

To simplify the notation, instead of $\mathcal{M}^{\mathrm{trop}}(\Delta_{\omega,\boldsymbol{\gamma}})$, in what follows we use $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma}}$ for the moduli space of $(\omega, \boldsymbol{\gamma})$-marked tropical curves in $M_{\mathbb{R}}$. It follows from the discussion after Definition 2.4 that $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma}}$ is naturally a finite polyhedral cone. For every cone σ of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma}}$, the relative interior $\mathrm{Int}(\sigma)$ of σ parametrizes tropical curves of a given type $(\Gamma_\sigma, \bar{u}_\sigma)$. Moreover, the dimension of σ is given by

$$\dim \sigma = d + |E(\Gamma_\sigma)|\,,$$

which can also be written using that Γ_σ has genus 0 as

$$\dim \sigma = d - 2 + r - \mathrm{ov}(\Gamma_\sigma)\,.$$

Here, $\mathrm{ov}(\Gamma_\sigma)$ is the *overvalence* of Γ_σ, which is the non-negative integer defined by

$$\mathrm{ov}(\Gamma_\sigma) := \sum_{v \in V(\Gamma)} \mathrm{ov}(v)\,, \tag{2.2}$$

where $\mathrm{ov}(v) := k - 3$ if v is a k-valent vertex. In particular, σ is of dimension at most $d-2+r$, and it is of dimension $d-2+r$ if and only if Γ_σ is trivalent.

2.3 Tropical constraints

Definition 2.9. A *$\boldsymbol{\gamma}$-constraint* is an r-tuple $\mathbf{A} = (A_1, \dots, A_r)$ of affine hyperplanes A_i in $M_\mathbb{R}$ whose associated linear hyperplanes are given by $\big(\gamma_i^\perp\big)_\mathbb{R} \subset M_\mathbb{R}$.

Choices of $\boldsymbol{\gamma}$-constraints $\mathbf{A}$ are in one-to-one correspondence with choices of points $[\mathbf{A}] = ([A_i])_i \in \prod_{i=1}^r M_\mathbb{R}/\big(\gamma_i^\perp\big)_\mathbb{R} \simeq \mathbb{R}^r$.

Definition 2.10. An $(\omega, \boldsymbol{\gamma})$-marked tropical curve $h\colon \Gamma \to M_\mathbb{R}$ *matches* a $\boldsymbol{\gamma}$-constraint $\mathbf{A}$ if $h(L_i) \subset A_i$ for all $1 \leqslant i \leqslant r$.

We denote by $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ the moduli space of $(\omega, \boldsymbol{\gamma})$-marked tropical curves matching the $\boldsymbol{\gamma}$-constraint $\mathbf{A}$. In other words, $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}} := (\mathrm{ev}^{\mathrm{trop}})^{-1}([\mathbf{A}])$, where $\mathrm{ev}^{\mathrm{trop}}$ is the evaluation map at the legs L_i, defined by

$$\mathrm{ev}^{\mathrm{trop}}\colon \mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma}} \longrightarrow \prod_{i=1}^r M_\mathbb{R}/\big(\gamma_i^\perp\big)_\mathbb{R} \simeq \mathbb{R}^r\,, \quad h \longmapsto ([h(L_i)])_i\,, \tag{2.3}$$

which is well defined because $\omega(\gamma_i, \gamma_i) = 0$ implies that $u_{L_i} = \iota_{\gamma_i}\omega \in \gamma_i^\perp$. As the evaluation map is affine in restriction to each cone of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma}}$, the moduli space $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ is naturally a finite polyhedral complex. For every face σ of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$, the relative interior $\mathrm{Int}(\sigma)$ of σ parametrizes tropical curves of a given type $(\Gamma_\sigma, \bar{u}_\sigma)$ and matching $\mathbf{A}$.

For every subset $I \subset \{1, \dots, r\}$, we set $\boldsymbol{\gamma}_I := (\gamma_i)_{i \in I}$, and we consider the $\boldsymbol{\gamma}_I$-constraint $\mathbf{A}_I := (A_i)_{i \in I}$.

Definition 2.11. A $\boldsymbol{\gamma}$-constraint $\mathbf{A} = (A_1, \dots, A_r)$ is *general* if the following conditions are satisfied:

(i) We have $A_i \neq A_j$ for $i \neq j$.

(ii) For every $I \subset \{1, \dots, r\}$, we have $\dim \mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma}_I,\mathbf{A}_I} \leqslant d-2$.

(iii) For every $I \subset \{1, \dots, r\}$, if σ is a face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma}_I,\mathbf{A}_I}$, then $\dim \sigma = d-2$ if and only if Γ_σ is trivalent.

Lemma 2.12. *The general $\boldsymbol{\gamma}$-constraints form an open dense subset in the space $\prod_{i=1}^r M_\mathbb{R}/\big(\gamma_i^\perp\big)_\mathbb{R} \simeq \mathbb{R}^r$ of $\boldsymbol{\gamma}$-constraints.*

Proof. As there are only finitely many subsets $I \subset \{1, \dots, r\}$, it is enough to prove that the set of $\mathbf{A}$ satisfying Definition 2.11(ii)–(iii) for a given I is open and dense. Let $\tau = (\Gamma, \bar{u})$ be the type of an $(\omega, \boldsymbol{\gamma}_I)$-marked tropical curve. As Γ is of genus 0, the curve h is not superabundant by [MR20, Proposition 2.5]. In this case, [MR20, Proposition 2.10] implies that, for $\mathbf{A}$ in an open

dense subset of the space of constraints, the space of $(\omega, \boldsymbol{\gamma})$-marked tropical curves of type τ matching $\mathbf{A}$ is a non-empty open convex polyhedron in a vector space of dimension

$$d + \overline{e} - \sum_{i \in I} \operatorname{codim}(A_i) = d + \overline{e} - |I|\,,$$

where by [MR20, § 2, equation (2)], using that Γ has $|I|+1$ legs, we have $\overline{e} = (|I|+1) - 3 - \operatorname{ov}(\Gamma)$, where the overvalence $\operatorname{ov}(\Gamma)$ is given by (2.2) and is a non-negative integer. Therefore, the space of $(\omega, \boldsymbol{\gamma}_I)$-marked tropical curves of type τ matching $\mathbf{A}$ is of dimension at most

$$d + (|I|+1) - 3 - \operatorname{ov}(\Gamma) - |I| = d - 2 - \operatorname{ov}(\Gamma)\,,$$

and it is of dimension $d-2$ if and only if $\operatorname{ov}(\Gamma) = 0$, that is, if Γ is trivalent. □

2.4 Properties of marked tropical curves

Let $\mathbf{A}$ be a $\boldsymbol{\gamma}$-constraint, and let σ be a face of the moduli space $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ of $(\omega, \boldsymbol{\gamma})$-marked tropical curves in $M_{\mathbb{R}}$ matching $\mathbf{A}$. For each vertex v of Γ_σ, we obtain a polyhedron $\mathfrak{j}^\sigma_v$ in $M_{\mathbb{R}}$ defined by

$$\mathfrak{j}^\sigma_v := \{h(v) \mid h \in \operatorname{Int}(\sigma)\}^{\mathrm{cl}} \subseteq M_{\mathbb{R}}\,, \tag{2.4}$$

and for each edge or leg E of Γ_σ, we obtain a polyhedron $\mathfrak{d}^\sigma_E$ in $M_{\mathbb{R}}$ defined by

$$\mathfrak{d}^\sigma_E := \Big(\bigcup_{h \in \operatorname{Int}(\sigma)} h(E) \Big)^{\mathrm{cl}} \subseteq M_{\mathbb{R}}\,; \tag{2.5}$$

see Figure 4. For every polyhedron $\mathfrak{d}$ in $M_{\mathbb{R}}$, we denote by $\Lambda_{\mathfrak{d}} \subset M$ the subspace of integral tangent vectors to $\mathfrak{d}$.

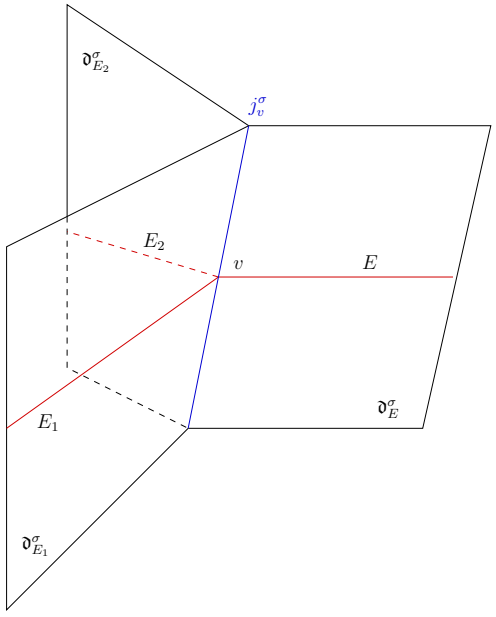


FIGURE 4. Polyhedra $\mathfrak{j}^\sigma_v$ and $\mathfrak{d}^\sigma_E$ for $d = 3$.

By Definition 2.9, the affine hyperplanes A_i of $\mathbf{A}$ have equations of the form $\gamma_i = \epsilon_i$ for some $\epsilon_i \in \mathbb{R}$. For every edge or leg E of Γ_σ, we denote by A_E the affine hyperplane in $M_{\mathbb{R}}$ with

equation

$$\gamma_E = \sum_i \epsilon_i \,, \tag{2.6}$$

where the sum is over the indices $1 \leqslant i \leqslant r$ such that the leg L_i is a descendant of E. We have $\dim A_E = d-1$ because $\gamma_E \neq 0$ by Lemma 2.8.

Lemma 2.13. *Let* $\mathbf{A}$ *be a* $\boldsymbol{\gamma}$*-constraint, and let* σ *be a face of* $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$*. Then,*

(i) *for every edge or leg* E *of* Γ_σ*, we have* $\mathfrak{d}^\sigma_E \subset A_E$*,* $\Lambda_{\mathfrak{d}^\sigma_E} \subset \gamma_E^\perp$*, and, in particular,* $\dim \mathfrak{d}^\sigma_E \leqslant d-1$*;*

(ii) *for every vertex* v *of* Γ_σ*, we have* $\mathfrak{j}^\sigma_v \subset \bigcap_{i=1}^k A_{E_i}$ *and* $\Lambda_{\mathfrak{j}^\sigma_v} \subset \bigcap_{i=1}^k \gamma_{E_i}^\perp$*, where* $E_1, \dots, E_k$ *are the children edges of* v*.*

Proof. We prove the result by induction following the flow on Γ_σ starting at the leaves and ending at the root.

For the initial step of the induction, we consider a leg $E = L_i$ of Γ_σ. Every tropical curve $h \in \mathrm{Int}(\sigma)$ matches the $\boldsymbol{\gamma}$-constraint $\mathbf{A}$, so $h(L_i) \subset A_i$. It follows that $\mathfrak{d}^\sigma_{L_i} \subset A_i$ and therefore $\Lambda_{\mathfrak{d}^\sigma_{L_i}} \subset \gamma_i^\perp$ because $\Lambda_{A_i} = \gamma_i^\perp$.

For the induction step, let v be a vertex of Γ_σ, E the parent edge of v, and $E_1, \dots, E_k$ the children edges of v. We assume that Lemma 2.13 holds for $E_1, \dots, E_k$, and we have to show that Lemma 2.13 holds for v and E. For every tropical curve $h \in \mathrm{Int}(\sigma)$, we have $h(v) \subset \cap_{i=1}^k h(E_i)$, so $\mathfrak{d}^\sigma_v \subset \bigcap_{i=1}^k \mathfrak{d}^\sigma_{E_1}$. By the induction hypothesis, we have $\mathfrak{d}^\sigma_{E_i} \subset A_{E_i}$ and $\Lambda_{\mathfrak{d}^\sigma_{E_i}} = \gamma_{E_i}^\perp$ for all $1 \leqslant i \leqslant k$, so we conclude that $\mathfrak{j}^\sigma_v \subset \bigcap_{i=1}^k A_{E_i}$ and $\Lambda_{\mathfrak{j}^\sigma_v} \subset \bigcap_{i=1}^k \gamma_{E_i}^\perp$. Similarly, we have $h(E) \subset h(v) + \mathbb{R}_{\geqslant 0} u_E$. Using Lemma 2.8, this implies that $\mathfrak{d}^\sigma_E \subset \mathfrak{j}^\sigma_v - \mathbb{R}_{\geqslant 0} \iota_{\gamma_E}\omega$. As we have already showed that $\Lambda_{\mathfrak{j}^\sigma_v} \subset \bigcap_{i=1}^k \gamma_{E_i}^\perp$, we conclude that $\mathfrak{d}^\sigma_E \subset \bigcap_{i=1}^k A_{E_i} - \mathbb{R}_{\geqslant 0}\iota_{\gamma_E}\omega \subset A_E$ and $\Lambda_{\mathfrak{d}^\sigma_E} \subset \bigcap_{i=1}^k \gamma_{E_i}^\perp - \mathbb{R}_{\geqslant 0}\iota_{\gamma_E}\omega \subset \gamma_E^\perp$. Indeed, we have $\bigcap_{i=1}^k \gamma_{E_i}^\perp \subset \gamma_E^\perp$ because $\gamma_E = \sum_{i=1}^k \gamma_{E_i}$, and we have $\iota_{\gamma_E}\omega \in \gamma_E^\perp$ because $\omega(\gamma_E, \gamma_E) = 0$. □

Let $(\Gamma, \bar{u})$ be a tropical type, and let $v \in V(\Gamma)$ be a vertex. We denote by $T_v\Gamma$ the $\mathbb{R}$-span in $M_\mathbb{R}$ of all the weighted directions u_E of the edges or legs of Γ adjacent to v. By the balancing condition in Definition 2.2(ii), it is also the span of the weighted directions u_{E_i} of the children edges or legs E_i of v.

Lemma 2.14. *Let* $\mathbf{A}$ *be a* $\boldsymbol{\gamma}$*-constraint, and let* σ *be a face of* $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$*. Let* $v \in V(\Gamma_\sigma)$ *be a vertex such that* $\dim \mathfrak{d}^\sigma_E = d-1$*, where* E *is the unique parent edge of* v*. Then,* $\dim T_v\Gamma_\sigma \leqslant 2$*.*

Proof. Let $E_1, \dots, E_k$ be the children edges of v. If $\dim T_v\Gamma_\sigma > 2$, then, as $u_{E_k} = -\iota_{\gamma_{E_k}}\omega$ by Lemma 2.8, the $\mathbb{R}$-span in $N_\mathbb{R}$ of the vectors γ_{E_k} is also of dimension greater than 2, so $\dim \bigcap_{i=1}^k \gamma_{E_k}^\perp < d-2$. By Lemma 2.13, we have $\mathfrak{j}^\sigma_v \subset \bigcap_{i=1}^k \gamma_{E_i}^\perp$, so $\dim \mathfrak{j}^\sigma_v < d-2$. Finally, as $\mathfrak{d}^\sigma_E \subset \mathfrak{j}^\sigma_v + \mathbb{R}_{\geqslant 0} u_E$, this implies that $\dim \mathfrak{d}^\sigma_E < d-1$, so we have a contradiction. □

The following lemma is similar to [KS14, Lemma 3.2.4] and [GHKK18, Lemma C.2].

Lemma 2.15. *Let* $\mathbf{A}$ *be a* $\boldsymbol{\gamma}$*-constraint, and let* σ *be a face of* $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$*. Let* $v \in V(\Gamma_\sigma)$ *be a vertex such that* $\dim T_v\Gamma_\sigma > 1$ *and* $\dim \mathfrak{d}^\sigma_E = d-1$*, where* E *is the unique parent edge of* v*. Then, we have* $\dim \mathfrak{d}^\sigma_{E_1} = d-1$ *for any child edge* E_1 *of* v*.*

Proof. By Lemma 2.13(i), it is enough to prove that $\dim \mathfrak{d}^\sigma_{E_1} \geqslant d-1$ in order to show that $\dim \mathfrak{d}^\sigma_{E_1} = d-1$.

Assume toward a contradiction that $\dim \mathfrak{d}^\sigma_{E_1} \leqslant d-2$ for a child edge or leg E_1 of v. Denote by $E_2, \dots, E_k$ the other child edges or legs of v. For every tropical curve $h \in \mathrm{Int}(\sigma)$, we have $h(v) \subset \bigcap_{i=1}^k h(E_i)$, so $\mathfrak{d}^\sigma_v \subset \bigcap_{i=1}^k \mathfrak{d}^\sigma_{E_i}$. By assumption, we have $\dim \mathfrak{d}^\sigma_{E_1} \leqslant d-2$. On the other hand, by Lemma 2.13(i), we also have $\dim \mathfrak{d}^\sigma_{E_i} \leqslant d-1$ for all $2 \leqslant i \leqslant k$. If $\Lambda_{\mathfrak{d}^\sigma_{E_1}}$ is not included in $\Lambda_{\mathfrak{d}^\sigma_{E_i}}$ for some $2 \leqslant i \leqslant k$, it follows that $\dim \mathfrak{j}^\sigma_v \leqslant d-3$, and so, as $\mathfrak{d}^\sigma_E \subset \mathfrak{j}^\sigma_v + \mathbb{R}_{\geqslant 0} u_E$, we conclude that $\dim \mathfrak{d}^\sigma_E \leqslant d-2$, and we have a contradiction.

Therefore, to end the proof, it is enough to consider the case where $\Lambda_{\mathfrak{d}^\sigma_{E_1}} \subset \bigcap_{i=2}^k \Lambda_{\mathfrak{d}^\sigma_{E_i}}$. In this case, the direction u_{E_1} is contained in $\bigcap_{i=2}^k \Lambda_{\mathfrak{d}^\sigma_{E_2}}$, hence in $\bigcap_{i=2}^k \gamma^\perp_{E_2}$ by Lemma 2.13(i). By Lemma 2.8, we have $u_{E_1} = -\iota_{\gamma_{E_1}} \omega$, so $u_{E_1} \in \bigcap_{i=2}^k \gamma^\perp_{E_i}$ implies $\omega(\gamma_{E_1}, \gamma_{E_i}) = 0$ for all $2 \leqslant i \leqslant k$.

As we are assuming that $\dim T_v \Gamma_\sigma > 1$ and $\dim \mathfrak{d}^\sigma_E = d-1$, we actually have $\dim T_v \Gamma_\sigma = 2$ by Lemma 2.14. Hence, there exists an $2 \leqslant i \leqslant k$ such that u_{E_1} and u_{E_i} generate $T_v \Gamma_\sigma$. Hence, $\omega(\gamma_{E_1}, \gamma_{E_i}) = 0$ implies that $T_v \Gamma_\sigma \subset \gamma^\perp_{E_1} \cap \gamma^\perp_{E_i}$. In particular, we obtain that $u_E \in \gamma^\perp_{E_1} \cap \gamma^\perp_{E_i}$. On the other hand, by Lemma 2.13(ii), we also have $\Lambda_{\mathfrak{j}^\sigma_v} \subset \gamma^\perp_{E_1} \cap \gamma^\perp_{E_i}$. We conclude that $\mathfrak{d}^\sigma_E \subset \mathfrak{j}^\sigma_v + \mathbb{R}_{\geqslant 0} u_E \subset \gamma^\perp_{E_1} \cap \gamma^\perp_{E_i}$. On the other hand, as we are assuming that u_{E_1} and u_{E_i} span the 2-dimensional space $T_v \Gamma_\sigma$, the directions u_{E_1} and u_{E_i} are not proportional, so the charges γ_{E_1} and γ_{E_i} are also not proportional. Hence, the hyperplanes $\gamma^\perp_{E_1}$ and $\gamma^\perp_{E_i}$ are distinct, so $\dim \mathfrak{d}^\sigma_E \leqslant d-2$, and we have a contradiction. □

LEMMA 2.16. *Let* $\mathbf{A}$ *be a* $\boldsymbol{\gamma}$*-constraint, and let* σ *be a face of* $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ *such that* $\dim \sigma = d-2$ *and* $\dim \mathfrak{d}^\sigma_{L_{\mathrm{out}}} = d-1$*. Then, we have* $\dim T_v \Gamma_\sigma = 2$ *for all vertices* $v \in V(\Gamma_\sigma)$*, and* $\dim \mathfrak{d}^\sigma_E = d-1$ *for all edges or legs* $E \in E(\Gamma_\sigma) \cup L(\Gamma_\sigma)$*.*

Proof. We first show that $\dim T_v \Gamma_\sigma > 1$ for all $v \in V(\Gamma_\sigma)$. Assume toward a contradiction that there exists a $v \in V(\Gamma_\sigma)$ such that $\dim T_v \Gamma_\sigma = 1$. Then, there exists at least one vertex $v' \in \Gamma_\sigma$ such that $\dim T_{v'} \Gamma_\sigma = 1$ and such that $\dim T_{v''} \Gamma_\sigma > 1$ for all vertices $v'' \neq v'$ on the unique path in Γ_σ from v' to the root.

If v' is the unique vertex adjacent to L_{out}, then, as $\dim T_{v'} \Gamma_\sigma = 1$, one can move $h(v')$ is the direction of $h(L_{\mathrm{out}})$, so $\mathfrak{j}^\sigma_{v'} = \mathfrak{d}^\sigma_{L_{\mathrm{out}}}$ and therefore $\dim \mathfrak{j}^\sigma_{v'} = d-1$. This contradicts the assumption that $\dim \sigma = d-2$ because $\mathfrak{j}^\sigma_{v'}$ is the image of σ by the evaluation map $h \mapsto h(v')$.

If v' is not adjacent at L_{out}, then v' admits a unique parent vertex v. Denote by E_1 the edge connecting v' and v. As $\dim T_{v''} \Gamma_\sigma > 1$ for every vertex v'' on the path from v to the root, applying Lemma 2.15 iteratively from the root down to v shows that the assumption $\dim \mathfrak{d}^\sigma_{L_{\mathrm{out}}} = d-1$ implies $\dim \mathfrak{d}^\sigma_{E_1} = d-1$, and so $\dim \mathfrak{j}^\sigma_v \geqslant d-2$. On the other hand, as $\dim T_{v'} \Gamma_\sigma = 1$, one can move $h(v')$ is the direction of $h(E_1)$ without moving $h(v)$, so $\dim \sigma \geqslant 1 + \dim \mathfrak{j}^\sigma_v \geqslant d-1$, contradicting the assumption that $\dim \sigma = d-2$.

Once one knows that $\dim T_v \Gamma_\sigma > 1$ for all $v \in V(\Gamma_\sigma)$, applying Lemma 2.15 iteratively starting from the root shows that the assumption $\dim \mathfrak{d}^\sigma_{L_{\mathrm{out}}} = d-1$ implies $\dim \mathfrak{d}^\sigma_E = d-1$ for all edges or legs E. We conclude that $\dim T_v \Gamma_\sigma = 2$ for all $v \in V(\Gamma_\sigma)$ by Lemma 2.14. □

THEOREM 2.17. *Let* $\mathbf{A}$ *be a* $\boldsymbol{\gamma}$*-constraint, and let* σ *be a* $(d-2)$*-dimensional face of* $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ *such that* $\dim \mathfrak{d}^\sigma_{L_{\mathrm{out}}} = d-1$*. Then,*

(i) *for every edge or leg* E *of* Γ_σ*, we have* $\dim \mathfrak{d}^\sigma_E = d-1$ *and* $\Lambda_{\mathfrak{d}^\sigma_E} = \gamma^\perp_E$*;*

(ii) *for every vertex* $v \in V(\Gamma_\sigma)$*, we have* $\dim T_v \Gamma_\sigma = 2$*,* $\dim \mathfrak{j}^\sigma_v = d-2$*, and* $\Lambda_{\mathfrak{j}^\sigma_v} = \bigcap_{i=1}^k \gamma^\perp_{E_i}$*, where* $E_1, \dots, E_k$ *are the children edges of* v*;*

(iii) *for every edge or leg* E *of* Γ_σ *with parent vertex* v*, we have* $u_E \notin \Lambda_{\mathfrak{j}^\sigma_v}$*.*

Proof. Item (i) follows from Lemmas 2.16 and 2.13(i). The first part of item (ii) also follows from Lemma 2.16. Moreover, denoting by E the parent edge of v, we have $\mathfrak{d}_E^\sigma = \mathfrak{j}_v^\sigma + \mathbb{R}_{\geqslant 0} u_E$, so the last part of item (ii) follows from item (i) and Lemma 2.13(ii). Finally, for item (iii), if one had $u_E \in \Lambda_{\mathfrak{j}_v^\sigma}$, one would have $\Lambda_{\mathfrak{d}_E^\sigma} = \Lambda_{\mathfrak{j}_v^\sigma}$, in contradiction with items (i) and (ii). □

2.5 Tropical multiplicities and coefficients

2.5.1 *Tropical multiplicities.* Let $\mathbf{A}$ be a $\boldsymbol{\gamma}$-constraint, and let σ be a face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$. For every leg $L \in L(\Gamma_\sigma)$, there exists a unique vertex $\partial^- L$ adjacent to L. For every edge $E \in E(\Gamma_\sigma)$, we denote by $\partial^+ E$ the parent vertex of E and by $\partial^- E$ the child vertex of E.

Definition 2.18. Let $\mathbf{A}$ be a $\boldsymbol{\gamma}$-constraint. For every face σ of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$, the *gluing map* gl_σ is the map

$$\begin{aligned} \mathrm{gl}_\sigma \colon \prod_{v \in V(\Gamma_\sigma)} M &\longrightarrow \prod_{e \in E(\Gamma_\sigma)} M/\mathbb{Z}u_E \times \prod_{i=1}^r M/\gamma_i^\perp \,, \\ (m_v)_{v \in V(\Gamma_\sigma)} &\longmapsto ((m_{\partial^+ E} - m_{\partial^- E})_{E \in E(\Gamma_\sigma)}, m_{\partial^- E}) \,. \end{aligned} \tag{2.7}$$

By construction, the kernel $T_\sigma := \ker \mathrm{gl}_\sigma$ is the integral tangent space to σ (see [MR20, Proposition 2.10]): An element $(m_v)_{v \in V(\Gamma_\sigma)} \in \prod_{v \in V(\Gamma_\sigma)} M$ contained in the kernel of gl_σ defines an integral way to move the vertices of Γ_σ in $M_\mathbb{R}$ while preserving the directions of the edges and the matching of the constraint $\mathbf{A}$.

