



# Automorphism groups of rigid affine surfaces: The identity component

Alexander Perepechko and Mikhail Zaidenberg

## ABSTRACT

It is known that the identity component of the automorphism group of a projective algebraic variety is an algebraic group. This is not true in general for quasi-projective varieties. In this note, we address the question as to when, given an affine algebraic surface  $Y$ , the identity component  $\text{Aut}^\circ(Y)$  of the automorphism group  $\text{Aut}(Y)$  is an algebraic group. We show that this occurs if and only if  $Y$  admits no effective action of the additive group  $\mathbb{G}_a$  of the field. In the latter case,  $\text{Aut}^\circ(Y)$  is an algebraic torus of rank at most 2.

## 1. Introduction

Let  $\mathbb{K}$  be an algebraically closed field of characteristic zero and  $\mathbb{G}_a$  (respectively,  $\mathbb{G}_m$ ) be the additive (respectively, the multiplicative) group of  $\mathbb{K}$ . If  $Y$  is an affine variety over  $\mathbb{K}$ , then the automorphism group  $\text{Aut}(Y)$  has a canonical structure of an ind-group equipped with its Zariski ind-topology; see for example [FK18]. In this paper, we concentrate on the identity component  $\text{Aut}^\circ(Y)$  of  $\text{Aut}(Y)$  with respect to the Zariski topology. Notice that, similarly to the case of a projective variety, the group of components  $\text{Aut}(Y)/\text{Aut}^\circ(Y)$  is countable; see [FK18, Proposition 1.7.1]. We address the following conjecture. We say that  $Y$  is *rigid* if  $Y$  admits no effective  $\mathbb{G}_a$ -action or, in other words,  $\text{Aut}^\circ(Y)$  contains no  $\mathbb{G}_a$ -subgroup, that is, no non-trivial unipotent element.

**CONJECTURE 1.1.** *Let  $Y$  be an affine algebraic variety over  $\mathbb{K}$ . If  $Y$  is rigid, then the group  $\text{Aut}^\circ(Y)$  is an algebraic torus of rank at most  $\dim Y$ .*

If Conjecture 1.1 is true, then the following one is true as well. (We thank Hanspeter Kraft, who suggested the second conjecture.)

**CONJECTURE 1.2.** *If an affine variety  $Y$  admits no effective  $\mathbb{G}_a$ - and  $\mathbb{G}_m$ -actions, then  $\text{Aut}(Y)$  is a discrete group.*

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Some partial results on these conjectures can be found for example in [AG17, Theorem 2.1], [Iit77, Proposition 5], [Jel15, JL14], [KPZ11, Theorems 4.4 and 4.7], [Kra17, Theorem 1.3 and Section 7] and [PR23, PR24]. Conjecture 1.1 holds, for instance, for toric affine varieties and rational affine varieties with a torus action of complexity 1; see [BG21, Theorem 3] and [BG25, Theorem 6.4]. In the present paper, we establish Conjecture 1.1 in the case  $\dim Y = 2$ .

**THEOREM 1.3.** *Let  $Y$  be a normal affine surface over  $\mathbb{K}$ . Then the following hold:*

- (a) *The identity component  $\text{Aut}^\circ(Y)$  of  $\text{Aut}(Y)$  is an algebraic group if and only if  $Y$  admits no effective  $\mathbb{G}_a$ -action, which holds if and only if  $\text{Aut}^\circ(Y)$  is an algebraic torus of rank at most 2.*
- (b) *Let  $(X, D)$  be a minimal completion of  $Y$  by a normal crossing divisor. Then  $\text{Aut}^\circ(Y)$  is an algebraic group if and only if every extremal linear segment of the weighted dual graph  $\Gamma(D)$  is admissible.*

See Definition 3.40 for the notion of a weighted dual graph. An *extremal linear segment* of a weighted graph  $\Gamma$  is a maximal linear subgraph that carries no branch vertex of  $\Gamma$  and contains a tip of  $\Gamma$ ; see Definition 3.4. It is *admissible* if all its weights are at most  $-2$ ; see Definition 4.1.

As a corollary, we establish the following combinatorial criterion of rigidity; cf. Corollary 6.2.

**COROLLARY 1.4.** *In the setup of Theorem 1.3(b), the surface  $Y$  is rigid if and only if every extremal linear segment of  $\Gamma(D)$  is admissible.*

Using Corollary 1.4, we show that the affine surface  $Y = \mathbb{P}^2 \setminus \text{supp } D$ , where  $D$  is a reduced effective divisor on  $\mathbb{P}^2$  with only nodes as singularities, is rigid if and only if  $\deg(D) \geq 3$ ; see Example 6.3.

Statement (a) of Theorem 1.3 was announced in [KPZ11, Proposition 4.7]. Our approach goes back to the work of V. I. Danilov and M. H. Gizatullin [GD75]. Namely, the automorphism group  $\text{Aut}(Y)$  can be realized as a group of birational transformations between the NC-completions of  $Y$ , that is, the completions  $(X, D)$  with a normal projective surface  $X$  and a normal crossing boundary divisor  $D$  contained in the smooth locus of  $X$ .

The opposite of rigidity is the property of generic flexibility. One says that  $Y$  is *generically flexible* if the subgroup  $\text{SAut}(Y) \subset \text{Aut}(Y)$  generated by all the  $\mathbb{G}_a$ -subgroups of  $\text{Aut}(Y)$  acts on  $Y$  with an open orbit. By a celebrated Gizatullin theorem, see [Giz71b], extended by A. Dubouloz to normal affine surfaces, see [Dub04], a normal affine surface  $Y$  non-isomorphic to  $\mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})$  is generically flexible if and only if the dual graph  $\Gamma(D)$  of a minimal completion  $(X, D)$  of  $Y$  by a normal crossing divisor  $D$  is linear.

In the proofs of our main results, our tools lie in the birational geometry of weighted graphs. Different aspects of this subject were developed, for example, in [Dai03, Dai08, GD75, GD77, EN85, FKZ07, FZ94, Fuj82, Giz71a, Hir95, Neu89, OZ93, Ram71, Rus02]. We consider, more generally, weighted graphs with a distinguished subset of marked vertices called *rational*. In the case where our graph  $\Gamma$  is the dual graph of a finite collection of curves on a surface, the rational vertices of  $\Gamma$  correspond to rational curves.

Section 2 contains preliminaries on NC-completions and their birational transformations. Section 3 deals with dual graphs and birational transformations of weighted graphs. For the sake of completeness, we provide detailed proofs of some known results.

In Section 4.2, we discuss birationally rigid weighted graphs. A weighted graph  $\Gamma$  is *minimal* if it has no rational vertex of degree at most 2 and of weight  $-1$ , and *birationally rigid* if it

is minimal and any birational transformation of weighted graphs  $\Gamma \dashrightarrow \Gamma'$  is an isomorphism, provided that  $\Gamma'$  also is minimal; see Definition 4.11.

In Section 5, we show that the rigidity of an affine surface can be read from the weighted dual graph of a minimal NC-completion of this surface.

The proof of Theorem 1.3 is done in Section 6; see Theorem 6.1. In the appendix, we classify all connected weighted graphs with an infinite number of minimal models; see Theorem A.8.

Let us indicate the main ingredients in the proof of Theorem 1.3.

- It is well known that any birational transformation between smooth surfaces can be decomposed into a sequence of blowups and blowdowns. The theory of birational transformations of weighted graphs is parallel to that of birational transformations of NC-pairs; see Definition 2.1.
- Namely, to every birational transformation of an NC-pair  $(X, D)$ , there corresponds a birational transformation of the dual graph  $\Gamma(D)$  and vice versa; see Proposition 3.44 and Lemma 3.47.
- The group  $\text{Aut}(Y)$  of automorphisms of a normal affine surface  $Y$  contains a  $\mathbb{G}_a$ -subgroup if and only if  $Y$  admits a minimal NC-completion  $(X, D)$  whose dual graph  $\Gamma(D)$  has a tip of weight zero; see Proposition 5.4 and its proof.
- An extremal linear segment  $S$  of a minimal weighted graph  $\Gamma$  is non-admissible if and only if  $S$  is birationally equivalent to a linear graph with a tip of weight zero; see for example [FKZ07, Examples 2.11].
- These results imply that  $Y = X \setminus D$  is rigid if and only if the dual graph  $\Gamma(D)$  contains no non-admissible extremal linear segment; see Corollary 1.4.
- A blowup of an NC-pair  $(X, D)$  is called *inner* if its center is a node of  $D$ . A blowdown is *inner* if it is the inverse of an inner blowup. A birational transformation is *inner* if it is composed of inner blowups, inner blowdowns and isomorphisms; see Definitions 2.4–2.5.
- Let  $\text{Bir}(X, D)$  stand for the subgroup of  $\text{Bir}(X)$  of birational transformations biregular on  $Y = X \setminus D$ . Then the inner birational transformations of  $(X, D)$  form a subgroup of  $\text{Bir}(X, D)$  denoted by  $\text{Inn}(X, D)$ .
- Similar definitions can be applied for birational transformations of weighted graphs; see Definitions 3.6 and 3.10. To every inner birational transformation of an NC-pair  $(X, D)$ , there corresponds a unique inner birational transformation of the dual graph  $\Gamma(D)$  and vice versa; see Proposition 3.44.
- If a minimal weighted graph  $\Gamma$  has only admissible extremal linear segments, then any birational transformation of  $\Gamma$  can be replaced by an inner birational transformation; see Proposition 4.15(c).
- Let  $(X, D)$  be a minimal NC-completion of a rigid affine surface  $Y$ . Then the identity component  $\text{Aut}^\circ(Y) =: \text{Bir}^\circ(X, D)$  of  $\text{Aut}(Y)$  coincides with the identity component  $\text{Inn}^\circ(X, D)$  of the group of inner birational transformations of the pair; see Definitions 2.4 and 2.5 and Theorem 4.17.
- In turn, the identity component  $\text{Inn}^\circ(X, D)$  coincides with the affine algebraic subgroup  $\text{Aut}^\circ(X, D)$ ; see Proposition 2.14.
- Because of the absence of  $\mathbb{G}_a$ -subgroups, the connected affine algebraic group  $\text{Aut}^\circ(Y) = \text{Aut}^\circ(X, D)$  must be an algebraic torus.

This proves Theorem 1.3. As a side result we obtain that for a rigid normal affine surface  $Y$ , the group  $\text{Aut}^\circ(X, D)$  does not depend on the choice of a minimal NC-completion  $(X, D)$  of  $Y$ .

## 2. NC-pairs

### 2.1 Birational transformations of NC-pairs

DEFINITION 2.1. Let  $X$  be a normal projective surface over  $\mathbb{K}$  and  $D$  be a reduced effective divisor on  $X$ . The pair  $(X, D)$  is called an *NC-pair* if  $D$  is a normal crossing divisor (that is, the only singularities of  $D$  are nodes) contained in the smooth locus of  $X$ . An NC-pair  $(X, D)$  is called an *SNC-pair* if  $D$  is a simple normal crossing divisor, that is, each component of  $D$  is smooth and any two components of  $D$  either are disjoint or intersect at a single point. An NC-pair  $(X, D)$  is called *minimal* if the contraction of a  $(-1)$ -curve on  $X$  that is a component of  $D$  does not lead to a new NC-pair.

Clearly, an NC-pair  $(X, D)$  is not an SNC-pair if and only if  $D$  has either a nodal component or a pair of components with more than one point in common. As an example, one can consider the pair  $(\mathbb{P}^2, C)$ , where  $C \subset \mathbb{P}^2$  is a nodal cubic curve. Any NC-pair is dominated by an SNC-pair via a birational morphism. For instance, in the example above such a domination is obtained after blowing  $\mathbb{P}^2$  up in the node of  $C$  and performing another blowup in a node of the resulting curve; cf. Example 3.41.

DEFINITION 2.2. Let  $(X, D)$  and  $(X', D')$  be NC-pairs. We denote by

- (a)  $\text{Bir}((X, D), (X', D'))$  the set of birational transformations  $X \dashrightarrow X'$  that restrict to biregular isomorphisms of the complements  $X \setminus \text{supp } D \xrightarrow{\simeq} X' \setminus \text{supp } D'$ ;
- (b)  $\text{Bir}(X, D)$  the group  $\text{Bir}((X, D), (X, D))$ ;
- (c)  $\text{Aut}(X, D)$  the group of biregular automorphisms of  $X$  that preserve  $D$ ;
- (d)  $\text{Aut}^\sharp(X, D)$  the subgroup of  $\text{Aut}(X, D)$  of automorphisms preserving every component and every node of  $D$ ;
- (e)  $\text{Aut}^\circ(X, D)$  the identity component of  $\text{Aut}(X, D)$ .

REMARK 2.3. The group  $\text{Bir}(X, D)$  is naturally isomorphic to the automorphism group  $\text{Aut}(Y)$  of the complement  $Y = X \setminus \text{supp } D$ . Indeed, for  $g \in \text{Bir}(X, D)$  the restriction  $g|_Y$  is an automorphism of  $Y$ , and any  $f \in \text{Aut}(Y)$  extends to a unique birational transformation  $(X, D) \dashrightarrow (X, D)$ .

Thus, the group  $\text{Bir}(X, D) = \text{Aut}(Y)$  does not depend on the choice of an NC-completion  $(X, D)$  of  $Y = X \setminus \text{supp } D$ . However, in general, the group  $\text{Aut}(X, D)$  depends on the chosen NC-completion. For instance, for the completion  $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$  of the affine plane, we have  $\text{Aut}(\mathbb{P}^2, \mathbb{P}^1) \simeq \text{Aff}(\mathbb{A}^2)$ , whereas for the completion  $\mathbb{A}^2 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$  we have

$$\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^1 \times \{\infty\} \cup \{\infty\} \times \mathbb{P}^1) \cong (\text{Aff}(\mathbb{A}^1))^2 \rtimes \mathbb{Z}_2.$$

### 2.2 Inner transformations

DEFINITION 2.4. A blowup of an NC-pair  $(X, D)$  with center  $x \in \text{supp } D$  again gives an NC-pair. Both pairs provide NC-completions of  $Y = X \setminus \text{supp } D$ . Such a blowup is called *inner* if  $x$  is a node of  $D$  and *outer* otherwise. We say that a blowdown of a component of  $D$ , producing a new NC-pair, is *inner* (respectively, *outer*) if the inverse blowup is inner (respectively, outer).

DEFINITION 2.5. A birational transformation  $(X, D) \dashrightarrow (X', D')$  between two NC-pairs is called *inner* if it can be decomposed into a sequence of inner blowups, inner blowdowns and

isomorphisms of NC-pairs. The subset of inner birational transformations forms a subgroup  $\text{Inn}(X, D) \subset \text{Bir}(X, D) = \text{Aut}(Y)$ .

LEMMA 2.6. *We have*

$$\text{Aut}^\circ(X, D) \subset \text{Aut}^{\natural}(X, D) \subset \text{Aut}(X, D) \subset \text{Inn}(X, D) \subset \text{Bir}(X, D) = \text{Aut}(Y).$$

*Proof.* This is immediate from our definitions.  $\square$

LEMMA 2.7. *The group  $\text{Aut}^\circ(X, D) = (\text{Aut}^{\natural}(X, D))^\circ$  is a (connected) algebraic group.*

*Proof.* It is well known that  $\text{Aut}^\circ(X)$  is a connected algebraic group, see for example [MO67, Mat58, Ram64], and so is its closed connected subgroup  $\text{Aut}^\circ(X, D)$ .  $\square$

LEMMA 2.8. *Every  $\phi \in \text{Inn}(X, D)$  can be written as*

$$\phi = \alpha \sigma_n^{\pm 1} \cdots \sigma_1^{\pm 1} = \sigma'_n{}^{\pm 1} \cdots \sigma'_1{}^{\pm 1} \beta, \quad (2.1)$$

where  $\alpha, \beta \in \text{Aut}(X, D)$  and the  $\sigma_i, \sigma'_i$  are inner blowdowns.

*Proof.* Consider a commutative diagram

$$\begin{array}{ccc} (X_1, D_1) & \xrightarrow{\sigma^{\pm 1}} & (X_2, D_2) \\ \alpha_1 \downarrow \simeq & & \simeq \downarrow \alpha_2 \\ (X'_1, D'_1) & \xrightarrow{\sigma'^{\pm 1}} & (X'_2, D'_2), \end{array}$$

where  $\sigma$  and  $\sigma'$  are inner blowdowns. Given a pair  $(\sigma', \alpha_1)$ , there exists a unique pair  $(\sigma, \alpha_2)$  fitting in the diagram, and vice versa. Now the claim follows.  $\square$

LEMMA 2.9. *Every inner birational transformation  $\phi: (X, D) \dashrightarrow (X', D')$  of NC-pairs induces an isomorphism*

$$\phi_*: \text{Aut}^{\natural}(X, D) \xrightarrow{\simeq} \text{Aut}^{\natural}(X', D'), \quad \alpha \mapsto \phi \circ \alpha \circ \phi^{-1}.$$

*In particular,  $\text{Aut}^{\natural}(X, D)$  is a normal subgroup of  $\text{Inn}(X, D)$ .*

*Proof.* The statement is definitely true for inner blowups and blowdowns. Hence it holds in general.  $\square$

Remark 2.10. The group  $\text{Aut}^{\natural}(X, D)$  is the kernel of the natural homomorphism  $\text{Aut}(X, D) \rightarrow \text{Aut}(\Gamma(D))$ , where  $\Gamma(D)$  is the dual graph of  $D$ ; see Definition 3.40.

PROPOSITION 2.11. *The group  $\text{Inn}(X, D)/\text{Aut}^{\natural}(X, D)$  is countable.*

*Proof.* Since  $\text{Aut}(X, D)/\text{Aut}^{\natural}(X, D)$  is a finite group, it suffices to show that the collection  $\text{Inn}(X, D)/\text{Aut}(X, D)$  of left cosets is countable.

By Lemma 2.8, any left coset in  $\text{Inn}(X, D)/\text{Aut}(X, D)$  has the form  $\psi \cdot \text{Aut}(X, D)$  for some  $\psi = \sigma'_n{}^{\pm 1} \cdots \sigma'_1{}^{\pm 1} \in \text{Inn}(X, D)$ . These cosets form a countable set.  $\square$

### 2.3 The ind-structure on $\text{Bir}(X, D)$

Let an NC-pair  $(X, D)$  be a completion of a normal affine surface  $Y = X \setminus \text{supp } D$ . By J. E. Goodman's criterion of affiness for surfaces [Goo69, Theorem 2], we can find an ample divisor  $H \subset X$

such that  $\text{supp } H = \text{supp } D$ . This defines an ample polarization on  $X$ . Replace  $H$  by a suitable high multiple; the very ample linear system  $|H|$  then defines an embedding  $\phi_{|H|}: X \hookrightarrow \mathbb{P}^n$  that sends the divisor  $H$  to a hyperplane section of  $X$ . The restriction of  $\phi_{|H|}$  to  $Y$  yields a proper embedding  $Y = X \setminus \text{supp } H \hookrightarrow \mathbb{A}^n = \mathbb{P}^n \setminus \mathbb{P}^{n-1}$ .

**DEFINITION 2.12** (cf. for example [CX20, (1.1)]). For  $\Phi \in \text{Bir}(X, D)$ , we define its degree as  $\deg \Phi = H \cdot \Phi^*(H)$ . For  $\Phi \in \text{Aut}(Y)$ , we understand  $\deg \Phi$  as the degree of the natural extension  $\hat{\Phi} \in \text{Bir}(X, D)$ .

*Remarks 2.13.* (1) The degree function  $\deg \Phi$  on  $\text{Bir}(X, D)$  coincides with the usual degree of  $\Phi|_Y \in \text{Aut}(Y)$  viewed as an automorphism of the closed subvariety  $Y \subset \mathbb{A}^n$ . Indeed, let  $\Phi$  be written in homogeneous coordinates on  $\mathbb{P}^n$  as

$$\Phi = (p_0 : \cdots : p_n)|_X,$$

where the  $p_i$  are homogeneous polynomials in  $n + 1$  variables of the same degree  $d$  that do not belong simultaneously to the homogeneous ideal of  $X$ . Then  $\deg \Phi = \min\{d\}$ , where the minimum is taken over all such presentations.

