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Deforming cyclic covers in towers

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Abstract

Obus, Wewers, and Pop recently resolved a long-standing conjecture of Oort, which asserts that every cyclic cover of a curve in characteristic p lifts to characteristic zero. Saïdi subsequently asked whether these covers are also "liftable in towers." We prove that the answer to the equal-characteristic version of this question is affirmative. Our proof uses the Hurwitz tree technique along with tools developed by Obus and Wewers.

1. Introduction

Throughout this paper, we assume that k is an algebraically closed field of characteristic $p > 0$. An *Artin–Schreier–Witt k-curve* is a smooth projective connected k-curve Y that is a $\mathbb{Z}/p^n$-cover of the projective line $\mathbb{P}^1_k$. When $n = 1$, we call Y an *Artin–Schreier* curve.

Classically, one may study an object in characteristic p by finding a "link" of it with characteristic zero. For instance, Grothendieck showed that every (smooth, projective, connected) curve Y over k "lifts" to a curve $\mathcal{Y}$ over a finite extension of the ring of Witt vectors $W(k)$, hence in characteristic zero. In addition, the prime-to-p part of the étale fundamental group of Y is equal to that of the generic fiber of $\mathcal{Y}$, see [Gro63, Corollaire XIII.2.12], which can be easily calculated from the topological fundamental group of its corresponding one over $\mathbb{C}$ due to the Riemann existence theorem (see, for example, [Ser56]). Note that the p-parts of the fundamental groups are not the same. For instance, when $Y = \mathbb{A}^1_k \cong \operatorname{Spec} k[x]$, the geometric fundamental group of its lift's generic fiber $\mathbb{A}^1_{\overline{\operatorname{Frac} W(k)}} \cong \operatorname{Spec} \overline{\operatorname{Frac} W(k)}[X]$ is trivial as $\pi_1\left(\mathbb{A}^1_{\mathbb{C}}\right)$ is. However, $\pi_1^{\text{ét}}\left(\mathbb{A}^1_k\right)$ is non-trivial as there always exists an étale $\mathbb{Z}/p$-cover defined by the equation $y^p - y = x$. In fact, $\pi_1^{\text{ét}}\left(\mathbb{A}^1_k\right)$ is infinitely generated as there exists for each $n \in \mathbb{Z}_{>0}$ a $\mathbb{Z}/p^n$-cover of $\mathbb{A}^1_k$ (see Section 2.2).

It is thus natural to ask whether one can lift a Galois cover of curves to characteristic zero. Thanks to a local-to-global principle [Gar96, § 3], one may restrict the study of Galois covers of curves to Galois extensions of power series (see also Section 2.3.2). The answer, in general, is *no* [Oor87, § 1.1]. We call a group G such that every local G-cover in characteristic p lifts a *local Oort* group for p. In fact, if a group G is a local Oort group for p, then G is either cyclic,

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dihedral of order $2p^n$, or the alternating group A_4 (with $p = 2$); see [CGH11]. One would naturally expect that the converse holds. When G is cyclic, it is known as the *Oort conjecture*, which first appeared in 1995 in a list of questions and conjecture published in [MR319, Appendix 1] and was settled recently. The first result is due to Oort, Sekiguchi, and Suwa, who showed that the conjecture holds for all $\mathbb{Z}/pm$-covers, where $(m, p) = 1$; see [SOS89]. Green and Matignon then proved that for the case $G \cong \mathbb{Z}/p^2m$; see [GM98]. Finally, a combined effort of Obus–Wewers and Pop resolved the conjecture in [OW14] and [Pop14]. Saïdi conjectured a more general version of this result [Saï12, Conj-0-Rev], which says that a lift of a subcover of a given G-cyclic cover can be extended to a lift of the cover itself. That conjecture holds given a definite answer to the following.

Question 1.1 (*The refined local lifting problem*). Let $k[[z]]/k[[x]]$ be a G-Galois extension, where G is cyclic. Suppose that we are given a discrete valuation ring R in characteristic zero and a lift $R[[S]]/R[[X]]$ of a subextension $k[[s]]/k[[x]]$. Do there exist a finite extension R' of R in characteristic zero with residue field k and a G-Galois extension $R'[[Z]]/R'[[X]]$ that lifts $k[[z]]/k[[x]]$ and contains $R'[[S]]/R'[[X]]$ as a subextension?

As in the standard local lifting problem for cyclic groups, one may also assume that $G = \mathbb{Z}/p^n$; see [Obu12, Proposition 6.3]. We are tackling Question 1.1 by following the approach from [OW14]. Let us briefly describe how the Oort conjecture was proved. Obus and Wewers first proved a general result stating that a cyclic cover lifts if it has no *essential ramification* ([OW14, Theorem 1.4], see also Definition 2.11). Pop completed the proof by showing that every cover that cannot be lifted by Obus and Wewers admits an equal-characteristic deformation whose generic fibers have no essential ramification and thus also lift to characteristic zero. The existence of these non-trivial equal-characteristic deformations, which change the number of branch points but fix the genus, is also a unique aspect of wildly ramified covers. That gives us another way to investigate a cover in characteristic p: finding a connection of it with a slightly different one via equal-characteristic deformation. The main result of this paper is a positive answer to the global equal-characteristic analog of Question 1.1.

THEOREM 1.2. *Suppose that $\phi\colon Z \to X$ is a cyclic G-Galois cover of curves over k and $\psi\colon Y \to X$ is its H-Galois subcover (where H is a quotient of G). Suppose, moreover, that $\Psi\colon \mathcal{Y}_R \to \mathcal{X}_R$ is a deformation of ψ over a complete discrete valuation ring R of characteristic p. Then there exist a finite extension R'/R and a deformation $\Phi\colon \mathcal{Z}_{R'} \to \mathcal{X}_{R'}$ of ϕ over R' that contains $\Psi \otimes_R R'\colon \mathcal{Y}_{R'} \to \mathcal{X}_{R'}$ as a subcover. That is, one can always fill in the following commutative diagram of cyclic Galois covers:*

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[d] & X \arrow[l] \arrow[d] & Y \arrow[l, "\psi"'] \arrow[d] & Z \arrow[l, "\lambda"'] \arrow[d, dotted] \\
\operatorname{Spec} R & \mathcal{X} \arrow[l] & \mathcal{Y} \arrow[l, "\Psi"'] & \mathcal{Z}, \arrow[l, dotted, "\Lambda"']
\end{tikzcd}$$

where λ is the factor of ϕ through Y, after maybe a finite extension of R.

To prove the theorem, we adapt the techniques from [OW14]. One may first reduce the problem to the case where ϕ is a one-point cover. Its deformation Φ can then be regarded as a cover of a rigid disc over R with good reduction (see Section 3.4.1). To achieve the right reduction on the disc's boundary, we continuously control the degeneration of the restriction of Φ from inside to outside. This information is kept track of in the *Hurwitz tree* of Φ. It is a combinatorial-differential object that has the shape of the dual graph of a semi-stable model of the cover, together with the degeneration data of some restrictions of that cover at the corresponding vertices. The degeneration datum of a cover is derived from its *refined Swan conductor*. The conductor was defined by Kato [Kat89] in 1989. Since then, it has been studied by many other authors (Matsuda, Tsuzuki, Saito, Abbes, Kedlaya, Xiao, Chiarelotto, Pulita, Leal, Thatte, . . .) from various points of view (see [Mat97, Mat95, Tsu98, AS02, Xia12, CP09, Lea18, Tha16]), and has plenty of applications.

Remark 1.3. The notion of a Hurwitz tree was initially formulated for covers in mixed characteristic to tackle the lifting problem for $\mathbb{Z}/p$-covers, by Henrio [Hen00]. It was later improved by Bouw, Brewis, and Wewers [BW06, BW09]. In [Dan20b], we characterize Hurwitz trees for $\mathbb{Z}/p$-covers of a rigid disc in *equal* characteristic and use them to classify equal-characteristic $\mathbb{Z}/p$-deformations [Dan20b, Theorem 1.2].

Remark 1.4. Understanding these deformations also equates to understanding the geometry of the moduli space of cyclic covers of fixed genus. Capitalizing on this fact, we show that the moduli space of Artin–Schreier covers of fixed genus g is connected when the integer g is sufficiently large, by explicitly constructing some local equal-characteristic $\mathbb{Z}/p$-deformations [Dan20a, Theorem 1.1]. Furthermore, knowing the geometry of a moduli space, in turn, allows the study of invariants over flat families of objects parameterized by the space. The most well-known among them are p-rank, a-number, Ekedahl–Oort type, and Newton polygons (see, for example, [MR319, Chapter 6]). Recently, there has been a consistent stream of papers about how these invariants behave for Galois covers (see, for example, [Zhu04, DWX16, KLS19, BC20]), especially cyclic covers. We will briefly discuss some of these applications along the way.

1.1 Outline of the proof of Theorem 1.2

Firstly, thanks to a local-to-global principle, we show that Theorem 1.2 holds if and only if its local version does (Theorem 2.20 and Proposition 2.21). This means that we can restrict ourselves to the case where ϕ is a cyclic extension of a power series over k. Furthermore, one may assume that $G \cong \mathbb{Z}/p^n$ for some $n \in \mathbb{Z}_{\geqslant 1}$ and $H \cong \mathbb{Z}/p^{n-1}$ (Theorem 2.34). In that setting, the G-extensions are described by Artin–Schreier–Witt (ASW) theory, which is briefly discussed in Section 2. Finally, a construction by Katz and Gabber allows us to go back to the global case, that is, where ϕ is a one-point cover of the projective line (Section 2.4), which further simplifies some calculations.

At this point, we can translate our problem to finding conditions for good reduction of a cover of a t-adic projective line as below.

PROPOSITION 1.5 (Proposition 2.35). *Suppose that $n \geqslant 2$ is an integer, $R = k[[t]]$, $\phi_n \colon \overline{Y}_n \to \mathbb{P}^1_k$ is a one-point $\mathbb{Z}/p^n$-cover, and $\phi_{n-1} \colon \overline{Y}_{n-1} \to \mathbb{P}^1_k$ is the $\mathbb{Z}/p^{n-1}$-subcover of ϕ_n. Suppose, moreover, that $\Phi_{n-1} \colon Y_{n-1} \to \mathbb{P}^1_R$ is a $\mathbb{Z}/p^{n-1}$-cover whose reduction (modulo t) is isomorphic to ϕ_{n-1}. Then there exist a finite extension R'/R and a $\mathbb{Z}/p^n$-cover $\Phi_n \colon Y_n \otimes_R R' \to \mathbb{P}^1_{R'}$ that extends Φ_{n-1} and has special fiber isomorphic to ϕ_n.*

One may then assume that the generic branch locus of ϕ_n lies inside a t-adic disc $D \subset \left(\mathbb{P}^1_{\operatorname{Frac}(R')}\right)^{\mathrm{an}}$. We prove the above proposition in four steps, as follows.

Step 1. We start by studying the refined Swan conductors of Artin–Schreier–Witt (ASW) covers of a t-adic disc over K (recall that $K = \operatorname{Frac} R$) (Section 3.5), which measure the degeneration of these covers. Those invariants usually have the form (δ, ω), where δ is a non-negative rational number and ω is a differential form over $k(x)$. A key result is Theorem 3.42, which gives some rules on the degeneration of Φ_n when that of Φ_{n-1} is known. That is the equal-characteristic analog of [Wew14, Theorem 1.2].

Step 2. The information from the previous step then allows us to define the Hurwitz tree $\mathcal{T}_{n-1}$ associated with Φ_{n-1} (Section 4). That is a directed tree that encodes in its structure the geometry of the branch points of Φ_{n-1}. Each vertex v of $\mathcal{T}_{n-1}$ is equipped with a pair (δ_v, ω_v), which is the refined Swan conductor of the restriction of Φ_{n-1} to some subdisc of D. Each leaf has a conductor, which is determined by the ramification datum of the generic fiber of Φ_{n-1}. This generalizes the Hurwitz tree construction for an Artin–Schreier cover in [Dan20b]. We further show how the information from a Hurwitz tree can tell whether the corresponding cover has good reduction or not (Proposition 4.15 and Remark 4.16).

Step 3. In this step, we show that one can always construct explicitly a $\mathbb{Z}/p^n$-tree $\mathcal{T}_n$ that "extends" $\mathcal{T}_{n-1}$ in the sense of the criteria from Step 1 (Proposition 4.22). Moreover, the structure of $\mathcal{T}_n$ is specially designed to be used in the final step.

Step 4. Finally, using $\mathcal{T}_n$ as a frame, we construct a cover Φ_n that proves Proposition 1.5. Roughly speaking, we start from a candidate for Φ_n that has the degeneration data asserted by $\mathcal{T}_n$ on certain subdiscs of D corresponding to the "leaves" of $\mathcal{T}_n$, where it is much easier to achieve (Section 5.6). We then continuously modify that cover by multiplying it with certain "controlling characters" (Section 5.5) along $\mathcal{T}_n$ (Section 5.7) until we get to its "root" (Section 5.10), which corresponds to the boundary of the t-adic disc. We then obtain a $\mathbb{Z}/p^n$-cover whose degeneration data coincide with those of $\mathcal{T}_n$. The fact that Φ_n has a good reduction isomorphic to ϕ_n is then immediate from what we learn in Step 2.

Remark 1.6. The modifying process in Step 4 is influenced by Obus and Wewers [OW14], which in turn is inspired by [GM98]. For those familiar with that work, we will present a comparison between the two techniques as well as provide more details of this step in Section 1.1.1. An overview of the construction can also be found in Section 5.4.