Lemma 2.19. *Let $\mathbf{A}$ be a general $\boldsymbol{\gamma}$-constraint as in Definition 2.11. For every $(d-2)$-dimensional face σ of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$, the cokernel $\operatorname{coker} \mathrm{gl}_\sigma$ of the gluing map gl_σ is finite.*

Proof. It is enough to show that gl_σ is surjective after tensoring with $\mathbb{Q}$. This follows from a dimension count: The domain of $\mathrm{gl}_\sigma \otimes \mathbb{Q}$ is of dimension $d|V(\Gamma_\sigma)|$, its codomain is of dimension $(d-1)|E(\Gamma_\sigma)| + r$, and its kernel is of dimension $d-2$. Hence,

$$\begin{aligned} \dim \operatorname{coker} \mathrm{gl}_\sigma \otimes \mathbb{Q} &= (d-1)|E(\Gamma_\sigma)| + r - d|V(\Gamma_\sigma)| + d - 2 \\ &= d(|E(\Gamma_\sigma)| - |V(\Gamma_\sigma)| + 1) - |E(\Gamma_\sigma)| + r - 2 \,. \end{aligned}$$

As Γ_σ is of genus 0, we have $|E(\Gamma_\sigma)| - |V(\Gamma_\sigma)| + 1 = 0$, so

$$\dim \operatorname{coker} \mathrm{gl}_\sigma \otimes \mathbb{Q} = -|E(\Gamma_\sigma)| + r - 2 \,. \tag{2.8}$$

Moreover, as $\mathbf{A}$ is general and $\dim \sigma = d-2$, the graph Γ_σ is trivalent by Definition 2.11(iii), with $r+1$ legs, and so we have $3|V(\Gamma_\sigma)| = 2|E(\Gamma_\sigma)| + r + 1$. Combining these two relations, we obtain

$$3(|E(\Gamma_\sigma)| + 1) = 2|E(\Gamma_\sigma)| + r + 1 \,,$$

so $|E(\Gamma_\sigma)| = r - 2$, which implies that $\dim \operatorname{coker} \mathrm{gl}_\sigma \otimes \mathbb{Q} = 0$ by (2.8). □

Definition 2.20. Let $\mathbf{A}$ be a general $\boldsymbol{\gamma}$-constraint. For every $(d-2)$-dimensional face σ of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$, the *tropical multiplicity* N^{trop}_σ is the cardinality of the cokernel of the gluing map gl_σ, which is finite by Lemma 2.19:

$$N^{\mathrm{trop}}_\sigma := |\operatorname{coker} \mathrm{gl}_\sigma| \,. \tag{2.9}$$

2.5.2 *Tropical coefficients.* Let $\mathbf{A}$ be a $\boldsymbol{\gamma}$-constraint and σ be a $(d-2)$-dimensional face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ such that $\dim \mathfrak{d}^\sigma_{L_{\mathrm{out}}} = d-1$. Let $p \colon T_\sigma \longrightarrow M$ be the composition of the inclusion $T_\sigma = \ker \mathrm{gl}_\sigma \subset \prod_{v \in V(\Gamma_\sigma)} M$ with the projection $\prod_{v \in V(\Gamma_\sigma)} M \to M$ on the factor corresponding

to the vertex $v_{\mathrm{out}} \in V(\Gamma_\sigma)$ adjacent to L_{out}. In other words, p is the integral derivative of the evaluation map

$$\sigma \longrightarrow \mathfrak{j}^\sigma_{v_{\mathrm{out}}}\,, \quad h \longmapsto h(v_{\mathrm{out}})\,.$$

This evaluation map is an affine map that is surjective by definition. By Theorem 2.17(ii), we have $\dim \mathfrak{j}^\sigma_{v_{\mathrm{out}}} = d-2$. Hence, as we also have $\dim \sigma = d-2$, this evaluation map is in fact bijective, and we have an inclusion $p(T_\sigma) \subset \Lambda_{\mathfrak{j}^\sigma_{v_{\mathrm{out}}}}$ of finite index. By Theorem 2.17(iii), we have $u_{L_{\mathrm{out}}} \notin \Lambda_{\mathfrak{j}^\sigma_{v_{\mathrm{out}}}}$, so the lattice

$$\mathcal{L} := p(T_\sigma) + \mathbb{Z} u_{L_{\mathrm{out}}}$$

is of rank $d-1$, and the natural inclusion $\mathcal{L} \subset \Lambda_{\mathfrak{d}^\sigma_{L_{\mathrm{out}}}}$ is of finite index.

DEFINITION 2.21. Let $\mathbf{A}$ be a $\boldsymbol{\gamma}$-constraint and σ be a $(d-2)$-dimensional face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ such that $\dim \mathfrak{d}^\sigma_{L_{\mathrm{out}}} = d-1$. The *tropical coefficient* k_σ is the index of the inclusion $\mathcal{L} \subset \Lambda_{\mathfrak{d}^\sigma_{L_{\mathrm{out}}}}$; that is,

$$k_\sigma := |\Lambda_{\mathfrak{d}^\sigma_{L_{\mathrm{out}}}}/\mathcal{L}|\,. \tag{2.10}$$

Note that $\Lambda_{\mathfrak{d}^\sigma_{L_{\mathrm{out}}}}$ is also equal to the saturation $\mathcal{L}^{\mathrm{sat}}$ of $\mathcal{L}$ in M, so one also have $k_\sigma = |\mathcal{L}^{\mathrm{sat}}/\mathcal{L}|$.

2.6 Product formula for the tropical multiplicities

In this section, we prove, in Lemmas 2.23 and 2.24, two product formulas for the tropical multiplicity N^{trop}_σ introduced in Definition 2.20.

Let $\mathbf{A}$ be a general $\boldsymbol{\gamma}$-constraint and σ a $(d-2)$-dimensional face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ such that $\dim \mathfrak{d}^\sigma_{L_{\mathrm{out}}} = d-1$. By the definition of a general $\boldsymbol{\gamma}$-constraint (Definition 2.11(iii)), the graph Γ_σ is trivalent. In particular, every vertex $v \in \Gamma_\sigma$ has two children edges or legs, which we denote by $E_{1,v}$ and $E_{2,v}$, and one parent edge $E_{\mathrm{out},v}$. We denote by $I_{k,v}$ (respectively, $I_{\mathrm{out},v}$) the subset of $\{1,\dots,r\}$ consisting of the indices $1 \leqslant i \leqslant r$ such that L_i is a descendant leg of $E_{k,v}$ (respectively, $E_{\mathrm{out},v}$).

For $k = 1, 2$, if $E_{k,v}$ is an edge, we denote by $\tau_{k,v}$ the type of tropical curves obtained from tropical curves $h \in \mathrm{Int}(\sigma)$ by removing the vertex v and extending the connected component containing $E_{k,v}$ to infinity. We denote by $\sigma_{k,v}$ the face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma}_{I_{k,v}},\mathbf{A}_{I_{k,v}}}$ whose relative interior parametrizes tropical curves of type $\tau_{k,v}$ matching $\mathbf{A}_{I_{k,v}}$. Similarly, we denote by $\tau_{\mathrm{out},v}$ the type of tropical curves obtained from tropical curves $h \in \mathrm{Int}(\sigma)$ by cutting the edge (or leg) $E_{\mathrm{out},v}$ and extending the connected component containing $E_{\mathrm{out},v}$ to infinity. We denote by $\sigma_{\mathrm{out},v}$ the face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma}_{I_{\mathrm{out},v}},\mathbf{A}_{I_{\mathrm{out},v}}}$ whose relative interior parametrizes tropical curves of type $\tau_{\mathrm{out},v}$ matching $\mathbf{A}_{I_{\mathrm{out},v}}$. By a small abuse of notation, we still denote by $E_{k,v}$ and $E_{\mathrm{out},v}$ the new legs of $\Gamma_{\sigma_{k,v}}$ and $\Gamma_{\sigma_{\mathrm{out},v}}$ obtained by cutting and extending the edges $E_{k,v}$ and $E_{\mathrm{out},v}$ of Γ_σ.

LEMMA 2.22. *Let $\mathbf{A}$ be a general $\boldsymbol{\gamma}$-constraint and σ a $(d-2)$-dimensional face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ such that $\dim \mathfrak{d}^\sigma_{L_{\mathrm{out}}} = d-1$. Then, for every vertex $v \in V(\Gamma)$,*

(i) *for $k = 1, 2$, if $E_{k,v}$ is an edge, then $\dim \sigma_{k,v} = d-2$ and $\dim \mathfrak{d}^{\sigma_{k,v}}_{E_{k,v}} = d-1$;*

(ii) *we have $\dim \sigma_{\mathrm{out},v} = d-2$ and $\dim \mathfrak{d}^{\sigma_{\mathrm{out},v}}_{E_{\mathrm{out},v}} = d-1$.*

Proof. As the graph $\Gamma_{\sigma_{k,v}}$ is cut out from the trivalent graph Γ_σ, it is also trivalent, so $\dim \sigma_{k,v} = d-2$ by the definition of a general $\boldsymbol{\gamma}$-constraint (Definition 2.11(iii)). In particular, $\dim \mathfrak{d}^{\sigma_{k,v}}_{E_{k,v}} \leqslant d-1$. But we also have $\mathfrak{d}^\sigma_{E_{k,v}} \subset \mathfrak{d}^{\sigma_{k,v}}_{E_{k,v}}$ and $\dim \mathfrak{d}^\sigma_{E_{k,v}} = d-1$ by Theorem 2.17(i), so we conclude that $\dim \mathfrak{d}^{\sigma_{k,v}}_{E_{k,v}} = d-1$. This proves case (i) of Lemma 2.22. The proof of case (ii) is identical. □

If $E_{k,v}$ is an edge, then by Lemma 2.22(i), the face $\sigma_{k,v}$ satisfies the assumptions of Section 2.5.2. In particular, following Section 2.5.2, we obtain a map $p_{k,v}\colon T_{\sigma_{k,v}} \to M$, a rank $(d-1)$ lattice

$$\mathcal{L}_{k,v} := p_{k,v}(T_{\sigma_{k,v}}) + \mathbb{Z}u_{E_{k,v}} \subset M$$

of finite index in $\Lambda_{\mathfrak{d}_{E_{k,v}}^{\sigma_{k,v}}}$, and a coefficient

$$k_{\sigma_{k,v}} = |\Lambda_{\mathfrak{d}_{E_{k,v}}^{\sigma_{k,v}}}/\mathcal{L}_{k,v}| = |\mathcal{L}_{k,v}^{\mathrm{sat}}/\mathcal{L}_{k,v}|\,. \tag{2.11}$$

If $E_{k,v}$ is a leg L_i of Γ_σ for some $1 \leqslant i \leqslant r$, then we define

$$\mathcal{L}_{k,v} := \gamma_i^\perp \subset M\,.$$

By Theorem 2.17(ii), we have $\dim T_v\Gamma_\sigma = 2$. In particular, the lattice $\mathcal{L}_{1,v} + \mathcal{L}_{2,v}$ is of rank d, and so of finite index in M.

Similarly, by Lemma 2.22(ii), the face $\sigma_{\mathrm{out},v}$ satisfies the assumptions of Section 2.5.2. In particular, we obtain a map $p_{\mathrm{out},v}\colon T_{\sigma_{\mathrm{out},v}} \to M$, a rank $(d-1)$-lattice

$$\mathcal{L}_{\mathrm{out},v} := p_{\mathrm{out},v}(T_{\sigma_{\mathrm{out},v}}) + \mathbb{Z}u_{E_{\mathrm{out},v}} \subset M$$

of finite index in $\Lambda_{\mathfrak{d}_{E_{\mathrm{out},v}}^{\sigma_{\mathrm{out},v}}}$, and a coefficient

$$k_{\sigma_{\mathrm{out},v}} = |\Lambda_{\mathfrak{d}_{E_{\mathrm{out},v}}^{\sigma_{\mathrm{out},v}}}/\mathcal{L}_{\mathrm{out},v}| = |\mathcal{L}_{\mathrm{out},v}^{\mathrm{sat}}/\mathcal{L}_{\mathrm{out},v}|\,.$$

As a tropical curve in $\sigma_{\mathrm{out},v}$ is obtained by gluing at v a tropical curve in $\sigma_{1,v}$ with a tropical curve in $\sigma_{2,v}$, we have $p_{\mathrm{out},v}(T_{\sigma_{\mathrm{out},v}}) = \mathcal{L}_{1,v} \cap \mathcal{L}_{2,v}$, so

$$\mathcal{L}_{\mathrm{out},v} := \mathcal{L}_{1,v} \cap \mathcal{L}_{2,v} + \mathbb{Z}u_{E_{\mathrm{out},v}} \subset M\,. \tag{2.12}$$

Lemma 2.23. *Let $\mathbf{A}$ be a general $\boldsymbol{\gamma}$-constraint and σ a $(d-2)$-dimensional face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ such that $\dim \mathfrak{d}^\sigma_{L_{\mathrm{out}}} = d-1$. Then, we have*

$$N^{\mathrm{trop}}_\sigma = \prod_{v \in V(\Gamma_\sigma)} |M/(\mathcal{L}_{1,v} + \mathcal{L}_{2,v})|\,.$$

Proof. We prove by induction, following the flow on Γ_σ starting at the leaves and ending at the root, that, for every vertex $v \in V(\Gamma_\sigma)$, we have

$$N^{\mathrm{trop}}_{\sigma_{\mathrm{out},v}} = \prod_{v \in V(\Gamma_{\sigma_{\mathrm{out},v}})} |M/(\mathcal{L}_{1,v} + \mathcal{L}_{2,v})|\,.$$

To shorten the notation, write Γ_{σ_k}, E_k, σ_k, σ_{out}, p_k, $\mathcal{L}_k$ for $\Gamma_{\sigma_{k,v}}$, $E_{k,v}$, $\sigma_{k,v}$, $\sigma_{\mathrm{out},v}$, $p_{k,v}$, $\mathcal{L}_{k,v}$, respectively. If E_1 and E_2 are both edges, we have to show that

$$N^{\mathrm{trop}}_{\sigma_{\mathrm{out}}} = |M/(\mathcal{L}_1 + \mathcal{L}_2)| N^{\mathrm{trop}}_{\sigma_1} N^{\mathrm{trop}}_{\sigma_2}\,. \tag{2.13}$$

Denote the domains of the gluing maps $\mathrm{gl}_{\sigma_{\sigma_{\mathrm{out}}}}$, gl_{σ_1}, gl_{σ_2} of Definition 2.18 by $\mathcal{M}$, $\mathcal{M}_1$, $\mathcal{M}_2$ and the codomains by $\mathcal{K}$, $\mathcal{K}_1$, $\mathcal{K}_2$. We have $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus M_v$, where M_v is the factor M in $\mathcal{M}$ corresponding to the vertex v, and $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus M/\mathbb{Z}u_{E_1} \oplus M/\mathbb{Z}u_{E_2}$. With respect to these decompositions, one can write

$$\mathrm{gl}_{\sigma_{\mathrm{out}}}(m_1, m_2, m_v) = (\mathrm{gl}_{\sigma_1}(m_1), \mathrm{gl}_{\sigma_2}(m_2), m_{v_1} - m_v, m_{v_2} - m_v)\,,$$

where v_1 (respectively, v_2) is the vertex of Γ_{σ_1} (respectively, Γ_{σ_2}) adjacent to E_1 (respectively, E_2).

In other words, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_v & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}_1 \oplus \mathcal{M}_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \mathrm{gl}_{\sigma_{\mathrm{out}}}} & & \downarrow{\scriptstyle \mathrm{gl}_{\sigma_1} \oplus \mathrm{gl}_{\sigma_2}} & & \\ 0 & \longrightarrow & M/\mathbb{Z}u_{E_1} \oplus M/\mathbb{Z}u_{E_2} & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{K}_1 \oplus \mathcal{K}_2 & \longrightarrow & 0\,, \end{array}$$

where the rows are short exact sequences of abelian groups, and where ψ is given by $\psi(m) = (-m, -m)$. By Theorem 2.17(ii), we have $\dim T_v\Gamma_\sigma = 2$, and so the vectors u_{E_1} and u_{E_2} are linearly independent. This implies that the map ψ is injective. Hence, by the snake lemma, we obtain a long exact sequence

$$0 \longrightarrow T_{\sigma_{\mathrm{out}}} \longrightarrow T_{\sigma_1} \oplus T_{\sigma_2} \xrightarrow{\xi} \operatorname{coker}\psi \longrightarrow \operatorname{coker}\mathrm{gl}_{\sigma_{\mathrm{out}}} \longrightarrow \operatorname{coker}\mathrm{gl}_{\sigma_1} \oplus \operatorname{coker}\mathrm{gl}_{\sigma_2} \longrightarrow 0$$

and so, using (2.9),

$$N^{\mathrm{trop}}_{\sigma_{\mathrm{out}}} = |\operatorname{coker}\xi| N^{\mathrm{trop}}_{\sigma_1} N^{\mathrm{trop}}_{\sigma_2}\,. \tag{2.14}$$

Moreover, the connecting homomorphism ξ is given by $\xi(m_1, m_2) = (\bar{p}_1(m_1), \bar{p}_2(m_2))$, where $\bar{p}_k(m_k)$ is the image of $p_k(m_k)$ in $M/\mathbb{Z}u_{E_k}$. As $\operatorname{coker}\psi = M/(\mathbb{Z}u_{E_1} + \mathbb{Z}u_{E_2})$, we deduce that $\operatorname{coker}\xi = M/(\mathcal{L}_1 + \mathcal{L}_2)$, so (2.14) implies (2.13).

Now consider the case where one of E_1 and E_2 is an edge and the other is a leg. By symmetry, one can assume that E_1 is an edge, with adjacent vertex v_1 in Γ_{σ_1}, and E_2 is a leg L_i for some $1 \leqslant i \leqslant r$. In this case, we have to show that

$$N^{\mathrm{trop}}_{\sigma_{\mathrm{out}}} = |M/(\mathcal{L}_1 + \mathcal{L}_2)| N^{\mathrm{trop}}_{\sigma_1}\,. \tag{2.15}$$

We have $\mathcal{M} = \mathcal{M}_1 \oplus M_v$ and $\mathcal{K} = \mathcal{K}_1 \oplus M/\mathbb{Z}u_{E_1} \oplus M/\gamma_i^\perp$, and so a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_v & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \mathrm{gl}_{\sigma}} & & \downarrow{\scriptstyle \mathrm{gl}_{\sigma_1}} & & \\ 0 & \longrightarrow & M/\mathbb{Z}u_{E_1} \oplus M/\gamma_i^\perp & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{K}_1 & \longrightarrow & 0\,, \end{array}$$

where ψ is given by $\psi(m) = (-m, m)$. By Theorem 2.17(ii), we have $\dim T_v\Gamma_\sigma = 2$, and therefore $u_{E_1} \notin \gamma_i^\perp$. Hence, ψ is injective, so by the snake lemma, we obtain a long exact sequence

$$0 \longrightarrow T_{\sigma_{\mathrm{out}}} \longrightarrow T_{\sigma_1} \xrightarrow{\xi} \operatorname{coker}\psi \longrightarrow \operatorname{coker}\mathrm{gl}_{\sigma_{\mathrm{out}}} \longrightarrow \operatorname{coker}\mathrm{gl}_{\sigma_1} \longrightarrow 0$$

and so, using (2.9),

$$N^{\mathrm{trop}}_{\sigma} = |\operatorname{coker}\xi| N^{\mathrm{trop}}_{\sigma_1}\,. \tag{2.16}$$

Moreover, we have $\operatorname{coker}\psi = M/\big(\mathbb{Z}u_{E_1} + \gamma_i^\perp\big)$, and the connecting homomorphism ξ is given by $\xi(m_1) = \bar{p}_1(m_1)$. As $\mathcal{L}_1 = p_1(T_{\sigma_1}) + \mathbb{Z}u_{E_1}$ and $\mathcal{L}_2 = \gamma_i^\perp$, we deduce that $\operatorname{coker}\xi = M/(\mathcal{L}_1 + \mathcal{L}_2)$, so (2.16) implies (2.15).

Finally, we consider the case where both E_1 and E_2 are legs, say L_i and L_j, of Γ for some $1 \leqslant i, j \leqslant r$. In this case, we have to show that $N^{\mathrm{trop}}_{\sigma} = |M/(\mathcal{L}_1 + \mathcal{L}_2)|$. This follows because in this case the gluing map $\mathrm{gl}_{\sigma_{\mathrm{out}}}$ is simply given by the projection map

$$M \longrightarrow M/\gamma_i^\perp \oplus M/\gamma_j^\perp\,,$$

and $\mathcal{L}_1 = \gamma_i^\perp$ and $\mathcal{L}_2 = \gamma_j^\perp$. □

For every non-zero $n \in N$, we denote by $|n|$ the divisibility of n in N.

Lemma 2.24. *Let $\mathbf{A}$ be a general $\boldsymbol{\gamma}$-constraint. Then, for every $(d-2)$-dimensional face σ of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ such that $\dim \mathfrak{d}^{\sigma}_{L_{\mathrm{out}}} = d-1$, we have*

$$k_\sigma N^{\mathrm{trop}}_\sigma = \frac{|\gamma|}{\prod_{i=1}^r |\gamma_i|} \prod_{v \in V(\Gamma_\sigma)} |\omega(\gamma_{E_{1,v}}, \gamma_{E_{2,v}})|\,, \tag{2.17}$$

where for every vertex v, we denote by $E_{1,v}$ and $E_{2,v}$ the two children edges of v.

Proof. We prove by induction, following the flow on Γ_σ starting at the leaves and ending at the root, that, for every vertex $v \in V(\Gamma_\sigma)$, we have

$$k_{\sigma_{\mathrm{out},v}} N^{\mathrm{trop}}_{\sigma_{\mathrm{out},v}} = \frac{|\gamma_{E_{\mathrm{out},v}}|}{\prod_{i\in I_v} |\gamma_i|} \prod_{v' \in V(\Gamma_{\sigma_{\mathrm{out},v}})} |\omega(\gamma_{E_{1,v'}}, \gamma_{E_{2,v'}})|\,.$$

Given a vertex $v \in V(\Gamma_\sigma)$, we use the same simplified notation E_1, E_2, E_{out} and so on as in the proof of Lemma 2.23. If E_1 and E_2 are both edges, we have to show that

$$k_{\sigma_{\mathrm{out}}} N^{\mathrm{trop}}_{\sigma_{\mathrm{out}}} = \frac{|\gamma_{E_{\mathrm{out}}}|}{|\gamma_{E_1}| \cdot |\gamma_{E_2}|} |\omega(\gamma_{E_1}, \gamma_{E_2})|\, k_{\sigma_1} N^{\mathrm{trop}}_{\sigma_1} k_{\sigma_2} N^{\mathrm{trop}}_{\sigma_2}\,. \tag{2.18}$$

By Lemma 2.23, we have $N^{\mathrm{trop}}_{\sigma_{\mathrm{out}}} = |M/(\mathcal{L}_1+\mathcal{L}_2)| N^{\mathrm{trop}}_{\sigma_1} N^{\mathrm{trop}}_{\sigma_2}$, so we have to show that

$$|M/(\mathcal{L}_1+\mathcal{L}_2)| = \frac{k_{\sigma_1} k_{\sigma_2}}{k_{\sigma_{\mathrm{out}}}} \frac{|\gamma_{E_{\mathrm{out}}}|}{|\gamma_{E_1}| \cdot |\gamma_{E_2}|} |\omega(\gamma_{E_1}, \gamma_{E_2})|\,. \tag{2.19}$$

First of all, we have

$$|M/(\mathcal{L}_1+\mathcal{L}_2)| = |M/(\mathcal{L}_1^{\mathrm{sat}}+\mathcal{L}_2^{\mathrm{sat}})| \cdot |(\mathcal{L}_1^{\mathrm{sat}}+\mathcal{L}_2^{\mathrm{sat}})/(\mathcal{L}_1+\mathcal{L}_2)|\,. \tag{2.20}$$

Then, the exact sequence

$$0 \longrightarrow \frac{\mathcal{L}_1^{\mathrm{sat}} \cap \mathcal{L}_2^{\mathrm{sat}}}{\mathcal{L}_1 \cap \mathcal{L}_2} \longrightarrow \mathcal{L}_1^{\mathrm{sat}}/\mathcal{L}_1 \oplus \mathcal{L}_2^{\mathrm{sat}}/\mathcal{L}_2 \longrightarrow \frac{\mathcal{L}_1^{\mathrm{sat}} + \mathcal{L}_2^{\mathrm{sat}}}{\mathcal{L}_1 + \mathcal{L}_2} \longrightarrow 0$$

and (2.11) imply that

$$|(\mathcal{L}_1^{\mathrm{sat}}+\mathcal{L}_2^{\mathrm{sat}})/(\mathcal{L}_1+\mathcal{L}_2)| = \frac{k_{\sigma_1} k_{\sigma_2}}{|(\mathcal{L}_1^{\mathrm{sat}} \cap \mathcal{L}_2^{\mathrm{sat}})/(\mathcal{L}_1 \cap \mathcal{L}_2)|}\,. \tag{2.21}$$

On the other hand, we have $\mathcal{L}_{\mathrm{out}} = \mathcal{L}_1 \cap \mathcal{L}_2 + \mathbb{Z} u_{E_{\mathrm{out}}}$ by (2.12), so

$$k_{\sigma_{\mathrm{out}}} = |\mathcal{L}_{\mathrm{out}}^{\mathrm{sat}}/\mathcal{L}_{\mathrm{out}}| = |(\mathcal{L}_1^{\mathrm{sat}} \cap \mathcal{L}_2^{\mathrm{sat}})/(\mathcal{L}_1 \cap \mathcal{L}_2)| \cdot |\overline{u}_{E_{\mathrm{out}}}|\,, \tag{2.22}$$

where for every $u \in M$, we denote by $\overline{u}$ the image of u in the rank 2 lattice $\overline{M} := M/(\mathcal{L}_1^{\mathrm{sat}} \cap \mathcal{L}_2^{\mathrm{sat}})$. Similarly, we have

$$|M/(\mathcal{L}_1^{\mathrm{sat}}+\mathcal{L}_2^{\mathrm{sat}})| = \frac{|\overline{u}_{E_1} \wedge \overline{u}_{E_2}|}{|\overline{u}_{E_1}| \cdot |\overline{u}_{E_2}|}\,.$$

Combining (2.20), (2.21), and (2.22), we obtain

$$|M/(\mathcal{L}_1+\mathcal{L}_2)| = \frac{k_{\sigma_1} k_{\sigma_2}}{k_{\sigma_{\mathrm{out}}}} \frac{|\overline{u}_{E_{\mathrm{out}}}|}{|\overline{u}_{E_1}| \cdot |\overline{u}_{E_2}|} |\overline{u}_{E_1} \wedge \overline{u}_{E_2}|\,.$$

Hence, comparing with (2.19), it remains to show that

$$\frac{|\overline{u}_{E_{\mathrm{out}}}|}{|\overline{u}_{E_1}| \cdot |\overline{u}_{E_2}|} |\overline{u}_{E_1} \wedge \overline{u}_{E_2}| = \frac{|\gamma_{E_{\mathrm{out}}}|}{|\gamma_{E_1}| \cdot |\gamma_{E_2}|} |\omega(\gamma_{E_1}, \gamma_{E_2})|\,. \tag{2.23}$$

Set $\overline{N} := (\mathbb{Z}\gamma_{E_1} + \mathbb{Z}\gamma_{E_2})^{\mathrm{sat}}$; it is a rank 2 saturated sublattice of N. As $\mathcal{L}_1 = \gamma_{E_1}^{\perp}$ and $\mathcal{L}_2^{\perp} = \gamma_{E_2}^{\perp}$, the duality between N and M implies that $\overline{N}$ and $\overline{M}$ are naturally dual to each other.

