Algebraic Geometry 13 (4) (2026) 588–597
doi:10.14231/AG-2026-017

Polynomiality of the double ramification cycle

Pim Spelier

Abstract

Let $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ be a sequence with sum $k(2g-2+n)$. The double ramification cycle $\mathrm{DR}_g(A) \in \mathrm{CH}^g\big(\overline{\mathcal{M}}_{g,n}\big)$ is the virtual class of the locus of curves $(C, p_1, \ldots, p_n)$ where the line bundle $\big(\omega_C^{\log}\big)^{-k}\big(\sum a_i p_i\big)$ is trivial. Although there has long been a formula for $\mathrm{DR}_g(A)$, the exact dependence on A was unknown for a long time, though it was conjectured to be polynomial in A. A proof was announced by Janda, Pandharipande, Pixton, and Zvonkine in 2017, and Pixton gave a proof incorporating ideas of Zagier in 2023. Here we present an alternative proof of the polynomiality of the double ramification cycle.

1. Introduction

Let $\mathcal{M}_{g,n}$ be the moduli space of smooth curves $(C/S, p_1, \ldots, p_n)$ of genus g with n distinct markings. Let $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ be a sequence with sum 0. Then the *double ramification locus* is

$$\mathrm{DRL}_g(A) = \left\{(C/S, p_1, \ldots, p_n) : \mathcal{O}_C\left(\sum a_i p_i\right) \text{ is fiberwise trivial}\right\} \subset \mathcal{M}_{g,n}\,.$$

Equivalently, this is the locus of curves C with a rational function $f\colon C \to \mathbb{P}^1$ with specified ramification profile above 0 and ∞.

In 2001 Eliashberg asked the question of how to compactify this substack and how to compute the compactification. Compactifying the substack was first done in [GV03], using stable maps to $\mathbb{P}^1$ modulo the $\mathbb{G}_m$-action. The authors defined the *double ramification cycle*

$$\mathrm{DR}_g(A) \in \mathrm{CH}^g\big(\overline{\mathcal{M}}_{g,n}\big)$$

as the virtual class of this compactification. There are other equivalent definitions, using birational geometry of $\overline{\mathcal{M}}_{g,n}$, see [Hol21], or logarithmic geometry [MW20]. These latter definitions also generalise to the *twisted double ramification cycle* $\mathrm{DR}_g(A)$ for a sequence of integers $A \in \mathbb{Z}^n$ summing to $k(2g-2+n)$ for some $k \in \mathbb{Z}$. This double ramification cycle is the virtual class of

Received 2 February 2024, accepted in final form 24 January 2025.
2020 Mathematics Subject Classification 14H10, 14H40, 14H51,14N35.
Keywords: moduli space of curves, enumerative geometry, double ramification cycle.

The author is supported by NWO grant VI.Vidi.193.006.

the compactification of the locus

$$\Big\{(C/S, p_1, \dots, p_n) : \big(\omega_C^{\log}\big)^{-k}\Big(\sum a_i p_i\Big) \text{ is fiberwise trivial}\Big\}.$$

The double ramification cycle has connections with Gromov–Witten invariants through the localisation formula [JPPZ17, RK24] and with orbifold Gromov–Witten invariants [FWY21]. It also has ties with the world of partial differential equations and integrable systems [Bur15, BR21] and with gauge theory [FTT16], and it has been used to find relations in the tautological ring [CJ18].

The twisted double ramification cycle has played an important role in the study of the Hodge bundle on $\overline{\mathcal{M}}_{g,n}$. The twisted double ramification locus parametrises differentials with prescribed zeros and poles, and the twisted double ramificiation cycle thence relates to strata of differentials inside the Hodge bundle. This link was first mentioned in [FP18]; see [CGH$^+$25] for an overview of the connection. Through the connection to the Hodge bundle, it has also been used to prove results on λ_g on $\overline{\mathcal{M}}_{g,n}$ and $\mathcal{A}_g$; see [MPS23].

Pixton conjectured a formula for the double ramification cycle, and in 2014 this was proven in [JPPZ17] for the case $k = 0$, thereby answering the second part of Eliashberg's question. The authors of [JPPZ17] define, for every integer $r \in \mathbb{Z}_{\geqslant 1}$, an explicit class

$$P_g^r(A) \in \mathrm{CH}^g\big(\overline{\mathcal{M}}_{g,n}\big)$$

in terms of decorated strata multiplied by certain numbers associated with the combinatorics of the graph and the integers $a_1, \dots, a_n$ and r.

