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A log resolution for the theta divisor of a hyperelliptic curve

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Abstract

In this paper, we prove that the theta divisor of a smooth hyperelliptic curve has a natural and explicit embedded resolution of singularities using iterated blowups of Brill–Noether subvarieties. We also show that the Brill–Noether stratification of the hyperelliptic Jacobian is a Whitney stratification.

1. Introduction

Let C be a smooth projective curve of genus $g \geqslant 1$. Let $\mathrm{Jac}(C)$ be the Jacobian of C, and let $\Theta \subseteq \mathrm{Jac}(C)$ be the theta divisor. The purpose of this paper is to give a natural and explicit log resolution of the pair $(\mathrm{Jac}(C), \Theta)$ when C is a hyperelliptic curve.

Recall that the Brill–Noether variety $W^r_{g-1}(C)$ parametrizes line bundles $L \in \mathrm{Pic}^{g-1}(C)$ of degree $g-1$ with $h^0(L) \geqslant r+1$. According to a theorem by Riemann, we can choose an isomorphism $\mathrm{Jac}(C) \cong \mathrm{Pic}^{g-1}(C)$ so that the theta divisor Θ becomes identified with $W_{g-1}(C) := W^0_{g-1}(C)$. The Abel–Jacobi map from the symmetric product $C_{g-1} := \mathrm{Sym}^{g-1}(C)$ gives a resolution of singularities of Θ, which is useful for answering many geometric questions about Jacobian varieties. However, if one wants to investigate the geometry of the embedding $\Theta \subseteq \mathrm{Jac}(C)$, one needs instead a log resolution of the pair $(\mathrm{Jac}(C), \Theta)$. Inspired by a global study of the vanishing cycle functor for divisors [SY26], we are lead to the question of finding an explicit log resolution in the case of hyperelliptic theta divisors. Since the log resolution is of a purely geometric nature, we leave the actual computation of vanishing cycles to another paper [SY26].

When C is a hyperelliptic curve of genus $g \geqslant 1$, we have a lot of very precise information about the chain of subvarieties

$$\Theta = W_{g-1}(C) \supseteq W^1_{g-1}(C) \supseteq \cdots \supseteq W^n_{g-1}(C)\,, \tag{1.1}$$

where $n = \lfloor (g-1)/2 \rfloor$ is the maximal integer such that $W^n_{g-1}(C) \neq \varnothing$. First, the dimension of $W^r_{g-1}(C)$ is equal to $g-1-2r$, and $W^r_{g-1}(C)$ is reduced (see Proposition A.3). Second, the

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singular locus of $W^r_{g-1}(C)$ is exactly $W^{r+1}_{g-1}(C)$. Third, the multiplicity of the theta divisor at a point $L \in \mathrm{Pic}^{g-1}(C)$ is equal to $h^0(L)$ by the Riemann singularity theorem, so $W^r_{g-1}(C)$ is exactly the set of points of multiplicity at least $r+1$ on Θ (see [ACGH85, § IV.4] for details).

These facts immediately suggest that one might be able to get a log resolution of the pair $(\mathrm{Jac}(C), \Theta)$ by successively blowing up the Brill–Noether subvarieties $W^r_{g-1}(C)$ in the order from smallest to largest. This guess turns out to be correct, but it requires quite a bit of work to prove rigorously that it works.

More precisely, we use the following iterative procedure, consisting of n steps. In the first step, we blow up $\mathrm{Jac}(C)$ along the smallest subvariety $W^n_{g-1}(C)$ and denote the blowup by $\pi_1\colon \mathrm{bl}_1(\mathrm{Jac}(C)) \to \mathrm{Jac}(C)$. In the second step, we blow up $\mathrm{bl}_1(\mathrm{Jac}(C))$ along the strict transform of $W^{n-1}_{g-1}(C)$ and denote the new blowup by $\pi_2\colon \mathrm{bl}_2(\mathrm{Jac}(C)) \to \mathrm{Jac}(C)$. In the ith step, we blow up $\mathrm{bl}_{i-1}(\mathrm{Jac}(C))$ along the strict transform of $W^{n+1-i}_{g-1}(C)$ and denote the new blowup by $\pi_i\colon \mathrm{bl}_i(\mathrm{Jac}(C)) \to \mathrm{Jac}(C)$. This process stops after the nth step. The strict transforms of the exceptional divisor give us a sequence of divisors $Z_0, Z_1, \dots, Z_{n-1}$, with Z_i sitting over the locus $W^{n-i}_{g-1}(C)$. Let $\tilde{\Theta}$ denote the strict transform of the theta divisor. With this notation, our main result is the following.

Theorem A. *The morphism $\pi_n\colon \mathrm{bl}_n(\mathrm{Jac}(C)) \to \mathrm{Jac}(C)$ is a log resolution of $(\mathrm{Jac}(C), \Theta)$, where*

$$\pi_n^*(\Theta) = \tilde{\Theta} + \sum_{i=0}^{n-1} (n+1-i) Z_i$$

is a divisor with simple normal crossing support. Moreover, the strict transform of $W^{n-i}_{g-1}(C)$ in $\mathrm{bl}_i(\mathrm{Jac}(C))$ is smooth. In other words, each blowup in the sequence is a blowup along a smooth center.

We can also describe the generic structure of the exceptional divisors.

Corollary B. *For $r = 1, \dots, n$, every fiber of the projection*

$$Z_{n-r} \setminus \Bigl(\tilde{\Theta} \cup \bigcup_{\substack{0 \leqslant i \leqslant n-1 \\ i \neq n-r}} Z_i \Bigr) \longrightarrow W^r_{g-1}(C) \setminus W^{r+1}_{g-1}(C)$$

is isomorphic to the complement of a hypersurface of degree $r+1$ in $\mathbf{P}^{2r}$; the hypersurface is the $(r-1)$th secant variety of a rational normal curve of degree $2r$ in $\mathbf{P}^{2r}$.

There are a few other examples in the literature where this simple-minded procedure of successive blowups along singular loci produces a log resolution.

(1) Let X be the affine space of n-by-n matrices, and let D be the hypersurface defined by the vanishing of the determinant function. Let $D_i \subseteq D$ be the set of matrices of rank at most $n-i$. According to [ACGH85, § II.2, p. 69, Proposition], one has $(D_i)_{\mathrm{sing}} = D_{i+1}$, and D_i is exactly the set of points of multiplicity at least i on D. It is proved in [Joh03, Chapter 4] and also in [Vai84] that one can construct a log resolution of the pair (X, D) by successively blowing up $D_n, D_{n-1}, \dots, D_2$.

(2) Let $X = \mathbf{P}H^0(C, M)$, and let $D = \mathrm{Sec}^n(C)$ be the nth secant variety of a smooth projective curve C, embedded by a line bundle M with $h^0(M) = 2n+3$ that separates $2n+2$ points. Setting $D_i = \mathrm{Sec}^{n-i+1}(C)$, Bertram [Ber92, § 1, Corollary, p. 440] proved that $(D_i)_{\mathrm{sing}} = D_{i+1}$ and that D_i is again the set of points of multiplicity at least i on D. He also showed [Ber92,

Corollary 2.4] that successively blowing up $D_n, D_{n-1}, \ldots, D_2$ produces a log resolution of the pair (X, D).

The subvarieties in (1.1) induce a stratification

$$\operatorname{Jac}(C) = (\operatorname{Jac}(C) - \Theta) \sqcup \bigsqcup_{0 \leqslant r \leqslant n} \left(W^r_{g-1}(C) - W^{r+1}_{g-1}(C)\right),$$

which is named the Brill–Noether stratification. Inspired by the proof of Theorem A, we also find the following.

Proposition 1.1. *If C is a smooth hyperelliptic curve, then the Brill–Noether stratification of* $\operatorname{Jac}(C)$ *is a Whitney stratification.*

Ideas of the proof

The main tool is Bertram's blowup construction for a chain of maps [Ber92]. One inconvenient point in the process described above is that the Brill–Noether varieties $W^r_{g-1}(C)$ are not smooth, which makes it hard to keep track of conormal bundles and exceptional divisors in the various blowups. Fortunately, on a hyperelliptic curve, each $W^r_{g-1}(C)$ has a natural resolution of singularities by C_{g-1-2r}, viewed as the space of effective divisors of degree $g-1-2r$ on C. Let g^1_2 be the line bundle corresponding to the hyperelliptic map $h\colon C \to \mathbf{P}^1$. For $0 \leqslant r \leqslant n = \lfloor (g-1)/2 \rfloor$, the resolution of singularities is given by the Abel–Jacobi mapping

$$\delta_{n-r}\colon C_{g-1-2r} \longrightarrow W^r_{g-1}(C), \quad D \longmapsto r g^1_2 \otimes \mathcal{O}_C(D).$$

Since it is easier to blow up smooth varieties, instead of $W^r_{g-1}(C)$, we work with the chain of maps $\{\delta_i\}_{i=0}^n$. The advantage is that we do not need to analyze the singularities of the proper transforms of $W^r_{g-1}(C)$ and how they intersect with exceptional divisors; instead, we transform the problem into checking that certain maps are embeddings (see Proposition 2.18), which eventually reduces to the calculation of certain conormal bundles. The projectivized conormal bundles that show up as exceptional divisors are closely related to secant bundles over symmetric products of $\mathbf{P}^1$; for that reason, Bertram's result about these secant bundles is another crucial input.

Outline of the paper

In Section 2, we recall Bertram's blowup construction in detail. In Section 3, we set up notation for the Abel–Jacobi maps and reduce the proof of Theorem A to two propositions (Propositions 3.1 and 3.2), which deal with the properties of two chains of maps between symmetric products and Jacobians. In Section 4, we review the construction of secant bundles and describe Bertram's results. In Section 5, we study some basic properties of Abel–Jacobi maps and the addition maps among symmetric products. In Sections 7 and 6, we prove Propositions 3.1 and 3.2, and thereby complete the proof of Theorem A for hyperelliptic curves of odd genus. The proof of Corollary B can be found at the end of Section 7. In Section 8, we outline a proof for hyperelliptic curves of even genus, which goes along the same line but requires a few changes in the notation. In Section 9, we prove Proposition 1.1. In Section 10, we propose some questions in the direction of this paper.

Notation

- If V is a vector space, $\mathbf{P}(V)$ stands for the projective space of 1-dimensional quotients of V. We use the same notation for vector bundles.

- Let $f\colon X \to Y$ be a morphism between smooth projective varieties. Let $Y_1 \subseteq Y$ be a subvariety. We use the notation
$$f^{-1}(Y_1) := Y_1 \times_Y X$$
exclusively for the scheme-theoretic preimage, which is the fiber product of the two morphisms $X \to Y$ and $Y_1 \to Y$.
- Let $f\colon X \to Y$ be a morphism between smooth varieties. We denote by
$$df\colon f^*T_Y^* \longrightarrow T_X^*$$
the induced morphism between cotangent bundles, and by
$$N_f^* = \operatorname{Ker}\big(df\colon f^*T_Y^* \to T_X^*\big) \tag{1.2}$$
the conormal bundle of the morphism. In the case of a closed embedding $X \subseteq Y$, we also use the notation $N_{X|Y}^*$.

2. Bertram's blowup construction

In this section, we review [Ber92, § 2] and provide more details for the benefit of readers. The main result is Proposition 2.18, which is an inductive criterion for constructing a log resolution out of a sequence of morphisms.

2.1 Chains and maps of chains

Let X be a projective variety, not necessarily smooth.

Definition 2.1. A *proper chain* is a sequence of morphisms $\{f_i\colon X_i \to X\}_{i=0}^n$ from projective varieties X_i with the property that for each $0 \leqslant i < j \leqslant n$, there exists a commutative diagram
$$\begin{array}{ccc} X_{i,j} & \xrightarrow{g_{i,j}} & X_i \\ \big\downarrow{\scriptstyle h_{i,j}} & & \big\downarrow{\scriptstyle f_i} \\ X_j & \xrightarrow{f_j} & X \end{array}$$
such that $g_{i,j}$ is surjective and there is a proper inclusion $f_i(X_i) \subsetneq f_j(X_j)$.

Remark 2.2. It is sufficient to take $X_{i,j}$ to be the fiber product of f_i and f_j. However, in practice, $X_{i,j}$ is usually not the fiber product and will become a fiber product after sufficiently many blowups; see Remark 3.3 for an example.

Definition 2.3. Let $\{f_j\colon X_j \to X\}$ be a proper chain. Assume that f_0 is an *embedding*. We identify X_0 with its image and define
$$\begin{aligned} \operatorname{bl}_1(X) &:= \text{the blowup of } X \text{ along } X_0\,, \\ \operatorname{bl}_1(X_j) &:= \text{the blowup of } X_j \text{ along } f_j^{-1}(X_0)\,, \\ \operatorname{bl}_1(f_j) &:= \text{the unique lift of } f_j \text{ to a map } \operatorname{bl}_1(X_j) \to \operatorname{bl}_1(X)\,. \end{aligned}$$
Assume that for some $i \geqslant 1$, the objects $\operatorname{bl}_i(X)$, $\operatorname{bl}_i(X_j)$ and $\operatorname{bl}_i(f_j)$ have already been defined for all $j \geqslant i$, and that the map
$$\operatorname{bl}_i(f_i)\colon \operatorname{bl}_i(X_i) \longrightarrow \operatorname{bl}_i(X)$$

is an *embedding*. Then we can identify $\mathrm{bl}_i(X_i)$ with its image and set

$$\begin{aligned} \mathrm{bl}_{i+1}(X) &:= \text{the blowup of } \mathrm{bl}_i(X) \text{ along } \mathrm{bl}_i(X_i)\,, \\ \mathrm{bl}_{i+1}(X_j) &:= \text{the blowup of } \mathrm{bl}_i(X_j) \text{ along } \mathrm{bl}_i(f_j)^{-1}(\mathrm{bl}_i(X_i))\,, \\ \mathrm{bl}_{i+1}(f_j) &:= \text{the unique lift of } \mathrm{bl}_i(f_j) \text{ to a map } \mathrm{bl}_{i+1}(X_j) \to \mathrm{bl}_{i+1}(X)\,. \end{aligned}$$

Notation 2.4. To have a uniform notation, we set

$$\mathrm{bl}_0(f_i) := f_i\,, \quad \mathrm{bl}_0(X_i) := X_i\,, \quad \mathrm{bl}_0(X) := X\,.$$

Definition 2.5. Let $\{f_i\colon X_i \to X\}_{i=0}^n$ be a proper chain. If $\mathrm{bl}_{n+1}(X)$ is defined in Definition 2.3, we say that $\{f_i\}$ is a *chain of centers*. Concretely, this amounts to the (recursive) condition that the $n+1$ morphisms f_0, $\mathrm{bl}_1(f_1)$, …, $\mathrm{bl}_n(f_n)$ are closed embeddings. If X and $\{\mathrm{bl}_i(X_i)\}_{i=0}^n$ are all smooth, then we say that $\{f_i\}$ is a *chain of smooth centers*.

We formulate an additional definition, which is not in [Ber92] and ensures that the exceptional divisors in the final blowup $\mathrm{bl}_{n+1}(X)$ form a simple normal crossing divisor.

Definition 2.6. Let $\{f_i\colon X_i \to X\}_{i=0}^n$ be a chain of smooth centers. We define divisors

$$E_{i,j} \subseteq \mathrm{bl}_j(X) \quad \text{for } j \geqslant i+1\,,$$

as follows. First, $E_{i,i+1} \subseteq \mathrm{bl}_{i+1}(X)$ is the exceptional divisor for the blowing-up of $\mathrm{bl}_i(X)$ along $\mathrm{bl}_i(X_i)$. For each $j \geqslant i+1$, let $E_{i,j} \subseteq \mathrm{bl}_j(X)$ be the inverse image of $E_{i,i+1}$, with the following Cartesian diagrams:

$$\begin{array}{ccccc} E_{i,j} & \longrightarrow & E_{i,i+1} & \longrightarrow & \mathrm{bl}_i(X_i) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathrm{bl}_j(X) & \longrightarrow & \mathrm{bl}_{i+1}(X) & \longrightarrow & \mathrm{bl}_i(X)\,. \end{array}$$

Moreover, set

$$E_i := E_{i,n+1} \subseteq \mathrm{bl}_{n+1}(X)\,.$$

We say that $\{E_i\}_{i=0}^n$ is *the set of exceptional divisors* of the chain $\{f_i\}_{i=0}^n$.

A chain of smooth centers $\{f_i\}_{i=0}^n$ is called an *NCD chain* if, for each $j \leqslant n$, the intersection

$$\mathrm{bl}_j(X_j) \cap E_{i,j} \subseteq \mathrm{bl}_j(X)$$

is transverse for all $i < j$ and the divisor $E_{0,j} + \cdots + E_{j-1,j} \subseteq \mathrm{bl}_j(X)$ has simple normal crossings.

Remark 2.7. Let $\{f_i\colon X_i \to X\}_{i=0}^n$ be a chain of smooth centers. Then there is a natural embedding of $X - f_n(X_n)$ into $\mathrm{bl}_{n+1}(X)$ such that

$$\mathrm{bl}_{n+1}(X) - \bigcup_{0 \leqslant i \leqslant n} E_i = X - f_n(X_n)\,.$$

For induction purposes, the following notation becomes convenient.

Notation 2.8. Let S be a smooth variety, and let $\{f_i\colon X_i \to X\}_{i=0}^n$ be a proper chain. It induces a new proper chain $\{f_i \times \mathrm{id}\colon X_i \times S \to X \times S\}_{i=0}^n$, that is, the collection of maps that are f_i on the first factor and the identity on S.

Lemma 2.9. *Let $\{f_i\colon X_i \to X\}_{i=0}^n$ be an NCD chain. For each $0 \leqslant i < j \leqslant n+1$, let $E_{i,j} \subseteq \mathrm{bl}_j(X)$ and $F_{i,j} \subseteq \mathrm{bl}_j(X \times S)$ be the exceptional divisors of the chains $\{f_i\}_{i=0}^n$ and $\{f_i \times \mathrm{id}\}_{i=0}^n$. Then* *there are natural isomorphisms*

$$\begin{aligned} \mathrm{bl}_i(X \times S) &= \mathrm{bl}_i(X) \times S\,, \quad \mathrm{bl}_i(X_j \times S) = \mathrm{bl}_i(X_j) \times S\,, \\ \mathrm{bl}_i(f_j \times \mathrm{id}) &= \mathrm{bl}_i(f_j) \times \mathrm{id}\,, \quad F_{i,j} = E_{i,j} \times S\,. \end{aligned}$$

Moreover, $\{f_i \times \mathrm{id}\}_{i=0}^{n}$ *is again an NCD chain.*

Proof. This follows from the fact that blowup maps commute with taking the Cartesian product with the smooth variety S, and that transversality is also preserved under product with S. □

To prove that a chain of centers is an NCD chain, it is useful to have the following notion.

Definition 2.10. Suppose that $\{f_i\colon X_i \to X\}_{i=0}^{n}$ and $\{g_i\colon Y_i \to Y\}_{i=0}^{n}$ are two chains of centers. We say that a map $\phi\colon X \to Y$ is a *map of chains of centers* if it satisfies the following conditions:

- First, $\phi^{-1}(Y_0) = X_0$, so $\mathrm{bl}_1(X) \to \mathrm{bl}_1(Y)$ is defined.
- Inductively, assume that, for some $0 < i \leqslant n$, the map $\mathrm{bl}_i(\phi)\colon\ \mathrm{bl}_i(Y) \to \mathrm{bl}_i(X)$ exists; then one has $\mathrm{bl}_i(\phi)^{-1}(\mathrm{bl}_i(Y_i)) = \mathrm{bl}_i(X_i)$. Consequently, one can define

  $$\mathrm{bl}_{i+1}(\phi)\colon\ \mathrm{bl}_{i+1}(X) \longrightarrow \mathrm{bl}_{i+1}(Y)$$

  to be the unique lift of $\mathrm{bl}_i(\phi)$.

We say that the map ϕ is an *injective* map of chains of centers if, in addition, $\mathrm{bl}_{n+1}(\phi)$ is injective.

The following lemma gives an inductive criterion for a chain to be NCD.

Lemma 2.11. *Let* $\{\phi_j\colon X_j \to X\}_{j=0}^{n}$ *be a proper chain, with the following diagrams for* $0 \leqslant i < j \leqslant n$*:*

$$\begin{array}{ccc} X_{i,j} & \twoheadrightarrow & X_i \\ \big\downarrow{\scriptstyle f_{i,j}} & & \big\downarrow{\scriptstyle \phi_i} \\ X_j & \xrightarrow{\ \phi_j\ } & X\,. \end{array}$$

Suppose that X*,* X_i*,* $X_{i,j}$ *are all smooth projective varieties, and assume that the following conditions are satisfied:*

(a) *For each* $j \leqslant n$*, the chain* $\{f_{i,j}\colon X_{i,j} \to X_j\}_{i=0}^{j-1}$ *is NCD.*

(b) *The chain* $\{\phi_j\}_{j=0}^{n}$ *is a chain of smooth centers.*

(c) *Each* ϕ_j *is a map of chains of centers.*

Then the chain $\{\phi_j\}_{j=0}^{n}$ *is an NCD chain.*

Proof. This follows from [Ber92, Lemma 2.1] and its proof. □

In the rest of this section, we discuss how one can replace conditions (b) and (c) in Lemma 2.11 with certain conditions on exceptional divisors and their complements, in the presence of condition (a), more precisely, the transversality condition from $\{f_{i,j}\}$ being NCD chains. By extracting these conditions from the proof of [Ber92, Proposition 2.2], we hope to make our proof of Theorem A more transparent.

2.2 Criteria for maps of chains of centers

Given a map $\phi\colon X \to Y$ between chains $\{f_i\colon X_i \to X\}$ and $\{g_i\colon Y_i \to Y\}$, to show that it is a map of chains of centers, one needs to check $\mathrm{bl}_i(\phi)^{-1}(\mathrm{bl}_i(Y_i)) = \mathrm{bl}_i(X_i)$ for each i. In the presence of certain transversality conditions, we can apply the lemma below.

Notation 2.12. For a sequence of divisors $\{E_i\}_{i=0}^n$, set

$$E_i^\circ := E_i - (E_0 \cup E_1 \cup \cdots \cup E_{i-1}) .$$

Note that we are removing *only* the intersections with the previous divisors.