Remark 1.7. One major difference between this paper and [OW14], besides the characteristic of the ring R, is that the tree $\mathcal{T}_n$ of the latter has no "branches" that one needs to control (that follows from the ramification breaks hypothesis of [OW14, Theorem 1.4]). Therefore, the Hurwitz tree technique is not utilized in the work, even though they used it to acquire the intuition and the idea for the main strategy. We, however, prove Proposition 5.4, which basically says that one can "partition" Φ_n at each vertex of $\mathcal{T}_n$ so that it is sufficient to modify the "part" of the cover corresponding to a subtree of $\mathcal{T}_n$. Hence, we can alter many techniques from Obus and Wewers to fit our situation.

Remark 1.8. There are three main obstacles that prevent us from answering Question 1.1 using the strategy from this manuscript:

(i) Firstly, it is hard to compute the Swan conductors of a Kummer cover (with given equation) of a disc in mixed characteristic, unlike in the equal-characteristic case, as discussed in Section 3.9.

(ii) Adapting Step 3 is also an issue, as finding the differential forms to fit in the tree $\mathcal{T}_n$ is no longer as natural, as shown in Section 6.1. We are able to do so when $p = 2$ and $n = 2$, though.

(iii) Finally, Step 4 also becomes much more complicated because the controlling characters of Section 5.5 are no longer straightforward to build. Our current method, which is developed from [OW14], requires showing that certain square matrices, whose sizes can be arbitrarily large, are invertible to prove the existences of such characters.

1.1.1 *Step* 4. As discussed above, we may restrict ourselves to the situation of Proposition 1.5. By Artin–Schreier–Witt theory (Section 2.2), the cover ϕ_n (respectively, Φ_{n-1}) could be presented by a length n (respectively, length $n-1$) Witt vector $\underline{g}_n := (g^1, g^2, \dots, g^n)$ in $W_n(k(x))$ (respectively, $\underline{G}_{n-1} = (G^1, \dots, G^{n-1}) \in W_{n-1}(K(X))$, where $K := \operatorname{Frac} R$). In addition, one may assume that a cover Φ_n as in Proposition 1.5 should have the form $\underline{G}_n := (G^1, G^2, \dots, G^{n-1}, G^n) \in W_n(K(X))$. Therefore, it suffices to show the existence of such a rational $G^n \in K(X)$. Corollary 3.24 then asserts that it suffices to construct a G^n such that

(C1) the cover Φ_n has an étale reduction (Definition 3.7),

(C2) its generic fiber's covering space has the right genus, and

(C3) the Witt vector representing the special fiber is in the same *ASW class* as $\underline{g}_n$.

The construction of such a G^n can be summarized as follows.

Step 4.1. We first assume that $\underline{G}_{n-1}$ is best (Definition 3.36) and set $G^n = 0$. That makes the genus of the generic fiber of Φ_n "minimal." This fact allows us to more easily manage the generic ramification.

Step 4.2. Recall from Step 3 that $\mathcal{T}_{n-1}$ is a Hurwitz tree arising from Φ_{n-1} and $\mathcal{T}_n$ is one that "extends" $\mathcal{T}_{n-1}$. Suppose that v is a vertex of $\mathcal{T}_n$. Then, by construction, there is a matching one in $\mathcal{T}_{n-1}$, which we also call v. That vertex corresponds to a closed subdisc $\mathcal{D}_v$ of the t-adic disc $\operatorname{Spec} R[[X]]$. Suppose moreover that there are m edges or leaves $e_1, e_2, \dots, e_m$ pointing out from v in $\mathcal{T}_{n-1}$. Each e_i represents an open annulus or an open disc A_{e_i} contained in $\mathcal{D}_v$ and has the same outer radius. We then break down

$$\underline{G}_{n-1} =: \underline{G}_{n-1,v,\infty} + \sum_{i=1}^{m} \underline{G}_{n-1,v,i}\,, \tag{1.1}$$

where $\underline{G}_{n-1,v,\infty}$ (respectively, $\underline{G}_{n-1,v,i}$) has no poles inside $\mathcal{D}_v$ (respectively, outside the disc formed by the outer radius of A_{e_i}). In addition, the datum on each vertex of $\mathcal{T}_{n-1}$ starting from e_i coincides with that of one formed by $\underline{G}_{n-1,v,i}$ (see Proposition 5.4). This fact allows us to modify G^n, hence $\underline{G}_n$, inductively along the tree $\mathcal{T}_n$ starting from the leaves and ending at the root. This process equates to controlling the degeneration of Φ_n from particular subdiscs to the whole disc. The four following steps will construct G^n by doing inductions on the vertices and edges of $\mathcal{T}_n$.

Step 4.3. Consider a "final vertex" v, which is adjacent to the leaves $\{b_1, \dots, b_m\}$ with conductors $\iota_{1,n}, \dots, \iota_{m,n}$, respectively, of $\mathcal{T}_n$. One can modify the part of G^n at v by adding polynomials with poles corresponding to the $b_i \in B_v$ and with degree at most $\iota_{i,n} - 1$. The result is a cover whose degeneration on $\mathcal{D}_v$ coincides with that of $\mathcal{T}_n$ at v. That is induction step *Ind* 1. in Section 5.4. Note that condition (C2) is achieved in this step. The items below correspond to induction steps *Ind* 2. through *Ind* 5..

Step 4.4. Look at an edge e adjacent to the final vertex v. Note that Step 4.3 gives the existence of a G^n giving rise to the right degeneration at v. Furthermore, Step 4.2 allows us to assume that G^n has only poles inside $\mathcal{D}_v$. It hence can be represented by a power series that converges outside $\mathcal{D}_v$. Using an analytical technique and the comparison tool developed in Section 3.8, we show that one can modify G^n in such a way that the right degeneration occurs at the starting vertex of e, settling the base case. The method is parallel to one in [OW14, § 6.4].

Step 4.5. Let v be a non-final vertex, which is the target of e and the start of $e_1, \dots, e_m$. Recall that we may apply (1.1) to partition $\underline{G}_{n-1}$ into a sum, where each summand $\underline{G}_{n-1,v,i}$

correspond to a $\mathbb{Z}/p^{n-1}$-cover whose branch points lie inside the disc D_{e_i} formed by the outer radius of A_{e_i}. By induction, we obtain, for each i, a $G^n_{v,i} \in K(X)$ that extends $\underline{G}_{n-1,v,i}$ and whose sum gives the right degeneration at v.

Step 4.6. This step is just a repetition of Step 4.4 to an edge e that is adjacent to the vertex v above, completing the inductive step. We hence obtain, by induction, the right degeneration everywhere but at the root of $\mathcal{T}_n$. Condition (C1) is hence fulfilled.

Step 4.7. Finally, we adapt [OW14, § 6.5] to obtain the right reduction at the root of $\mathcal{T}_n$, hence on the whole disc. Condition (C3) is thus satisfied.

1.2 Notation and conventions

The letter K will always be a field of characteristic p that is complete with respect to a discrete valuation $\nu\colon K^\times \to \mathbb{Q}$. The residue field k of K is algebraically closed of characteristic p. One example to keep in mind is $K = k((t))$, the field of Laurent series over k, and $\nu(t) = 1$ defines the discrete valuation. The ring of integers of K will be denoted by R.

We fix an algebraically closure $\overline{K}$ of K, and whenever necessary, we will replace K with a suitable finite extension within $\overline{K}$, without changing the above notation. The symbol $\mathbb{K}$ usually denotes a function field over K (for example, $\mathbb{K} = K(X)$).

Below is some unusual notation used in this paper, with the corresponding locations.

Notation	Description	Location
$\mathfrak{K}_n(\underline{a})$	The character defined by the Witt vector $\underline{a}$ of length n	Section 3.4
$\mathcal{D}[r, z]$	Closed disc of radius p^{-r}, centered at z	Section 3.2
$D[r, z]$	Open disc of radius p^{-r}, centered at z	Section 3.2
$W_n(\mathbb{K})$	Witt vector of length n over $\mathbb{K}$	Section 2
$(-)_r$	$- \cdot \pi^{-pr}$, where $- \in \mathbb{K}$ and π is a uniformizer of a discrete valuation ring	Section 3.2
$[\,-\,]_{r,z}$	Reduction of $((-) - z)_r$	Section 3.2
$\mathcal{G}_v$	A collection of extensions with right depth at v	Definition 5.8
$\mathcal{H}_v$	A collection of extensions with right branching datum at v	Definition 5.6
$\mathcal{W}_v$	A collection of extensions giving rise to right differentials at v	Definition 5.14
$\lambda_e(G)$	The kink of the character that G gives rise to on an edge e	Section 5.7
$\lambda_m(\chi)$	The largest kink of χ that satisfies the conditions in Section 3.8	Proposition 3.31
$\mu_m(\chi)$	A function that detects the kink of χ of Section 3.8	Proposition 3.31
$\delta_\chi(r, z)$	The depth conductor of $\chi\vert_{D[pr,z]}$	Section 3.5.2
$\omega_\chi(r, z)$	The differential conductor of $\chi\vert_{D[pr,z]}$	Section 3.5.2
$\mathrm{sw}_\chi(\overline{x})$	The boundary conductor of χ at the boundary associated with $\overline{x}$	Section 3.5
$\mathfrak{C}_\chi(r, z, \overline{w})$	The sum of the conductors on the $\overline{w}$-direction in $\mathcal{D}[pr, z]$	Definition 3.19
$\mathfrak{C}_{\mathcal{T}}(e)$	The sum of conductors of the leaves succeeding an edge e of $\mathcal{T}$	Definition 4.7
$\nu_{s,z}$	The Gauss valuation associated with $\mathcal{D}[s, z]$	Section 3.2
$\mathbb{B}(\chi)$	The branch locus of the character χ	Section 3.2

$\mathbb{B}_{\mathcal{T}}(e)$	The leaves of $\mathcal{T}$ succeeding an edge e	Definition 3.19
$\mathcal{T}(e)$	The subtree of $\mathcal{T}$ that contains only the data after e	Section 4.4.1

2. Artin–Schreier–Witt covers

Recall that the prime-to-p part of the fundamental group of a curve in characteristic p is equal to that of its lift to characteristic zero. However, the p-part of $\pi_1^{\text{et}}(\mathbb{A}_k^1)$ is no longer trivial as there always exists an étale $\mathbb{Z}/p$-cover defined by the equation $y^p - y = x$. That cover, also known as an Artin–Schreier cover, is the simplest example of a *wildly ramified* Galois cover (p divides the order of an inertia group). Furthermore, $\mathbb{Z}/p^n$-covers of a projective line can be described using Witt vectors [Lan02, Chapter VIII, Exercises, p. 330]; hence it is easy to construct examples and study them using explicit methods. Therefore, understanding these covers is usually the first step in developing a theory for all wildly ramified covers.

2.1 Artin–Schreier–Witt theory

In this section, we give a quick overview of Artin–Schreier–Witt theory. For more details, see, for example, [Lor08, § 26] or [Lan02, Chapter IV]. Suppose that M/K is a $\mathbb{Z}/p^n$-extension of fields in characteristic $p > 0$. Then $M = K(\alpha^1, \dots, \alpha^n)$, where $\alpha^i \in K^{\text{sep}}$ is a solution of an *Artin–Schreier–Witt equation*

$$\wp(\alpha^1, \dots, \alpha^n) = \left(f^1, \dots, f^n\right), \tag{2.1}$$

where $(f^1, \dots, f^n)$ lies in the ring $W_n(K)$ of truncated Witt vectors of length n and $\wp(\alpha := (\alpha^1, \dots, \alpha^n)) = F(\alpha) - \alpha$ is the Artin–Schreier–Witt isogeny, where F is the Frobenius morphism of $W_n(K)$. Moreover, the extension is unique up to adding an element of the form $\wp(b^1, \dots, b^n)$, where $(b^1, \dots, b^n) \in W_n(K)$, to $(f^1, \dots, f^n)$; see [Lor08, § 26, Theorem 5]. In other words, we have the following bijection, which is an application of Hilbert's Theorem 90:

$$\mathrm{H}^1(K, \mathbb{Z}/p^n) \xrightarrow{\simeq} W_n(K)/\wp(W_n(K)) .$$

We call $\underline{f} := (f^1, \dots, f^n)$ the *defining Witt vector* of the extension M/K and the process of adding $\wp(b^1, \dots, b^n)$ to the right-hand side of (2.1) an *Artin–Schreier–Witt operation*. If $\underline{f}$ and $\underline{g}$ differ by an Artin–Schreier–Witt operation, we say that they are in the same *Artin–Schreier–Witt (ASW) class.*

2.2 Cyclic covers of curves

An *Artin–Schreier–Witt curve of level* n is a smooth, projective, connected curve that is a $G \cong \mathbb{Z}/p^n$-cover of the projective line over k. Recall that the category of normal projective k-curves and non-constant morphisms is equivalent to that of finitely generated field extensions K/k of transcendence degree 1; see [Har77, Corollary I.6.12]. Therefore, it follows from the previous section that an arbitrary $\mathbb{Z}/p^n$-cover $\phi_n \colon Y_n \to \mathbb{P}_k^1 = \operatorname{Proj} k[x, v]$ can be represented by an ASW equation as follows:

$$\wp\left(y^1, \dots, y^n\right) = \left(f^1(x), \dots, f^n(x)\right), \tag{2.2}$$

where $\underline{f} := (f^1, \dots, f^n)$ lies in the ring $W_n(k(x))$. For the rest of Section 2.2, we set $K := k(x)$.