(2) We have  $\text{Bir}(X, D) = \varinjlim \text{Bir}(X, D)_{\leq d}$ , where

$$\text{Bir}(X, D)_{\leq d} = \{\Phi \in \text{Bir}(X, D) \mid \max\{\deg \Phi, \deg \Phi^{-1}\} \leq d\}$$

is an affine algebraic variety and  $\text{Bir}(X, D)_{\leq d} \subset \text{Bir}(X, D)_{\leq d+1}$  is a closed embedding. This yields an affine ind-structure on  $\text{Bir}(X, D) = \text{Aut}(Y)$ . It is easily seen that this ind-structure coincides with the usual ind-structure on  $\text{Aut}(Y)$  defined via the induced closed embedding  $Y \hookrightarrow \mathbb{A}^n$ ; see for example [FK18, Section 5.1] or [KPZ11, Section 2.1]. On the other hand, in the case of a rational surface  $X$ , the group  $\text{Bir}(X)$  carries no natural ind-structure; see [BF13].

(3) The very ample linear system  $|H|$  is stable under the action of  $\text{Aut}^{\natural}(X, D)$  on  $(X, D)$ . Hence the embedding

$$\phi_{|H|}: (X, \text{supp } D) = (X, \text{supp } H) \hookrightarrow (\mathbb{P}^n, \mathbb{P}^{n-1})$$

induces an embedding of  $\text{Aut}^{\natural}(X, D)$  onto a closed subgroup of  $\text{Aut}(\mathbb{P}^n, \mathbb{P}^{n-1})$ , so  $\text{Aut}^{\natural}(X, D)$  is an affine algebraic group.

Recall the following definitions; see for example [FK18]. One says that a subgroup  $G$  of  $\text{Bir}(X, D)$  is connected in the Zariski ind-topology if every element  $g \in G$  belongs to a connected algebraic subset of  $G$  that contains the identity. The identity component  $G^{\circ}$  of a subgroup  $G \subset \text{Bir}(X, D)$  is the maximal connected subgroup of  $G$ .

**PROPOSITION 2.14.** *We have  $\text{Inn}^{\circ}(X, D) = \text{Aut}^{\circ}(X, D)$ . In particular,  $\text{Inn}^{\circ}(X, D)$  is a connected affine algebraic group.*

*Proof.* First assume that the ground field  $\mathbb{K}$  is uncountable. Recall that  $\text{Aut}^{\natural}(X, D)$  is a normal affine algebraic subgroup of  $\text{Inn}(X, D)$  of countable index, closed in  $\text{Bir}(X, D)$ ; see Lemma 2.9 and Proposition 2.11. Therefore,  $\text{Inn}(X, D)$  is a countable union of mutually disjoint closed algebraic subvarieties of  $\text{Bir}(X, D)$  of the form  $h_k \circ \text{Aut}^{\natural}(X, D)$  with  $h_k \in \text{Inn}(X, D)$  for  $k \in \mathbb{N}$ . Let  $A$  be an irreducible algebraic subvariety of  $\text{Bir}(X, D) = \text{Aut}(Y)$  of positive dimension contained in  $\text{Inn}(X, D)$ . Since the set of  $\mathbb{K}$ -points of  $A$  is uncountable,  $A$  is actually contained in one of the left cosets  $h_k \circ \text{Aut}^{\natural}(X, D)$ ; see [FK18, Lemma 1.3.1]. If  $\text{id}_{(X, D)} \in A$ , then  $A \subset (\text{Aut}^{\natural}(X, D))^{\circ} = \text{Aut}^{\circ}(X, D)$ . Now the equality  $\text{Inn}^{\circ}(X, D) = \text{Aut}^{\circ}(X, D)$  follows.

Now suppose that  $\mathbb{K}$  is countable. We embed  $\mathbb{K}$  in the uncountable algebraically closed field  $\mathbb{L}$ ; for instance, one can take for  $\mathbb{L}$  the algebraic closure of the function field  $\mathbb{K}(x_t \mid t \in \mathbb{R})$ . Let  $\text{Gal} =$

$\text{Gal}(\mathbb{L}/\mathbb{K})$  stand for the Galois group of the field extension  $\mathbb{K} \subset \mathbb{L}$ . Then  $\mathbb{K} = \mathbb{L}^{\text{Gal}}$ . Moreover, for any algebraic  $\mathbb{K}$ -variety  $Z$ , we have  $Z = Z(\mathbb{L})^{\text{Gal}}$ , and  $Z(\mathbb{L})$  is irreducible if and only if  $Z$  is; see [FK18, Section 1.11] and [Sta24, Lemma 037K]. In particular,  $(X, D) = (X(\mathbb{L}), D(\mathbb{L}))^{\text{Gal}}$ . Furthermore,

$$\text{Aut}^{\natural}(X(\mathbb{L}), D(\mathbb{L}))^{\text{Gal}} = (\text{Aut}^{\natural}(X, D)(\mathbb{L}))^{\text{Gal}} = \text{Aut}^{\natural}(X, D),$$

and likewise for  $\text{Aut}^{\circ}(X, D) = (\text{Aut}^{\natural}(X, D))^{\circ}$ . Since  $\text{Inn}(X, D) = \bigcup_k h_k \circ \text{Aut}^{\natural}(X, D)$ , we have  $\text{Inn}(X(\mathbb{L}), D(\mathbb{L}))^{\text{Gal}} = \text{Inn}(X, D)$ .

Applying the first part of the proof, we get  $\text{Inn}^{\circ}(X(\mathbb{L}), D(\mathbb{L})) = \text{Aut}^{\circ}(X(\mathbb{L}), D(\mathbb{L}))$ . Passing to the Gal-fixed-point loci in the latter equality, we obtain  $\text{Inn}^{\circ}(X, D) = \text{Aut}^{\circ}(X, D)$ .  $\square$

### 3. Birational geometry of weighted graphs

The birational geometry of NC-pairs has its combinatorial counterpart, namely, the birational geometry of weighted graphs; see the references in the introduction.

#### 3.1 Weighted graphs

**DEFINITION 3.1.** Let  $\Gamma$  be a (non-empty) finite weighted graph with set of vertices  $\text{Vert}(\Gamma)$  and set of edges  $\text{Edge}(\Gamma)$ , where vertices are weighted by integer numbers, and let  $\text{Ratio}(\Gamma) \subseteq \text{Vert}(\Gamma)$  be a distinguished subset of vertices which we call *rational vertices*. A rational vertex of weight  $-1$  (respectively, of weight  $0$ ) is called a  $(-1)$ -vertex (respectively,  $(0)$ -vertex). The graph  $\Gamma$  may be disconnected and may contain loops, cycles and multiple edges, that is, several edges joining the same pair of vertices.

*Notation 3.2.* Following [FKZ07], we let  $[[a_1, \dots, a_n]]$  be a linear weighted graph with  $n$  ordered vertices of weights  $a_1, \dots, a_n$ . We also let  $((a_1, \dots, a_n))$  be a circular weighted graph with  $n$  cyclically ordered vertices of weights  $a_1, \dots, a_n$ .

**DEFINITION 3.3.** • Given a weighted graph  $\Gamma$ , the *degree*  $\deg_{\Gamma}(v)$  of a vertex  $v \in \text{Vert}(\Gamma)$  is the number of incident edges of  $\Gamma$  at  $v$ , where every loop  $[v, v]$  is counted twice. We simply write  $\deg(v)$  when the graph  $\Gamma$  has been fixed.

- A rational vertex  $v$  of  $\Gamma$  is called a *tip* or an *end vertex* if  $\deg(v) = 1$  and an *at most linear vertex* if  $\deg(v) \leq 2$  and no loop of  $\Gamma$  is incident with  $v$ . We let  $\text{Tip}(\Gamma)$  be the set of tips of  $\Gamma$ .
- A vertex  $v$  is called a *branch vertex* if either  $v$  is non-rational or  $\deg(v) \geq 3$ . We let  $\text{Br}(\Gamma)$  be the set of all branch vertices of  $\Gamma$ . Thus,  $\text{Vert}(\Gamma) \setminus \text{Ratio}(\Gamma) \subseteq \text{Br}(\Gamma)$ .
- For a proper subset  $V$  of  $\text{Vert}(\Gamma)$ , we let  $\Gamma \ominus V$  be the subgraph of  $\Gamma$  obtained by deleting  $V$  and all its incident edges, including the incident loops.
- Given a vertex  $v \in \Gamma$ , the connected components of  $\Gamma \ominus \{v\}$  linked to  $v$  are the *branches* of  $\Gamma$  at  $v$ . A branch linked to  $v$  via exactly one edge is called *simple*.

**DEFINITION 3.4.** • In the case where  $V = \text{Br}(\Gamma)$  is the set of branch vertices of  $\Gamma$ , the connected components of  $\Gamma \ominus \text{Br}(\Gamma)$  are called *segments*. A connected graph with no branch vertex coincides with its only segment.

- A segment can be either linear or circular. A circular segment is a cycle in  $\Gamma$  which contains no branch vertex; it is a connected component of  $\Gamma$ . A linear segment can consist of a single

vertex. For example, this is the case for a vertex of degree 2 linked to a branch vertex by two edges.

- A linear segment  $L$  is called *extremal* if  $L$  contains one or two tips of  $\Gamma$  and *inner* if  $L$  contains no tip of  $\Gamma$ .

DEFINITION 3.5. Let  $\Gamma$  be a weighted graph with vertices  $v_1, \dots, v_n$ . The *intersection form*  $I(\Gamma)$  is a bilinear form on  $\mathbb{R}^n$  given by the symmetric square matrix  $A(\Gamma) = (a_{i,j})$ , where  $a_{i,i}$  is the weight of  $v_i$  and for  $j \neq i$ , the entry  $a_{i,j}$  equals the number of edges joining  $v_i$  and  $v_j$ . The *discriminant*  $\text{discr}(\Gamma)$  is the determinant  $\det(-A(\Gamma))$ . We let  $i_+(\Gamma)$ ,  $i_-(\Gamma)$  and  $i_0(\Gamma)$  be the inertia indices of the associate quadratic form  $I(\Gamma)$ .

DEFINITION 3.6. Let  $[v_1, v_2]$  (where possibly  $v_1 = v_2$ ) be an edge of  $\Gamma$ . The *inner blowup* in  $[v_1, v_2]$  consists in adding a new  $(-1)$ -vertex  $v$ , replacing the edge  $[v_1, v_2]$  by two new edges  $[v_1, v]$  and  $[v, v_2]$  and decreasing the weights of  $v_1$  and  $v_2$  by 1 if  $v_1 \neq v_2$  and by 4 if  $v_1 = v_2$  (that is,  $[v_1, v_2]$  is a loop).

The *outer blowup* at a vertex  $v$  in  $\Gamma$  consists in introducing a new  $(-1)$ -vertex  $v'$  along with a new edge  $[v, v']$  and decreasing the weight of  $v$  by 1.

The modifications inverse to inner and outer blowups are called *inner* and *outer blowdowns*, respectively.

Remark 3.7. The definition of inner birational transformations of weighted graphs in [FKZ07, Definitions 2.3 and 2.8] has a broader meaning. Indeed, the inverse of an outer blowup is considered in loc. cit. to be inner.

PROPOSITION 3.8 (cf. [Neu89, Proposition 1.1], [Rus02, Proposition 1.14], [FKZ07, Lemma 4.6]). A blowup of a weighted graph  $\Gamma$  adds 1 to  $i_-(\Gamma)$ , while  $i_+(\Gamma)$  and  $i_0(\Gamma)$  remain unchanged.

Proof. Letting  $\text{Vert}(\Gamma) = \{v_1, \dots, v_n\}$ , consider the  $\mathbb{R}$ -vector space  $V = \bigoplus_i \mathbb{R}v_i$  with inner product given by the intersection form  $I(\Gamma)$ . The symmetric matrix  $M_n$  of  $I(\Gamma)$  is diagonal in a suitable orthogonal basis  $\{e_1, \dots, e_n\}$  of  $V$ .

A blowup  $\delta: \Gamma' \dashrightarrow \Gamma$  creating a vertex  $v_{n+1}$  induces a map  $\delta^*: V \hookrightarrow V' = \bigoplus_{i=1}^{n+1} \mathbb{R}v_i$  defined by

$$\delta^*: v \mapsto v + I(\Gamma')(v, v_{n+1})v_{n+1}.$$

The vector  $v_{n+1}$  is an eigenvector of the matrix  $M_{n+1}$  of  $I(\Gamma')$  with eigenvalue  $-1$ . Indeed, one can check that  $I(\Gamma')$  restricted on  $\delta^*(V) \subset V'$  coincides with  $\delta^*I(\Gamma)$  and  $v_{n+1}$  is orthogonal to  $\delta^*(V)$ . The orthogonal basis in  $V'$  of the eigenvectors

$$\delta^*(e_1), \dots, \delta^*(e_n), v_{n+1}$$

is diagonalizing for  $M_{n+1}$ , which proves the claim.  $\square$

Remark 3.9. In the case where  $\Gamma = \Gamma(D)$  is the dual graph of an NC-divisor  $D$  on a smooth surface  $X$  and  $\delta: X' \rightarrow X$  is the inverse of the blowup of a point on  $D$ , the homomorphism  $\delta^*$  sends an  $\mathbb{R}$ -divisor  $N$  supported on  $\text{supp } D$  to its total transform  $\delta^*(N)$  on  $X'$ , and  $v_{n+1}$  represents the exceptional  $(-1)$ -curve of  $\delta$ . The properties of  $\delta^*$  used in the proof follow in this case from the projection formula.

DEFINITION 3.10. A *birational transformation of weighted graphs*  $\phi: \Gamma \dashrightarrow \Gamma'$  is a finite sequence of blowups, blowdowns and isomorphisms. If all these blowups and blowdowns are inner, then  $\phi$  is called *inner*. Two weighted graphs are called *birationally equivalent* if one can be transformed into the other by means of a birational transformation.

From Proposition 3.8 we deduce the following.

COROLLARY 3.11. *The inertia indices  $i_+(\Gamma)$  and  $i_0(\Gamma)$  are birational invariants.*

REMARKS 3.12. (1) Consider a weighted graph  $\Gamma$  and a sequence  $\phi$  of blowups, blowdowns and isomorphisms resulting in a final graph  $\Gamma'$ . If, besides isomorphisms,  $\phi$  also includes blowups or blowdowns, then one can ignore the isomorphisms that participate in  $\phi$  by simply renaming the created graphs, including the final one. Anyway, one can reduce  $\phi$  to a sequence of blowups and blowdowns preceded or followed by a single isomorphism; see formula (2.1) in the proof of Proposition 2.11.

(2) A birational transformation of weighted graphs  $\phi: \Gamma \dashrightarrow \Gamma'$  is not a mapping of graphs: in general, it is not well defined either on the sets of vertices or on the sets of edges. Nonetheless, given a second birational transformation  $\phi': \Gamma' \dashrightarrow \Gamma''$ , the composition  $\phi' \circ \phi: \Gamma \dashrightarrow \Gamma''$  is a well-defined birational transformation of weighted graphs. The identity  $\text{id}: \Gamma \rightarrow \Gamma$  and the inverse birational transformation  $\phi^{-1}: \Gamma' \dashrightarrow \Gamma$  are also well defined.

### 3.2 Contractible graphs

DEFINITION 3.13. A birational transformation  $\phi: \Gamma \dashrightarrow \Gamma'$  is a *birational morphism* (or a *domination*) if no blowup participates in  $\phi$ . In this case, we write  $\phi: \Gamma \rightarrow \Gamma'$ . If, moreover,  $\phi$  is composed only of blowdowns, then we say that  $\phi$  is a *contraction*. So, contractions are birational morphisms, but an isomorphism of graphs is not a contraction.

REMARK 3.14. The composition of birational morphisms (respectively, contractions) of weighted graphs is a birational morphism (respectively, a contraction).

DEFINITION 3.15. A weighted graph  $\Gamma$  is called *minimal* if any birational morphism  $\Gamma \rightarrow \Gamma'$  is in fact an isomorphism. A graph  $\Gamma$  is minimal if and only if it has no at most linear  $(-1)$ -vertex. Clearly, any weighted graph  $\Gamma$  dominates a minimal one called a *minimal model* of  $\Gamma$ .

A simple example of a minimal weighted graph is a graph consisting of a single vertex  $v$  and a loop; cf. Example 3.41. Indeed, an at most linear  $(-1)$ -vertex has no incident loop; see Definition 3.3.

DEFINITION 3.16. A connected weighted graph  $\Gamma$  with at least two vertices is said to be *contractible* if there exists a contraction of  $\Gamma$  to a graph  $\Gamma'$  that consists of a single  $(-1)$ -vertex with no incident loop; the latter graph  $\Gamma'$  also is considered to be contractible.

The following lemma is well known.

LEMMA 3.17. *Let  $\Gamma$  be a weighted graph.*

- (a) *If  $\Gamma$  is contractible, then  $\Gamma$  is a tree with only rational vertices. Furthermore, the intersection form  $I(\Gamma)$  is negative definite of discriminant 1.*
- (b) *Conversely, if  $\Gamma$  is a tree with only rational vertices and negative-definite intersection form  $I(\Gamma)$  of discriminant 1, then  $\Gamma$  is contractible.*
- (c) *Contracting an at most linear  $(-1)$ -vertex of a contractible graph with at least two vertices yields a contractible graph, and so does a blowup of a contractible graph. Thus, the contractibility is a birational invariant.*

*Proof.* To show the first assertion in statement (a), it suffices to look at the reconstruction process of blowups starting from the graph which consists of a single isolated  $(-1)$ -vertex. The

second assertion follows from [Hir95, Section 3, Proposition and Theorem]. We address [Rus02, Proposition 1.20] and [FKZ07, Remark 4.8, formula (17)] for statement (b). Statement (c) follows, for example, from statement (b) due to [FKZ07, Remark 4.8, formula (17)]; cf. also [Hir95, Section 3, Proposition].  $\square$

The next corollary is immediate.

**COROLLARY 3.18.** *For a contractible weighted graph  $\Gamma$ , the following hold:*

- (a) *The weight of every vertex  $v$  of  $\Gamma$  is negative.*
- (b) *No two  $(-1)$ -vertices of  $\Gamma$  are neighbors.*
- (c) *Every  $(-1)$ -vertex  $v$  of  $\Gamma$  is at most linear.*
- (d) *If  $\Gamma$  contains at least two vertices, then every  $(-1)$ -vertex of  $\Gamma$  is contracted in any process of contraction of  $\Gamma$  to a graph that consists of a single  $(-1)$ -vertex with no incident loop.*

### 3.3 Contraction of a subgraph

Below, we use the following notions. Let  $\Gamma$  be a weighted graph and  $\Delta$  be a proper *induced subgraph* of  $\Gamma$ . The latter means that every edge of  $\Gamma$  linking two vertices of  $\Delta$  is an edge of  $\Delta$ .

**DEFINITION 3.19.** Let  $\Delta$  be an induced subgraph of  $\Gamma$  and  $p: \Gamma \rightarrow \Gamma'$  be a contraction. We say that  $p$  *contracts*  $\Delta$  if every vertex of  $\Delta$  is contracted under  $p$ . We say that  $p$  is a *contraction of  $\Delta$*  if  $p$  contracts every vertex of  $\Delta$  and no other vertices.

*Remarks 3.20.* (1) The fact that  $p$  contracts  $\Delta$  does not imply, in general, that  $\Delta$  is contractible; see for instance Example 4.6. In this example, the contractions  $p_1$  and  $p_2$  contract the branching vertex of weight  $-2$ , even though this vertex is not contractible. On the other hand, if  $\Delta$  is contractible, this does not imply, in general, that there exists a contraction  $p: \Gamma \rightarrow \Gamma'$  that contracts  $\Delta$ . For instance, a  $(-1)$ -vertex  $v$  of a minimal graph  $\Gamma$  survives every contraction  $\Gamma \rightarrow \Gamma'$ . Indeed, a minimal graph admits no contraction, and so  $v$  cannot be an at most linear vertex.

(2) It follows by recursion from Lemma 3.17(c) that a contraction of a proper induced subgraph of a contractible graph results in a contractible graph.

**DEFINITION 3.21.** Consider the blowdown  $\delta: \Gamma \rightarrow \Gamma'$  of an at most linear vertex  $v_0 \in \text{Vert}(\Gamma)$ . The image  $\delta(\Gamma \ominus \{v_0\})$  in  $\Gamma'$  is a well-defined subgraph isomorphic to  $\Gamma \ominus \{v_0\}$  up to a change of weights. Likewise, there is a well-defined image of any subgraph  $\Delta$  of  $\Gamma \ominus \{v_0\}$ . For a connected subgraph  $\Delta$  of  $\Gamma$  such that  $v_0 \in \text{Vert}(\Delta)$  and  $\text{Vert}(\Delta) \neq \{v_0\}$ , the image of  $\Delta$  in  $\Gamma'$  coincides with  $\delta(\Delta \ominus \{v_0\})$  if  $v_0$  is a tip of  $\Delta$ ; otherwise,  $\delta(\Delta) = \delta(\Delta \ominus \{v_0\}) \cup [\delta(u), \delta(v)]$ , where  $u$  and  $v$  are the vertices of  $\Delta$  linked to  $v_0$  in  $\Delta$  (it is possible that  $u = v$ ).