Lemma 2.13. *Let $\phi\colon X \to Y$ be a morphism between smooth projective varieties. Let $\{E_i\}_{i=0}^{j-1}$ and $\{F_i\}_{i=0}^{j-1}$ be two sequences of smooth divisors in X and Y such that*

$$\phi^{-1}(F_i) = E_i , \quad \forall 0 \leqslant i \leqslant j-1 .$$

Let $X_1 \subseteq X$ and $Y_1 \subseteq Y$ be two smooth subvarieties. Assume that for each $0 \leqslant i \leqslant j-1$,

(a) *the intersections $X_1 \cap E_i$ and $Y_1 \cap F_i$ are transverse,*

(b) $\phi^{-1}(Y_1 \cap F_i^\circ) = X_1 \cap E_i^\circ$,

(c) $\phi^{-1}(Y_1 - \bigcup_i F_i) = X_1 - \bigcup_i E_i$.

Then $\phi^{-1}(Y_1) = X_1$.

Proof. From the assumption, we know that the set-theoretic preimage of Y_1 under ϕ is X_1. In order to show the scheme-theoretical statement, we need to know that $\phi^*\mathcal{I}_{Y_1} \to \mathcal{I}_{X_1}$ is surjective. Recall that for a closed embedding of smooth varieties $i\colon A \hookrightarrow B$, the conormal bundle is denoted by

$$N^*_{A|B} = \mathrm{Ker}(di\colon i^*T^*_B \to T^*_A) .$$

Since $N^*_{X_1|X} = \mathcal{I}_{X_1}/\mathcal{I}^2_{X_1}$ and $N^*_{Y_1|Y} = \mathcal{I}_{Y_1}/\mathcal{I}^2_{Y_1}$, by Nakayama's lemma, this is equivalent to the surjectivity of

$$d\phi\colon \phi^* N^*_{Y_1|Y} \longrightarrow N^*_{X_1|X} .$$

This can be checked over $X_1 - \bigcup_i E_i = X_1 - \bigcup_i E_i^\circ$ and $X_1 \cap E_i^\circ$ separately. Condition (c) implies that $d\phi$ is surjective over $X_1 - \bigcup_i E_i$. On the other hand, using condition (a), we have the following commutative diagram:

$$\begin{array}{ccc} \phi^* N^*_{Y_1|Y}\big|_{Y_1\cap F_i^\circ} & \xrightarrow{d\phi|_{X_1\cap E_i^\circ}} & N^*_{X_1|X}\big|_{X_1\cap E_i^\circ} \\ \downarrow{\cong} & & \downarrow{\cong} \\ \phi^* N^*_{Y_1\cap F_i^\circ|F_i^\circ} & \xrightarrow{d(\phi|_{E_i^\circ})} & N^*_{X_1\cap E_i^\circ|E_i^\circ} . \end{array}$$

The bottom map is induced by $\phi|_{E_i^\circ}\colon E_i^\circ \to F_i^\circ$. The vertical maps are isomorphisms because of the transversality condition. Therefore, condition (b) implies that $d\phi$ is also surjective over $X_1 \cap E_i^\circ$ for each i. $\square$

2.3 Criterion for a chain of centers

Let $\{h_i\colon Z_i \to Z\}$ be a proper chain where Z_i and Z are all smooth. Assume that for some i, the map

$$\mathrm{bl}_i(h_i)\colon\ \mathrm{bl}_i(Z_i) \longrightarrow \mathrm{bl}_i(Z) \tag{2.1}$$

exists. In this subsection, we explain how to check that it is an embedding under suitable transversality conditions.

Lemma 2.14. *Let $f\colon X \to Y$ be a morphism between two smooth projective varieties. Let $F \subseteq Y$ be a smooth divisor. If $E := f^{-1}(F)$ is also a smooth divisor, then*

$$f^* N^*_{F|Y} = N^*_{E|X}\,,$$

where $N^*_{E|X}$ is the conormal bundle of E in X.

Proof. Since F is a smooth divisor, we have $N^*_{F|Y} = \mathcal{O}_Y(-F)|_F$ by the conormal sequence; the same holds for E. Therefore,

$$f^* N^*_{F|Y} = f^*\mathcal{O}_Y(-F)|_F = \mathcal{O}_X(-E)|_E = N^*_{E|X}\,. \qquad \square$$

Lemma 2.15. *Let $f\colon X \to Y$ be a morphism between two smooth projective varieties. Let $Z \subseteq Y$ be a smooth subvariety such that $W := f^{-1}(Z)$ is smooth and properly contained in X. Set $\tilde{Y} = \mathrm{bl}_Z\, Y$ and $\tilde{X} = \mathrm{bl}_W\, X$, with the following diagram:*

$$\begin{array}{ccccccc} E & \hookrightarrow & \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} & \hookleftarrow & F \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ W & \hookrightarrow & X & \xrightarrow{f} & Y & \hookleftarrow & Z\,, \end{array}$$

where $\tilde{f}\colon \tilde{X} \to \tilde{Y}$ is the induced morphism and F and E are the exceptional divisors. Then

$$\tilde{f}^{-1}(F) = E\,, \quad \tilde{f}^* N^*_{F|\tilde{Y}} = N^*_{E|\tilde{X}}\,. \tag{2.2}$$

Furthermore, suppose that

(a) *the map $\tilde{f}\big|_E\colon E \to F$ is an embedding,*
(b) *the map $f\colon X - W \to Y - Z$ is an embedding.*

Then $\tilde{f}$ is an embedding.

Proof. It is proved in [Ber92, § 1.2, p. 442, Fact A] that $\tilde{f}^{-1}(F) = E$. By assumption, Z and W are smooth. Therefore, E and F are smooth divisors, and we can apply Lemma 2.14 to obtain (2.2).

By condition (b), the map $\tilde{f}$ is an embedding away from E. Condition (a) implies that $\tilde{f}$ is set-theoretically injective over E. Therefore, it suffices to show that

$$d\tilde{f}\colon \tilde{f}^* T^*_{\tilde{Y}} \longrightarrow T^*_{\tilde{X}}$$

is surjective over E. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{f}^* N^*_{F|\tilde{Y}} & \longrightarrow & \tilde{f}^* T^*_{\tilde{Y}}\big|_E & \longrightarrow & \tilde{f}^* T^*_F & \longrightarrow & 0 \\ & & \downarrow{\cong} & & \downarrow{d\tilde{f}} & & \downarrow{d(\tilde{f}|_E)} & & \\ 0 & \longrightarrow & N^*_{E|\tilde{X}} & \longrightarrow & T^*_{\tilde{X}}\big|_E & \longrightarrow & T^*_E & \longrightarrow & 0\,. \end{array}$$

By (2.2), the left vertical arrow is an isomorphism. Since $\tilde{f}\big|_E$ is an embedding, the right vertical arrow is surjective. By the snake lemma, we conclude that the middle vertical arrow is also surjective and conclude that $\tilde{f}$ is an embedding. $\square$

In the situation of (2.1), to prove that $\mathrm{bl}_i(h_i)$ is an embedding, we usually also know that $h_i\colon Z_i \to Z$ is a map of chains of centers. We thus give a generalization of Lemma 2.15 in such situations. Let $\{f_i\colon X_i \to X\}_{i=0}^{j-1}$ and $\{g_i\colon Y_i \to Y\}_{i=0}^{j-1}$ be two chains of centers, and let $\phi\colon X \to Y$ be a map of chains of centers. Then the map $\mathrm{bl}_i(\phi)\colon \mathrm{bl}_i(X) \to \mathrm{bl}_i(Y)$ exists for each $i \leqslant j$, and

$$\mathrm{bl}_i(\phi)^{-1}\big(\mathrm{bl}_i(X_i)\big) = \mathrm{bl}_i(Y_i)\,, \quad \forall 0 \leqslant i \leqslant j-1\,. \tag{2.3}$$

For each i, consider the following diagram:

$$\begin{array}{ccccccc}
E_i & \hookrightarrow & \mathrm{bl}_j(X) & \xrightarrow{\mathrm{bl}_j(\phi)} & \mathrm{bl}_j(Y) & \hookleftarrow & F_i \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{bl}_i(X_i) & \hookrightarrow & \mathrm{bl}_i(X) & \xrightarrow{\mathrm{bl}_i(\phi)} & \mathrm{bl}_i(Y) & \hookleftarrow & \mathrm{bl}_i(Y_i)\,.
\end{array}$$

Here E_i and F_i are exceptional divisors. Repeatedly applying (2.2) and (2.3), one has

$$\mathrm{bl}_j(\phi)^{-1}(F_i) = E_i \quad \forall 0 \leqslant i \leqslant j-1\,. \tag{2.4}$$

Lemma 2.16. *Assume that X, Y are smooth projective varieties and that*

(a) *the two chains $\{f_i\}_{i=0}^{j-1}$, $\{g_i\}_{i=0}^{j-1}$ are NCD chains;*

(b) *for every $0 \leqslant i \leqslant j-1$, the induced map*

$$\mathrm{bl}_j(\phi)\colon E_i^\circ \longrightarrow F_i^\circ$$

is an embedding, where $E_i^\circ = E_i - \bigcup_{\ell<i} E_\ell$, the same for F_i°;

(c) *the map $\phi\colon X - f_{j-1}(X_{j-1}) \to Y$ is an embedding.*

Then $\mathrm{bl}_j(\phi)$ is an embedding.

Proof. The transversality condition (a) guarantees that F_i and E_i are smooth divisors in smooth projective varieties. We combine (2.4) with Lemma 2.14 to obtain

$$\mathrm{bl}_j(\phi)^* N^*_{F_i|\,\mathrm{bl}_j(Y)} = N^*_{E_i|\,\mathrm{bl}_j(X)}\,.$$

Restricting to E_i° and F_i° gives

$$\mathrm{bl}_j(\phi)^* N^*_{F_i^\circ|\,\mathrm{bl}_j(Y)} = N^*_{E_i^\circ|\,\mathrm{bl}_j(X)}\,. \tag{2.5}$$

Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{bl}_j(\phi)^* N^*_{F_i^\circ|\,\mathrm{bl}_j(Y)} & \longrightarrow & \mathrm{bl}_j(\phi)^* T^*_{\mathrm{bl}_j(Y)}\big|_{E_i^\circ} & \longrightarrow & \mathrm{bl}_j(\phi)^* T^*_{F_i^\circ} & \longrightarrow & 0 \\
 & & \downarrow{\cong} & & \downarrow{d\,\mathrm{bl}_j(\phi)} & & \downarrow{d(\mathrm{bl}_j(\phi)|_{E_i^\circ})} & & \\
0 & \longrightarrow & N^*_{E_i^\circ|\,\mathrm{bl}_j(X)} & \longrightarrow & T^*_{\mathrm{bl}_j(X)}\big|_{E_i^\circ} & \longrightarrow & T^*_{E_i^\circ} & \longrightarrow & 0\,.
\end{array}$$

Using (2.5), condition (b) and the snake lemma, we see that

$$d\,\mathrm{bl}_j(\phi)\colon\ \mathrm{bl}_j(\phi)^* T^*_{\mathrm{bl}_j(Y)} \longrightarrow T^*_{\mathrm{bl}_j(X)}$$

is surjective over each E_i°. By Remark 2.7, the set $X - f_n(X_n)$ naturally embeds into $\mathrm{bl}_j(X)$ with complement $\bigcup_i E_i$, and condition (c) says that $d\,\mathrm{bl}_j(\phi)$ is surjective away from $\bigcup_i E_i$. Since $\bigcup_i E_i = \bigcup_i E_i^\circ$, we conclude that $\mathrm{bl}_j(\phi)$ is an embedding. □

Remark 2.17. In condition (b), we can ask $\mathrm{bl}_j(\phi)\colon E_i \to F_i$ to be an embedding, but in practice it is much easier to check this condition inductively on open subsets.

2.4 Criterion for a proper chain to be NCD

Putting everything together, we have the following inductive criterion for an NCD chain.

Proposition 2.18. *Let k be an integer. Let $\{\phi_j\colon X_j \to X\}_{j=0}$ be a proper chain with diagrams*

$$\begin{array}{ccc} X_{j,k} & \twoheadrightarrow & X_j \\ \downarrow{\scriptstyle f_{j,k}} & & \downarrow{\scriptstyle \phi_j} \\ X_k & \xrightarrow{\phi_k} & X \end{array}$$

for each $j < k$. Assume that the following conditions are satisfied:

(I) *The chains $\{\phi_j\}_{j=0}^{k-1}$ and $\{f_{j,k}\}_{j=0}^{k-1}$ are NCD.*

(II) *The map $\phi_k\colon X_k - f_{k-1,k}(X_{k-1,k}) \to X$ is an embedding, and $\phi_k^{-1}(X_0) = X_{0,k}$.*

(III) *Let $j < k$. Suppose that if the blowup spaces $\mathrm{bl}_i(X_k)$ associated with $\{f_{i,k}\}_{i=0}^{j-1}$ and $\{\phi_i\}_{i=0}^{j-1}$ coincide for all i, so the following diagram exists for $i < j$:*

$$\begin{array}{ccccccc} & & & & \mathrm{bl}_j(X_j) & & \\ & & & & \downarrow{\scriptstyle \mathrm{bl}_j(\phi_j)} & & \\ E_i & \hookrightarrow & \mathrm{bl}_j(X_k) & \xrightarrow{\mathrm{bl}_j(\phi_k)} & \mathrm{bl}_j(X) & \hookleftarrow & F_i \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{bl}_i(X_{i,k}) & \hookrightarrow & \mathrm{bl}_i(X_k) & \xrightarrow{\mathrm{bl}_i(\phi_k)} & \mathrm{bl}_i(X) & \hookleftarrow & \mathrm{bl}_i(X_i)\,, \end{array}$$

where E_i, F_i are exceptional divisors, then

$$\mathrm{bl}_j(\phi_k)^{-1}\Big(\mathrm{bl}_j(X_j) - \bigcup_{i<j} F_i\Big) = \mathrm{bl}_j(X_{j,k}) - \bigcup_{i<j} E_i\,, \tag{$*$}$$

$$\mathrm{bl}_j(\phi_k)^{-1}(\mathrm{bl}_j(X_j) \cap F_i^\circ) = \mathrm{bl}_j(X_{j,k}) \cap E_i^\circ\,, \quad \forall i < j\,. \tag{$**$}$$

Moreover, suppose that if the spaces $\mathrm{bl}_k(\phi_k)$ for the chains $\{f_{i,k}\}_{i=0}^{k-1}$ and $\{\phi_i\}_{i=0}^{k-1}$ coincide, then

$$\mathrm{bl}_k(\phi_k)\colon E_i^\circ \to F_i^\circ \text{ is an embedding for all } i < k\,, \tag{$***$}$$

where $E_i^\circ = E_i - \bigcup_{h<i} E_h$.

Then ϕ_k is a map of chains of centers, and the chain $\{\phi_j\}_{j=0}^{k}$ is an NCD chain.

Remark 2.19. Condition (I) is inductive. Condition (II) consists of properties of proper chains, and condition (III) consists of certain compatibility conditions of blowups associated with the chains $\{f_{i,k}\}_{i=0}^{k-1}$ and $\{\phi_i\}_{i=0}^{k-1}$.

Proof. It suffices to prove the following statements:

(a) We have $\mathrm{bl}_j(\phi_k)^{-1}(\mathrm{bl}_j(X_j)) = \mathrm{bl}_j(X_{j,k})$ for all $j < k$.

(b) The blowup $\mathrm{bl}_k(X_k)$ is smooth.

(c) The map $\mathrm{bl}_k(\phi_k)\colon \mathrm{bl}_k(X_k) \to \mathrm{bl}_k(X)$ is an embedding.

We prove statement (a) by induction on j. The base case follows from condition (II). Suppose that it holds for all $i < j$, that is,

$$\mathrm{bl}_i(\phi_k)^{-1}(\mathrm{bl}_i(X_i)) = \mathrm{bl}_i(X_{i,k})\,, \quad \forall i < j\,.$$

It follows that $\mathrm{bl}_j(\phi_k)^{-1}(F_i) = E_i$ for all $i < j$. Furthermore, this says that the spaces $\mathrm{bl}_i(X_k)$ associated with $\{f_{i,k}\}_{i=0}^{j-1}$ and $\{\phi_i\}_{i=0}^{j-1}$ coincide for all $i \leqslant j-1$, so ($*$) and ($**$) of condition (III) hold. On the other hand, condition (I) implies that $\mathrm{bl}_j(X_j)$, $\{E_i\}_{i=0}^{j-1}$ and $\mathrm{bl}_j(X_{j,k})$, $\{F_i\}_{i=0}^{j-1}$ are smooth subvarieties of $\mathrm{bl}_j(X)$ and $\mathrm{bl}_j(X_k)$, respectively. Using condition (I), and ($*$) and ($**$) from condition (III), we can apply Lemma 2.13 to the map $\mathrm{bl}_j(\phi_k)$ to conclude that

$$\mathrm{bl}_j(\phi_k)^{-1}(\mathrm{bl}_j(X_j)) = \mathrm{bl}_j(X_{j,k})\,.$$

This finishes the inductive proof.

Statement (a) implies that the spaces $\mathrm{bl}_k(\phi_k)$ for the chains $\{f_{i,k}\}_{i=0}^{k-1}$ and $\{\phi_i\}_{i=0}^{k-1}$ coincide, so ($***$) holds. Since the chain $\{f_{j,k}\}_{j=0}^{k-1}$ is NCD by condition (I), the blowup $\mathrm{bl}_k(X_k)$ must be smooth. Using conditions (I) and (II), and ($***$) from condition (III), we see that Lemma 2.16 implies that $\mathrm{bl}_k(\phi_k)$ is an embedding. This concludes the proof. □

Since the exceptional divisors in smooth blowups are projective bundles, to verify the assumptions for ($**$) and ($***$) in Proposition 2.18 in practice, we need relative versions of some lemmas in previous subsections.

Lemma 2.20. *Let $\phi\colon X \to Y$ be a B-morphism of smooth algebraic varieties over a smooth variety B, such that f and g in the following diagram are smooth morphisms:*

$$\begin{array}{ccc} X & \xrightarrow{\ \phi\ } & Y \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & B\,. & \end{array}$$

Denote the induced map over a closed point $t \in B$ by $\phi_t\colon X_t \to Y_t$. Then the following hold:

(a) *If ϕ_t is an embedding for each t, then ϕ is an embedding.*
(b) *Let $X_1 \subseteq X$ and $Y_1 \subseteq Y$ be smooth subvarieties. If $Y_t \cap Y_1$ and $X_t \cap X_1$ are smooth and $\phi_t^{-1}(Y_t \cap Y_1) = X_t \cap X_1$ for each t, then $\phi^{-1}(Y_1) = X_1$.*
(c) *Suppose that ϕ is an embedding; then $\mathrm{bl}_X(Y)$ is a B-variety, and the fiber over $t \in B$ is $\mathrm{bl}_{X_t}(Y_t)$.*

Proof. For statement (a), it suffices to check that the differential $d\phi\colon \phi^*T_Y^* \to T_X^*$ is surjective, which can be checked by restriction to each X_t; the proof is similar to that of Lemma 2.15 by using the isomorphism

$$\phi^* N^*_{Y_t|Y} \cong N^*_{X_t|X}\,.$$

For statement (b), it suffices to check the surjectivity of the induced map

$$d\phi\colon \phi^* N^*_{Y_1|Y} \longrightarrow N^*_{X_1|X}\,.$$

The proof is similar to that of Lemma 2.13 and uses the isomorphism

$$N^*_{Y_1|Y}\big|_{Y_1\cap Y_t} \cong N^*_{Y_1\cap Y_t|Y_t}$$

because $Y_1 \cap Y_t$ is smooth, and likewise for $N^*_{X_1|X}$. Statement (c) follows from local computations. □

3. Reduction of the proof of Theorem A

In this section, we reduce the proof of Theorem A to two propositions about certain natural proper chains being NCD chains, using the framework developed in Section 2. Let C be a hyperelliptic curve of odd genus $g = 2n + 1$. The even-genus case is similar and will be treated separately in Section 8. Let g^1_2 be the line bundle corresponding to the hyperelliptic map $h\colon C \to \mathbf{P}^1$, and denote by $C_j := \mathrm{Sym}^j(C)$ the jth symmetric product of C; we view the closed points of C_j as effective divisors of degree j on the curve C. Let C_0 be the scheme parametrizing the trivial divisor on C. Consider the following sequence of morphisms for all $0 \leqslant j \leqslant n$:

$$\begin{aligned} \delta_j\colon C_{2j} &\longrightarrow \mathrm{Jac}(C) = \mathrm{Pic}^{g-1}(C)\,, \\ D &\longmapsto (n-j)g^1_2 \otimes \mathcal{O}_C(D)\,. \end{aligned} \tag{3.1}$$

By the Abel–Jacobi theorem, we have $\delta_j(C_{2j}) = W^{n-j}_{g-1}$ and a natural embedding

$$\mathbf{P}^j = \delta_j^{-1}\big(ng^1_2\big) \hookrightarrow C_{2j}\,.$$

For $i < j$, we have a commutative diagram

$$\begin{array}{ccc} C_{2i} \times \mathbf{P}^{j-i} & \xrightarrow{\ p_1\ } & C_{2i} \\ \big\downarrow{\scriptstyle \gamma_{i,j}} & & \big\downarrow{\scriptstyle \delta_i} \\ C_{2j} & \xrightarrow{\ \delta_j\ } & \mathrm{Jac}(C)\,, \end{array} \tag{3.2}$$

where p_1 denotes the projection to the first factor and

$$\gamma_{i,j}\colon C_{2i} \times \mathbf{P}^{j-i} \hookrightarrow C_{2i} \times C_{2j-2i} \longrightarrow C_{2j}\,, \quad \forall 0 \leqslant i \leqslant j-1\,, \tag{3.3}$$

is the composition of the natural embedding $\mathbf{P}^{j-i} \hookrightarrow C_{2j-2i}$ and the addition map on symmetric products. Since

$$\delta_i(C_{2i}) = W^{n-i}_{g-1} \subsetneq W^{n-j}_{g-1} = \delta_j(C_{2j})\,,$$

the chain $\{\delta_j\}_{j=0}^n$ is proper.