Let (e_1, e_2) be a basis of $\overline{N}$ and $(e_1^\star, e_2^\star)$ the corresponding dual basis of $\overline{M}$. Then, we have $\iota_{e_1}\omega = \omega(e_1,e_2)e_2^\star$ and $\iota_{e_2}\omega = -\omega(e_1,e_2)e_1^\star$. In particular, for every $n = \alpha e_1 + \beta e_2 \in \overline{N}$, we have $\overline{\iota_n\omega} = \omega(e_1,e_2)(\alpha e_2^\star - \beta e_1^\star)$, so $|\iota_n\omega| = |\omega(e_1,e_2))| \gcd(\alpha,\beta) = |\omega(e_1,e_2))| \cdot |n|$. As $\overline{u}_{E_1} = \iota_{\gamma_{E_1}}\omega$, $\overline{u}_{E_2} = \iota_{\gamma_{E_2}}\omega$, and $\overline{u}_{E_{\mathrm{out}}} = \iota_{\gamma_{E_{\mathrm{out}}}}\omega$, we obtain

$$|\overline{u}_{E_1}| = \omega(e_1,e_2)|\gamma_{E_1}|\,, \quad |\overline{u}_{E_2}| = \omega(e_1,e_2)|\gamma_{E_2}|\,, \quad \text{and} \quad |\overline{u}_{E_{\mathrm{out}}}| = \omega(e_1,e_2)|\gamma_{E_{\mathrm{out}}}|\,. \tag{2.24}$$

On the other hand, writing $\gamma_{E_1} = ae_1 + be_2$ and $\gamma_{E_2} = ce_1 + de_2$, we have $|\overline{u}_{E_1} \wedge \overline{u}_{E_2}| = \omega(e_1,e_2)^2|ad-bc|$, whereas $|\omega(\gamma_{E_1},\gamma_{E_2})| = |\omega(e_1,e_2)| \cdot |ad-bc|$, so

$$|\overline{u}_{E_1} \wedge \overline{u}_{E_2}| = \omega(e_1,e_2)|\omega(\gamma_{E_1},\gamma_{E_2})|\,. \tag{2.25}$$

Combining (2.24) and (2.25), we obtain (2.23); this concludes the proof of (2.18).

If E_1 is an edge and E_2 is a leg L_i for some $1 \leqslant i \leqslant r$, then we similarly show that

$$k_{\sigma_{\mathrm{out}}} N^{\mathrm{trop}}_{\sigma_{\mathrm{out}}} = \frac{|\gamma_{E_{\mathrm{out}}}|}{|\gamma_{E_1}| \cdot |\gamma_{E_2}|} |\omega(\gamma_{E_1},\gamma_{E_2})|\, k_{\sigma_1} N^{\mathrm{trop}}_{\sigma_1}\,.$$

The only difference is that, in this case, we have $\mathcal{L}_2 = \gamma_i^\perp$, so $|\mathcal{L}_2^{\mathrm{sat}}/\mathcal{L}_2| = 1$.

Finally, if E_1 and E_2 are both legs, say L_i and L_j, respectively, for some $1 \leqslant i,j \leqslant r$, then we similarly show that

$$k_{\sigma_{\mathrm{out}}} N^{\mathrm{trop}}_{\sigma_{\mathrm{out}}} = \frac{|\gamma_{E_{\mathrm{out}}}|}{|\gamma_{E_1}| \cdot |\gamma_{E_2}|} |\omega(\gamma_{E_1},\gamma_{E_2})|$$

using that $\mathcal{L}_1 = \gamma_i^\perp$ and $\mathcal{L}_2 = \gamma_j^\perp$; consequently, $|\mathcal{L}_1^{\mathrm{sat}}/\mathcal{L}_1| = 1$ and $|\mathcal{L}_2^{\mathrm{sat}}/\mathcal{L}_2| = 1$. □

Remark 2.25. The tropical problem, the coefficient k_σ, and the tropical multiplicity N^{trop}_σ only depend on the vectors $\iota_{\gamma_i}\omega$ and on the hyperplanes $\gamma_i^\perp$. In particular, they are invariant under a common rescaling of γ_i and ω of the form $\gamma_i \mapsto t\gamma_i$ and $\omega \mapsto t^{-1}\omega$. As a consistency check, one can verify directly that the right-hand side of (2.17) in Lemma 2.24 is invariant under such rescaling: A trivalent tree with $r+1$ legs has $r-1$ vertices, so the right-hand side of (2.17) scales as $t(t^{-1}t^2)^{r-1}/t^r = 1$.

2.7 Polyhedral decompositions and tropical multiplicities

Until now, we considered tropical curves in $M_{\mathbb{R}}$. In this subsection, we introduce tropical curves in $M_{\mathbb{R}}$ endowed with a polyhedral decomposition.

DEFINITION 2.26 ([NS06, Definition 3.1]). A *polyhedral decomposition* of $M_{\mathbb{R}}$ is a covering $\mathscr{P} = \{\Xi\}$ of $M_{\mathbb{R}}$ by a finite number of strongly convex polyhedra satisfying the following properties:

(i) If $\Xi \in \mathscr{P}$ and $\Xi' \subset \Xi$ is a face, then $\Xi' \in \mathscr{P}$.

(ii) If $\Xi, \Xi' \in \mathscr{P}$, then $\Xi \cap \Xi'$ is a common face of Ξ and Ξ'.

For the remainder of this section, we fix a polyhedral decomposition $\mathscr{P}$ of $M_{\mathbb{R}}$.

DEFINITION 2.27. A *parametrized tropical curve in* $(M_{\mathbb{R}}, \mathscr{P})$ is a weighted graph Γ together with a proper continuous map $h\colon \Gamma \to M_{\mathbb{R}}$ satisfying the following conditions:

(i) For every edge or leg E, the restriction $h|_E$ is an embedding with image contained in an affine line with rational slope.

(ii) For every vertex $v \in V(\Gamma)$, the following *balancing condition* holds: Let $E_1, \dots, E_m \in E(\Gamma) \cup L(\Gamma)$ be the edges or legs adjacent to v, and let $\bar{u}_i \in M$ be the primitive integral vector emanating from $h(v)$ in the direction of $h(E_i)$. Then $\sum_{i=1}^m w(E_i)\bar{u}_i = 0$.

(iii) For every edge or leg E, there exists a cell of $\mathscr{P}$ containing $h(E)$.

(iv) For every divalent vertex $v \in V(\Gamma)$, denoting by $\boldsymbol{\sigma}(v)$ the smallest cell of $\mathscr{P}$ containing $h(v)$, there exists an open subset U of Γ containing v such that $h(U) \cap \boldsymbol{\sigma}(v) = \{h(v)\}$.

We define as in Section 1.1 a *tropical curve in* $(M_{\mathbb{R}}, \mathscr{P})$ as an isomorphism class of parametrized tropical curves in $(M_{\mathbb{R}}, \mathscr{P})$. Compared with the tropical curves in $M_{\mathbb{R}}$ of Definition 2.2, tropical curves in $(M_{\mathbb{R}}, \mathscr{P})$ are required to be compatible with the polyhedral decomposition $\mathscr{P}$ in the sense of Definition 2.27(iii) and can have divalent vertices, which are required to satisfy Definition 2.27(iv).

Definition 2.28. A *tropical type* for $(M_{\mathbb{R}}, \mathscr{P})$ is the data $(\Gamma, \bar{u}, \boldsymbol{\sigma})$ of a weighted graph Γ, a map $\bar{u}\colon F(\Gamma) \to M$ as in Definition 2.3, and a map $\boldsymbol{\sigma}\colon V(\Gamma) \cup E(\Gamma) \cup L(\Gamma) \to \mathscr{P}$ assigning a cell of $\mathscr{P}$ to every vertex, edge, or leg of Γ.

Definition 2.29. The type of a tropical curve $h\colon \Gamma \to M_{\mathbb{R}}$ in $(M_{\mathbb{R}}, \mathscr{P})$ is the tropical type $(\Gamma, \bar{u}, \boldsymbol{\sigma})$, where for every $(v, E) \in F(\Gamma)$, the image $\bar{u}(v, E) = \bar{u}_{v,E}$ is the primitive integral in M emanating from $h(v)$ in the direction of $h(E)$, and for every vertex, edge, or leg x of Γ, the image $\boldsymbol{\sigma}(x)$ is the smallest cell of $\mathscr{P}$ containing x.

One defines as in Sections 2.2 and 2.3 the notion of $(\omega, \boldsymbol{\gamma})$-tropical curves in $(M_{\mathbb{R}}, \mathscr{P})$, the notion of $(\omega, \boldsymbol{\gamma})$-tropical curves in $(M_{\mathbb{R}}, \mathscr{P})$ matching a $\boldsymbol{\gamma}$-constraint $\mathbf{A}$, and the moduli space $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P})$ of $(\omega, \boldsymbol{\gamma})$-tropical curves in $(M_{\mathbb{R}}, \mathscr{P})$ matching $\mathbf{A}$. Like the moduli space $\mathcal{M}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ of tropical curves in $M_{\mathbb{R}}$, the moduli space $\mathcal{M}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P})$ is a polyhedral complex: The relative interior $\mathrm{Int}(\sigma)$ of a face $\sigma \in \mathcal{M}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P})$ parametrizes $(\omega, \boldsymbol{\gamma})$-tropical curves $h\colon \Gamma_\sigma \to M_{\mathbb{R}}$ in $(M_{\mathbb{R}}, \mathscr{P})$ matching $\mathbf{A}$ and of a given type for $(M_{\mathbb{R}}, \mathscr{P})$.

In fact, there is a natural homeomorphism $\mathcal{M}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P}) \simeq \mathcal{M}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ realizing $\mathcal{M}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P})$ as a polyhedral refinement of $\mathcal{M}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$: Every face of $\mathcal{M}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ is a union of faces $\mathcal{M}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P})$. Indeed, given a tropical curve in $(M_{\mathbb{R}}, \mathscr{P})$, we obtain a tropical curve in $M_{\mathbb{R}}$ by "erasing" the divalent vertices, and, conversely, given a tropical curve in $M_{\mathbb{R}}$, there is a unique minimal way to add divalent vertices so that Definition 2.27(iii) is satisfied and we obtain a tropical curve in $(M_{\mathbb{R}}, \mathscr{P})$.

However, the identification $\mathcal{M}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P}) \simeq \mathcal{M}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ does not preserve the integral structures in general: If $\tilde{\sigma}$ is a face of $\mathcal{M}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P})$ contained in a face σ of $\mathcal{M}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ of the same dimension, then there is a natural lattice embedding of integral tangent spaces $T_{\tilde{\sigma}} \subset T_\sigma$ that can have a non-trivial finite cokernel. The issue is that when adding a divalent vertex on a edge in a family of tropical curves, the lengths of the newly created edges are affine functions on the base of the family that might not be integral with respect to the original integral structure on the base: One needs in general to coarsen the integral structure on the base to make all length functions integral affine. In the general context of tropicalizations of stable log maps, this phenomenon is discussed in [Joh26, § 4]. Let $\mathbf{A}$ be a general $\boldsymbol{\gamma}$-constraint, σ be a $(d-2)$-dimensional face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ such that $\dim \mathfrak{d}^\sigma_{L_{\mathrm{out}}} = d-1$, and $\tilde{\sigma}$ be a $(d-2)$-dimensional face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P})$ contained in σ. We give below an explicit description of the difference between $T_{\tilde{\sigma}}$ and T_σ.

Recall from (2.4) and (2.5) that we defined, for every vertex $v \in V(\Gamma_\sigma)$, the locus $\mathfrak{j}^\sigma_v$ as the closure in $M_{\mathbb{R}}$ of the locus of $h(v)$ for $h \in \mathrm{Int}(\sigma)$, and for every edge or leg E of Γ_σ, we define the locus $\mathfrak{d}^\sigma_E$ as the closure in $M_{\mathbb{R}}$ of the locus of $h(E)$ for $h \in \mathrm{Int}(\sigma)$. We define similarly $\mathfrak{j}^{\tilde{\sigma}}_v$ and $\mathfrak{d}^{\tilde{\sigma}}_E$ for every vertex v and edge or leg E of $\Gamma_{\tilde{\sigma}}$. To simplify the notation, we write T_v for the integral tangent space to $\mathfrak{j}^\sigma_v$ or $\mathfrak{j}^{\tilde{\sigma}}_v$ depending on whether $v \in V(\Gamma_\sigma)$ or $v \in V(\Gamma_{\tilde{\sigma}})$, and T_E for the integral tangent space to $\mathfrak{d}^\sigma_E$ or $\mathfrak{d}^{\tilde{\sigma}}_E$ depending on whether $E \in E(\Gamma_\sigma) \cup L(\Gamma_\sigma)$ or $E \in E(\Gamma_{\tilde{\sigma}}) \cup V(\Gamma_{\tilde{\sigma}})$.

To compare $T_{\tilde{\sigma}}$ and T_σ, we first give an alternative description of T_σ. Consider the map

$$\Psi_\sigma\colon \prod_{v\in V(\Gamma_\sigma)} T_v \times \prod_{E\in E(\Gamma_\sigma)} \mathbb{Z} \longrightarrow \prod_{E\in E(\Gamma_\sigma)} T_E \tag{2.26}$$

given by

$$\Psi_\sigma\big((m_v)_v, (\ell_E)_E\big) = (m_{\partial^+ E} - m_{\partial^- E} + \ell_E u_E)_E\,.$$

Then $T_\sigma = \ker \Psi_\sigma$. Recall that we also have $T_\sigma = \ker \mathrm{gl}_\sigma$, with the gluing map gl_σ defined by (2.7), and that we defined the tropical multiplicity $N_\sigma^{\mathrm{trop}} = |\operatorname{coker} \mathrm{gl}_\sigma|$ in Definition 2.20. However, N_σ^{trop} is in general distinct from $|\operatorname{coker} \Psi_\sigma|$. Rather, using the notation introduced in Section 2.6, we have the following lemma.

LEMMA 2.30. *Let $\mathbf{A}$ be a general $\boldsymbol{\gamma}$-constraint and σ a $(d-2)$-dimensional face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ such that $\dim \mathfrak{d}^\sigma_{L_{\mathrm{out}}} = d-1$. Then, we have*

$$|\operatorname{coker} \Psi_\sigma| = \prod_{v\in V(\Gamma_\sigma)} |(\mathcal{L}^{\mathrm{sat}}_{1,v} + \mathcal{L}^{\mathrm{sat}}_{2,v})/(\mathcal{L}_{1,v} + \mathcal{L}_{2,v})|\,.$$

Proof. We prove by induction, following the flow on Γ_σ starting at the leaves and ending at the root, that for every vertex $v \in V(\Gamma_\sigma)$, we have

$$|\operatorname{coker} \Psi_{\sigma_{\mathrm{out},v}}| = \prod_{v'\in V(\Gamma_{\sigma_{\mathrm{out}},v})} |(\mathcal{L}^{\mathrm{sat}}_{1,v'} + \mathcal{L}^{\mathrm{sat}}_{2,v'})/(\mathcal{L}_{1,v'} + \mathcal{L}_{2,v'})|\,.$$

Given a vertex $v \in V(\Gamma_\sigma)$, we use the same simplified notation E_1, E_2, E_{out}, $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_{\mathrm{out}}$, and so on as in the proofs of Lemmas 2.23 and 2.24. If E_1 and E_2 are both edges, we have to show that

$$|\operatorname{coker} \Psi_{\sigma_{\mathrm{out}}}| = |(\mathcal{L}^{\mathrm{sat}}_1 + \mathcal{L}^{\mathrm{sat}}_2)/(\mathcal{L}_1 + \mathcal{L}_2)| \cdot |\operatorname{coker} \Psi_{\sigma_1}| \cdot |\operatorname{coker} \Psi_{\sigma_2}|\,. \tag{2.27}$$

For $k = 1$ and $k = 2$, let

$$M_k = \prod_{v\in V(\Gamma_{\sigma_k})} T_v \times \prod_{E\in E(\Gamma_{\sigma_k})} \mathbb{Z}\,, \quad K_k = \prod_{E\in E(\Gamma_{\sigma_k})} T_E\,,$$

and consider the commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_v \times \mathbb{Z}^2 & \longrightarrow & M_1 \times M_2 \times T_v \times \mathbb{Z}^2 & \longrightarrow & M_1 \times M_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \Psi_{\sigma_{\mathrm{out}}}} & & \downarrow{\scriptstyle \Psi_{\sigma_1}\times\Psi_{\sigma_2}} & & \\
0 & \longrightarrow & T_{E_1} \times T_{E_2} & \longrightarrow & K_1 \times K_2 \times T_{E_1} \times T_{E_2} & \longrightarrow & K_1 \times K_2 & \longrightarrow & 0\,,
\end{array}$$

where the map θ is given by

$$\theta(m, \ell_1, \ell_2) = (\ell_1 u_{E_1} - m, \ell_2 u_{E_2} - m)\,.$$

As $\ker \Psi_{\sigma_{\mathrm{out}}} = T_{\sigma_{\mathrm{out}}}$ and $\ker \Psi_{\sigma_k} = T_{\sigma_k}$, by the snake lemma, we obtain a long exact sequence

$$T_{\sigma_{\mathrm{out}}} \longrightarrow T_{\sigma_1} \times T_{\sigma_2} \xrightarrow{\xi} \operatorname{coker} \theta \longrightarrow \operatorname{coker} \Psi_{\sigma_{\mathrm{out}}} \longrightarrow \operatorname{coker} \Psi_{\sigma_1} \times \operatorname{coker} \Psi_{\sigma_2} \longrightarrow 0\,.$$

As $T_v = T_{E_1} \cap T_{E_2}$, the cokernel of the map $T_v \to T_{E_1} \times T_{E_2}$ given by $m \mapsto (-m, -m)$ can be identified with $T_{E_1} + T_{E_2} \subseteq M$, so

$$\operatorname{coker} \theta = (T_{E_1} + T_{E_2})/(\mathbb{Z}u_{E_1} + \mathbb{Z}u_{E_2}) = (\mathcal{L}^{\mathrm{sat}}_1 + \mathcal{L}^{\mathrm{sat}}_2)/(\mathbb{Z}u_{E_1} + \mathbb{Z}u_{E_2})\,.$$

On the other hand, the connecting map $\xi\colon T_{\sigma_1}\times T_{\sigma_2}\to\operatorname{coker}\theta$ is given by $\xi(t_1,t_2)=(p_1(t_1),p_2(t_2))$, so

$$\operatorname{coker}\xi=(\mathcal{L}_1^{\mathrm{sat}}+\mathcal{L}_2^{\mathrm{sat}})/(\mathcal{L}_1+\mathcal{L}_2)\,,$$

because $\mathcal{L}_k=p_k(T_{\sigma_k})+\mathbb{Z}u_{E_k}$ by definition. This conclude the proof of (2.27).

Now consider the case where one of E_1 and E_2 is an edge and the other is a leg. By symmetry, one can assume that E_1 is an edge, with adjacent vertex v_1 in Γ_{σ_1}, and E_2 is a leg L_i for some $1\leqslant i\leqslant r$. We have to show that

$$|\operatorname{coker}\Psi_{\sigma_{\mathrm{out}}}|=|(\mathcal{L}_1^{\mathrm{sat}}+\mathcal{L}_2^{\mathrm{sat}})/(\mathcal{L}_1+\mathcal{L}_2)|\cdot|\operatorname{coker}\Psi_{\sigma_1}|\,. \tag{2.28}$$

In this case, we consider the commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_v\times\mathbb{Z} & \longrightarrow & M_1\times T_v\times\mathbb{Z} & \longrightarrow & M_1 & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\Psi_{\sigma_{\mathrm{out}}}} & & \downarrow{\scriptstyle\Psi_{\sigma_1}} & & \\
0 & \longrightarrow & T_{E_1} & \longrightarrow & K_1\times T_{E_1} & \longrightarrow & K_1 & \longrightarrow & 0\,,
\end{array}$$

where the map θ is given by

$$\theta(m,\ell)=(\ell u_{E_1}-m)\,.$$

As $\ker\Psi_{\sigma_{\mathrm{out}}}=T_{\sigma_{\mathrm{out}}}$ and $\ker\Psi_{\sigma_1}=T_{\sigma_1}$, by the snake lemma, we obtain a long exact sequence

$$T_{\sigma_{\mathrm{out}}}\longrightarrow T_{\sigma_1}\xrightarrow{\xi}\operatorname{coker}\theta\longrightarrow\operatorname{coker}\Psi_{\sigma_{\mathrm{out}}}\longrightarrow\operatorname{coker}\Psi_{\sigma_1}\longrightarrow 0\,.$$

As $T_v=T_{E_1}\cap\gamma_i^\perp$ and $\mathcal{L}_2=\mathcal{L}_2^{\mathrm{sat}}=\gamma_i^\perp$, we have the equality $\operatorname{coker}\theta=T_{E_1}/(T_v+\mathbb{Z}u_{E_1})=\mathcal{L}_1^{\mathrm{sat}}/(\mathcal{L}_1^{\mathrm{sat}}\cap\mathcal{L}_2^{\mathrm{sat}}+\mathbb{Z}u_{E_1})$. On the other hand, the connecting map $\xi\colon T_{\sigma_1}\to\operatorname{coker}\theta$ is given by $\xi(t)=p_1$, so

$$\operatorname{coker}\xi=\mathcal{L}_1^{\mathrm{sat}}/(\mathcal{L}_1+\mathcal{L}_1^{\mathrm{sat}}\cap\mathcal{L}_2^{\mathrm{sat}})\,,$$

which is equal to $(\mathcal{L}_1^{\mathrm{sat}}+\mathcal{L}_2^{\mathrm{sat}})/(\mathcal{L}_1+\mathcal{L}_2)$ because $\mathcal{L}_2=\mathcal{L}_2^{\mathrm{sat}}$. This concludes the proof of (2.28).

Finally, if both E_1 and E_2 are legs L_i and L_j, then $\Psi_{\sigma_{\mathrm{out}}}$ is the zero map $T_v\to 0$, so $\operatorname{coker}\Psi_{\sigma_{\mathrm{out}}}=0$. Moreover, $\mathcal{L}_1=\mathcal{L}_1^{\mathrm{sat}}=\gamma_i^\perp$ and $\mathcal{L}_2=\mathcal{L}_2^{\perp}=\gamma_j^\perp$, so $(\mathcal{L}_1^{\mathrm{sat}}+\mathcal{L}_2^{\mathrm{sat}})/(\mathcal{L}_1+\mathcal{L}_2)=0$. In particular, we have $\operatorname{coker}\Psi_{\sigma_{\mathrm{out}}}=(\mathcal{L}_1^{\mathrm{sat}}+\mathcal{L}_2^{\mathrm{sat}})/(\mathcal{L}_1+\mathcal{L}_2)$. This ends the proof of Lemma 2.30. □

Similarly, the integral tangent space $T_{\tilde\sigma}$ can be viewed as the kernel of the map

$$\Psi_{\tilde\sigma}:\prod_{v\in V(\Gamma_{\tilde\sigma})}T_v\times\prod_{E\in E(\Gamma_{\tilde\sigma})}\mathbb{Z}\longrightarrow\prod_{E\in E(\Gamma_{\tilde\sigma})}T_E \tag{2.29}$$

with $\Psi_{\tilde\sigma}$ defined analogously to Ψ_σ in (2.26). The comparison we need between T_σ and $T_{\tilde\sigma}$ is provided by the following lemma. We denote by B the set of divalent vertices of $\Gamma_{\tilde\sigma}$, and for every $v\in B$, we set $w_v:=|\overline{u}_E|$, where E is an edge adjacent to v and $\overline{u}_E$ is the image of u_E in the rank 2 lattice M/T_v. The definition of w_v is independent of the choice of E by the balancing condition at v.