We extend by recursion the notion of image of a subgraph to any birational morphism  $\phi: \Gamma \rightarrow \Gamma'$ .

*Remark 3.22.* If  $\delta$  is an inner blowdown, then  $v_0$  has degree 2 in  $\Gamma$  and exactly two incident edges, say  $[u, v_0]$  and  $[v_0, v]$ , that are replaced by a single edge  $e' = [\delta(u), \delta(v)]$  of  $\Gamma'$ ; see Definition 3.6. If  $\delta$  is outer, then  $v_0$  is a tip of  $\Gamma$ , and the unique edge  $[v_0, v]$  of  $\Gamma$  at  $v_0$  disappears under  $\delta$ .

For a birational morphism  $\delta: \Gamma \rightarrow \Gamma'$  and an induced subgraph  $\Delta' \subset \Gamma'$ , we define the partial preimage  $\delta^{-1}(\Delta') \subset \Gamma$  and the total preimage  $\delta^*(\Delta') \subset \Gamma$  as follows.

DEFINITION 3.23. Let  $\Delta' \subset \Gamma'$  be an induced subgraph and  $\delta: \Gamma \rightarrow \Gamma'$  be the blowdown of an at most linear vertex  $v_0 \in \text{Vert}(\Gamma)$ . If  $\delta$  is inner (respectively, outer) and  $v_0$  is linked to vertices  $v_1$  and  $v_2$  of  $\Gamma$  (respectively, to a vertex  $v_1 \in \text{Vert}(\Gamma)$ ), then we define the *partial preimage*

$$\delta^{-1}(\Delta') := \delta_0^{-1}(\Delta' \setminus [v_1, v_2]) \quad (\text{respectively, } \delta^{-1}(\Delta') := \delta_0^{-1}(\Delta')),$$

where

$$\delta_0: \Gamma \ominus \{v_0\} \xrightarrow{\cong} \Gamma' \setminus [v_1, v_2] \quad (\text{respectively, } \delta_0: \Gamma \ominus \{v_0\} \xrightarrow{\cong} \Gamma')$$

stands for the induced isomorphism of unweighted graphs; cf. Definition 3.21.

If  $\delta$  is inner, respectively outer, then we define the *total preimage*

$$\delta^*(\Delta') := \begin{cases} \delta^{-1}(\Delta') \cup [v_1, v_0] \cup \{v_0\} \cup [v_0, v_2] & \text{if } [v_1, v_2] \in \text{Edge}(\Delta'), \\ \delta^{-1}(\Delta') \cup \{v_0\} \cup [v_1, v_0] & \text{if } v_1 \in \text{Vert}(\Delta') \text{ and } v_2 \notin \text{Vert}(\Delta'), \\ \delta^{-1}(\Delta') & \text{if } v_1, v_2 \notin \text{Vert}(\Delta'), \end{cases}$$

respectively

$$\delta^*(\Delta') := \begin{cases} \delta^{-1}(\Delta') \cup [v_1, v_0] \cup \{v_0\} & \text{if } v_1 \in \text{Vert}(\Delta'), \\ \delta^{-1}(\Delta') & \text{if } v_1 \notin \text{Vert}(\Delta'). \end{cases}$$

We extend by recursion the notions of partial and total preimages of an induced subgraph to any birational morphism  $\phi: \Gamma \rightarrow \Gamma'$ .

Remark 3.24. For a vertex  $v \in \text{Vert}(\Gamma')$ , its partial preimage  $\delta^{-1}(v)$  is a vertex of  $\Gamma$  that is often denoted by the same letter  $v$ . Given an induced subgraph  $\Delta' \subset \Gamma'$ , the subgraph  $\delta^{-1}(\Delta')$  is not necessarily induced, while  $\delta^*(\Delta')$  is. So,  $\delta^*(\Delta')$  is uniquely defined by the set of its vertices, that is, by the subset  $\text{Vert}(\delta^{-1}(\Delta')) \subset \text{Vert}(\Gamma)$  plus the vertex  $v_0$  in certain cases.

If  $\Delta'$  is connected, then  $\delta^*(\Delta')$  also is. Moreover,  $\delta^*(\Delta')$  is a connected component of the subgraph  $\Gamma \ominus \bigcup_{v \in \text{Vert}(\Gamma') \setminus \text{Vert}(\Delta')} \{\delta^{-1}(v)\}$ .

We will usually apply Definitions 3.21 and 3.23 to induced subgraphs of  $\Gamma$ .

Convention 3.25. Given a birational morphism  $\phi: \Gamma \rightarrow \Gamma'$ , we consider  $\text{Vert}(\Gamma')$  as a subset of  $\text{Vert}(\Gamma)$  by identifying  $v' \in \text{Vert}(\Gamma')$  with  $\phi^{-1}(v') \in \text{Vert}(\Gamma)$ .

Remark 3.26. A path  $\gamma$  in  $\Gamma$  is called *simple* if  $\gamma$  contains no loop and has no self-intersection. It is easily seen that, under a birational morphism  $p: \Gamma \rightarrow \Gamma'$ , the partial preimage of a simple path  $\gamma$  connecting vertices  $u', v' \in \text{Vert}(\Gamma')$  is a simple path in  $\Gamma$  connecting  $p^{-1}(u')$  and  $p^{-1}(v')$ . Moreover, the partial preimage  $p^{-1}(\Delta')$  of a subgraph  $\Delta' \subset \Gamma'$  is homeomorphic to  $\Delta'$ ; that is the associated simplicial 1-complexes viewed as topological spaces are homeomorphic. Also,  $\Delta'$  and the total preimage  $p^*(\Delta')$  are homotopically equivalent; cf. [FKZ07, Section 2.2].

DEFINITION 3.27. Given a weighted graph  $\Gamma$ , we say that two birational morphisms  $p_i: \Gamma \rightarrow \Gamma_i$ ,  $i = 1, 2$ , are *equivalent* if there is a (uniquely defined) isomorphism of weighted graphs  $\iota: \Gamma_1 \rightarrow \Gamma_2$  such that  $p_1^{-1}(\Delta) = p_2^{-1}(\iota(\Delta))$  for any subgraph  $\Delta \subset \Gamma_1$ . If this holds for  $\Gamma_1 = \Gamma_2$  with  $\iota = \text{id}$ , then we say that  $p_1$  and  $p_2$  are *equal* and we write  $p_1 = p_2$ . We say that a diagram of weighted graphs

$$\begin{array}{ccc} & \Gamma & \\ p_1 \swarrow & & \searrow p_2 \\ \Gamma_1 & \xrightarrow{p_3} & \Gamma_2 \end{array}$$

where the  $p_i$  are birational morphisms of graphs, commutes if  $p_3 \circ p_1 = p_2$ . In the latter case, the diagram

$$\begin{array}{ccc} & \Gamma & \\ p_2 \swarrow & & \searrow p_1 \\ \Gamma_2 & \dashrightarrow & \Gamma_1 \\ & p_3^{-1} & \end{array}$$

is also called commutative. Square commutative diagrams of birational morphisms and more complicated commutative diagrams are defined in a similar way.

*Notation 3.28.* Given a birational morphism  $\phi: \Gamma \rightarrow \Gamma'$  and a vertex  $v$  of  $\Gamma$ , we let  $\phi^v$  be the maximal initial subsequence of blowdowns and isomorphisms in  $\phi$  preserving  $v$ . Thus,  $\phi^v = \phi$  if and only if  $v$  is not contracted under  $\phi$ . In particular,  $\phi^v = \phi$  if  $v$  is non-rational.

**LEMMA 3.29.** (a) *Given a birational morphism  $\phi: \Gamma \rightarrow \Gamma'$  that is not an isomorphism, there is a contraction  $p: \Gamma \rightarrow \Gamma''$  equivalent to  $\phi$ .*

(b) *Consider a composition  $\Gamma \xrightarrow{\phi} \Gamma_1 \xrightarrow{\sigma} \Gamma_2$  of a birational morphism  $\phi$  and a blowdown  $\sigma$  of a vertex  $v$  that is not adjacent in  $\Gamma$  to any vertex contracted by  $\phi$ . Then there exists a birational morphism  $\phi': \Gamma \rightarrow \Gamma_2$  equal to  $\sigma \circ \phi$  and starting with the blowdown of  $v$ .*

(c) *Consider a birational morphism  $\phi: \Gamma \rightarrow \Gamma_1$  and an at most linear  $(-1)$ -vertex  $v \in \text{Vert}(\Gamma)$  contracted by  $\phi$ . Then there exists a birational morphism  $\phi': \Gamma \rightarrow \Gamma_2$  equal to  $\phi$  and starting with the blowdown  $\sigma_v$  of  $v$ .*

*Proof.* (a) We obtain the desired contraction  $p: \Gamma \rightarrow \Gamma''$  by blowing down the vertices of  $\Gamma$  blown down by  $\phi$  in the same order as this is done under  $\phi$ .

(b) The weight of  $v$  is  $-1$  both in  $\Gamma$  and  $\Gamma_1$ , and the weight of any vertex contracted by  $\phi$  is the same in  $\Gamma$  and after the blowdown of  $v$  in  $\Gamma$ . Thus, we obtain  $\phi'$  by contracting  $v$  first and then contracting all the vertices contracted by  $\phi$  in the same order. The weights of the non-contracted vertices are the same under  $\sigma \circ \phi$  and  $\phi'$ . The assertion follows.

(c) The vertex  $v$  does not change its weight under  $\phi^v$ ; hence it is adjacent to no vertex contracted by  $\phi^v$ . The claim now follows from statement (b) applied to  $\phi^v$  and  $\sigma_v$ .  $\square$

**PROPOSITION 3.30.** *Two birational morphisms  $\phi_i: \Gamma \rightarrow \Gamma_i$ ,  $i = 1, 2$ , are equivalent if and only if the subsets of vertices of  $\Gamma$  contracted by  $\phi_1$  and  $\phi_2$  coincide.*

*Proof.* The “only if” part is clear. To show the “if” part, assume that each vertex contracted by  $\phi_1$  is contracted by  $\phi_2$  and vice versa. If there are no contracted vertices in  $\Gamma$ , then both  $\phi_1$  and  $\phi_2$  are isomorphisms and it suffices to set  $\iota = \phi_2 \circ \phi_1^{-1}$ ; see Definition 3.27. Now let  $v \in \text{Vert}(\Gamma)$  be a  $(-1)$ -vertex contracted by the  $\phi_i$ . By Lemma 3.29(c), up to equivalence we may suppose that both  $\phi_1$  and  $\phi_2$  start with the blowdown  $\sigma_v$  of  $v$ . This reduces the assertion to the one for the birational morphisms  $\phi'_i: \sigma_v(\Gamma) \rightarrow \Gamma_i$ . Induction on the number of contracted vertices ends the proof.  $\square$

**LEMMA 3.31** (see [FKZ07, Lemma 2.4(a)]). *Consider a contraction  $p: \Gamma \rightarrow \Gamma'$ .*

- (a) *Assume that a branch  $W$  of  $\Gamma$  at a vertex  $v \in \text{Vert}(\Gamma)$  is contracted under  $p^v: \Gamma \rightarrow \Gamma^v$ . Then  $W$  is contractible, and there is a factorization  $p^v = r \circ q$ , where  $q: \Gamma \rightarrow \Gamma''$  is a contraction of  $W$  and  $r: \Gamma'' \rightarrow \Gamma'$  is a birational morphism. Furthermore,  $W$  is either simple (that is, linked to  $v$  by a single edge, see Definition 3.3) or linked to  $v$  by exactly two edges.*

In the former case,  $p^v$  decreases the degree of  $v$  by 1, while in the latter case  $p^v$  replaces the branch  $W$  by a loop at  $v$ , so the degree of  $v$  does not change. The latter case cannot happen provided  $p$  contracts  $v$ .

- (b) Assume that a branch vertex  $v \in \text{Br}(\Gamma)$  is blown down under  $p$ . Then  $v$  is rational and has no incident loop. Furthermore, at least  $\deg(v) - 2 > 0$  branches of  $\Gamma$  at  $v$  are simple and contracted by  $p^v$ .

*Proof.* (a) The first assertion is immediate from Lemma 3.29(b) and (c). Consider the contraction  $q: \Gamma \rightarrow \Gamma''$  of  $W$  induced by  $p^v$  and its decomposition  $q = \sigma_n \circ \cdots \circ \sigma_1$  into a sequence of blowdowns. Then  $\sigma_1^{-1}$  is a blowup of  $\Gamma''$  either at  $v$  or at an incident loop at  $v$ . So,  $W$  is a tree linked to  $v$  by a single edge in the former case and by two edges in the latter case; cf. Lemma 3.17(a).

(b) Consider the contraction  $p^v: \Gamma \rightarrow \Gamma^v$ . Since  $p$  contracts  $v$ , the image  $p^v(v)$  is an at most linear  $(-1)$ -vertex of  $\Gamma^v$ . In particular,  $v$  is rational and has no incident loop; see Definition 3.3.

The contraction of a non-simple branch of  $\Gamma$  at  $v$  creates a loop of  $\Gamma^v$  incident to  $p^v(v)$ ; see statement (a). Since  $p^v(v)$  has no incident loop, all contracted branches at  $v$  are simple. Since  $p^v(v)$  is at most linear, there are at least  $\deg(v) - 2$  such branches.  $\square$

LEMMA 3.32. (a) Let  $p: \Gamma \rightarrow \Gamma'$  be a contraction and  $L$  be an extremal linear segment of  $\Gamma$  that is not contracted under  $p$ . Then the image  $p(L)$  is contained in an extremal linear segment of  $\Gamma'$  and contains a tip of  $\Gamma'$ .

(b) Let  $\Gamma$  be a contractible graph. Then  $\Gamma$  contains a contractible extremal linear segment.

*Proof.* (a) The assertion is true if  $p$  is a blowdown. Now the general case follows by induction on the number of blowdowns in a decomposition of  $p$ .

(b) The assertion is evident if  $\Gamma$  is linear. Otherwise,  $\Gamma$  contains branch vertices. Let  $\text{Br}(\Gamma) = \{v_1, \dots, v_k\}$ . The number  $\nu(\Gamma) = \sum_{i=1}^k (\deg(v_i) - 2)$  is called the *total branching number* of  $\Gamma$ ; see [FKZ07, Section 2.1]. Clearly,  $\nu(\Gamma) = 0$  if and only if  $\Gamma$  is linear. In each step of the contraction, the total branching number can only decrease. At the first moment when  $\nu(\Gamma)$  actually drops, an extremal linear branch of  $\Gamma$  has been contracted.  $\square$

### 3.4 Relatively minimal diagrams

Recall the following well-known fact.

LEMMA 3.33. Any birational transformation of NC-pairs  $\Phi: (X_1, D_1) \dashrightarrow (X_2, D_2)$  fits in a commutative diagram

$$\begin{array}{ccc} & (\tilde{X}, \tilde{D}) & \\ P_1 \swarrow & & \searrow P_2 \\ (X_1, D_1) & \dashrightarrow & (X_2, D_2) \\ & \Phi & \end{array} \quad (3.1)$$

where every  $P_i$  is a composition of an isomorphism and blowdowns of smooth rational  $(-1)$ -components of boundary divisors and all the intermediate pairs are NC-pairs.

*Proof.* Any birational transformation of smooth projective surfaces  $\Phi: X_1 \rightarrow X_2$  fits in a com-

mutative diagram

$$\begin{array}{ccc} & \tilde{X} & \\ P_1 \swarrow & & \searrow P_2 \\ X_1 & \dashrightarrow & X_2, \\ & \Phi & \end{array}$$

where each  $P_i$  is a composition of an isomorphism and blowdowns of smooth rational  $(-1)$ -curves; see for example [Mum95, Corollaries (8.10)–(8.11)] or [Bea96, Corollary II.12]. This can be applied as well in our setting after a simultaneous resolution of singularities of  $X_1$  and  $X_2$ . Indeed,  $\Phi|_{X_1 \setminus \text{supp } D_1} : X_1 \setminus \text{supp } D_1 \rightarrow X_2 \setminus \text{supp } D_2$  is a biregular isomorphism, and  $X_i$  is smooth near  $D_i$ ; see Definitions 2.1 and 2.2.

Note that  $P_i^{-1}$  can be written as a composition of an isomorphism and a sequence of blowups with smooth centers located at points of  $\text{supp } D_i$  and infinitesimally near points. A blowup of an NC-pair  $(X, D)$  in a point of  $\text{supp } D$  yields an NC-pair. Hence all the intermediate pairs in the decomposition

$$P_i : (\tilde{X}, \tilde{D}) \xrightarrow{\sigma_{i,n_i}} (X_{i,n_i-1}, D_{i,n_i-1}) \longrightarrow \cdots \longrightarrow (X_{i,2}, D_{i,2}) \xrightarrow{\sigma_{i,2}} (X_{i,1}, D_{i,1}) \simeq (X_i, D_i)$$

are NC-pairs.  $\square$

To formulate an analog of this lemma for birational transformations of weighted graphs, one needs to explain the meaning of the corresponding commutative diagram.

DEFINITION 3.34. Consider a diagram of weighted graphs

$$\begin{array}{ccc} & \Gamma & \\ p_1 \swarrow & & \searrow p_2 \\ \Gamma_1 & & \Gamma_2, \end{array} \quad (3.2)$$

where the  $p_i$  are birational morphisms of graphs. Also consider a birational transformation  $\phi : \Gamma_1 \dashrightarrow \Gamma_2$  written as

$$\phi : \Gamma_1 = \Delta_1 \xrightarrow{\psi_1} \Delta_2 \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_{n-2}} \Delta_{n-1} \xrightarrow{\psi_{n-1}} \Delta_n = \Gamma_2, \quad (3.3)$$

where the  $\Delta_i$  are weighted graphs and exactly one of  $\psi_i$  and  $\psi_{i+1}$  is a birational morphism and the other one is the inverse of a birational morphism. We say that  $\phi$  fits in (3.2) if (3.3) is included in a sequence of commutative diagrams

$$\begin{array}{ccc} & \Gamma & \\ \eta_i \swarrow & & \searrow \eta_{i+1} \\ \Delta_i & \dashrightarrow & \Delta_{i+1}, \\ & \psi_i & \end{array} \quad (3.4)$$

where  $\eta_1 = p_1$  and  $\eta_n = p_2$ ; cf. Definition 3.27.

Remark 3.35. The graphs  $\Gamma_1$  and  $\Gamma_2$  in diagram (3.2) are birationally equivalent via the birational transformation  $\phi' = p_2 \circ p_1^{-1} : \Gamma_1 \dashrightarrow \Gamma_2$ . Thus, starting with a diagram (3.2), we come to a birational transformation  $\phi'$ . In the next proposition, we show that, conversely, given a birational transformation of weighted graphs, one can construct a corresponding diagram (3.2).

PROPOSITION 3.36 (see [FZ94, Remark A1(1)]). Let  $\phi : \Gamma_1 \dashrightarrow \Gamma_2$  be a birational transformation of weighted graphs. Then  $\phi$  fits in a diagram (3.2) for some  $\Gamma$ ,  $p_1$ ,  $p_2$ .

*Proof.* The assertion is evidently true if  $\phi$  is an isomorphism. Assume that this is not the case. Up to an automorphism of  $\Gamma_2$ , we can write  $\phi$  as in (3.3), where  $\psi_i$  for  $1 \leq i \leq n-2$  is either a product of blowdowns following by a product  $\psi_{i+1}$  of blowups, or such a product in the reverse order; see Remark 3.12(1). Let us proceed by recursion. For  $n=1$ , our assertion is obvious.

Now assume that  $n=2$  and  $\phi = \sigma_2^{-1}\sigma_1$ , where the  $\sigma_i: \Gamma_i \rightarrow \Gamma_0$  are blowdowns. If both  $\sigma_1^{-1}$  and  $\sigma_2^{-1}$  are inner blowups of the same edge of  $\Gamma_0$ , then the graph  $\Gamma = \Gamma_1 = \Gamma_2$  dominates both  $\Gamma_1$  and  $\Gamma_2$  via isomorphisms. Otherwise, applying both of them simultaneously, one transforms  $\Gamma_0$  into a graph  $\Gamma$  with two distinct vertices  $v_1$  and  $v_2$  such that  $\Gamma$  dominates  $\Gamma_1$  (respectively,  $\Gamma_2$ ) via the contraction  $p_1$  of  $v_2$  (respectively,  $p_2$  of  $v_1$ ). This produces a desired diagram (3.2).