To understand the chain $\{\delta_j\}_{j=0}^n$, we need several auxiliary chains. The first such chain is

$$\{\gamma_{i,k}\colon C_{2i} \times \mathbf{P}^{k-i} \to C_{2k}\}_{i=0}^{k-1}\,. \tag{3.4}$$

For $i < j < k$, we have the following commutative diagram:

$$\begin{array}{ccc} C_{2i} \times \mathbf{P}^{j-i} \times \mathbf{P}^{k-j} & \xrightarrow{\ \mathrm{id}\times r\ } & C_{2i} \times \mathbf{P}^{k-i} \\ \big\downarrow{\scriptstyle \gamma_{i,j}\times\mathrm{id}} & & \big\downarrow{\scriptstyle \gamma_{i,k}} \\ C_{2j} \times \mathbf{P}^{k-j} & \xrightarrow{\ \gamma_{j,k}\ } & C_{2k}\,. \end{array} \tag{3.5}$$

Here r is the restriction of the addition map $C_{2(j-i)} \times C_{2(k-j)} \to C_{2(k-i)}$, which we can see coincides with the addition map for symmetric products of $\mathbf{P}^1$ by thinking of $\mathbf{P}^\ell$ as $\mathrm{Sym}^\ell\,\mathbf{P}^1$. It is easy to see that $\gamma_{i,k}(C_{2i} \times \mathbf{P}^{k-i})$ parametrizes effective divisors D of degree $2k$ such that $h^0(\mathcal{O}_C(D)) \geqslant k - i + 1$. It follows that

$$\gamma_{i,k}\big(C_{2i} \times \mathbf{P}^{k-i}\big) \subsetneq \gamma_{j,k}\big(C_{2j} \times \mathbf{P}^{k-j}\big)\,, \quad \forall i < j\,,$$

and so $\{\gamma_{i,k}\}_{i=0}^{k-1}$ is a proper chain.

The second auxiliary chain lives over $C_{2j} \times \mathbf{P}^{k-j}$ for a fixed tuple (j,k) with $j < k$. Consider the chain

$$\big\{\gamma_{i,j} \times \mathrm{id}\colon \big(C_{2i} \times \mathbf{P}^{j-i}\big) \times \mathbf{P}^{k-j} \to C_{2j} \times \mathbf{P}^{k-j}\big\}_{i=0}^{j-1}$$

that is induced by taking the product of the chain $\{\gamma_{i,j}\}_{i=0}^{j-1}$ with $\mathbf{P}^{k-j}$, as in Notation 2.8. Fortunately, because of the product structure of this chain, the inductive process stops here, and no further auxiliary chains are needed.

Denote by $\mathrm{bl}_i(C_{2j} \times \mathbf{P}^{j-k})$ and $\mathrm{bl}_i(C_{2k})$ the spaces associated with the chain $\{\gamma_{j,k}\}_{j=0}^{k-1}$. Denote by $\mathrm{bl}_i(C_{2i} \times \mathbf{P}^{j-i} \times \mathbf{P}^{k-j})$ the spaces associated with the chain $\{\gamma_{i,j} \times \mathrm{id}\}_{i=0}^{j-1}$. In Section 6, the following will be proved.

Proposition 3.1. *Let k be an integer such that $1 \leqslant k \leqslant n$. Then the proper chain $\{\gamma_{j,k}\}_{j=0}^{k-1}$ is an NCD chain, and for each $j < k$, the map*

$$\gamma_{j,k} \colon C_{2j} \times \mathbf{P}^{k-j} \longrightarrow C_{2k}$$

is a map of chains of centers from $\{\gamma_{i,j} \times \mathrm{id}\}_{i=0}^{j-1}$ to $\{\gamma_{i,k}\}_{i=0}^{j-1}$. Concretely, this means the following:

(a) *For $0 \leqslant i < k$, there is a closed embedding*

$$\mathrm{bl}_i(\gamma_{i,k}) \colon \mathrm{bl}_i\big(C_{2i} \times \mathbf{P}^{k-i}\big) \lhook\joinrel\longrightarrow \mathrm{bl}_i(C_{2k}),$$

whose image intersects the union of all the exceptional divisors in $\mathrm{bl}_i(C_{2k})$ transversely.

(b) *There is a natural embedding $C_{2k} - \gamma_{k-1,k}(C_{2k-2} \times \mathbf{P}^1) \hookrightarrow \mathrm{bl}_k(C_{2k})$, whose complement has k smooth components with normal crossings.*

(c) *The blowup $\mathrm{bl}_i(C_{2j} \times \mathbf{P}^{j-k})$ coincides with the space associated with $\{\gamma_{i,j} \times \mathrm{id}\}_{i=0}^{j-1}$. and one has a natural identification*

$$\mathrm{bl}_i\big(C_{2i} \times \mathbf{P}^{j-i} \times \mathbf{P}^{k-j}\big) = \mathrm{bl}_i(C_{2i}) \times \mathbf{P}^{j-i} \times \mathbf{P}^{k-j}\,,$$

so for each $i < j < k$, one has a Cartesian diagram

$$\begin{array}{ccc}
\mathrm{bl}_i(C_{2i}) \times \mathbf{P}^{j-i} \times \mathbf{P}^{k-j} & \xrightarrow{\mathrm{id} \times r} & \mathrm{bl}_i(C_{2i}) \times \mathbf{P}^{k-i} \\
\Big\downarrow{\scriptstyle \mathrm{bl}_i(\gamma_{i,j}) \times \mathrm{id}} & & \Big\downarrow{\scriptstyle \mathrm{bl}_i(\gamma_{i,k})} \\
\mathrm{bl}_i(C_{2j}) \times \mathbf{P}^{k-j} & \xrightarrow{\mathrm{bl}_i(\gamma_{j,k})} & \mathrm{bl}_i(C_{2k})\,,
\end{array}$$

where $r \colon \mathbf{P}^{j-i} \times \mathbf{P}^{k-j} \to \mathbf{P}^{k-i}$ is the addition map.

With the help of Proposition 3.1, in Section 7 we will prove the following proposition.

Proposition 3.2. *The proper chain $\{\delta_j \colon C_{2j} \to \mathrm{Jac}(C)\}_{j=0}^{n}$ is an NCD chain, and the map $\delta_k \colon C_{2k} \to \mathrm{Jac}(C)$ is a map of chains of centers between the chains $\{\gamma_{j,k}\}_{j=0}^{k-1}$ and $\{\delta_j\}_{j=0}^{k-1}$. Concretely, this means the following:*

(a) *For $0 \leqslant j \leqslant n$, the maps $\mathrm{bl}_j(\delta_k)$ associated with the chain $\{\delta_j\}_{j=0}^{n}$ exist, and*

$$\mathrm{bl}_j(\delta_j) \colon \mathrm{bl}_j(C_{2j}) \longrightarrow \mathrm{bl}_j(\mathrm{Jac}(C))$$

is an embedding such that its image intersects the union of all the exceptional divisors in $\mathrm{bl}_j(\mathrm{Jac}(C))$ transversely, where $\mathrm{bl}_j(C_{2k})$ coincides with the blowup space associated with the chain $\{\delta\}_{j=0}^{k-1}$.

(b) *There is a natural embedding $\mathrm{Jac}(C) - \delta_n(C_{2n}) \hookrightarrow \mathrm{bl}_{n+1}(\mathrm{Jac}(C))$, whose complement has $n+1$ smooth components with normal crossings.*

Finally, there is a natural identification

$$\mathrm{bl}_j\big(C_{2j} \times \mathbf{P}^{k-j}\big) = \mathrm{bl}_j(C_{2j}) \times \mathbf{P}^{k-j}\,,$$

which for $j < k$ induces a Cartesian diagram

$$\begin{array}{ccc} \mathrm{bl}_j(C_{2j}) \times \mathbf{P}^{k-j} & \xrightarrow{p_1} & \mathrm{bl}_j(C_{2j}) \\ \Big\downarrow{\scriptstyle \mathrm{bl}_j(\gamma_{j,k})} & & \Big\downarrow{\scriptstyle \mathrm{bl}_j(\delta_j)} \\ \mathrm{bl}_j(C_{2k}) & \xrightarrow{\mathrm{bl}_j(\delta_k)} & \mathrm{bl}_j(\mathrm{Jac}(C))\,. \end{array}$$

Assuming Proposition 3.2 for now, we can easily deduce the main theorem for hyperelliptic curves of odd genus $g = 2n+1$.

Proof of Theorem A. For the sake of clarity, let us denote by $\{\mathrm{bl}'_i(\mathrm{Jac}(C))\}_{i=1}^n$ the sequence of blowups described in the introduction, where at the ith stage, to get $\mathrm{bl}'_i(\mathrm{Jac}(C))$, we blow up the strict transform of the Brill–Noether variety $W_{g-1}^{n+1-i}(C)$. We are going to argue that, in fact,

$$\mathrm{bl}'_i(\mathrm{Jac}(C)) = \mathrm{bl}_i(\mathrm{Jac}(C))\,.$$

First, since $g = 2n+1$, the image of $\mathrm{bl}_n(C_{2n})$ in $\mathrm{bl}_n(\mathrm{Jac}(C))$ is a divisor, and so we have $\mathrm{bl}_{n+1}(\mathrm{Jac}(C)) = \mathrm{bl}_n(\mathrm{Jac}(C))$. Therefore, both sequences really have only n steps. Note that $\delta_i(C_{2i})$ equals $W_{g-1}^{n-i}(C)$, which is reduced by Proposition A.3. Proposition 3.2 gives an embedding

$$\mathrm{bl}_i(C_{2i}) \hookrightarrow \mathrm{bl}_i(\mathrm{Jac}(C))\,.$$

By induction on i, it follows easily that $\mathrm{bl}'_i(\mathrm{Jac}(C)) = \mathrm{bl}_i(\mathrm{Jac}(C))$ and that the proper transform of $W_{g-1}^{n-i}(C)$ in $\mathrm{bl}_i(\mathrm{Jac}(C))$ is equal to the image of $\mathrm{bl}_i(C_{2i}) \hookrightarrow \mathrm{bl}_i(\mathrm{Jac}(C))$, hence is smooth.

The conclusion is that $\mathrm{bl}'_n(\mathrm{Jac}(C)) = \mathrm{bl}_n(\mathrm{Jac}(C))$ is smooth and that the strict transform $\tilde{\Theta}$ is the image of $\mathrm{bl}_n(C_{2n}) \hookrightarrow \mathrm{bl}_n(\mathrm{Jac}(C))$, hence is also smooth. Since $\{\delta_i\}_{i=0}^n$ is an NCD chain, the pullback $\pi_n^*\Theta$ is a divisor with simple normal crossings. The multiplicity of the exceptional divisor Z_i equals the multiplicity of Θ at a point in $W_{g-1}^{n-i} - W_{g-1}^{n+1-i}$, which is $n+1-i$ by the Riemann singularity theorem. □

Remark 3.3. We show that $\{\delta_j\}_{j=0}^n$ is a proper chain using the diagram

$$\begin{array}{ccc} C_{2i} \times \mathbf{P}^{j-i} & \xrightarrow{p_1} & C_{2i} \\ \Big\downarrow{\scriptstyle \gamma_{i,j}} & & \Big\downarrow{\scriptstyle \delta_i} \\ C_{2j} & \xrightarrow{\delta_j} & \mathrm{Jac}(C)\,. \end{array}$$

However, it is easy to check that this diagram is not Cartesian, and we will prove later that it is only Cartesian over suitable open subsets (see Lemma 5.8(a)). On the other hand, Proposition 3.2 shows that it becomes Cartesian after sufficiently many blowups.

4. Secant bundles and maps between them

The proofs of Propositions 3.1 and 3.2 rely on the geometry of secant bundles over symmetric products of curves. In this section, we review the necessary definitions and results, following the notation in Bertram's work [Ber92].

Let C be a smooth projective curve of genus $g \geqslant 0$, let M be a line bundle on C, and let $j \geqslant 0$ be an integer. We denote by $C_j = \mathrm{Sym}^j C$ the jth symmetric product of the curve. Consider the following diagram:

$$\begin{array}{ccccc} & \mathscr{D}_{j+1} & \hookrightarrow & C \times C_{j+1} & \\ & & \swarrow^{p_1} & & \searrow^{p_2} \\ C & & & & C_{j+1}\,. \end{array}$$

Here $\mathscr{D}_{j+1} = C \times C_j$ is the universal divisor of degree $j+1$ over C_{j+1}, embedded via $(p, D) \mapsto (p, p+D)$. We say that M separates d points if

$$h^0(C, M) = h^0(C, M(-D)) + d\,, \quad \forall D \in C_d\,.$$

If M separates $j+1$ points, then the sequence of sheaves

$$0 \longrightarrow p_1^* M \otimes \mathcal{O}(-\mathscr{D}_{j+1}) \longrightarrow p_1^* M \longrightarrow p_1^* M \otimes \mathcal{O}_{\mathscr{D}_{j+1}} \longrightarrow 0$$

on $C \times C_{j+1}$ remains exact when pushed down to C_{j+1}.

Definition 4.1. The secant bundle of j-planes over C_{j+1}, with respect to M, is

$$B^j(M) := \mathbf{P}(p_2)_*(p_1^* M \otimes \mathcal{O}_{\mathscr{D}_{j+1}})\,.$$

This is a $\mathbf{P}^j$-bundle over the symmetric product C_{j+1}; for $j = 0$, we have $B^0(M) = C$. If M separates $j+1$ points, the natural map to $\mathbf{P}H^0(C, M)$ is

$$\beta_j \colon B^j(M) \longrightarrow \mathbf{P}(p_2)_*(p_1^* M) = \mathbf{P}H^0(C, M) \times C_{j+1} \longrightarrow \mathbf{P}H^0(C, M)\,, \tag{4.1}$$

where the last map is the projection to $\mathbf{P}H^0(C, M)$.

Assuming that M separates $m+1$ points, we get a proper chain

$$\bigl\{\beta_j \colon B^j(M) \to \mathbf{P}H^0(C, M)\bigr\}_{j=0}^{m}\,, \tag{4.2}$$

using the diagram

$$\begin{array}{ccc} B^i(M) \times C_{j-i} & \xrightarrow{p_1} & B^i(M) \\ \downarrow{\scriptstyle \alpha_{i,j}} & & \downarrow{\scriptstyle \beta_i} \\ B^j(M) & \xrightarrow{\beta_j} & \mathbf{P}H^0(C, M)\,. \end{array}$$

Here p_1 is again the projection to the first coordinate. For $i < j$, the map

$$\alpha_{i,j} \colon B^i(M) \times C_{j-i} \longrightarrow B^j(M) \tag{4.3}$$

is induced by the addition map $r \colon C_{i+1} \times C_{j-i} \to C_{j+1}$ (see [Ber92, § 1, p. 432, second definition]). This chain is proper because the image $\beta_j(B^j(M))$ is exactly the usual secant variety $\operatorname{Sec}^j(C)$ of j-planes through $j+1$ points of C inside $\mathbf{P}H^0(C, M)$; see [Ber92, § 1, p. 432].

In order to study this chain, Bertram introduced certain auxiliary chains, just as in the previous section. Fix $k < m$; one can show that the chain

$$\bigl\{\alpha_{i,k} \colon B^i(M) \times C_{k-i} \to B^k(M)\bigr\}_{i=0}^{k-1}$$

is a proper chain using the diagram

$$\begin{array}{ccc} B^i(M) \times C_{j-i} \times C_{k-j} & \xrightarrow{\mathrm{id} \times r} & B^i(M) \times C_{k-i} \\ \downarrow{\scriptstyle \alpha_{i,j} \times \mathrm{id}} & & \downarrow{\scriptstyle \alpha_{i,k}} \\ B^j(M) \times C_{k-j} & \xrightarrow{\alpha_{j,k}} & B^k(M)\,, \end{array}$$

where $r\colon C_{j-i} \times C_{k-j} \to C_{k-i}$ is the addition map on symmetric products. The chain $\{\alpha_{i,k}\}_{i=0}^{k-1}$ is proper because the image $\alpha_{i,k}(B^i(M) \times C_{k-i})$ can be thought of as the relative secant variety of i-planes in $B^k(M)$; see [Ber92, Lemma 1.3(d)].

Lastly, using Notation 2.8, for each pair (j,k) with $j<k$, we have a proper chain

$$\left\{\alpha_{i,j} \times \mathrm{id}\colon (B^i(M) \times C_{j-i}) \times C_{k-j} \longrightarrow B^j(M) \times C_{k-j}\right\}_{i=0}^{j-1}.$$

In [Ber92, Propositions 2.2 and 2.3], Bertram proved the following result.

Proposition 4.2 (Bertram). *Let M be a line bundle on C.*

(a) *If M separates $m+1$ points, then for $j<k<m$, both $\{\alpha_{i,k}\}_{i=0}^{k-1}$ and $\{\alpha_{i,j} \times \mathrm{id}\}_{i=0}^{j-1}$ are chains of smooth centers, and the map*

$$\alpha_{j,k}\colon B^j(M) \times C_{k-j} \longrightarrow B^k(M)$$

is an injective map of chains from $\{\alpha_{i,j} \times \mathrm{id}\}_{i=0}^{j-1}$ to $\{\alpha_{i,k}\}_{i=0}^{j-1}$.

(b) *If M separates $2k+2$ points, then $\{\beta_j\}_{j=0}^{k}$ is a chain of smooth centers, and*

$$\beta_k\colon B^k(M) \longrightarrow \mathbf{P}H^0(C,M)$$

is an injective map of chains from $\{\alpha_{j,k}\}_{j=0}^{k-1}$ to $\{\beta_j\}_{j=0}^{k-1}$.

Remark 4.3. Bertram's proof actually shows that $\{\alpha_{j,k}\}_{j=0}^{k-1}$ and $\{\beta_j\}_{j=0}^{k}$ are NCD chains. But this fact will not be used later.

Let us spell out in detail what Bertram's theorem says in the case of $\mathbf{P}^1$, where the images of the secant bundles for $\mathcal{O}_{\mathbf{P}^1}(d)$ are the secant varieties to the rational normal curve of degree d in $\mathbf{P}^d$.

Corollary 4.4. *Let $d \geqslant 2k+1$, and consider the line bundle $M = \mathcal{O}_{\mathbf{P}^1}(d)$ on $\mathbf{P}^1$. Then $\{\beta_j\}_{j=0}^{k}$ is a chain of smooth centers. Fix $0 \leqslant i < j < k$.*

(a) *The diagram*

$$\begin{array}{ccc} \mathrm{bl}_i\big(B^i(M) \times \mathbf{P}^{j-i}\big) \times \mathbf{P}^{k-j} & \twoheadrightarrow & \mathrm{bl}_i\big(B^i(M) \times \mathbf{P}^{k-i}\big) \\ \big\downarrow{\scriptstyle \mathrm{bl}_i(\alpha_{i,j})\times\mathrm{id}} & & \big\downarrow{\scriptstyle \mathrm{bl}_i(\alpha_{i,k})} \\ \mathrm{bl}_i\, B^j(M) \times \mathbf{P}^{k-j} & \xrightarrow{\mathrm{bl}_i(\alpha_{j,k})} & \mathrm{bl}_i\, B^k(M) \end{array}$$

is Cartesian, and the two vertical arrows are embeddings. In particular, $\alpha_{j,k}$ is a map of chains of centers.

(b) *The diagram*

$$\begin{array}{ccc} \mathrm{bl}_i\big(B^i(M) \times \mathbf{P}^{j-i}\big) & \twoheadrightarrow & \mathrm{bl}_i\, B^i(M) \\ \big\downarrow{\scriptstyle \mathrm{bl}_i(\alpha_{i,j})} & & \big\downarrow{\scriptstyle \mathrm{bl}_i(\beta_i)} \\ \mathrm{bl}_i\, B^j(M) & \xrightarrow{\mathrm{bl}_i(\beta_j)} & \mathrm{bl}_i\, \mathbf{P}^d \end{array}$$

is Cartesian, and the two vertical arrows are embeddings. In particular, β_j is a map of chains of centers.

(c) *There is a natural isomorphism*

$$\mathrm{bl}_i\big(B^i(M) \times \mathbf{P}^{j-i}\big) \cong \mathrm{bl}_i\, B^i(M) \times \mathbf{P}^{j-i}.$$

Proof. Since $M = \mathcal{O}_{\mathbf{P}^1}(d)$ separates $d+1$ points on $\mathbf{P}^1$, we can apply Proposition 4.2 and use the isomorphisms $\mathbf{P}^k \cong \operatorname{Sym}^k \mathbf{P}^1$ and $\mathbf{P}^d \cong \mathbf{P}H^0(\mathbf{P}^1, M)$. The last statement follows from Lemma 2.9. □

5. Properties of Abel–Jacobi maps and addition maps

In this section, let C be a hyperelliptic curve of genus $g = 2n+1$. As a preparation for the proofs of Propositions 3.1 and 3.2, we establish some basic properties of the map $\gamma_{i,j}\colon C_{2i} \times \mathbf{P}^{j-i} \to C_{2j}$ from (3.3) and of the Abel–Jacobi map $\delta_j\colon C_{2j} \to \operatorname{Jac}(C)$. In particular, their conormal bundles are calculated in terms of the secant bundles over symmetric products of $\mathbf{P}^1$. In fact, it is known that the conormal bundle of the Abel–Jacobi map of any curve can be described in terms of Steiner bundles (see [Ein91, Theorem 1.1]). For our purpose, it is more natural to use secant bundles.

We start by collecting some basic facts about hyperelliptic curves. From [ACGH85, §I.2, p. 13], we know that for each divisor D with $h^0(\mathcal{O}_C(D)) = r+1$ and degree $d \leqslant g(C)$, there is a unique decomposition

$$D = E + \sum_{\ell=1}^{r} (p_\ell + q_\ell)$$

such that $p_\ell + q_\ell$ are hyperelliptic pairs and E is a degree $d-2r$ divisor with $h^0(\mathcal{O}_C(E)) = 1$. Similarly, for any $L \in \operatorname{Jac}(C)$ with $h^0(L) = r+1$, there is a unique decomposition

$$L = rg_2^1 \otimes L' \tag{5.1}$$

such that $h^0(L') = 1$.

Notation 5.1. For each j, we define $U_{2j} := C_{2j} - \gamma_{j-1,j}(C_{2j-2} \times \mathbf{P}^1)$. In other words, U_{2j} consists of divisors D where none of the degree 2 subdivisors of D form a hyperelliptic pair and $h^0(\mathcal{O}_C(D)) = 1$.

Notation 5.2. By definition, any $D \in U_{2j}$ gives a degree $2j$ divisor on $\mathbf{P}^1$ by pushforward along the hyperelliptic map $h\colon C \to \mathbf{P}^1$. We denote this divisor on $\mathbf{P}^1$ by the symbol h_*D and define

$$\mathcal{O}_{\mathbf{P}^1}(g-1-h_*D) := \mathcal{O}_{\mathbf{P}^1}(g-1) \otimes \mathcal{O}_{\mathbf{P}^1}(-h_*D),$$

which is a line bundle of degree $g-1-2j$.