Lemma 2.31. *Let $\mathbf{A}$ be a general $\boldsymbol{\gamma}$-constraint, σ a $(d-2)$-dimensional face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ such that $\dim\mathfrak{d}^\sigma_{L_{\mathrm{out}}}=d-1$, and $\tilde\sigma$ a $(d-2)$-dimensional face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P})$ contained in σ. Then, there is a long exact sequence*

$$0\longrightarrow T_{\tilde\sigma}\longrightarrow T_\sigma\longrightarrow\prod_{v\in B}\mathbb{Z}/w_v\mathbb{Z}\longrightarrow\operatorname{coker}\Psi_{\tilde\sigma}\longrightarrow\operatorname{coker}\Psi_\sigma\longrightarrow 0\,. \tag{2.30}$$

Proof. We first introduce some notation. For every divalent vertex $v \in V(\Gamma_{\tilde{\sigma}})$, we denote by $E_1(v)$ and $E_2(v)$ the two edges of $\Gamma_{\tilde{\sigma}}$ with vertex v, with $E_1(v)$ oriented toward v and $E_2(v)$ oriented away from v. We consider the map

$$\delta\colon \prod_{v\in B} T_v \times \prod_{v\in B} \mathbb{Z} \longrightarrow \prod_{v\in V(\Gamma_{\tilde{\sigma}})} T_v \times \prod_{E\in E(\Gamma_{\tilde{\sigma}})} \mathbb{Z}\,,$$

which is the obvious inclusion on $\prod_{v\in B} T_v \to \prod_{v\in V(\Gamma_{\tilde{\sigma}})} T_v$ and is such that the component of $\delta\big((m_v),(\ell_v)\big)$ indexed by $E \in E(\Gamma_{\tilde{\sigma}})$ is given by

$$\sum_{\substack{v\in B\\ E_1(v)=E}} l_v - \sum_{\substack{v\in B\\ E_2(v)=E}} l_v\,.$$

We also define a map

$$\delta'\colon \prod_{v\in B} T_{E_1(v)} \longrightarrow \prod_{E\in E(\Gamma_{\tilde{\sigma}})} T_E$$

by $\delta'((m_v)_{v\in B}) = (n_E)_{E\in E(\Gamma_{\tilde{\sigma}})}$, where

$$n_E = \sum_{\substack{v\in B\\ E_1(v)=E}} m_v - \sum_{\substack{v\in B\\ E_2(v)=E}} m_v\,.$$

Finally, we define

$$\begin{aligned}\pi\colon \prod_{v\in V(\Gamma_{\tilde{\sigma}})} T_v \times \prod_{E\in E(\Gamma_{\tilde{\sigma}})} \mathbb{Z} &\longrightarrow \prod_{v\in V(\Gamma_\sigma)} T_v \times \prod_{E\in E(\Gamma_\sigma)} \mathbb{Z}\,,\\ \big((n_v)_{v\in V(\Gamma_{\tilde{\sigma}})},(\ell_E)_{E\in E(\Gamma_{\tilde{\sigma}})}\big) &\longmapsto \left((n_v)_{v\in V(\Gamma_\sigma)}, \left(\sum_{\substack{E'\subset E\\ E'\in E(\Gamma_{\tilde{\sigma}})}} \ell_{E'}\right)_{E\in E(\Gamma_\sigma)}\right)\end{aligned}$$

and

$$\pi'\colon \prod_{E\in E(\Gamma_{\tilde{\sigma}})} T_E \longrightarrow \prod_{E\in E(\Gamma_\sigma)} T_E\,,\quad (n_E)_{E\in E(\Gamma_{\tilde{\sigma}})} \longmapsto \left(\sum_{\substack{E'\subset E\\ E'\in E(\Gamma_{\tilde{\sigma}})}} n_{E'}\right)_{E\in E(\Gamma_\sigma)},$$

where we view $\Gamma_{\tilde{\sigma}}$ as a refinement of Γ_σ so that each edge of $\Gamma_{\tilde{\sigma}}$ is contained in some edge of Γ_σ. These maps fit into a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod_{v\in B} T_v \times \prod_{v\in B}\mathbb{Z} & \xrightarrow{\delta} & \prod_{v\in V(\Gamma_{\tilde{\sigma}})} T_v \times \prod_{E\in E(\Gamma_{\tilde{\sigma}})}\mathbb{Z} & \xrightarrow{\pi} & \prod_{v\in V(\Gamma_\sigma)} T_v \times \prod_{E\in E(\Gamma_\sigma)}\mathbb{Z} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \Psi'} & & \downarrow{\scriptstyle \Psi_{\tilde{\sigma}}} & & \downarrow{\scriptstyle \Psi_\sigma} & & \\
0 & \longrightarrow & \prod_{v\in B} T_{E_1(v)} & \xrightarrow[\delta']{} & \prod_{E\in E(\Gamma_{\tilde{\sigma}})} T_E & \xrightarrow[\pi']{} & \prod_{E\in E(\Gamma_\sigma)} T_E & \longrightarrow & 0\,,
\end{array}$$

where Ψ' is defined by $\Psi'\big((m_v),(\ell_v)\big) = (\ell_v u_{E_1(v)} - m_v)$. The map Ψ' is injective, and the cokernel of $T_v \times \mathbb{Z} \to T_{E_1(v)}$, $(m,\ell) \mapsto -m + \ell u_{E_1(v)}$, is just $\mathbb{Z}/w_v\mathbb{Z}$. Therefore, the snake lemma gives the desired long exact sequence (2.30). □

3. Counts of marked stable log maps in toric varieties

In this section, after briefly reviewing, in Sections 3.1–3.2, the theory of stable log maps, we define in Sections 3.3–3.4 a particular class of genus 0 log Gromov–Witten invariants of toric varieties.

3.1 Log schemes and their tropicalizations

We assume basic familiarity with log geometry throughout this section; see [Kat89, Ogu18]. We nonetheless roughly review definitions to establish notation.

A *log structure* on a scheme X is a sheaf of commutative monoids $\mathcal{M}_X$ together with a homomorphism of sheaves of multiplicative monoids $\alpha_X\colon \mathcal{M}_X \to \mathcal{O}_X$ inducing an isomorphism $\alpha_X^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$, allowing us to identify $\mathcal{O}_X^\times$ as a subsheaf of $\mathcal{M}_X$. The standard notation we use for a log scheme is $X := (\underline{X}, \mathcal{M}_X, \alpha_X)$, where by $\underline{X}$ we denote the underlying scheme. By abuse of notation, we also denote the underlying scheme just by X when it is clear from the context. Throughout this paper, we assume that all log schemes are fine and saturated (fs); see [Ogu18, § I.1.3]. A *morphism of log schemes* $f\colon X \to Y$ consists of an ordinary morphism $\underline{f}\colon \underline{X} \to \underline{Y}$ of schemes along with a map $f^\flat\colon f^{-1}\mathcal{M}_Y \to \mathcal{M}_X$ that is compatible with $f^\#\colon f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ via the structure homomorphisms α_X and α_Y.

The *ghost sheaf*, defined by $\overline{\mathcal{M}}_X := \mathcal{M}_X/\mathcal{O}_X^\times$, captures the key combinatorial information about the log structure. In particular, it leads to the description of the *tropicalization* $\Sigma(X)$ of a log scheme X as an abstract polyhedral cone complex; see [ACGS20, § 2.1] for details. Briefly, $\Sigma(X)$ is a collection of cones along with face maps between them. There is one cone $\sigma_{\bar{x}} := \mathrm{Hom}\big(\overline{\mathcal{M}}_{X,\bar{x}}, \mathbb{R}_{\geqslant 0}\big)$ for every geometric point $\bar{x} \to X$, and if $\bar{x}$ specializes to $\bar{y}$, there is a generization map $\overline{\mathcal{M}}_{X,\bar{y}} \to \overline{\mathcal{M}}_{X,\bar{x}}$ that leads dually to a map $\sigma_{\bar{x}} \to \sigma_{\bar{y}}$. The condition of being fine and saturated implies that this is an inclusion of faces. Note that each cone $\sigma_{\bar{x}} \in \Sigma(X)$ comes with a tangent space of integral tangent vectors

$$\Lambda_{\sigma_{\bar{x}}} := \mathrm{Hom}\big(\overline{\mathcal{M}}_{X,\bar{x}}, \mathbb{Z}\big)$$

and a set of integral points

$$\sigma_{\bar{x},\mathbb{Z}} := \mathrm{Hom}\big(\overline{\mathcal{M}}_{X,\bar{x}}, \mathbb{N}\big)\,.$$

Tropicalization is functorial, with $f\colon X \to Y$ inducing $f_{\mathrm{trop}}\colon \Sigma(X) \to \Sigma(Y)$, with a map of cones $\sigma_{\bar{x}} \to \sigma_{f(\bar{x})}$ induced by $\bar{f}^\flat\colon \overline{\mathcal{M}}_{Y,f(\bar{x})} \to \overline{\mathcal{M}}_{X,x}$. In the cases we consider in this paper, after identifying $\sigma_{\bar{x}}$ and $\sigma_{\bar{y}}$ whenever $\sigma_{\bar{x}} \to \sigma_{\bar{y}}$ is an isomorphism, we obtain an ordinary polyhedral cone complex.

Example 3.1. Let X be a toric variety associated with a fan Σ in $M_{\mathbb{R}}$. There is a canonical *divisorial log structure* $\mathcal{M}_X = \mathcal{M}_{(X,D)}$, where $D \subset X$ is the toric boundary divisor [Gro11, Example 3.8]. Furthermore, the tropicalization $\Sigma(X_\Sigma)$ is naturally identified with $M_{\mathbb{R}}$ endowed with the fan Σ.

Example 3.2. Let X be a log point, that is, a log scheme whose underlying scheme is a point $\underline{X} = \mathrm{Spec}\,\Bbbk$ for some field $\Bbbk$. Then, there exists a monoid Q with $Q^\times = \{0\}$ such that $\mathcal{M}_X = Q \oplus \Bbbk^\star$ and α_X is the map

$$Q \oplus \Bbbk^\times \longrightarrow \Bbbk\,, \quad (q,a) \longmapsto \begin{cases} a\,, & q = 0\,, \\ 0\,, & q \neq 0\,. \end{cases}$$

One recovers Q as the ghost sheaf of X, that is, $\overline{\mathcal{M}}_X = Q$, and the tropicalization $\Sigma(X)$ of X is the cone $Q^\vee_{\mathbb{R}} := \mathrm{Hom}(Q, \mathbb{R}_{\geqslant 0})$.

3.2 Stable log maps

Following [AC14, GS13], a *prestable log map* with target $(X, \mathcal{M}_X) \to (S, \mathcal{M}_S)$ is a commutative diagram in the category of log schemes

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ W & \longrightarrow & S\,, \end{array} \tag{3.1}$$

where $W = (W, \mathcal{M}_W)$ is a log point,[1] π is a log smooth family of curves that is required to be an integral morphism all of whose geometric fibers are reduced curves. In this situation, one can describe locally the log structure on the log curve C; see [Kat00]. In particular, C/W comes with a set of disjoint sections $p_1, \dots, p_n \colon \underline{W} \to \underline{C}$, referred to as the *marked points*, disjoint from the nodal locus of C, and such that away from the nodal locus,

$$\overline{\mathcal{M}}_C = \pi^* \overline{\mathcal{M}}_W \oplus \bigoplus_{i=1}^{n} p_{i*}\underline{\mathbb{N}}\,.$$

A *stable log map* is a prestable log map whose underlying prestable marked map is a stable map.

If $p \in C$ is a marked point, we have

$$\bar{f}^\flat \colon P_p := \overline{\mathcal{M}}_{X,f(p)} \longrightarrow \overline{\mathcal{M}}_{C,p} = \overline{\mathcal{M}}_{W,\pi(p)} \oplus \mathbb{N} \xrightarrow{\mathrm{pr}_2} \mathbb{N}\,,$$

which can be viewed as an element $u_p \in P_p^\vee := \mathrm{Hom}(P_p, \mathbb{N}) \subseteq \sigma_{f(p)}$, called the *contact order at* p. Similarly, if $x = q$ is a node, there exists a homomorphism

$$u_q \colon P_q := \overline{\mathcal{M}}_{X,f(q)} \longrightarrow \mathbb{Z}\,,$$

called the *contact order at* q; see [GS13, equation (1.8)] or [ACGS20, § 2.3.4]. In the case where the target space is (X, D), the contact order records tangency information with the irreducible components of D.

A key point in log Gromov–Witten theory is the tropical interpretation of stable log maps. Let $f \colon C \to X$ be a stable log map as in (3.1), where W is the log point $(\mathrm{Spec}\,\Bbbk, \Bbbk^\star \oplus Q)$, as in Example 3.2. Then by the functoriality of tropicalization, we obtain a diagram

$$\begin{array}{ccc} \Gamma := \Sigma(C) & \xrightarrow{h = f_{\mathrm{trop}}} & \Sigma(X) \\ {\scriptstyle\pi_{\mathrm{trop}}}\downarrow & & \downarrow \\ \Sigma(W) & \longrightarrow & \Sigma(S)\,. \end{array}$$

Here $\Sigma(W) = \mathrm{Hom}(Q, \mathbb{R}_{\geqslant 0}) = Q^\vee_{\mathbb{R}}$ is a rational polyhedral cone, and for $q \in \mathrm{Int}(Q^\vee_{\mathbb{R}})$, the inverse image $\pi^{-1}_{\mathrm{trop}}(q)$ can be identified with the dual graph G of the curve C, which is the graph with vertices corresponding to irreducible components of C, edges corresponding to nodes of C, and legs corresponding to marked points.

The map $h \colon \Sigma(C) \to \Sigma(X)$ defines a family of tropical maps to $\Sigma(X)$ as defined in [ACGS20, Definition 2.21]. For $s \in \mathrm{Int}\big(Q^\vee_{\mathbb{R}}\big)$, write

$$h_s \colon G \longrightarrow \Sigma(X)$$

for the restriction of h to $G = \pi^{-1}_{\mathrm{trop}}(s)$.

[1] For families of stable log maps, W can be an arbitrary scheme.

Associated with any family of tropical maps to $\Sigma(X)$ is the type, recording which cones of $\Sigma(X)$ vertices, edges, and legs of G are mapped to, as well as tangent vectors to the images of the edges and legs.

Definition 3.3. A *type* of tropical map to $\Sigma(X)$ is the data of a triple $\tau = (G, \boldsymbol{\sigma}, \mathbf{u})$, where G is a graph and $\boldsymbol{\sigma}$ is a map

$$\boldsymbol{\sigma} \colon V(G) \cup E(G) \cup L(G) \longrightarrow \Sigma(X)$$

with the property that if v is a vertex of an edge or leg E, then $\boldsymbol{\sigma}(v) \subseteq \boldsymbol{\sigma}(E)$. Next, $\mathbf{u}$ associates with each oriented edge $E \in E(G)$ a tangent vector $\mathbf{u}(E) \in \Lambda_{\boldsymbol{\sigma}(E)}$ and with each leg $L \in L(G)$ a tangent vector $\mathbf{u}(L) \in \Lambda_{\boldsymbol{\sigma}(L)}$.

Associated with a type is a moduli space of tropical maps of the given type, and this dually defines a monoid called the *basic monoid*.

Definition 3.4. Given a type $\tau = (G, \boldsymbol{\sigma}, \mathbf{u})$, we define the *basic monoid* $Q_\tau = \mathrm{Hom}(Q_\tau^\vee, \mathbb{N})$ of τ by defining its dual

$$Q_\tau^\vee := \left\{ \left((p_v)_{v \in V(G)}, (\ell_E)_{E \in E(G)} \right) \mid p_{v'} - p_v = \ell_E \mathbf{u}(E) \text{ for all } E \in E(G) \right\},$$

a submonoid of

$$\prod_{v \in V(G)} \boldsymbol{\sigma}(v)_{\mathbb{Z}} \times \prod_{E \in E(G)} \mathbb{N}.$$

Here $\boldsymbol{\sigma}(v)_{\mathbb{Z}}$ denotes the set of integral points of the cone $\boldsymbol{\sigma}(v)$, and v' and v are taken to be the endpoints of E consistent with the chosen orientation of the edge.

Given a stable log map $f \colon C/W \to X/S$, there is a canonical map $Q_\tau \to \overline{M}_W$, where Q_τ is the basic monoid defined by the combinatorial type of f; see [GS13, AC14]. A stable log map $f \colon C/W \to X/S$ is said to be *basic* if the natural map of monoids $Q_\tau \to \overline{\mathcal{M}}_W$ is an isomorphism. When $X \to S$ is log smooth and projective, [AC14, GS13] show that the moduli space of basic stable log maps with given genus, number of marked points, and degree, is a Deligne–Mumford stack, proper over S, and carrying a natural virtual fundamental class that can be used to define log Gromov–Witten invariants. From now on, we use stable log maps to mean basic stable log maps.

3.3 Moduli spaces of marked stable log maps

Let $(\omega, \boldsymbol{\gamma})$ be as in Section 2.2: $\omega \in \bigwedge^2 M$ is a skew-symmetric form on N, and $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_r)$ is an r-tuple of elements $\gamma_i \in N$ such that $\iota_{\gamma_i}\omega \neq 0$ for all $1 \leqslant i \leqslant r$ and $\iota_\gamma \omega \neq 0$, where $\gamma := \sum_{i=1}^r \gamma_i$.

Definition 3.5. A *$\boldsymbol{\gamma}$-fan* is a fan Σ in $M_{\mathbb{R}}$ of a d-dimensional projective toric variety X_Σ over $\mathbb{C}$ such that

(i) for every $1 \leqslant i \leqslant r$, the space $\mathbb{R}_{\geqslant 0}\iota_{\gamma_i}\omega$ is a ray of Σ;

(ii) for every $1 \leqslant i \leqslant r$, the hyperplane $\left(\gamma_i^\perp\right)_{\mathbb{R}}$ is a union of $(d-1)$-dimensional cones of Σ.

Given a $\boldsymbol{\gamma}$-fan Σ, for every $1 \leqslant i \leqslant r$, we denote by D_i the toric divisor of X_Σ corresponding to the ray $\mathbb{R}_{\geqslant 0}\iota_{\gamma_i}\omega$.

In what follows, we focus on particular stable log maps, referred to as $(\omega, \boldsymbol{\gamma})$-marked stable log maps, defined as follows.

DEFINITION 3.6. Let Σ be a $\boldsymbol{\gamma}$-fan. An $(\omega, \boldsymbol{\gamma})$-marked stable log map to X_Σ is a genus 0 stable log map $f\colon C \to X_\Sigma$ endowed with a bijection between its set of marked points and

$$\{x_i \mid 1 \leqslant i \leqslant r\} \cup \{x_{\mathrm{out}}\}\,,$$

and such that $u_{x_i} = \iota_{\gamma_i}\omega$ for all $1 \leqslant i \leqslant r$ and $u_{x_{\mathrm{out}}} = -\iota_\gamma \omega$.

Let Σ be a $\boldsymbol{\gamma}$-fan. We denote by $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(X_\Sigma)$ the moduli space of $(\omega, \boldsymbol{\gamma})$-marked stable log maps to X_Σ. This moduli space is a proper Deligne–Mumford log stack [GS13]. By [Ran17, Proposition 3.3.1], the space $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(X_\Sigma)$ is log smooth of the expected dimension $d-3+(r+1) = d-2+r$. Moreover, $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(X_\Sigma)$ is also irreducible by [Ran17, Proposition 3.3.5].

It follows from Definition 3.5 that, for every $1 \leqslant i \leqslant r$, the projection

$$M_{\mathbb{R}}/\mathbb{R}\iota_{\gamma_i}\omega \longrightarrow M_{\mathbb{R}}/\big(\gamma_i^\perp\big)_{\mathbb{R}} \simeq \mathbb{R}$$

is a morphism of fans from the fan $\Sigma(D_i)$ of D_i in $M_{\mathbb{R}}/\mathbb{R}\iota_{\gamma_i}\omega$ to the fan $\Sigma\big(\mathbb{P}^1\big)$ of $\mathbb{P}^1$ in $M_{\mathbb{R}}/\big(\gamma_i^\perp\big)_{\mathbb{R}} \simeq \mathbb{R}$ (consisting of the cones $\mathbb{R}_{\leqslant 0}$, $\{0\}$, $\mathbb{R}_{\geqslant 0}$). Hence, there exists a corresponding toric morphism $\pi_i\colon D_i \to \mathbb{P}^1$, which in restriction to the dense torus orbit of D_i is simply the monomial $z^{\gamma_i/|\gamma_i|}$. For every point $[\mathbf{H}] = ([H_1], \ldots, [H_r]) \in \big(\mathbb{P}^1\big)^r$, we set $\mathbf{H} := (H_1, \ldots, H_r)$, where, for every $1 \leqslant i \leqslant r$, the element H_i is the hypersurface in D_i given by the fiber of π_i over $[H_i] \in \mathbb{P}^1$, that is, $H_i := \pi_i^{-1}([H_i])$. For $[H_i] = c_i \in \Bbbk^\star \subset \mathbb{P}^1$, the space H_i is the closure in D_i of the hypersurface $-c_i + z^{\gamma_i/|\gamma_i|} = 0$ in $(\Bbbk^\star)^{d-1} \subset D_i$.

For every $1 \leqslant i \leqslant r$, we have a (scheme) evaluation morphism at the marked point x_i:

$$\underline{\mathrm{ev}}_i\colon \mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(X_\Sigma) \longrightarrow D_i\,, \quad f \longmapsto f(x_i)\,.$$

Composing with $\pi_i\colon D_i \to \mathbb{P}^1$, we obtain a (scheme) morphism

$$\underline{\nu}\colon \mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(X_\Sigma) \longrightarrow \big(\mathbb{P}^1\big)^r\,, \quad f \longmapsto (\pi_i(f(x_i)))_i\,.$$

For every $[\mathbf{H}] \in \big(\mathbb{P}^1\big)^r$, the moduli space $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma},\mathbf{H}}(X_\Sigma)$ of $(\omega, \boldsymbol{\gamma})$-stable log maps matching $\mathbf{H}$ is defined by the fiber diagram (in the category of schemes)

$$\begin{array}{ccc} \mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma},\mathbf{H}}(X_\Sigma) & \longrightarrow & \mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(X_\Sigma) \\ \downarrow & & \downarrow{\scriptstyle (\underline{\mathrm{ev}}_i)_i} \\ \prod_{i=1}^r H_i & \xrightarrow{\iota_H} & \prod_{i=1}^r D_i\,, \end{array} \tag{3.2}$$

where the bottom horizontal arrow ι is defined by the inclusion morphisms $H_i \subset D_i$, or equivalently by the fiber diagram

$$\begin{array}{ccc} \mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma},\mathbf{H}}(X_\Sigma) & \longrightarrow & \mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(X_\Sigma) \\ \downarrow & & \downarrow{\scriptstyle \underline{\nu}} \\ [\mathbf{H}] & \xrightarrow{\iota_{[\mathbf{H}]}} & \big(\mathbb{P}^1\big)^r\,, \end{array} \tag{3.3}$$

where $\iota_{[\mathbf{H}]}$ is the inclusion of the point $[\mathbf{H}]$ in $\big(\mathbb{P}^1\big)^r$.

LEMMA 3.7. *For every $1 \leqslant i \leqslant r$, the evaluation scheme morphism $\underline{\mathrm{ev}}_i$ lifts naturally to a log morphism*

$$\mathrm{ev}_i\colon \mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(X_\Sigma) \longrightarrow D_i\,,$$

where D_i is given its toric log structure (as opposed to the log structure restricted from X_Σ).

Proof. Let $(\pi\colon C \to T, f\colon C \to X_\Sigma)$ be an $(\omega,\boldsymbol{\gamma})$-marked stable log map. Evaluation at the marked point x_i defines a scheme evaluation morphism $\underline{\mathrm{ev}}_i\colon T \to D_i$, and we want to show that $\underline{\mathrm{ev}}_i$ naturally lifts to a log morphism $\mathrm{ev}_i\colon T \to D_i$, where D_i is given its toric log structure.

Let T_i be the log scheme with underlying scheme $\underline{T}$ and with log structure $x_i^\star\mathcal{M}_C$, where the marked point x_i is viewed as a section $\underline{T} \to \underline{C}$. Then x_i lifts as a log morphism $T_i \to C$, and composing with $f\colon C \to X_\Sigma$, we obtain a log lift of $\underline{\mathrm{ev}}_i$ as a log morphism $\mathrm{ev}_i\colon T_i \to D_i'$, where D_i' is the log scheme with underlying scheme $\underline{D}_i$ and log structure restricted from X_Σ. We have $\mathcal{M}_T \subset \mathcal{M}_{T_i}$ and $\mathcal{M}_{D_i} \subset \mathcal{M}_{D_i'}$, so it is sufficient to show that $\mathrm{ev}_i^\flat\colon \mathrm{ev}_i^*\mathcal{M}_{D_i'} \to \mathcal{M}_{T_i}$ takes $\mathrm{ev}_i^*\mathcal{M}_{D_i}$ into $\mathcal{M}_T$. To do this, it is sufficient to show this at the level of stalks of ghost sheaves.

For any $p \in C$ in the image of the section x_i, we obtain an induced map

$$\overline{\mathrm{ev}}_i^\flat\colon P_p := \overline{\mathcal{M}}_{X_\Sigma,f(p)} = \overline{\mathcal{M}}_{D_i',f(p)} \longrightarrow \overline{\mathcal{M}}_{C,p} = Q \oplus \mathbb{N}\,,$$

where $Q = \overline{\mathcal{M}}_{T,\pi(p)}$, and where the composition with the second projection is given by the contact order $u_{x_i} = \iota_{\gamma_i}\omega \in P_p^\vee$. Necessarily, $\operatorname{Spec}\Bbbk[P_p]$ is an affine toric chart of X_Σ containing $f(p)$, while $\operatorname{Spec}\Bbbk\big[P_p \cap u_{x_i}^\perp\big]$ is the intersection of this toric affine chart with D_i. In particular, $\overline{\mathcal{M}}_{D_i,f(p)} = P_p \cap u_{x_i}^\perp$, so $\overline{\mathrm{ev}}_i^\flat$ induces a homomorphism between submonoids $\overline{\mathcal{M}}_{D_i,f(p)} \to Q$, as desired. □

The toric morphisms $\pi_i\colon D_i \to \mathbb{P}^1$ naturally lifts to log morphisms when D_i and $\mathbb{P}^1$ are endowed with their toric log structures, so by composition with the log lifts ev_i of the evaluation morphisms $\underline{\mathrm{ev}}_i$ given by Lemma 3.7, we obtain a log lift of the scheme morphism $\underline{\nu}$ introduced in (3.4),

$$\nu\colon \mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(X_\Sigma) \longrightarrow \big(\mathbb{P}^1\big)^r\,, \tag{3.4}$$

where $(\mathbb{P}^1)^r$ is endowed with its toric log structure. For every point $[\mathbf{H}] \in (\mathbb{P}^1)^r$, we endow $[\mathbf{H}]$ with the log structure restricted from the toric log structure on $(\mathbb{P}^1)^r$. In particular, this log structure is trivial if $[\mathbf{H}] \in (\Bbbk^\star)^r \subset (\mathbb{P}^1)^r$. Similarly, we endow H_i with the log structure restricted from the toric log structure on D_i. Then, the inclusions ι_H and $\iota_{[\mathbf{H}]}$ become strict morphisms of log schemes, so the fiber diagrams of schemes (3.2) and (3.3) lift to fiber diagrams of fs log schemes.