Let further  $n=2$  and  $\phi = \psi_2^{-1}\psi_1$ , where this time the  $\psi_i$  stand for birational morphisms. We can write

$$\psi_i = \sigma_{i,1} \cdots \sigma_{i,m_i}: \Gamma_i = \Gamma_{i,m_i} \longrightarrow \Gamma_0 = \Gamma_{i,0},$$

where  $\sigma_{i,j}: \Gamma_{i,j} \rightarrow \Gamma_{i,j-1}$ ,  $i=1,2$ , is a blowdown. Applying the preceding case to the composition  $\sigma_{2,1}^{-1}\sigma_{1,1}: \Gamma_{1,1} \dashrightarrow \Gamma_{2,1}$ , we can find a weighted graph  $\Gamma_{0,1,1}$  which dominates both  $\Gamma_{1,1}$  and  $\Gamma_{2,1}$ , making the square diagram commutative; see Figure 1. The same procedure applied to the pairs  $\{\Gamma_{2,2}, \Gamma_{0,1,1}\}$  and  $\{\Gamma_{0,1,1}, \Gamma_{1,2}\}$  yields two new graphs  $\Gamma_{0,1,2}$  and  $\Gamma_{0,2,1}$  which dominate the corresponding pairs. Continuing in this way, we fill in a commutative diagram in the form of a lattice parallelogram consisting of  $(m_1+1)(m_2+1)$  weighted graphs and their morphisms. This parallelogram has pairs of opposite vertices  $\{\Gamma_1, \Gamma_2\}$  and  $\{\Gamma_0, \Gamma\}$ , where  $\Gamma = \Gamma_{0,m_1,m_2}$  dominates both  $\Gamma_1$  and  $\Gamma_2$  via sequences of blowdowns inducing  $\phi$ ; see Figure 1 for  $m_1=2$  and  $m_2=3$ . This yields the assertion for  $n=2$ .

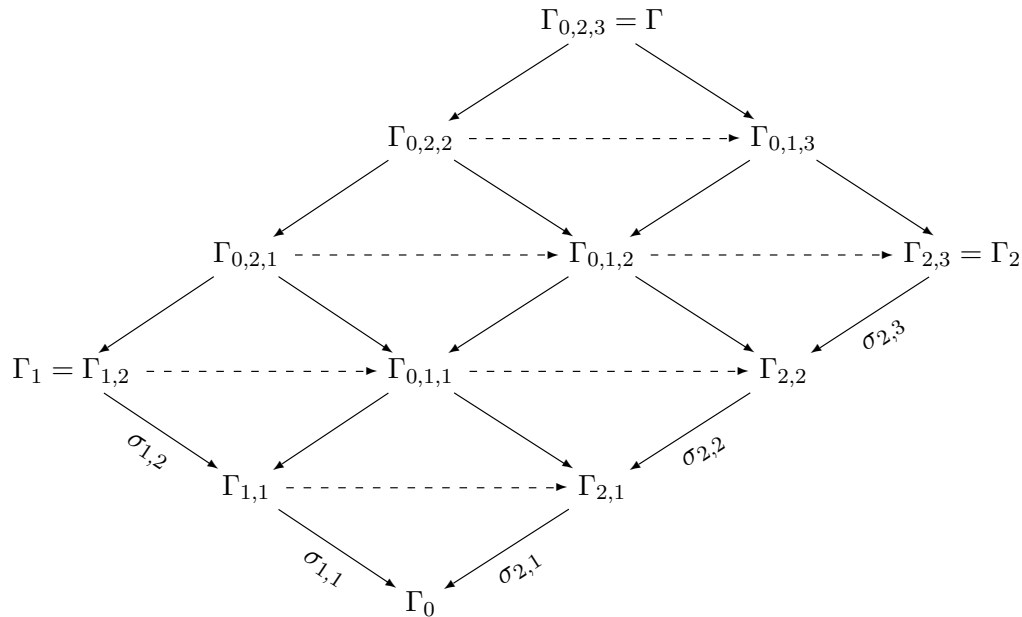


FIGURE 1. Construction of diagram (3.2) for  $m_1=2$  and  $m_2=3$ .

Proceeding by induction on  $n$ , we assume that  $n \geq 3$ , and we choose an  $i \geq 2$  such that  $\psi_i^{-1}$  and  $\psi_{i-1}$  in (3.3) are morphisms. By the preceding, the product  $\psi_i\psi_{i-1}$  can be replaced by  $\rho_i\rho_{i-1}$ , where this time  $\rho_i$  and  $\rho_{i-1}^{-1}$  are morphisms. Now replacing  $\psi_{i+1}\rho_i$  by  $\psi'_{i+1}$  provided  $i \leq n-1$  and  $\rho_{i-1}\psi_{i-2}$  by  $\psi'_{i-2}$  provided  $i \geq 3$ , we transform (3.3) into a shorter decomposition of the same type. This gives the inductive step.

The graph  $\Gamma$  in diagram (3.2) constructed in the proof dominates every pair of intermediate graphs obtained under our procedure. In particular,  $\phi$  fits in this diagram; see Definition 3.34.  $\square$

DEFINITION 3.37. A diagram (3.2) with two minimal weighted graphs  $\Gamma_1$  and  $\Gamma_2$  is called *relatively minimal* if no  $(-1)$ -vertex of  $\Gamma$  is contracted in both  $\Gamma_1$  and  $\Gamma_2$ .

Remark 3.38. Given a birational map  $\phi: \Gamma_1 \dashrightarrow \Gamma_2$  fitting in a relatively minimal diagram (3.2), the graph  $\Gamma$  is not uniquely defined, in general; see Figure 2 for a simple example; cf. also [FKZ07, Example 2.6].

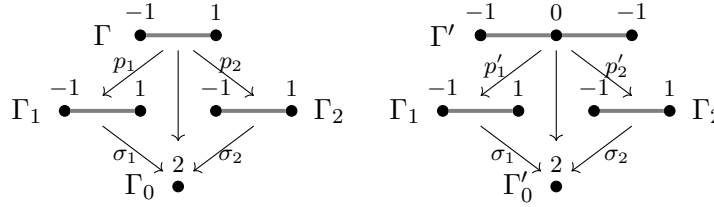


FIGURE 2. Two different dominations of the same weighted graphs.

In both diagrams, the linear graphs  $\Gamma_1$  and  $\Gamma_2$  are isomorphic, the  $\sigma_i$  are blowdowns of  $(-1)$ -vertices, and  $\phi = \sigma_2^{-1} \circ \sigma_1: \Gamma_1 \dashrightarrow \Gamma_2$  is a birational transformation. In the left diagram,  $\Gamma \simeq \Gamma_1 \simeq \Gamma_2$ , while in the right diagram,  $p'_1$  and  $p'_2$  are blowdowns of distinct  $(-1)$ -vertices of  $\Gamma'$  and  $\Gamma_1 \simeq \Gamma_2 \not\simeq \Gamma$ . The vertical arrows in both diagrams correspond to the left arrow in diagram (3.4) with  $i = 2$ .

According to the following lemma, for any pair of birationally equivalent minimal weighted graphs  $\Gamma_1$  and  $\Gamma_2$ , there exist a weighted graph  $\Gamma$  and birational morphisms  $p_i: \Gamma \rightarrow \Gamma_i$  that form a relatively minimal diagram (3.2).

LEMMA 3.39 (cf. [FZ94, Remark A1(1)]). *Given a diagram (3.2) with minimal weighted graphs  $\Gamma_1$  and  $\Gamma_2$ , there exist birational morphisms*

$$\Gamma \xrightarrow{\psi} \Gamma' \xrightarrow{p'_i} \Gamma_i, \quad i = 1, 2,$$

such that  $p_i$  is equivalent to  $p'_i \circ \psi$  for  $i = 1, 2$  and  $\Gamma'$ ,  $p'_i$ ,  $\Gamma_i$ ,  $i = 1, 2$ , fit in a relatively minimal diagram (3.2).

*Proof.* Let  $v \in \text{Vert}(\Gamma)$  be a  $(-1)$ -vertex contracted by both  $p_1$  and  $p_2$ . Replacing the  $p_i$  by equivalent birational morphisms  $\Gamma \rightarrow \Gamma_i$  as in Lemma 3.29(c), we may suppose that both  $p_1$  and  $p_2$  start with the same blowdown  $\sigma_v: \Gamma \rightarrow \Gamma_v$  of  $v$  and there are factorizations

$$p_i: \Gamma \xrightarrow{\sigma_v} \Gamma_v \xrightarrow{p'_{v,i}} \Gamma_i,$$

where  $\Gamma_v$ ,  $p'_{v,1}$  and  $p'_{v,2}$  form a diagram (3.2). Now recursion on the number of vertices of  $\Gamma$  ends the proof.  $\square$

### 3.5 Dual graphs of NC-pairs

Let us recall the correspondence between NC-pairs and weighted graphs.

DEFINITION 3.40. Given an NC-pair  $(X, D)$ , the *dual graph*  $\Gamma(D)$  is a weighted graph whose vertices are in bijection with the irreducible components of  $D$  and edges are in bijection with

the nodes of  $D$ . The loops of  $\Gamma(D)$  at a vertex  $C$  are in bijection with the self-intersection points of the component  $C$  of  $D$ , and the edges joining two different vertices  $C_1$  and  $C_2$  of  $\Gamma(D)$  are in bijection with the points of  $C_1 \cap C_2$ . The weight of  $C$  in  $\Gamma(D)$  is the self-intersection index  $C^2$  of the component  $C$  in  $X$ . The rational vertices in  $\text{Ratio}(\Gamma(D))$  correspond to the rational components of  $D$ .

An NC-pair  $(X, D)$  is an SNC-pair if and only if  $\Gamma(D)$  contains no loops and multiple edges; cf. Definition 2.1. Clearly, any NC-pair is dominated by an SNC-pair. An NC-pair  $(X, D)$  is minimal if the dual graph  $\Gamma(D)$  is minimal.

The notions of inner and outer blowups of an NC-pair are consistent with the notions of inner and outer blowups of the dual graph  $\Gamma(D)$ , respectively.

EXAMPLE 3.41. Recall that any non-complete normal algebraic surface  $Y$  admits an SNC-completion  $(X, D)$ , where  $Y = X \setminus \text{supp } D$ , and an NC-completion with a minimal dual graph.

For example, if  $X = \mathbb{P}^2$  and  $D$  is a nodal cubic, then  $\Gamma(D)$  is minimal and consists of a single vertex of weight 9 and a loop. So the pair  $(\mathbb{P}^2, D)$  is an NC-completion of the affine surface  $Y = \mathbb{P}^2 \setminus D$  with a minimal dual graph. There exists a non-minimal SNC-completion  $(X, D_1)$  whose dual graph  $\Gamma(D_1)$  is a cycle with three rational vertices and a cyclically ordered sequence of weights  $((-2, -1, 4))$ . Moreover, there exists a minimal such completion with sequence of weights  $((0, 0, -2, -2, -2, -2, -3))$ . Thus, two minimal cyclic graphs with rational vertices and cyclically ordered sequences of weights  $((9))$  and  $((0, 0, -2, -2, -2, -2, -3))$ , respectively, are birationally equivalent.

Similarly, one can show that for any  $a, b > 0$  the minimal linear graphs with sequences of weights

$$[[a]], \quad \underbrace{[-2, -2, \dots, -2, 0, 0]}_{a+1} \quad \text{and} \quad \underbrace{[-2, \dots, -2, -3]}_{a-1}, \underbrace{[-2, \dots, -2, 0, 1]}_{b+1}$$

are birationally equivalent; cf. Theorem A.8.

The assumption that the surface  $Y = X \setminus \text{supp } D$  is affine imposes severe restrictions on the dual graph  $\Gamma(D)$ .

LEMMA 3.42. *Let an NC-pair  $(X, D)$  be a completion of a normal affine surface  $Y = X \setminus \text{supp } D$ . Then the following hold:*

- (a) *The support  $\text{supp } D$  is connected and supports an effective ample divisor.*
- (b) *The dual graph  $\Gamma(D)$  is connected, the intersection form  $I(\Gamma(D))$  is not seminegative definite, and  $D$  and  $\Gamma(D)$  are not contractible.*
- (c) *If  $\text{supp } D = C$  is an irreducible, reduced curve, then  $C^2 > 0$  and the linear system  $|C|$  is ample. If, moreover,  $C$  is smooth and rational, then  $X$  is rational.*

*Proof.* According to the Goodman criterion of affiness for surfaces [Goo69, Theorem 2] (cf. also [Giz70, Lemma 2]),  $\text{supp } D$  coincides with the support of an ample divisor  $H$  on  $X$ . The connectedness of  $\text{supp } D = \text{supp } H$  now follows from the Lefschetz hyperplane section theorem; see for example [Goo69, p. 166, Corollary] or [Har70, Corollary II.6.2]. Hence the graph  $\Gamma(D)$  is connected. This proves statement (a).

Since  $H^2 > 0$ , the intersection form  $I(\Gamma(D))$  cannot be seminegative definite. By Lemma 3.17, the dual graph  $\Gamma(D)$  is not contractible; hence  $D$  also is not contractible. This proves statement (b).

Now assume that  $\text{supp } D = C$  is a reduced irreducible curve. Then  $C^2 > 0$ , and the linear system  $|C|$  is ample by the Nakai–Moishezon criterion. If  $C$  is smooth and rational, then  $X$  is rational as well; see [Giz70, Remarks 2 and 3] or [BHP<sup>+</sup>04, Proposition V.4.3]. This shows statement (c).  $\square$

Lemma 3.42 justifies the following convention.

*Convention 3.43.* In the following, we consider only weighted graphs  $\Gamma$  with no contractible connected component.

DEFINITION 3.44. Every decomposition of  $\Phi \in \text{Bir}((X_1, D_1), (X_2, D_2))$  into a sequence of isomorphisms, blowups and blowdowns induces a birational transformation of dual graphs  $\phi: \Gamma(D_1) \dashrightarrow \Gamma(D_2)$  called a *representation* of  $\Phi$ .

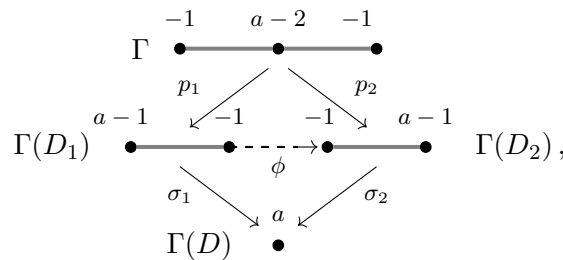
In the opposite direction, we have the following result.

PROPOSITION 3.45 (cf. [FKZ07, Proposition 3.34]). *Given an NC-pair  $(X, D)$  and a birational transformation of the dual graph  $\phi: \Gamma(D) \dashrightarrow \Gamma'$  consisting of a sequence of blowups and blowdowns, one can find a new NC-pair  $(X', D')$  and a birational transformation of NC-pairs  $\Phi: (X, D) \dashrightarrow (X', D')$  such that  $\Phi|_{X \setminus \text{supp } D}: X \setminus \text{supp } D \rightarrow X' \setminus \text{supp } D'$  is an isomorphism and  $\Phi$  is represented by  $\phi$ ; in particular,  $\Gamma(D') = \Gamma'$ . If  $\phi$  contains no outer blowup, then  $\Phi$  is uniquely defined up to isomorphism of NC-pairs.*

*Proof.* The first statement clearly holds for a single blowup and a single blowdown of  $\Gamma(D)$ . By recursion, it holds in the general case. The uniqueness follows by recursion from the facts that to an inner blowup of  $\Gamma(D)$  there corresponds the blowup of  $X$  at a uniquely defined node of  $D$ , and to a blowdown of a  $(-1)$ -vertex of  $\Gamma(D)$  there corresponds the contraction of a uniquely defined  $(-1)$ -component of  $D$ .  $\square$

Remark 3.46. Let  $(X, D)$  be an NC-pair, and let  $C$  be a smooth component of  $D$  that corresponds to a  $(0)$ -vertex  $v$  of  $\Gamma(D)$  of degree 1 or 2. Fix a point  $P \in C$  which is a node of  $D$  in the case where  $\deg_{\Gamma(D)}(v) = 2$ . Blowing  $P$  up and contracting the proper transform of  $C$  yields a birational transformation  $\Phi: (X, D) \dashrightarrow (X', D')$  called an *elementary transformation*. The corresponding birational transformation  $\phi$  of  $\Gamma(D)$  is also called *elementary*; it affects the weights of the neighbors of  $C$  in  $\Gamma(D)$ . See the proof of Lemma A.7 for an example.

Now let  $D = C$  be a smooth rational curve, so  $\Gamma(D)$  consists of a single isolated vertex  $C$  of weight  $a = C^2$  with no incident loop. Take two distinct points  $P_1$  and  $P_2$  of  $C$ . Blowing  $P_i$  up produces an NC-pair  $(X_i, D_i)$  whose dual graph  $\Gamma(D_i)$  has a sequence of weights  $[[ -1, a - 1 ]]$ . Let  $\sigma_i: X_i \rightarrow X$  be the blowdown of the exceptional curve. The pairs  $(X_1, D_1)$  and  $(X_2, D_2)$  are not isomorphic, in general, while the dual graphs  $\Gamma(D_1)$  and  $\Gamma(D_2)$  are. The corresponding birational transformation  $\Phi = \sigma_2^{-1}\sigma_1: (X_1, D_1) \dashrightarrow (X_2, D_2)$  is not an isomorphism of pairs. It restricts to the identity on  $Y = X \setminus \text{supp } D$ . On the level of dual graphs, the birational map  $\phi = \sigma_2^{-1}\sigma_1: \Gamma(D_1) \dashrightarrow \Gamma(D_2)$  in the induced representation of  $\Phi$  fits in the commutative diagram



where  $\Gamma$  is the dual graph of the pair  $(\tilde{X}, \tilde{D})$  obtained from  $(X, D)$  by blowing up the points  $P_1$  and  $P_2$  on  $X$ . Varying the positions of  $P_1$  and  $P_2$  on  $C$  yields non-trivial deformations of the participating pairs. Thus, once outer blowups are involved,  $\Phi$  cannot be reconstructed in general from its representation  $\phi = p_2 \circ p_1^{-1}$ .

We have the following geometric analog of Lemma 3.39.

**LEMMA 3.47.** *Any birational transformation  $\Phi: (X_1, D_1) \dashrightarrow (X_2, D_2)$  between minimal NC-pairs  $(X_i, D_i)$  fits in a commutative diagram (3.1) such that a representation  $\phi$  of  $\Phi$  fits into the corresponding relatively minimal diagram (3.2).*

*Proof.* Starting with an arbitrary commutative diagram (3.1), we repeat the procedure from the proof of Lemma 3.39 on the geometric level. Namely, to each at most linear  $(-1)$ -vertex of  $\Gamma(\tilde{D})$ , there corresponds a  $(-1)$ -component  $C$  of  $\tilde{D}$  which meets the union of other components transversally in at most two points. If  $C$  is contracted under the  $P_i: X \rightarrow X_i$ ,  $i = 1, 2$ , then there are factorizations

$$P_i: \tilde{X} \xrightarrow{\sigma_C} \tilde{X}' \xrightarrow{P'_i} X_i,$$

where  $\sigma_C$  stands for the contraction of  $C$  in  $\tilde{X}$ ; see [Bea96, Proposition II.8]. This produces a new NC-pair  $(\tilde{X}', \tilde{D}')$  which still dominates both  $(X_i, D_i)$  fitting in a new diagram (3.1) with the same  $\Phi: (X_1, D_1) \dashrightarrow (X_2, D_2)$ . A simultaneous resolution of singularities of  $X_1$ ,  $X_2$  and  $\tilde{X}$  does not affect our procedure. Hence we may assume that all these surfaces are smooth. We have  $\rho(\tilde{X}') = \rho(\tilde{X}) - 1$ , where  $\rho$  stands for the Picard rank; see for example [Bea96, Proposition II.3]. Induction on the Picard rank shows that this procedure ends with a relatively minimal diagram. Finally, we contract simultaneously the exceptional divisors of the resolutions and arrive at the same conclusion for the original surfaces.  $\square$

#### 4. The graph lemma and birational rigidity of weighted graphs

In this section, we elaborate criteria as to when  $\text{Bir}(X, D) = \text{Inn}(X, D)$  and, on the combinatorial level, as to when every relatively minimal birational transformation between two weighted graphs is inner; see Theorem 4.17 and Proposition 4.15, respectively. To this end, we apply an important tool called the graph lemma.

##### 4.1 The graph lemma

We use the following terminology.

**DEFINITION 4.1.** Let  $\Gamma$  be a minimal weighted graph. A circular segment  $S$  of  $\Gamma$  is called *admissible* if either the weights of its vertices are at most  $-2$ , or it consists of a single vertex of weight at most 2 and a loop. A linear segment  $L$  of  $\Gamma$  is called *admissible* if the weights of its vertices are at most  $-2$ .