Since $h\colon C \to \mathbf{P}^1$ has $g+1$ branch points, one has $h_*\mathcal{O}_C \cong \mathcal{O}_{\mathbf{P}^1}(-1-g) \oplus \mathcal{O}_{\mathbf{P}^1}$. As we have $\omega_C \cong h^*\mathcal{O}_{\mathbf{P}^1}(g-1)$, the projection formula gives

$$h_*\omega_C \cong \omega_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(g-1).$$

Consider the short exact sequence $0 \to \mathcal{O}_C(-D) \to \mathcal{O}_C \to \mathcal{O}_D \to 0$ for $D \in U_{2j}$. Because h is finite, pushing this forward along h gives

$$0 \longrightarrow h_*\mathcal{O}_C(-D) \longrightarrow h_*\mathcal{O}_C \longrightarrow h_*\mathcal{O}_D \longrightarrow 0.$$

Since h is an isomorphism over D, we have $h_*\mathcal{O}_D = \mathcal{O}_{h_*D}$. Moreover, as h_*D is supported away from the branch points of h, the map $h_*\mathcal{O}_C \to h_*\mathcal{O}_D$ is the composition

$$h_*\mathcal{O}_C \longrightarrow \mathcal{O}_{\mathbf{P}^1} \longrightarrow \mathcal{O}_{h_*D} = h_*\mathcal{O}_D,$$

where the first map is the projection induced by $h_*\mathcal{O}_C = \mathcal{O}_{\mathbf{P}^1}(-1-g) \oplus \mathcal{O}_{\mathbf{P}^1}$. It follows that

$h_*\mathcal{O}_C(-D) = \mathcal{O}_{\mathbf{P}^1}(-1-g) \oplus \mathcal{O}_{\mathbf{P}^1}(-h_*D)$. Consequently, using the projection formula again gives

$$h_*\omega_C(-D) \cong \omega_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)\,. \tag{5.2}$$

In the following statement, see (1.2) for the notation of conormal bundle N^*_f or $N^*_{X|Y}$.

Lemma 5.3. *For $0 \leqslant i < j \leqslant n$ and $\gamma_{i,j}$ from (3.3), the following hold:*

(a) *The map $d\gamma_{i,j}\colon \gamma^*_{i,j}T^*_{C_{2j}} \to T^*_{C_{2i}\times \mathbf{P}^{j-i}}$ is surjective when restricted to $U_{2i} \times \mathbf{P}^{j-i}$.*

(b) *For $i = 0$, we have an isomorphism*

$$\mathbf{P}N^*_{\mathbf{P}^j|C_{2j}} \cong B^{j-1}(\mathcal{O}_{\mathbf{P}^1}(g-1))\,;$$

the right-hand side is the secant bundle over $\operatorname{Sym}^j \mathbf{P}^1 = \mathbf{P}^j$ with respect to $\mathcal{O}_{\mathbf{P}^1}(g-1)$.

(c) *For $i \geqslant 1$, the space $\mathbf{P}N^*_{\gamma_{i,j}}\big|_{U_{2i}\times\mathbf{P}^{j-i}}$ is smooth over U_{2i}, and over $D \in U_{2i}$, we have an isomorphism*

$$\mathbf{P}N^*_{\gamma_{i,j}}\big|_{\{D\}\times\mathbf{P}^{j-i}} \cong B^{j-i-1}(\mathcal{O}_{\mathbf{P}^1}(g-1-h_*D))\,, \tag{5.3}$$

where the right-hand side is the secant bundle over $\mathbf{P}^{j-i}$ with respect to $\mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)$.

Proof. As a warm-up, let us calculate the conormal bundle of $\mathbf{P}^j$ inside C_{2j}. Recall that for any divisor $D \in C_{2j}$, there is a canonical identification [ACGH85, § IV.2, p. 160]

$$T^*_{C_{2j}}\big|_D \cong H^0(C, \omega_C \otimes \mathcal{O}_D)\,.$$

Under the isomorphism $\mathbf{P}^j \cong \operatorname{Sym}^j \mathbf{P}^1$, the morphism $\mathbf{P}^j \to C_{2j}$ associates an effective divisor E of degree j on $\mathbf{P}^1$ with an effective divisor h^*E of degree $2j$ on C, and we have also

$$T^*_{\mathbf{P}^j}\big|_E \cong H^0\big(\mathbf{P}^1, \omega_{\mathbf{P}^1} \otimes \mathcal{O}_E\big)\,.$$

Moreover, $T^*_{C_{2j}}\big|_{h^*E}$ and $T^*_{\mathbf{P}^j}\big|_E$ can be related in the following way. One has

$$\begin{aligned} T^*_{C_{2j}}\big|_{h^*E} &\cong H^0(C, \omega_C \otimes h^*\mathcal{O}_E) \\ &\cong H^0(\mathbf{P}^1, h_*\omega_C \otimes \mathcal{O}_E) \\ &\cong H^0\big(\mathbf{P}^1, \omega_{\mathbf{P}^1} \otimes \mathcal{O}_E\big) \oplus H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1) \otimes \mathcal{O}_E\big) \\ &= T^*_{\mathbf{P}^j}\big|_E \oplus H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1)\otimes\mathcal{O}_E\big)\,. \end{aligned}$$

It follows that the map $T^*_{C_{2j}}\big|_{h^*E} \to T^*_{\mathbf{P}^j}\big|_E$ induced by $\mathbf{P}^j \hookrightarrow C_{2j}$ is surjective with kernel

$$N^*_{\mathbf{P}^j|C_{2j}}\big|_E \cong H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1)\otimes\mathcal{O}_E\big)\,.$$

Varying this isomorphism over $E \in \mathbf{P}^j$ gives

$$N^*_{\mathbf{P}^j|C_{2j}} \cong (p_2)_*\big(p_1^*\mathcal{O}_{\mathbf{P}^1}(g-1) \otimes \mathcal{O}_{\mathscr{E}_j}\big)\,, \tag{5.4}$$

where $\mathscr{E}_j$ denotes the universal divisor over $\mathbf{P}^j = \operatorname{Sym}^j \mathbf{P}^1$, as follows:

$$\begin{array}{ccccc} & \mathscr{E}_j \hookrightarrow & \mathbf{P}^1 \times \operatorname{Sym}^j \mathbf{P}^1 & & \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ \mathbf{P}^1 & & & & \operatorname{Sym}^j \mathbf{P}^1\,. \end{array}$$

In particular, the right-hand side of (5.4) is the secant bundle $B^{j-1}(\mathcal{O}_{\mathbf{P}^1}(g-1))$, and we have proved statement (b).

For statements (a) and (c), consider $D \in U_{2i}$ and $E \in \mathbf{P}^{j-i}$. The morphism $\gamma_{i,j} \colon U_{2i} \times \mathbf{P}^{j-i} \to C_{2j}$ takes the pair (D, E) to the divisor $D + h^*E$ of degree $2j$ on C. Because D is in U_{2i}, we have $H^0(C, \mathcal{O}_C(D + h^*E)) = H^0(C, \mathcal{O}_C(h^*E))$ by (5.1). After a little bit of diagram chasing, this gives us a diagram involving a short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(C, \omega_C(-D) \otimes \mathcal{O}_{h^*E}) & \longrightarrow & H^0(C, \omega_C \otimes \mathcal{O}_{D+h^*E}) & \longrightarrow & H^0(C, \omega_C \otimes \mathcal{O}_D) & \longrightarrow & 0\,, \\ & & & & \| & & \| & & \\ & & & & T^*_{C_{2j}}\big|_{D+h^*E} & \longrightarrow & T^*_{U_{2i}}\big|_D & & \end{array}$$

where the bottom map is induced by the cotangent map of

$$\begin{aligned} U_{2i} &\hookrightarrow U_{2i} \times \mathbf{P}^{j-i} \longrightarrow C_{2j}\,, \\ D &\longmapsto (D, E) \longmapsto \gamma_{i,j}(D, E)\,. \end{aligned}$$

Using (5.2), one has

$$\begin{aligned} \operatorname{Ker}\bigl(T^*_{C_{2j}}\big|_{D+h^*E} \to T^*_{U_{2i}}\big|_D\bigr) &\cong H^0(C, \omega_C(-D) \otimes h^*\mathcal{O}_E) \\ &\cong H^0\bigl(\mathbf{P}^1, h_*\omega_C(-D) \otimes \mathcal{O}_E\bigr) \\ &\cong H^0\bigl(\mathbf{P}^1, \omega_{\mathbf{P}^1} \otimes \mathcal{O}_E\bigr) \oplus H^0\bigl(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D) \otimes \mathcal{O}_E\bigr) \\ &= T^*_{\mathbf{P}^{j-i}}\big|_E \oplus H^0\bigl(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D) \otimes \mathcal{O}_E\bigr)\,, \end{aligned}$$

and the morphism to the cotangent space of $\mathbf{P}^{j-i}$ is the projection to the first summand. Hence, we deduce that

$$d\gamma_{i,j} \colon T^*_{C_{2j}}\big|_{D+h^*E} \longrightarrow T^*_{U_{2i}}\big|_D \oplus T^*_{\mathbf{P}^{j-i}}\big|_E$$

is surjective, proving statement (a) and that the kernel of $d\gamma_i, j$ can be identified with

$$N^*_{\gamma_{i,j}}\big|_{(D,E)} \cong H^0\bigl(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D) \otimes \mathcal{O}_E\bigr)\,. \tag{5.5}$$

This isomorphism is natural in $E \in \mathbf{P}^{j-i}$, and therefore

$$N^*_{\gamma_{i,j}}\big|_{\{D\} \times \mathbf{P}^{j-i}} \cong (p_2)_*\bigl(p_1^*\mathcal{O}_{\mathbf{P}^1}(g-1-h_*D) \otimes \mathcal{O}_{\mathscr{E}_{j-i}}\bigr)\,.$$

The right-hand side is a vector bundle on $\mathbf{P}^{j-i}$ because the line bundle $\mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)$ separates $j-i$ points (on account of the inequality $g-1-2i = 2n-2i > j-i$). Varying $D \in U_{2i}$, we see that the projectivized conormal bundle $N^*_{\gamma_{i,j}}|_{U_{2i} \times \mathbf{P}^{j-i}}$ is a projective bundle over U_{2i}, hence is smooth over U_{2i}. Moreover, its fiber over $D \in U_{2i}$ is the secant bundle $B^{j-i-1}(\mathcal{O}_{\mathbf{P}^1}(g-1-h_*D))$, which proves statement (c). ☐

Remark 5.4. Lemma 5.3 is parallel to [Ber92, Lemma 1.3], with the difference that the relevant divisor is h_*D, not $2h_*D$ as in [Ber92].

From the proof of Lemma 5.3, we can deduce one additional useful fact. For $0 \leqslant i < j < k \leqslant n$, consider the commutative diagram induced by (3.5) via restriction

$$\begin{array}{ccc} (U_{2i} \times \mathbf{P}^{j-i}) \times \mathbf{P}^{k-j} & \xrightarrow{\ \mathrm{id} \times r\ } & U_{2i} \times \mathbf{P}^{k-i} \\ \big\downarrow{\scriptstyle \gamma_{i,j} \times \mathrm{id}} & & \big\downarrow{\scriptstyle \gamma_{i,k}} \\ C_{2j} \times \mathbf{P}^{k-j} & \xrightarrow{\ \gamma_{j,k}\ } & C_{2k}\,. \end{array}$$

Corollary 5.5. *For $D \in U_{2i}$, the induced map of conormal bundles*

$$\epsilon\colon (\mathrm{id} \times r)^* N^*_{\gamma_{i,k}}\big|_{\{D\}\times \mathbf{P}^{k-i}} \longrightarrow N^*_{\gamma_{i,j}\times \mathrm{id}}\big|_{\{D\}\times \mathbf{P}^{j-i}\times \mathbf{P}^{k-j}}$$

on $\{D\} \times \mathbf{P}^{j-i} \times \mathbf{P}^{k-j}$ is surjective, and the diagram

$$\begin{array}{ccc} \mathbf{P}N^*_{\gamma_{i,j}\times \mathrm{id}}\big|_{\{D\}\times \mathbf{P}^{j-i}\times \mathbf{P}^{k-j}} & \xrightarrow{\alpha} & \mathbf{P}N^*_{\gamma_{i,k}}\big|_{\{D\}\times \mathbf{P}^{k-i}} \\ \big\downarrow {\scriptstyle (5.3)\times \mathrm{id}_{\mathbf{P}^{k-j}}} & & \big\downarrow {\scriptstyle (5.3)} \\ B^{j-i-1}(M) \times \mathbf{P}^{k-j} & \xrightarrow{\alpha_{j-i-1,k-i-1}} & B^{k-i-1}(M) \end{array}$$

commutes. Here α is induced by ϵ and the projection to $\mathbf{P}N^*_{\gamma_{i,k}}\big|_{\{D\}\times \mathbf{P}^{k-i}}$, both vertical arrows are isomorphisms, and $\alpha_{j-i-1,k-i-1}$ is the map in (4.3) for the curve $\mathbf{P}^1$ and the line bundle $M = \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)$.

Proof. To simplify the notation, fix a point $D \in U_{2i}$, and set $M = \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)$. For $E_1 \in \mathbf{P}^{j-i}$ and $E_2 \in \mathbf{P}^{k-j}$, according to (5.5), the map

$$\epsilon\big|_{(D,E_1,E_2)}\colon N^*_{\gamma_{i,k}}\big|_{(D,E_1+E_2)} \longrightarrow N^*_{\gamma_{i,j}}\big|_{(D,E_1)}$$

is identified with the map

$$H^0\big(\mathbf{P}^1, M \otimes \mathcal{O}_{E_1+E_2}\big) \longrightarrow H^0\big(\mathbf{P}^1, M \otimes \mathcal{O}_{E_1}\big)\,,$$

which is obviously surjective. The remaining assertion is clear from (4.3). □

Lemma 5.6. *For $0 \leqslant j \leqslant n$, consider the map δ_j from (3.1); then the following hold:*

(a) *The map $d\delta_j\colon \delta_j^* T^*_{\mathrm{Jac}(C)} \to T^*_{C_{2j}}$ is surjective when restricted to U_{2j}.*

(b) *The fiber of $N^*_{\delta_j}$ over $D \in U_{2j}$ is $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D))$.*

Proof. We only sketch the proof, as it is similar to that of Lemma 5.3. The cotangent bundle $T^*_{\mathrm{Jac}(C)}$ is a trivial bundle with fibers isomorphic to $H^0(C, \omega_C)$, and the map $d\delta_j$ over $D \in C_{2j}$ can be identified with

$$H^0(C, \omega_C) \longrightarrow H^0(C, \omega_C \otimes \mathcal{O}_D)\,. \tag{5.6}$$

The cokernel of this map is the kernel of $H^1(\omega_C(-D)) \to H^1(\omega_C)$; dually, one has the natural map $H^0(\mathcal{O}_C) \to H^0(\mathcal{O}_C(D))$. If $D \in U_{2j}$, then $h^0(\mathcal{O}_C(D)) = 1$ and the last map is an isomorphism. Therefore, the map (5.6) is surjective, and we deduce

$$N^*_{\delta_j}\big|_D \cong H^0(C, \omega_C(-D)) \cong H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)\big) \tag{5.7}$$

using (5.2). □

Let us record one additional useful fact. Consider the commutative diagram induced by (3.2) via restriction ($i < j$):

$$\begin{array}{ccc} U_{2i} \times \mathbf{P}^{j-i} & \xrightarrow{p_1} & U_{2i} \\ \big\downarrow {\scriptstyle \gamma_{i,j}} & & \big\downarrow {\scriptstyle \delta_i} \\ C_{2j} & \xrightarrow{\delta_j} & \mathrm{Jac}(C)\,. \end{array}$$

Corollary 5.7. *Fix $0 \leqslant i < j \leqslant n$. For $D \in U_{2i}$, the induced map of conormal bundles*

$$\epsilon\colon p_1^* N^*_{\delta_i}\big|_D \longrightarrow N^*_{\gamma_{i,j}}\big|_{\{D\}\times \mathbf{P}^{j-i}}$$

over $\{D\} \times \mathbf{P}^{j-i}$ *is surjective, and the diagram*

$$\begin{array}{ccc} \mathbf{P}N^*_{\gamma_{i,j}}\big|_{\{D\}\times\mathbf{P}^{j-i}} & \xrightarrow{\ \beta\ } & \mathbf{P}N^*_{\delta_i}\big|_D \\ \big\downarrow{\scriptstyle (5.3)} & & \big\downarrow{\scriptstyle \cong} \\ B^{j-i-1}(M) & \xrightarrow{\ \beta_{j-i-1}\ } & \mathbf{P}H^0\big(\mathbf{P}^1, M\big) \end{array}$$

commutes. Here the first vertical map is an isomorphism by Lemma 5.3(c), the second isomorphism comes from Lemma 5.6, the map β_{j-i-1} *is (4.1) for the curve* $\mathbf{P}^1$ *and the line bundle* $M = \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)$*, and* β *is induced by* ϵ *and the projection to* $\mathbf{P}N^*_{\delta_i}\big|_D$*.*

Proof. Fix a point $D \in U_{2i}$, and set $M = \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)$. For $E \in \mathbf{P}^{j-i}$, by (5.5) and (5.7), the map

$$\epsilon\big|_{(D,E)} \colon N^*_{\delta_i}\big|_D \longrightarrow N^*_{\gamma_{i,j}}\big|_{(D,E)}$$

between the fibers of the two conormal bundles is identified with the map

$$H^0(\mathbf{P}^1, M) \longrightarrow H^0\big(\mathbf{P}^1, M \otimes \mathcal{O}_E\big)\,,$$

which is surjective by degree reasons ($\deg M = g-1-2i > j-i = \deg E$). The remaining assertion is clear from (4.1). □

We end this section by some properties of δ_j and $\gamma_{i,j}$, restricting to suitable open subsets.

Lemma 5.8. *Fix* $0 \leqslant i < j \leqslant n$.

(a) *The maps* $\delta_j \colon U_{2j} \to \operatorname{Jac}(C)$ *and* $\gamma_{i,j} \colon U_{2i} \times \mathbf{P}^{j-i} \longrightarrow C_{2j}$ *are embeddings, and the following restriction of the diagram (3.2) is Cartesian:*

$$\begin{array}{ccc} U_{2i} \times \mathbf{P}^{j-i} & \xrightarrow{\ p_1\ } & U_{2i} \\ \big\downarrow{\scriptstyle \gamma_{i,j}} & & \big\downarrow{\scriptstyle \delta_i} \\ C_{2j} & \xrightarrow{\ \delta_j\ } & \operatorname{Jac}(C)\,. \end{array}$$

Equivalently, we have

$$\delta_j^{-1}(U_{2i}) = U_{2i} \times \mathbf{P}^{j-i}.$$

(b) *For* $0 \leqslant i < j < k \leqslant n$*, the following restriction of the diagram (3.5) is Cartesian:*

$$\begin{array}{ccc} \big(U_{2i} \times \mathbf{P}^{j-i}\big) \times \mathbf{P}^{k-j} & \xrightarrow{\ \mathrm{id}\times r\ } & U_{2i} \times \mathbf{P}^{k-i} \\ \big\downarrow{\scriptstyle \gamma_{i,j}\times\mathrm{id}} & & \big\downarrow{\scriptstyle \gamma_{i,k}} \\ C_{2j} \times \mathbf{P}^{k-j} & \xrightarrow{\ \gamma_{j,k}\ } & C_{2k}\,. \end{array}$$

Equivalently, we have

$$\gamma_{j,k}^{-1}\big(U_{2i} \times \mathbf{P}^{k-i}\big) = U_{2i} \times \mathbf{P}^{j-i} \times \mathbf{P}^{k-j}.$$

(c) *In particular, for* $j < k \leqslant n$ *and* $i = 0$*, we have* $\gamma_{j,k}^{-1}\big(C_0 \times \mathbf{P}^k\big) = C_0 \times \mathbf{P}^j \times \mathbf{P}^{k-j}$.

Proof. As $\delta_i(D) = (n-i)g^1_2 \otimes \mathcal{O}_C(D)$ by (3.1), it follows immediately from the uniqueness of the decomposition (5.1) that $\delta_i \colon U_{2i} \to \operatorname{Jac}(C)$ is injective and that the image $\delta_i(U_{2i})$ consists of line bundles $L \in \operatorname{Jac}(C)$ such that $h^0(L) = n-i+1$. Similarly, one can show that the map

$\gamma_{i,j}\colon U_{2i} \times C_{j-i} \to C_{2j}$ is injective, and any divisor in its image can be written as

$$D = E + \sum_{\ell=1}^{j-i} (p_\ell + q_\ell)\,,$$

where E is a degree $2i$ divisor with $h^0(\mathcal{O}_C(E)) = 1$ and the $p_\ell + q_\ell$ are hyperelliptic pairs. Therefore, the image $\gamma_{i,j}(U_{2i} \times C_{j-i})$ consists of divisors D of degree $2j$ such that $h^0(\mathcal{O}_C(D)) = j - i + 1$. To show that the restricted maps δ_j and $\gamma_{i,j}$ are embeddings, one needs the surjectivity of $d\delta_j$ and $d\gamma_{i,j}$, which follows from Lemmas 5.3(a) and 5.6(a).

Now we want to argue that the diagrams in statements (a) and (b) are Cartesian. We will identify U_{2i} and $U_{2i} \times \mathbf{P}^{j-i}$ with their images in $\operatorname{Jac}(C)$ and C_{2j}. We want to show that $\delta_j^{-1}(U_{2i}) = U_{2i} \times \mathbf{P}^{j-i}$. Let us first prove this set-theoretically. Suppose $D \in \delta_j^{-1}(U_{2i})$; this means that

$$h^0(\delta_j(D)) = h^0\big((n-j)g_2^1 \otimes \mathcal{O}_C(D)\big) = n - i + 1$$

by the characterization of the image of U_{2i} in $\operatorname{Jac}(C)$. Using (5.1), we must have

$$h^0(\mathcal{O}_C(D)) = (n-i+1) - (n-j) = j - i + 1$$

and conclude that $D \in U_{2i} \times \mathbf{P}^{j-i}$, by the characterization of its image in C_{2j}. The argument for the set-theoretic part of statements (b) and (c) is similar. To finish the proof of statement (a), we need to show the surjectivity of

$$d\delta_j\colon \delta_j^* N^*_{U_{2i}|\operatorname{Jac}(C)} \longrightarrow N^*_{U_{2i} \times \mathbf{P}^{j-i}|C_{2j}}\,,$$

which follows from Corollary 5.7. Similarly, the fact that the diagram in statement (b) is Cartesian follows from Corollary 5.5. □

6. The proof of Proposition 3.1

Let C be a smooth hyperelliptic curve of genus $g = 2n+1$. Let $t \leqslant n$ be an integer, and consider the proper chain $\big\{\gamma_{k,t}\colon C_{2k} \times \mathbf{P}^{t-k} \to C_{2t}\big\}_{k=0}^{t-1}$ from (3.4) with the diagram (3.5) for $j < k < t$:

$$\begin{array}{ccc} C_{2j} \times \mathbf{P}^{k-j} \times \mathbf{P}^{t-k} & \xrightarrow{\ \mathrm{id}\times r\ } & C_{2j} \times \mathbf{P}^{t-j} \\ \big\downarrow{\scriptstyle \gamma_{j,k}\times \mathrm{id}} & & \big\downarrow{\scriptstyle \gamma_{j,t}} \\ C_{2k} \times \mathbf{P}^{t-k} & \xrightarrow{\ \gamma_{k,t}\ } & C_{2t}\,. \end{array}$$

Denote by $\mathrm{bl}_j\big(C_{2k} \times \mathbf{P}^{t-k}\big)$ and $\mathrm{bl}_j(C_{2t})$ the spaces associated with the chain $\{\gamma_{k,t}\}_{k=0}^{t-1}$, and denote by $\mathrm{bl}_i\big(C_{2j} \times \mathbf{P}^{k-j} \times \mathbf{P}^{t-k}\big)$ the space associated with the chain $\{\gamma_{j,k} \times \mathrm{id}\}_{j=0}^{k-1}$.