For $[\mathbf{H}] \in (\Bbbk^\star)^r \subset (\mathbb{P}^1)^r$, the tropicalization of $[\mathbf{H}]$ is the origin 0 in the tropicalization $\Sigma((\mathbb{P}^1)^r) \simeq \mathbb{R}^r$ of $(\mathbb{P}^1)^r$. Hence, taking the tropicalization of (3.3) viewed as a fiber diagram of fs log schemes, we obtain the fiber diagram of cone complexes

$$\begin{array}{ccc} \Sigma\big(\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma},\mathbf{H}}(X_\Sigma)\big) & \longrightarrow & \Sigma\big(\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(X_\Sigma)\big) \\ \downarrow & & \downarrow{\scriptstyle \Sigma(\nu)} \\ 0 & \longrightarrow & \mathbb{R}^r\,, \end{array} \tag{3.5}$$

where $\Sigma(\mathcal{M}_{\omega,\boldsymbol{\gamma},\mathbf{H}}(X_\Sigma))$ and $\Sigma\big(\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(X_\Sigma)\big)$ are the respective tropicalizations of $\mathcal{M}_{\omega,\boldsymbol{\gamma},\mathbf{H}}(X_\Sigma)$ and $\Sigma\big(\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(X_\Sigma)\big)$.

As discussed in Section 3.2, the tropicalization of an $(\omega,\boldsymbol{\gamma})$-marked stable log map to X_Σ is a family of $(\omega,\boldsymbol{\gamma})$-marked tropical curve in $(M_\mathbb{R},\Sigma)$ as in Definition 2.27, where one views the fan Σ as a particular polyhedral decomposition of $M_\mathbb{R}$. This induces a map of cone complexes

$$T\colon \Sigma(\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(X_\Sigma)) \longrightarrow \mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma}}(\Sigma)\,.$$

Moreover, the map $\Sigma(\nu)$ in (3.5) is the composition of T with the tropical evaluation map at the

legs $\mathrm{ev}^{\mathrm{trop}}\colon \mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma}}(\Sigma) \to \prod_{i=1}^r M_{\mathbb{R}}/\big(\gamma_i^{\perp}\big)_{\mathbb{R}} \simeq \mathbb{R}^r$ given in (2.3):

$$\Sigma(\nu) = \mathrm{ev}^{\mathrm{trop}} \circ T\,.$$

Recall from Definition 2.9 that $\prod_{i=1}^r M_{\mathbb{R}}/\big(\gamma_i^{\perp}\big)_{\mathbb{R}} \simeq \mathbb{R}^r$ is naturally the space of $\boldsymbol{\gamma}$-constraints. In particular, the origin $0 \in \mathbb{R}^r$ corresponds to the $\boldsymbol{\gamma}$-constraint

$$\mathbf{A}^0 := \big(A_1^0, \ldots, A_r^0\big)\,, \tag{3.6}$$

where A_i^0 is the linear hyperplane $\big(\gamma_i^{\perp}\big)_{\mathbb{R}}$ in $M_{\mathbb{R}}$. Therefore, it follows from Diagram (3.5) that the restriction of T to $\Sigma\big(\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma},\mathbf{H}}(X_\Sigma)\big)$ defines a map

$$T_H\colon \Sigma\big(\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma},\mathbf{H}}(X_\Sigma)\big) \longrightarrow \mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}(\Sigma)\,, \tag{3.7}$$

where $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}(\Sigma)$ is the moduli space of $(\omega,\boldsymbol{\gamma})$-marked tropical curves to $(M_{\mathbb{R}},\Sigma)$ matching $\mathbf{A}^0$. In other words, the tropicalization of a stable log map matching $\mathbf{H}$ is a tropical curve matching $\mathbf{A}^0$.

3.4 Log Gromov–Witten invariants

Let ρ be a $(d-2)$-dimensional face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}(\Sigma)$. The space $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}(\Sigma)$ is a sub-cone complex of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma}}(\Sigma)$, so ρ is also a face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma}}(\Sigma)$. The relative interior $\mathrm{Int}(\rho)$ parametrizes $(\omega,\boldsymbol{\gamma})$-marked tropical curves in $(M_{\mathbb{R}},\Sigma)$ of a given type τ_ρ. Let $\mathcal{M}^{\log}_{\rho}(X_\Sigma)$ be the closure in $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(X_\Sigma)$ of the locus of $(\omega,\boldsymbol{\gamma})$-marked stable log maps of type τ_ρ.

As reviewed in Section 3.3, the space $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(X_\Sigma)$ is log smooth of dimension $d-2+r$. By definition, $\mathcal{M}^{\log}_{\rho}(X_\Sigma)$ is a union of log strata of $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(X_\Sigma)$ where the ghost monoid is generically given by the basic monoid Q_{τ_ρ}. As $\mathrm{rk}\, Q_{\tau_\rho}^{\mathrm{gp}} = d-2$, it follows that $\mathcal{M}^{\log}_{\rho}(X_\Sigma)$ is of pure dimension r. In particular, the virtual fundamental class on $\mathcal{M}^{\log}_{\rho}(X_\Sigma)$ constructed by log Gromov–Witten theory [GS13] coincides with the usual fundamental class $\big[\mathcal{M}^{\log}_{\rho}(X_\Sigma)\big]$.

For every $[\mathbf{H}] \in (\mathbb{P}^1)^r$, let $\mathcal{M}^{\log}_{\rho,\mathbf{H}}(X_\Sigma)$ be the closure in $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma},\mathbf{H}}(X_\Sigma)$ of the locus of $(\omega,\boldsymbol{\gamma})$-marked stable log maps of type τ_ρ. Restricting the morphism ν of (3.4) to $\mathcal{M}^{\log}_{\rho}(X_\Sigma)$, we obtain a fiber diagram

$$\begin{array}{ccc} \mathcal{M}^{\log}_{\rho,\mathbf{H}}(X_\Sigma) & \longrightarrow & \mathcal{M}^{\log}_{\rho}(X_\Sigma) \\ \downarrow & & \downarrow{\scriptstyle \nu} \\ [\mathbf{H}] & \xrightarrow{\ \iota_{[\mathbf{H}]}\ } & \big(\mathbb{P}^1\big)^r\,, \end{array}$$

and we define a 0-dimensional virtual fundamental class on $\mathcal{M}^{\log}_{\rho,\mathbf{H}}(X_\Sigma)$ by

$$\big[\mathcal{M}^{\log}_{\rho,\mathbf{H}}(X_\Sigma)\big]^{\mathrm{vir}} := \iota^!_{[\mathbf{H}]}\big[\mathcal{M}^{\log}_{\rho}(X_\Sigma)\big]\,, \tag{3.8}$$

where $\iota^!_{[\mathbf{H}]}$ is the Gysin pullback [Ful98, Chapter 6] defined by the regular embedding $\iota_{[\mathbf{H}]}$ of codimension r. As the moduli space $\mathcal{M}^{\log}_{\rho,\mathbf{H}}(X_\Sigma)$ is proper, one can define a log Gromov–Witten invariant $N^{\mathrm{toric}}_{\rho}$ as the degree of this class:

$$N^{\mathrm{toric}}_{\rho}(X_\Sigma) = \deg\big[\mathcal{M}^{\log}_{\rho,\mathbf{H}}(X_\Sigma)\big]^{\mathrm{vir}}\,. \tag{3.9}$$

By the deformation invariance of the Gysin pullback $\iota^!_{[\mathbf{H}]}$, the log Gromov–Witten invariant $N^{\mathrm{toric}}_{\rho}$ is independent of the choice of $[\mathbf{H}] \in (\mathbb{P}^1)^r$.

Theorem 3.8. *For general* $[\mathbf{H}] \in (\mathbb{P}^1)^l$, *the moduli stack* $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma},\mathbf{H}}(X_\Sigma)$ *is log smooth of dimension* $d-2$, *and, for every* $(d-2)$*-dimensional face of* $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}(\Sigma)$, *the moduli stack* $\mathcal{M}^{\log}_{\rho,\mathbf{H}}(X_\Sigma)$ *is reduced and of pure dimension* 0. *In particular, the virtual fundamental class is in this case given by the usual fundamental class,*

$$N^{\mathrm{toric}}_\rho(X_\Sigma) = \deg\left[\mathcal{M}^{\log}_{\rho,\mathbf{H}}(X_\Sigma)\right],$$

and $N^{\mathrm{toric}}_\rho(X_\Sigma)$ *is an enumerative count with automorphisms.*

Proof. As $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(X_\Sigma)$ is log smooth of dimension $d-2+r$ over the trivial log point, it follows from the generic log smoothness result in Theorem A.1 that $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma},\mathbf{H}}(X_\Sigma)$ is also log smooth of dimension $d-2$ over the trivial log point for general $[\mathbf{H}] \in (\mathbb{P}^1)^l$. In particular, as $\mathcal{M}^{\log}_{\rho,\mathbf{H}}(X_\Sigma)$ is a union of log strata of $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma},\mathbf{H}}(X_\Sigma)$ where the ghost monoid is generically given by the basic monoid Q_{τ_ρ} with $\mathrm{rk}\, Q^{\mathrm{gp}}_{\tau_\rho} = d-2$, this implies that $\mathcal{M}^{\log}_\rho(X_\Sigma)$ is a union of 0-dimensional strata of $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma},\mathbf{H}}(X_\Sigma)$, which are necessarily reduced. □

4. The log-tropical correspondence

In this section, we establish our main correspondence result between the tropical multiplicities N^{trop}_σ introduced in Section 2 and the log Gromov–Witten invariants $N^{\mathrm{toric}}_\rho(X_\Sigma)$ defined in Section 3.

4.1 Good polyhedral decompositions and toric degenerations

As in the work of Nishinou–Siebert [NS06], we obtain our log-tropical correspondence theorem by the study of a toric degeneration defined by an appropriately chosen polyhedral decomposition of $M_\mathbb{R}$. In this section, we define the notion of a "good" polyhedral decomposition, and we prove that good polyhedral decompositions exist. The main difference with the setup of [NS06] is that we are considering families of tropical curves and not rigid tropical curves in general.

We fix a skew-symmetric form $\omega \in \bigwedge^2 M$ on N and an r-tuple $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_r)$ of elements $\gamma_i \in N$ such that $\iota_{\gamma_i}\omega \neq 0$ for all $1 \leqslant i \leqslant r$ and $\iota_\gamma \omega \neq 0$, where $\gamma := \sum_{i=1}^r \gamma_i$ as in Section 2.2. We also fix a general $\boldsymbol{\gamma}$-constraint $\mathbf{A}$ defined as in Definitions 2.9–2.11. In what follows, the tuple

$$(\omega, \boldsymbol{\gamma}, \mathbf{A}) \tag{4.1}$$

is referred to as the *tropical data*. Moreover, we say that $\mathbf{A}$ is *rational* if the affine hyperplanes A_i are defined over $\mathbb{Q}$, that is, $[\mathbf{A}] \in \mathbb{Q}^r \subset \mathbb{R}^r$. Recall that we reviewed the notion of polyhedral decomposition in Definition 2.26. Then, a good polyhedral decomposition for a given tropical data is defined as follows.

Definition 4.1. Let $(\omega, \boldsymbol{\gamma}, \mathbf{A})$ be a tropical data with rational $\mathbf{A}$. A *good polyhedral decomposition* $\mathscr{P}$ of $M_\mathbb{R}$ for $(\omega, \boldsymbol{\gamma}, \mathbf{A})$ is a rational polyhedral decomposition of $M_\mathbb{R}$ that satisfies the following conditions:

(i) For each $(d-2)$-dimensional face σ of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$, each $\mathfrak{j}^\sigma_v$ defined as in (2.4) is a union of $(d-2)$-dimensional faces of $\mathscr{P}$, and each $\mathfrak{d}^\sigma_E$ defined as in (2.5) is a union of $(d-1)$-dimensional faces of $\mathscr{P}$.

(ii) For every $1 \leqslant i \leqslant r$, the constraint affine hyperplane A_i is a union of $(d-1)$-dimensional faces of $\mathscr{P}$,

To show that a good polyhedral decomposition exists, we first need the following lemma.

LEMMA 4.2. *Let P be a k-dimensional polyhedron in $\mathbb{R}^n$. Then, there exists a polyhedral decomposition $\mathscr{P}$ of $\mathbb{R}^n$ such that P is a union of k-dimensional cell of $\mathscr{P}$.*

Proof. We first remark that is enough to prove the result for $n = k$. Indeed, if $\mathscr{P}'$ is a polyhedral decomposition of the k-dimensional hull of P such that P is a union of k-dimensional cells of $\mathscr{P}'$, then, choosing an $(n-k)$-dimensional linear subspace V transverse to the affine hull of P, $\mathscr{P} = \mathscr{P}' + \Sigma_{\mathbb{P}^{n-k}}$ is a polyhedral decomposition of $\mathbb{R}^n$ with the same property, where $\Sigma_{\mathbb{P}^{n-k}}$ is the fan of $\mathbb{P}^{n-k}$ viewed as a fan in V.

We prove the result for k-dimensional polytopes in $\mathbb{R}^k$ by induction on k. For $k = 0$, there is nothing to prove. We now treat the induction step. By the induction hypothesis, for every codimension 1 face F of P, there exists a polyhedral decomposition $\mathscr{P}_F$ of the $(k-1)$-dimensional affine hull of F such that F is a union of $(k-1)$-dimensional cells of $\mathscr{P}_F$. For every such F, choose an $n_F \in \mathbb{Z}^k$ not tangent to F, and define the polyhedral decomposition $\widetilde{\mathscr{P}_F}$ of $\mathbb{R}^k$ with cells $\sigma + \mathbb{R}_{\geqslant 0} n_F$ and $\sigma + \mathbb{R}_{\leqslant 0} n_F$ for all cells σ of $\mathscr{P}_F$. Then, P is a union of k-dimensional cells of the polyhedral decomposition $\mathscr{P} := \bigcap_F \widetilde{\mathscr{P}_F}$ of $\mathbb{R}^k$, where the intersection is taken over the codimension 1 faces F of P. □

Remark 4.3. We gave an elementary proof of Lemma 4.2 for completeness. Much stronger results exist: For example, one could assume that P is actually a k-cell of $\mathscr{P}$ by considering the cone over $P \times \{1\}$ in $\mathbb{R}^n \times \mathbb{R}$ and then using the rather difficult result that a cone can always be completed in a complete fan; see [Roh11] and references there. We will not use these non-trivial results.

LEMMA 4.4. *For every tropical data $(\omega, \boldsymbol{\gamma}, \mathbf{A})$ with rational $\mathbf{A}$, a good polyhedral decomposition exists.*

Proof. By Theorem 2.17, we have $\dim \mathfrak{j}_v^\sigma = d-2$ and $\dim \mathfrak{d}_E^\sigma = d-1$ for every $\mathfrak{j}_v^\sigma$ and $\mathfrak{d}_E^\sigma$ as in Definition 4.1. Moreover, by Lemma 4.2, for every $\mathfrak{j}_v^\sigma$, $\mathfrak{d}_E^\sigma$, or A_i as in Definition 4.1, there exists polyhedral decompositions of $M_\mathbb{R}$ containing, respectively, $\mathfrak{j}_v^\sigma$, $\mathfrak{d}_E^\sigma$, or A_i as a face. Taking the intersection of all these polyhedral decompositions for every $\mathfrak{j}_v^\sigma$, $\mathfrak{d}_E^\sigma$, and A_i, we obtain a good polyhedral decomposition. □

Let $(\omega, \boldsymbol{\gamma}, \mathbf{A})$ be a tropical data with rational $\mathbf{A}$. Given a good polyhedral decomposition $\mathscr{P}$ for $(\omega, \boldsymbol{\gamma}, \mathbf{A})$, whose existence is guaranteed by Lemma 4.4, one constructs a toric degeneration as in [NS06, § 3]. Let $\overline{M} := M \oplus \mathbb{Z}$, and let $\overline{\Sigma}_\mathscr{P}$ be the fan in $\overline{M}_\mathbb{R} := \overline{M} \otimes \mathbb{R}$ whose faces are the cones over the cells of $\mathscr{P}$ viewed in $M_\mathbb{R} \times \{1\} \subset \overline{M}_\mathbb{R}$. We denote by $X_\mathscr{P}$ the corresponding $(d+1)$-dimensional toric variety. The projection on the $\mathbb{R}$-factor of $\overline{M}_\mathbb{R} = M_\mathbb{R} \oplus \mathbb{R}$ defines a map of fans $\overline{\Sigma}_\mathscr{P} \to \mathbb{R}_{\geqslant 0} = \Sigma(\mathbb{A}^1)$, and so a toric morphism

$$\pi_\mathscr{P} \colon \mathcal{X}_\mathscr{P} \longrightarrow \mathbb{A}^1$$

whose fiber $\pi_\mathscr{P}^{-1}(z)$ for $z \in \mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$ is the toric variety $X_{\Sigma_\mathscr{P}}$ associated with the *asymptotic fan* $\Sigma_\mathscr{P}$ of $\mathscr{P}$, defined by

$$\Sigma_\mathscr{P} := \left\{ \lim_{a \to 0} a\Xi \subset M_\mathbb{R} \mid \Xi \in \mathscr{P} \right\}.$$

The irreducible components of the central fiber $\pi_\mathscr{P}^{-1}(0)$ are in one-to-one correspondence with the vertices of $\mathscr{P}$. By rescaling $M_\mathbb{R}$ if necessary, we can assume that all vertices of $\mathscr{P}$ lie in M, in which case the central fiber $\pi_\mathscr{P}^{-1}(0)$ is reduced.

We also obtain a degeneration of the constraints $\mathbf{H} = (H_1, \dots, H_r)$ determined by the choice of the tropical constraints $\mathbf{A} = (A_1, \dots, A_r)$. Up to rescaling $\mathscr{P}$ if necessary, one can assume that $M \cap A_i \neq \emptyset$ for every $1 \leqslant i \leqslant r$. Then, one chooses a point $P_i \in M \cap A_i$ for every $1 \leqslant i \leqslant r$. The point $P_i \in M$ determines a point $(P_i, 1) \in \overline{M}$ and hence a one-parameter subgroup $\mathbb{G}_m(P_i, 1) \subseteq \overline{M} \otimes \mathbb{G}_m$. This can be viewed as a section σ_i of the projection $\overline{M} \otimes \mathbb{G}_m \to \mathbb{G}_m$ onto the last coordinate. For every $z \in \mathbb{G}_m$, the image $\sigma_i(z)$ acts on the fiber of $\mathcal{X}_{\mathscr{P}} \to \mathbb{A}^1$ over $z \in \mathbb{G}_m \subset \mathbb{A}^1$. Recall that the inverse image of $\mathbb{G}_m$ in $\mathcal{X}_{\mathscr{P}}$ is $X_{\Sigma_{\mathscr{P}}} \times \mathbb{G}_m$. We then define $\overline{H_i}$ to be the closure of the subset

$$\bigcup_{z \in \mathbb{G}_m} (\sigma_i(z) H_i, z) \subseteq X_{\Sigma_{\mathscr{P}}} \times \mathbb{G}_m$$

in $\mathcal{X}_{\mathscr{P}}$. By construction, $\overline{H_i}$ is a hypersurface in the divisor $\overline{D_i}$ of $\mathcal{X}_{\mathscr{P}}$ corresponding to the ray $\mathbb{R}_{\geqslant 0}(\iota_{\gamma_i}\omega, 1)$ in $\overline{\Sigma}$. We set $\overline{\mathbf{H}} := \big(\overline{H}_1, \dots, \overline{H}_r\big)$.

4.2 Decomposition formula

We fix a good polyhedral decomposition $\mathscr{P}$ for a tropical data $(\omega, \boldsymbol{\gamma}, \mathbf{A})$ with rational $\mathbf{A}$ as in Section 4.1. In this section, we express the log Gromov–Witten invariants $N_\rho^{\mathrm{toric}}(X_{\Sigma_{\mathscr{P}}})$ of the general fiber of the toric degeneration $\pi_{\mathscr{P}} \colon \mathcal{X}_{\mathscr{P}} \to \mathbb{A}^1$ in terms of log Gromov–Witten invariants of the special fiber.

We define as in Definition 3.6 the notion of $(\omega, \boldsymbol{\gamma})$-marked stable maps to $\mathcal{X}_{\mathscr{P}}$, the only difference being that the contact orders at the marked points are now $(\iota_{\gamma_i}\omega, 0)$ and $(\iota_\gamma \omega, 0)$ in $\overline{M} = M \oplus \mathbb{Z}$. We denote by $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$ the moduli space of $(\omega, \boldsymbol{\gamma})$-marked stable maps to $\pi_{\mathscr{P}} \colon \mathcal{X}_{\mathscr{P}} \to \mathbb{A}^1$. We have a natural morphism $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1) \to \mathbb{A}^1$ whose fiber over $z \in \mathbb{G}_m$ is $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(X_{\Sigma_{\mathscr{P}}})$.

Lemma 4.5. *The moduli space $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$ is log smooth over $\mathbb{A}^1$ and is of pure dimension $d - 2 + r + 1$.*

Proof. The relative log tangent bundle to $\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1$ is trivial, given by $T^{\log}_{\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1} = M \otimes \mathcal{O}_{\mathcal{X}_{\mathscr{P}}}$, so we have $H^1\big(C, f^\star T^{\log}_{\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1}\big) = 0$ for every genus 0 stable log map $f \colon C \to \mathcal{X}_{\mathscr{P}}/\mathbb{A}^1$. It follows from [GS13] that relative deformations of genus 0 stable log maps to $\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1$ with fixed domain are unobstructed. Hence, it is enough to show that the moduli stack $\mathfrak{M}^{\log}(\mathbb{A}^1)$ of not necessarily basic prestable log maps to $\mathbb{A}^1$ is log smooth over $\mathbb{A}^1$. We have $\mathfrak{M}^{\log}(\mathbb{A}^1) = \mathrm{Log}_{\mathfrak{M} \times \mathbb{A}^1}$, where $\mathfrak{M}$ is the moduli stack of prestable curves and $\mathrm{Log}_{\mathfrak{M} \times \mathbb{A}^1}$ is Olsson's stack [Ols03] of log schemes over $\mathfrak{M} \times \mathbb{A}^1$. As $\mathfrak{M}$ is log smooth over the trivial log point, $\mathfrak{M} \times \mathbb{A}^1$ is log smooth over $\mathbb{A}^1$, so the result follows because $\mathrm{Log}_{\mathfrak{M} \times \mathbb{A}^1} \to \mathbb{A}^1$ factors into $\mathrm{Log}_{\mathfrak{M} \times \mathbb{A}^1} \to \mathfrak{M} \times \mathbb{A}^1 \to \mathbb{A}^1$, and for any log stack S, the natural morphism $\mathrm{Log}_S \to S$ is log étale by [Ols03, Theorem 4.6(iii)] (the corresponding morphism of stacks $\mathrm{Log}_S \to \mathrm{Log}_S$ is the identity and so is in particular étale). □

We also define as in (3.2) the moduli space $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma},\overline{\mathbf{H}}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$ of $(\omega, \boldsymbol{\gamma})$-marked stable maps to $\mathcal{X}_{\mathscr{P}} \to \mathbb{A}^1$ matching the constraint $\overline{\mathbf{H}}$ by the fiber diagram (both in the schemes and fs log schemes categories)

$$\begin{array}{ccc} \mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma},\overline{\mathbf{H}}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1) & \longrightarrow & \mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1) \\ \downarrow & & \downarrow \\ \prod_{i=1}^r \overline{H}_i & \xrightarrow{\iota_{\overline{H}}} & \prod_{i=1}^r \overline{D}_i\,. \end{array}$$

We have a natural morphism $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma},\overline{\mathbf{H}}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1) \to \mathbb{A}^1$ whose fiber over $z \in \mathbb{G}_m$ is the space $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma},\{\sigma_i(z)\cdot H_i\}_i}(X_{\Sigma_{\mathscr{P}}})$.

For every $1 \leqslant i \leqslant r$, denote by $\overline{A}_i$ the half-hyperplane in $\overline{M}_{\mathbb{R}}$ obtained as the closure in $\overline{M}_{\mathbb{R}}$ of $\mathbb{R}_{\geqslant 0}(A_i, 1)$. By construction, one has a natural projection $\overline{A}_i \to \mathbb{R}_{\geqslant 0}$ whose fiber over 1 is A_i, and whose fiber over 0 is $A_i^0 = \big(\gamma_i^{\perp}\big)_{\mathbb{R}}$. As in Section 2.3, we define the moduli space $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\overline{\mathbf{A}}}\big(\overline{\Sigma}_{\mathscr{P}}\big)$ of $(\omega,\boldsymbol{\gamma})$-marked tropical curves in $\big(\overline{M}_{\mathbb{R}}, \overline{\Sigma}_{\mathscr{P}}\big)$ matching the constraint

$$\overline{\mathbf{A}} := \big(\overline{A}_1, \dots, \overline{A}_r\big)\,.$$

There is a natural map $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\overline{\mathbf{A}}}\big(\overline{\Sigma}_{\mathscr{P}}\big) \to \mathbb{R}_{\geqslant 0}$ whose fiber over 1 is $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P})$ and whose fiber over 0 is $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}(\Sigma_{\mathscr{P}})$. Using this, we obtain a natural map

$$\tilde{\Phi}\colon \big\{\text{faces of } \mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P})\big\} \longrightarrow \big\{\text{faces of } \mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}(\Sigma_{\mathscr{P}})\big\}\,, \tag{4.2}$$

defined as follows. Given a face $\tilde{\sigma}$ of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P})$, its image $\tilde{\Phi}(\tilde{\sigma})$ is the intersection of the fiber $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}(\Sigma_{\mathscr{P}})$ over 0 with the face $\overline{\mathbb{R}_{\geqslant 0}(\tilde{\sigma}, 1)}$ of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\overline{\mathbf{A}}}\big(\overline{\Sigma}_{\mathscr{P}}\big)$ obtained as the closure of the cone over $\tilde{\sigma}$.