The following graph lemma, Lemma 4.2, extends [FZ94, Proposition A1] and [FKZ07, Lemma 2.4(b)]. For the reader's convenience, we provide a proof. Recall that, for a contraction  $p: \Gamma \rightarrow \Gamma'$ , we consider  $\text{Vert}(\Gamma')$  as a subset of  $\text{Vert}(\Gamma)$ ; that is, we identify every vertex of  $\Gamma'$  with its proper preimage in  $\Gamma$ ; see Definition 3.23. For a vertex  $v \in \Gamma$ , we write  $p^v: \Gamma \rightarrow \Gamma^v$  for the maximal subsequence of blowdowns in  $p$  that preserve  $v$ ; see Notation 3.28.

LEMMA 4.2 (Graph lemma). *Consider a relatively minimal diagram (3.2) with connected weighted graphs  $\Gamma$ ,  $\Gamma_1$  and  $\Gamma_2$  and with birational morphisms  $p_i: \Gamma \rightarrow \Gamma_i$ , where the  $\Gamma_i$  are minimal for  $i = 1, 2$ . Then the following hold:*

(a) *For the sets of branch vertices, we have*

$$\text{Br}(\Gamma_1) = \text{Br}(\Gamma) \cap \text{Vert}(\Gamma_1) = \text{Br}(\Gamma) \cap \text{Vert}(\Gamma_2) = \text{Br}(\Gamma_2).$$

*Furthermore, the degrees of vertices in  $B := \text{Br}(\Gamma_1) = \text{Br}(\Gamma_2)$  do not change under the contractions  $p_1$  and  $p_2$ .*

(b) *Let  $\mathcal{C}$  be a connected component of  $\Gamma \ominus B$ . Then for  $i = 1, 2$  the image  $S_i = p_i(\mathcal{C})$  is a (non-empty) segment of  $\Gamma_i$ . Conversely, for any segment  $S_i$  of  $\Gamma_i$ , the total preimage  $p_i^*(S_i)$  is a connected component of  $\Gamma \ominus B$ . The induced correspondence  $S_1 \rightsquigarrow S_2$  between the segments of  $\Gamma_1$  and of  $\Gamma_2$  is a bijection that preserves the sets of circular segments, of inner linear segments and of extremal linear segments.*

(c) *Assume that  $\text{Br}(\Gamma) \neq B$ . Then for every vertex  $v \in \text{Br}(\Gamma) \setminus B$ , the following hold:*

(c<sub>1</sub>) *The vertex  $v$  is rational with no incident loop, and all branches of  $\Gamma$  at  $v$  are simple. There are two distinct branches  $W_1$  and  $W_2$  of  $\Gamma$  at  $v$  such that  $p_1$  contracts  $W_1$  and  $v$  and sends  $W_2$  into a non-admissible extremal linear segment of  $\Gamma_1$ , and symmetrically for  $p_2$ .*

(c<sub>2</sub>) *We have  $\deg_\Gamma(v) \in \{3, 4\}$ .*

(i) *If  $\deg_\Gamma(v) = 3$ , then  $p_i^v$  contracts at least one branch  $W_i$  at  $v$ ,  $i = 1, 2$ .*

(ii) *If  $\deg_\Gamma(v) = 4$ , then exactly two branches of  $\Gamma$  at  $v$ , say  $W_1$  and  $W'_1$ , are contracted in  $\Gamma_1$  and the other two, say  $W_2$  and  $W'_2$ , are contracted in  $\Gamma_2$ . The branches  $W_i$  and  $W'_i$ ,  $i = 1, 2$ , are linear, and the minimal graphs  $\Gamma_i$  are linear and non-admissible.*

See Proposition 4.9 below for a more detailed description of statement (c<sub>2</sub>)(ii).

*Proof.* (a) We start with the following claims.

CLAIM 1. *Let  $v \in \text{Vert}(\Gamma)$ .*

- (i) *Assume that a simple branch  $W_1$  of  $\Gamma$  at  $v$  is contracted by  $p_1^v$ . Then  $v$  is a rational vertex contracted by  $p_2$ . Furthermore,  $p_2^v$  induces an isomorphism of weighted graphs  $W_1 \cong p_2^v(W_1)$ , and all branches of  $\Gamma$  at  $v$  are simple.*
- (ii) *Let  $W_1$  and  $W_2$  be simple branches of  $\Gamma$  at  $v$  such that  $W_i$  is contracted by  $p_i^v$  for  $i = 1, 2$ . Then  $W_1 \neq W_2$  and  $v$  is contracted in both  $\Gamma_1$  and  $\Gamma_2$ .*

*Proof.* (i) All  $(-1)$ -vertices of  $W_1$  are contracted in  $\Gamma_1$ . By the assumption of relative minimality, no  $(-1)$ -vertex of  $W_1$  is contracted in  $\Gamma_2$ . Since  $\Gamma_2$  is supposed to be minimal,  $p_2(W_1)$  contains no  $(-1)$ -vertex. This can only happen provided  $p_2$  contracts  $v$  prior to the contraction of any vertex of  $W_1$ . Since  $p_2$  contracts no  $(-1)$ -vertex of  $W_1$ , the map  $p_2^v$  contracts no vertex of  $W_1$ , and so  $W_1 \cong p_2^v(W_1)$  as weighted graphs.

Suppose to the contrary that there is a non-simple branch  $W$  of  $\Gamma$  at  $v$ . Then  $p_2^v(W)$  is either a non-simple branch of  $p_2^v(\Gamma)$  at  $v$  different from the simple branch  $p_2^v(W_1) \cong W_1$ , or a loop incident to  $v' := p_2^v(v)$ ; see Lemma 3.31(a). It follows that  $v'$  is a branch vertex of  $p_2^v(\Gamma)$ . Hence  $v'$  cannot be contracted by  $p_2$  in the next steps; that is,  $p_2 = p_2^v$  does not contract  $v$ , contrary to the preceding conclusion. Statement (i) follows.

(ii) By statement (i), we have  $W_1 \neq W_2$ , and  $v$  is contracted in both  $\Gamma_1$  and  $\Gamma_2$ . □

CLAIM 2. We have  $\text{Br}(\Gamma_i) = \text{Br}(\Gamma) \cap \text{Vert}(\Gamma_i)$ .

*Proof.* Clearly,  $\text{Br}(\Gamma_i) \subset \text{Br}(\Gamma) \cap \text{Vert}(\Gamma_i)$ . Let us show the opposite inclusion. By symmetry, we can take  $i = 1$ . According to Definition 3.3, for a non-rational vertex  $v \in \text{Br}(\Gamma)$ , we have  $v \in \text{Br}(\Gamma_i)$  for  $i = 1, 2$ . Suppose to the contrary that there is a rational vertex  $v \in \text{Br}(\Gamma) \cap \text{Vert}(\Gamma_1) \setminus \text{Br}(\Gamma_1)$ . Then  $\deg_\Gamma(v) \geq 3$ , the vertex  $v$  is not contracted in  $\Gamma_1$ , and  $\deg_{\Gamma_1}(v) \leq 2$ . It follows that  $p_1 = p_1^v$  contracts at least  $\deg_\Gamma(v) - 2 > 0$  simple branches of  $\Gamma$  at  $v$ ; see Lemma 3.31(b). In particular, there is a simple branch  $W_1$  of  $\Gamma$  at  $v$  contracted by  $p_1^v$ . By Claim 1(i), the vertex  $v$  is contracted in  $\Gamma_2$ . Therefore,  $p_2^v$  also contracts at least  $\deg_\Gamma(v) - 2 > 0$  branches of  $\Gamma$  at  $v$ . So, there is a simple branch  $W_2$  of  $\Gamma$  at  $v$  contracted in  $\Gamma_2$ . Due to Claim 1(ii), the map  $p_1$  contracts  $v$  too, contrary to our assumption that  $v \in \text{Vert}(\Gamma_1)$ .  $\square$

CLAIM 3. We have  $\text{Br}(\Gamma_1) = \text{Br}(\Gamma_2)$ .

*Proof.* Assume to the contrary that  $\text{Br}(\Gamma_1) \neq \text{Br}(\Gamma_2)$ . Using Claim 2, we can consider by symmetry that there exists a rational vertex  $v \in \text{Br}(\Gamma) \cap \text{Vert}(\Gamma_1) \setminus \text{Vert}(\Gamma_2)$ . Since  $v$  is contracted in  $\Gamma_2$ , some simple branch  $W_2$  of  $\Gamma$  at  $v$  is contracted in  $\Gamma_2$ . According to Claim 1(i), the vertex  $v$  is contracted in  $\Gamma_1$ . This contradicts our choice of  $v \in \text{Vert}(\Gamma_1)$ .  $\square$

Claims 2 and 3 prove the equalities in statement (a). To show the last assertion in statement (a), we recall that the degree of a branch vertex  $v$  of  $\Gamma$  can drop only under a contraction of a simple branch of  $\Gamma$  at  $v$ ; see Lemma 3.31(a). According to Claim 1(i), if a simple branch at  $v \in B$  is contracted under  $p_1^v$  (respectively,  $p_2^v$ ), then  $v \notin \text{Vert}(\Gamma_2)$  (respectively,  $v \notin \text{Vert}(\Gamma_1)$ ). The latter contradicts our choice of  $v \in B = \text{Br}(\Gamma_1) = \text{Br}(\Gamma_2)$ .  $\square_{(a)}$

(b) Let  $\mathcal{C}$  be a connected component of  $\Gamma \ominus B$ . We proceed with the following claim.

CLAIM 4. The image  $S_i = p_i(\mathcal{C})$  is a (non-empty) connected component of  $\Gamma_i \ominus \text{Br}(\Gamma_i)$  for  $i = 1, 2$ .

*Proof.* Every vertex  $v \in \text{Vert}(\Gamma) \setminus \text{Vert}(\mathcal{C})$  linked to  $\mathcal{C}$  belongs to  $B = \text{Br}(\Gamma_i)$  for  $i = 1, 2$  by statement (a). In particular,  $v$  is not contracted in  $\Gamma_i$ ,  $i = 1, 2$ . It follows that there is a decomposition  $p_i = p'_i \circ p''_i = p''_i \circ p'_i$ , where  $p'_i$  (respectively,  $p''_i$ ) contracts only vertices from  $\text{Vert}(\mathcal{C})$  (respectively, from  $\text{Vert}(\Gamma) \setminus \text{Vert}(\mathcal{C})$ ).

Assume to the contrary that, say,  $p'_1$  contracts  $\mathcal{C}$ . Then  $S_2 \neq \emptyset$  due to the relative minimality assumption. Since  $\mathcal{C}$  is contractible, so is  $S_2 = p_2(\mathcal{C})$ ; see Remark 3.20(2). The latter contradicts the minimality of  $\Gamma_2$ . Thus,  $S_i \neq \emptyset$  for  $i = 1, 2$ .

By the argument above, every vertex of  $\text{Vert}(\Gamma_i) \setminus \text{Vert}(S_i)$  linked to a vertex of  $S_i$  belongs to  $\text{Br}(\Gamma_i)$ . Since  $S_i$  is connected, it follows that  $S_i$  is a connected component of  $\Gamma_i \ominus \text{Br}(\Gamma_i)$ .  $\square$

Let us return to the proof of statement (b). Since  $B = \text{Br}(\Gamma_i)$ , the component  $S_i$  contains no branch vertex of  $\Gamma_i$ . By Claim 4, we know that  $S_i$  is a segment of  $\Gamma_i$ . To show the converse, take a segment  $S_i$  of  $\Gamma_i$ . By definition,  $S_i$  is a connected component of  $\Gamma_i \ominus \text{Br}(\Gamma_i)$ . Therefore,  $\mathcal{C} := p_i^*(S_i)$  is a connected component of  $\Gamma \ominus B$ ; see Remark 3.24.

Thus,  $p_i$  induces a one-to-one correspondence between the segments of  $\Gamma_i$  and the connected components of  $\Gamma \ominus B$ . This yields a one-to-one correspondence between the sets of segments of  $\Gamma_1$  and of  $\Gamma_2$ .

The number of tips of  $S_i$  linked to  $\text{Br}(\Gamma_i)$  equals the number of vertices of  $\mathcal{C}$  linked to  $B$ . So, it is the same for  $i = 1, 2$ . This number equals 0 if  $S_i$  (respectively,  $\mathcal{C}$ ) is a connected component of  $\Gamma_i$  (respectively, of  $\Gamma$ ), equals 1 if  $S_i$  is a non-isolated extremal linear segment of  $\Gamma_i$ , and equals 2

if  $S_i$  is an inner linear segment of  $\Gamma_i$ . Therefore, the above correspondence preserves the subsets of circular, inner linear and extremal linear segments.  $\square_{(b)}$

(c<sub>1</sub>) Every non-rational vertex of  $\Gamma$  belongs to  $B = \text{Br}(\Gamma_i)$ ; see Definition 3.3. Any vertex  $v \in \text{Br}(\Gamma) \setminus B$  is rational; given such a vertex, there are simple branches  $W_1$  and  $W_2$  of  $\Gamma$  at  $v$  contracted by  $p_1^v$  and  $p_2^v$ , respectively; see Lemma 3.31(b). According to Claims 1(i) and (ii), the vertex  $v$  has no incident loop and is contracted by both  $p_1$  and  $p_2$ . Besides,  $W_1 \neq W_2$  and all branches of  $\Gamma$  at  $v$  are simple.

Since  $W_i$  is contracted in  $\Gamma_i$ , it contains no vertex of  $B = \text{Br}(\Gamma_1) = \text{Br}(\Gamma_2)$ ; see statement (a). Therefore, for  $j \neq i$  the image  $p_j(W_i)$  in  $\Gamma_j$  contains no branch vertex of  $\Gamma_j$ . It follows that  $p_j(W_i)$  is contained in a segment, say  $S_i$ , of  $\Gamma_j$ .

According to Lemma 3.32(b), the contractible tree  $W_i$  has a contractible extremal linear branch, say  $L_i$ . Let  $v_i$  be a  $(-1)$ -vertex of  $L_i$ . By the relative minimality assumption,  $v_i$  is not contracted in  $\Gamma_j$  for  $j \neq i$ . In particular,  $p_j(W_i)$  is non-empty. For  $j \neq i$ , the image  $p_j(L_i)$  contains a tip  $t$  of  $\Gamma_j$ ; see Lemma 3.32(a). Since  $t \in S_i$ , it follows that  $S_i$  is an extremal linear segment of  $\Gamma_j$ .

The weight of  $v_i$  in  $\Gamma$  can only increase under the contraction  $p_j$ , and it must indeed increase because of the minimality of  $\Gamma_j$ . Therefore, the weight of  $v_i$  in  $\Gamma_j$  is at least 0, and so the segment  $S_i$  is not admissible.  $\square_{c_1}$

(c<sub>2</sub>) We use the notation from statement (c<sub>1</sub>). According to Claim 1(ii), the branches of  $\Gamma$  at  $v$  contracted by  $p_1^v$ , and those contracted by  $p_2^v$  are distinct. In total, there are at least  $2 \deg_\Gamma(v) - 4 \leq \deg_\Gamma(v)$  contractible simple branches of  $\Gamma$  at  $v$ ; see Lemma 3.31(b). It follows that  $3 \leq \deg_\Gamma(v) \leq 4$ , and there are exactly  $\deg_\Gamma(v)$  (simple) branches of  $\Gamma$  at  $v$ .

(i) Suppose that  $\deg_\Gamma(v) = 3$ . By statement (a), the image  $p_i^v(v)$  is not a branch vertex of  $\Gamma_i^v$  for  $i = 1, 2$ . Hence  $p_i^v$  contracts at least one branch, say  $W_i$ , of  $\Gamma$  at  $v$ . By the preceding,  $W_1 \neq W_2$ .

(ii) Now suppose that  $\deg_\Gamma(v) = 4$ . Since  $p_i^v$  contracts at least two branches for  $i = 1, 2$ , and these four branches are distinct, two branches, say  $W_1$  and  $W'_1$ , of  $\Gamma$  at  $v$  are contracted by  $p_1^v$  and two others, say  $W_2$  and  $W'_2$ , are contracted by  $p_2^v$ .

According to statement (c<sub>1</sub>), for  $j \neq i$  the images  $p_j(W_i)$  and  $p_j(W'_i)$  are contained in extremal linear segments of  $\Gamma_j$ . Hence these images are linear chains, and their union equals  $\Gamma_j$ . None of these chains contain a branch vertex of  $\Gamma_j$ ; see the proof of statement (c<sub>1</sub>). Therefore, both  $\Gamma_1$  and  $\Gamma_2$  are linear graphs. Since the  $(-1)$ -vertices of  $W_i$  and  $W'_i$  are not contracted in  $\Gamma_j$ , their images in  $\Gamma_j$  have non-negative weights. So,  $\Gamma_1$  and  $\Gamma_2$  are not admissible.

Suppose to the contrary that there exists a second branch vertex, say  $v' \neq v$ , of  $\Gamma$ , and let  $V_1, \dots, V_k$ ,  $k \geq 3$ , be the branches of  $\Gamma$  at  $v'$ . We may assume that  $v' \in W_1$  and  $W'_1 \subset V_1$ . Then we have  $V_2 \cup \dots \cup V_k \subset W_1$ .

Since the branch  $W'_1$  is contractible and is contracted under  $p_1$ , it contains a  $(-1)$ -vertex that survives the contraction  $p_2$ . Hence the branch  $V_1$  of  $\Gamma$  at  $v'$  cannot be contracted under  $p_2$ .

Since  $V_2, \dots, V_k$  are contracted in  $\Gamma_1$  together with  $W_1$ , no  $(-1)$ -vertex of  $V_2 \cup \dots \cup V_k$  can be contracted under  $p_2$ . It follows that there is an isomorphism of weighted graphs  $V_i \cong p_2^{v'}(V_i)$  for  $i = 2, \dots, k$ . Since  $V_1$  is also not contracted under  $p_2^{v'}$ , the image of  $v'$  in  $p_2^{v'}(\Gamma)$  is a branch vertex of degree  $k \geq 3$ . Hence it cannot be contracted in the next step of contraction  $p_2$ . It follows that  $p_2 = p_2^{v'}$ , and so  $v' \in \text{Br}(\Gamma_2)$ . However, by the preceding,  $\Gamma_2$  is a linear graph. This gives a contradiction.  $\square_{c_2} \quad \square$

*Remark 4.3.* For a vertex  $v \in \text{Br}(\Gamma) \setminus B$  with  $\deg_\Gamma(v) = 3$ , it can happen that both  $p_1^v$  and  $p_2^v$  contract exactly one branch of  $\Gamma$  at  $v$ , as occurs in Example 4.4 below. It can also happen that,

say,  $p_1^v$  contracts a single (non-linear) branch of  $\Gamma$  at  $v$ , while two other branches of  $\Gamma$  at  $v$  are contracted by  $p_2^v$ ; see Example 4.6.

EXAMPLE 4.4. We present in Figure 3 a graph  $\Gamma$  that dominates two minimal linear graphs  $\Gamma_1$  and  $\Gamma_2$ . For  $i = 1, 2$ , the birational morphism  $p_i$  contracts the branch  $W_i$  of  $\Gamma$  and the vertex  $v$ . We have  $B = \emptyset$  and  $\deg_\Gamma(v) = 3$ . The simple branch  $T$  of  $\Gamma$  at  $v$  that consists of a single vertex  $a$  is not contractible. In fact, it can be replaced by any minimal weighted graph  $T$  linked to  $v$  by a single edge.

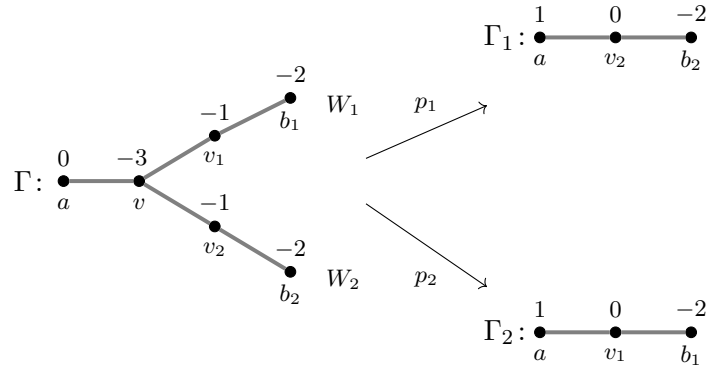


FIGURE 3. A weighted graph  $\Gamma$  with branches  $W_1$  and  $W_2$  at the vertex  $v$  that are contracted in  $\Gamma_1$  and  $\Gamma_2$ , respectively.