Let k be an integer with $0 \leqslant k < t$. We break the proof of Proposition 3.1 down into the following claims. (We apologize to the reader for the fact that this looks so complicated; the problem is that the threefold induction needs a lot of notation just to state everything correctly.)

Claim 6.1. *The chain $\{\gamma_{j,t}\}_{j=0}^{k}$ is NCD.*

Claim 6.2. *Let $j \leqslant k < t$; then*

$$\mathrm{bl}_j(\gamma_{k,t})^{-1}\big(\mathrm{bl}_j\big(C_{2j} \times \mathbf{P}^{t-j}\big)\big) = \mathrm{bl}_j\big(C_{2j} \times \mathbf{P}^{k-j} \times \mathbf{P}^{t-k}\big)\,. \tag{6.1}$$

In other words, $\gamma_{k,t}$ is a map of chains of centers.

As a consequence, the blowup $\mathrm{bl}_j\bigl(C_{2k} \times \mathbf{P}^{t-k}\bigr)$ coincides with the space associated with the chain $\{\gamma_{j,k} \times \mathrm{id}\}_{j=0}^{k-1}$, and by Lemma 2.9 there are natural isomorphisms

$$\begin{aligned} \mathrm{bl}_j\bigl(C_{2k} \times \mathbf{P}^{t-k}\bigr) &= \mathrm{bl}_j(C_{2k}) \times \mathbf{P}^{t-k}\,, \\ \mathrm{bl}_i\bigl(C_{2j} \times \mathbf{P}^{k-j} \times \mathbf{P}^{t-k}\bigr) &= \mathrm{bl}_i(C_{2j}) \times \mathbf{P}^{k-j} \times \mathbf{P}^{t-k}\,, \quad \forall i < j\,. \end{aligned}$$

Claim 6.3. *Let $0 \leqslant i < j \leqslant k < t$, and consider the following diagram:*

$$\begin{array}{ccccccc} & & & & \mathrm{bl}_j\bigl(C_{2j} \times \mathbf{P}^{t-j}\bigr) & & \\ & & & & \big\downarrow{\scriptstyle \mathrm{bl}_j(\gamma_{j,t})} & & \\ G_{i,j} & \hookrightarrow & \mathrm{bl}_j\bigl(C_{2k} \times \mathbf{P}^{t-k}\bigr) & \xrightarrow{\mathrm{bl}_j(\gamma_{k,t})} & \mathrm{bl}_j(C_{2t}) & \hookleftarrow & E_{i,j} \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathrm{bl}_i\bigl(C_{2i} \times \mathbf{P}^{k-i} \times \mathbf{P}^{t-k}\bigr) & \xhookrightarrow{\mathrm{bl}_i(\gamma_{i,k}\times \mathrm{id})} & \mathrm{bl}_i\bigl(C_{2k} \times \mathbf{P}^{t-k}\bigr) & \xrightarrow{\mathrm{bl}_i(\gamma_{k,t})} & \mathrm{bl}_i(C_{2t}) & \xhookleftarrow{\mathrm{bl}_i(\gamma_{i,t})} & \mathrm{bl}_i\bigl(C_{2i} \times \mathbf{P}^{t-i}\bigr)\,, \end{array}$$

where $G_{i,j}$, $E_{i,j}$ are the exceptional divisors over $\mathrm{bl}_i\bigl(C_{2i} \times \mathbf{P}^{k-i} \times \mathbf{P}^{t-k}\bigr)$ and $\mathrm{bl}_i\bigl(C_{2i} \times \mathbf{P}^{t-i}\bigr)$, respectively. Then

$$\mathrm{bl}_j(\gamma_{k,t})^{-1}(E_{i,j}) = G_{i,j}\,, \tag{6.2}$$

$$\mathrm{bl}_j\bigl(C_{2j} \times \mathbf{P}^{t-j}\bigr) - \bigcup_{i<j} E_{i,j} = U_{2j} \times \mathbf{P}^{t-j}\,, \tag{6.3}$$

where $U_{2j} = C_{2j} - \gamma_{j-1,j}\bigl(C_{2j-2} \times \mathbf{P}^1\bigr)$.

Claim 6.4. *Let $0 \leqslant i < j \leqslant k < t$. Set $E^{\circ}_{i,j} = E_{i,j} - \bigcup_{h<i} E_{h,j}$ and similarly for $G^{\circ}_{i,j}$. Then the map*

$$\mathrm{bl}_j(\gamma_{k,t})\colon G^{\circ}_{i,j} \longrightarrow E^{\circ}_{i,j}$$

induced from (6.2) is a morphism of U_{2i}-varieties, whose fiber over $D \in U_{2i}$ is the map

$$\mathrm{bl}_{j-i-1}(\alpha_{k-i-1,t-i-1})\colon \mathrm{bl}_{j-i-1}\bigl(B^{k-i-1}(M) \times \mathbf{P}^{t-k}\bigr) \longrightarrow \mathrm{bl}_{j-i-1} B^{t-i-1}(M)\,,$$

associated with the chain

$$\bigl\{\alpha_{\ell,t-i-1}\colon B^{\ell}(M) \times \mathbf{P}^{t-i-1-\ell} \to B^{t-i-1}(M)\bigr\}_{\ell=0}^{k-i-1}$$

from (4.3) with respect to the line bundle $M = \mathscr{O}_{\mathbf{P}^1}(g-1-h_*D)$ (see Notation 5.2).

Claim 6.5. *Let $0 \leqslant i < j \leqslant k < t$; then*

$$\mathrm{bl}_j(\gamma_{k,t})^{-1}\Bigl(\mathrm{bl}_j\bigl(C_{2j} \times \mathbf{P}^{t-j}\bigr) - \bigcup_{i<j} E_{i,j}\Bigr) = \mathrm{bl}_j\bigl(C_{2j} \times \mathbf{P}^{k-j} \times \mathbf{P}^{t-k}\bigr) - \bigcup_{i<j} G_{i,j}\,, \tag{$*$}$$

$$\mathrm{bl}_j(\gamma_{k,t})^{-1}\bigl(\mathrm{bl}_j\bigl(C_{2j} \times \mathbf{P}^{t-j}\bigr) \cap E^{\circ}_{i,j}\bigr) = \mathrm{bl}_j\bigl(C_{2j} \times \mathbf{P}^{k-j} \times \mathbf{P}^{t-k}\bigr) \cap G^{\circ}_{i,j}\,, \tag{$**$}$$

$$\mathrm{bl}_k(\gamma_{k,t})\colon G^{\circ}_{i,k} \to E^{\circ}_{i,k} \text{ is an embedding}\,. \tag{$***$}$$

We prove these claims by induction on the triples (j,k,t), representing the integers referenced in the statements of the claims. In what follows, when we say something like "*Claim 6.3 holds for a triple (j,k,t)*," we mean that the statements in the claim hold for all $0 < i < j \leqslant k < t$, in the notation used in Claim 6.3. This principle applies similarly to the other claims.

The base case consists of the following two cases:

(1) $k = 0$ and arbitrary t. Claim 6.1 follows from the fact that $\gamma_{0,t}\colon C_0 \times \mathbf{P}^t \hookrightarrow C_{2t}$ is an embedding of smooth varieties. There is nothing to check for the other claims.

(2) $j = 0$, arbitrary $k < t$ (except Claim 6.1). Lemma 5.8 (c) says that

$$\gamma_{k,t}^{-1}(C_0 \times \mathbf{P}^t) = C_0 \times \mathbf{P}^k \times \mathbf{P}^{t-k}\,.$$

Therefore, Claim 6.2 holds. The statements for Claims 6.3–6.5 are empty.

For the sake of clarity, let us restate the assumptions that we are allowed to use in the induction in the following form.

Assumption. *Let j, k, t be integers such that $j \leqslant k < t \leqslant n$. Suppose the following:*

- *Claim 6.1 holds for all the tuples (k', t') such that $k' < t' \leqslant t-1$, or $k' \leqslant k-1$, $t' = t$.*
- *Claims 6.2–6.5 hold for all the triples (j', k', t') such that $j' \leqslant k' < t' \leqslant t-1$, or $j' \leqslant k' \leqslant k-1$, $t' = t$.*
- *Claim 6.2 holds for all (j', k, t) such that $j' \leqslant j-1$.*

The goal is to prove that Claim 6.1 holds for the tuple (k, t) and Claims 6.2– 6.5 hold for the triple (j, k, t).

Proof of Claim 6.3. Let i be an integer such that $i < j$. By the inductive Claims 6.1 and 6.2, the diagram in Claim 6.3 exists and the spaces $\mathrm{bl}_i\big(C_{2i} \times \mathbf{P}^{t-i}\big)$ and $\mathrm{bl}_i\big(C_{2i} \times \mathbf{P}^{k-i} \times \mathbf{P}^{t-k}\big)$ are smooth. The inductive equality (6.1) implies that

$$\mathrm{bl}_i(\gamma_{k,t})^{-1}\big(\mathrm{bl}_i\big(C_{2i} \times \mathbf{P}^{t-i}\big)\big) = \mathrm{bl}_i\big(C_{2i} \times \mathbf{P}^{k-i} \times \mathbf{P}^{t-k}\big)\,.$$

Since $G_{i,j}$ and $E_{i,j}$ are the corresponding exceptional divisors, a repeated application of (2.2) gives (6.2).

To prove (6.3), consider the diagram

$$\begin{array}{ccccccc}
G'_{i,j} & \hookrightarrow & \mathrm{bl}_j\big(C_{2j} \times \mathbf{P}^{t-j}\big) & \xrightarrow{\mathrm{bl}_j(\gamma_{j,t})} & \mathrm{bl}_j(C_{2t}) & \hookleftarrow & E_{i,j} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{bl}_i\big(C_{2i} \times \mathbf{P}^{j-i} \times \mathbf{P}^{t-j}\big) & \xhookrightarrow{\mathrm{bl}_i(\gamma_{i,j}\times\mathrm{id})} & \mathrm{bl}_i\big(C_{2j} \times \mathbf{P}^{t-j}\big) & \xrightarrow{\mathrm{bl}_i(\gamma_{j,t})} & \mathrm{bl}_i(C_{2t}) & \xhookleftarrow{\mathrm{bl}_i(\gamma_{i,t})} & \mathrm{bl}_i\big(C_{2i} \times \mathbf{P}^{t-i}\big)\,,
\end{array}$$

where $G'_{i,j}$ is the exceptional divisor over $\mathrm{bl}_i\big(C_{2i} \times \mathbf{P}^{j-i} \times \mathbf{P}^{t-j}\big)$. This diagram exists because $\{\gamma_{i,t}\}_{i=0}^{j-1}$ and $\{\gamma_{i,j} \times \mathrm{id}\}_{i=0}^{j-1}$ are NCD chains by the inductive Claim 6.1 and Lemma 2.9, respectively, as well as the consequence of the inductive Claim 6.2 that$\mathrm{bl}_i\big(C_{2j} \times \mathbf{P}^{t-j}\big)$ coincides with the space associated with the chain $\{\gamma_{i,j} \times \mathrm{id}\}_{i=0}^{j-1}$. Because $i < j$, the inductive Claim 6.2 gives

$$\mathrm{bl}_i(\gamma_{j,t})^{-1}\big(\mathrm{bl}_i\big(C_{2i} \times \mathbf{P}^{t-i}\big)\big) = \mathrm{bl}_i\big(C_{2i} \times \mathbf{P}^{j-i} \times \mathbf{P}^{t-j}\big)\,.$$

Then using (2.2), we have

$$\mathrm{bl}_j\big(C_{2j} \times \mathbf{P}^{t-j}\big) \cap E_{i,j} = \mathrm{bl}_j(\gamma_{j,t})^{-1}(E_{i,j}) = G'_{i,j}\,.$$

Since $\{G'_{i,j}\}_{i=0}^{j-1}$ is the set of exceptional divisors associated with the NCD chain $\{\gamma_{i,j} \times \mathrm{id}\}_{i=0}^{j-1}$, using Remark 2.7, one has

$$\begin{aligned}\mathrm{bl}_j\big(C_{2j}\times\mathbf{P}^{t-j}\big)-\bigcup_{i<j}E_{i,j} &= \mathrm{bl}_j\big(C_{2j}\times\mathbf{P}^{t-j}\big)-\bigcup_{i<j}\big(\mathrm{bl}_j\big(C_{2j}\times\mathbf{P}^{t-j}\big)\cap E_{i,j}\big)\\ &= \mathrm{bl}_j\big(C_{2j}\times\mathbf{P}^{t-j}\big)-\bigcup_{i<j}G'_{i,j}\\ &= C_{2j}\times\mathbf{P}^{t-j}-(\gamma_{j-1,j}\times\mathrm{id})\big(C_{2j-2}\times\mathbf{P}^1\times\mathbf{P}^{t-j}\big)=U_{2j}\times\mathbf{P}^{t-j}\,. \qquad \square\end{aligned}$$

Proof of Claim 6.4. Let i be an integer such that $i<j$. We prove Claim 6.4 for all quadruples (i,ℓ,k,t) such that $0\leqslant i<\ell\leqslant k<t$ and $\ell\in[i+1,j]$, by induction on ℓ.

Before the proof, we need some preparation. By the inductive Claim 6.1, the following diagram associated with the chain $\{\gamma_{i,k}\}_{i=0}^{j-1}$ exists:

$$\begin{array}{ccc} E^k_{i,j} & \hookrightarrow & \mathrm{bl}_j(C_{2k}) \\ \downarrow & & \downarrow \\ \mathrm{bl}_i\big(C_{2i}\times\mathbf{P}^{k-i}\big) & \xhookrightarrow{\mathrm{bl}_i(\gamma_{i,k})} & \mathrm{bl}_i(C_{2k})\,, \end{array}$$

where $E^k_{i,j}$ is the exceptional divisor over $\mathrm{bl}_i(C_{2i}\times\mathbf{P}^{k-i})$. Since $i\leqslant j-1$, we can use the inductive Claim 6.2 to conclude that $\mathrm{bl}_i(C_{2k}\times\mathbf{P}^{t-k})$ coincides with the space associated with the chain $\{\gamma_{j,k}\times\mathrm{id}\}_{j=0}^{k-1}$ for $i\leqslant j$. As the $G_{i,j}\subseteq\mathrm{bl}_i(C_{2k}\times\mathbf{P}^{t-k})$ are exceptional divisors associated with the chain $\{\gamma_{j,k}\times\mathrm{id}\}_{j=0}^{k-1}$, using Lemma 2.9 we have

$$G_{i,j}=E^k_{i,j}\times\mathbf{P}^{t-k}\,. \tag{6.4}$$

Now we start the inductive proof of Claim 6.4. Let $D\in U_{2i}$, and set

$$M:=\mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)\,.$$

The base case is $\ell=i+1$. Let $h<i$. Because the chain $\{\gamma_{h,t}\}_{h=0}^{k-1}$ is NCD and $i\leqslant k-1$, we know that $\mathrm{bl}_i\big(C_{2i}\times\mathbf{P}^{t-i}\big)$ intersects $E_{h,i}$ transversally. Hence $E^\circ_{i,i+1}$ is the exceptional divisor over

$$\mathrm{bl}_i\big(C_{2i}\times\mathbf{P}^{t-i}\big)-\bigcup_{h<i}E_{h,i}\overset{(6.3)}{=}U_{2i}\times\mathbf{P}^{t-i}\subseteq\mathrm{bl}_i(C_{2t})\,;$$

thus we may identify $E^\circ_{i,i+1}$ as the exceptional divisor for blowing up C_{2t} along $U_{2i}\times\mathbf{P}^{t-i}$. By Lemma 5.3(c), we know that $E^\circ_{i,i+1}$ is a U_{2i}-variety with fiber over D equal to $B^{t-i-1}(M)$. On the other hand, it follows from the inductive Claim 6.4 and (6.4) that $G^\circ_{i,i+1}$ is a U_{2i}-variety with fiber over D equal to $B^{k-i-1}(M)\times\mathbf{P}^{t-k}$. Moreover, Corollary 5.5 shows that $\mathrm{bl}_{i+1}(\gamma_{k,t})\colon G^\circ_{i,i+1}\to E^\circ_{i,i+1}$ is a U_{2i}-morphism and the fiber over D is

$$\alpha_{k-i-1,t-i-1}\colon B^{k-i-1}(M)\times\mathbf{P}^{t-k}\longrightarrow B^{t-i-1}(M)\,.$$

This concludes the base case.

Assume that Claim 6.4 holds for all $\ell' \leqslant \ell - 1$. Consider the diagram

$$\begin{array}{ccccccc}
G^\circ_{i,\ell} & \hookrightarrow & \mathrm{bl}_\ell\big(C_{2k} \times \mathbf{P}^{t-k}\big) & \xrightarrow{\mathrm{bl}_\ell(\gamma_{k,t})} & \mathrm{bl}_\ell(C_{2t}) & \hookleftarrow & E^\circ_{i,\ell} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
G^\circ_{i,\ell-1} & \hookrightarrow & \mathrm{bl}_{\ell-1}\big(C_{2k} \times \mathbf{P}^{t-k}\big) & \xrightarrow{\mathrm{bl}_{\ell-1}(\gamma_{k,t})} & \mathrm{bl}_{\ell-1}(C_{2t}) & \hookleftarrow & E^\circ_{i,\ell-1} \\
 & & \uparrow{\scriptstyle \mathrm{bl}_{\ell-1}(\gamma_{\ell,k}\times\mathrm{id})} & & \uparrow{\scriptstyle \mathrm{bl}_{\ell-1}(\gamma_{\ell-1,t})} & & \\
 & & \mathrm{bl}_{\ell-1}\big(C_{2\ell-2} \times \mathbf{P}^{k-(\ell-1)} \times \mathbf{P}^{t-k}\big) & & \mathrm{bl}_{\ell-1}\big(C_{2\ell-2} \times \mathbf{P}^{t-(\ell-1)}\big)\,. & &
\end{array}$$

Using the inductive Claim 6.1, we know that the chain $\{\gamma_{i,t}\}_{i=0}^{j-1}$ is NCD, so $E^\circ_{i,\ell}$ is the blowup of $E^\circ_{i,\ell-1}$ along $E^\circ_{i,\ell-1} \cap \mathrm{bl}_{\ell-1}(C_{2\ell-2} \times \mathbf{P}^{t-(\ell-1)})$. By the inductive Claim 6.4 and Lemma 2.20(c), we deduce that $E^\circ_{i,\ell}$ is a U_{2i}-variety and the fiber over D is $\mathrm{bl}_{\ell-i-1} B^{t-i-1}(M)$. Similarly, using (6.4), the inductive Claim 6.4 and Lemma 2.9, one can deduce that $G^\circ_{i,\ell}$ as a U_{2i}-variety with fiber $\mathrm{bl}_{\ell-i-1}(B^{k-i-1}(M) \times \mathbf{P}^{t-k})$. Moreover, by Corollary 4.4(a), the map $\mathrm{bl}_\ell(\gamma_{k,t})\colon G^\circ_{i,\ell} \to E^\circ_{i,\ell}$ is a U_{2i}-morphism with fiber over D equal to

$$\mathrm{bl}_{\ell-i-1}(\alpha_{k-i-1,t-i-1})\colon\ \mathrm{bl}_{\ell-i-1} B^{k-i-1}(M) \times \mathbf{P}^{t-k} \longrightarrow \mathrm{bl}_{\ell-i-1} B^{t-i-1}(M)\,.$$

This finishes the inductive proof of Claim 6.4. □

Proof of Claim 6.5. For equality ($*$), using (6.4) and the inductive (6.3), we can show that

$$\mathrm{bl}_j\big(C_{2j} \times \mathbf{P}^{k-j} \times \mathbf{P}^{t-k}\big) - \bigcup_{i<j} G_{i,j} = \big(U_{2j} \times \mathbf{P}^{k-j}\big) \times \mathbf{P}^{t-k}\,.$$

On the other hand, from the proof of Claim 6.4 we know that $U_{2j} \times \mathbf{P}^{t-j}$ is not touched by any blowup associated with $\mathrm{bl}_j(\gamma_{k,t})$. Using Lemma 5.8(b), we obtain

$$\mathrm{bl}_j(\gamma_{k,t})^{-1}\big(U_{2j} \times \mathbf{P}^{t-j}\big) = \gamma_{k,t}^{-1}\big(U_{2j} \times \mathbf{P}^{t-j}\big) = \big(U_{2j} \times \mathbf{P}^{k-j}\big) \times \mathbf{P}^{t-k}\,.$$

Together with (6.3), this proves equality ($*$).

As for equality ($**$), the fiberwise equality over $D \in U_{2i}$ follows from Claim 6.4 and Corollary 4.4(a). Then we can apply Lemma 2.20(b) to obtain equality ($**$).