They prove that $P_g^r(A)$ is polynomial in r for r large enough. Then they define $P_g^0(A)$ to be the constant term of this polynomial expression, and they prove the equality

$$\mathrm{DR}_g(A) = P_g^0(A)\,.$$

In [BHP$^+$23] this result was extended to a formula for the twisted double ramification cycle.

These two results were major breakthroughs, but a very simple question remained open: what is the behaviour of $\mathrm{DR}_g(A)$ in terms of A? In [JPPZ17] the double ramification cycle was conjectured to be polynomial in A, but this is a deceptively difficult question, as the formula itself has no obvious polynomial dependency on A. A proof was announced in [JPPZ17]. A proof incorporating ideas of Zagier was made public in 2023; see [Pix23]. In this paper we present a different proof of the same theorem.

Theorem A (Theorem 5.1). *Fix g, n. The function*

$$\Big\{(a_1, \dots, a_n) \in \mathbb{Z}^n : \exists k, \sum a_i = k(2g-2+n)\Big\} \longrightarrow \mathrm{CH}^g\big(\overline{\mathcal{M}}_{g,n}\big)\,, \quad A \longmapsto \mathrm{DR}_g(A)$$

is polynomial.

We use different techniques to prove this statement. Our methods are explicit and can be used recursively to give a polynomial expression for the double ramification cycle.

In Section 2 we recall the combinatorics necessary to state Pixton's formula. In Section 3 we give a brief recap of Pixton's formula. In Section 4 we prove the main technical result, a certain polynomiality statement for sums of weightings on graphs. In Section 5 we finally give the proof of Theorem A.

2. Definitions

Definition 2.1. We denote by $G_{g,n}$ the set of graphs

$$\Gamma = \big(V, H, L = (\ell_i)_{i=1}^n,\ \mathrm{rt}\colon H \to V,\ i\colon H \to H,\ g\colon V \to \mathbb{N}\big)\,,$$

where

(i) V is the set of vertices;
(ii) H is the set of half edges;
(iii) i is an involution on H;
(iv) rt assigns to every half edge the vertex it is incident to;
(v) $L = (\ell_i)_{i=1}^n \subset H$ is a list of the legs, the fixed points of the involution i;
(vi) (V, H) is a connected graph;
(vii) $g(v)$ is the genus of vertex v;
(viii) for every vertex v we have $2g(v) - 2 + n(v) > 0$, where $n(v)$ is the number of half edges incident to v;
(ix) the genus $\sum_{v\in V} g(v) + h^1(\Gamma)$ of the graph is g.

For such a graph, we denote by E the set of edges, that is, the set of pairs $\{h, h'\}$ of half edges with $h = i(h')$ and $h \neq h'$.

Then $G_{g,n}$ parametrises the strata of $\overline{\mathcal{M}}_{g,n}$. Every graph $\Gamma \in G_{g,n}$ corresponds to a moduli space $\overline{\mathcal{M}}_\Gamma = \prod_{v\in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)}$ together with a glueing map $\zeta_\Gamma\colon \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}$.

Definition 2.2. We fix a vector $A \in \mathbb{Z}^n$ with sum $k(2g - 2 + n)$. A *weighting for A* on a graph $\Gamma \in G_{g,n}$ is a map $w\colon H(\Gamma) \to \mathbb{Z}$ satisfying the following three conditions:

(i) For $i = 1, \dots, n$ we have $w(\ell_i) = a_i$.
(ii) For $h \in H \setminus L$ we have $w(h) + w(i(h)) = 0$.
(iii) For every vertex v we have $\sum_{h\colon \mathrm{rt}(h)=v} w(h) = k(2g(v) - 2 + n(v))$.

For $r \in \mathbb{N}_{\geqslant 1}$ a *weighting for A* modulo r is a map $w\colon H(\Gamma) \to \{0, \dots, r-1\} \subset \mathbb{Z}$ satisfying these three conditions modulo r. We write $W^\Gamma_{A,r}$ for the finite set of weightings for A mod r.

We fix g, n, and we fix a graph $\Gamma \in G_{g,n}$. Let $Q \in \mathbb{Z}[x_h : h \in H \setminus L]$ be a polynomial.

Definition 2.3. For $A = (a_1, \dots, a_n)$ with $\sum a_i = k(2g - 2 + n)$, consider the sum

$$S^\Gamma_{A,r}(Q) = r^{-h_1(\Gamma)} \sum_{w\in W^\Gamma_{A,r}} Q((w(h))_{h\in H\setminus L})\,. \tag{2.1}$$

In [JPPZ17, Appendix A] the authors prove the following property of these sums.

Proposition 2.4 ([JPPZ17, Proposition 3″]). *The sums $S^\Gamma_{A,r}(Q)$ are eventually polynomial in r.*

This allows them to define the following quantity.