Example 2.1. Suppose $p = 5$. The cover $Y \to \mathbb{P}^1_K$ defined by

$$y^5 - y = \frac{1}{x^5} + \frac{1}{(x-1)^7} \tag{2.3}$$

is an Artin–Schreier curve. Note that the term $1/x^5$ is a 5th power. Hence, by the above discussion, adding $(-1/x)^5 - (-1/x)$ to the right-hand side of (2.3) does not change the cover it defines. The result is an Artin–Schreier equation of the form

$$y^5 - y = -\frac{1}{x} + \frac{1}{(x-1)^7}\,. \tag{2.4}$$

Example 2.2. Suppose $p = 2$. The following equation defines a $\mathbb{Z}/4$-cover of $\mathbb{P}^1_k$:

$$\wp(y^1, y^2) = \left(\frac{1}{x^2}, \frac{1}{x^4} + x^2\right). \tag{2.5}$$

By adding $\wp(1/x, x + 1/x)$ to right-hand side of (2.5), one obtains an alternative representation

$$\wp(z^1, z^2) = \left(\frac{1}{x}, \frac{1}{x} + x\right). \tag{2.6}$$

Remark 2.3. We say that the ASW equations (2.4) and (2.6) have *reduced forms*, or the defining Witt vectors are reduced. This means that the partial-fraction decomposition of each entry of the defining Witt vector only consists of prime-to-p-degree terms. One can show that every ASW cover can be represented by a unique Witt vector of reduced form.

2.2.1 *Branching datum.* A reduced defining Witt vector tells us everything about the ramification datum of the cover it defines. Particularly, suppose $\underline{f} \in W_n(K)$ (recall that $K = k(x)$) is *reduced*, and let $\mathscr{P} := \{P_1, \ldots, P_r\}$ be the set of poles of the f^i. Then $\mathscr{P}$ is also the branch locus of ϕ_n. Furthermore, for each ramified point Q_j above P_j, the cover ϕ_n induces an exponent p cyclic extension of complete local ring $\hat{\mathcal{O}}_{Y_n,Q_j}/\hat{\mathcal{O}}_{\mathbb{P}^1,P_j}$ with perfect residue field. Hence, it makes sense to talk about the ramification filtration of ϕ_n at a branch point P_j (see [Ser79, § IV.3]). Suppose that the inertia group of Q_j is $\mathbb{Z}/p^m$ (where $n \geqslant m$). We say that the *ith ramification break* of ϕ_n at P_j is -1 for $i \leqslant n - m$. When $i > n - m$, the ith ramification break of ϕ_n at P_j is the $(i - n + m)$th one of $\hat{\mathcal{O}}_{Y_n,Q_j}/\hat{\mathcal{O}}_{\mathbb{P}^1,P_j}$. We denote by $u_{j,i}$ the ith upper ramification break of ϕ_n at P_j. We call $e_{j,i} = u_{j,i} + 1$ the *ith conductor of ϕ at P_j*. The following formula explicitly computes the ramification filtration of ϕ_n in terms of $\underline{f}$.

Theorem 2.4 ([Gar02, Theorem 1]). *With the assumptions and the notation as above, we have*

$$u_{j,i} = \max\bigl\{p^{i-l} \deg_{(x-P_j)^{-1}}\bigl(f^l\bigr) \mid l = 1, \ldots, i\bigr\} \tag{2.7}$$

for $i > n - m$.

Remark 2.5. From equation (2.7), one deduces that, if the inertia group of Q_j is $\mathbb{Z}/p^i$, then $i = \min\bigl\{l \mid \deg_{(x-P_j)^{-1}}(f^l) \neq 0\bigr\}$. In addition, if $p \nmid u_{j,n-m+1} \neq 0$, then $u_{j,i} \geqslant p u_{j,i-1}$ for $n - m + 2 \leqslant i \leqslant n$, and if $p \mid u_{j,i}$, then $u_{j,i} = p u_{j,i-1}$. In particular, when $n = 1$, that is, when ϕ_1 is an Artin–Schreier cover, the unique ramification break at P_i is equal to the order of the pole of f^1 at P_i.

Example 2.6. The cover from Example 2.2 has breaks $(1, 2)$ at $x = 0$ and $(-1, 1)$ at $x = \infty$.

Remark 2.7. One can easily derive from the above discussion that the length n Witt vector $(x, 0, \ldots, 0)$ defines an étale $\mathbb{Z}/p^n$-cover of the affine line over k. That proves the infinitely generatedness of the fundamental group of $\mathbb{A}^1_k$ discussed in Section 1.

Furthermore, it follows from [Pop14, Fact 2.3] that the degree of the different at the branch point P_j is

$$\delta_{P_j} = \sum_{i=1}^{n} (u_{j,i}+1)\big(p^i - p^{i-1}\big). \tag{2.8}$$

Set $e_{j,i} := u_{j,i}+1$. We denote by ϕ_l the $\mathbb{Z}/p^l$-subcover of ϕ_n. Then $\phi_l \colon Y_l \to \mathbb{P}^1_k$ corresponds to the length l Witt vector $(f^1, \dots, f^l) \in W_l(K)$. We denote by $\delta^l_{P_j}$ the degree of the different at P_j of ϕ_l. The following result gives the genus of Y_l in terms of the ramification breaks.

Proposition 2.8. *In the above notation, the genus of the $\mathbb{Z}/p^l$-covering curve Y_l is*

$$g_{Y_l} = \frac{2(1-p^l) + \sum_{i=1}^{l}\big(\sum_{j=1}^{r} e_{j,i}\big)\big(p^i - p^{i-1}\big)}{2}. \tag{2.9}$$

Proof. Applying the Riemann–Hurwitz formula [Har77, Corollary IV.2.4], we obtain

$$g_{Y_l} = \frac{2\big(1-p^l\big) + \sum_{j=1}^{r} \delta^l_{P_j}}{2}.$$

Using (2.8) to calculate the degree of the different at each branch point P_j, we immediately realize (2.9). ☐

Therefore, $\mathbb{Z}/p^n$-covers of the same genus on each of its subcovers have the same $e_i := \sum_{j=1}^{r} e_{j,i}$ for $1 \leqslant i \leqslant n$. We thus use an $r \times n$ matrix as follows to record the ramification datum of the cover:

$$M = \begin{bmatrix} e_{1,1} & e_{1,2} & \dots & e_{1,n} \\ e_{2,1} & e_{2,2} & \dots & e_{n,n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{r,1} & e_{r,2} & \dots & e_{r,n} \end{bmatrix}.$$

We call the above matrix M the *branching datum* of ϕ_n. We call the divisor $\sum_{j=1}^{r} e_{j,l} P_j$ the *level l branching divisor* or, when $l = n$, just the *branching divisor* of ϕ_n.

The forward direction of the following corollary is immediate from Remark 2.5.

Corollary 2.9. *An $r \times n$ matrix $M = (e_{j,i})$ with positive integer entries is a branching datum of a $\mathbb{Z}/p^n$-cover if and only if the following hold:*

(i) *For $J = \min\{i \mid e_{j,i} \neq 0\}$, we have $e_{j,J} \not\equiv 1 \pmod p$ for $j = 1, \dots, r$.*

(ii) *For $J < i \leqslant n$, we have $e_{j,i} \geqslant p e_{j,i-1} - p + 1$. Equality holds if and only if $e_{j,i} \equiv 1 \pmod p$. If $e_{j,i} > p e_{i,j-1} - p + 1$, then $e_{j,i} = p e_{i,j-1} - p + a_{j,i} + 1$ for an integer $a_{j,i}$ prime to p.*

Proof. Suppose that $M = (e_{j,i})$ is an $r \times n$ matrix whose entries satisfy conditions (i) and (ii). Let $x_i, \dots, x_r$ be distinct points on k. Consider the length n Witt vector

$$F = \big(f^1, \dots, f^n\big) \in W_n(k(x)),$$

where $f^i = \sum_{j=1}^{r} a_{j,i}/(x-x_j)^{e_{j,i}-1}$ and the $a_{j,i}$ are defined as follows:

$$a_{j,i} := \begin{cases} 0, & e_{j,i} \equiv 1 \pmod p, \\ 1, & e_{j,i} \not\equiv 1 \pmod p. \end{cases}$$

It then follows from Theorem 2.4 that the cyclic cover defined by F has branching datum coinciding with M, proving the necessity of the conditions. ☐

DEFINITION 2.10. Suppose that $(e_1, \dots, e_n)$, as a $1 \times n$ matrix, satisfies the conditions of Corollary 2.9. We define $\Omega_{e_1,\dots,e_n}$ to be the collection of $r \times n$ matrices that partition $(e_1, \dots, e_n)$ and are such that each row satisfies the conditions of that same corollary.

DEFINITION 2.11. We use the notation above and set $e_{0,i} = 0$ for $1 \leqslant j \leqslant r$. We say that ϕ_n has no *essential jump* from level $i-1$ to i at P_j if $e_{j,i} = pe_{j,i-1} - p + 1$ or $e_{j,i} = pe_{j,i-1} - p + a_{j,i} + 1$, where $1 \leqslant a_{j,i} \leqslant p-1$. If this holds for all P_j, we say that ϕ_n has no essential jumps from level $i-1$ to i. We say that ϕ_n has no essential jump if this is true for all $1 \leqslant i \leqslant n$.

Example 2.12. The cover in Example 2.1 has branching datum $[2, 8]^{\top}$, hence has an essential jump at $x = 1$. The cover from Example 2.2 has branching datum $\left[\begin{smallmatrix} 2 & 3 \\ 0 & 2 \end{smallmatrix}\right]$, so has no essential jump.

Remark 2.13. When $n = 1$, the genus of an Artin–Schreier cover $Y_1 \xrightarrow{\phi} \mathbb{P}^1_k$ is

$$g := g_{Y_1} = \frac{\left(\sum_{i=1}^{r}(e_{i,1}+1) - 2\right)(p-1)}{2}.$$

Hence, all Artin–Schreier k-curves with the same genus g have the same sum of conductors $d+2$, where $d := 2g/(p-1)$. Thus, it is natural to classify Artin–Schreier covers of the same genus by their branching data. This idea is utilized by Pries and Zhu to partition the moduli space $\mathcal{AS}_g$ of Artin–Schreier curves of genus g into irreducible strata [PZ12, Theorem 1.1]. By explicitly constructing some local deformations, we show that $\mathcal{AS}_g$ is connected when g is sufficiently large [Dan20a, Theorem 1.1]. In [DH24], we carry the idea from [PZ12] and [Dan20a] further to all cyclic covers. In particular, we show that the moduli space $\mathcal{ASW}\mathrm{cov}_{(g^1,\dots,g^n)}$ of $\mathbb{Z}/p^n$-covers whose $\mathbb{Z}/p^i$-subcovers have genus g^i can be partitioned into irreducible strata, where each stratum is represented by a suitable branching datum matrix.

2.3 Deformations of cyclic covers

2.3.1 *Deformation of Galois covers.* Suppose that $Y \xrightarrow{\phi} C$ is a G-Galois cover over k, where C and Y are smooth, projective, connected k-curves. Suppose, moreover, that A is a local, Noetherian, complete k-algebra with residue field k, which is also a domain. Let

$$\mathrm{Def}_{\phi}\colon \mathrm{Alg}/k \longrightarrow \mathrm{Set}$$

be the functor which, with any $A \in \mathrm{Alg}/k$, associates classes of G-Galois covers $\mathscr{Y}_A \xrightarrow{\Phi} \mathscr{C}_A$ of smooth proper curves that make the Cartesian diagram

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "\phi"] & \mathscr{Y}_A \arrow[d, "\Phi"] \\
C \arrow[r] \arrow[d] & \mathscr{C}_A \arrow[d] \\
\operatorname{Spec} k \arrow[r, "f"] & \operatorname{Spec} A
\end{tikzcd} \tag{2.10}$$

commute and are so that the G-action on $\mathscr{Y}_A$ induces the original action on Y. More precisely,

- the special fiber of Φ is isomorphic to ϕ, and
- the isomorphism $Y \cong \mathscr{Y}_A \otimes_A k$ is G-equivariant.

We call Φ a *deformation* of ϕ over A. For more details, see [BM00, § 2]. In this paper, we focus on the case where G is cyclic and A is a finite extension of a power series $k[[t]]$, which is a complete discrete valuation ring of characteristic p with residue k.

Remark 2.14. One application of deformation is reducing hard, general problems to ones that are easier to study. For instance, Norman and Oort showed that every abelian variety in characteristic $p > 0$ can deform to an ordinary one [NO80], hence lifts to characteristic zero by Serre–Tate [Kat81, Del81]. A similar strategy was applied to prove that every cyclic cover in characteristic p lifts to characteristic zero. Namely, Pop showed that every cyclic cover equal-characteristically deforms to one with no essential ramification jumps. This cover always lifts by Obus and Wewers [OW14], hence so does the original one. This approach was first discussed in [OM68].

2.3.2 *A local-to-global principle.* Let R be a complete discrete valuation ring as before. The *local-to-global principle* below allows us to study the deformation problem by way of the local deformation problem.

Theorem 2.15. *Let Y be a smooth projective curve over k, with a faithful action of G by k-automorphisms. Let $y_1, \ldots, y_r \in Y$ be the points where G acts with non-trivial inertia. For each $1 \leqslant j \leqslant r$, let G_j be the inertia group of y_j in G, and let $\iota_j \colon G_j \to \operatorname{Aut}_k k[[u_j]]$ be the induced local action on the complete local ring of y_j. Then the deformation of Y with G-action over R is determined by the deformation over R of each of the local G_j-actions.*

Proof. See [Saï12, § 1.2] or [Gar96, § 3]. ☐

We thus can reduce the study of deformation to the case where ϕ is local, namely, where ϕ is a G-Galois extension of power series $k[[x]] \to k[[z]]$. In that case, Bertin and Mézard showed that the functor Def_ϕ is represented by (the spectrum of) a versal deformation ring [BM00, Theorem 2.1], which we denote by R_ϕ or R_G. The *tangent space* of Def_ϕ is $\mathrm{H}^1(G, \operatorname{Der}_k k[[z]])$. Its *obstruction space* is $\mathrm{H}^2(G, \operatorname{Der}_k k[[z]])$. Note that, when G is cyclic, the dimension of the tangent space $\mathrm{H}^2(G, \operatorname{Der}_k k[[z]])$ is usually positive [BM00, Théorème 4.1.1]. The deformation problem is hence non-trivial.