DEFINITION 4.6. A good polyhedral decomposition $\mathscr{P}$ of $M_{\mathbb{R}}$ is *very good* if

(i) every vertex of $\mathscr{P}$ lies in M;
(ii) for every $(d-2)$-dimensional face $\tilde{\sigma}$ of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P})$, the natural projection $T_{\overline{\mathbb{R}_{\geqslant 0}(\tilde{\sigma},1)}} \to \mathbb{Z}$ is surjective, where $T_{\overline{\mathbb{R}_{\geqslant 0}(\tilde{\sigma},1)}}$ is the integral tangent space to the face $\overline{\mathbb{R}_{\geqslant 0}(\tilde{\sigma}, 1)}$ of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\overline{\mathbf{A}}}\big(\overline{\Sigma}_{\mathscr{P}}\big)$.

LEMMA 4.7. *For every tropical data $(\omega,\boldsymbol{\gamma},\mathbf{A})$ with rational $\mathbf{A}$, there exists a very good polyhedral decomposition for $(\omega,\boldsymbol{\gamma},\mathbf{A})$.*

Proof. By Lemma 4.4, good polyhedral decompositions exist. The result follows because a good polyhedral decomposition can always be rescaled into a very good one. □

From now on, we assume that $\mathscr{P}$ is a very good polyhedral decomposition. Let $\tilde{\sigma}$ be a $(d-2)$-dimensional face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P})$. The relative interior of the face $\overline{\mathbb{R}_{\geqslant 0}(\tilde{\sigma}, 1)}$ of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\overline{\mathbf{A}}}\big(\overline{\Sigma}_{\mathscr{P}}\big)$ parametrizes $(\omega,\boldsymbol{\gamma})$-marked tropical curves in $\big(\overline{M}_{\mathbb{R}}, \overline{\Sigma}_{\mathscr{P}}\big)$ of a given type $\tau_{\tilde{\sigma}}$. We define the space $\mathcal{M}^{\log}_{\tilde{\sigma}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$ (respectively, the space $\mathcal{M}^{\log}_{\tilde{\sigma},\overline{\mathbf{H}}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$) as the closure in $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$ (respectively, $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma},\overline{\mathbf{H}}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$) of the locus of stable log maps of type $\tau_{\tilde{\sigma}}$. These moduli spaces fit in a fiber diagram (both in the schemes and fs log schemes categories)

$$\begin{array}{ccc} \mathcal{M}^{\log}_{\tilde{\sigma},\overline{\mathbf{H}}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1) & \longrightarrow & \mathcal{M}^{\log}_{\tilde{\sigma}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1) \\ \downarrow & & \downarrow \\ \prod_{i=1}^r \overline{H}_i & \xrightarrow{\iota_{\overline{H}}} & \prod_{i=1}^r \overline{D}_i\,. \end{array}$$

Similarly, given a $(d-2)$-dimensional face ρ of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}(\Sigma_{\mathscr{P}})$, we denote by $\mathcal{M}^{\log}_{\rho}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$ (respectively, $\mathcal{M}^{\log}_{\rho,\overline{\mathbf{H}}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$) the closure in $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$ (respectively, $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma},\overline{\mathbf{H}}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$) of the locus of stable log maps of type determined by the interior of the face ρ. There is a natural map

$$\mathcal{M}^{\log}_{\rho,\overline{\mathbf{H}}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1) \longrightarrow \mathbb{A}^1$$

whose fiber over 1 is the moduli space $\mathcal{M}^{\log}_{\rho,\mathbf{H}}(X_{\Sigma_\mathscr{P}})$ defined in Section 3.4. Moreover, the fiber over 0 is the union of the moduli spaces $\mathcal{M}^{\log}_{\tilde\sigma,\overline{\mathbf{H}}}(\mathcal{X}_\mathscr{P}/\mathbb{A}^1)$, where the $\tilde\sigma$ are the faces of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P})$ such that $\tilde\Phi(\tilde\sigma)=\rho$.

Lemma 4.8. *The moduli spaces $\mathcal{M}^{\log}_{\tilde\sigma}(\mathcal{X}_\mathscr{P}/\mathbb{A}^1)$ and $\mathcal{M}^{\log}_{\rho}(\mathcal{X}_\mathscr{P}/\mathbb{A}^1)$ are log smooth over $\mathbb{A}^1$. Moreover, $\mathcal{M}^{\log}_{\tilde\sigma}(\mathcal{X}_\mathscr{P}/\mathbb{A}^1)$ is of pure dimension r, and $\mathcal{M}^{\log}_{\rho}(\mathcal{X}_\mathscr{P}/\mathbb{A}^1)$ is of pure dimension $r+1$.*

Proof. By Lemma 4.5, the moduli space $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(\mathcal{X}_\mathscr{P}/\mathbb{A}^1)$ is log smooth over $\mathbb{A}^1$, of dimension $d-2+r+1$. The result follows from the fact that $\mathcal{M}^{\log}_{\tilde\sigma}(\mathcal{X}_\mathscr{P}/\mathbb{A}^1)$ and $\mathcal{M}^{\log}_{\rho}(\mathcal{X}_\mathscr{P}/\mathbb{A}^1)$ are log strata of $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma}}(\mathcal{X}_\mathscr{P}/\mathbb{A}^1)$. □

By Lemma 4.8, one can consider the r-dimensional fundamental class $\big[\mathcal{M}^{\log}_{\tilde\sigma}(\mathcal{X}_\mathscr{P}/\mathbb{A}^1)\big]$. We define a 0-dimensional virtual fundamental class on $\mathcal{M}^{\log}_{\tilde\sigma,\overline{\mathbf{H}}}(\mathcal{X}_\mathscr{P}/\mathbb{A}^1)$ by

$$\big[\mathcal{M}^{\log}_{\tilde\sigma,\overline{\mathbf{H}}}(\mathcal{X}_\mathscr{P}/\mathbb{A}^1)\big]^{\mathrm{vir}} := \iota^!_{\overline{H}}\big[\mathcal{M}^{\log}_{\tilde\sigma}(\mathcal{X}_\mathscr{P}/\mathbb{A}^1)\big] \tag{4.3}$$

and the corresponding log Gromov–Witten invariant by

$$N^{\mathrm{toric}}_{\tilde\sigma}(\mathcal{X}_\mathscr{P}/\mathbb{A}^1) := \deg\big[\mathcal{M}^{\log}_{\tilde\sigma,\overline{\mathbf{H}}}(\mathcal{X}_\mathscr{P}/\mathbb{A}^1)\big]^{\mathrm{vir}}. \tag{4.4}$$

In the following theorem, we express the invariants $N^{\mathrm{toric}}_{\rho}(X_{\Sigma_\mathscr{P}})$ defined in (3.9) in terms of the invariants $N^{\mathrm{toric}}_{\tilde\sigma}(\mathcal{X}_\mathscr{P}/\mathbb{A}^1)$ defined in (4.4). For every $(d-2)$-dimensional face ρ of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}(\Sigma)$, denote by $\tilde S_\rho$ the set of $(d-2)$-dimensional faces $\tilde\sigma$ of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P})$ such that $\tilde\Phi(\tilde\sigma)=\rho$.

Theorem 4.9. *Let $\mathscr{P}$ be a very good polyhedral decomposition for $(\omega,\boldsymbol{\gamma},\mathbf{A})$, and let ρ be a $(d-2)$-dimensional face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}(\Sigma)$. Then, we have*

$$N^{\mathrm{toric}}_{\rho}(X_{\Sigma_\mathscr{P}}) = \sum_{\tilde\sigma\in\tilde S_\rho} N^{\mathrm{toric}}_{\tilde\sigma}\big(\mathcal{X}_\mathscr{P}/\mathbb{A}^1\big). \tag{4.5}$$

Proof. By Lemma 4.8, the moduli space $\mathcal{M}^{\log}_{\rho}(\mathcal{X}_\mathscr{P}/\mathbb{A}^1)$ is log smooth over $\mathbb{A}^1$ and of dimension $r+1$. The general fiber of $\mathcal{M}^{\log}_{\rho}(\mathcal{X}_\mathscr{P}/\mathbb{A}^1)$ is the r-dimensional moduli space $\mathcal{M}^{\log}_{\rho}(X_{\Sigma_\mathscr{P}})$, whereas the irreducible components of the central fiber over $0\in\mathbb{A}^1$ are exactly the r-dimensional moduli spaces $\mathcal{M}^{\log}_{\tilde\sigma}(\mathcal{X}_\mathscr{P}/\mathbb{A}^1)$ for $\tilde\sigma\in\tilde S_\rho$. Hence, it follows from the decomposition result of [ACGS20, Corollary 3.2] for the fibers of log smooth morphisms that

$$\big[\mathcal{M}^{\log}_{\rho}(X_{\Sigma_\mathscr{P}})\big] = \sum_{\tilde\sigma\in\tilde S_\rho}\big[\mathcal{M}^{\log}_{\tilde\sigma}\big(\mathcal{X}_\mathscr{P}/\mathbb{A}^1\big)\big]. \tag{4.6}$$

The multiplicities present in the general formula of [ACGS20, Corollary 3.2] are all equal to 1 in our situation because we are assuming that $\mathscr{P}$ is a very good polyhedral decomposition and so Definition 4.6(ii) is satisfied (geometrically, the central fiber of $\mathcal{M}^{\log}_{\rho}(\mathcal{X}_\mathscr{P}/\mathbb{A}^1)\to\mathbb{A}^1$ is reduced). Now, applying $\iota^!_{\overline{H}}$ to both sides of (4.6), we obtain by (3.8) and (4.3) that

$$\big[\mathcal{M}^{\log}_{\rho,\mathbf{H}}(X_{\Sigma_\mathscr{P}})\big]^{\mathrm{vir}} = \sum_{\tilde\sigma\in\tilde S_\rho}\big[\mathcal{M}^{\log}_{\tilde\sigma,\overline{\mathbf{H}}}\big(\mathcal{X}_\mathscr{P}/\mathbb{A}^1\big)\big]^{\mathrm{vir}}. \tag{4.7}$$

Taking the degree on both sides gives (4.5) by (3.9) and (4.4). □

4.3 The log-tropical correspondence for a fixed tropical type

In this subsection, we relate the log Gromov–Witten invariants $N^{\mathrm{toric}}_{\tilde{\sigma}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$ defined in (4.4) with the tropical multiplicities $N^{\mathrm{trop}}_{\sigma}$ defined in (2.9).

Let $\tilde{\sigma}$ be a $(d-2)$-dimensional face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P})$ such that $\dim \mathfrak{d}^{\tilde{\sigma}}_{L_{\mathrm{out}}} = d-1$ and $\dim \tilde{\Phi}(\tilde{\sigma}) = d-2$. Tropical curves in the relative interior of $\tilde{\sigma}$ are of type $\tau_{\tilde{\sigma}}$, and we defined in Section 4.2 the moduli spaces $\mathcal{M}^{\log}_{\tilde{\sigma}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$ and $\mathcal{M}^{\log}_{\tilde{\sigma},\overline{\mathbf{H}}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$ as closures of the loci of stable log maps of type $\tau_{\tilde{\sigma}}$. Lemma 4.10 shows that stable log maps in $\mathcal{M}^{\log}_{\tilde{\sigma},\overline{\mathbf{H}}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$ are automatically of type $\tau_{\tilde{\sigma}}$, so taking the closure was actually unnecessary in this case. Denote by $\mathcal{M}^{\log,0}_{\tilde{\sigma}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$ the open dense locus in $\mathcal{M}^{\log}_{\tilde{\sigma}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$ consisting of the stable log maps of type $\tau_{\tilde{\sigma}}$.

Lemma 4.10. *Stable log maps in $\mathcal{M}^{\log}_{\tilde{\sigma},\overline{\mathbf{H}}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$ are of type $\tau_{\tilde{\sigma}}$; that is, we have*

$$\mathcal{M}^{\log}_{\tilde{\sigma},\overline{\mathbf{H}}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1) \subset \mathcal{M}^{\log,0}_{\tilde{\sigma}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)\,.$$

Proof. If there were a stable log map in $\mathcal{M}^{\log}_{\tilde{\sigma},\overline{\mathbf{H}}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$ of type $\tau \neq \tau_{\tilde{\sigma}}$, then the closure of the tropical curves of type τ would be a face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P})$ containing strictly $\tau_{\tilde{\sigma}}$, and so in particular of dimension greater than $d-2$. But as the $\boldsymbol{\gamma}$-constraint $\mathbf{A}$ is general, every face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ is of dimension at most $d-2$ by Definition 2.11(ii), and the same is true for $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P})$ because $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P})$ is a polyhedral refinement of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ by Section 2.7. Hence, we have obtained a contradiction. □

By Lemma 4.10, it will be enough to study stable log maps of type $\tau_{\tilde{\sigma}}$. We give below an explicit description of stable log maps of type $\tau_{\tilde{\sigma}}$.

By the definition of a good polyhedral decomposition (Definition 4.1), for each vertex v of $\Gamma_{\tilde{\sigma}}$, the polyhedron $\mathfrak{j}^{\tilde{\sigma}}_v$ lies in the interior of an $(n-2)$-dimensional cell $\mathcal{P}_v$ of $\mathscr{P}$, and for each edge or leg E, the polyhedron $\mathfrak{d}^{\tilde{\sigma}}_E$ lies in the interior of an $(n-1)$-dimensional cell $\mathcal{P}_E$ of $\mathscr{P}$. Let $Z_v, Z_E \subseteq \mathcal{X}_{\mathscr{P}}$ be the closed toric strata corresponding to $\mathcal{P}_v$, $\mathcal{P}_E$, respectively. The strata Z_v and Z_E are toric subvarieties of $\mathcal{X}_{\mathscr{P}}$ with $\dim Z_v = 2$ and $\dim Z_E = 1$. Denote by T_v and T_E the integral tangent spaces to $\mathcal{P}_v$ and $\mathcal{P}_E$, respectively. Then, Z_v is a toric surface given by a fan in the lattice

$$M_v := M/T_v\,,$$

and we have $Z_E \simeq \mathbb{P}^1$ with fan naturally in $M_E := M/T_E$.

If $f\colon C/W \to \mathcal{X}_{\mathscr{P}}/\mathbb{A}^1$ is a stable log map of type $\tau_{\tilde{\sigma}}$, then the dual graph of C is given by the graph $\Gamma_{\tilde{\sigma}}$. We denote by $\underline{C}_v \simeq \mathbb{P}^1$ the irreducible component corresponding to $v \in V(\Gamma_{\tilde{\sigma}})$. Moreover, for every edge or leg E of $\Gamma_{\tilde{\sigma}}$, we denote by x_E the corresponding special point (node or marked point) on C. For every $v \in V(\Gamma_{\tilde{\sigma}})$, we have $f(C_v) \subset Z_v$, and the only points of C_v mapped by f in the toric boundary of Z_v are the points x_E for E an edge or leg adjacent to v. In addition, the point $f(x_E)$ is contained in the interior of the big torus orbit $\mathbb{G}_m$ of $Z_E \simeq \mathbb{P}^1$, and the contact order of $f(C_v)$ to Z_E at $f(x_E)$ is specified by the image $\bar{u}_{v,E}$ in $M_v = M/T_v$ of the weighted direction $u_{v,E} \in M$. In particular, the morphism of schemes $\underline{f}_v\colon \underline{C}_v \to Z_v$ obtained by restricting f to $\underline{C}_v$ is torically transverse as in [NS06, Definition 4.1]. In fact, as the vertices of $\Gamma_{\tilde{\sigma}}$ are either divalent or trivalent, $\underline{f}_v\colon \underline{C}_v \to Z_v$ is a line in the sense of [NS06, Definition 5.1]. Moreover, if E is the leg L_i of $\Gamma_{\tilde{\sigma}}$ for some $1 \leqslant i \leqslant r$, then the intersection $Z_{L_i} \cap \overline{H}_i$ consists of a single point, because the defining equation of $\overline{H}_i$ is $z^{\gamma_i/|\gamma_i|} = c_i$ and $\gamma_i/|\gamma_i|$ is primitive. The stable log map $f\colon C/W \to \mathcal{X}_{\mathscr{P}}/\mathbb{A}^1$ matches $\overline{\mathbf{H}}$ if and only if $\{f(x_{L_i})\} = Z_{L_i} \cap \overline{H}_i$ for all $1 \leqslant i \leqslant r$.

Lemma 4.11. *The moduli stack $\mathcal{M}^{\log}_{\tilde{\sigma},\overline{\mathbf{H}}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$ is reduced and 0-dimensional.*

Proof. The deformation theory of stable log maps to $\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1$ is controlled by the log tangent bundle $T^{\log}_{\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1} = M \otimes \mathcal{O}_{\mathcal{X}_{\mathscr{P}}}$. Let $f\colon C/W \to \mathcal{X}_{\mathscr{P}}/\mathbb{A}^1$ be a stable log map in $\mathcal{M}^{\log}_{\tilde{\sigma},\overline{\mathbf{H}}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$. The map f is of type $\tau_{\tilde{\sigma}}$ by Lemma 4.10. We have $H^1\big(C, f^\star T^{\log}_{\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1}\big) = 0$ as C is of genus 0, so the tangent space to f in $\mathcal{M}^{\log}_{\omega,\boldsymbol{\gamma},\mathbf{H}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$ is $T_f := H^0(C, N_f)$, where $N_f := f^\star T^{\log}_{\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1}/T^{\log}_{C/W}$. As f is a union of lines $\underline{f}_v\colon \underline{C}_v \to Z_v$, one checks as in the proof of [NS06, Theorem 7.3] that $T_f = \ker \mathrm{gl}_{\tilde{\sigma}} \otimes \Bbbk$, where $\mathrm{gl}_{\tilde{\sigma}}$ is the gluing map

$$\mathrm{gl}_{\tilde{\sigma}}\colon \prod_{v\in V(\Gamma_{\tilde{\sigma}})} M \longrightarrow \prod_{E\in E(\Gamma_{\tilde{\sigma}})} M/\mathbb{Z}u_E \times \prod_{i=1}^{r} M/\gamma_i^\perp$$

as in (2.7). On the other hand, the tangent space $\overline{T}_f$ to f in $\mathcal{M}^{\log}_{\tilde{\sigma},\overline{\mathbf{H}}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$ is obtained by restricting the infinitesimal deformations to those tangent to the strata Z_v and Z_E, and so is the kernel of the restricted map $\overline{\mathrm{gl}}_{\tilde{\sigma}} \otimes \Bbbk$, where

$$\overline{\mathrm{gl}}_{\tilde{\sigma}}\colon \prod_{v\in V(\Gamma_{\tilde{\sigma}})} M_v \longrightarrow \prod_{E\in E(\Gamma_{\tilde{\sigma}})} M_E \times \prod_{i=1}^{r} M/\gamma_i^\perp\,.$$

As $\dim \tilde{\sigma} = d-2$, the map $\mathrm{gl}_{\tilde{\sigma}} \otimes \Bbbk$ is surjective by Lemma 2.19. The natural projections $M \to M_v$ and $M \to M_E$ induce a commutative diagram

$$\begin{array}{ccc} \prod_{v\in V(\Gamma_{\tilde{\sigma}})} M & \xrightarrow{\mathrm{gl}_{\tilde{\sigma}}} & \prod_{E\in E(\Gamma_{\tilde{\sigma}})} M \times \prod_{i=1}^{r} M/\gamma_i^\perp \\ \downarrow & & \downarrow \\ \prod_{v\in V(\Gamma_{\tilde{\sigma}})} M_v & \xrightarrow{\overline{\mathrm{gl}}_{\tilde{\sigma}}} & \prod_{E\in E(\Gamma_{\tilde{\sigma}})} M_E \times \prod_{i=1}^{r} M/\gamma_i^\perp\,. \end{array}$$

As the right vertical arrow of this diagram is surjective, the surjectivity of $\mathrm{gl}_{\tilde{\sigma}} \otimes \Bbbk$ implies the surjectivity of $\overline{\mathrm{gl}}_{\tilde{\sigma}} \otimes \Bbbk$. Finally, because the trivalent graph obtained from $\Gamma_{\tilde{\sigma}}$ by erasing the divalent vertices has $r+1$ legs, $r-1$ vertices, and $r-2$ edges, the difference between the dimensions of the domain and the codomain of $\overline{\mathrm{gl}}_{\tilde{\sigma}} \otimes \Bbbk$ is $2(r-1)-(r-2+r) = 0$, so $\overline{\mathrm{gl}}_{\tilde{\sigma}} \otimes \Bbbk$ is actually an isomorphism. In particular, $\overline{T}_f = 0$, so $\mathcal{M}^{\log}_{\tilde{\sigma},\overline{\mathbf{H}}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$ is reduced of dimension 0 at the point f. □

Lemma 4.12. *The log Gromov–Witten invariant $N^{\mathrm{toric}}_{\tilde{\sigma}}$ is equal to the count with automorphisms of the $(\omega,\boldsymbol{\gamma})$-marked stable log maps to $\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1$ of type $\tau_{\tilde{\sigma}}$ and matching $\overline{\mathbf{H}}$.*

Proof. By Lemma 4.11, the moduli space $\mathcal{M}^{\log}_{\tilde{\sigma},\overline{\mathbf{H}}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$ is a 0-dimensional scheme, so (4.3) implies that $\big[\mathcal{M}^{\log}_{\tilde{\sigma},\overline{\mathbf{H}}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)\big]^{\mathrm{vir}} = \big[\mathcal{M}^{\log}_{\tilde{\sigma},\overline{\mathbf{H}}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)\big]$, and therefore $N^{\mathrm{trop}}_{\tilde{\sigma}} = \deg\big[\mathcal{M}^{\log}_{\tilde{\sigma},\overline{\mathbf{H}}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)\big]$ by (4.4). By the same lemma, the space $\mathcal{M}^{\log}_{\tilde{\sigma},\overline{\mathbf{H}}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$ is also reduced, so $N^{\mathrm{trop}}_{\tilde{\sigma}}$ is equal to the count with automorphisms of points in $\mathcal{M}^{\log}_{\tilde{\sigma},\overline{\mathbf{H}}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$. □

Recall from Section 2.7 that $\tilde{\sigma}$ is contained in a $(d-2)$-dimensional face σ of $\mathcal{M}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$, that we denote by T_σ and $T_{\tilde{\sigma}}$ the integral tangent spaces to σ and $\tilde{\sigma}$, respectively, and that the natural inclusion $T_{\tilde{\sigma}} \subset T_\sigma$ is of finite index by Lemma 2.31. We can now state the log-tropical correspondence theorem for a fixed face $\tilde{\sigma}$.

THEOREM 4.13. *Fix a tropical data $(\omega, \boldsymbol{\gamma}, \mathbf{A})$ with rational $\mathbf{A}$ as in (4.1), and let $\mathscr{P}$ be a very good polyhedral decomposition for $(\omega, \boldsymbol{\gamma}, \mathbf{A})$. Let $\tilde{\sigma}$ be a $(d-2)$-dimensional face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P})$ such that $\dim \mathfrak{d}^{\tilde{\sigma}}_{L_{\mathrm{out}}} = d-1$ and $\dim \tilde{\Phi}(\tilde{\sigma}) = d-2$, and let σ be the $(d-2)$-dimensional face of $\mathcal{M}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ containing $\tilde{\sigma}$. Then, we have*

$$N^{\mathrm{toric}}_{\tilde{\sigma}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1) = \frac{N^{\mathrm{trop}}_{\sigma}}{|T_\sigma/T_{\tilde{\sigma}}|},$$

where $N^{\mathrm{toric}}_{\tilde{\sigma}}(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1)$ is the log Gromov–Witten invariant defined in (4.4) and $N^{\mathrm{trop}}_{\sigma}$ is the tropical multiplicity defined in (2.9).

Proof. By Lemma 4.12, the invariant $N^{\mathrm{toric}}_{\tilde{\sigma}}$ is equal to the count with automorphisms of the $(\omega, \boldsymbol{\gamma})$-marked stable log maps to $\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1$ of type $\tau_{\tilde{\sigma}}$ and matching $\overline{\mathbf{H}}$. Let $\underline{C}$ be the scheme-theoretic domain curve of stable log maps of type $\tau_{\tilde{\sigma}}$, with irreducible components $\underline{C}_v \simeq \mathbb{P}^1$ for $v \in V(\Gamma_{\tilde{\sigma}})$ and special points (nodes and marked points) x_E for $E \in E(\Gamma_{\tilde{\sigma}}) \cup L(\Gamma_{\tilde{\sigma}})$. Imitating the terminology of [NS06, Definition 4.3], we will first count the number of pre-log curves to $\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1$ of type $\tau_{\tilde{\sigma}}$ and matching $\overline{\mathbf{H}}$, that is, the number of scheme morphisms $\underline{f}\colon \underline{C} \to \mathcal{X}_{\mathscr{P}}$ such that

(i) $\underline{f}(\underline{C}_v) \subset Z_v$ for every $v \in V(\Gamma_{\tilde{\sigma}})$,

(ii) the only points of $\underline{C}_v$ mapped by $\underline{f}$ in the toric boundary of Z_v are the points x_E for E an edge or leg adjacent to v,

(iii) the contact order of $f(C_v)$ to Z_E at $f(x_E)$ is specified by the image $\bar{u}_{v,E}$ in $M_v = M/T_v$ of the weighted direction $u_{v,E} \in M$,

(iv) the point $f(x_E)$ is contained in the interior of the big torus orbit $\mathbb{G}_m$ of $Z_E \simeq \mathbb{P}^1$ for every $E \in E(\Gamma_{\tilde{\sigma}}) \cap L(\Gamma_{\tilde{\sigma}})$,

(v) $\{\underline{f}(x_{L_i})\} = Z_{L_i} \cap \overline{H}_i$ for every $1 \leqslant i \leqslant r$.

We will then count the number of ways to lift a pre-log curve $\underline{f}\colon \underline{C} \to \mathcal{X}_{\mathscr{P}}$ to a stable log curve $f\colon C/W \to \mathcal{X}_{\mathscr{P}}/\mathbb{A}^1$. To count the number of pre-log curves of type $\tau_{\tilde{\sigma}}$ and matching $\mathbf{H}$, we proceed by induction following the flow on $\Gamma_{\tilde{\sigma}}$ starting at the leaves and ending at the root. We use the notation introduced in the proofs of Lemmas 2.23, 2.24, and 2.30 but also applied to $\Gamma_{\tilde{\sigma}}$, so that a vertex v may also be divalent. In particular, there may be only a $\Gamma_{\tilde{\sigma}_1}$ and not both a $\Gamma_{\tilde{\sigma}_1}$ and a $\Gamma_{\tilde{\sigma}_2}$ attached to a vertex v.