Definition 2.5. We denote the constant term of the polynomial expression for $S^\Gamma_{A,r}(Q)$ for large r by $S^\Gamma_{A,0}(Q)$.

3. Pixton's formula for the double ramification cycle

Fix $g, n \in \mathbb{N}$, $k \in \mathbb{Z}$ with $2g-2+n>0$. Fix a sequence $A \in \mathbb{Z}^n$ with sum $k(2g-2+n)$. In this section we present Pixton's formula for the double ramification cycle $\mathrm{DR}_g(A) \in \mathrm{CH}^*\big(\overline{\mathcal{M}}_{g,n}\big)$. The first version of this, proven for $k=0$, was [JPPZ17, Theorem 1]. A formula for any k was conjectured in [JPPZ17, Section 1.1] and proven in [BHP+23, Theorem 7].

For $r \in \mathbb{N}_{\geqslant 1}$ we define the term $P_r^\Gamma \in \mathrm{CH}\big(\overline{\mathcal{M}}_\Gamma\big)$ as follows:

$$P_r^\Gamma = r^{-h^1(\Gamma)} \sum_{w \in W_{A,r}^\Gamma} \prod_{e=\{h,h'\}\in E(\Gamma)} \frac{1-\exp\big(-\frac{1}{2}w(h)w(h')(\psi_h+\psi_{h'})\big)}{\psi_h+\psi_{h'}}.$$

We observe that the factor

$$\frac{1-\exp\big(-\frac{1}{2}w(h)w(h')(\psi_h+\psi_{h'})\big)}{\psi_h+\psi_{h'}}$$

is well defined, as the denominator formally divides the numerator.

For every choice of monomial in the factors $(\psi_h+\psi_h')$ for edges $\{h,h'\}$, the corresponding coefficient of P_r^Γ is of the form $S_{A,r}^\Gamma(Q)$ for some polynomial Q, where $S_{A,r}^\Gamma(Q)$ is as defined in Definition 2.3. Thus P_r^Γ is eventually polynomial per Proposition 2.4. We define

$$P_0^\Gamma \in \mathrm{CH}^g\big(\overline{\mathcal{M}}_\Gamma\big)$$

to be the element obtained by taking the constant term of the polynomial expression.

Then by [BHP+23, Theorem 7] we have the formula

$$\mathrm{DR}_g(A) = \left[\exp\left(-\frac{1}{2}\left(k^2\kappa_1 - \sum_i a_i^2\psi_i\right)\right) \sum_{\Gamma\in G_{g,n}} \frac{1}{|\operatorname{Aut}(\Gamma)|}\zeta_{\Gamma,*}P_0^\Gamma\right]^g, \tag{3.1}$$

where $[\cdot]^g$ means the codimension g term.

4. Sums over weightings

We fix g, n as in the introduction, and we fix a graph $\Gamma \in G_{g,n}$. Write $H \setminus L = \{h_1, \dots, h_m\}$. Let $Q \in \mathbb{Z}[x_h : h \in H \setminus L]$ be a polynomial.

Fix $A \in \mathbb{Z}^n$ with $\sum a_i = 0$. We recall the sum

$$S_{A,r}^\Gamma(Q) = r^{-h_1(\Gamma)} \sum_{w\in W_{A,r}^\Gamma} Q((w(h))_{h\in H\setminus L})$$

from (2.1) and the constant term $S_{A,0}^\Gamma(Q)$ of the polynomial expression for the sum for large r.

The main theorem in this section is the following polynomiality statement.

THEOREM 4.1. *For every polynomial $Q \in \mathbb{Z}[x_h : h \in H \setminus L]$, the function*

$$\Big\{A \in \mathbb{Z}^n : \sum a_i = 0\Big\} \longrightarrow \mathbb{Q}, \quad A \longmapsto S_{A,0}^\Gamma(Q)$$

is polynomial in A.

We will prove this by induction on the number of edges of Γ. In Section 4.1 we treat the case where Γ has a separating edge. In Section 4.2 we prove a preliminary result in the case where Γ has a non-separating edge. In Section 4.3 we put everything together and finish the proof of Theorem 4.1. We will then prove Corollary 4.7, generalising Theorem 4.1 to the domain $\{A \in \mathbb{Z}^n : (2g-2+n) \mid \sum a_i\}$.

We first make some general observations and give some definitions.

Remark 4.2. We will use the notation $w \bmod r$ to denote the unique element in $\{0, \ldots, r-1\}$ that is equivalent to w modulo r. We use the notation $w \equiv w' \pmod r$ to denote that w and w' are equivalent modulo r.