Little is known about R_G when ϕ is wildly ramified, that is, when $p \mid |G|$. When $G \cong \mathbb{Z}/p$ and ϕ has "conductor" 1 (that is, when the corresponding HKG cover [Har80] (see also Section 2.4) of the projective line over k has genus 0), the ring R_G is completely described by [BM00, Théorème 4.2.8]. It is also known for all conductors when $G = \mathbb{Z}/2$; see [BM00, Théorème 4.3.7]. To generalize these results, one would naturally ask the following Galois theory-type question.

Question 2.16. Suppose $n \geqslant m \in \mathbb{Z}_{\geqslant 0}$ and that we are given a tower of Galois extensions

$$\begin{tikzcd} k[[x]] \arrow[r, "\phi_m"] \arrow[rr, bend right, "\phi_n"'] & k[[y_m]] \arrow[r, "\phi_{n/m}"] & k[[y_n]] \, . \end{tikzcd}$$

How do the rings R_{ϕ_m}, $R_{\phi_{n/m}}$, and R_{ϕ_n} relate?

There is a natural map $\operatorname{ind} \colon \operatorname{Spec} R_{\phi_n} \to \operatorname{Spec} R_{\phi_m}$; see [Bys11]. In particular, when ϕ_n is a $\mathbb{Z}/p^n$-cover defined by a length n Witt vector $\underline{f}_n := (f^1, \ldots, f^n)$ and ϕ_m is its $\mathbb{Z}/p^m$-subcover, the morphism ind maps $\underline{f}_n$ to $\underline{f}_m := (f^1, \ldots, f^m)$. Let ϕ_m denote the mth *level* of ϕ_n. We conjecture the following.

Conjecture 2.17. *In the notation of Question 2.16, if $G := \operatorname{Gal}(k[[y_n]]/k[[x]])$ is cyclic, then $\operatorname{ind} \colon \operatorname{Spec} R_{\phi_n} \to \operatorname{Spec} R_{\phi_m}$ is surjective. That is, given an arbitrary complete discrete valuation* *ring R with residue field k, one can always fill in the commutative diagram*

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[d] & \operatorname{Spec} k[[x]] \arrow[l] \arrow[d] & \operatorname{Spec} k[[y_m]] \arrow[l, "\phi_m"'] \arrow[d] & \operatorname{Spec} k[[y_n]] \arrow[l, "\phi_{n/m}"'] \arrow[d, dotted] \\
\operatorname{Spec} R' & \operatorname{Spec} R'[[X]] \arrow[l] & \operatorname{Spec} R'[[Y_m]] \arrow[l, "\Phi_m"'] & \operatorname{Spec} R'[[Y_n]] \arrow[l, dotted, "\Phi_{n/m}"']
\end{tikzcd}$$

after a finite extension R' of R. If this holds, we say that ϕ_n is deformable in towers.

Remark 2.18. When R has characteristic zero, the conjecture is precisely the refined local lifting problem (Question 1.1).

Remark 2.19. One may want to approach Question 2.16 on the level of tangent spaces, in other words, investigate the relations between the groups $\mathrm{H}^1(\mathrm{Gal}(k[[y_n]]/k[[x]]), \mathrm{Der}_k\, k[[y_n]])$ and $\mathrm{H}^1(\mathrm{Gal}(k[[y_m]]/k[[x]]), \mathrm{Der}_k\, k[[y_m]])$. So far, by generalizing some results from [BM00], we have computed the dimensions of $R_{\mathbb{Z}/p^n}$, generalizing [BM00, Theorem 5.3.3].

Note that a cyclic extension in characteristic zero is described by Kummer theory, which is "multiplicative" compared with the "additive" nature of ASW theory. It is hence natural to first study the case where R has equal characteristic, as both the generic and the special fiber are additive in that situation. That is exactly the local version of Theorem 1.2.

Theorem 2.20 (Local deformation of Artin–Schreier–Witt covers). *Suppose that H is a (nontrivial) quotient of a finite cyclic group G and ϕ_n is a G-extension $k[[y_n]]/k[[x]]$, hence branches at exactly one point. Suppose, moreover, that Φ_m is an H-extension $R[[Y_m]]/R[[X]]$ that deforms the unique H-subcover $\phi_m\colon k[[x]] \to k[[y_m]]$ of ϕ_n over $R := k[[t]]$. Then there exist a finite extension R'/R and a deformation $R'[[Y_n]]/R'[[X]]$ of ϕ_n over R' that contains $R'[[Y_{n-1}]]/R'[[X]]$ as a subcover.*

In the following proposition, we show that the local result implies the global one. The proof of the "⇒" direction is postponed to Section 2.4.

Proposition 2.21. *Theorem 1.2 holds if and only if Theorem 2.20 does.*

Sketch of the proof of the "⇐" direction. This proof is developed from that of [Obu12, Theorem 3.1]. Let us assume that each of the local covers deforms in towers. With the assumptions of Theorem 1.2, let $\mathfrak{X}$ (respectively, $\mathfrak{Y}$) be the formal completion of $\mathcal{X}_R$ (respectively, $\mathcal{Y}_R$) at X (respectively, at Y). Let $B \subsetneq X$ be the branch locus of ψ, and set $U := X \setminus B$, $V := \psi^{-1}(U)$, and $W := \phi^{-1}(U) = Y \setminus \{y_1, \ldots, y_s\}$. Let $\mathfrak{U} \subseteq \mathfrak{X}$ be the formal subscheme associated with $U \subseteq X$. By Grothendieck's theory of étale lifting [Gro63, § IX.1.10], the G-cover $\phi|_W\colon W \to U$ deforms over R to an étale G-cover of formal schemes $\mathfrak{W} \to \mathfrak{U}$ with the deformation $\mathfrak{V} \to \mathfrak{U}$ of $\psi|_V\colon V \to U$ as the H-subcover. The boundary of $\mathfrak{W}$ is isomorphic to a disjoint union $\bigsqcup_{j=1}^{s} \mathcal{B}_j$, where each $\mathcal{B}_j$, which corresponds to the point y_j, is isomorphic to the boundary of a disc (see Section 3.2). For each j, there exists a canonical action of the inertia group $G_j \leqslant G$ of y_j on $\mathcal{B}_j$.

By assumption, each local G_j-extension $\hat{\mathcal{O}}_{Z,y_j}/\hat{\mathcal{O}}_{X,\phi(y_j)}$ deforms over R in towers to a G_j-cover of an open disc $D_j \cong \operatorname{Spec} R[[W_j]] \to \operatorname{Spec} R[[V_j]] \to \operatorname{Spec} R[[U_j]]$. The action of G_j on D_j induces an action on its boundary ∂D_j. In addition, the theory of étale lifting asserts that the G_j-action on ∂D_j is isomorphic to the action on $\mathcal{B}_j$. Thus, by identifying $\mathcal{B}_j$ and ∂D_j, we can use formal patching to "glue" each of these discs D_j to $\mathfrak{W}$ in a G_j-equivariant way. This yields a formal curve with G-action and projective special fiber. By Grothendieck's existence theorem

[GD61, § 5], this formal curve is the projective completion of a smooth projective curve $\mathcal{Z}_R$ with G-action such that $\mathcal{Z}_R/G \cong \mathcal{X}_R$ and $\mathcal{Z}_R/H \cong \mathcal{Y}_R$. See [Hen00] or [Dan20b] for concrete examples of formal patching. This is the cover we are looking for. □

On the level of tangent spaces, the following result follows immediately from Theorem 2.20.

Corollary 2.22. *In the notation of Question 2.16, when ϕ_n is cyclic, the following group homomorphism is surjective:*

$$\mathrm{ind}\colon \mathrm{H}^1(\mathrm{Gal}\, k[[y_n]]/k[[x]], \mathrm{Der}_k\, k[[y_n]]) \longrightarrow \mathrm{H}^1(\mathrm{Gal}\, k[[y_m]]/k[[x]], \mathrm{Der}_k\, k[[y_m]]).$$

We first explore the case where G is a cyclic p-group, that is, $G \cong \mathbb{Z}/p^n$ for some $n > 1$.

2.3.3 *Deformation of Artin–Schreier–Witt covers.* In the notation of Theorem 2.20, suppose $G = \mathbb{Z}/p^n$. We assume that R is a complete discrete valuation ring of equal characteristic. Thanks to the local-to-global principle (Theorem 2.15), one might assume that ϕ_n branches at exactly one point while studying its deformations. Thus, a deformation Φ_n of ϕ_n over R can be described by the following expression (see Section 2.2.1):

$$[e_1, e_2, \dots, e_n] \longrightarrow \begin{bmatrix} e_{1,1} & e_{1,2} & \cdots & e_{1,n} \\ e_{2,1} & e_{2,2} & \cdots & e_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{r,1} & e_{r,2} & \cdots & e_{r,n} \end{bmatrix}. \tag{2.11}$$

We call the expression in (2.11) the *type* of the deformation Φ_n. The second matrix indicates that the generic fiber of Φ_n branches at r points $Q_1, \dots, Q_r$, which correspond to rows $1, \dots, r$, with upper ramification jumps $(e_{i,1} - 1, e_{i,2} - 1, \dots, e_{i,n} - 1)$ at Q_i.

Proposition 2.23. *Suppose that we are given a deformation of a $\mathbb{Z}/p^n$-cover over R of type (2.11). Then $\sum_{j=1}^{r} e_{j,i} = e_i$ for $i = 1, 2, \dots, n$.*

Proof. A deformation of a $\mathbb{Z}/p^n$-cover induces for each $1 \leqslant i < n$ a $\mathbb{Z}/p^i$-deformation whose fibers have the same genus. The rest follows immediately from Proposition 2.8. □

In the following section, we present a well-known tool to realize a deformation and some explicit examples.

2.3.4 *Birational deformations.* When dealing with $k[[x]]$, we will often want to think of its Galois extension in terms of the associated extension of fraction fields.

Definition 2.24. Suppose that $A/k[[x]]$ is a G-extension. Suppose, moreover, that the quotient $M/\operatorname{Frac}(R[[X]])$, where $R/k[[t]]$ finite, is a G-extension, and A_R is the integral closure of $R[[X]]$ in M. We say that $M/\operatorname{Frac}(R[[X]])$ is a *birational deformation* of $A/k[[x]]$ if

(i) the integral closure of $A_R \otimes_R k$ is isomorphic to A, and

(ii) the G-action on $\operatorname{Frac}(A) = \operatorname{Frac}(A_R \otimes_R k)$ induced from that on A_R restricts to the given G-action on A.

Adapting the strategy from [Gar96], which proves that any local G-extension lifts, one can show that any local G-extension can also be birationally deformed in towers! The following criterion is extremely useful for seeing when a birational deformation is actually a deformation (that is, when $A_R \otimes_R k$ is already integrally closed, thus isomorphic to A).

PROPOSITION 2.25 (The different criterion [GM98, § I.3.4]). *Suppose that $A_R/R[[X]]$ is a birational deformation of the G-Galois extension $A/k[[x]]$. Let $K := \operatorname{Frac} R$, let δ_η be the degree of the different of $(A_R \otimes_R K)/(R[[X]] \otimes_R K)$, and let δ_s be the degree of the different of $A/k[[x]]$. Then $\delta_s \leqslant \delta_\eta$, and equality holds if and only if $A_R/R[[X]]$ is a deformation of $A/k[[x]]$.*

Example 2.26. One may use the above criterion to check that the $\mathbb{Z}/5$-extension Φ given by

$$Y^5 - Y = \frac{X + 2t^{10}}{X^5\big(X - t^{10}\big)^2\big(X - t^5\big)^5}$$

is a deformation of $y^5 - y = 1/x^{11}$ over $k[[t]]$. An easy computation shows that the generic branch points of Φ are 0, t^{10}, and t^5, which have conductors 4, 3, and 5, respectively. Hence, it is a deformation of type $[12] \to [4, 3, 5]^\top$. See [Dan20b] and [Dan20a] for more explicit examples of $\mathbb{Z}/p$-deformations.

Example 2.27. Let us consider the $\mathbb{Z}/4$-extension χ_2 of $k[[t, X]]$ (where $\operatorname{char} k = 2$) defined by

$$\wp(Y_1, Y_2) = \left(\frac{1}{X^2\big(X - t^8\big)}, \frac{1}{X^3\big(X - t^8\big)^2\big(X - t^2\big)^4}\right). \tag{2.12}$$

Its fiber $\overline{\chi}_2$ at $t = 0$ is birationally equivalent to the following $\mathbb{Z}/4$-extension of $k[[x]]$:

$$\wp(y_1, y_2) = \left(\frac{1}{x^3}, \frac{1}{x^9}\right).$$

By converting the Witt vector in (2.12) to a reduced form as in Example 2.2, one sees that the generic fiber has upper jumps $(1, 2)$ at 0, $(1, 2)$ at t^8, and $(-1, 3)$ at t^2. Therefore, the deformation χ_2 has type

$$\begin{bmatrix} 4 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 2 & 0 \\ 3 & 3 & 4 \end{bmatrix}^\top .$$

Note that the $\mathbb{Z}/2$-subextension of χ_2 is a deformation of type $[4] \to [2, 2]^\top$ with generic branch points 0 and t^8.