Finally, Claim 6.4 together with Corollary 4.4(a) and Lemma 2.20(a) give property ($***$). □

Proof of Claim 6.2. We apply Proposition 2.18 to the chain $\{\gamma_{j,t}\}_{j=0}^{k}$ with

$$X = C_{2t}\,, \quad X_k = C_{2k} \times \mathbf{P}^{t-k}\,, \quad X_{j,k} = C_{2j} \times \mathbf{P}^{k-j} \times \mathbf{P}^{t-k}\,, \quad f_{j,k} = \gamma_{j,k} \times \mathrm{id}\,, \quad \phi_k = \gamma_{k,t}\,.$$

The assumptions in Proposition 2.18 can be checked as follows:

- The chain $\{\gamma_{j,k} \times \mathrm{id}\}_{j=0}^{k-1}$ is NCD, by the inductive Claim 6.1 and Lemma 2.9.
- The map

  $$\gamma_{k,t}\colon \big(C_{2k} \times \mathbf{P}^{t-k} - \big(\gamma_{k-1,k} \times \mathrm{id}\big)\big(C_{2k-2} \times \mathbf{P}^1 \times \mathbf{P}^{t-k}\big)\big) = U_{2k} \times \mathbf{P}^{t-k} \longrightarrow C_{2t}$$

  is an embedding, by Lemma 5.8 and Remark 2.7.
- The chain $\{\gamma_{j,t}\}_{j=0}^{k-1}$ is NCD by the inductive Claim 6.1.
- The inductive Claim 6.2 gives

  $$\mathrm{bl}_i(\gamma_{k,t})^{-1}\big(\mathrm{bl}_i\big(C_{2i} \times \mathbf{P}^{t-i}\big)\big) = \mathrm{bl}_i\big(C_{2i} \times \mathbf{P}^{k-i} \times \mathbf{P}^{t-k}\big)\,, \quad \forall i \leqslant j-1\,.$$

  Moreover, properties ($*$) and ($**$) hold by Claim 6.5.

Consequently, Proposition 2.18 gives

$$\mathrm{bl}_j(\gamma_{k,t})^{-1}\big(\mathrm{bl}_j(C_{2j}\times \mathbf{P}^{t-j})\big) = \mathrm{bl}_j\big(C_{2j}\times \mathbf{P}^{k-j}\times \mathbf{P}^{t-k}\big)\,.$$

This proves Claim 6.2. □

Proof of Claim 6.1. Running the arguments above for all $j \leqslant k$ and using property ($***$) from Claim 6.5, we see that Proposition 2.18 also implies that $\{\gamma_{j,t}\}_{j=0}^{k}$ is an NCD chain. This proves Claim 6.1. □

Therefore, we have finished the inductive proofs of Claims 6.1–6.5. As a consequence, we have proved Proposition 3.1.

7. The proofs of Proposition 3.2 and Corollary B

In this section, we prove Proposition 3.2 following a similar argument to the one for Proposition 3.1. As a byproduct of the proof, we deduce Corollary B.

Let C be a smooth hyperelliptic curve of odd genus $g = 2n+1$. Consider the proper chain $\{\delta_k\colon C_{2k}\to \mathrm{Jac}(C)\}_{k=0}^{n}$ from (3.1) with the diagram from (3.2) for $j<k$:

$$\begin{array}{ccc} C_{2j}\times \mathbf{P}^{k-j} & \xrightarrow{p_1} & C_{2j} \\ \big\downarrow{\scriptstyle \gamma_{j,k}} & & \big\downarrow{\scriptstyle \delta_j} \\ C_{2k} & \xrightarrow{\delta_k} & \mathrm{Jac}(C)\,. \end{array}$$

Let k be an integer such that $0\leqslant k\leqslant n$. Denote by $\mathrm{bl}_j(C_{2k})$ and $\mathrm{bl}_j(\mathrm{Jac}(C))$ the spaces associated with the chain $\{\delta_k\}_{k=0}^{n}$. Denote by $\mathrm{bl}_i(C_{2j}\times \mathbf{P}^{k-j})$ the space associated with the chain $\{\gamma_{j,k}\}_{j=0}^{k-1}$. We break the proof of Proposition 3.2 down into the following statements.

Claim 7.1. *The chain $\{\delta_j\}_{j=0}^{k}$ is NCD.*

Claim 7.2. *Let $j\leqslant k$; then $\mathrm{bl}_j(\delta_k)^{-1}(\mathrm{bl}_j(C_{2j})) = \mathrm{bl}_j(C_{2j}\times \mathbf{P}^{k-j})$. In other words, δ_k is a map of chains of centers.*

Consequently, $\mathrm{bl}_j(C_{2k})$ coincides with the blowup space associated with the chain $\{\gamma_{j,k}\}_{j=0}^{k-1}$, and there is a natural isomorphism $\mathrm{bl}_j(C_{2j}\times \mathbf{P}^{k-j}) = \mathrm{bl}_j(C_{2j})\times \mathbf{P}^{k-j}$.

Claim 7.3. *Let $0\leqslant i<j\leqslant k$, and consider the diagram*

$$\begin{array}{ccccccc} & & & & \mathrm{bl}_j(C_{2j}) & & \\ & & & & \big\downarrow{\scriptstyle \mathrm{bl}_j(\delta_j)} & & \\ E_{i,j} & \hookrightarrow & \mathrm{bl}_j(C_{2k}) & \xrightarrow{\mathrm{bl}_j(\delta_k)} & \mathrm{bl}_j(\mathrm{Jac}(C)) & \hookleftarrow & F_{i,j} \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathrm{bl}_i\big(C_{2i}\times \mathbf{P}^{k-i}\big) & \xhookrightarrow{\mathrm{bl}_i(\gamma_{i,k})} & \mathrm{bl}_i(C_{2k}) & \xrightarrow{\mathrm{bl}_i(\delta_k)} & \mathrm{bl}_i(\mathrm{Jac}(C)) & \xhookleftarrow{\mathrm{bl}_i(\delta_i)} & \mathrm{bl}_i(C_{2i})\,, \end{array}$$

where $F_{i,j}$, $E_{i,j}$ are the exceptional divisors over $\mathrm{bl}_i(C_{2i})$ and $\mathrm{bl}_i(C_{2i}\times \mathbf{P}^{k-i})$, respectively. Then

$$\mathrm{bl}_j(\delta_k)^{-1}(F_{i,j}) = E_{i,j}\,, \tag{7.1}$$

$$\mathrm{bl}_j(C_{2j}) - \bigcup_{i<j} F_{i,j} = U_{2j}\,, \tag{7.2}$$

where $U_{2j} = C_{2j} - \gamma_{j,j-1}(C_{2j-2} \times \mathbf{P}^1)$.

Claim 7.4. *For* $0 \leqslant i < j \leqslant k$, *set* $E^\circ_{i,j} = E_{i,j} - \bigcup_{h<i} E_{h,j}$ *and similarly for* $F^\circ_{i,j}$. *Then* (7.1) *induces a morphism of* U_{2i}*-varieties*

$$\mathrm{bl}_j(\delta_k)\colon E^\circ_{i,j} \longrightarrow F^\circ_{i,j}\,.$$

Moreover, over $D \in U_{2i}$ *the morphism is*

$$\mathrm{bl}_{j-i-1}(\beta_{k-i-1})\colon \mathrm{bl}_{j-i-1} B^{k-i-1}(M) \longrightarrow \mathrm{bl}_{j-i-1} \mathbf{P}H^0(M)\,,$$

which is associated with the chain $\{\beta_\ell\}_{\ell=0}^{k-i-1}$ *in* (4.2) *with respect to the line bundle* $M = \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)$ *(see Notation* 5.2*).*

Claim 7.5. *Let* $0 \leqslant i < j \leqslant k$; *then*

$$\mathrm{bl}_j(\delta_k)^{-1}\Big(\mathrm{bl}_j(C_{2j}) - \bigcup_{i<j} F_{i,j}\Big) = \mathrm{bl}_j\big(C_{2j} \times \mathbf{P}^{k-j}\big) - \bigcup_{i<j} E_{i,j}\,, \tag{$*$}$$

$$\mathrm{bl}_j(\delta_k)^{-1}\big(\mathrm{bl}_j(C_{2j}) \cap F^\circ_{i,j}\big) = \mathrm{bl}_j\big(C_{2j} \times \mathbf{P}^{k-j}\big) \cap E^\circ_{i,j}\,, \tag{$**$}$$

$$\mathrm{bl}_k(\delta_k)\colon E^\circ_{i,k} \to F^\circ_{i,k} \text{ is an embedding}\,. \tag{$***$}$$

As in the proof of Proposition 3.1, we prove these claims by induction on the tuples (j,k). The base case contains the following cases:

- $k = 0$. All the claims are clear.
- $j = 0$ and arbitrary $k \geqslant 0$ (except Claim 7.1). For Claim 7.2, this is clear, as the Abel–Jacobi theorem gives
$$\mathrm{bl}_0\big(\delta_k^{-1}\big)(C_0) = \delta_k^{-1}(C_0) = C_0 \times \mathbf{P}^k\,.$$
The statements for the other claims are empty.

As in the previous section, we summarize the assumptions that we are allowed to use during the inductive step in the following way.

Assumption. *Let* j, k *be integers such that* $j \leqslant k$. *Suppose the following:*

- *Claim* 7.1 *holds for all* k' *such that* $k' \leqslant k-1$.
- *Claims* 7.2–7.5 *hold for all the tuples* (j', k') *such that* $j' \leqslant k' \leqslant k-1$.
- *Claim* 7.2 *holds for all* (j', k) *with* $j' \leqslant j-1$.

We will prove that Claim 7.1 holds for k and Claims 7.2–7.5 hold for the tuple (j,k).

Proof of Claim 7.3. By the inductive hypothesis, all maps in the diagram are well defined. Let i be an integer such that $i < j$. The inductive Claim 7.2 gives

$$\mathrm{bl}_i(\delta_k)^{-1}(\mathrm{bl}_i(C_{2i})) = \mathrm{bl}_i\big(C_{2i} \times \mathbf{P}^{k-i}\big)\,.$$

Using the inductive Claims 7.1 and 6.1 that state that $\{\delta_j\}$ and $\{\gamma_{i,k}\}$ are chains of smooth centers, we know that $\mathrm{bl}_i(C_{2i})$ and $\mathrm{bl}_i(C_{2i} \times \mathbf{P}^{k-i})$ are smooth varieties. Then a repeated application of (2.2) yields (7.1).

Regarding (7.2), to understand the intersection $\mathrm{bl}_j(C_{2j}) \cap F_{i,j}$, let us consider the following diagram:

$$\begin{array}{ccccccc}
E'_{i,j} & \hookrightarrow & \mathrm{bl}_j(C_{2j}) & \xhookrightarrow{\mathrm{bl}_j(\delta_j)} & \mathrm{bl}_j(\mathrm{Jac}(C)) & \hookleftarrow & F_{i,j} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{bl}_i\big(C_{2i} \times \mathbf{P}^{j-i}\big) & \xhookrightarrow{\mathrm{bl}_i(\gamma_{i,j})} & \mathrm{bl}_i(C_{2j}) & \xrightarrow{\mathrm{bl}_i(\delta_j)} & \mathrm{bl}_i(\mathrm{Jac}(C)) & \xhookleftarrow{\mathrm{bl}_i(\delta_i)} & \mathrm{bl}_i(C_{2i}) \,,
\end{array}$$

where $E'_{i,j}$ is the exceptional divisor over $\mathrm{bl}_i(C_{2i} \times \mathbf{P}^{j-i})$. This diagram exists because $\{\delta_i\}_{i=0}^{j-1}$ and $\{\gamma_{i,j}\}_{i=0}^{j-1}$ are NCD chains by the inductive Claim 7.1 and Proposition 3.1, respectively. As $i < j$, the inductive Claim 7.2 gives

$$\mathrm{bl}_i(\delta_j)^{-1}(\mathrm{bl}_i(C_{2i})) = \mathrm{bl}_i\big(C_{2i} \times \mathbf{P}^{j-i}\big) \,.$$

As a consequence, using (2.2) again, we have

$$\mathrm{bl}_j(C_{2j}) \cap F_{i,j} = \mathrm{bl}_j(\delta_j)^{-1}(F_{i,j}) = E'_{i,j} \,.$$

By construction, $\{E'_{i,j}\}_{i=0}^{j-1}$ is the set of exceptional divisors associated with the NCD chain $\{\gamma_{i,j}\}_{i=0}^{j-1}$; thus it follows from Remark 2.7 that

$$\mathrm{bl}_j(C_{2j}) - \bigcup_{i<j} F_{i,j} = \mathrm{bl}_j(C_{2j}) - \bigcup_{i<j} E'_{i,j} = C_{2j} - \gamma_{j-1,j}\big(C_{2j-2} \times \mathbf{P}^1\big) = U_{2j} \,. \qquad \square$$

Proof of Claim 7.4. Let i be an integer with $i < j$. We prove Claim 7.4 for all triples (i, ℓ, k) such that $0 < i < \ell \leqslant k$ and $\ell \in [i+1, j]$, by an additional induction on ℓ.

The base case is $\ell = i + 1$. Let $h < i$. By construction, $F_{i,i+1}$ and $F_{h,i+1}$ are the exceptional divisors over $\mathrm{bl}_i(C_{2i})$ and $F_{h,i}$, respectively. To understand

$$F^\circ_{i,i+1} = F_{i,i+1} - \bigcup_{h<i} F_{h,i+1} \,,$$

we need to know how $F_{h,i}$ intersects $\mathrm{bl}_i(C_{2i})$. By the inductive Claim 7.1, the chain $\{\delta_h\}_{h=0}^{k-1}$ is NCD, and hence we have transverse intersections

$$\mathrm{bl}_i(C_{2i}) \cap F_{h,i} \subseteq \mathrm{bl}_i(\mathrm{Jac}(C)) \,.$$

It follows that $F^\circ_{i,i+1}$ is the exceptional divisor for the blowup of $\mathrm{bl}_i(\mathrm{Jac}(C))$ along

$$\mathrm{bl}_i(C_{2i}) - \bigcup_{h<i} F_{h,i} = U_{2i} \,,$$

which follows from the inductive (7.2). Since U_{2i} is away from all the exceptional divisors $F_{h,i}$, the divisor $F^\circ_{i,i+1}$ can also be identified with the exceptional divisor for the blowup of $\mathrm{Jac}(C)$ along U_{2i}, so we can use Bertram's results as follows.

Let $D \in U_{2i}$, and write $M = \mathcal{O}_{\mathbf{P}^1}(g - 1 - h_* D)$. It follows from Lemma 5.6(b) that $F^\circ_{i,i+1}$ is a U_{2i}-variety with fiber over D equal to $\mathbf{P}H^0(M)$. On the other hand, Claim 6.4 says that $E^\circ_{i,i+1}$ is a U_{2i}-variety with fiber over D equal to $B^{k-i-1}(M)$. Moreover, by Corollary 5.7, the map $\mathrm{bl}_{i+1}(\delta_k) \colon E^\circ_{i,i+1} \to F^\circ_{i,i+1}$ is a U_{2i}-morphism, whose fiber over D is

$$\beta_{k-i-1} \colon B^{k-i-1}(M) \longrightarrow \mathbf{P}H^0(M) \,.$$

We conclude the base case $\ell = i + 1$.

Assume that Claim 7.4 holds for all $\ell' \leqslant \ell - 1$. Let us look at the diagram

$$\begin{array}{ccccccc}
E^\circ_{i,\ell} & \hookrightarrow & \mathrm{bl}_\ell(C_{2k}) & \xrightarrow{\mathrm{bl}_\ell(\delta_k)} & \mathrm{bl}_\ell(\mathrm{Jac}(C)) & \hookleftarrow & F^\circ_{i,\ell} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E^\circ_{i,\ell-1} & \hookrightarrow & \mathrm{bl}_{\ell-1}(C_{2k}) & \xrightarrow{\mathrm{bl}_{\ell-1}(\delta_k)} & \mathrm{bl}_{\ell-1}(\mathrm{Jac}(C)) & \hookleftarrow & F^\circ_{i,\ell-1} \\
 & & \uparrow & & \uparrow & & \\
 & & \mathrm{bl}_{\ell-1}\big(C_{2\ell-2} \times \mathbf{P}^{k-(\ell-1)}\big) & & \mathrm{bl}_{\ell-1}(C_{2\ell-2})\,. & &
\end{array}$$

Moreover, (7.1) gives the diagram

$$\begin{array}{ccc}
E^\circ_{i,\ell} & \xrightarrow{\mathrm{bl}_\ell(\delta_k)} & F^\circ_{i,\ell} \\
\downarrow & & \downarrow \\
E^\circ_{i,\ell-1} & \xrightarrow{\mathrm{bl}_{\ell-1}(\delta_k)} & F^\circ_{i,\ell-1}\,.
\end{array} \tag{7.3}$$

Let $D \in U_{2i}$. We would like to understand the fiber of the top horizontal map over D. To do this, let us compute the other parts of the diagram over D.

Part 1: $F^\circ_{i,\ell-1}$. As $\ell - 1 \leqslant k - 1$, the inductive Claim 7.1 implies that there is an embedding $\mathrm{bl}_{\ell-1}(\delta_{\ell-1})\colon \mathrm{bl}_{\ell-1}(C_{2\ell-2}) \hookrightarrow \mathrm{bl}_{\ell-1}(\mathrm{Jac}(C))$, and the intersection

$$\mathrm{bl}_{\ell-1}(C_{2\ell-2}) \cap F^\circ_{i,\ell-1}$$

is transverse. Consequently, $F^\circ_{i,\ell}$ is the blowup of $F^\circ_{i,\ell-1}$ along $\mathrm{bl}_{\ell-1}(C_{2\ell-2}) \cap F^\circ_{i,\ell-1}$. Note that $M = \mathcal{O}_{\mathbf{P}^1}(g - 1 - h_*D)$ has degree

$$g - 1 - 2i = 2n - 2i \geqslant 2(k - i - 1) + 1\,,$$

as $g(C) = 2n + 1$ and $k \leqslant n$. By the inductive hypothesis,

$$F^\circ_{i,\ell-1} \quad \text{and} \quad \mathrm{bl}_{\ell-1}(C_{2\ell-2}) \cap F^\circ_{i,\ell-1} = \mathrm{bl}_{\ell-1}(\delta_{\ell-1})^{-1}(F^\circ_{i,\ell-1})$$

are U_{2i}-varieties whose fibers over D are

$$\mathrm{bl}_{\ell-i-2}\,\mathbf{P}H^0(M) \quad \text{and} \quad \mathrm{bl}_{\ell-i-2}\,B^{\ell-i-2}(M)\,.$$

We can use Lemma 2.20(c) to obtain that $F^\circ_{i,\ell}$ is a U_{2i}-variety with fiber over D equal to the blowup of $\mathrm{bl}_{\ell-i-2}\,\mathbf{P}H^0(M)$ along $\mathrm{bl}_{\ell-i-2}\,B^{\ell-i-2}(M)$, which is by definition $\mathrm{bl}_{\ell-i-1}\,\mathbf{P}H^0(M)$.

Part 2: $E^\circ_{i,\ell-1}$. A similar argument using Claim 6.4 says that $E^\circ_{i,\ell}$ is a U_{2i}-variety with fiber over D equal to $\mathrm{bl}_{\ell-i-1}\,B^{k-i-1}(M)$, the blowup of $\mathrm{bl}_{\ell-i-2}\,B^{k-i-1}(M)$ along the product $\mathrm{bl}_{\ell-i-2}(B^{\ell-i-2}(M) \times \mathbf{P}^{k-\ell+1})$.

Part 3: maps in (7.3). By the inductive hypothesis, over D, the map $\mathrm{bl}_{\ell-1}(\delta_k)\colon E^\circ_{i,\ell-1} \to F^\circ_{i,\ell-1}$ is $\mathrm{bl}_{\ell-i-2}(\beta_{k-i-1})$. The vertical maps over D are the blowups described above.

Consequently, we can draw the corresponding diagram of (7.3) over D

$$\begin{array}{ccc} \mathrm{bl}_{\ell-i-1} B^{k-i-1}(M) & \longrightarrow & \mathrm{bl}_{\ell-i-1} \mathbf{P}H^0(M) \\ \downarrow & & \downarrow \\ \mathrm{bl}_{\ell-i-2} B^{k-i-1}(M) & \xrightarrow{\mathrm{bl}_{\ell-i-2}(\beta_{k-i-1})} & \mathrm{bl}_{\ell-i-2} \mathbf{P}H^0(M)\,. \end{array}$$

Since β_{k-i-1} is a map of chains (Proposition 4.2), the top horizontal map must be $\mathrm{bl}_{\ell-i-1}(\beta_{k-i-1})$, which is the fiber of $\mathrm{bl}_\ell(\delta_k)\colon E_{i,\ell}^\circ \to F_{i,\ell}^\circ$ over D. This finishes the inductive proof of Claim 7.4. □

Proof of Claim 7.5. For property $(*)$, we know from the discussion in the proof of Claim 7.4 that U_{2j} is not touched by all the blowups associated with $\mathrm{bl}_j(\delta_k)$; thus

$$\mathrm{bl}_j(\delta_k)^{-1}(U_{2j}) = \delta_k^{-1}(U_{2j}) = U_{2j} \times \mathbf{P}^{k-j}\,,$$

where the last equality comes from Lemma 5.8(a). Using (6.3) in Claim 6.3, we have

$$\mathrm{bl}_j\bigl(C_{2j} \times \mathbf{P}^{k-j}\bigr) - \bigcup_{i<j} E_{i,j} = U_{2j} \times \mathbf{P}^{k-j}\,.$$

Putting these two equations together with (7.2), we obtain property $(*)$.

Let us turn to property $(**)$. Let $D \in U_{2i}$. By Claims 7.4 and 6.4, the diagram

$$\begin{array}{ccc} \mathrm{bl}_j(C_{2j} \times \mathbf{P}^{k-j}) \cap E_{i,j}^\circ & \longrightarrow & \mathrm{bl}_j(C_{2j}) \cap F_{i,j}^\circ \\ \big\downarrow & & \big\downarrow \\ E_{i,j}^\circ & \xrightarrow{\mathrm{bl}_j(\delta_k)} & F_{i,j}^\circ \end{array} \tag{7.4}$$

is a diagram of U_{2i}-varieties, whose fiber over D is

$$\begin{array}{ccc} \mathrm{bl}_{j-i-1}\bigl(B^{j-i-1}(M) \times \mathbf{P}^{k-j}\bigr) & \longrightarrow & \mathrm{bl}_{j-i-1} B^{j-i-1}(M) \\ \big\downarrow {\scriptstyle \mathrm{bl}_{j-i-1}(\alpha_{j-i-1,k-i-1})} & & \big\downarrow {\scriptstyle \mathrm{bl}_{j-i-1}(\beta_{j-i-1})} \\ \mathrm{bl}_{j-i-1} B^{k-i-1}(M) & \xrightarrow{\mathrm{bl}_{j-i-1}(\beta_{k-i-1})} & \mathrm{bl}_{j-i-1} \mathbf{P}H^0(M)\,, \end{array} \tag{7.5}$$

where M denotes the line bundle $\mathscr{O}_{\mathbf{P}^1}(g-1-h_*D)$ and the top horizontal map is induced by the natural projection

$$B^{j-i-1}(M) \times \mathbf{P}^{k-j} \longrightarrow B^{j-i-1}(M)\,.$$

By Corollary 4.4(b), the diagram (7.5) is Cartesian, that is,

$$\mathrm{bl}_{j-i-1}(\beta_{k-i-1})^{-1}\bigl(\mathrm{bl}_{j-i-1} B^{j-i-1}(M)\bigr) = \mathrm{bl}_{j-i-1}\bigl(B^{j-i-1}(M) \times \mathbf{P}^{k-j}\bigr)\,.$$

As $(\mathrm{bl}_j(C_{2j}) \times \mathbf{P}^{k-j}) \cap E_{i,j}^\circ$ and $\mathrm{bl}_j(C_{2j}) \cap F_{i,j}^\circ$ are both smooth, we can apply Lemma 2.20(b) to the U_{2i}-morphism $\mathrm{bl}_j(\delta_k)\colon E_{i,j}^\circ \to F_{i,j}^\circ$ to conclude that the diagram (7.4) is also Cartesian. This proves that property $(**)$ holds.