Let v be a vertex of $\Gamma_{\tilde{\sigma}}$. If v is divalent, we are in a situation similar to [NS06, Proposition 5.5]: We have a $\mathbb{P}^1$-fibration structure $Z_v \to \mathbb{P}^1$ induced by the map of fans given by

$$M/T_v \longrightarrow M/\mathcal{L}^{\mathrm{sat}}_{1,v} \simeq \mathbb{Z}.$$

Recall from Section 2.7 that we set $w_v := |\bar{u}_E|$, where E is an edge adjacent to v and $\bar{u}_E$ is the image of u_E in the rank 2 lattice M/T_v. Then the vertex v corresponds to a component $\underline{C}_v$ of the pre-log curve that maps $w_v : 1$ to a fiber of $Z_v \to \mathbb{P}^1$, totally ramified over the two 1-dimensional toric strata of Z_v this fiber intersects. Because this stable map has automorphism group $\mathbb{Z}/w_v\mathbb{Z}$, its count with automorphisms is w_v^{-1}. If the child edge of v is not a leg, we deduce that $N_{\tilde{\sigma}_{\mathrm{out}}} = w_v^{-1} N_{\tilde{\sigma}_1}$. If the child edge of v is a leg L_i, then $f(x_{L_i})$ is constrained to equal the point $Z_{L_i} \cap \overline{H}_i$, so $N_{\tilde{\sigma}_{\mathrm{out}}} = w_v^{-1}$.

Next consider the case with v trivalent. First assume that E_1 and E_2 are not legs. Then Z_v is a toric surface, and the edges or legs E_1, E_2, and E_{out} determine three divisors $Z_1 = Z_{E_1}$, $Z_2 = Z_{E_2}$, and $Z_{\mathrm{out}} = Z_{E_{\mathrm{out}}}$ of Z_v. We then need to count lines, in the sense of [NS06, Definition 5.1],

that intersect the boundary of Z_v in three points, with maximal tangency order, and meeting Z_j at one of the $N_{\tilde{\sigma}_j}$ points where the pre-log curves of type $\tau_{\tilde{\sigma}_j}$ meet Z_v. Given fixed input locations, the number of such curves is

$$N_v := \left| \frac{u_{E_1}}{|u_{E_1}|} \wedge \frac{u_{E_2}}{|u_{E_2}|} \right|$$

by [NS06, Proposition 5.7]. This can be rewritten as

$$N_v := \left| M/(\mathcal{L}_{1,v}^{\mathrm{sat}} + \mathcal{L}_{2,v}^{\mathrm{sat}}) \right|. \tag{4.8}$$

Then we obtain $N_{\tilde{\sigma}_{\mathrm{out}}} = N_v N_{\tilde{\sigma}_1} N_{\tilde{\sigma}_2}$. If E_1 is an edge and E_2 is a leg, we similarly obtain $N_{\tilde{\sigma}_{\mathrm{out}}} = N_v N_{\tilde{\sigma}_1}$, and if both E_1 and E_2 are legs, we have $N_{\tilde{\sigma}_{\mathrm{out}}} = N_v$. Putting this all together, the count with automorphisms of pre-log curves of type $\tau_{\tilde{\sigma}}$ and matching $\mathbf{H}$ is

$$\Big(\prod_{v \text{ divalent}} w_v^{-1} \Big) \Big(\prod_{v \text{ trivalent}} N_v \Big). \tag{4.9}$$

We may now calculate the number of logarithmic enhancements of each pre-log curve, that is, the number of lifts of each pre-log curve to a stable log curve. We use the theory of *punctured log curves*, introduced in [ACGS25]. Examples of punctured log curves are obtained by restricting the log structure of a log curve to one of its irreducible components. Conversely, punctured log curves can be glued together to produce log curves. First note that the cones corresponding to Z_v and Z_E in the fan of $\mathcal{X}_{\mathscr{P}}$ are the cones $\mathbf{C}\mathscr{P}_v$ and $\mathbf{C}\mathscr{P}_E$ over $\mathscr{P}_v$ and $\mathscr{P}_E$, respectively. We will denote by P_v and P_E the monoids of integral points in the cones $\mathrm{Hom}(\mathbf{C}\mathscr{P}_v, \mathbb{Z})$ and $\mathrm{Hom}(\mathbf{C}\mathscr{P}_E, \mathbb{Z})$, dual to $\mathbf{C}\mathscr{P}_v$ and $\mathbf{C}\mathscr{P}_E$, respectively. Then, the monoids P_v and P_E are stalks of the ghost sheaf $\overline{\mathcal{M}}_{\mathcal{X}_{\mathscr{P}}}$ at generic points of Z_v and Z_E, respectively.

Now, consider one of the pre-log curves $\underline{f}\colon \underline{C} \to \mathcal{X}_{\mathscr{P}}$ constructed above. For every vertex $v \in V(\Gamma_{\tilde{\sigma}})$, we write $\underline{f}_v\colon \underline{C}_v \to \mathcal{X}_{\mathscr{P}}$ for the restriction of $\underline{f}$ to the irreducible component $\underline{C}_v \simeq \mathbb{P}^1$ of $\underline{C}$ corresponding to v.

Note that we can enhance the stable map $\underline{f}_v$ to a basic punctured log map $f_v\colon C_v^\circ/W_v \to \mathcal{X}_{\mathscr{P}}$ in a unique way so that the punctured points have contact orders given by the weighted tangent vectors to the edges adjacent to v (and pointing away from v). The base W_v needs to be the log point $W_v := \mathrm{Spec}(Q_v \to \Bbbk)$ with ghost sheaf Q_v given by the basic monoid. By definition, the basic monoid is the dual of the moduli space of tropical maps with domain the dual graph of $\underline{C}_v$: Here the only modulus is the location of the vertex, which is constrained to lie in $\mathbf{C}\mathcal{P}_v$ and trace out a $((d-2)+1)$-dimensional subcone $\mathcal{C}_v$ of that cone. The basic monoid Q_v is the monoid of integral points of this cone $\mathcal{C}_v$. Note that the inclusion $\mathcal{C}_v \subset \mathbf{C}\mathcal{P}_v$ induces a projection

$$\phi_v\colon P_v \longrightarrow Q_v\,. \tag{4.10}$$

Consider the curve $\underline{C}_v$, and recall that we denote by x_E the marked point corresponding to an edge or leg E adjacent to v. As in [Kat00], the curve $\underline{C}_v$ lifts uniquely to a log curve, which is log smooth over W_v with marked points x_E. We use the notation $(C_v, \mathcal{M}_{C_v})$ to denote this log curve.

On the complements of the marked points on C_v, the map $\underline{f}_v$ admits a unique lift to a log map f_v. Indeed, for a p in $\underline{C}_v$ that is not a marked point, $\underline{f}_v(p)$ lies in the interior of Z_v, so we have $\overline{\mathcal{M}}_{X_{\mathscr{P}}, \underline{f}_v(p)} = P_v$. Then, locally f_v is given by the map

$$\begin{aligned} \mathcal{M}_{X_{\mathscr{P}}, \underline{f}_v(p)} = \mathcal{O}^*_{\underline{X}_{\mathscr{P}}, f(p)} \oplus P_v &\longrightarrow \mathcal{M}_{C_v,p} = \mathcal{O}^*_{\underline{C}_v,p} \oplus Q_v \\ (s,r) &\longmapsto (\underline{f}_v^\star(s), \phi_v(r))\,. \end{aligned}$$

For a marked point x_E, we will show in a moment that locally near x_E, there exists a unique puncturing C_v° of C_v, and the map $\underline{f}_v$ admits a unique lift to a punctured log map $f_v\colon C_v^\circ \to X_\mathscr{P}$. First note that the inclusion $\mathbf{C}\mathcal{P}_v \subset \mathbf{C}\mathcal{P}_E$ induces a projection $P_E \to P_v$ whose composition with ϕ_v in (4.10) gives a map

$$\phi_{v,E}\colon P_E \longrightarrow Q_v\,. \tag{4.11}$$

Moreover, the weighted direction u_E of the edge E is an integral tangent vector to $\mathbf{C}\mathcal{P}_E$, hence can be viewed as an element in $P_E^* := \mathrm{Hom}(P_E, \mathbb{Z})$. Denote by $\mathcal{M}_{(\underline{C}_v, x_E)}$ the divisorial log structure on $\underline{C}_v$, defined by the marked point x_E viewed as a divisor. Then, for the stalk of the ghost sheaf, we have $\overline{\mathcal{M}}_{(\underline{C}_v,x_E),x_E} = \mathbb{N}$. So, from [Kat00] we have

$$\mathcal{M}_{C_v,x_E} = (\mathcal{O}^*_{\underline{C}_v,x_E} \oplus Q_v) \oplus_{\mathcal{O}^*_{\underline{C}_v,x_E}} \mathcal{M}_{(C_v,x_E),x_E}\,.$$

Let $x = 0$ be a local equation of x_E in $\underline{C}_v$. One can naturally view x as an element of $\mathcal{M}_{(C_v,x_E),x_E}$. Denote by $\mathcal{M}_{C_v^\circ,x_E}$ the submonoid of $(\mathcal{O}^*_{\underline{C}_v,x_E}\oplus Q_v)\oplus_{\mathcal{O}^*_{\underline{C}_v,x_E}} \mathcal{M}^{\mathrm{gp}}_{(C_v,x_E),x_E}$ generated by $\mathcal{O}^*_{\underline{C}_v,x_E}\oplus Q_v$ and elements of the form $\left(s, \phi_{v,E}(r), x^{u_E(r)}\right)$ for $s \in \mathcal{O}^*_{\underline{C}_v,x_E}$ and $r \in P_E$. Then, $\mathcal{M}_{C_v^\circ,x_E}$ is the unique puncturing of C_v at x_E such that locally $\underline{f}_v$ lifts to a punctured log map $f_v\colon C_v^0 \to \mathcal{X}_\mathscr{P}$. This map is given by

$$\begin{aligned}\mathcal{M}_{X_\mathscr{P},\underline{f}_v(x_E)} = \mathcal{O}^*_{\underline{X}_\mathscr{P},\underline{f}_v(x_E)} \oplus P_E &\longrightarrow \mathcal{M}_{C_v^\circ,x_E} \subset (\mathcal{O}^*_{\underline{C}_v,x_E} \oplus Q_v) \oplus_{\mathcal{O}^*_{\underline{C}_v,x_E}} \mathcal{M}^{\mathrm{gp}}_{(C_v,x_E),x_E}\\ (s,r) &\longmapsto \left(\underline{f}_v^\star(s), \phi_{v,E}(r), x^{u_E(r)}\right).\end{aligned}$$

Here we use that $\underline{f}(x_E)$ is contained in the interior of Z_E, so $\bar{\mathcal{M}}_{X_\mathscr{P},\underline{f}_v(x_E)} = P_E$.

This shows that the restriction of $\underline{f}\colon \underline{C} \to \underline{\mathcal{X}}_\mathscr{P}$ to any irreducible component $\underline{C}_v$ lifts uniquely to a punctured log map. In the remaining part of the proof, we will count the number of ways to glue these punctured maps to a stable log map $f\colon C \to \mathcal{X}_\mathscr{P}$. For this we will follow [Gro23].

We consider punctured log maps $f_v\colon C_v^\circ/W_v \to X_\mathscr{P}$ over the log points $W_v = \mathrm{Spec}(Q_v \to \Bbbk)$, which we want to glue at marked points x_E such that $\overline{\mathcal{M}}_{X_\mathscr{P},\underline{f}_v(x_E)} = P_E$. Let $Q_v^* = \mathrm{Hom}(Q_v, \mathbb{Z})$ and $P_E^* = \mathrm{Hom}(P_E, \mathbb{Z})$. As in [Gro23, Definition 4.2], there is a gluing map

$$\overline{\Psi}\colon \prod_{v\in V(\Gamma_{\tilde\sigma})} Q_v^* \times \prod_{E\in E(\Gamma_{\tilde\sigma})} \mathbb{Z} \longrightarrow \prod_{E\in E(\Gamma_{\tilde\sigma})} P_E^*\,,$$

given by

$$\overline{\Psi}\left((q_v),(\ell_E)\right) = \left(\phi^\star_{v_E,E}(q_{v_E}) + \ell_E u_E - \phi^\star_{v'_E,E}(q_{v'_E})\right)_{E\in E(\tilde\Gamma)}\,,$$

where for every vertex v adjacent to an edge E, the map $\phi^\star_{v,E}\colon Q_v^\star \to P_E^\star$ is the dual of the map $\phi_{v,E}$ in (4.11), where each edge E is oriented from endpoints v_E to v'_E and u_E is the weighted tangent vector to the edge E pointing from v_E to v'_E.

We now explain how to compare the map $\overline{\Psi}$ with the map $\Psi_{\tilde\sigma}$ in (2.29), following [Gro23, § 5.1]. The morphism $\mathcal{X}_\mathscr{P} \to \mathbb{A}^1$ induces maps $Q_v^* \to \mathbb{Z}$ and $P_E^* \to \mathbb{Z}$ with kernels isomorphic to T_v and T_E, respectively. We then obtain a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \prod_v T_v \times \prod_E \mathbb{Z} & \longrightarrow & \prod_v Q_v^* \times \prod_E \mathbb{Z} & \longrightarrow & \prod_v \mathbb{Z} & \longrightarrow & 0\\ & & \downarrow{\scriptstyle \Psi_{\tilde\sigma}} & & \downarrow{\scriptstyle \overline{\Psi}} & & \downarrow{\scriptstyle \partial} & & \\ 0 & \longrightarrow & \prod_E T_E & \longrightarrow & \prod_E P_E^* & \longrightarrow & \prod_E \mathbb{Z} & \longrightarrow & 0\,. \end{array}$$

Here the maps to $\prod \mathbb{Z}$ are induced by the above maps $Q_v^*, P_E^* \to \mathbb{Z}$, the map $\Psi_{\tilde\sigma}$ is the restriction

of $\overline{\Psi}$ to the kernels, hence the previously defined $\Psi_{\tilde{\sigma}}$, and ∂ can be viewed as the coboundary map for calculating simplicial cohomology of $\Gamma_{\tilde{\sigma}}$, so $\ker\partial = \mathbb{Z}$ and $\operatorname{coker}\partial = 0$. Thus by the snake lemma, we obtain a long exact sequence

$$0 \longrightarrow \ker\Psi_{\tilde{\sigma}} \longrightarrow \ker\overline{\Psi} \longrightarrow \mathbb{Z} \longrightarrow \operatorname{coker}\Psi_{\tilde{\sigma}} \longrightarrow \operatorname{coker}\overline{\Psi} \longrightarrow 0\,.$$

Note that $\ker\partial$ is generated by $(1,\cdots,1)$. As the kernel $\ker\overline{\Psi}$ is precisely the integral tangent space $T_{\overline{\mathbb{R}_{\geqslant 0}(\tilde{\sigma},1)}}$ to $\overline{\mathbb{R}_{\geqslant 0}(\tilde{\sigma},1)}$, the map $\ker\overline{\Psi} \to \ker\partial$ is actually surjective by Definition 4.6(ii) because we are assuming that $\mathscr{P}$ is a very good polyhedral decomposition. Thus $\operatorname{coker}\Psi_{\tilde{\sigma}} \simeq \operatorname{coker}\overline{\Psi}$.

By Lemmas 2.30 and 2.31, the cokernel $\operatorname{coker}\Psi_{\tilde{\sigma}} \simeq \operatorname{coker}\overline{\Psi}$ is finite, so we are in a tropically transverse gluing situation in the sense of [Gro23, Definition 4.8]. Hence, by [Gro23, Theorem 4.9], the moduli space W of glued stable log maps is non-empty. Therefore, one can apply [Gro23, Theorem 4.4] and obtain that W has $|\operatorname{coker}\Psi_{\tilde{\sigma}}|$ connected components. Lemma 4.11 implies that W is reduced of dimension 0, and hence there are $|\operatorname{coker}\Psi_{\tilde{\sigma}}|$ choices of log gluings.

Now by Lemmas 2.31 and 2.30, we have

$$|\operatorname{coker}\Psi_{\tilde{\sigma}}| = |\operatorname{coker}\Psi_{\sigma}|\frac{\prod_v w_v}{|T_\sigma/T_{\tilde{\sigma}}|} = \frac{\prod_{v\in\Gamma}|(\mathcal{L}_{1,v}^{\mathrm{sat}}+\mathcal{L}_{2,v}^{\mathrm{sat}})/(\mathcal{L}_{1,v}+\mathcal{L}_{2,v})|\cdot\prod_v w_v}{|T_\sigma/T_{\tilde{\sigma}}|}\,.$$

Combining this with (4.9) and (4.8) and Lemma 2.23, the total count with automorphisms of the $(\omega,\boldsymbol{\gamma})$-marked stable log maps to $\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1$ of type $\tau_{\tilde{\sigma}}$ and matching $\overline{\mathbf{H}}$ is

$$\frac{\prod_{v\in\Gamma}|M/(\mathcal{L}_{1,v}^{\mathrm{sat}}+\mathcal{L}_{2,v}^{\mathrm{sat}})|\cdot|(\mathcal{L}_{1,v}^{\mathrm{sat}}+\mathcal{L}_{2,v}^{\mathrm{sat}})/(\mathcal{L}_{1,v}+\mathcal{L}_{2,v})|}{|T_\sigma/T_{\tilde{\sigma}}|} = \frac{\prod_{v\in\Gamma}|M/(\mathcal{L}_{1,v}+\mathcal{L}_{2,v})|}{|T_\sigma/T_{\tilde{\sigma}}|} = \frac{N_\sigma^{\mathrm{trop}}}{|T_\sigma/T_{\tilde{\sigma}}|}\,.$$

This concludes the proof of Theorem 4.13. □

In the purely toric setup, a general gluing formula for log Gromov–Witten invariants is also proven in [Wu24]. However, in the proof Theorem 4.13, we used rather an explicit toric degeneration and then the gluing formula of [Gro23]. The main motivation for this choice is that we expect the explicit description of the log curves in the special fiber of the toric degeneration to be useful to define and study the higher-dimensional version of the higher-genus log Gromov–Witten invariants considered in [Bou19], which should be related to refined DT invariants.

4.4 The proof of the log-tropical correspondence

In this subsection, we prove our main log-tropical correspondence result comparing the log Gromov–Witten invariants $N_\rho^{\mathrm{toric}}(X_\Sigma)$ with the tropical multiplicities N_σ^{trop}.

Recall that choosing a $\boldsymbol{\gamma}$-constraint $\mathbf{A}$ as in Definition 2.9 amounts to choosing a point in $[\mathbf{A}] = ([A_i])_i \in \prod_{i=1}^r M_\mathbb{R}/\left(\gamma_i^\perp\right)_\mathbb{R} \simeq \mathbb{R}^r$. The constraint $\mathbf{A}^0$ defined in (3.6) corresponds to the origin $0\in\mathbb{R}^r$. We have a natural map

$$\Phi\colon \left\{\text{faces of } \mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}\right\} \longrightarrow \left\{\text{faces of } \mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}\right\} \tag{4.12}$$

defined as follows. Given a face σ of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$, the relative interior $\mathrm{Int}(\sigma)$ of σ parametrizes $(\omega,\boldsymbol{\gamma})$-tropical curves in $M_\mathbb{R}$ of a given type τ_σ matching the constraint $\mathbf{A}$. Let σ' be the face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma}}$ such that $\mathrm{Int}(\sigma')$ parametrizes $(\omega,\boldsymbol{\gamma})$-tropical curves of type τ_σ. As in (2.3), we have an evaluation

map at the legs

$$\mathrm{ev}\colon \sigma' \longrightarrow \prod_{i=1}^{r} M_{\mathbb{R}}/\big(\gamma_i^{\perp}\big)_{\mathbb{R}} \simeq \mathbb{R}^r$$

such that $\sigma = \mathrm{ev}^{-1}([\mathbf{A}])$, and we set $\Phi(\sigma) := \mathrm{ev}^{-1}(0)$, which is naturally a face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}$. Intuitively, $\Phi(\sigma)$ is obtained from σ by rescaling the point $[\mathbf{A}] \in \mathbb{R}^r$ in the space of $\boldsymbol{\gamma}$-constraints by a parameter t and then taking the limit $t \to 0$. As a face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}$ is necessarily a cone, we say that σ is *asymptotically equivalent* to the cone $\Phi(\sigma)$.

If Σ is a $\boldsymbol{\gamma}$-fan and ρ is a $(d-2)$-dimensional face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}(\Sigma)$, we denote by S_ρ the set of faces σ of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ such that $\rho \subset \Phi(\sigma)$. As $\dim \rho = d-2$ and $\mathbf{A}$ is a general $\boldsymbol{\gamma}$-constraint as in Definition 2.11, the condition $\sigma \in S_d$ automatically implies $\dim \sigma = d-2$. If in addition $\dim \mathfrak{d}^{\rho}_{L_{\mathrm{out}}} = d-1$, then $\dim \mathfrak{d}^{\sigma}_{L_{\mathrm{out}}} = d-1$, so one can consider the tropical multiplicity $N^{\mathrm{trop}}_{\sigma}$ as in (2.9) and the tropical coefficient k_σ as in (2.10).

THEOREM 4.14 (Theorem A). *Fix a tropical data $(\omega, \boldsymbol{\gamma}, \mathbf{A})$ as in (4.1) and a $\boldsymbol{\gamma}$-fan Σ as in Definition 3.5. For every $(d-2)$-dimensional face ρ of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}(\Sigma)$ such that $\dim \mathfrak{d}^{\rho}_{L_{\mathrm{out}}} = d-1$, we have*

$$k_\rho N^{\mathrm{toric}}_{\rho}(X_\Sigma) = \sum_{\sigma \in S_\rho} k_\sigma N^{\mathrm{trop}}_{\sigma}\,, \tag{4.13}$$

where $N^{\mathrm{toric}}_{\rho}(X_\Sigma)$ is the log Gromov–Witten invariant of the toric variety X_Σ as in (3.9), the $N^{\mathrm{trop}}_{\sigma}$ are the tropical multiplicities as in (2.9), and the tropical coefficients k_ρ and k_σ are defined as in (2.10).

Proof. If Σ' is a refinement of Σ, then, as in Section 2.7, the space $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}(\Sigma')$ is a polyhedral refinement of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}(\Sigma)$. If ρ' is a $(d-2)$-dimensional face of $\mathcal{M}_{\omega,\boldsymbol{\gamma}}(\Sigma')$ contained in ρ, then, by [Joh26, Theorem 1.4] on the behavior of log Gromov–Witten invariants under log-étale transformations, we have $N^{\mathrm{toric}}_{\rho'}(X_{\Sigma'}) = (1/|T_\rho/T_{\rho'}|)N^{\mathrm{toric}}_{\rho}(X_\Sigma)$, so, using (2.10), we have $k_{\rho'}N^{\mathrm{toric}}_{\rho'}(X_{\Sigma'}) = k_\rho N^{\mathrm{toric}}_{\rho}(X_\Sigma)$. Hence, it is enough to prove (4.13) for a particular $\boldsymbol{\gamma}$-fan to have the result for all $\boldsymbol{\gamma}$-fans.

As $\mathbb{Q}$ is dense in $\mathbb{R}$, we can assume without loss of generality that $\mathbf{A}$ is rational. Let $\mathscr{P}$ be a very good polyhedral decomposition of $M_{\mathbb{R}}$ for $(\omega, \boldsymbol{\gamma}, \mathbf{A})$. Such a $\mathscr{P}$ exists by Lemma 4.7. We will prove (4.13) for the $\boldsymbol{\gamma}$-fan $\Sigma_{\mathscr{P}}$ asymptotic to $\mathscr{P}$. By Theorem 4.9, we have

$$N^{\mathrm{toric}}_{\rho}(X_{\Sigma_{\mathscr{P}}}) = \sum_{\tilde{\sigma} \in \tilde{S}_\rho} N^{\mathrm{toric}}_{\tilde{\sigma}}\big(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1\big)\,,$$

where $\tilde{S}_\rho$ is the set of $(d-2)$-dimensional faces $\tilde{\sigma}$ of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P})$ such that $\tilde{\Phi}(\tilde{\sigma}) = \rho$, where $\tilde{\Phi}$ is given by (4.2). Every $(d-2)$-dimensional face $\tilde{\sigma}$ of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P})$ is contained in a unique $(d-2)$-dimensional face σ of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$, and we have

$$N^{\mathrm{toric}}_{\tilde{\sigma}}\big(\mathcal{X}_{\mathscr{P}}/\mathbb{A}^1\big) = \frac{N^{\mathrm{trop}}_{\sigma}}{|T_\sigma/T_{\tilde{\sigma}}|}$$

by Theorem 4.13. The maps $\tilde{\Phi}$ in (4.2) and Φ in (4.12) are compatible with the polyhedral refinements of the moduli spaces induced by $\mathscr{P}$ and $\Sigma_{\mathscr{P}}$, so the conditions $\tilde{\Phi}(\tilde{\sigma}) = \rho$ and $\tilde{\sigma} \subset \sigma$ imply $\rho \subset \Phi(\sigma)$. Conversely, if σ is a $(d-2)$-dimensional face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P})$ such that $\rho \subset \Phi(\sigma)$, there exists a unique $(d-2)$-dimensional face $\tilde{\sigma}$ of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}(\mathscr{P})$ such that $\Phi(\tilde{\sigma}) = \rho$. Hence, we

obtain

$$N_\rho^{\mathrm{toric}}(X_{\Sigma_{\mathscr{P}}}) = \sum_{\sigma \in S_\rho} \frac{N_\sigma^{\mathrm{trop}}}{|T_\sigma / T_{\tilde{\sigma}}|}.$$

Finally, we have $|T_\sigma/T_{\tilde{\sigma}}| = k_{\tilde{\sigma}}/k_\sigma$ by (2.10), and $k_\rho = k_{\tilde{\sigma}}$ by the definition of a very good polyhedral decomposition (Definition 4.6(ii)). □

5. Quiver DT invariants, flow trees, and log curves

In this section, after briefly reviewing in Sections 5.1 and 5.2 how to calculate quiver DT invariants tropically using attractor flow trees as in [AB22], we prove our quiver DT-log Gromov–Witten correspondence in Section 5.3.