2.3.5 *A family of non-trivial deformations.* Suppose that ϕ_n is a $\mathbb{Z}/p^n$-extension $k[[z_n]]/k[[x]]$ with upper ramification jumps $\iota_1 < \iota_2 < \cdots < \iota_n$. Set $\iota_0 = 0$. Then, for each $1 \leqslant j \leqslant n$, we may write

$$\iota_j - p\iota_{j-1} = pq_j + \epsilon_j \,,$$

with $0 \leqslant q_j$, $0 \leqslant \epsilon_j < p$, and $q_j = 0$ if $\epsilon_j = 0$, as asserted by Corollary 2.9. Thus $0 < \epsilon_j$ if and only if $(p, \iota_j) = 1$, which holds if and only if $p\iota_{j-1} < \iota_j$. We call q_j the *essential part* of the upper jump at level j, and if $0 < q_j$, we say that ι_j is an *essential upper jump.* These terminologies were introduced in [Pop14]. Let $r := 1 + \sum_{j \text{ essential}} q_j$. According to Pop, one may partition $(\iota_1 + 1, \ldots, \iota_n + 1)$ into an $r \times n$ matrix $M := (e_{j,i})$ as follows:

(i) Set $e_{1,i} = pe_{1,i-1} + \epsilon_i$ for $1 \leqslant i \leqslant n$ and $e_{1,0} = 0$.

(ii) Add q_j rows of the following form to M for each essential place j:

$$\big(0, \ldots, 0, p - 1, p^2 - 1, \ldots, p^{n-j+1} - 1\big) .$$

Furthermore, Pop shows that there always exists a deformation of type M as above. We restate his result using the conventions in this paper as follows.

LEMMA 2.28 ([Pop14, Key Lemma 3.2]). *Let $A = k[[x]] \hookrightarrow k[[z]] := B$ be a cyclic $\mathbb{Z}/p^n$-extension with upper ramification jumps $\iota_1, \dots, \iota_n$. In the above notation, let $x_1, \dots, x_r \in tk[[t]]$ be distinct elements. Then there exists a $\mathbb{Z}/p^n$-deformation of $A \hookrightarrow B$ over $k[[t]]$ of type M that has $x_1, \dots, x_r$ as branch points.*

Example 2.29. Suppose that $\overline{\chi}$ is a $\mathbb{Z}/5^2$-extension of $k[[x]]$ that is defined by

$$\wp(y_1, y_2) = \left(\frac{1}{x^8}, \frac{1}{x^{52}} + \frac{1}{x^{46}}\right) = \left(\frac{1}{x^8}, \frac{1+x^6}{x^{52}}\right).$$

It branches at $x = 0$ with jumps $(\iota_1, \iota_2) = (8, 52)$, hence branching datum $[9, 53]$. Observe that

$$\begin{cases} \iota_1 - 5\iota_0 = 5 \cdot 1 + 3\,, \\ \iota_2 - 5\iota_1 = 5 \cdot 2 + 2\,. \end{cases} \tag{2.13}$$

Based on the above data, one can "split" $[9, 53]$ into the following 4×2 matrix:

$$\begin{bmatrix} 9 & 53 \end{bmatrix} \longrightarrow \begin{bmatrix} 3+1 & 3 \cdot 5 + 2 + 1 \\ 5 & 25 \\ 0 & 5 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 18 \\ 5 & 25 \\ 0 & 5 \\ 0 & 5 \end{bmatrix}. \tag{2.14}$$

One can show, using the strategy from Example 2.27, that the extension χ defined by

$$\wp(Y_1, Y_2) = \left(\frac{1}{x^3(x-t_1)^5}, \frac{1+x^6}{x^{17}(x-t_1)^{25}(x-t_2)^5(x-t_3)^5}\right), \tag{2.15}$$

where $t_1, t_2, t_3 \in tk[[t]]$ are distinct, is a deformation of $\overline{\chi}$ over $k[[t]]$ with type (2.14). Equation (2.15) is derived from [Pop14, proof of Key Lemma 3.2].

Remark 2.30. The splitting in (2.14) and the explicit equation (2.15) easily generalize to arbitrary $\mathbb{Z}/p^n$-covers of $\mathbb{P}^1_k$. We call these types of deformations *Oort–Sekiguchi–Suwa* (OSS) deformations. They are generalizations of (the p-fiber of) ones for Artin–Schreier covers introduced in [BM00, § 4.3]. See [Dan20a, § 3.1.1] for a detailed discussion regarding how the p-fibers of the deformations in [BM00, § 4.3] are OSS deformations of Artin–Schreier covers. In addition, when the cover is of order p, Bertin and Mézard show that these deformations form a dominant component of the local deformation ring's spectrum. One thus would expect that this also holds for the versal deformation rings of $\mathbb{Z}/p^n$-covers when $n > 1$.

In the next section, we show that it suffices to answer Theorem 2.20 for the case $G \cong \mathbb{Z}/p^n$.

2.3.6 *Reduction to the case of cyclic p-groups.* We first state a well-known result, which suggests that the deformations of local cyclic tamely ramified extensions are not very interesting.

PROPOSITION 2.31. *Tamely ramified cyclic covers are deformable over $k[[t]]$ in towers.*

Proof. Suppose that ϕ is a tamely ramified cyclic cover of $k[[x]]$ with Galois group $\mathbb{Z}/m$, where m is prime to p. It follows from Kummer theory that, after a change of variables, ϕ is given by $z^m = x$. Suppose $m = nr$, where $n, r \neq 1$. Then the unique $\mathbb{Z}/n$-subcover τ of ϕ is defined by $y^n = x$. Furthermore, a deformation $\mathscr{T}$ of τ over $k[[t]]$, after a change of variables, can be defined generically by $Y^n = X - h(t)$, where $h(t) \in t \cdot k[[t]]$ and X is a lift of x to $k[[t]]$. It is then easy to verify, using the different criterion, that $Y^m = X - h(t)$ defines the deformation of ϕ that we want. $\square$

Proposition 2.32. *Let $G \cong \mathbb{Z}/mp^n$, where $p \nmid m$. If $\mathbb{Z}/p^n$ is deformable in towers, then so is G. In particular, it suffices to prove Theorem 2.20 for the case $G \cong \mathbb{Z}/p^n$.*

Proof. Let H be a subgroup of G. Then H is isomorphic to $\mathbb{Z}/lp^r$, where $l|m$ and $0 \leqslant r \leqslant n$. Given a G-cover $f\colon Y \to \operatorname{Spec} k[[x]]$, let $g\colon Z \to \operatorname{Spec} k[[x]]$ (respectively, $h\colon X \to \operatorname{Spec} k[[x]]$) be the unique $\mathbb{Z}/m$-subcover (respectively, $\mathbb{Z}/p^n$-subcover). Then the normalization $Z \times_{\operatorname{Spec} k[[x]]} X$ is isomorphic to Y. Similarly, the H-subcover of f can be identified with the H-cover $f'\colon Y' \cong Z' \times_{\operatorname{Spec} k[[x]]} X' \to \operatorname{Spec} k[[x]]$, where $g'\colon Z' \to \operatorname{Spec} k[[x]]$ (respectively, $h'\colon X' \to \operatorname{Spec} k[[x]]$) is the $\mathbb{Z}/l$-subcover over g (respectively, the $\mathbb{Z}/p^r$-subcover over h). Suppose that $F'_R\colon \mathcal{Y}' \to \operatorname{Spec} R[[X]]$ is a deformation of f' over R. Then $\mathcal{Y}' \cong \mathcal{X}' \times_{\operatorname{Spec} R[[X]]} \mathcal{Z}'$, where $G'_R\colon \mathcal{Z}' \to \operatorname{Spec} R[[X]]$ deforms g' and $H'_R\colon \mathcal{X}' \to \operatorname{Spec} R[[X]]$ deforms h'. Moreover, the unique branch point of the generic fiber G'_K of G'_R has index p^l in the generic fiber H'_K of H'_R. By assumption, H'_R extends to $H_R\colon \mathcal{X} \to \operatorname{Spec} R[[X]]$. Furthermore, by Proposition 2.31, one can extend G'_R to $G_R\colon \mathcal{Z} \to \operatorname{Spec} R[[X]]$.

Finally, let $\mathcal{Y}''_R$ be the normalization of $\mathcal{X}_R \times_{\mathcal{D}} \mathcal{Z}_R$. Then the canonical map $F_R\colon \mathcal{Y}''_R \to \mathcal{D}$ is a birational deformation of f. The degree of the different of g (and of G_K) is $m-1$. Let δ be the degree of the different of h (and of H_K). Using our assumption on the branch loci of F'_K and G'_K, one shows that the degrees of the differents of the generic fibers of F_R and f are both $m\delta + m - 1$. It thus follows from Proposition 2.25 that F_R is a deformation of f over R. $\square$

Let $L_n = k[[y_n]]/k[[x]]$ be a $\mathbb{Z}/p^n$-extension. From the above discussion, Theorem 2.20 is an immediate result of the following.

Corollary 2.33. *Suppose that $k[[y_n]]/k[[x]]$ is cyclic Galois of order p^n, and $R[[Y_m]]/R[[X]]$ is a deformation of the $\mathbb{Z}/p^m$-subextension $k[[y_m]]/k[[x]]$ over a finite extension R of $k[[t]]$. Then, there exist a finite extension R'/R and a deformation $R'[[Y_n]]/R'[[X]]$ of $k[[y_n]]/k[[x]]$ over R' that extends $R'[[Y_m]]/R'[[X]]$.*

We will prove Corollary 2.33 inductively using the result below.

Theorem 2.34. *Suppose that $k[[y_n]]/k[[x]]$ is cyclic Galois of order p^n, and $R[[Y_{n-1}]]/R[[X]]$ is a deformation of the $\mathbb{Z}/p^{n-1}$-subextension $k[[y_{n-1}]]/k[[x]]$ over a finite extension R of $k[[t]]$. Then, there exist a finite extension R'/R and a deformation $R'[[Y_n]]/R'[[X]]$ of $k[[y_n]]/k[[x]]$ over R' that extends $R'[[Y_{n-1}]]/R'[[X]]$.*

Assuming Theorem 2.34, Corollary 2.33 easily follows.

2.4 Deformation of one-point covers

Let us once more consider the $\mathbb{Z}/p^n$-extension $L_n/k[[x]]$. Harbater, Katz, and Gabber show that there exists a unique $\mathbb{Z}/p^n$-cover $\overline{Y}_n \to \overline{C} := \mathbb{P}^1_k$, which is usually known as the HKG cover of $L_n/k[[x]]$, that is étale outside $x = 0$, totally ramified at $x = 0$, and such that the formal completion of $\overline{Y} \to \overline{C}$ at $x = 0$ yields the extension $L_n/k[[x]]$ (see [Har80, Kat86]). This allows one to go from a local back to a global situation. Specifically, Theorem 2.34 is equivalent to the following version for one-point covers (of curves), which is compatible with the language used in [OW14] and allows us to only deal with rational functions instead of Laurent series. The proof of Proposition 2.35, hence of Theorem 2.34, is deferred to Section 5.

Proposition 2.35. *Suppose that $k[[y_n]]/k[[x]]$ is a $G \cong \mathbb{Z}/p^n$-Galois extension. Let $\psi_{n-1}\colon Y_{n-1} \to C := \mathbb{P}^1_K$ be a $\mathbb{Z}/p^{n-1}$-cover with the following properties:*

(i) *The cover ψ_{n-1} has good reduction with respect to the standard model $\mathbb{P}^1_R$ of C and reduces to a $\mathbb{Z}/p^{n-1}$-cover $\overline{\psi}_{n-1}\colon \overline{Y}_{n-1} \to \overline{C} \cong \mathbb{P}^1_k$ that is totally ramified above $x = 0$ and étale everywhere else.*

(ii) *The completion of $\overline{\psi}_{n-1}$ at $x = 0$ yields $k[[y_{n-1}]]/k[[x]]$, the unique $\mathbb{Z}/p^{n-1}$-subextension of $k[[y_n]]/k[[x]]$. We may thus assume that $k[[y_n]]/k[[x]]$ is given by $\underline{g}_n = (g^1, \dots, g^n) \in W_n(k(x))$; that is, all the entries are rational.*

Then, after possibly a finite extension of R, there exists a G-Galois cover $\psi_n\colon Y_n \to C$ with good reduction that extends ψ_{n-1} and satisfies the following:

(a) *Its reduction $\overline{\psi}_n\colon \overline{Y}_n \to \overline{C}$ is totally ramified above $\overline{x} = 0$ and étale elsewhere.*

(b) *The completion of $\overline{\psi}_n$ at $x = 0$ yields $k[[y_n]]/k[[x]]$.*

Remark 2.36. Items (a) and (b) of Proposition 2.35 can be reformulated as follows:

(1) The cover $Y_n \to C$ is étale outside the open disc

$$D := \{X \in K \mid |X| < 1\}\,.$$

(2) The inverse image of D in Y_n is an open disc.

(3) If $A = R[[X]]\{X^{-1}\}$ is the ring

$$\left\{\sum_{j\in\mathbb{Z}} a_j X^j \mid a_j \in R,\ a_j \to 0 \text{ as } j \to -\infty\right\} \quad \text{(see Section 3.2)}\,,$$

the cover $Y_n \to C$ is unramified when base changed to $\operatorname{Spec} A$, which can be thought of as the boundary of the disc D. The extension of residue fields is isomorphic to the extension of fraction fields coming from $k[[y_n]]/k[[x]]$.