LEMMA 4.3. *Fix A. Then for r large enough and coprime to a fixed integer, we have*

$$S^{\Gamma}_{A,0}(Q) \equiv S^{\Gamma}_{A,r}(Q) \pmod r\,.$$

Proof. We know that for r large enough $S^{\Gamma}_{A,r}(Q)$ is a polynomial in r, with rational coefficients. If the product of the denominators is coprime to r, this means that $S^{\Gamma}_{A,r}(Q) \equiv S^{\Gamma}_{A,0}(Q) \pmod r$. □

DEFINITION 4.4. Let $e = \{h_1, h_2\}$ be an edge of Γ. We let Γ_e denote the (possibly disconnected) graph Γ with the edge e removed and with two legs ℓ_{n+1}, ℓ_{n+2} added at the roots of the half edges $\mathrm{rt}(h_1)$, $\mathrm{rt}(h_2)$.

4.1 Separating edge case

In this subsection we prove Theorem 4.1 in the case where Γ has a separating edge.

PROPOSITION 4.5. *Assume that Γ has a separating edge. Then Theorem 4.1 holds, assuming that it holds for graphs with fewer edges than Γ.*

Proof. By linearity we can assume that Q is a monomial, and we write

$$Q = \prod_{h \in H \setminus L} x_h^{c_h}\,.$$

Let $e = \{h_1, h_2\}$ be a separating edge of Γ.

Then Γ_e is a disjoint union of two graphs, denoted by $\Gamma_1 \sqcup \Gamma_2$. Let A_1 denote the subsequence of A consisting of elements whose corresponding vertex lies in Γ_1, and similarly for A_2. We denote the sums $\sum_{a \in A_i} a$ by s_i for $i \in \{1,2\}$. For $i \in \{1,2\}$ let $Q_i \in \mathbb{Z}[x_h : h \in H_i \setminus L_i]$ denote the monomial Q with all half edges not in $H_i \setminus L_i$ set to 1, and let c_i denote c_{h_i}.

Note that $W^{\Gamma}_{A,r}$ then splits as $W^{\Gamma_1}_{(A_1,-s_1),r} \times W^{\Gamma_2}_{(A_2,-s_2),r}$. Denote the summands by W_1 and W_2, respectively. Then we find the formula

$$\begin{aligned} S^{\Gamma}_{A,r}(Q) &= r^{-h_1(\Gamma)} \sum_{w_1 \in W_1} \sum_{w_2 \in W_2} Q_1(w_1)(-s_1 \bmod r)^{c_1}(-s_2 \bmod r)^{c_2} Q_2(w_2) \\ &= \left(r^{-h_1(\Gamma_1)} \sum_{w_1 \in W_1} Q_1(w_1) \right) \cdot (-s_1 \bmod r)^{c_1}(-s_2 \bmod r)^{c_2} \cdot \left(r^{-h_1(\Gamma_2)} \sum_{w_2 \in W_2} Q_2(w_2) \right) \\ &= S^{\Gamma_1}_{(A_1,-s_1),r}(Q_1) \cdot (-s_1 \bmod r)^{c_1}(-s_2 \bmod r)^{c_2} \cdot S^{\Gamma_2}_{(A_2,-s_2),r}(Q_2)\,. \end{aligned}$$

By Lemma 4.3 we know that modulo r the first and last factors are equal to $S^{\Gamma_1}_{(A_i,-s_i),0}(Q_i)$ with $i = 1$ and 2, respectively (for r large enough and coprime to a fixed integer). In total we see that for such r we have the following equalities:

$$\begin{aligned} S^{\Gamma}_{A,0}(Q) &\equiv S^{\Gamma}_{A,r}(Q) && \pmod r \\ &= S^{\Gamma_1}_{(A_1,-s_1),r}(Q_1) \cdot (-s_1 \bmod r)^{c_1}(-s_2 \bmod r)^{c_2} \cdot S^{\Gamma_2}_{(A_2,-s_2),r}(Q_2) \\ &\equiv S^{\Gamma_1}_{(A_1,-s_1),0}(Q_1) \cdot (-s_1)^{c_1}(-s_2)^{c_2} \cdot S^{\Gamma_2}_{(A_2,-s_2),0}(Q_2) && \pmod r\,. \end{aligned}$$

By our assumption that Theorem 4.1 holds for graphs with fewer edges than Γ, the term

$$S^{\Gamma_1}_{(A_1,-s_1),0}(Q_1)\cdot(-s_1)^{c_1}(-s_2)^{c_2}\cdot S^{\Gamma_2}_{(A_2,-s_2),0}(Q_2)$$

is polynomial in A. As it agrees with $S^{\Gamma}_{A,0}$ modulo an infinite number of integers, we have

$$S^{\Gamma}_{A,0}(Q)=S^{\Gamma_1}_{(A_1,-s_1),0}(Q_1)\cdot(-s_1)^{c_1}(-s_2)^{c_2}\cdot S^{\Gamma_2}_{(A_2,-s_2),0}(Q_2)\,, \tag{4.1}$$

and, in particular, $S^{\Gamma}_{A,0}(Q)$ is polynomial in A. □

4.2 Non-separating case

In this subsection we treat the case where Γ has a non-separating edge.