EXAMPLE 4.5. Figure 4 presents a graph  $\Gamma$  that dominates two minimal linear graphs  $\Gamma_1$  and  $\Gamma_2$ . The birational morphism  $p_1$  (respectively,  $p_2$ ) contracts the vertices  $b_1$  and  $b_2$  (respectively,  $a_1$  and  $a_2$ ) of  $\Gamma$  and the vertex  $v$ . We have  $B = \emptyset$  and  $\deg_\Gamma(v) = 4$ .

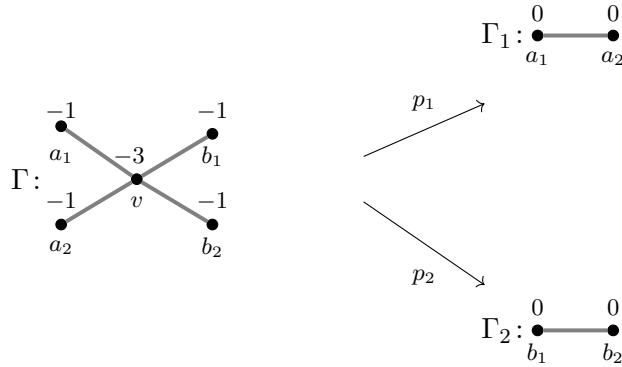
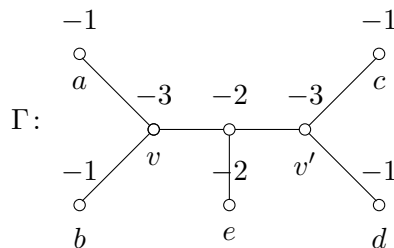


FIGURE 4. A weighted graph  $\Gamma$  with four branches at the vertex  $v$  that are contracted, two by two, in  $\Gamma_1$  and  $\Gamma_2$ , respectively.

EXAMPLE 4.6 (See [FKZ07, Example 2.6]). The graph



admits three different contractions  $p_i: \Gamma \rightarrow \Gamma_i$ ,  $i = 1, 2, 3$ , to the minimal linear graphs  $\Gamma_1 = \Gamma_2 := [[0, 0]]$  and  $\Gamma_3 := [[0, -2]]$ , respectively. The morphisms  $p_1$  and  $p_2$  contract the non-linear subgraphs  $\Gamma \ominus \{a, b\}$  and  $\Gamma \ominus \{c, d\}$ , respectively, and  $p_3$  contracts two linear subgraphs of  $\Gamma$  with tips  $a, b$  and  $c, d$ , respectively.

The contractions  $p_1$  and  $p_3$  (respectively,  $p_2$  and  $p_3$ ) fit in a diagram (3.2) that is not relatively minimal, while the diagram (3.2) with  $p_1$  and  $p_2$  is. In the latter case, we have  $B = \emptyset$  and  $\text{Br}(\Gamma)$  consists of three vertices of degree 3, including  $v$  and  $v'$  of weight  $-3$  and the third vertex, say  $v_0$ , of weight  $-2$ . The morphism  $p_1^v$  contracts the branch  $\Gamma \ominus \{a, b\}$  of  $\Gamma$  at  $v$ , while  $p_2^v$  contracts two other branches of  $\Gamma$  at  $v$  that consist of the tips  $a$  and  $b$ . The branch  $T$  of  $\Gamma$  at  $v_0$  consisting of the single vertex  $e$  is not contractible. Nevertheless,  $T$  is contracted by the  $p_i$  for  $i = 1, 2$ ; hence  $e \notin p_1^{-1}(\Gamma_1) \cup p_2^{-1}(\Gamma_2)$ .

In Proposition 4.9, we provide a description of all possible graphs  $\Gamma$  as in Lemma 4.2(c<sub>2</sub>)(ii). We use the following notation and results.

*Notation 4.7.* For a sequence of integer numbers  $k_1, \dots, k_n$  with  $k_1, \dots, k_n \geq 2$ , we let  $[k_1, \dots, k_n]$  be the minus continued fraction defined inductively via  $[k_1] = k_1$  and  $[k_1, \dots, k_n] = k_1 - 1/[k_2, \dots, k_n]$ . If  $[k_1, \dots, k_n] = m/e$ , where  $0 < e < m$  and  $\gcd(e, m) = 1$ , then we let

$$\frac{e}{m} \square = [[-k_1, \dots, -k_n]] \quad \text{and} \quad \frac{(e/m)^*}{\square} = [[-k_n, \dots, -k_1]].$$

For the subgraph  $L' = [[-k_2, \dots, -k_n]]$  of  $L = [[-k_1, \dots, -k_n]]$ , we have  $m = \text{discr}(L)$  and  $e = \text{discr}(L')$ ; see [FKZ07, Lemma 4.4].

LEMMA 4.8 ([FKZ07, Proposition 4.9(b)]). *A contractible weighted linear graph  $L$  has a unique  $(-1)$ -vertex, and the weights of other vertices are at most  $-2$ .*

Now let  $L = [[a_1, \dots, a_n]]$  be a weighted linear graph with  $a_k = -1$  for some  $k \in \{1, \dots, n\}$  and  $a_i \leq -2$  for all  $i \neq k$ . If  $1 < k < n$ , then  $L$  is contractible if and only if  $\text{discr}(L) = 1$  or, equivalently,

$$\frac{e_1}{m_1} + \frac{e_2}{m_2} = 1 - \frac{1}{m_1 m_2},$$

where

$$\frac{m_1}{e_1} = [-a_{k-1}, \dots, -a_1] \quad \text{and} \quad \frac{m_2}{e_2} = [-a_{k+1}, \dots, -a_n]$$

with  $e_i, m_i > 0$  and  $\gcd(e_i, m_i) = 1$  for  $i = 1, 2$ .

If  $k = 1$  (respectively,  $k = n$ ), then  $L$  is contractible if and only if  $L = [[-1, -2, \dots, -2]]$  (respectively,  $L = [[-2, \dots, -2, -1]]$ ).

PROPOSITION 4.9. (a) *Consider a relatively minimal diagram (3.2) with a connected weighted graph  $\Gamma$  and minimal weighted graphs  $\Gamma_1$  and  $\Gamma_2$  such that  $\Gamma$  has a vertex  $v \in \text{Br}(\Gamma) \setminus B$  of degree 4. Let  $L_{3-i} = p_i^v(\Gamma)$  for  $i = 1, 2$  be the induced weighted linear subgraph of  $\Gamma$  spanned by  $W_i, W'_i$  and  $v$ , where the weight of  $v$  is replaced by  $-1$ . Then  $L_{3-i}$  has the form*

$$L_{3-i} = \begin{array}{ccccccc} (e_{1,i}/m_{1,i})^* & e_{2,i}/m_{2,i} & v & (e_{3,i}/m_{3,i})^* & e_{4,i}/m_{4,i} \\ \square & \square & \circ & \square & \square \\ & & -1 & & -1 & & -1 \end{array}, \quad (4.1)$$

where

$$\frac{e_{j,i}}{m_{j,i}} + \frac{e_{j+1,i}}{m_{j+1,i}} = 1 - \frac{1}{m_{j,i}m_{j+1,i}} \quad \text{for } i = 1, 2 \text{ and } j = 1, 2, 3. \quad (4.2)$$

(b) Conversely, let  $\Gamma$  be a weighted graph with a unique branch vertex  $v$  of degree 4 and with four contractible linear branches at  $v$ , say,  $W_1, W'_1, W_2, W'_2$ . Suppose that for  $i = 1, 2$ , after the contraction of  $W_i$  and  $W'_i$ , the weight of  $v$  becomes equal to  $-1$  and the resulting graph  $L_{3-i}$  satisfies (4.1) and (4.2). Then  $\Gamma$  fits in a relatively minimal diagram (3.2) with non-admissible minimal linear graphs  $\Gamma_1$  and  $\Gamma_2$  and with birational morphisms  $p_i: \Gamma \rightarrow \Gamma_i$  such that  $p_i^v$  contracts  $W_i$  and  $W'_i$  for  $i = 1, 2$ .

*Proof.* (a) After the contraction of  $W_2$  and  $W'_2$  under  $p_2^v$ , the weight of  $v$  becomes  $-1$ , because  $v$  is contracted in the next step; see Lemma 4.2(c<sub>1</sub>). The image  $L_1$  of  $\Gamma$  under  $p_2^v$  is a linear graph with contractible linear branches  $W_1$  and  $W'_1$  at  $v$  that remain unchanged under  $p_2^v$ ; see Lemma 4.2(c<sub>1</sub>). Since a contractible linear graph has a unique  $(-1)$ -vertex,  $L_1$  has the desired form (4.1). Both  $W_1$  and  $W'_1$  are contracted under  $p_1$ ; hence we have the equalities in (4.2) for  $i = 1, j = 1$  and  $j = 3$ . By the relative minimality assumption, the contraction  $p_2$  of  $\Gamma$  to a minimal graph  $\Gamma_2$  does not contract the  $(-1)$ -vertices of  $W_1$  and  $W'_1$  but changes their weights. Therefore, the connected subgraph of  $L_1$  that connects the leftmost and rightmost  $(-1)$ -vertices of  $L_1$  is also contractible. By Proposition 4.8, this implies the last equality in (4.2) for  $i = 1$  and  $j = 2$ . By symmetry, the same conclusions hold for the linear graph  $L_2$ .

(b) The converse assertion follows easily from Lemma 4.8.  $\square$

*Remark 4.10.* Consider a normal affine surface  $Y$  and its NC-completion  $(X, D)$  with dual graph  $\Gamma(D)$ . Suppose that  $\Gamma(D)$  satisfies the assumptions of Proposition 4.9(a). According to Lemma 4.2(c<sub>2</sub>)(ii), in this case the dual graph of a minimal NC-completion of  $Y$  is linear. By Gizatullin's characterization of generic flexibility [Giz71b] extended by Dubouloz [Dub04], the surface  $Y$  is generically flexible; see the definition in the introduction. In particular,  $Y$  admits two  $\mathbb{G}_a$ -actions with distinct general orbits.

Moreover,  $Y$  admits an NC-completion  $(X_0, D_0)$  with an irreducible and smooth boundary divisor  $D_0$ . Indeed, it suffices to contract the four branches of  $D$  at the branch vertex  $v$  of  $\Gamma(D)$ ; see Lemma 4.2(c<sub>2</sub>)(ii). If  $Y$  is smooth, then by [Giz70, Proposition 1] the pair  $(X_0, D_0)$  is isomorphic to one of the following pairs:

$$(\mathbb{P}^2, L), \quad (\mathbb{P}^2, C) \quad \text{and} \quad (\mathbb{F}_n, S), \quad n \geq 0, \quad (4.3)$$

where  $L$  (respectively,  $C$ ) is a line (respectively, a conic) in  $\mathbb{P}^2$  and  $S$  is an ample section of the Hirzebruch surface  $\mathbb{F}_n \rightarrow \mathbb{P}^1$ . By a Danilov–Gizatullin theorem [GD77, Theorem 5.8.1], see also [FKZ09], two affine surfaces  $Y = \mathbb{F}_n \setminus S$  and  $Y' = \mathbb{F}_m \setminus S'$  are isomorphic if and only if  $S^2 = S'^2$ . Thus, up to isomorphism, the smooth affine surfaces that admit a minimal completion with the dual graph as in Proposition 4.9(a) form an infinite countable set.

Actually, every smooth affine surface from Gizatullin's list (4.3), and, more generally, every normal affine surface  $Y$  that can be completed by a single smooth rational curve  $D_0$ , also admits a completion as in Lemma 4.2(c<sub>2</sub>)(ii), that is, with a dual graph  $\Gamma$  that has two pairs of contractible linear branches at a single branch vertex  $v$  such that the corresponding minimal graphs  $\Gamma_i$ ,  $i = 1, 2$ , are linear and fit in a relatively minimal diagram (3.2). Indeed, let  $D_0^2 = k > 0$ . Take four distinct points  $P_1, \dots, P_4$  on  $D_0$ . Blowing up the  $P_j$  and their infinitesimally near points successively, we can transform the pair  $(X_0, D_0)$  into an SNC-pair  $(X, D)$  with  $Y = X \setminus D$  such that the weighted dual graph  $\Gamma(D)$  has a single branch vertex  $v$  of degree 4 that corresponds to

the proper transform of  $D_0$  on  $X$ , and four linear branches  $W_1, W'_1, W_2, W'_2$  at  $v$  of the form

$$W_1 \cong W_2 = [[-1, -2, \dots, -2]] \text{ of length } k \text{ and } W'_1 \cong W'_2 = [[-1]],$$

where all four  $(-1)$ -vertices are linked to  $v$ . After the contraction of all four branches, the weight of  $v$  becomes equal to  $k$ . For  $i = 1, 2$ , after the contraction of two branches  $W_i$  and  $W'_i$ , the weight of  $v$  becomes equal to  $-1$ ; the further blowing down of  $v$  yields a minimal linear graph  $\Gamma_i$ . Hence the graph  $\Gamma = \Gamma(D)$  satisfies the assumptions of Proposition 4.9(a).

## 4.2 Birationally rigid graphs

DEFINITION 4.11. A minimal weighted graph  $\Gamma_1$  is said to be *birationally rigid* if in any relatively minimal diagram (3.2) with a minimal graph  $\Gamma_2$ , no vertex of  $\text{Vert}(\Gamma_1) \subset \text{Vert}(\Gamma)$  is contracted in  $\Gamma_2$ . In other words,  $\text{Vert}(\Gamma_1) \subset \text{Vert}(\Gamma_2)$  in  $\Gamma$ .

We say that  $\Gamma_1$  is *admissible* if all its segments are admissible; see Definition 4.1.

EXAMPLES 4.12. The graph  $\Gamma_1 = [[0, 0]]$  is neither admissible, nor birationally rigid; see Examples 4.5 and 4.6. The same holds for the circular graph  $\Gamma_1 = ((3))$  with a single vertex  $v$  of weight 3 and a loop. Indeed, there is a relatively minimal diagram (3.2), where  $\Gamma$  is the circular graph  $((-1, -1))$  with two vertices  $v_1$  and  $v_2$ . It dominates, for  $i = 1, 2$ , the minimal graph  $\Gamma_i = ((3))$  with a single vertex  $v_{3-i}$  via the blowdown  $p_i$  of  $v_i$ .

LEMMA 4.13. A minimal weighted graph  $\Gamma_1$  is birationally rigid if and only if for every relatively minimal diagram (3.2) with a minimal graph  $\Gamma_2$ , the birational morphisms  $p_1$  and  $p_2$  are isomorphisms.

*Proof.* The “if” part is immediate. Let us prove the “only if” part. Suppose to the contrary that  $\Gamma_1$  is birationally rigid and that there exists a  $(-1)$ -vertex  $v \in \text{Vert}(\Gamma) \setminus \text{Vert}(\Gamma_2)$ . By the assumption of birational rigidity, we have  $v \in \text{Vert}(\Gamma) \setminus \text{Vert}(\Gamma_2) \subset \text{Vert}(\Gamma) \setminus \text{Vert}(\Gamma_1)$ . It follows that  $v$  is blown down by both  $p_1$  and  $p_2$ . The latter contradicts the relative minimality assumption. Thus,  $p_2$  is an isomorphism, and so  $\Gamma \cong \Gamma_2$  is minimal. Therefore,  $p_1$  is an isomorphism too.  $\square$

We also have the following criterion of birational rigidity; cf. [FZ94, Corollaries A.3 and A.4]. Notice that our birationally rigid graphs correspond to absolutely minimal graphs of [FZ94]. For the reader’s convenience, we provide a proof.

PROPOSITION 4.14. A minimal weighted graph  $\Gamma_1$  is birationally rigid if and only if it is admissible.

*Proof.* Assume to the contrary that all segments of  $\Gamma_1$  are admissible, but  $\Gamma_1$  is not birationally rigid. Then there is a relatively minimal diagram (3.2) with a minimal graph  $\Gamma_2$  and a vertex of  $\Gamma_1$  that is contracted in  $\Gamma_2$ . Since  $p_2$  is not an isomorphism, there is a  $(-1)$ -vertex  $v \in \text{Vert}(\Gamma)$  blown down the first under the morphism  $p_2: \Gamma \rightarrow \Gamma_2$ . Due to the relative minimality of diagram (3.2), the vertex  $v$  is not contracted under  $p_1: \Gamma \rightarrow \Gamma_1$ ; that is,  $v \in \text{Vert}(\Gamma_1) \subset \text{Vert}(\Gamma)$ .

Since  $v$  is an at most linear vertex of  $\Gamma$  and  $\text{Br}(\Gamma_1) \subset \text{Br}(\Gamma)$ , there is a segment  $S_1$  of  $\Gamma_1$  that contains  $v$ . Let  $w_1(v)$  be the weight of  $v$  in  $\Gamma_1$  and  $w(v)$  be its weight in  $\Gamma$ . Clearly,  $-1 = w(v) \leq w_1(v)$ . If the admissible segment  $S_1$  is either linear of any length, or circular with at least two vertices, then we have  $-1 = w(v) \leq -2$ , which gives a contradiction.

Now let  $S_1$  be a circular segment of  $\Gamma_1$  with a single vertex  $v$ . Since  $S_1$  is admissible, we have  $w_1(v) \leq 2$ ; see Definition 4.1. Let  $S$  be the total preimage of  $S_1$  in  $\Gamma$ . Assume that  $S$  has

at least two vertices. Since  $v$  is at most linear in  $\Gamma$ , no outer blowup was done at  $v$  under  $p_1^{-1}$ . If the loop of  $S_1$  at  $v$  was blown up under  $p_1^{-1}$ , then again  $-1 = w(v) \leq -2$  hold, and we have a contradiction. Otherwise,  $S$  contains a single vertex  $v$  and has an incident loop in  $\Gamma$ ; hence  $v$  cannot be contracted in  $\Gamma_2$ . Once again, this gives a contradiction.

To show the converse, assume that  $\Gamma_1$  has a non-admissible segment  $S_1$ . Both  $\Gamma_1$  and  $S_1$  are minimal, so they are not contractible. Being non-admissible,  $S_1$  has a vertex  $v$  of weight  $a$ , where  $a \geq 0$  if  $S_1$  is different from a circular segment with a single vertex and  $a \geq 3$  otherwise. Let us construct a relatively minimal diagram (3.2) with a minimal graph  $\Gamma_2$  such that  $v$  is contracted in  $\Gamma_2$ .

Blowing  $\Gamma_1$  up successively, we can reduce the weight of  $v$  to  $-1$  while keeping  $v$  at most linear. In more detail, first we blow up at an edge  $[u, v]$  incident to  $v$  in  $\Gamma_1$  unless  $\Gamma_1$  consists of a single isolated vertex  $v$ ; in the latter case we start with an outer blowup at  $v$ . Let  $e_1$  be the  $(-1)$ -vertex created in the first step. In the second step, we blow up the edge  $[e_1, v]$ , creating a new vertex  $e_2$ , etc. In the  $i$ th step, we blow up the edge  $[e_{i-1}, v]$  by adding a new vertex  $e_i$ . After  $k > 0$  successive blowups, where  $k = a + 1$  if  $S_1$  is different from a circular segment with a single vertex and  $k = a - 2$  otherwise, we obtain a weighted graph  $\Gamma$  with the vertices  $e_1, \dots, e_k$  appearing in this order under our procedure. For  $i = 1, \dots, k - 1$ , the weight of  $e_i$  in  $\Gamma$  equals  $-2$ , while  $e_k$  and  $v$  are at most linear  $(-1)$ -vertices of  $\Gamma$ . Notice that  $e_k$  is the unique  $(-1)$ -vertex of  $\Gamma$  blown down under the contraction  $p_1: \Gamma \rightarrow \Gamma_1$  of  $e_k, \dots, e_1$  in this order.

Starting with the blowdown of  $v$  in  $\Gamma$ , we continue to blow down the new at most linear  $(-1)$ -vertices until we reach a minimal graph  $\Gamma_2$ ; this yields a birational morphism  $p_2: \Gamma \rightarrow \Gamma_2$ . The graph  $\Gamma$  and the minimal graphs  $\Gamma_i$ ,  $i = 1, 2$ , along with the contractions  $p_i: \Gamma \rightarrow \Gamma_i$  form a diagram (3.2) (cf. for instance Example 4.12).

We claim that this diagram is relatively minimal. Indeed, after the first blowdown of  $v$  under  $p_2$ , the weight of  $e_k$  becomes 0, and it remains non-negative during the successive blowdowns which form  $p_2$ . Thus, the only  $(-1)$ -vertex  $e_k$  of  $\Gamma$  blown down under  $p_1$  is not blown down under  $p_2$ . This proves our claim. Since the vertex  $v$  of  $\Gamma_1$  is contracted in  $\Gamma_2$ , the graph  $\Gamma_1$  is not birationally rigid; see Definition 4.11.  $\square$

Summarizing, we get the following proposition.