2.4.1 *Proof of the "$\Rightarrow$" direction of Proposition* 2.21. With the assumptions of Theorem 2.20, there exists an H-cover $\psi_m\colon Y_m \to \mathbb{P}^1_K = \operatorname{Proj} K[X, V]$ that

(i) has properties (i) and (ii) of Proposition 2.35, where H is in place of $\mathbb{Z}/p^{n-1}$ (G is in place of $\mathbb{Z}/p^n$), and

(ii) is such that its standard model's completion at $X = 0$ is isomorphic to $R[[Y_m]]/R[[X]]$,

as discussed in Section 2.4. It thus follows from Theorem 1.2 that one can extend ψ_m to a G-cover ψ_n over K whose standard model's completion at $X = 0$ is a deformation Φ_n of ϕ_n over R that contains $R[[Y_m]]/R[[X]]$ as the H-subcover. The extension Φ_n is exactly what we seek. □

3. Degeneration of cyclic covers

In this section, we study the degeneration of $G := \mathbb{Z}/p^n$-covers of a curve C as in Section 2.4.

3.1 Setup

Throughout the section, R is a complete discrete valuation ring of characteristic p with uniformizer π, valuation ν, and residue field k (for example, $R = k[[t]]$ and $\pi = t$). Normalize the valuation on R so that $\nu(\pi) = 1$. Let K be the fraction field of R. Let C be a smooth, projective, irreducible curve over K. Note that, for the purpose of this paper, we only need to consider the case $C \cong \mathbb{P}^1_K$. We denote by $\mathbb{K}$ the function field of C (for example, $\mathbb{K} = \operatorname{Frac} K[X]$). We may fix a smooth R-model C_R of C over R. Fix a rational point x_0 on C and write $\overline{x}_0 \in \overline{C}$ for its specialization.

We denote by C^{an} the rigid analytic space associated with C. The *residue class* of x_0 with respect to the model C_R, denoted by

$$D :=]\overline{x}_0[_{C_R} \subset C^{\mathrm{an}},$$

is the set of points of C^{an} specializing to $\overline{x}_0$. It is an open subspace of C^{an} and is isomorphic to the open unit disc.

The central focus of this section is examining the degeneration of cyclic covers of C^{an} that are étale outside D. In the next subsection, we briefly recall some non-archimedean geometry notions that are crucial for the study of the degeneration.

3.2 Discs and annuli

Suppose $\epsilon \in \mathbb{Q}_{\geqslant 0}$, $r = p^{-\epsilon}$, $z \in R$, and let $a \in K$ be such that $\nu(a) = \epsilon$. In Table 1, we list some usual rigid geometry conventions. For more details, see [BL93, BL85, BL84].

Notion	Algebra	Geometry
Open unit disc	$\operatorname{Spec} R[[X]]$	$D = \{u \in (\mathbb{A}^1_K)^{\mathrm{an}} \mid \nu(u) > 0\}$
Closed unit disc	$\operatorname{Spec} R\{X\}$	$\mathcal{D} = \left\{u \in \left(\mathbb{A}^1_K\right)^{\mathrm{an}} \mid \nu(u) \geqslant 0\right\}$
Boundary of a unit disc	$\operatorname{Spec} R[[X]]\{X^{-1}\}$	$\left\{u \in \left(\mathbb{A}^1_K\right)^{\mathrm{an}} \mid \nu(u) = 0\right\}$
Open disc of radius r center z	$\operatorname{Spec} R\left[\left[a^{-1}(X-z)\right]\right]$	$D[\epsilon, z] := \left\{u \in \left(\mathbb{A}^1_K\right)^{\mathrm{an}} \mid \nu(u-z) > \epsilon\right\}$
Open disc of radius r center 0	$\operatorname{Spec} R\left[\left[a^{-1}X\right]\right]$	$D[\epsilon] := D[\epsilon, 0]$
Open annulus of thickness ϵ	$\operatorname{Spec} R[[X, U]]/(XU - a)$	$\left\{u \in \left(\mathbb{A}^1_K\right)^{\mathrm{an}} \mid 0 < \nu(u) < \epsilon\right\}$

Table 1. Non-archimedean geometry notions

Recall that $R\{X\} = \left\{\sum_{i \geqslant 0} a_i X^i \in R[[X]] \mid \lim_{i \to \infty} \nu(a_i) = \infty\right\}$. Let $z \in R$ and $s \in \mathbb{Q}_{\geqslant 0}$. One can associate with $\mathcal{D}[s, z]$ the "Gauss valuation" $\nu_{s,z}$ defined by

$$\nu_{s,z}(f) = \inf_{a \in \mathcal{D}[s,z]} (\nu(f(a)))$$

for each $f \in \mathbb{K}^{\times}$. This is a discrete valuation on $\mathbb{K}$ that extends the valuation ν on K and has the property that $\nu_{s,z}(X - z) = s$. We denote by κ_s the function field of $\mathbb{K}$ with respect to the valuation $\nu_{s,z}$. That is the function field of the canonical reduction $\overline{\mathcal{D}}[s, z]$ of $\mathcal{D}[s, z]$. In fact, $\overline{\mathcal{D}}[s, z]$ is isomorphic to the affine line over k with function field $\kappa_{s,z} := k(x_{s,z})$, where $x_{s,z}$ is the image of $\pi^{-s}(X - z)$ in $\kappa_{s,z}$. For a closed point $\overline{x} \in \overline{\mathcal{D}}[s, z]$, we let $\operatorname{ord}_{\overline{x}} \colon \kappa_{s,z}^{\times} \to \mathbb{Z}$ denote the normalized discrete valuation corresponding to the specialization of $\overline{x}$ on $\overline{\mathcal{D}}[s, z]$. We let $\operatorname{ord}_{\infty}$ denote the unique normalized discrete valuation on $\kappa_{s,z}$ associated with the "point at infinity."

For $b \in \mathbb{K}$ and $r \in \mathbb{Q}_{\geqslant 0}$, we define $(b)_r := b\pi^{-pr}$. We will usually write $X_r := X\pi^{-pr}$. For $F \in \mathbb{K}$, $z \in \left(\mathbb{A}^1_K\right)^{\mathrm{an}}$, and $s \in \mathbb{Q}_{\geqslant 0}$, we let $[F]_{s,z}$ stand for the image of $\pi^{-\nu_{s,z}(F)} F$ in $\kappa_{s,z}$.

3.3 Semi-stable models and a partition of a disc

Consider the open unit disc $D \subset C^{\mathrm{an}} \cong \left(\mathbb{P}^1_K\right)^{\mathrm{an}}$, which we may associate with $\operatorname{Spec} R[[X]]$ for some $X \in \mathbb{K}$. Suppose that we are given $x_{1,K}, \ldots, x_{r,K}$ in $D(K)$, with $r \geqslant 2$. We can think of $x_{1,K}, \ldots, x_{r,K}$ as elements of the maximal ideal of R. Let C^{st} be a blow-up of C_R such that

- the exceptional divisor $\overline{C}$ of the blow-up is a semi-stable curve over k,
- the fixed points $x_{b,K}$ specialize to pairwise-distinct smooth points x_b on $\overline{C}$, and
- if x_0 (which we usually denote by $\overline{\infty}$) denotes the unique point on C that lies in the closure of $C^{\mathrm{st}} \otimes k \setminus \overline{C}$, then $\bigl(\overline{C}, (x_b), x_0\bigr)$ is stably marked in the sense of [Knu83].

Then we call C^{st} the *stable model of C corresponding to the marked disc* $(D; x_1, \dots, x_r)$. See, for example, [Ake15, § 2.5.3] for more details. Note that the set of points of $(\mathbb{P}^1_K)^{\mathrm{an}}$ that specialize away from x_0 form the closed unit disc $\mathcal{D}$. The dual graph of $\overline{C}$ is a tree whose

- *leaves* correspond to the marked points,
- *root* corresponds to x_0,
- *vertices* correspond to the punctured discs,
- *edges* correspond to the annuli that partition D.

For each edge e of the tree, the *source* (respectively, the *target*) of e is the unique vertex $s(e) \in V$ (respectively, $t(e) \in V$) adjacent to e that lies in the direction toward the root (respectively, in the direction away from the root). For each vertex v of the dual graph, we denote by U_v the corresponding closed punctured disc.

Example 3.1. Let us consider the $\mathbb{Z}/4$-cover in Example 2.27. Recall that χ_2 branches at four points 0, t^8, t^2, and $t^2(1+t^2)$, all contained in the open disc D associated with $\operatorname{Spec} R[[X]]$. The left graph in Figure 1 represents a semi-stable model of $\mathbb{P}^1_K$ corresponding to the disc D marked by the branch points of the cover. The model is obtained by first blowing up $\operatorname{Spec} R[[X]]$ with respect to the ideal (t^2, X), which separates the reductions of t^2 and t^2+t^4 from those of 0 and t^8. We can then distinguish t^2 and t^2+t^4 by doing the same for $\operatorname{Spec} R[[X_1^{-1}]]$, where $X_1 := X/t^2$, with respect to $(t^2, X_1^{-1}-1)$.

The tree on the right in Figure 1 is the special fiber's dual graph together with the leaves (labeled by $\bigl[\overline{t^2}\bigr]$, $\bigl[\overline{t^2+t^4}\bigr]$, $\bigl[\overline{t^8}\bigr]$, and $\bigl[\overline{0}\bigr]$) that are associated with the corresponding marked points. See [Dan20b, Example 3.1] for more explanation.

The vertex v_1 represents the punctured disc associated with $\operatorname{Spec} R\bigl\{X_1^{-1}, X_1, \bigl(X_1^{-1}-1\bigr)^{-1}\bigr\}$. The points t^2 and t^2+t^4 (respectively, 0 and t^8) of D lie inside $\operatorname{Spec} R\bigl(X_1^{-1}-1\bigr)$ (respectively, $\operatorname{Spec} R(X_1)$) and reduce to 1 (respectively, 0) on its special fiber. Table 2 illustrates where the $\overline{K}$-points of D specialize.

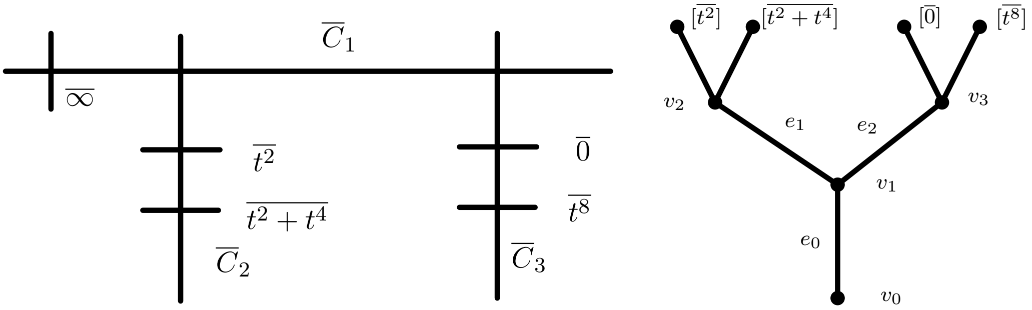


Figure 1. The special fiber $\overline{C}$ and its dual graph

Subscheme $\overline{V}$ of $\overline{C}$	Points of $C(K)$ that specialize to $\overline{V}$	Associated algebraic object
$\overline{\infty}$	$\{Y \mid 0 < \nu(Y) < 2\}$	$R[[X, X_1]]/\big(XX_1 - t^2\big)$
$\overline{C}_3 \cap \overline{C}_1$	$\{Y \mid 2 < \nu(Y) < 8\}$	$R\big[\big[X_1^{-1}, X_2\big]\big]/\big(X_1^{-1}X_2 - t^6\big)$
$\overline{C}_3 \setminus \overline{C}_1$	$\{Y \mid \nu(Y) \geqslant 8\}$	$R\big\{X_2^{-1}\big\}$
$\overline{C}_2 \cap \overline{C}_1$	$\big\{Y \mid 2 < \nu\big(Y - t^2\big) < 4\big\}$	$R\big[\big[X_1^{-1} - 1, V\big]\big]\big(V\big(X_1^{-1} - 1\big) - t^2\big)$
$\overline{C}_2 \setminus \overline{C}_1$	$\big\{Y \mid \nu\big(Y - t^2\big) \geqslant 4 \wedge \nu(Y) = 2\big\}$	$R\big\{V^{-1}\big\}$
$\overline{C}_1 \setminus (\overline{C}_3 \cup \overline{C}_2 \cup \{\overline{\infty}\})$	$\big\{Y \mid \nu(Y) = 2 \wedge \nu\big(Y - t^2\big) = 2\big\}$	$R\big\{X_1^{-1}, X_1, \big(X_1^{-1} - 1\big)^{-1}\big\}$

Table 2. Partitions of $C(K)$

Remark 3.2. In Example 3.1, we say that the directions from v_1 toward e_2, e_1, and e_0, are the 0-, 1-, and ∞-*directions*, respectively.

3.3.1 *A coordinate system for a marked disc.* With the notation of Section 3.3, we define a 1-dimensional "coordinate system" for each edge e of the dual graph of the special fiber of $\overline{C}$, as follows. One may assume that e corresponds to the annulus $\{Y \in K \mid pr_1 < \nu(Y - x_j) < pr_2\}$. We identify e with a rational line segment $[r_1, r_2] \cap \mathbb{Q}$ and assign the value r_1 to $s(e)$ and the value r_2 to $t(e)$. We then naturally associate a rational number $r \in [r_1, r_2]$ with the circle $\{Y \mid \nu(Y - x_j) = pr\}$. We refer to r as a *rational place* on e. Suppose that r is a rational place on an edge e, and r' is a rational place on a succeeding edge of e. Then we say $r < r'$. By abuse of notation, we usually write $t(e)$ in place of r_2 and $s(e)$ in place of r_1. The reason we scale the coordinate by $1/p$ is to make it compatible with the calculation of the depth Swan conductor introduced in Section 3.5, which, in turn, is also modified to be consistent with [Saï07] and [Dan20b].