Proposition 4.6. *Let $e=\{h_1,h_2\}$ be a non-separating edge, and denote by $v_i=\mathrm{rt}(h_i)$ the root of h_i for $i\in\{1,2\}$. Assume that for each $i\in\{1,2\}$ that there is a leg ℓ_i with insertion a_i and adjacent to v_i. Let $A\in\mathbb{Z}^n$ be a vector with total sum zero. For $a\in\mathbb{Z}$ let A_a be the vector $(a_1-a,a_2+a,a_3,a_4,\dots,a_n)$. Assume that Theorem 4.1 holds for graphs with fewer edges than Γ.*

Then there exist a polynomial $\Psi_{\Gamma,e,Q}(x_1,\dots,x_n,y,z_1,\dots,z_m)$ and polynomials $R_1,\dots,R_m$ in $\mathbb{Z}[x_h : h\in H\setminus L]$ such that

$$S^{\Gamma}_{A_a,0}(Q)=\Psi_{\Gamma,e,Q}\big(a_1,\dots,a_n,a,S^{\Gamma}_{A,0}(R_1),\dots,S^{\Gamma}_{A,0}(R_m)\big)\,.$$

Proof. By linearity we can assume that Q is a monomial, and we write $Q=\prod_{h\in H\setminus L}x_h^{c_h}$. For $i\in\{1,2\}$ we let c_i denote c_{h_i}.

We first prove the case where $a\geqslant 0$. We fix an r and let W_a denote $W^{\Gamma}_{A_a,r}$. We first rewrite $S^{\Gamma}_{A_a,r}(Q)$. Note that there is a bijection

$$\begin{aligned}\varphi_a\colon W_0&\longrightarrow W_a\\ w&\longmapsto\left(h_i\longmapsto\begin{cases}(w(h_1)+a)\bmod r & \text{if } i=1\,,\\ (w(h_2)-a)\bmod r & \text{if } i=2\,,\\ w(h_i) & \text{if } i>2\end{cases}\right).\end{aligned}$$

Let $Q_0(w)$ denote the monomial Q with x_{h_1} and x_{h_2} substituted by 1. We then have the formula

$$\begin{aligned}S^{\Gamma}_{A_a,r}(Q)&=r^{-h_1(\Gamma)}\sum_{w\in W_0}Q(\varphi_a(w))\\ &=r^{-h_1(\Gamma)}\sum_{w\in W_0}((w(h_1)+a)\bmod r)^{c_1}((w(h_2)-a)\bmod r)^{c_2}Q_0(w)\,.\end{aligned}$$

Now we will rewrite the factor $((w(h_1)+a)\bmod r)^{c_1}((w(h_2)-a)\bmod r)^{c_2}$ using some facts about $m\bmod r$ for integers m.

For any $m\not\equiv 0\pmod r$ we have $(r-m)\bmod r=r-(m\bmod r)$ and hence

$$(m\bmod r)^{c_1}((r-m)\bmod r)^{c_2}=(m\bmod r)^{c_1}(r-(m\bmod r))^{c_2}\,.$$

For $m\equiv 0\pmod r$ we have

$$(m\bmod r)^{c_1}((r-m)\bmod r)^{c_2}=(m\bmod r)^{c_1}(r-(m\bmod r))^{c_2}-(1-\delta_{c_2,0})0^{c_1}r^{c_2}\,,$$

where δ is the Kronecker delta. (The term $(1-\delta_{c_2,0})0^{c_1}r^{c_2}$ is usually 0, but it can be non-zero in the case $c_1=0<c_2$). Also, $m\bmod r=m-r\lfloor m/r\rfloor$ is a polynomial in m and $r\lfloor m/r\rfloor$.