Last, we prove that property $(*{*}*)$ holds. Claim 7.4 implies that the fiber of the map $\mathrm{bl}_k(\delta_k)\colon E_{i,k}^\circ \to F_{i,k}^\circ$ over $D \in U_{2i}$ is $\mathrm{bl}_{k-i-1}(\beta_{k-i-1})$, which is an embedding by Corollary 4.4(b). As $E_{i,k}^\circ$ and $F_{i,k}^\circ$ are both smooth, we conclude from Lemma 2.20(a) that property $(*{*}*)$ holds. □

Proof of Claim 7.2. We apply Proposition 2.18 to the chain $\{\delta_j\}_{j=0}^k$ with

$$X = \mathrm{Jac}(C)\,, \quad X_k = C_{2k}\,, \quad X_{j,k} = C_{2j} \times \mathbf{P}^{k-j}\,, \quad f_{j,k} = \gamma_{j,k}\,, \quad \phi_k = \delta_k\,.$$

The assumptions in Proposition 2.18 can be checked as follows:

- The chain $\{\gamma_{j,k}\}_{j=0}^{k-1}$ is NCD, by Proposition 3.1.
- The map $\delta_k\colon C_{2k}-\gamma_{k-1,k}(C_{2k-2}\times\mathbf{P}^1)\to\mathrm{Jac}(C)$ is an embedding: By Notation 5.1, we have $C_{2k}-\gamma_{k-1,k}(C_{2k-2}\times\mathbf{P}^1)=U_{2k}$, and $U_{2k}\hookrightarrow\mathrm{Jac}(C)$ is an embedding by Lemma 5.8(a).
- The chain $\{\delta_j\}_{j=0}^{k-1}$ is NCD, by the inductive Claim 7.1.
- We have $\mathrm{bl}_i(\delta_k)^{-1}(\mathrm{bl}_i(C_{2i}))=\mathrm{bl}_i(C_{2i}\times\mathbf{P}^{k-i})$ for all $i\leqslant j-1$, by the inductive Claim 7.2.
- Properties $(*)$ and $(**)$ hold thanks to Claim 7.5.

Hence Proposition 2.18 shows that

$$\mathrm{bl}_j(\delta_k)^{-1}(\mathrm{bl}_j(C_{2j}))=\mathrm{bl}_j\big(C_{2j}\times\mathbf{P}^{k-j}\big)\,;$$

that is, Claim 7.2 holds. □

Proof of Claim 7.1. Running the arguments above for all $j\leqslant k$ and using property $(***)$ from Claim 7.5, Proposition 2.18 implies that $\{\delta_j\}_{j=0}^{k}$ is a NCD chain. This proves Claim 7.1. □

Therefore, we have finished the inductive proof of all the claims above. Proposition 3.2 immediately follows.

Proof of Corollary B. For $0\leqslant i\leqslant n-1$, the exceptional divisor $Z_i\subseteq\mathrm{bl}_{n+1}(\mathrm{Jac}(C))$ is the exceptional divisor $F_{i,n+1}\subseteq\mathrm{bl}_{n+1}(\mathrm{Jac}(C))$, defined as the preimage of $F_{i,n}\subseteq\mathrm{bl}_n(\mathrm{Jac}(C))$ (see the notation in Claim 7.3). The proper transform $\tilde\Theta$ is $F_{n,n+1}$, the exceptional divisor for the blowup of $\mathrm{bl}_n(\mathrm{Jac}(C))$ along $\mathrm{bl}_n(C_{2n})$. Since $\delta_i(C_{2i})$ equals W_{g-1}^{n-i} and δ_i is an isomorphism over $U_{2i}\subseteq C_{2i}$ by Lemma 5.8, this induces a natural identification

$$U_{2i}\cong W_{g-1}^{n-i}(C)-W_{g-1}^{n-i+1}(C)\,.$$

Let $D\in U_{2i}$ and set $M=\mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)$. Claim 7.4 implies that the fiber of the projection

$$F_{i,n}^\circ=F_{i,n}-\bigcup_{h<i}F_{h,n}\longrightarrow W_{g-1}^{n-i}(C)-W_{g-1}^{n-i+1}(C)\cong U_{2i}$$

over D is $\mathrm{bl}_{n-i-1}\,\mathbf{P}H^0(\mathbf{P}^1,M)$, the blowup space associated with the chain

$$\big\{\beta_{k-i-1}\colon B^{k-i-1}(M)\to\mathbf{P}H^0(M)\big\}_{k=i+1}^{n}\,.$$

Blowing up one more time, we can argue as in the proof of Claim 7.4 that the fiber of $F_{i,n+1}^\circ=F_{i,n+1}-\bigcup_{h<i}F_{h,n+1}$ over D is $\mathrm{bl}_{n-i}\,\mathbf{P}H^0(\mathbf{P}^1,M)$. Let k be an integer such that $i+1\leqslant k\leqslant n$; Claim 7.4 implies that

$$F_{i,k}^\circ\cap\mathrm{bl}_k(C_{2k})=E_{i,k}^\circ\subseteq\mathrm{bl}_k(\mathrm{Jac}(C))$$

is a U_{2i}-variety whose fiber over D is $\mathrm{bl}_{k-i-1}\,B^{k-i-1}(M)$. Then in the final blowup space,

$$F_{i,n+1}^\circ\cap F_{k,n+1}\subseteq\mathrm{bl}_{n+1}(\mathrm{Jac}(C))$$

is a U_{2i}-variety whose fiber over D is the exceptional divisor

$$H_{k-i-1}\subseteq\mathrm{bl}_{n-i}\,\mathbf{P}H^0\big(\mathbf{P}^1,M\big)$$

over $\mathrm{bl}_{k-i-1}\,B^{k-i-1}(M)$, associated with the chain $\{\beta_{k-i-1}\}_{k=i+1}^{n}$.

As $\{\beta_{k-i-1}\}_{k=i+1}^{n}$ is a smooth chain by Corollary 4.4(b), Remark 2.7 implies that the fiber of

$$Z_i-\bigcup_{\substack{0\leqslant j\leqslant n-1\\ j\neq i}}Z_j-\tilde\Theta=F_{i,n+1}^\circ-\bigcup_{i+1\leqslant k\leqslant n}F_{k,n+1}$$

over D is

$$\mathrm{bl}_{n-i}\,\mathbf{P}H^0\big(\mathbf{P}^1, M\big) - \bigcup_{i+1\leqslant k\leqslant n} H_{k-i-1} \cong \mathbf{P}H^0\big(\mathbf{P}^1, M\big) - \beta_{n-i-1}\big(B^{n-i-1}(M)\big)\,.$$

By definition, the image $\beta_{n-i-1}(B^{n-i-1}(M))$ is the $(n-i-1)$th secant variety

$$\mathrm{Sec}^{n-i-1}\big(\mathbf{P}^1\big) \subseteq \mathbf{P}H^0\big(\mathbf{P}^1, M\big) = \mathbf{P}^{2(n-i)}\,,$$

where $\mathbf{P}^1$ is the rational normal curve $\beta_0(B^0(M))$. Note that the secant variety $\mathrm{Sec}^{n-i-1}(\mathbf{P}^1)$ has degree $(n-i)+1$ and the rational curve has degree $2(n-i)$. Taking $i = n-r$, we obtain Corollary B. □

The following result is an (almost immediate) consequence of the proof above.

Corollary 7.6. *Let $L \in W^r_{g-1}(C) - W^{r+1}_{g-1}(C)$ for some $0 \leqslant r \leqslant n$ so that $L = \delta_{n-r}(D)$ for a unique $D \in U_{2(n-r)}$. Set $M = \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)$; then the fiber of the map*

$$\mathrm{bl}_{n+1}(\mathrm{Jac}(C)) \longrightarrow \mathrm{Jac}(C)$$

over L is $\mathrm{bl}_r\,\mathbf{P}H^0(\mathbf{P}^1, M)$. In particular, the fiber of the map $\pi_n\colon \mathrm{bl}_n(\mathrm{Jac}(C)) \to \mathrm{Jac}(C)$ in Theorem A over L is $\mathrm{bl}_r\,\mathbf{P}H^0(\mathbf{P}^1, M)$, using Bertram's notation from Section 2.

Proof. Set $i = n-r$. The fiber we are interested in is the fiber of $F_{i,n+1}$ over D, using the notation above. From the proof of Corollary B, we know that the fiber of

$$F^\circ_{i,n+1} = F_{i,n+1} - \bigcup_{h<i} F_{h,n+1}$$

over D is $\mathrm{bl}_{n-i}\,\mathbf{P}H^0(\mathbf{P}^1, M)$. We claim that the fiber of $F_{i,n+1}$ over D is the same as the fiber of $F^\circ_{i,n+1}$ over D. The reason is as follows. Let $h < i$; then (7.2) gives $\mathrm{bl}_i(C_{2i}) - \bigcup_{h<i} F_{h,i} = U_{2i}$. Hence the image of $U_{2i} \hookrightarrow \mathrm{bl}_i(C_{2i}) \hookrightarrow \mathrm{bl}_i(\mathrm{Jac}(C))$ is away from $F_{h,i}$. As $F_{h,n+1}$ is the exceptional divisor over $F_{h,i}$, we conclude that the fiber of $F_{h,n+1}$ over D is empty. This proves the desired result.

The last statement follows from the fact that the image of $\mathrm{bl}_n(C_{2n})$ in $\mathrm{bl}_n(\mathrm{Jac}(C))$ is a divisor, so $\mathrm{bl}_{n+1}(\mathrm{Jac}(C)) = \mathrm{bl}_n(\mathrm{Jac}(C))$. □

8. Even-genus case

In this section, let C be a smooth hyperelliptic curve of even genus $g = 2n+2$. We sketch a proof of Theorem A for C. The ideas are essentially the same, by reducing to the calculation of conormal bundles, but for parity reasons, the corresponding maps need some modification. First, we have a chain of maps $\{\delta_j\}_{j=0}^n$ to $\mathrm{Jac}(C)$, where for $0 \leqslant j \leqslant n$,

$$\begin{aligned} \delta_j\colon C_{2j+1} &\longrightarrow \mathrm{Pic}^{g-1}(C) = \mathrm{Jac}(C)\,,\\ D &\longmapsto (n-j)g^1_2 \otimes \mathcal{O}_C(D)\,. \end{aligned}$$

The image $\delta_j(C_{2j+1})$ is W^{n-j}_{g-1}; hence this is a proper chain. By the Abel–Jacobi theorem, for each $\ell \geqslant 1$, we have $\mathbf{P}^\ell \subseteq C_{2\ell}$. Then for each $j \geqslant 1$, there is a proper chain of maps $\{\gamma_{i,j}\}_{i=0}^{j-1}$ induced by the addition maps:

$$\gamma_{i,j}\colon C_{2i+1} \times \mathbf{P}^{j-i} \lhook\joinrel\longrightarrow C_{2i+1} \times C_{2j-2i} \longrightarrow C_{2j+1}\,, \quad \forall 0 \leqslant i < j\,.$$

The even-genus case of Theorem A is reduced to the following analog of Propositions 3.1 and 3.2.

Proposition 8.1. (a) *For each* $1 \leqslant j \leqslant n$, *the chain*

$$\left\{\gamma_{i,j} \colon C_{2i+1} \times \mathbf{P}^{j-i} \to C_{2j+1}\right\}_{i=0}^{j-1}$$

is an NCD chain, and for each $1 \leqslant i < j$, *the map* $\gamma_{i,j}$ *is a map of chains of centers.*

(b) *The chain* $\{\delta_j \colon C_{2j+1} \to \mathrm{Jac}(C)\}_{j=0}^{n}$ *is an NCD chain, and for each* $1 \leqslant j \leqslant n$, *the map* $\delta_j \colon C_{2j+1} \to \mathrm{Jac}(C)$ *is a map of chains of centers.*

Like the proofs of Propositions 3.1 and 3.2, the proof of Proposition 8.1 relies on a parallel statement for Abel–Jacobi maps and conormal bundles as in Lemmas 5.8 and 5.3. We will only give the statements and leave the proofs to the interested reader.

For each $j \geqslant 0$, set

$$U_{2j+1} := C_{2j+1} - \gamma_{j-1,j}\left(C_{2j-1} \times \mathbf{P}^1\right).$$

Note that for $j = 0$, we have $U_1 = C$. Any divisor $D \in U_{2j+1}$ gives a degree $2j+1$ divisor on $\mathbf{P}^1$ via the hyperelliptic map $C \to \mathbf{P}^1$, which is denoted by h_*D, and the associated line bundle is

$$\mathcal{O}_{\mathbf{P}^1}(g - 1 - h_*D) := \mathcal{O}_{\mathbf{P}^1}(g-1) \otimes \mathcal{O}_{\mathbf{P}^1}(-h_*D).$$

Lemma 8.2. *Let* C *be a hyperelliptic curve of genus* $g = 2n+2$. *Then*

(a) *for any* $0 \leqslant i < j$, *the map* $\gamma_{i,j} \colon C_{2i+1} \times \mathbf{P}^{j-i} \to C_{2j+1}$ *is an embedding when restricting to* $U_{2i+1} \times \mathbf{P}^{j-i}$, *and for* $0 \leqslant \ell < i < j$, *we have*

$$\gamma_{i,j}^{-1}\left(U_{2\ell+1} \times \mathbf{P}^{j-\ell}\right) = U_{2\ell+1} \times \mathbf{P}^{i-\ell} \times \mathbf{P}^{j-i};$$

(b) *for* $0 \leqslant j \leqslant n$, *the map* $\delta_j \colon C_{2j+1} \to \mathrm{Jac}(C)$ *is an embedding when restricting to* U_{2j+1}, *and for* $0 \leqslant i < j$, *we have*

$$\delta_j^{-1}(U_{2i+1}) = U_{2i+1} \times \mathbf{P}^{j-i}.$$

In particular, since $U_1 = C$, *we have*

$$\delta_j^{-1}(C) = C \times \mathbf{P}^j \subseteq C_{2j+1}.$$

Lemma 8.3. *For* $0 \leqslant i < j$, *consider the map* $\gamma_{i,j} \colon C_{2i+1} \times \mathbf{P}^{j-i} \to C_{2j+1}$, *and let* $D \in U_{2i+1}$. *Then*

(a) $d\gamma_{i,j} \colon \gamma_{i,j}^* T^*_{C_{2j+1}} \to T^*_{C_{2i+1} \times \mathbf{P}^{j-i}}$ *is surjective when restricted to* $U_{2i+1} \times \mathbf{P}^{j-i}$;

(b) $\mathbf{P}N^*_{\gamma_{i,j}}|_{U_{2i+1} \times \mathbf{P}^{j-i}}$ *is a smooth variety over* U_{2i+1} *such that over* $D \in U_{2i+1}$ *we have an isomorphism*

$$\mathbf{P}N^*_{\gamma_{i,j}}|_{\{D\} \times \mathbf{P}^{j-i}} \cong B^{j-i-1}(\mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)).$$

Furthermore, for $\ell < i$, *consider the commutative diagram*

$$\begin{array}{ccc} \left(C_{2\ell+1} \times \mathbf{P}^{i-\ell}\right) \times \mathbf{P}^{j-i} & \xrightarrow{\mathrm{id} \times r} & C_{2\ell+1} \times \mathbf{P}^{j-\ell} \\ \downarrow{\scriptstyle \gamma_{\ell,i} \times \mathrm{id}} & & \downarrow{\scriptstyle \gamma_{\ell,j}} \\ C_{2i+1} \times \mathbf{P}^{j-i} & \xrightarrow{\gamma_{i,j}} & C_{2j+1}. \end{array}$$

Here r *is the addition map. For any* $D \in U_{2\ell+1}$, *there is an induced map of conormal bundles on* $\{D\} \times \mathbf{P}^{i-\ell} \times \mathbf{P}^{j-i}$:

$$\epsilon \colon (\mathrm{id} \times r)^* N^*_{\gamma_{\ell,j}}|_{\{D\} \times \mathbf{P}^{j-\ell}} \longrightarrow N^*_{(\gamma_{\ell,i} \times \mathrm{id})}|_{\{D\} \times \mathbf{P}^{i-\ell} \times \mathbf{P}^{j-i}}.$$

Then

(c) ϵ *is surjective;*

(d) *the following diagram commutes:*

$$\begin{array}{ccc} \mathbf{P}N^*_{(\gamma_{\ell,i}\times \mathrm{id})}|_{\{D\}\times \mathbf{P}^{i-\ell}\times \mathbf{P}^{j-i}} & \xrightarrow{\alpha} & \mathbf{P}N^*_{\gamma_{\ell,i}}|_{\{D\}\times \mathbf{P}^{j-\ell}} \\ \downarrow \cong & & \downarrow \cong \\ B^{i-\ell-1}(M)\times \mathbf{P}^{j-i} & \xrightarrow{\alpha_{i-\ell-1,j-\ell-1}} & B^{j-\ell-1}(M)\,. \end{array}$$

Here α is the map induced by ϵ composed with a projection to $\mathbf{P}N^*_{\gamma_{\ell,j}}(D\times \mathbf{P}^{j-\ell})$, and $\alpha_{i-\ell-1,j-\ell-1}$ is the map (4.3) for the curve $\mathbf{P}^1$ and the line bundle $M = \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)$.

LEMMA 8.4. *Keep the notation of Lemma 8.3. For $0 \leqslant j \leqslant n$, consider the map $\delta_j \colon C_{2j+1} \to \mathrm{Jac}(C)$. Then*

(a) $d\delta_j \colon \delta_j^* T^*_{\mathrm{Jac}(C)} \to T^*_{C_{2j+1}}$ *is surjective when restricted to U_{2j+1};*

(b) *the fiber of $N^*_{\delta_j}$ over $D \in U_{2j+1}$ is $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D))$.*

Furthermore, for $i < j$, consider the diagram

$$\begin{array}{ccc} C_{2i+1}\times \mathbf{P}^{j-i} & \xrightarrow{p_1} & C_{2i+1} \\ \downarrow \gamma_{i,j} & & \downarrow \delta_i \\ C_{2j+1} & \xrightarrow{\delta_j} & \mathrm{Jac}(C)\,. \end{array}$$

Then for $D \in U_{2i+1}$, we get the induced map of conormal bundles over $\{D\}\times \mathbf{P}^{j-i}$

$$\epsilon \colon p_1^* N^*_{\delta_i}|_D \longrightarrow N^*_{\gamma_{i,j}}|_{\{D\}\times \mathbf{P}^{j-i}}\,,$$

and

(c) ϵ *is surjective;*

(d) *the following diagram commutes:*

$$\begin{array}{ccc} \mathbf{P}N^*_{\gamma_{i,j}}|_{\{D\}\times \mathbf{P}^{j-i}} & \xrightarrow{\beta} & \mathbf{P}N^*_{\delta_i}|_D \\ \downarrow \cong & & \downarrow \cong \\ B^{j-i-1}(M) & \xrightarrow{\beta_{j-i-1}} & \mathbf{P}H^0\big(\mathbf{P}^1, M\big)\,. \end{array}$$

The first vertical isomorphism comes from Lemma 8.3(b) and β_{j-i-1} is the map (4.1) for the curve $\mathbf{P}^1$ and the line bundle $M = \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)$. The map β is the map induced from ϵ composed with a projection to $\mathbf{P}N^*_{\delta_i}|_D$.

9. Brill–Noether stratifications are Whitney

In this section, let C be a smooth projective hyperelliptic curve of genus g. We show that the Brill–Noether stratification of $\mathrm{Jac}(C)$ determined by

$$\mathrm{Jac}(C) \supseteq \Theta = W_{g-1}(C) \supseteq W^1_{g-1}(C) \supseteq \cdots \supseteq W^n_{g-1}(C)$$

is a Whitney stratification, where $n = \lfloor (g-1)/2 \rfloor$. We will assume $g = 2n+1$; the even-genus case is similar.

9.1 Whitney stratifications

Let Z be a smooth real manifold, and let $X, Y \subseteq Z$ be two embedded smooth real submanifolds such that X and Y are from a stratification of Z. Suppose $Y \subseteq \overline{X}$, where the closure is taken inside Z with respect to the Euclidean topology.

Definition 9.1. We say that the pair (X, Y) satisfies the *Whitney conditions* if for any point $y \in Y$, the following two conditions hold:

(A) If $\{x_i\} \subseteq X$ is a sequence of points converging to y, and if the sequence of tangent spaces $T_{x_i}X$ converges to a linear space T of the same dimension, then $T_yY \subseteq T$.

(B) If $\{x_i\} \subseteq X$ and $\{y_i\} \subseteq Y$ are two sequences of points that both converge to y, if the sequence of real secant lines between x_i and y_i converges to a real line L, and if the sequence of tangent spaces $T_{x_i}X$ converges to a linear subspace T of the same dimension, then $L \subseteq T$.

The Whitney condition (B) involves real secant lines (in local coordinates) and is therefore not so easy to verify in practice. Instead, in the case of complex algebraic varieties, there is a condition (W) introduced by Kuo [Kuo71] and Verdier [Ver76], which implies the Whitney conditions and is easier to work with in our situation. It is proved by Teissier that, for complex analytic stratifications, the Whitney conditions are equivalent to condition (W), but we will not need this fact.

Definition 9.2 (Distance). Let V be a complex vector space, and let $A, B \subseteq V$ be two linear subspaces. Fix an inner product $(-,-)$ on V. The *distance from A to B* is defined to be

$$d(A, B) := \sup_{\substack{a \in A, \\ a \neq 0}} \inf_{\substack{b \in B, \\ b \neq 0}} \sin \theta(a, b) \,.$$

Here $\theta(a, b)$ is the angle between two vectors a, b determined by the inner product $(-,-)$.