3.4 Characters

Fix $n \geqslant 1$. We set

$$\mathrm{H}^1_{p^n}(\mathbb{K}) := \mathrm{H}^1(\mathbb{K}, \mathbb{Z}/p^n) \overset{\mathrm{ASW}}{\cong} W_n(\mathbb{K})/\wp(W_n(\mathbb{K}))\,. \tag{3.1}$$

The "$\overset{\mathrm{ASW}}{\cong}$" in (3.1) is just Artin–Schreier–Witt theory (Section 2.1). We call the identification ASW the *Artin–Schreier–Witt map*. The elements of $\mathrm{H}^1_{p^n}(\mathbb{K})$ are called the *characters* on C. Given an element $\underline{F}_n \in W_n(\mathbb{K})$, we let $\mathfrak{K}_n(\underline{F}_n) \in \mathrm{H}^1_{p^n}(\mathbb{K})$ denote the character corresponding to the class of $\underline{F}_n$ in $W_n(\mathbb{K})/\wp(W_n(\mathbb{K}))$.

For $i = 1, \ldots, n$, the homomorphism

$$W_i(\mathbb{F}_p) \cong \mathbb{Z}/p^i\mathbb{Z} \xhookrightarrow{V^{n-i}} \mathbb{Z}/p^n \cong W_n(\mathbb{F}_p)\,, \quad \big(a^1, \ldots, a^i\big) \longmapsto \big(0, \ldots, 0, a^1, \ldots, a^i\big)$$

induces an injective homomorphism

$$\mathrm{H}^1_{p^i}(\mathbb{K}) \xrightarrow{\phi_{i,n}} \mathrm{H}^1_{p^n}(\mathbb{K})\,. \tag{3.2}$$

Its image consists of all characters killed by p^i. We consider $\mathrm{H}^1_{p^i}(\mathbb{K})$ as a subgroup of $\mathrm{H}^1_{p^n}(\mathbb{K})$ via this embedding.

A character $\chi \in \mathrm{H}^1_{p^n}(\mathbb{K})$ gives rise to a branched Galois cover $Y \to C$. In particular, if $\chi = \mathfrak{K}_n(\underline{F})$ for some $\underline{F} \in W_n(\mathbb{K})$, then Y is a connected component of the smooth projective curve given generically by the ASW equation $\wp(y) = \underline{F}$. If χ has order p^i as an element of $\mathrm{H}^1_{p^n}(\mathbb{K})$, then the Galois group of $Y \to C$ is the unique quotient of $\mathbb{Z}/p^n\mathbb{Z}$ of order p^i.

Remark 3.3. If one identifies $\mathrm{H}^1_{p^i}(\mathbb{K})$ with $W_i(\mathbb{K})/\wp(W_i(\mathbb{K}))$, then the map $\phi_{i,n}$ assigns to the class of a length i Witt vector $(a^1, \ldots, a^i)$ the class of the length n Witt vector $(0, \ldots, 0, a^1, \ldots, a^i)$.

Remark 3.4. If $\chi := \mathfrak{K}((a^1, \ldots, a^n)) \in \mathrm{H}^1_{p^n}(\mathbb{K})$, then $\chi^{p^{n-i}} \cong \mathfrak{K}((a^1, \ldots, a^i)) \in \mathrm{H}^1_{p^i}(\mathbb{K})$.

A point $x \in C$ is called a *branch point* for the character $\chi \in \mathrm{H}^1_{p^n}(\mathbb{K})$ if it is a branch point for the cover $Y \to C$. The *branching index* of x is the order of the inertia group for some point $y \in Y$ above x. The set of all branch points is called the *branch locus* of χ and is denoted by $\mathbb{B}(\chi)$.

Definition 3.5. A character $\chi \in \mathrm{H}^1_{p^n}(\mathbb{K})$ is called *admissible* if its branch locus $\mathbb{B}(\chi)$ is contained in an open unit disc $D \subset C^{\mathrm{an}}$. We call its associated cover an *admissible cover*.

Definition 3.6. Suppose that $\chi := \mathfrak{K}((f^1, \ldots, f^n)) \in \mathrm{H}^1_{p^n}(\mathbb{K})$ is an admissible character and $\mathbb{B}(\chi) = \{b_1, \ldots, b_l\}$. We call the dual graph of the semi-stable model of $\mathbb{P}^1_K$ corresponding to the marked disc $(D; b_1, \ldots, b_l)$ (discussed in Section 3.3) the *branching geometry* of χ, or the *geometry of the branch points* of χ, or the *geometry of the poles* of $(f^1, \ldots, f^n)$.

3.4.1 *Reduction of characters.* Let $\chi \in \mathrm{H}^1_{p^n}(\mathbb{K})$ be an admissible character of order p^n, and let $Y \to C$ be the corresponding cyclic Galois cover. Let Y_R be the normalization of C_R in Y. Then Y_R is a normal R-model of Y, and we have $C_R = Y_R/(\mathbb{Z}/p^n)$.

After enlarging our ground field K, we may assume that the character χ is weakly *unramified* with respect to the valuation ν_0 (where X has valuation 0); see [Epp73] (there is an error, corrected in [Kuh03]). By definition, this means that for all extension ω of ν_0 to the function field of Y, the ramification index $e(\omega/\nu_0)$ is equal to 1. It then follows that the special fiber $\overline{Y} = Y_R \otimes_R k$ is reduced [AW12, § 2.2].

Definition 3.7. We say that the character χ has *étale reduction* if the map $\overline{Y} \to \overline{C}$ is generically étale. It has *good reduction* if, in addition, $\overline{Y}$ is smooth.

Remark 3.8. In the language of Galois cohomology, a character χ has étale reduction if and only if its restriction to the completion $\hat{\mathbb{K}}_0$ of $\mathbb{K}$ with respect to the valuation ν_0 is unramified. This means that $\chi|_{\hat{\mathbb{K}}_0}$ belongs to the image of the cospecialization map

$$\mathrm{H}^1_{p^n}(\kappa_0) \longrightarrow \mathrm{H}^1_{p^n}\big(\hat{\mathbb{K}}_0\big),$$

which is the restriction map induced by the projection of the absolute Galois groups $\mathrm{Gal}_{\hat{\mathbb{K}}_0} \to \mathrm{Gal}_{\kappa_0}$. Since the cospecialization map is injective, there is a unique character $\overline{\chi} \in \mathrm{H}^1_{p^n}(\kappa_0)$ whose image in $\mathrm{H}^1_{p^n}\big(\hat{\mathbb{K}}_0\big)$ matches $\chi|_{\hat{\mathbb{K}}_0}$. The Galois cover of $\overline{C}$ associated with $\overline{\chi}$ is, by construction, isomorphic to an irreducible component of the normalization of $\overline{Y}$.

Definition 3.9. If χ has étale reduction, we call $\overline{\chi}$ the *reduction* of χ, and χ a *deformation* of $\overline{\chi}$ over R.

With the new definition of good reduction, we can reformulate Proposition 2.35 as follows.

Proposition 3.10. *Proposition 2.35 holds if there exists an admissible extension $\psi_n \colon Y_n \to C$ of $\psi_{n-1} \colon Y_{n-1} \to C$ with the following properties:*

(i) *The map ψ_n has good reduction.*

(ii) *The completion of the reduction $\overline{\psi}_n$ at $x = 0$ is birationally equivalent to $k[[y_n]]/k[[x]]$.*

3.5 Swan conductors

Suppose that $\chi \in \mathrm{H}^1_{p^n}(\mathbb{K})$ is a character associated with a cyclic p^n-exponent cover of C. Suppose, moreover, that $\mathcal{D} \subset C^{\mathrm{an}}$ is a closed disc equipped with the topology corresponding to the canonical valuation ν_0 (see Section 3.2). After enlarging R, we may assume that the restriction $\chi|_{\mathcal{D}}$ is weakly unramified with respect to ν_0. As usual, the residue field of $\mathrm{Frac}(\mathcal{D})$ is denoted by κ.

As in [Dan20b, § 3.4], we define two invariants that measure the degeneration of $\chi|_{\mathcal{D}}$ with respect to the valuation ν_0. The *depth* is

$$\delta_{\chi|_{\mathcal{D}}} := \mathrm{sw}(\chi|_{\mathcal{D}})/p \in \mathbb{Q}_{\geqslant 0}\,,$$

where $\mathrm{sw}(\chi|_{\mathcal{D}})$ is the classical Swan conductor [Kat89, Definition 3.3] of $\chi|_{\mathcal{D}}$. Note that the rational number $\delta_{\chi|_{\mathcal{D}}}$ is equal to 0 if and only if $\chi|_{\mathcal{D}}$ is unramified. If this is the case, then its reduction $\overline{\chi}|_{\mathcal{D}}$ is well defined. In particular, if $\chi|_{\mathcal{D}}$ is of order p^n and $\delta_{\chi|_{\mathcal{D}}} = 0$, then there exists an $\underline{f} \in W_n(\kappa)$ such that $\overline{\chi}|_{\mathcal{D}}$ is defined by $\wp(y) = \underline{f}$. We call $\underline{f}$ a *reduction* of $\chi|_{\mathcal{D}}$. Also note that, as discussed in Section 2.1, a reduction $\underline{f}$ is unique up to adding an element of the form $\wp(a)$, where $a \in W_n(\kappa)$, and there exists a unique $\underline{f}^{\mathrm{red}} = \underline{f} + \wp(b)$ for some $b \in W_n(\kappa)$, which we call the *reduced reduction* of $\chi|_{\mathcal{D}}$. We say that $\chi|_{\mathcal{D}}$ is *radical* if $\delta_{\chi|_{\mathcal{D}}} > 0$.

Suppose $\delta_{\chi|_{\mathcal{D}}} > 0$. Then we can define the *differential Swan conductor* or *differential conductor*

$$\omega_{\chi|_{\mathcal{D}}} := \mathrm{dsw}(\chi|_{\mathcal{D}}) \in \Omega^1_{\kappa}$$

identically to how it is defined in the mixed-characteristic case in [Kat89, Definition 3.9] (see also [Wew14, § 3.2]). It is derived from the *refined Swan conductor* $\mathrm{rsw}^{\mathrm{ab}}(\chi|_{\mathcal{D}})$, see [Kat89], and depends on the choice of the uniformizer π for R. In particular, we have the relation

$$\mathrm{rsw}^{\mathrm{ab}}(\chi|_{\mathcal{D}}) = \pi^{-sw(\chi|_{\mathcal{D}})} \otimes \mathrm{dsw}(\chi|_{\mathcal{D}}) \in \mathfrak{m}^{-\mathrm{sw}(\chi|_{\mathcal{D}})} \otimes_{\mathcal{O}_{\mathbb{K}}} \Omega^1_{\kappa}\,, \tag{3.3}$$

where $\mathfrak{m}$ is the maximal ideal of R. Note that $\mathrm{rsw}^{\mathrm{ab}}(\chi|_{\mathcal{D}})$ does not depend on the choice of π.

We call the pair $(\delta_{\chi|_{\mathcal{D}}}, \omega_{\chi|_{\mathcal{D}}})$ when $\delta_{\chi|_{\mathcal{D}}} > 0$, or $(\delta_{\chi|_{\mathcal{D}}}, \underline{f})$ when $\delta_{\chi|_{\mathcal{D}}} = 0$ and $\underline{f}$ is the reduced reduction of $\chi|_{\mathcal{D}}$, the *degeneration type* or the *reduction type* of the restriction $\chi|_{\mathcal{D}}$.

Suppose that $\overline{x}$ is a point on the canonical reduction of $\mathcal{D}$ or the point at infinity, which we write $\overline{x} = \infty$, and let $\mathrm{ord}_{\overline{x}} \colon \kappa^\times \to \mathbb{Z}$ be the normalized discrete valuation (whose restriction to k is trivial) corresponding to $\overline{x}$. Then the composite of ν_0 with $\mathrm{ord}_{\overline{x}}$ is a valuation on K of rank 2, which we denote by $\mathbb{K}^\times \to \mathbb{Q}\times\mathbb{Z}$. In [Kat87b], Kato defined a Swan conductor $\mathrm{sw}^K_{\chi|_{\mathcal{D}}}(\overline{x}) \in \mathbb{Q}_{\geqslant 0}\times\mathbb{Z}$. Its first component is equal to $p\delta_{\chi|_{\mathcal{D}}}$. We define the *boundary Swan conductor with respect to* $\overline{x}$,

$$\mathrm{sw}_{\chi|_{\mathcal{D}}}(\overline{x}) \in \mathbb{Z}\,,$$

as the second component of $\mathrm{sw}^K_{\chi|_{\mathcal{D}}}(\overline{x})$. Geometrically, it gives the instantaneous rate of change of the depth in the direction (see Remark 3.2) corresponding to $\overline{x}$. We will discuss this phenomenon in detail in Section 3.5.2.