We now use this for $m = w(h_1) + a$. All in all, the factor

$$((w(h_1) + a) \bmod r)^{c_1}((w(h_2) - a) \bmod r)^{c_2}$$

can be rewritten as

$$p\left(w(h_1), a, r, r\left\lfloor \frac{w(h_1)+a}{r} \right\rfloor\right) - \delta_{w(h_1)+a \bmod r, 0} \cdot (1 - \delta_{c_2,0}) \cdot 0^{c_1} r^{c_2}$$

for the polynomial

$$p(x_0, x_1, x_2, x_3) = (x_0 + x_1 - x_3)^{c_1} \cdot (x_2 - x_0 - x_1 + x_3)^{c_2}\,.$$

Now we assume that $r > a$. Then we have $0 \leqslant w(h_1) + a < 2r$ and so

$$\left\lfloor \frac{w(h_1)+a}{r} \right\rfloor = \begin{cases} 0 & \text{if } r - w(h_1) > a\,, \\ 1 & \text{if } r - w(h_1) \leqslant a\,. \end{cases} \tag{4.2}$$

Define $p_0(x_0, x_1, x_2) = p(x_0, x_1, x_2, 0)$ and $p_1(x_0, x_1, x_2, x_3) = (p - p_0)/x_3$. Then we can write

$$p = p_0(x_0, x_1, x_2) + x_3 p_1(x_0, x_1, x_2, x_3)\,.$$

Write

$$S_1 := r^{-h_1(\Gamma)} \sum_{w \in W_0} p_0(w(h_1), a, r) Q_0(w)\,,$$

$$S_2 := r^{-h_1(\Gamma_e)} \sum_{w \in W_0} \left\lfloor \frac{w(h_1)+a}{r} \right\rfloor p_1\left(w(h_1), a, r, r\left\lfloor \frac{w(h_1)+a}{r} \right\rfloor\right) Q_0(w)\,,$$

$$S_3 := r^{-h_1(\Gamma)} \sum_{w \in W_0 \colon w(h_1) = -a \pmod r} (1 - \delta_{c_2,0}) \cdot 0^{c_1} r^{c_2} Q_0(w)\,,$$

so that $S^{\Gamma}_{A_a,r}(Q) = S_1 + S_2 - S_3$. Here Γ_e is the graph defined in Definition 4.4, and we have used that $r^{-h_1(\Gamma)} \cdot r = r^{-h_1(\Gamma_e)}$. We will show that modulo r each of S_1, S_2, S_3 is of the required form.

Write $p_0(x_0, x_1, x_2) = \sum_{i=0}^{d} p_{0,i}(x_0) x_1^i + x_2 p_2(x_0, x_1, x_2)$ for some polynomials $p_{0,i}$, p_2. Then we get

$$S_1 \equiv \sum_{i=0}^{d} a^i S^{\Gamma}_{A,r}(p_{0,i}(w(h_1)) \cdot Q_0) \pmod r\,,$$

Hence for r large enough and coprime to a finite set of integers, S_1 is equivalent modulo r to a polynomial in a and terms $S^{\Gamma}_{A,0}(R)$ for some polynomials R, and, in particular, modulo r it is of the required form.

By (4.2) we can rewrite the sum S_2 as

$$\begin{aligned} S_2 &= r^{-h_1(\Gamma_e)} \sum_{w \in W_0 \colon 1 \leqslant r - w(h_1) \leqslant a} p_1(w(h_1), a, r, r) Q_0(w) \\ &= \sum_{j=1}^{a} \left(p_1(r-j, a, r, r) r^{-h_1(\Gamma_e)} \sum_{w \in W^{\Gamma_e}_{(A, r-j, j-r), r}} Q_0(w) \right) \\ &= \sum_{j=1}^{a} p_1(r-j, a, r, r) S^{\Gamma_e}_{(A,-j,j),r}(Q_0)\,. \end{aligned}$$

Taking r large enough with respect to a and coprime to a finite set of integers, for every j the factor $S^{\Gamma_e}_{(A,-j,j),r}(Q_0)$ is equivalent modulo r to $S^{\Gamma_e}_{(A,-j,j),0}(Q_0)$. By our induction hypothesis applied to the graph Γ_e, the term $q(A,j) \coloneqq S^{\Gamma_e}_{(A,-j,j),0}(Q_0)$ is a polynomial in A and j. Then

$$S_2 \equiv \sum_{j=1}^{a} p_1(-j,a,0,0)q(A,j) \pmod{r},$$

and $\sum_{j=1}^{a} p_1(-j,a,0,0)q(A,j)$ is a polynomial in A and a for $a \geqslant 0$.

Finally, S_3 is 0 unless $c_1 = 0 < c_2$, so assume $c_1 = 0 < c_2$. Then

$$S_3 = r^{c_2-1} S^{\Gamma_e}_{(A,-a,a),r}(Q_0)\,.$$

Modulo r this is either 0 or $S^{\Gamma_e}_{(A,-a,a),0}(Q_0)$, which by the induction hypothesis is polynomial in A and a.