Here are some basic properties of $d(A, B)$. Note that it is *not* symmetric in A and B (which is why we do not call it the distance "between" A and B).

Fact 9.3. (a) We have $d(A, B) = 0$ if and only if $A \subseteq B$.

(b) Let $A \subseteq A'$ be two subspaces; then $d(A, B) \leqslant d(A', B)$.

(c) Identify V with the conjugate dual space V^* via the inner product $(-,-)$ so that the kernel $\operatorname{Ker}(V^* \to B^*)$ is identified with the orthogonal complement $B^\perp$. Then

$$d(\operatorname{Ker}(V^* \to B^*), \operatorname{Ker}(V^* \to A^*)) = d\big(B^\perp, A^\perp\big) = d(A, B) \,.$$

After choosing an orthonormal basis, this comes down to the fact that a linear operator and its adjoint (between two finite-dimensional Hilbert spaces) have the same operator norm.

From now on, let Z be a complex manifold. Let X, Y be two embedded smooth complex submanifolds of Z such that $Y \subseteq \overline{X}$.

Definition 9.4. We say that the pair (X, Y) satisfies *condition* (W) if for any point $y \in Y$, and for any sequence of points $\{x_i\} \subseteq X$ converging to y, there exists a constant $C > 0$ such that

$$d(T_yY, T_{x_i}X) \leqslant C \cdot d(y, x_i) \,, \quad \forall i \,,$$

where we view $T_{x_i}X$ as a subspace of T_yZ using a local trivialization of the tangent bundle T_Z, and $d(y, x_i)$ is the Euclidean distance between y and x_i in a local coordinate chart.

Kuo [Kuo71] (see also Verdier [Ver76, Théorème 1.5]) proved the following.

Theorem 9.5. *Let Z be a complex manifold. Let X, Y be two embedded smooth complex submanifolds of Z such that $Y \subseteq \overline{X}$ and X, Y are from a stratification of Z. If the pair (X, Y) satisfies condition (W), then the pair (X, Y) satisfies the Whitney conditions (A) and (B).*

We are going to use this result in the following form.

Lemma 9.6. *Keep the same assumptions as above. Assume that the pair (X, Y) satisfies the Whitney condition (A). Let $y \in Y$ be an arbitrary point, and let $\{x_i\} \subseteq X$ be a sequence of points converging to y such that $T = \lim_{i\to\infty} T_{x_i}X$ exists. Suppose that there is a constant $C > 0$ (which is allowed to depend on y and x_i) such that*

$$d(T, T_{x_i}X) \leqslant C \cdot d(y, x_i)\,, \quad \forall i\,;$$

an equivalent formulation is that there is a constant $C > 0$ such that

$$d((N^*_{X|Z})_{x_i}, \lim_{i\to\infty}(N^*_{X|Z})_{x_i}) \leqslant C \cdot d(y, x_i)\,, \quad \forall i\,,$$

where $N^*_{X|Z}$ denotes the conormal bundle of X inside Z. Then the pair (X, Y) satisfies the Whitney condition (B).

Proof. By the Whitney condition (A), we know that $T_yY \subseteq T$. By property (b) of the distance function in Fact 9.3, we conclude that

$$d(T_yY, T_{x_i}X) \leqslant d(T, T_{x_i}X) \leqslant C \cdot d(y, x_i)\,.$$

This verifies condition (W) and thus gives the Whitney condition (B) by Theorem 9.5. The last statement uses the property (c) of the distance function in Fact 9.3. □

Definition 9.7. Let X be a complex algebraic variety, and suppose that there is a finite algebraic stratification

$$X = \bigsqcup S_i$$

by connected algebraic varieties whose irreducible components are smooth. We say that this is a Whitney stratification if for any $S_j \subseteq \overline{S_i}$, the pair (S_i, S_j) satisfies the Whitney conditions (A) and (B).

9.2 Brill–Noether stratifications are Whitney

Recall that C is a genus $2n+1$ smooth hyperelliptic curve. For each $0 \leqslant r \leqslant n$, set

$$W^r_{g-1}(C)^\circ := W^r_{g-1}(C) - W^{r+1}_{g-1}(C)\,;$$

this is a connected smooth algebraic variety and parametrizes degree $g-1$ line bundles with exactly $r+1$ independent sections. The subvariety $\mathrm{Jac}(C) - \Theta$ is also smooth and parametrizes degree $g-1$ lined bundles with no sections. The Brill–Noether stratification of $\mathrm{Jac}(C)$ is defined to be

$$\mathrm{Jac}(C) = (\mathrm{Jac}(C) - \Theta) \sqcup \bigsqcup_{0\leqslant r\leqslant n} W^r_{g-1}(C)^\circ\,.$$

Proposition 9.8. *The Brill–Noether stratification of* $\mathrm{Jac}(C)$ *is a Whitney stratification.*

Proof. Note that $\overline{W^i_{g-1}(C)^\circ} = W^i_{g-1}(C)$ and for $i < j$, we have

$$W^j_{g-1}(C)^\circ \subseteq W^j_{g-1}(C) \subseteq W^i_{g-1}(C)\,.$$

We also have $\overline{\mathrm{Jac}(C) - \Theta} = \mathrm{Jac}(C)$. By Definition 9.7, it suffices to show that for each $i < j$, the pair $(W^i_{g-1}(C)^\circ, W^j_{g-1}(C)^\circ)$ satisfies the Whitney conditions, and the same holds for the pair $(\mathrm{Jac}(C) - \Theta, W^i_{g-1}(C)^\circ)$. To apply Lemma 9.6, we need to understand the conormal bundles of the Brill–Noether strata. Recall that for each $0 \leqslant r \leqslant n$, the Abel–Jacobi map $\delta_{(g-1-2r)/2} = \delta_{n-r}$ induces an isomorphism

$$\delta_{(g-1-2r)/2} \colon U_{g-1-2r} \xrightarrow{\sim} W^r_{g-1}(C)^\circ , \quad D \longmapsto \mathcal{O}_C(D) \otimes r g^1_2 , \tag{9.1}$$

where U_{g-1-2r} is defined in Notation 5.1 and is the open subset of C_{g-1-2r} consisting of divisors D such that $h^0(C, \mathcal{O}_C(D)) = 1$. By Lemma 5.6, for any $D \in U_{g-1-2r}$ and $L := \mathcal{O}_C(D) \otimes r g^1_2$, we have

$$\big(N^*_{W^r_{g-1}(C)^\circ | \, \mathrm{Jac}(C)}\big)_L = \big(N^*_{\delta_{(g-1-2r)/2}}\big)_D = H^0(C, \omega_C(-D)) \cong H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)\big) ,$$

where the last isomorphism is induced by $h \colon C \to \mathbf{P}^1$, the hyperelliptic map determined by the unique g^1_2 and h_*D is the degree $g-1-2r$ divisor defined in Notation 5.2.

For each $i < j$, let $\{L_k\} \subseteq W^i_{g-1}(C)^\circ$ be a sequence of line bundles converging to an element $L \in W^j_{g-1}(C)^\circ$. Using the isomorphism (9.1), we can write

$$L_k = \mathcal{O}_C(D_k) \otimes i g^1_2 \quad \text{and} \quad L = \mathcal{O}_C(D) \otimes j g^1_2$$

with $D_k \in U_{g-1-2i}$ and $D \in U_{g-1-2j}$. From the discussion above, we know that

$$\begin{aligned}\big(N^*_{W^i_{g-1}(C)^\circ | \, \mathrm{Jac}(C)}\big)_{L_k} &= H^0(C, \omega_C(-D_k)) \cong H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D_k)\big) , \\ \big(N^*_{W^j_{g-1}(C)^\circ | \, \mathrm{Jac}(C)}\big)_{L} &= H^0(C, \omega_C(-D)) \cong H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)\big) .\end{aligned}$$

If we let $\overline{D} := \lim_{k\to\infty} D_k \in C_{g-1-2i}$ be the limit divisor, then since $\lim_{k\to\infty} L_k = L$, we see that D is an effective subdivisor of $\overline{D}$. Therefore,

$$\begin{aligned}\lim_{k\to\infty} \big(N^*_{W^i_{g-1}(C)^\circ | \, \mathrm{Jac}(C)}\big)_{L_k} &= H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}\big(g-1-h_*\overline{D}\big)\big) \\ &\subseteq H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D)\big) = \big(N^*_{W^j_{g-1}(C)^\circ | \, \mathrm{Jac}(C)}\big)_{L} ,\end{aligned}$$

where the first equality uses the fact that $H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(k)) = 0$ for any $k \geqslant 0$ and hence we can take limits. This verifies the Whitney condition (A) for the pair $(W^i_{g-1}(C)^\circ, W^j_{g-1}(C)^\circ)$, by going to the dual spaces. Now by Lemma 9.6, in order to prove the Whitney condition (B), we just need to show that there exists a constant A such that

$$d\big(H^0\big(C, \omega_C\big(-\overline{D}\big)\big), H^0(C, \omega_C(-D_k))\big) \leqslant A \cdot d(L, L_k) = A \cdot d\big(\overline{D}, D_k\big) ,$$

where the distance function on the left is induced by an inner product on the vector space $H^0(C, \omega_C)$ and the distance functions on the right are induced by the Euclidean norm on a neighborhood of $\overline{D}$ in C_{g-1-2i} and a neighborhood of L in $\mathrm{Jac}(C)$, respectively. The last equality comes from the fact that $D_k \in U_{g-1-2r}$ and $\delta_{(g-1-2r)/2}$ is an isomorphism over U_{g-1-2r}. Since the hyperelliptic map $h \colon C \to \mathbf{P}^1$ is either a local isomorphism (off the branch locus) or locally of the form $t \mapsto t^2$ (on the branch locus), we can push everything down to $\mathbf{P}^1$; there, it suffices to prove that there exists a constant A such that

$$d\big(H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}\big(g-1-h_*\overline{D}\big)\big), H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D_k)\big)\big) \leqslant A \cdot d\big(h_*\overline{D}, h_*D_k\big) ,$$

which follows from the interpretation of the space $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1-h_*D))$ as the space of degree $g-1$ homogeneous polynomials vanishing along the divisor h_*D and explicit computations.[1]

[1]Botong Wang pointed out that one can view this as a Lipschitz property of the map between compact manifolds

For the pair $(\mathrm{Jac}(C)-\Theta, W^i_{g-1}(C)^\circ)$, Condition ($W$) is vacuous because $\mathrm{Jac}(C)$ is a complex manifold (using the property (a) of the distance function in Fact 9.3). ☐

10. Questions and open problems

The log resolution of the hyperelliptic theta divisor is rather intricate. To have a better understanding of it, we ask the following.

Question 10.1. Is there a modular interpretation of the log resolution in Theorem A?

Remark 10.2. Let C be a Brill–Noether general curve. The Brill–Noether varieties $W^r_{g-1}(C)$ behave like generic determinantal varieties. It is natural to ask whether an extension of our results Theorem A and Proposition 1.1 hold for such a curve C. This problem has been solved by Budur in [Bud25].

Appendix. Reducedness of $W^r_d(C)$

In this appendix, we provide the proof of Proposition A.3. This result is not needed for the paper because in Theorem A we can always take $W^r_d(C)$ with its induced reduced structures. While this result is not essential, we include it here in response to a question posed by Budur [BD23].

Let C be a smooth projective curve. Let r, d be nonnegative integers. Set-theoretically, $W^r_d(C)$ is the space of line bundles $L \in \mathrm{Pic}^d(C)$ such that $h^0(L) \geqslant r+1$. Its scheme structure can be defined as follows, which is a variant of the method in [ACGH85, § IV.3]. Let $L \in \mathrm{Pic}^d(C)$ be a closed point. As in the proof of Lemma A.5 below, one can produce a vector bundle map $A\colon E^0 \to E^1$ over $\mathrm{Pic}^d(C)$, where E^0, E^1 are vector bundles of rank $h^0(L)$ and $h^1(L)$ (see (A.1)). Then $W^r_d(C)$ is defined, in a neighborhood of L, as the $(h^0(L)-r)$th determinantal variety associated with A; equivalently, it is cut out by all $(h^0(L)-r+1)\times(h^0(L)-r+1)$ minors of A.

Proposition A.3. *Let C be a smooth hyperelliptic curve of genus g. Let $d, r \in \mathbb{N}$ be integers such that $0 \leqslant r \leqslant d \leqslant g$. Then the Brill–Noether variety $W^r_d(C)$ is reduced.*

We recall the following result saying that reducedness can be checked on the level of tangent cones.

Lemma A.4. *Let $Z \subseteq X$ be a closed subscheme of a smooth variety X and $x \in Z$ be a closed point. If the tangent cone $TC_xZ \subseteq T_xX$ is reduced, then Z is reduced in an open neighborhood of x.*

Proof. Equip the tangent cone TC_xZ with its natural scheme structure; then there is a flat specialization of (a neighborhood of x in) Z to TC_xZ. The desired result follows from the fact that reducedness is an open condition in flat families; cf. [GD66, Theorem 12.1.1(vii)]. ☐

By Lemma A.4, Proposition A.3 is reduced to the following.

Lemma A.5. *The tangent cone $TC_LW^r_d(C)$ is reduced for any $L \in W^r_d(C)$.*

Proof. To simplify the notation, we set $W^r_d = W^r_d(C)$. Fix $\mathcal{L}$ a Poincaré line bundle on the product $C \times \mathrm{Pic}^d(C)$, and let $\mathrm{pr}_2\colon C\times\mathrm{Pic}^d(C) \to \mathrm{Pic}^d(C)$ be the second projection. Let $L \in W^r_d$ be a line bundle of degree d, and assume $h^0(L) = s+1$ with $s \geqslant r$. In a neighborhood of L

$\mathrm{Sym}^{g-1-2i}\,\mathbf{P}^1 \to \mathrm{Grass}(H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1)), 2i)$ which sends E to $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(g-1-E))$.

in $\mathrm{Pic}^d(C)$, we can produce a minimal complex computing W^r_d, by a variant of the method in [ACGH85, § IV.3]. Note that we can always pick a point $p \in C$ such that $h^1(L(p)) = h^1(L) - 1$ and $H^0(L) = H^0(L(p))$. Iterating this, we can pick an effective divisor D of degree $h^1(L) = g - d + s$ with the property that $H^1(L(D)) = 0$ and that in the short exact sequence

$$0 \longrightarrow L \longrightarrow L(D) \longrightarrow L(D) \otimes \mathcal{O}_D \longrightarrow 0\,,$$

the induced connecting map $H^0(D, L(D) \otimes \mathcal{O}_D) \to H^1(C, L)$ is an isomorphism (equivalently, $H^0(C, L) \to H^0(C, L(D))$ is an isomorphism). Denote by $\mathcal{D} = \mathrm{pr}_2^* D$ the effective divisor on $C \times \mathrm{Pic}^d(C)$. Then on some neighborhood of the point L, we have a short exact sequence

$$0 \longrightarrow \mathrm{pr}_{2,*}\mathcal{L}(\mathcal{D}) \longrightarrow \mathrm{pr}_{2,*}(\mathcal{L}(\mathcal{D}) \otimes \mathcal{O}_{\mathcal{D}}) \longrightarrow R^1\mathrm{pr}_{2,*}\mathcal{L} \longrightarrow 0\,,$$

where $R^1\mathrm{pr}_{2,*}\mathcal{L}(\mathcal{D})$ vanishes on the neighborhood in question. Here $\mathrm{pr}_{2,*}\mathcal{L} = 0$ because it is torsion-free and vanishes at a general point in the neighborhood of L. This gives us a presentation

$$0 \longrightarrow E^0 \xrightarrow{A} E^1 \longrightarrow R^1\mathrm{pr}_{2,*}\mathcal{L} \longrightarrow 0\,, \tag{A.1}$$

where E^0 and E^1 are vector bundles of rank $h^0(L) = s + 1$, respectively $h^1(L) = g - d + s$. Moreover, the differential, viewed as a matrix A, vanishes at the point L. Let A_1 be the linear part of A; it has entries in $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal at L.

Now the Brill–Noether variety W^r_d is defined, in a neighborhood of the point L, by the vanishing of all the $(s - r + 1) \times (s - r + 1)$ minors of A because for $L' \in W^r_d$, the condition is

$$\begin{aligned} h^0(L') \geqslant r + 1 &\iff h^1(L') \geqslant g - d + r \\ &\iff \mathrm{rank}(A)_{L'} \leqslant (g - d + s) - (g - d + r) = (s - r)\,. \end{aligned}$$

It follows from the tangent cone theorem in generic vanishing theory (cf. [GL87, Theorem 4]) that one has the following containments:

$$\mathcal{I}_1 \subseteq \mathcal{I}_{TC_L W^r_d} \subseteq \sqrt{\mathcal{I}_{TC_L W^r_d}}\,,$$

where the first ideal is generated by all the $(s-r+1) \times (s-r+1)$ minors of A_1. If one knows that the first ideal $\mathcal{I}_1$ is a radical ideal, and that both $\mathcal{I}_1$ and $\sqrt{\mathcal{I}_{TC_L W^r_d}}$ define the same conical subset in $T_L \mathrm{Pic}^d(C)$ (forgetting about the scheme structure), then the tangent cone $TC_L W^r_d$ is reduced.

Since we have $W^d_r \cong W_{d-2r}$ as sets, one has

$$\dim TC_L W^r_d = \dim W^r_d = d - 2r\,.$$

By the discussion above, it suffices to show that the $(s - r + 1) \times (s - r + 1)$ minors of the matrix A_1 define a reduced, irreducible subscheme of $T_L \mathrm{Pic}^d(C)$ of dimension $d - 2r$. This boils down to the following two claims.

Claim A.6. *The matrix A_1 is a Hankel/Catalecticant matrix, that is,*

$$A_1 = \begin{pmatrix} x_1 & x_2 & \cdots & x_{g-d+s} \\ x_2 & x_3 & \cdots & x_{g-d+s+1} \\ \cdots & \cdots & \cdots & \cdots \\ x_{s+1} & \cdots & \cdots & x_{g-d+2s} \end{pmatrix}$$

up to a change of local coordinates.

Proof. We learned this argument from Nero Budur; see [BD23, Proposition 8.17]. By [ACGH85], the matrix A_1 is the one given by the map $H^0(L) \to H^1(L) \otimes H^0(\omega_C)$, which is equivalent to the Petri map

$$\pi_L \colon H^0(L) \otimes H^0\big(\omega_C \otimes L^{-1}\big) \longrightarrow H^0(\omega_C)\,.$$

Since C is hyperelliptic and $h^0(L) = s+1$, we can write

$$L = s g^1_2 + p_1 + \cdots + p_{d-2s}\,,$$
$$\omega_C \otimes L^{-1} = (g-1-s-(d-2s))g^1_2 + q_1 + \cdots + q_{d-2s}\,,$$

where $p_i + q_i$ is a hyperelliptic pair for each $1 \leqslant i \leqslant d-2s$ and no two p_i lie in the same fiber of the hyperelliptic involution $C \to \mathbf{P}^1$. Then the Petri map corresponds to

$$H^0\big(\mathbf{P}^1, \mathcal{O}(s)\big) \otimes H^0\big(\mathbf{P}^1, \mathcal{O}(g-1-d+s)\big) \longrightarrow H^0\big(\mathbf{P}^1, \mathcal{O}(g-1-d+2s)\big) \longrightarrow H^0\big(\mathbf{P}^1, \mathcal{O}(g-1)\big)\,.$$

The last map is the tensor product with the section $\eta \in H^0(\mathbf{P}^1, \mathcal{O}(d-2s))$, where η is the product of all linear forms defining the image of p_i in $\mathbf{P}^1$ for $1 \leqslant i \leqslant d-2s$. Write $V = H^0(\mathbf{P}^1, \mathcal{O}(1))$, then the Petri map is the natural multiplication map

$$\operatorname{Sym}^s V \otimes \operatorname{Sym}^{g-1-d+s} V \longrightarrow \operatorname{Sym}^{g-1-d+2s} V\,,$$

which clearly gives a Catalecticant matrix since $\dim V = 2$. □

Claim A.7. *Let $C_{v,w}$ be a $v \times w$ Catalecticant matrix with $v \geqslant w$, that is,*

$$C_{v,w} = \begin{pmatrix} x_1 & x_2 & \cdots & x_w \\ x_2 & x_3 & \cdots & x_{w+1} \\ \cdots & \cdots & \cdots & \cdots \\ x_v & \cdots & \cdots & x_{v+w-1} \end{pmatrix}.$$

Then for $k < w$, the ideals of $(k+1)\times(k+1)$ minors of $C_{v,w}$ define a reduced irreducible subscheme Z of dimension $2k$ in $\mathbb{C}^{v+w-1}$.

Proof. We use notation from [Eis88]. Let $M = \operatorname{Cat}(v,w) \subseteq \mathbf{P}\mathbb{C}^{vw}$ be the Catalecticant space, which is of dimension $v+w-2$ (cf. [Eis88, § 4, p. 561]). Let M_k be the subscheme of matrices of rank at most k in M; this linear space corresponds to all the minors of $C_{v,w}$ of size $k+1$. By [Eis88, Proposition 4.3], one has

$$\operatorname{codim}_M M_k = v+w-1-2k\,,$$

and M_k is the k-secant variety of a rational normal curve. Thus

$$\dim M_k = \dim M - (v+w-1-2k) = (v+w-2) - (v+w-1-2k) = 2k-1\,,$$

and M_k is irreducible. Moreover, Eisenbud [Eis88, Proposition 4.3 and after] observes that M_k is reduced. Therefore, the corresponding space $Z \subseteq \mathbb{C}^{v+w-1}$ is reduced, irreducible and has dimension $2k$. □

Now we can finish the proof of this lemma. If $d \geqslant g-1$, then $s-r < s+1 \leqslant g-d+s$; if $d = g$, then we can assume $r \geqslant 1$ (note that $W^0_g(C)$ is reduced by a theorem of Kempf) and still get $s-r < s = g-d+s \leqslant s+1$. Therefore, we can apply Claims A.6 and A.7 to obtain that the $(s-r+1)\times(s-r+1)$ minors of the matrix A_1 define a reduced, irreducible subscheme in $T_L \operatorname{Pic}^d(C) = \mathbb{C}^g$ of dimension $2(s-r)+(d-2s)$ (because only the variables $x_1, \dots, x_{g-d+2s}$ show up in the matrix A_1, and the other variables provide an additional $d-2s$ dimensions). This gives what we want, and therefore we have finished the proof that $TC_L W^r_d$ is reduced.

As a consequence, $W^r_d(C)$ is reduced for any $0 \leqslant r \leqslant d \leqslant g$. □

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