5.1 Attractor flow trees and tropical curves

Let ω be a skew-symmetric form on N. We fix an r-tuple $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_r)$ of elements $\gamma_i \in N$ such that $\iota_{\gamma_i}\omega \neq 0$ for every $1 \leqslant i \leqslant r$ and $\iota_\gamma \omega \neq 0$, where $\gamma := \sum_{i=1}^r \gamma_i$.

Definition 5.1. A *rooted tree* is a genus 0 graph without divalent vertices and with a single univalent vertex, referred to as the *root* and denoted by R_T.

Given a rooted tree T with r legs $L_1, \dots, L_r$ decorated by $\gamma_1, \dots, \gamma_r$, for every edge or leg E of T, we define the class γ_E of E by $\gamma_E = \sum_i \gamma_i$, where the sum is over the indices $1 \leqslant i \leqslant r$ such that the leg L_i is a descendant of E, that is, such that E is contained in the path from the root to L_i.

Definition 5.2. Fix a point $\theta \in \gamma^\perp$ and a $\boldsymbol{\gamma}$-constraint $\mathbf{A}$. A parametrized $(\omega, \boldsymbol{\gamma})$-marked *attractor flow tree* with root θ and matching $\mathbf{A}$ is a pair (T, h), where

(i) T is a rooted tree with r legs decorated by $\gamma_1, \dots, \gamma_r$ such that, for every edge of leg E, we have $\iota_{\gamma_E}\omega \neq 0$, and for every vertex $v \in V(E)$ distinct from the root R_T, there exist edges or legs E_1 and E_2 adjacent to v such that $\omega(\gamma_{E_1}, \gamma_{E_2}) \neq 0$;

(ii) $h\colon T \to M_\mathbb{R}$ is a proper continuous map such that $h(R_T) = \theta$, $h(L_i) \subset A_i$ for every $1 \leqslant i \leqslant r$, and for every edge or leg E, the restriction $h|_E$ is an embedding with image contained in an affine line of oriented direction $\iota_{\gamma_E}\omega$, where the orientation is defined following the flow starting at the root of T and ending at the leaves.

An isomorphism of parameterized attractor flow tree $h\colon T \to M_\mathbb{R}$ and $h_0\colon T_0 \to M_\mathbb{R}$ is a homeomorphism $\Psi\colon T \to T_0$ respecting the marking of the legs, the weights of the edges and legs, and such that $h = h_0 \circ \Psi$. An *attractor flow tree* is an isomorphism class of parameterized attractor flow trees. We denote by $\mathscr{T}^\theta_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ the set of $(\omega, \boldsymbol{\gamma})$-marked attractor flow trees with root θ and matching $\mathbf{A}$.

Let $(h\colon T \to M_\mathbb{R})$ be an $(\omega, \boldsymbol{\gamma})$-marked attractor flow tree with root θ and matching $\mathbf{A}$. We denote by $\overline{T}$ the graph obtained from T by removing the root and viewing the edge E_{out} of T adjacent to the root as a leg L_{out} of $\overline{T}$. We extend the map $h\colon T \to M_\mathbb{R}$ to a map $\overline{h}\colon \overline{T} \to M_\mathbb{R}$ by extending the segment $h(E_{\mathrm{out}})$ to a half-line in the direction of $-\iota_\gamma\omega$. By construction, $\overline{h}\colon \overline{T} \to M_\mathbb{R}$ is an $(\omega, \boldsymbol{\gamma})$-marked tropical curve in $M_\mathbb{R}$ matching $\mathbf{A}$ and such that $\theta \in \mathrm{Int}\big(\overline{h}(L_{\mathrm{out}})\big)$.

Definition 5.3. Fix a point $\theta \in \gamma^\perp$ and a $\boldsymbol{\gamma}$-constraint $\mathbf{A}$. Let $(h\colon T \to M_\mathbb{R}) \in \mathscr{T}^\theta_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ be an $(\omega, \boldsymbol{\gamma})$-marked attractor flow tree with root θ and matching $\mathbf{A}$. The *family of tropical*

curves corresponding to h is the face ρ_h of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ whose relative interior parametrizes tropical curves with the same type as the tropical curve $\overline{h}\colon \overline{T} \to M_{\mathbb{R}}$.

LEMMA 5.4. *Fix a* γ*-constraint* $\mathbf{A}$ *and a general point* $\theta \in \left(\gamma^{\perp}\right)_{\mathbb{R}}$ *Then, for every* $(\omega,\boldsymbol{\gamma})$*-marked attractor flow tree* $(h\colon T \to M_{\mathbb{R}}) \in \mathcal{T}^{\theta}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$*, the family* ρ_h *of tropical curves corresponding to* h *satisfies the following conditions:*

(i) *For every edge or leg* E *of* Γ_{ρ_h}*, we have* $\dim \mathfrak{d}_E^{\rho_h} = d-1$.

(ii) *For every vertex* v *of* Γ_{ρ_h}*, we have* $\dim T_v\Gamma_{\rho_h} = 2$ *and* $\dim \mathfrak{j}_v^{\rho_h} = d-2$.

Proof. We prove the result by induction following the flow on Γ_{ρ_h} starting at the root leg and ending at the leaves.

For the initial step of the induction, note that $\dim \mathfrak{d}_{L_{\mathrm{out}}}^{\rho_h} = d-1$. Indeed, we have the inequality $\dim \mathfrak{d}_{L_{\mathrm{out}}}^{\rho_h} \leqslant d-1$ by Lemma 2.13(i), and if we had $\dim \mathfrak{d}_{L_{\mathrm{out}}}^{\rho_h} \leqslant d-2$, then θ would be constrained to lie in a strict hyperplane of $\left(\gamma^{\perp}\right)_{\mathbb{R}}$, in contradiction with the assumption that θ is a general point of $\left(\gamma^{\perp}\right)_{\mathbb{R}}$.

For the induction step, let E be a leg or vertex of Γ_{ρ_h} such that $\dim \mathfrak{d}_E^{\rho_h} = d-1$. Denote by v the child vertex of E and by $E_1,\dots,E_k$ the children edges of v. By Lemma 2.14, the assumption that $\dim \mathfrak{d}_E^{\rho_h} = d-1$ implies that $\dim T_v\Gamma_{\rho_h} \leqslant 2$. On the other hand, by the definition of an attractor flow tree (Definition 5.2(i)), there exist edges or legs E' and E'' adjacent to v such that $\omega(\gamma_{E'},\gamma_{E''}) \neq 0$, and so in particular $\dim T_v\Gamma_{\rho_h} \geqslant 2$. We conclude that $\dim T_v\Gamma_{\rho_h} = 2$ and so $\dim \mathfrak{j}_v^{\rho_h} = d-2$. Finally, $\dim T_v\Gamma_{\rho_h} = 2$ and $\dim \mathfrak{d}_E^{\rho_h} = d-1$ imply that $\dim \mathfrak{d}_{E_i}^{\rho_h} = d-1$ for every child edge E_i of v by Lemma 2.15. □

LEMMA 5.5. *Fix a* γ*-constraint* $\mathbf{A}$ *and a general point* $\theta \in \left(\gamma^{\perp}\right)_{\mathbb{R}}$ *Let* $(h\colon T \to M_{\mathbb{R}}) \in \mathcal{T}^{\theta}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ *be an* $(\omega,\boldsymbol{\gamma})$*-marked attractor flow tree with root* θ *and matching* $\mathbf{A}$*. Then for every edge* E *of* T*, with parent vertex* v *and child vertex* v'*, the point* $h(v')$ *in* $M_{\mathbb{R}}$ *is the unique intersection point of the half-line* $h(v) + \mathbb{R}_{\geqslant 0}\iota_{\gamma_E}\omega$ *with the affine codimension* 2 *plane* $\bigcap_{i=1}^k A_{E_j}$*, where* $E_1,\dots,E_k$ *are the children edges of* v' *and the affine hyperplanes* A_{E_j} *are defined as in* (2.6).

Proof. By Definition 5.2(ii), we have $h(v') \in h(v) + \mathbb{R}_{\geqslant 0}\iota_{\gamma_E}\omega$. On the other hand, we have $h(v') \in \bigcap_{i=1}^k A_{E_i}$ by Lemma 2.13(ii). We have $\iota_{\gamma_E}\omega \neq 0$, so $h(v) + \mathbb{R}_{\geqslant 0}\iota_{\gamma_E}\omega$ is a half-line in A_E. As $\dim T_v\Gamma_{\rho_h} = 2$ by Lemma 5.4, the intersection $\bigcap_{i=1}^k A_{E_i}$ is an affine hyperplane in A_E. Finally, as $\dim \mathfrak{d}_E^{\rho_h} = d-1$ by Lemma 5.4, the half-line $h(v) + \mathbb{R}_{\geqslant 0}\iota_{\gamma_E}\omega$ is not contained in $\bigcap_{i=1}^k A_{E_i}$, so the intersection $(h(v) + \mathbb{R}_{\geqslant 0}\iota_{\gamma_E}\omega) \cap \left(\bigcap_{i=1}^k A_{E_i}\right)$ consists of at most one point. One concludes that $(h(v) + \mathbb{R}_{\geqslant 0}\iota_{\gamma_E}\omega) \cap \left(\bigcap_{i=1}^k A_{E_i}\right) = \{h(v')\}$. □

LEMMA 5.6. *Fix a* γ*-constraint* $\mathbf{A}$ *and a general point* $\theta \in \left(\gamma^{\perp}\right)_{\mathbb{R}}$*. Let* $(h\colon T \to M_{\mathbb{R}}) \in \mathcal{T}^{\theta}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ *be an* $(\omega,\boldsymbol{\gamma})$*-marked attractor flow tree with root* θ *and matching* $\mathbf{A}$*. Let* v_{out} *be the unique vertex of* Γ_{ρ_h} *adjacent to the root leg* L_{out}*. Then, the evaluation map*

$$\rho_h \longrightarrow \mathfrak{j}_{v_{\mathrm{out}}}^{\rho_h}, \quad \overline{h} \longmapsto \overline{h}(v_{\mathrm{out}})$$

is a bijection. In particular, the dimension of the family ρ_h *of tropical curves corresponding to* h *is* $d-2$.

Proof. Following the flow on Γ_{ρ_h} starting at the root leg and ending at the leaves, $\overline{h}$ is inductively uniquely determined by $\overline{h}(v_{\mathrm{out}})$ as in Lemma 5.5. Moreover, we have $\dim \mathfrak{j}_{v_{\mathrm{out}}}^{\rho_h} = d-2$ by Lemma 5.4(ii). □

5.2 The flow tree formula

Let (Q,W) be a quiver with potential as in Section 1.1. Let ω be the corresponding skew-symmetric form on $N=\mathbb{Z}^{Q_0}$ as in (1.3). We fix a dimension vector $\gamma\in N$ and a general stability parameter $\theta\in\left(\gamma^\perp\right)_{\mathbb{R}}$. As reviewed in (1.5), the DT invariant Ω^θ_γ can be expressed in terms of the simpler attractor DT invariants in terms of universal coefficients $F^\theta_r(\gamma_1,\dots,\gamma_r)$ indexed by decompositions $\gamma=\sum_{i=1}^r\gamma_i$. These coefficients further decompose as

$$F^\theta_r(\gamma_1,\dots,\gamma_r)=\sum_{h\in\mathcal{T}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}}F^\theta_{r,h}(\gamma_1,\dots,\gamma_r)\,, \tag{5.1}$$

where $\boldsymbol{\gamma}=(\gamma_1,\dots,\gamma_r)$, $\mathbf{A}^0$ is the $\boldsymbol{\gamma}$-constraint defined by the linear hyperplanes $\left(\gamma_i^\perp\right)_{\mathbb{R}}$, and the sum is over the $(\omega,\boldsymbol{\gamma})$-marked attractor flow trees with root θ and matching $\mathbf{A}^0$ as in Definition 5.2. The decomposition (5.1) is obtained by iterative use of the wall-crossing formula. In general, the attractor flow trees $h\in\mathcal{T}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}$ may contain vertices of arbitrary high valency, as in [KS14, § 3]. In this section, following [AB22], we reformulate the flow tree formula of [AB22] expressing the coefficients $F^\theta_{r,h}(\gamma_1,\dots,\gamma_r)$ in terms of simpler trivalent tropical curves matching constraints $\mathbf{A}$ that are generic small perturbations of $\mathbf{A}^0$.

For a $\boldsymbol{\gamma}$-constraint $\mathbf{A}$, recall from (4.12) that we have a map

$$\Phi\colon\left\{\text{faces of }\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}\right\}\longrightarrow\left\{\text{faces of }\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}\right\}. \tag{5.2}$$

For every face σ of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$, the image $\Phi(\sigma)$ is the face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}$ obtained from σ by rescaling $\mathbf{A}$ to $\mathbf{A}^0$ in the space $\prod_{i=1}^r M_{\mathbb{R}}/\left(\gamma_i^\perp\right)_{\mathbb{R}}\simeq\mathbb{R}^r$ of $\boldsymbol{\gamma}$-constraints. Finally, recall that for every attractor tree $h\in\mathcal{T}^\theta_{\omega,\boldsymbol{\gamma},\mathbf{A}}$, we defined in Definition 5.3 the corresponding family of tropical curves ρ_h, which is a face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}$. By Lemma 5.6, we have $\dim\rho_h=d-2$. Hence, if $\mathbf{A}$ is a general $\boldsymbol{\gamma}$-constraint as in Definition 2.11 and σ is a face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ such that $\Phi(\sigma)=\rho_h$, then $\dim\sigma=d-2$ and Γ_σ is trivalent.

Theorem 5.7. *Fix a general point $\theta\in\left(\gamma^\perp\right)_{\mathbb{R}}$ and an attractor flow tree $(h\colon T\to M_{\mathbb{R}})\in\mathcal{T}^\theta_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}$. Then, for every general $\boldsymbol{\gamma}$-constraint $\mathbf{A}$, we have*

$$F^\theta_{r,h}(\gamma_1,\dots,\gamma_r)=\sum_{\sigma\in\Phi^{-1}(\rho_h)}\prod_{v\in V(\Gamma_\sigma)}\left|\omega(\gamma_{E_{1,v}},\gamma_{E_{2,v}})\right|, \tag{5.3}$$

where $E_{1,v}$ and $E_{2,v}$ are the two children edges of v for every vertex $v\in V(\Gamma_\sigma)$.

Proof. We explain below how Theorem 5.7 follows from the flow tree formula given in [AB22, Theorem 5.6]. In [AB22], we introduce a bigger lattice $\mathcal{N}:=\bigoplus_{i=1}^r\mathbb{Z}e_i$, and we consider the map $p\colon\mathcal{N}\to N$ defined by $e_i\mapsto\gamma_i$. We also define the pullback skew-symmetric form η on $\mathcal{N}$ by $\eta(e_i,e_j):=\omega(\gamma_i,\gamma_j)$. By duality, we have a map $q\colon M_{\mathbb{R}}\to\mathcal{M}_{\mathbb{R}}=\mathrm{Hom}(\mathcal{N},\mathbb{R})$. Let $\tilde\theta$ be a small generic perturbation of $q(\theta)$ in $\left(\sum_{i=1}^r e_i\right)^\perp$. Then, [AB22, Theorem 5.6] states that

$$[F^\theta_r(\gamma_1,\dots,\gamma_r):=\sum_{T_r}\prod_{v\in V^\circ_{T_r}}-\epsilon^{\tilde\theta,\omega}_{T_r,v}\eta(e_{v'},e_{v''}), \tag{5.4}$$

where the sum is over rooted binary trees T_r with r leaves decorated by $\{e_1,\dots,e_r\}$; $V^\circ_{T_r}$ denotes the set of vertices of T_r distinct from the root; for any $v\in V^\circ_{T_r}$, $e_v\in\mathcal{N}$ is the sum of the e_i attached to leaves descendant from v; and $\epsilon^{\tilde\theta,\omega}_{T_r,v}\in\{-1,0,1\}$ is defined in [AB22, equation (1.11)] in terms of the discrete attractor flow [AB22, equation (1.10)] embedding the trees T_r in $\mathcal{M}_{\mathbb{R}}$. Note that [AB22, Theorem 5.6] is actually stated for the refined DT invariants, in which case

$F_r^\theta(\gamma_1,\dots,\gamma_r)$ is a function of an additional variable y, and we obtained (5.4) by taking the limit $y \to -1$ to recover the case of numerical DT invariants $\Omega_\gamma^{+,\theta}$ defined in (1.1) (the limit $y \to 1$ corresponds to the signed DT invariants Ω_γ^θ).

As explained in [AB22, Remark 4.25], the trees embedded in $\mathcal{M}_\mathbb{R}$ by the discrete attractor flow are contained in a copy of $M_\mathbb{R}$ in $\mathcal{M}_\mathbb{R}$. Moreover, it follows from the proof of [AB22, equation (4.57)] and from Lemma 5.5 that these embedded trees are exactly attractor flow trees as in Definition 5.2, with root at a general point $\tilde\theta$ of $A_{L_{\mathrm{out}}}$ close enough to θ and matching a small generic perturbation $\mathbf{A}$ of the constraint $\mathbf{A}^0$ if and only if $\epsilon_{T_r,v}^{\tilde\theta,\omega} \neq 0$, and that in this case, $-\epsilon_{T_r,v}^{\tilde\theta,\omega}$ has the same sign as $\eta(e_{v'},e_{v''})$ (see the paragraph above [AB22, equation (4.52)]). In particular, a tree T_r with a non-zero contribution to (5.4) contributes $\prod_{v\in V_{T_r}^\circ}|\eta(e_{v'},e_{v''})|$, so (5.3) follows from (5.4). □

Remark 5.8. Theorem 5.7 could also be deduced from [CM20, Theorem 4.4], which expresses the functions attached to the walls of a scattering diagram in terms of tropical disks. The proof of [AB22, Theorem 5.6] is also based on the study of a scattering diagram in $\mathcal{M}_\mathbb{R}$ that can be viewed as a universal family for the perturbed scattering diagrams in $M_\mathbb{R}$ considered in [CM20, Theorem 4.4] (see [AB22, Remark 4.25]).

5.3 The quiver DT-log Gromov–Witten correspondence

Let (Q,W) be a quiver with potential as in Section 1.1. Let ω be the corresponding skew-symmetric form on $N=\mathbb{Z}^{Q_0}$. We fix a dimension vector $\gamma\in N$ such that $\iota_\gamma\omega\neq 0$ and a decomposition $\gamma=\sum_{i=1}^r\gamma_i$ such that $\iota_{\gamma_i}\omega\neq 0$ for all $1\leqslant i\leqslant r$. We set $\boldsymbol{\gamma}=(\gamma_1,\dots,\gamma_r)$. We also fix a general stability parameter $\theta\in(\gamma^\perp)_\mathbb{R}$. In this section, we prove our main result relating the coefficients $F_r^\theta(\gamma_1,\dots,\gamma_r)$ in (1.5) and (1.6), expressing DT invariants in terms of attractor DT invariants and log Gromov–Witten invariants of toric varieties.

Recall that given an attractor flow tree $(h\colon T\to M_\mathbb{R})\in\mathcal{T}^\theta_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}$, we defined in Definition 5.3 the corresponding family of tropical curves ρ_h: It is a face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}$ such that $\dim\rho=d-2$ by Lemma 5.6 and $\dim\mathfrak{d}^{\rho_h}_{L_{\mathrm{out}}}=d-1$ by Lemma 5.4. Given a $\boldsymbol{\gamma}$-fan Σ as in Definition 3.5, we denote by $\widetilde{\rho}_h$ the $(d-2)$-dimensional face of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}(\Sigma)$ contained in ρ_h and containing θ. In particular, one can consider as in (3.9) the genus 0 log Gromov–Witten invariant $N^{\mathrm{toric}}_{\widetilde{\rho}_h}(X_\Sigma)$ of the toric variety X_Σ of fan Σ and the tropical coefficient $k_{\widetilde{\rho}_h}$ defined as in (2.10). By Theorem 3.8, the invariant $N^{\mathrm{toric}}_{\widetilde{\rho}_h}(X_\Sigma)$ is actually an enumerative count of log curves.

Theorem 5.9 (Theorem B). *Fix a general stability parameter $\theta\in(\gamma^\perp)_\mathbb{R}$ and an attractor flow tree $(h\colon T\to M_\mathbb{R})\in\mathcal{T}^\theta_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}$. Then, for every $\boldsymbol{\gamma}$-fan Σ, the coefficient $F_r^\theta(\gamma_1,\dots,\gamma_r)$ in (1.5) and (1.6) satisfies*

$$F^\theta_{r,h}(\gamma_1,\dots,\gamma_r)=\frac{\prod_{i=1}^r|\gamma_i|}{|\gamma|}k_{\widetilde{\rho}_h}N^{\mathrm{toric}}_{\widetilde{\rho}_h}(X_\Sigma)\,. \tag{5.5}$$

Proof. Fix a general $\boldsymbol{\gamma}$-constraint $\mathbf{A}$, which exists by Lemma 2.12. Then, Theorem 5.7 expresses the coefficient $F^\theta_{r,h}(\gamma_1,\dots,\gamma_r)$ in terms of the $(d-2)$-dimensional faces σ of the moduli space $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ of $(\omega,\boldsymbol{\gamma})$-marked tropical curves matching $\mathbf{A}$ as

$$F^\theta_{r,h}(\gamma_1,\dots,\gamma_r)=\sum_{\sigma\in\Phi^{-1}(\rho_h)}\prod_{v\in V(\Gamma_\sigma)}|\omega(\gamma_{E_{1,v}},\gamma_{E_{2,v}})|\,.$$

Using the product formula for tropical multiplicities in Lemma 2.24, we can rewrite this in terms

of the tropical multiplicities N_σ^{trop} defined in (2.9) and of the tropical coefficients k_σ defined in (2.10), as

$$F_{r,h}^{\theta}(\gamma_1,\dots,\gamma_r) = \frac{\prod_{i=1}^r |\gamma_i|}{|\gamma|} \sum_{\sigma\in\Phi^{-1}(\rho_h)} k_\sigma N_\sigma^{\mathrm{trop}}\,. \tag{5.6}$$

On the other hand, by the log-tropical correspondence of Theorem 4.14, we have

$$k_{\widetilde{\rho}_h} N_{\widetilde{\rho}_h}^{\mathrm{toric}}(X_\Sigma) = \sum_{\sigma\in S_{\widetilde{\rho}_h}} k_\sigma N_\sigma^{\mathrm{trop}}\,, \tag{5.7}$$

where $S_{\widetilde{\rho}_h}$ is the set of faces σ of $\mathcal{M}^{\mathrm{trop}}_{\omega,\boldsymbol{\gamma},\mathbf{A}}$ such that $\widetilde{\rho}_h \subset \Phi(\sigma)$. As ρ_h is the $(d-2)$-dimensional face of $\mathcal{M}_{\omega,\boldsymbol{\gamma},\mathbf{A}^0}$ containing $\widetilde{\rho}_h$, the set $\Phi^{-1}(\rho_h)$ of faces σ such that $\Phi(\sigma)=\rho_h$ coincides with the set $S_{\tilde{\rho}_h}$ of faces σ such that $\tilde{\rho}_h \subset \Phi(\sigma)$, so (5.5) follows by combining (5.6) and (5.7). □

Appendix. Generic log smoothness

The classical generic smoothness (or Bertini–Sard) theorem states that given two varieties X and Y over a field $\Bbbk$ of characteristic 0 and a morphism $f\colon X\to Y$, if X is smooth over $\Bbbk$, then there exists an open dense subset U of Y such that for every $u\in U$, the fiber $f^{-1}(u)$ is smooth over $\Bbbk$ (see for example [Har77, Corollary 10.7]). In the following theorem, we give a logarithmic version of this result.

Theorem A.1. *Let $\Bbbk$ be a field of characteristic 0. Let $f\colon X\to Y$ be a morphism between fine and saturated log schemes over $\Bbbk$. Assume that X is log smooth over the trivial log point $(\operatorname{Spec}\Bbbk,\Bbbk^\star)$, that the scheme underlying Y is integral, and that the log structure on Y is generically trivial. Then, there exists an open dense subset U of Y such that for every $u\in U$, the fiber $f^{-1}(u)$, endowed with the log structure restricted from X, is log smooth over the trivial log point $(\operatorname{Spec}\Bbbk(u),\Bbbk(u)^\star)$, where $\Bbbk(u)$ is the residue field of u.*

Proof. As it is log smooth over the trivial log point $(\operatorname{Spec}\Bbbk,\Bbbk^\star)$, the variety X is log regular in the sense of [Kat94, equation (2.1)] by [Kat94, Proposition 8.3(1)]. By [Kat94, Proposition 7.2], the localization of a log regular log scheme is log regular. So, the generic fiber X_η of X over the generic point η of Y is also log regular. As $\Bbbk$ is of characteristic 0, the function field $\Bbbk(\eta)$ of Y is also of characteristic 0, so X_η being log regular implies that X_η is log smooth over the trivial log point $(\operatorname{Spec}\Bbbk(\eta),\Bbbk(\eta)^\star)$ by [Kat94, Proposition 8.3(2)]. As the log structure on Y is assumed to be generically trivial, $(\operatorname{Spec}\Bbbk(\eta),\Bbbk(\eta)^\star)$ is the generic point of Y endowed with the log structure restricted from Y, so there exists an open dense subset V of Y such that $f|_{f^{-1}(V)}$ is log smooth by the generic smoothness result of [Lod13, § 1.8]. On the other hand, applying [Lod13, § 1.8] to Y over $(\operatorname{Spec}\Bbbk,\Bbbk^\star)$, one deduces that there exists an open dense subset W of Y such that the log structure of Y is trivial in restriction to W. The intersection $U:=V\cap W$ is an open dense subset of Y with trivial log structure such that $f|_{f^{-1}(U)}$ is log smooth. Hence, for every $u\in U$, the fiber $f^{-1}(u)$ is log smooth over the trivial log point $(\operatorname{Spec}\Bbbk(u),\Bbbk(u)^\star)$. □

Acknowledgements

Some of the material in this paper was originally written with the intention of appearing as part of [AG22] in an earlier form. We thank Mark Gross for his generosity in letting us use this material. We also thank the anonymous referee for the time they spared to carefully read our manuscript and for the very useful comments they provided toward its improvement.

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