In total $S^{\Gamma}_{A_a,r}(Q)$ is equivalent modulo r to a fixed polynomial $\Psi_{\Gamma,e,Q}$ in A and a and terms $S^{\Gamma}_{A,0}(R_i)$ for polynomials $R_1,\dots,R_m$, for every r large enough and coprime to a fixed integer. This means that for $a \geqslant 0$ we find

$$S^{\Gamma}_{A_a,0}(Q) = \Psi_{\Gamma,e,Q}\big(a_1,\dots,a_n,a,S^{\Gamma}_{A,0}(R_1),\dots,S^{\Gamma}_{A,0}(R_m)\big)\,.$$

This finishes the proof for $a \geqslant 0$.

Now we apply this to the edge with the opposite orientation and to A_a with a positive. We find that for $a,b \geqslant 0$ we have that $S^{\Gamma}_{A_{a-b},0}(Q)$ is a polynomial in A, a, b and terms $S^{\Gamma}_{A_a,0}(R)$ for polynomials R. The latter are polynomials in A, a, $S^{\Gamma}_{A,0}(R')$ for polynomials R', so in total for $a,b \geqslant 0$ the term $S^{\Gamma}_{A_{a-b},0}(Q)$ is a polynomial in A, a, b and terms $S^{\Gamma}_{A,0}(R)$. As it is also a function of A, $a-b$ and terms $S^{\Gamma}_{A,0}(R)$, it is a polynomial in A, $a-b$ and terms $S^{\Gamma}_{A,0}(R)$, and we find that

$$S^{\Gamma}_{A_a,0}(Q) = \Psi_{\Gamma,e,Q}\big(a_1,\dots,a_n,a,S^{\Gamma}_{A,0}(R_1),\dots,S^{\Gamma}_{A,0}(R_m)\big)$$

holds for any $a \in \mathbb{Z}$. $\square$

4.3 Proof of Theorem 4.1

Now we have assembled the ingredients to prove Theorem 4.1.

Proof of Theorem 4.1. We will prove this by induction on the number of edges of Γ. If Γ is a graph without edges, this is immediately clear. Now we assume that Theorem 4.1 holds for graphs with fewer edges than Γ.

If Γ has a separating edge, the result follows immediately from Proposition 4.5. We now assume that there are no separating edges.

We can assume that there is exactly one leg at every vertex. We number the vertices 1 through n, in accordance with the attached leg. Let T be a spanning tree of Γ. Without loss of generality, our numbering is such that for every $k = 1,\dots,n$ the vertices $\{1,\dots,k\}$ form a subtree T_k of T. We write

$$P_k \coloneqq \left\{A \in \mathbb{Z}^n : \sum_{i=1}^{k} a_i = 0 \wedge a_{k+1} = \dots = a_n = 0\right\}.$$

Then we will prove by induction on k that for every polynomial Q the map $A \mapsto S^{\Gamma}_{A,0}(Q)$ is polynomial when restricted to P_k.

We start with a subtree with one vertex. By the total sum zero condition, we have $P_1 = \{0\}$, and then for every polynomial Q the map $P_1 \to \mathbb{Q}$, $A \mapsto S^{\Gamma}_{A,0}(Q)$ is a constant map, and in particular polynomial.

Now assume that we have proven the result for k and want to prove it for $k+1$. Let $j \leqslant k$ be the vertex that in T is adjacent to vertex $k+1$. Note that $P_{k+1} = P_k + \{a(e_k - e_j) : a \in \mathbb{Z}\}$, where e_i is the ith basis vector in $\mathbb{Z}^n$. The induction step follows from Proposition 4.6. As $P_n = \left\{A \in \mathbb{Z}^n : \sum_i a_i = 0\right\}$, we are done. $\square$

We immediately find the following corollary.

Corollary 4.7. *The function*

$$\left\{A \in \mathbb{Z}^n : (2g-2+n) \mid \sum a_i\right\} \longrightarrow \mathbb{Q}\,, \quad A \longmapsto S^{\Gamma}_{A,0}(Q)$$

is polynomial in A.

Proof. We will prove this by reducing to the $k = 0$ case.

Let $\Gamma' \in G_{g,n+\#V}$ be the graph Γ with one leg added at every vertex. Let A' denote the vector $(a_1, \ldots, a_n, (-k(2g(v) - 2 + n(v)))_{v \in v})$. We see that the sum of the elements in A' is 0. Note that $S^{\Gamma}_{A,r}(Q) = S^{\Gamma'}_{A',r}(Q)$. The theorem follows by applying Theorem 4.1 to $S^{\Gamma'}_{A',r}(Q)$.