**PROPOSITION 4.15.** *Assume that we are given a relatively minimal diagram (3.2) with minimal weighted graphs  $\Gamma_1$  and  $\Gamma_2$  dominated by a graph  $\Gamma$ . Then the following hold:*

- (a) *The birational map  $\phi := p_2 \circ p_1^{-1}: \Gamma_1 \dashrightarrow \Gamma_2$  induces*
  - (i) *a bijection between the sets of branch vertices  $\text{Br}(\Gamma_1) \simeq \text{Br}(\Gamma_2)$ ;*
  - (ii) *a bijection between the sets of segments of  $\Gamma_1$  and  $\Gamma_2$  that preserves the subsets of admissible segments, of inner (respectively, extremal) linear segments and of circular segments;*
  - (iii) *isomorphisms between every pair of corresponding admissible linear (respectively, admissible circular) segments;*
  - (iv) *an inner birational transformation between every pair of corresponding non-admissible inner linear segments;*
  - (v) *a birational transformation between every pair of corresponding non-admissible extremal linear segments.*
- (b) *If the  $\Gamma_i$  in diagram (3.2) are circular graphs, then  $\Gamma$  is also circular and the birational morphisms  $p_i$  are inner.*

- (c) If all extremal linear segments in either  $\Gamma_1$  or  $\Gamma_2$  are admissible, then the birational transformation  $\phi = p_2 \circ p_1^{-1}: \Gamma_1 \dashrightarrow \Gamma_2$  is inner and induces a bijection between the sets of tips  $\text{Tip}(\Gamma_1) \simeq \text{Tip}(\Gamma_2)$ .
- (d) If  $\phi$  is inner and the  $\Gamma_i$  are linear graphs, then  $\Gamma$  also is a linear graph.
- (e) If  $\phi$  is inner and the  $\Gamma_i$  are linear graphs with two vertices, then the  $p_i: \Gamma \xrightarrow{\cong} \Gamma_i$  are isomorphisms of weighted graphs.

*Proof.* Statement (a)(i) follows from Lemma 4.2(a). By Lemma 4.2(b), we may restrict  $\phi$  to any segment of  $\Gamma_1$  extended by its incident edges and adjacent branch vertices. Such a restriction includes all the blowups and blowdowns in  $\phi$  that happen on this segment, its incident edges and their successive images. It yields again a relatively minimal diagram (3.2) with minimal dominated graphs. Thus, statement (a)(ii) (respectively, statement (a)(iii)) follows from Lemma 4.2(b) (respectively, from Lemma 4.13 and Proposition 4.14).

To prove statement (a)(iv), we have to show that a relatively minimal birational transformation between non-admissible inner linear segments is inner. Assume the contrary; then the connected component of  $\Gamma \setminus B$  corresponding to these segments contains a branch vertex  $v \in \text{Br}(\Gamma) \setminus B$ . Then by Lemma 4.2(c<sub>1</sub>) these segments are extremal, contrary to our assumption.

See [FKZ07, Lemma 2.7] for statement (b). Statement (c) follows immediately from statements (a)(iii), (a)(iv) and (b). To show statement (d), it suffices to observe that an inner blowup cannot create a new branch vertex.

To show statement (e), suppose that the linear segment  $\Gamma$  in diagram (3.2) contains more than two vertices. Since  $p_i$  is inner, it cannot contract a tip of  $\Gamma$ . Hence it contracts all vertices in  $\text{Vert}(\Gamma) \setminus \text{Tip}(\Gamma)$ . Therefore, there is a  $(-1)$ -vertex  $v \in \text{Vert}(\Gamma) \setminus \text{Tip}(\Gamma)$  contracted in both  $\Gamma_1$  and  $\Gamma_2$ . The latter is impossible since our diagram (3.2) is relatively minimal. Hence  $\Gamma$  contains just two vertices, and so the  $p_i$  are isomorphisms of weighted graphs.  $\square$

*Remark 4.16.* Given a relatively minimal diagram (3.2), where  $\Gamma_1$  and  $\Gamma_2$  are minimal non-admissible linear graphs, the morphisms  $p_i$  do not need to be inner, in general; see for instance Example 4.4 and [FKZ07, Examples 2.6 and 3.30].

Proposition 4.15 leads to the following theorem.

**THEOREM 4.17.** *Let  $(X, D)$  be an NC-pair with minimal dual graph  $\Gamma(D)$  such that all extremal linear segments of  $\Gamma(D)$  are admissible. Then  $\text{Bir}(X, D) = \text{Inn}(X, D)$ .*

*Proof.* By Lemma 3.47, any  $\Phi \in \text{Bir}(X, D)$  fits in a relatively minimal commutative diagram (3.1) of NC-pairs. Hence  $\Phi = P_2 \circ P_1^{-1}$ , where  $P_i^{-1}: (X, D) \dashrightarrow (\tilde{X}, \tilde{D})$  for  $i = 1, 2$  is composed of a sequence of blowups of  $X$  in points of  $D$  and infinitesimally near points followed by an isomorphism. Let  $p_i: \Gamma(\tilde{D}) \rightarrow \Gamma(D)$  be the birational morphism of dual graphs induced by  $P_i$ ,  $i = 1, 2$ . Then the induced birational transformation of dual graphs  $\phi = p_2 \circ p_1^{-1}: \Gamma(D) \dashrightarrow \Gamma(D)$  (see Definition 3.40) fits in a relatively minimal diagram (3.2). According to Proposition 4.15, the map  $\phi$  is an inner birational transformation of weighted graphs. Hence  $\Phi$  is an inner birational transformation of NC-pairs; that is,  $\Phi \in \text{Inn}(X, D)$ .  $\square$

## 5. $\mathbb{G}_a$ -actions, affine rulings and dual graphs

The following lemma is well known.

LEMMA 5.1 ([FZ05, Corollary 1.2]). *If an affine variety  $Y$  of dimension at least 2 admits a non-trivial  $\mathbb{G}_a$ -action, then  $\text{Aut}(Y)$  contains a connected infinite-dimensional commutative unipotent ind-subgroup. In particular, it contains algebraic unipotent subgroups of arbitrarily large dimensions.*

*Proof.* Indeed, let  $H \subset \text{Aut}^\circ(Y)$  be the one-parameter unipotent subgroup which corresponds to the given  $\mathbb{G}_a$ -action. Then  $H$  can be written as  $H = \exp(\mathbb{K}\partial)$ , where  $\partial$  is a non-zero locally nilpotent derivation of the algebra  $\mathcal{O}(Y)$ ; see [Fre06]. The ring of invariants  $\ker \partial = \mathcal{O}(Y)^H$  is an infinite-dimensional vector space, and  $\exp((\ker \partial)\partial)$  is a connected infinite-dimensional commutative unipotent subgroup of  $\text{Aut}^\circ(Y)$ . For any finite-dimensional subspace  $V \subset \ker \partial$ , the group  $\exp(V\partial)$  is an algebraic unipotent subgroup of  $\text{Aut}^\circ(Y)$ .  $\square$

Remark 5.2. The assumption in Lemma 5.1 that  $Y$  is affine is important. Indeed, an open surface  $Y = \mathbb{P}^1 \times \mathbb{A}^1$  admits an effective  $\mathbb{G}_a$ -action, and at the same time  $\text{Aut}(Y)$  is an algebraic group; see [Lam05] for a description of  $\text{Aut}(Y)$ .

Recall that an  $\mathbb{A}^1$ -fibration, or an *affine ruling*, on a normal affine surface  $Y$  is a morphism  $Y \rightarrow B$  to a smooth curve with general fiber isomorphic to the affine line  $\mathbb{A}^1$ . The following lemma is well known; see for example [FKZ13, Lemma 1.6]; cf. the proof of Proposition 5.4.

LEMMA 5.3. *Let  $Y$  be a normal affine surface. Given an  $\mathbb{A}^1$ -fibration  $\pi: Y \rightarrow B$  over a smooth affine curve  $B$ , there exists an SNC-completion  $(X_0, D_0)$  of  $Y$  with the following properties:*

- *The fibration  $\pi$  extends to a  $\mathbb{P}^1$ -fibration  $\bar{\pi}: X_0 \rightarrow \bar{B}$  over a smooth completion  $\bar{B}$  of  $B$ .*
- *There is a unique component  $S$  of  $D_0$  with  $S^2 = 0$  which is a section of  $\bar{\pi}$ .*
- *All other components of  $D_0$  are smooth rational curves contained in fibers of  $\bar{\pi}$ .*
- *The dual graph  $\Gamma(D_0)$  is a minimal tree.*
- *The (0)-vertices of  $\Gamma(D_0)$  different from  $S$  are tips; they correspond to the irreducible fibers of  $\bar{\pi}$  contained in  $D_0$ .*

The following proposition proves Corollary 1.4. The equivalence (a)  $\Leftrightarrow$  (b) is contained in [FZ05, Lemma 1.6]; see also [FKZ13, Remark 1.7]. For the convenience of the reader, we provide a proof.

PROPOSITION 5.4. *Let  $Y$  be a normal affine surface and  $(X, D)$  be a minimal NC-completion of  $Y$ . Then the following conditions are equivalent:*

- (a) *The surface  $Y$  admits an effective  $\mathbb{G}_a$ -action.*
- (b) *The surface  $Y$  admits an  $\mathbb{A}^1$ -fibration over a smooth affine curve.*
- (c) *The dual graph  $\Gamma(D)$  has a non-admissible extremal linear segment.*

*Proof.* Assume that statement (c) holds. The dual graph  $\Gamma(D)$  is birationally equivalent to a graph  $\Gamma_0$  with a rational (0)-vertex  $v$  of degree 1; see for instance [FKZ07, Examples 2.11]. By Proposition 3.45, we can replace the original NC-completion  $(X, D)$  with a new one  $(X_0, D_0)$  such that  $\Gamma(D_0) = \Gamma_0$ . The vertex  $v$  corresponds to a smooth rational component, say  $C_0$ , of  $D_0$  with  $C_0^2 = 0$ . Resolving singularities of  $X_0$ , we obtain a smooth projective surface  $\tilde{X}_0$  with the exceptional divisor  $E$  and a smooth rational curve  $C'_0$  disjoint with  $E$  (namely, the proper transform of  $C_0$  on  $\tilde{X}_0$ ) such that  $C'^2_0 = 0$ . Indeed, the singular points of  $X_0$  being points of  $Y = X_0 \setminus \text{Supp } D_0$ , the resolution does not affect the divisor  $D_0$ . Such a curve  $C'_0$  on  $\tilde{X}_0$  is a reduced member of a linear pencil of rational curves with no base point; see for example

[BHP<sup>+</sup>04, Proposition V.4.3]. This pencil defines a  $\mathbb{P}^1$ -fibration  $\tilde{\pi}: \tilde{X}_0 \rightarrow \bar{B}$  over a smooth projective curve  $\bar{B}$  such that  $C'_0$  is a reduced fiber of  $\tilde{\pi}$ . Since  $E \cdot C'_0 = 0$ , each connected component of  $E$  is properly contained in a fiber of  $\tilde{\pi}$ . The contraction of  $E$  yields a  $\mathbb{P}^1$ -fibration  $\pi_0: X_0 \rightarrow \bar{B}$  with a reduced fiber  $C_0$  that lies in the smooth locus of  $X_0$ .

The surface  $Y = X_0 \setminus \text{supp } D_0$  being affine,  $\text{supp } D_0$  is connected and supports an effective ample divisor; see Lemma 3.42(a). Furthermore,  $Y$  contains no complete curve, in particular, no entire fiber of  $\pi_0$ . It follows that  $\text{supp } D_0 \neq C_0$ . Since  $\deg(v) = 1$ , the curve  $C_0$  meets transversally just one component  $S \neq C_0$  of  $D_0$ . Since  $C_0 \cdot S = 1$ , the component  $S$  is a section of  $\pi_0$ . Hence  $\pi_0|_Y: Y \rightarrow B$  is an  $\mathbb{A}^1$ -fibration over a smooth affine curve  $B \subset \bar{B} \setminus \{\pi_0(C_0)\}$ . This proves the implication (c)  $\Rightarrow$  (b).

Now assume that statement (b) holds. By Lemma 5.3, the  $\mathbb{A}^1$ -fibration  $\pi: Y \rightarrow B$  over a smooth affine curve  $B$  admits an extension  $\tilde{\pi}: X_0 \rightarrow \bar{B}$  to an SNC-completion  $(X_0, D_0)$  of  $Y$ , where  $\bar{B}$  is a smooth completion of  $B$ . The places at infinity of general  $\mathbb{A}^1$ -fibers of  $\pi$  lie on a section, say  $S$ , of  $\tilde{\pi}$ .

A contraction of at most linear  $(-1)$ -vertex of  $\Gamma(D_0)$  corresponds to a contraction of an exceptional  $(-1)$ -component of  $D_0$  preserving the NC-condition. Every branch  $W$  of  $\Gamma(D_0)$  at the vertex  $[S]$  corresponds to the intersection  $D_0(P) := D_0 \cap \tilde{\pi}^{-1}(P)$  for a point  $P \in \bar{B}$ . Contracting subsequently at most linear  $(-1)$ -vertices of  $W$ , we transform the divisor  $D_0(P)$  into a minimal divisor, say  $C_P$ . If  $P \in \bar{B} \setminus B$ , then  $D_0(P)$  coincides with the full fiber  $\tilde{\pi}^{-1}(P)$  and the corresponding divisor  $C_P$  is a reduced  $\mathbb{P}^1$ -curve that corresponds to a  $(0)$ -vertex of the dual graph. We have  $C_P \cdot S = 1$ .

After all such birational transformations in the fibers of  $\tilde{\pi}$  that carry components of  $D_0$ , it might happen that  $[S]$  becomes an at most linear  $(-1)$ -vertex of the resulting dual graph. In this case we perform an elementary transformation at a point of a full fiber  $C_P$  with  $P \in \bar{B} \setminus B$ ; see Remark 3.46. This changes the weight of  $[S]$  by 1. In this way, we can finally arrive at a minimal NC-completion  $(X'_0, D'_0)$  of  $Y$  along with a  $\mathbb{P}^1$ -fibration  $\tilde{\pi}': X'_0 \rightarrow \bar{B}$  that restricts to  $\pi$  on  $Y$ . The birationally equivalent minimal dual graphs  $\Gamma_1 = \Gamma(D)$  and  $\Gamma_2 = \Gamma(D'_0)$  fit in a relatively minimal diagram (3.2); see Proposition 3.36.

For  $P \in \bar{B} \setminus B$ , the  $\mathbb{P}^1$ -component  $C_P$  of  $D'_0$  corresponds to a  $(0)$ -vertex  $v$  of the dual graph  $\Gamma(D'_0)$  that is a tip of a non-admissible extremal linear segment  $L'$  of  $\Gamma(D'_0)$ . By Proposition 4.15(a)(ii), the segment  $L'$  corresponds to a non-admissible extremal linear segment  $L$  of  $\Gamma(D)$ . This proves the equivalence (c)  $\Leftrightarrow$  (b).

Suppose that statement (a) holds, that is,  $Y$  admits an effective  $\mathbb{G}_a$ -action. Let  $\delta$  be the non-zero locally nilpotent derivation on  $\mathcal{O}_Y(Y)$  generating the given  $\mathbb{G}_a$ -action. The subalgebra  $\ker \delta \subset \mathcal{O}_Y(Y)$  of  $\mathbb{G}_a$ -invariants is finitely generated, see [Fie94, Lemma 1.1], and  $B = \text{Spec}(\ker \delta)$  is a smooth affine curve. The embedding  $\ker \delta \hookrightarrow \mathcal{O}_Y(Y)$  yields an  $\mathbb{A}^1$ -fibration  $Y \rightarrow B$  along the orbits of the  $\mathbb{G}_a$ -action. Thus, statement (b) holds.

Conversely, suppose that statement (b) holds, and let  $\pi: Y \rightarrow B$  be an  $\mathbb{A}^1$ -fibration over a smooth affine curve  $B$ . Consider as before an SNC-completion  $(X_0, D_0)$  of  $Y$  and an extension  $\tilde{\pi}: X_0 \rightarrow \bar{B}$  to a  $\mathbb{P}^1$ -fibration on  $X_0$  with a section  $S \subset D_0$ . Shrinking the base  $B$  suitably, we obtain a principal Zariski open cylinder  $U \cong Z \times \mathbb{A}^1$  on  $Y$  with an affine base  $Z \subset B$ , where  $U = \tilde{\pi}^{-1}(Z) \setminus S$ . By [KPZ11, Proposition 3.1.5], there is an effective  $\mathbb{G}_a$ -action on  $Y$ . This shows the equivalence (a)  $\Leftrightarrow$  (b).  $\square$

*Remark 5.5.* Under the assumptions of Proposition 5.4, suppose that  $\Gamma(D)$  has a non-admissible extremal linear segment. We claim that in this case  $\Gamma(D)$  is a tree with at most one non-rational

vertex. Indeed, let  $(X_0, D_0)$  and  $S$  be as in the proof of the proposition. Then  $\Gamma(D_0)$  is a tree with a root  $[S]$ , whose branches  $C_P$  at  $[S]$  are trees with only rational vertices. Since  $\Gamma(D)$  and  $\Gamma(D_0)$  are birationally equivalent, the same holds for  $\Gamma(D)$  as well.

## 6. Main results

In this section, we prove our main result, Theorem 1.3. In addition, we show that  $\text{Bir}(X, D) = \text{Inn}(X, D)$  if and only if the surface  $Y = X \setminus \text{supp } D$  admits no  $\mathbb{G}_a$ -action. In the latter case, the identity component  $\text{Aut}^\circ(X)$  is an algebraic torus. The next theorem yields Theorem 1.3 from the introduction.

**THEOREM 6.1.** *Let  $(X, D)$  be a minimal NC-completion of a normal affine surface  $Y$ . Then the following are equivalent:*

- (a) *All extremal linear segments of the dual graph  $\Gamma(D)$  are admissible.*
- (b) *The surface  $Y$  admits no effective  $\mathbb{G}_a$ -action.*
- (c) *We have  $\text{Bir}(X, D) = \text{Inn}(X, D)$ .*
- (d) *The group  $\text{Aut}^\circ(Y)$  is an algebraic torus of rank at most 2.*

*Proof.* Properties (a) and (b) are equivalent by Proposition 5.4. The implication (a)  $\Rightarrow$  (c) follows from Theorem 4.17.

To deduce the implication (c)  $\Rightarrow$  (b), assume that property (c) holds. If on the contrary  $Y$  admits an effective  $\mathbb{G}_a$ -action, then  $\text{Bir}(X, D) = \text{Aut}(Y)$  contains connected algebraic subgroups of arbitrary dimension; see Lemma 5.1. This contradicts the fact that every connected algebraic subgroup  $G$  of  $\text{Bir}(X, D) = \text{Inn}(X, D)$  is contained in the algebraic group  $\text{Aut}^\circ(X, D)$ ; see Proposition 2.14. Thus, conditions (a), (b) and (c) are equivalent.

The implication (d)  $\Rightarrow$  (b) is immediate. Now suppose that one of the equivalent conditions (a)–(c) is fulfilled. By condition (c), we have  $\text{Aut}(Y) = \text{Bir}(X, D) = \text{Inn}(X, D)$ . By Proposition 2.14, the closed connected ind-subgroup  $\text{Aut}^\circ(Y) = \text{Inn}^\circ(X, D)$  of  $\text{Bir}(X, D)$  coincides with the connected affine algebraic subgroup  $\text{Aut}^\circ(X, D)$ . Due to condition (b), the latter group contains no  $\mathbb{G}_a$ -subgroup, hence no unipotent element. Therefore,  $\text{Aut}^\circ(Y)$  is an algebraic torus. This proves the converse implication (b)  $\Rightarrow$  (d).  $\square$

Combining Theorem 6.1 and Proposition 5.4 and its proof, we arrive at the following conclusion. It implies Corollary 1.4 from the introduction; cf. [FKZ07, Proposition 3.34].

**COROLLARY 6.2.** *For a normal affine surface  $Y$ , the following are equivalent:*

- (a) *The surface  $Y$  admits an effective  $\mathbb{G}_a$ -action.*
- (b) *The surface  $Y$  admits an  $\mathbb{A}^1$ -fibration over a smooth affine curve.*
- (c) *For every minimal NC-completion  $(X, D)$  of  $Y$ , the dual graph  $\Gamma(D)$  has a non-admissible extremal segment.*
- (d) *There exists a minimal NC-completion  $(X_0, D_0)$  of  $Y$  such that  $\Gamma(D_0)$  has a (0)-vertex of degree 1.*
- (e) *We have  $\text{Bir}(X, D) \neq \text{Inn}(X, D)$  for every minimal NC-completion  $(X, D)$  of  $Y$ .*
- (f) *There exists a minimal NC-completion  $(X_0, D_0)$  of  $Y$  such that  $\text{Bir}(X_0, D_0) \neq \text{Inn}(X_0, D_0)$ .*

EXAMPLE 6.3. Consider the affine surface  $Y = \mathbb{P}^2 \setminus \text{supp } D$ , where  $D$  is a non-zero reduced effective divisor on  $\mathbb{P}^2$  with only nodes as singularities. Let us show, as a simple application of Corollary 6.2, that  $Y$  is rigid if and only if  $\deg(D) \geq 3$ .

First of all, consider the case  $\deg(D) \leq 2$ . In this case,  $\text{supp } D$  is either a projective line, or a pair of lines, or a smooth conic. In all three cases, there exists a pencil of conics  $\mathcal{L}$  on  $\mathbb{P}^2$  which includes  $D$  (respectively,  $2D$  in the former case) and has a unique base point. The restriction  $\mathcal{L}|_Y$  yields an  $\mathbb{A}^1$ -fibration over  $\mathbb{A}^1$ . Thus,  $Y$  admits an effective  $\mathbb{G}_a$ -action, and so is not rigid; see Corollary 6.2(a)–(b).

Suppose further that  $\deg(D) \geq 3$ . Using the Bezout theorem and analyzing separately the cases where  $D$  has one, two and at least three components, one can easily conclude that every component  $C$  of  $D$  corresponds either to a branch vertex of  $\Gamma(D)$  or to a vertex sitting on a cycle. Anyway,  $\text{Tip}(\Gamma(D)) = \emptyset$ . Therefore,  $\Gamma(D)$  has no extremal linear segment. By Corollary 6.2, this implies the rigidity.

As an alternative proof one can use the fact that  $\bar{k}(Y) = -\infty$  provided  $Y$  admits an effective  $\mathbb{G}_a$ -action, where  $\bar{k}$  stands for the logarithmic Kodaira dimension. One has  $K_{\mathbb{P}^2} + D = (\deg(D) - 3)H$ , where  $H \in \text{Pic}(\mathbb{P}^2)$  is the class of a line. Since  $D$  is nodal and reduced, we have  $\bar{k}(Y) = -\infty$  if and only if  $\deg(D) \leq 2$ .

A more direct approach is as follows.<sup>1</sup> Suppose that  $Y$  admits an effective  $\mathbb{G}_a$ -action. Consider an SNC-completion  $(X', D')$  as in Lemma 5.3. Since  $X$  is rational, the dual graph  $\Gamma(D')$  is a tree with only rational vertices; see Remark 5.5. The latter properties of  $\Gamma(D')$  are preserved under birational transformations between NC-pairs. Hence  $\Gamma(D)$  is also a tree with only rational vertices. By the Bezout theorem, this implies as before that  $\deg(D) \leq 2$ .

## Appendix. Minimal models of weighted graphs

All weighted graphs considered in this section are supposed to have only rational vertices. Among these graphs we classify those that have a unique-up-to-isomorphism minimal model, and show that any other weighted graph with only rational vertices has an infinite number of non-isomorphic minimal models; see Theorem A.8. Recall that  $[[w_1, \dots, w_n]]$  (respectively,  $((w_1, \dots, w_n))$ ) represents the linear (respectively, circular) graph with ordered (respectively cyclically ordered) sequence of weights  $(w_1, \dots, w_n)$ . We abbreviate by  $a_k$  the sequence  $\underbrace{(a, \dots, a)}_k$ .

The following corollary of Propositions 4.14 and 4.15(a)(i)–(iii) is immediate.

COROLLARY A.1. *Let  $\Gamma$  be a birationally rigid minimal weighted graph. Then up to isomorphism  $\Gamma$  is a unique minimal graph in its class of birational equivalence.*

We provide below examples of non-birationally rigid minimal weighted graphs with a unique minimal model; see Proposition A.4. We use the following definitions.

DEFINITION A.2 (Triangulation of a circular graph). Let  $\mathcal{C}$  be a simplicial 2-complex homeomorphic to a disc, with all vertices lying on the boundary of the disc. In this case, the incidence graph of triangles in  $\mathcal{C}$  is a tree. One associates with  $\mathcal{C}$  a weighted circular graph that consists of the 1-complex  $B = \partial\mathcal{C}$  by attributing the weight  $-k + 1$  to a vertex of degree  $k$  of  $\mathcal{C}$ . We call  $\mathcal{C}$  a *triangulation* of the weighted circular graph  $B$ , and we say that  $B$  is *triangulable*.

<sup>1</sup>We thank Shulim Kaliman, who suggested this approach.

DEFINITION A.3. We say that an inner segment  $S$  of a weighted graph  $\Gamma$  is a *charm earring* if  $S$  consists of a single rational (0)-vertex of degree 2 adjacent to a branch vertex via two edges. We say that a minimal graph  $\Gamma$  is *admissible modulo charm earrings* if every non-admissible segment of  $\Gamma$  is a charm earring; see Definition 4.1.

PROPOSITION A.4. Let  $\Gamma_1$  be a graph that either is admissible modulo charm earrings or coincides with one of the graphs

$$[[0]], \quad ((3)), \quad ((4)), \quad ((0,0)) \quad \text{and} \quad ((0,m)) \quad \text{with } m \leq -2.$$

Then up to isomorphism  $\Gamma_1$  is a unique minimal graph in its birational equivalence class.

*Proof.* Assume that  $\Gamma_2$  is a minimal graph birationally equivalent to  $\Gamma_1$  and not isomorphic to  $\Gamma_1$ . By Lemma 3.39, there exists a relatively minimal diagram (3.2) with  $\Gamma$  that dominates  $\Gamma_1$  and  $\Gamma_2$  via birational morphisms  $p_1$  and  $p_2$ , respectively. We may suppose that  $p_1$  is composed of  $n \geq 1$  blowdowns and  $p_2$  also is composed of blowdowns. Let  $p_1^{-1}: \Gamma_1 \dashrightarrow \Gamma$  add vertices  $v_1, \dots, v_n$  in this order, where  $n \geq 1$ . By the relative minimality assumption,  $p_2$  cannot contract any  $(-1)$ -vertex of  $\Gamma$  among  $v_1, \dots, v_n$ . In particular, it does not contract the  $(-1)$ -vertex  $v_n$  of  $\Gamma$ .

*Case 1:*  $\Gamma_1 = [[0]]$ . Let  $\text{Vert}(\Gamma_1) = \{v_0\}$ . We prove by induction that for every  $k = 1, \dots, n$  the following hold:

- (i<sub>k</sub>) The first  $k$  blowups in  $p_1^{-1}$  are outer and yield the graph  $[-1, -2_{k-1}, -1]$ , with vertices  $v_0, \dots, v_k$  appearing in this order.
- (ii<sub>k</sub>) The further blowups in  $p_1^{-1}$  do not change the weights of  $v_0, \dots, v_{k-1}$ .
- (iii<sub>k</sub>) The first  $k$  blowdowns in  $p_2$  contract the vertices  $v_0, \dots, v_{k-1}$  of  $\Gamma$  in this order.

For  $k = 1$ , statements (i<sub>1</sub>)–(iii<sub>1</sub>) hold because  $p_2$  contracts no  $(-1)$ -vertex of  $\Gamma$  different from  $v_0$  and  $\Gamma \ominus \{v_0\}$  is not minimal, while  $\Gamma_2$  is. Suppose by induction that for some  $k \in \{1, \dots, n-1\}$ , statements (i<sub>k</sub>)–(iii<sub>k</sub>) hold. By statement (ii<sub>k</sub>), the  $(k+1)$ st blowup in  $p_1^{-1}$  must be the outer blowup at  $v_k$ ; hence statement (i<sub>k+1</sub>) holds. Let  $\Gamma_{2,k}$  be the graph obtained from  $\Gamma$  under the first  $k$  blowdowns in  $p_2$ . By statements (ii<sub>k</sub>) and (iii<sub>k</sub>), we have a natural isomorphism of weighted graphs  $\Gamma_{2,k} \ominus \{v_k\} \cong \Gamma \ominus \{v_0, \dots, v_k\}$ . The latter graph contains the at most linear  $(-1)$ -vertex  $v_n$ . Hence it is not minimal, and none of its  $(-1)$ -vertices are contracted by  $p_2$ . Thus,  $v_k$  is the only vertex of  $\Gamma_{2,k}$  that can be contracted in the next blowdown in  $p_2$ . Its weight is  $-1$  in  $\Gamma_{2,k}$  and  $-2$  in  $\Gamma$ . Statements (ii<sub>k+1</sub>) and (iii<sub>k+1</sub>) follow. By recursion, statement (i<sub>n</sub>) holds; that is,  $\Gamma = [-1, -2_{n-1}, -1]$ . So  $\Gamma_2 = [[0]]$  due to the minimality of  $\Gamma_2$ .

*Case 2:*  $\Gamma_1 = ((m,0))$  with  $m \leq -2$ . Since  $\Gamma_1$  is circular,  $\Gamma$  is also circular, and  $p_1, p_2$  consist of inner blowdowns; see Proposition 4.15(b). Let  $\text{Vert}(\Gamma_1) = \{u, v_0\}$ , where  $v_0$  is the (0)-vertex. The same recursion as in Case 1 shows that for  $k = 1, \dots, n$  we have the following:

- (i<sub>k</sub>) The first  $k$  blowups in  $p_1^{-1}$  yield the graph  $((m-k, -1, -2_{k-1}, -1))$  with vertices  $u, v_0, \dots, v_k$  appearing in this order.
- (ii<sub>k</sub>) The further blowups in  $p_1^{-1}$  do not change the weights of  $v_0, \dots, v_{k-1}$ .
- (iii<sub>k</sub>) The first  $k$  blowdowns in  $p_2$  contract vertices  $v_0, \dots, v_{k-1}$ .

The only additional observation that we need is the following: the weight of  $u$  in  $\Gamma_{2,k-1}$  is at most  $m - k + (k-1) \leq -3$ , so  $u$  cannot be contracted in the next step. Thus, we conclude as before that  $\Gamma_2 \cong ((m,0))$ .

*Case 3:*  $\Gamma_1 = ((3))$ . After the first blowup we get  $\Gamma'_1 = ((-1, -1))$  with vertices  $v_0$  and  $v_1$ , where  $v_1$  is contracted in  $\Gamma_1$ . The second blowup would change the weight of  $v_0$  to  $-2$ . The latter is not possible, since the vertex  $v_0$  must be the first contracted under  $p_2$ . So  $\Gamma_2 \cong ((3))$ .

*Case 4:*  $\Gamma_1 = ((4))$ . After the first blowup we get  $\Gamma'_1 = ((-1, 0))$ . Letting now  $m = -1$ , the argument from Case 2 can be applied mutatis mutandis to  $\Gamma'_1$  instead of  $\Gamma_1$ .

*Case 5:*  $\Gamma_1$  is admissible modulo charm earrings. By Proposition 4.15, the composed map  $p_2 \circ p_1^{-1}: \Gamma_1 \dashrightarrow \Gamma_2$  induces a bijection between  $\text{Br}(\Gamma_1)$  and  $\text{Br}(\Gamma_2)$ , an isomorphism between the corresponding admissible segments of  $\Gamma_1$  and  $\Gamma_2$  and a birational transformation between every segment  $[[0]]$  coming from a charm earring  $S$  of  $\Gamma_1$  and the corresponding linear segment of  $\Gamma_2$ . Let  $v_0$  be the  $(0)$ -vertex of  $S$  adjacent to a branch vertex  $u \in \text{Br}(\Gamma_1)$ . Since  $u$  cannot be contracted by  $p_2$ , the argument used in Case 1 can be applied to the linear segment  $[[0]]$  with vertex  $v_0$ . This shows that  $p_2 \circ p_1^{-1}$  sends every charm earring of  $\Gamma_1$  isomorphically onto a charm earring of  $\Gamma_2$  and induces an isomorphism of weighted graphs  $\Gamma_1 \cong \Gamma_2$ .

*Case 6:*  $\Gamma_1 = ((0, 0))$ . According to Proposition 4.15(b), the graphs that appear in successive steps of the birational transformation  $\phi' = p_2 \circ p_1^{-1}$  are circular. We claim that such a circular graph  $B$  is triangulable, except for the initial graph  $\Gamma_1 = ((0, 0))$ ; see Definition A.2. Indeed, the first blowup in  $p_1^{-1}$  yields the circular graph  $((-1, -1, -1))$ , which has a triangulation with a single triangle. Suppose that a circular graph  $B$  is equipped with a triangulation  $\mathcal{C}$ . Then the inner blowup of  $B$  at the edge  $[v_i, v_{i+1}]$  that introduces a new vertex  $u$  results in gluing to  $\mathcal{C}$  a new triangle with vertices  $v_i, v_{i+1}$  and  $u$ . This adds a new leaf to the incidence tree of triangles. Conversely, a blowdown of a  $(-1)$ -vertex  $v$  of  $B$  results in removing the unique triangle in  $\mathcal{C}$  with vertex  $v$ , provided  $B$  is different from  $((-1, -1, -1))$ .

So  $\Gamma_2$  is either  $((0, 0))$  or triangulable. In the latter case,  $\Gamma_2$  is not minimal. Indeed, the incidence tree of triangles associated with the triangulation  $\mathcal{C}$  of  $\Gamma_2$  contains a leaf that corresponds to a triangle with a vertex of weight  $-1$ .  $\square$

*Remark A.5.* An alternative approach is based on the following observation. Consider again a relatively minimal diagram (3.2) with  $\Gamma$  that dominates minimal graphs  $\Gamma_1$  and  $\Gamma_2$ , where  $\Gamma_1$  as in Proposition A.4 is not admissible modulo charm earrings and  $\Gamma \neq ((-1, -1, -1))$ . We claim that  $\Gamma$  contains exactly two  $(-1)$ -vertices. Indeed, for the inertia indices of  $\Gamma_1$ , we have  $i_0(\Gamma_1) + i_+(\Gamma_1) = 1$ . Let  $v$  be a  $(-1)$ -vertex of  $\Gamma$  that belongs to  $\text{Vert}(\Gamma_1)$ . Then  $v$  is the unique  $(-1)$ -vertex of  $\Gamma$  contracted under  $p_2$ . Assume to the contrary that  $\Gamma$  contains two distinct  $(-1)$ -vertices  $u_1$  and  $u_2$  different from  $v$ . They are not adjacent; see Corollary 3.18(b), are not contained among the vertices of  $\Gamma_1$  and are not contracted under  $p_2$ . Since  $\Gamma_2$  is minimal, the weights of  $u_1$  and  $u_2$  in  $\Gamma_2$  are non-negative. One can show that  $u_1$  and  $u_2$  in  $\Gamma_2$  cannot be adjacent. Then the intersection form  $I(\Gamma_2)$  is semipositive definite on the vector subspace spanned by the mutually orthogonal vectors  $u_1$  and  $u_2$ . However, this contradicts the fact that  $i_0(\Gamma_2) + i_+(\Gamma_2) = i_0(\Gamma_1) + i_+(\Gamma_1) = 1$ ; see Corollary 3.11.

*Remark A.6.* A linear segment is called *standard* if it is one of the following:

$$[[0_{2k}, w_1, \dots, w_n]] \quad \text{and} \quad [[0_{2k+1}]],$$

where  $k, n \geq 0$  and  $w_i \leq -2$  for all  $i$ . Similarly, a circular segment is called *standard* if it is one of the following:

$$((0_{2k}, w_1, \dots, w_n)), \quad ((0_l, w)) \quad \text{and} \quad ((0_{2k}, -1, -1)),$$

where  $k, l \geq 0$ ,  $n > 0$ ,  $w \leq 0$  and  $w_i \leq -2$  for all  $i$ ; see [FKZ07, Definition 2.13]. Notice that for  $w = 0$  the second circular graph above becomes  $((0_{l+1}))$ . Any minimal segment is birationally equivalent to a standard one; see [FKZ07, Theorem 2.15(b)]. Moreover, for a minimal linear segment  $L$ , its birational equivalence class contains at most two standard graphs related by

a *reversion*

$$[[0_{2k}, w_1, \dots, w_n]] \mapsto [[0_{2k}, w_n, \dots, w_1]].$$

For a minimal circular segment  $C$ , the standard graph is unique in the birational equivalence class up to a cyclic permutation of its non-zero weights and reversion; see [FKZ07, Corrigendum, Corollary 3.33].

For instance, the circular graphs with sequences  $((-1, -1))$  and  $((0, -1))$ , respectively, are the unique standard graphs in the respective birational equivalence classes of  $((3))$  and  $((4))$ , while  $[[0]]$  is the unique standard graph in its birational equivalence class.

LEMMA A.7. *Let  $\Gamma$  be a (not necessarily minimal) non-contractible connected weighted graph. Assume that  $\Gamma$  has a  $(0)$ -vertex  $v$  of degree 1 or 2 with no incident loop and multiple edges. Then  $\Gamma$  admits an infinite number of non-isomorphic minimal models.*

*Proof.* By our assumption,  $\Gamma$  contains an edge  $[v, u]$  isomorphic to  $[[0, c]]$  for some  $c \in \mathbb{Z}$ . Applying iteratively elementary transformations at  $v$ ,

$$[[0, c]] \rightsquigarrow [[-1, -1, c]] \rightsquigarrow [[0, c + 1]],$$

which is inner if  $\deg(v) = 2$  and outer otherwise, see Remark 3.46, we obtain an infinite sequence of graphs  $\{\Gamma_n\}$  from the birational equivalence class of  $\Gamma$  with the same number of vertices, where  $\Gamma_n$  has a vertex of weight  $n$ . Since  $\Gamma_n$  is not contractible, it dominates a minimal graph  $\Gamma'_n$ . We claim that among the  $\Gamma'_n$  there exists an infinite number of non-isomorphic minimal graphs. Indeed, the contraction  $\Gamma_n \rightarrow \Gamma'_n$  consists of at most  $N = \text{card}(\text{Vert}(\Gamma))$  blowdowns. Hence it drops the maximal weight of vertices in  $\Gamma$  by at most  $4N$ . Therefore, the range of maximal weights of the graphs  $\Gamma'_n$  is unbounded, which proves our claim.  $\square$

THEOREM A.8. *Let  $\Gamma$  be a minimal connected weighted graph. Then the birational equivalence class of  $\Gamma$  contains an infinite number of non-isomorphic minimal models if and only if  $\Gamma$  is not admissible modulo charm earrings and is different from the graphs  $[[0]]$ ,  $((3))$ ,  $((4))$ ,  $((0, m))$  with  $m \leq 0$ . In the opposite case, up to isomorphism,  $\Gamma$  is a unique minimal graph in its birational equivalence class.*

*Proof.* The second assertion follows immediately from Proposition A.4. To show the first assertion, first consider the case where  $\Gamma$  is a non-admissible circular segment with  $N$  vertices. If  $N \geq 3$ , then there is a vertex  $v$  of weight  $a \geq 0$  that has degree 2 and two distinct neighbors. Performing  $a$  inner blowups near  $v$ , we drop the weight of  $v$  to 0, and then Lemma A.7 gives the result.

Now let  $N = 2$ . If  $\Gamma$  has a vertex  $v$  of weight  $a > 0$ , then after an inner blowup near  $v$ , we return to the previous case  $N \geq 3$ . Otherwise,  $\Gamma = ((0, m))$  with  $m \leq 0$ , which is excluded by our assumption. Finally, if  $N = 1$  and  $\Gamma = ((m))$ , where  $m \geq 5$  by our assumption, then an inner blowup yields the 2-cycle  $((m - 4, -1))$  with a positive weight, which returns us to a previous case.

Finally, let  $\Gamma$  be non-circular. Since by assumption  $\Gamma$  is not admissible modulo charm earrings,  $\Gamma \ominus \text{Br}(\Gamma)$  contains a non-admissible linear segment with a vertex  $v$  of weight  $a \geq 0$ , where  $a > 0$  if either  $\Gamma = [[a]]$  or  $v$  has two incident edges  $[v, u]$  with the same branch vertex  $u \in \text{Br}(\Gamma)$ . Performing  $a$  blowups near  $v$ , we reduce the setup to the one of Lemma A.7, which implies the assertion.  $\square$

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Alexander Perepechko [a@perrep.ru](mailto:a@perrep.ru)

HSE University, Faculty of Computer Science, 11 Pokrovsky blvd., Moscow, 109028 Russia

Mikhail Zaidenberg [mikhail.zaidenberg@gmail.com](mailto:mikhail.zaidenberg@gmail.com)

Institut Fourier, UMR 5582, Laboratoire de Mathématiques, Université Grenoble Alpes,  
CS 40700, 38058 Grenoble cedex 9, France