$\square$

5. Polynomiality of the double ramification cycle

In this section we prove the polynomiality of the double ramification cycle. With Theorem 4.1 and Pixton's formula (3.1), we conclude with the following theorem.

Theorem 5.1. *Fix g, n. The function*

$$\left\{(a_1, \ldots, a_n) \in \mathbb{Z}^n : \exists k, \sum a_i = k(2g-2+n)\right\} \longrightarrow \mathrm{CH}^g\left(\overline{\mathcal{M}}_{g,n}\right), \quad A \longmapsto \mathrm{DR}_g(A)$$

is polynomial.

Proof. By formula (3.1) we have that $\mathrm{DR}_g(a)$ is a polynomial in a finite set of decorated strata, the a_i and terms $S^{\Gamma}_{A,0}(Q)$. The result follows from Corollary 4.7. $\square$

References

BHP+23 Y. Bae, D. Holmes, R. Pandharipande, J. Schmitt, and R. Schwarz, *Pixton's formula and Abel–Jacobi theory on the Picard stack*, Acta Math. **230** (2023), no. 2, 205–319; doi:10.4310/acta.2023.v230.n2.a1.

Bur15 A. Buryak, *Double ramification cycles and integrable hierarchies*, Comm. Math. Phys. **336** (2015), no. 3, 1085–1107; doi:10.1007/s00220-014-2235-2.

BR21 A. Buryak and P. Rossi, *Quadratic double ramification integrals and the noncommutative KdV hierarchy*, Bull. London Math. Soc. **53** (2021), no. 3, 843–854; doi:10.1112/blms.12464.

CGH+25 D. Chen, S. Grushevsky, D. Holmes, M. Möller, and J. Schmitt, *A tale of two moduli spaces: logarithmic and multi-scale differentials*, Épijournal Géom. Algébrique **9** (2025), article no. 21; doi:10.46298/epiga.2025.11278.

CJ18 E. Clader and F. Janda, *Pixton's double ramification cycle relations*, Geom. Topol. **22** (2018), no. 2, 1069–1108; doi:10.2140/gt.2018.22.1069.

FWY21 H. Fan, L. Wu, and F. You, *Higher genus relative Gromov–Witten theory and double ramification cycles*, J. Lond. Math. Soc. (2) **103** (2021), no. 4, 1547–1576; doi:10.1112/jlms.12417.

FP18 G. Farkas and R. Pandharipande, *The moduli space of twisted canonical divisors*, J. Inst. Math. Jussieu **17** (2018), no. 3, 615–672; doi:10.1017/S1474748016000128.

FTT16 E. Frenkel, C. Teleman, and A. J. Tolland, *Gromov–Witten gauge theory*, Adv. Math. **288** (2016), 201–239; doi:10.1016/j.aim.2015.10.008.

GV03 T. Graber and R. Vakil, *Hodge integrals and Hurwitz numbers via virtual localization*, Compos. Math. **135** (2003), no. 1, 25–36; doi:10.1023/A:1021791611677.

Hol21 D. Holmes, *Extending the double ramification cycle by resolving the Abel-Jacobi map*, J. Inst. Math. Jussieu **20** (2021), no. 1, 331–359; doi:10.1017/S1474748019000252.

JPPZ17 F. Janda, R. Pandharipande, A. Pixton, and D. Zvonkine, *Double ramification cycles on the moduli spaces of curves*, Publ. Math. Inst. Hautes Études Sci. **125** (2017), 221–266; doi:10.1007/s10240-017-0088-x.

MW20 S. Marcus and J. Wise, *Logarithmic compactification of the Abel–Jacobi section*, Proc. Lond. Math. Soc. **121** (2020), no. 5, 1207–1250; doi:10.1112/plms.12365.

MPS23 S. Molcho, R. Pandharipande, and J. Schmitt, *The Hodge bundle, the universal* 0*-section, and the log Chow ring of the moduli space of curves*, Compos. Math. **159** (2023), no. 2, 306–354; doi:10.1112/S0010437X22007874.

Pix23 A. Pixton, *DR cycle polynomiality and related results*, 2023, available at https://public.websites.umich.edu/~pixton/papers/DRpoly.pdf, accessed 11 January 2024.

RK24 D. Ranganathan and A. U. Kumaran, *Logarithmic Gromov-Witten theory and double ramification cycles*, J. reine angew. Math. **809** (2024), 1–40; doi:10.1515/crelle-2023-0100.

Pim Spelier p.spelier@uu.nl
Department of Mathematics, Utrecht University, P.O. Box 80010, 3508 TA Utrecht, The Netherlands