Remark 3.11. The invariant $\mathrm{sw}_{\chi|_{\mathcal{D}}}(\overline{x})$ is determined by $\delta_{\chi|_{\mathcal{D}}}$ and $\omega_{\chi|_{\mathcal{D}}}$ as follows:

(i) If $\delta_{\chi|_{\mathcal{D}}} = 0$, then

$$\mathrm{sw}_{\chi|_{\mathcal{D}}}(\overline{x}) = \mathrm{sw}_{\overline{\chi}|_{\mathcal{D}}}(\overline{x})\,,$$

where $\overline{\chi}|_{\mathcal{D}}$ is the reduction of $\chi|_{\mathcal{D}}$ (see Remark 3.8), and $\mathrm{sw}_{\overline{\chi}|_{\mathcal{D}}}(\overline{x})$ is the usual Swan conductor of $\overline{\chi}|_{\mathcal{D}}$ with respect to the valuation $\mathrm{ord}_{\overline{x}}$; see [Ser79, § IV.2]. This follows immediately from the definitions of the conductors. We thus have $\mathrm{sw}_{\chi|_{\mathcal{D}}}(\overline{x}) \geqslant 0$ and $\mathrm{sw}_{\chi|_{\mathcal{D}}}(\overline{x}) = 0$ if and only if $\overline{\chi}|_{\mathcal{D}}$ is unramified with respect to $\mathrm{ord}_{\overline{x}}$.

(ii) If $\delta_{\chi|_{\mathcal{D}}} > 0$, then [Kat87b, Corollary 4.6] asserts that

$$\mathrm{sw}_{\chi|_{\mathcal{D}}}(\overline{x}) = -\operatorname{ord}_{\overline{x}}(\omega_{\chi|_{\mathcal{D}}}) - 1\,.$$

This fact will be used intensively later in this paper.

3.5.1 *Refined Swan conductors of a product of characters.* The Swan conductors behave somewhat like valuations.

LEMMA 3.12. *Let χ_i, with $i = 1, 2, 3$, be abelian characters on a disc satisfying the relation $\chi_3 = \chi_1 \cdot \chi_2$. Set $\delta_i := \mathrm{sw}(\chi_i)/p$ (with the canonical valuation) and $\omega_i := \mathrm{dsw}(\chi_i)$ for $i = 1, 2, 3$. Then the following hold:*

- (i) *If $\delta_1 \neq \delta_2$, then $\delta_3 = \max\{\delta_1, \delta_2\}$. Furthermore, we have $\omega_3 = \omega_1$ if $\delta_1 > \delta_2$ and $\omega_3 = \omega_2$ otherwise.*
- (ii) *If $\delta_1 = \delta_2 > 0$ and $\omega_1 + \omega_2 \neq 0$, then $\delta_1 = \delta_2 = \delta_3$ and $\omega_3 = \omega_1 + \omega_2$.*
- (iii) *If $\delta_1 = \delta_2 > 0$ and $\omega_1 + \omega_2 = 0$, then $\delta_3 < \delta_1$.*
- (iv) *If $\delta_1 = \delta_2 = 0$, then $\delta_3 = 0$ and $\overline{\chi}_3 = \overline{\chi}_1 \cdot \overline{\chi}_2$. Hence, if the reduction type of χ_1 (respectively, χ_2) is $\underline{f}$ (respectively, $\underline{g}$), then the reduction type of χ_3 is $\underline{f} + \underline{g}$.*

Proof. The proof is parallel to that of [OW14, Proposition 5.6]. □

Remark 3.13. Lemma 3.12 allows one to compute the conductors of one character by breaking it down into easier-to-calculate ones. This philosophy will be employed frequently later in this paper.

Remark 3.14. With the notation of Lemma 3.12, suppose $m \geqslant n \in \mathbb{Z}_{\geqslant 0}$, and let $\chi_1 \in \mathrm{H}^1_{p^n}(\mathbb{K})$ (respectively, $\chi_2 \in \mathrm{H}^1_{p^m}(\mathbb{K})$) be defined by the Witt vector $\underline{G} \in W_n(\mathbb{K})$ (respectively, $\underline{H} \in W_m(\mathbb{K})$). Then $\chi_1 \cdot \chi_2$ correspond to the Witt vector $\underline{H} + \phi_{n,m}(\underline{G}) \in W_m(\mathbb{K})$ (where $\phi_{n,m}$ is defined in (3.2)).

3.5.2 Fix a closed disc $\mathcal{D}$, which may be associated with the ring $R\{X\}$. Let z be a K-point of $\mathcal{D}$ and $r \in \mathbb{Q}_{\geqslant 0}$. We denote by $\delta_\chi(r, z)$ (respectively, $\delta_\chi(r)$) and $\omega_\chi(r, z)$ (respectively, $\omega_\chi(r)$) the depth and the differential conductors of the restriction of χ to $\mathcal{D}[pr, z]$ (respectively, $\mathcal{D}[pr]$). Following the notation in [OW14], we define $\nu_r \colon \mathbb{K}^\times \to \mathbb{Q}$ as the Gauss valuation associated with $D[pr]$, such that $\nu_r(X) = pr$. Furthermore, let κ_r represent the residue of $\mathbb{K}$ with respect to the valuation ν_r.

When the point z is fixed, we may regard $\delta_\chi(r, z)$ and $\omega_\chi(r, z)$ as functions of r. Suppose that $\overline{y}$ is a point on the reduction of $\mathcal{D}[pr, z]$, or a point at infinity $\overline{y} = \overline{\infty}$. Set $\mathrm{sw}_\chi(r, z, \overline{y}) := \mathrm{sw}_{\chi|_{\mathcal{D}[pr,z]}}(\overline{y})$. The following results show how understanding a character's differential conductor can give a lot of information about its depth. They are the exact analogs of results from [OW14, § 5.3.2] in mixed characteristic.

PROPOSITION 3.15. *Suppose that $z \in R$ is fixed. Then $\delta_\chi(-, z)$ extends to a continuous, piecewise-linear function*

$$\delta_\chi(-, z) \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}\,.$$

Furthermore,

- (i) *for $r \in \mathbb{Q}_{>0}$, the left (respectively, right) derivative of $\delta_\chi(-, z)$ at r is $-\mathrm{sw}_\chi(r, z, \overline{\infty})$ (respectively, $\mathrm{sw}_\chi(r, z, \overline{0})$);*

(ii) *if r is a kink of $\delta_\chi(-,z)$ (meaning that the left and the right derivatives do not agree), then $r \in \mathbb{Q}$.*

Proof. See, for example, [Wew05, Proposition 2.9] or [Bah20, Theorem 1.9]. Note that, in the language of [Bah20], a lisse étale sheaf of $\mathcal{F}_\ell$-module on a rigid disc $\mathcal{D}$ coincides with a connected étale cover of $\mathcal{D}$; see [dJo95, Theorem 2.10]. □

Corollary 3.16. *If $r \in \mathbb{Q}_{\geqslant 0}$ and $\delta_\chi(r,z) > 0$, then the left and right derivatives of δ_χ at r are given by $\operatorname{ord}_{\overline{\infty}}(\omega_\chi(r)) + 1$ and $-\operatorname{ord}_{\overline{0}}(\omega_\chi(r)) - 1$, respectively. In particular, $-\operatorname{ord}_{\overline{x}}(\omega_\chi(r,z)) - 1$ is the instantaneous rate of change of $\delta_\chi(r,z)$ in the direction with respect to $\overline{x}$.*

Proof. This is immediate from Proposition 3.15(i) and Remark 3.11(ii). □

Definition 3.17. Due to the above results, we can define the refined Swan conductors of the restriction of χ to an open disc $D[pr,z]$ as $\delta(\chi|_{D[pr,z]}) := \delta(\chi|_{\mathcal{D}[pr,z]})$ and $\omega(\chi|_{D[pr,z]}) := \omega(\chi|_{\mathcal{D}[pr,z]})$. We also set the boundary conductor of $\chi|_{D[pr,z]}$ to be that of $\chi|_{\mathcal{D}[pr,z]}$.

The result below describes how the refined Swan conductors vary under a change of coordinates. In particular, it shows that the depth Swan conductor of a cover (of a disc) does not depend on the choice of the parameter.

Proposition 3.18. *Suppose that z_1 and z_2 are two K-points in the interior of a closed unit disc $\mathcal{D}$. Suppose, moreover, that $\nu(z_1 - z_2) \geqslant r$ for some non-negative rational r, and $(z)_r := (z_1 - z_2)\pi^{-r}$. Then*

$$\delta_\chi(r,z_1) = \delta_\chi(r,z_2)\,.$$

Moreover, if $\omega_\chi(r,z_1) = f(x)dx$, then $\omega_\chi(r,z_2) = f\big(x + \overline{(z)_r}\big)dx$. In particular, if $\nu(z_1 - z_2) > r$, then $\omega_\chi(r,z_1)$ coincides with $\omega_\chi(r,z_2)$.

Proof. We first note that $\mathcal{D}[pr,z_1] = \mathcal{D}[pr,z_2]$ as $\nu(z_1 - z_2) \geqslant r$. The first part is then immediate from the definition of the depth Swan conductor. Suppose that X is a parameter of $\mathcal{D}[pr,z_1]$. Then $X + (z_1 - z_2)$ is a parameter of $\mathcal{D}[pr,z_2]$. The second part follows from the change of variables $X \mapsto X + (z_1 - z_2)$. □

3.6 A vanishing cycle formula

In this section, we discuss how the refined Swan conductors determine whether a character has good reduction. Most of the details closely follow [OW16, § 2.4], which examines the case of mixed characteristic.

We fix an admissible character $\chi \in \mathrm{H}^1_{p^n}(\mathbb{K})$ of order p^n, which may be identified with a cover $\phi\colon Y \to C$. Let $b_1, \dots, b_r \in K$ be the branch points of ϕ, which are contained in an open rigid disc $D \subset C^{\mathrm{an}}$ (as χ is admissible), and let $\iota_{i,n}$ be the nth conductor at b_i (see Section 2.2.1). Let us also fix $r \in \mathbb{Q}_{\geqslant 0}$.

Definition 3.19. With the notation as above, let z be a point inside D, and let $\overline{w}$ be a closed point of $\overline{\mathcal{D}}[pr,z]$. We denote by $U(r,z,\overline{w}) :=]\overline{w}[_{\mathcal{D}[pr,z]}$ the residue class of $\overline{w}$ in the affinoid $\mathcal{D}[pr,z]$. Set

$$\mathfrak{C}_\chi(r,z,\overline{w}) := \sum_{i \in I(r,z,\overline{w})} \iota_{i,n} \in \mathbb{Z}_{\geqslant 0}\,,$$

where $I(r,z,\overline{w}) = \{i \in \{1,2,\dots,r\} \mid b_i \in \mathbb{B}(\chi) \cap U(r,z,\overline{w})\}$.

Proposition 3.20. *With the notation introduced above, suppose $\overline{w} \neq \overline{\infty} \in \kappa$, considered as a closed point of $\overline{D}[pr, z]$. We have*

$$\operatorname{ord}_{\overline{w}}(\omega_{\chi}(r,z)) \geqslant -\mathfrak{C}_{\chi}(r,z,\overline{w})\,. \tag{3.4}$$

Moreover, equality holds if χ has good reduction. When $\overline{w} = \overline{\infty}$, let g_C be the genus of C. The following holds:

$$\operatorname{ord}_{\overline{\infty}}(\omega_{\chi}(r,z)) \geqslant -\frac{2pg_C}{p-1} - \mathfrak{C}_{\chi}(r,z,\overline{\infty})\,. \tag{3.5}$$

Proof. The proof is parallel to the mixed-characteristic proof of [OW16, Proposition 2.14], where $\mathfrak{C}_{\chi}(r,z,\overline{w})$ (respectively, $\mathfrak{C}_{\chi}(r,z,\overline{\infty})$) plays the role of $|\mathbb{B}(\chi) \cap U(r,z,\overline{w})|$ (respectively, $|\mathbb{B}(\chi) \cap U(r,z,\overline{\infty})|$). See [Dan20b, Proposition 3.8] for an explanation regarding the case $n = 1$ and $\overline{w} \neq \overline{\infty}$. The case $n > 1$ or $\overline{w} = \overline{\infty}$ is similar. □

Remark 3.21. We call (3.4) and (3.5) the local vanishing cycle formulas because they can be derived from Kato's vanishing cycle formula [Kat87b] (see [OW15]).

One easily derives the result below from Proposition 3.20.

Corollary 3.22. *With the notation as in the Proposition 3.20 and assuming $\overline{w} \neq \overline{\infty}$, we have the following inequality:*

$$\operatorname{sw}_{\chi}(r,z,\overline{w}) \leqslant \mathfrak{C}_{\chi}(r,z,\overline{w}) - 1\,. \tag{3.6}$$

Moreover, if χ has good reduction and $\overline{w} \neq \overline{\infty}$, then equality holds.

Remark 3.23. Corollary 3.22 and Proposition 3.20 indicate that, for χ to have good reduction, $\delta_{\chi}(r,z)$ is the smallest it could be, and its instantaneous rate of change at r depends only on the sum of the nth conductors of the branch points whose differences with z have valuations greater than r.

Corollary 3.24. (i) *Let $\chi \in H^1_{p^n}(\mathbb{K})$ be an admissible character of order p^n. Then*

$$\iota_n := \sum_{x \in \mathbb{B}(\chi)} \iota_{x,n} = \mathfrak{C}_{\chi}\big(0,0,\overline{0}\big) \geqslant \operatorname{sw}_{\chi}\big(0,0,\overline{0}\big) + 1\,. \tag{3.7}$$

Also, χ has good reduction if and only if $\delta_{\chi}(0) = 0$ and equality holds in (3.7). We call ι_n the conductor of the character χ.
(ii) *Suppose that χ has good reduction with upper breaks $(m_1, \ldots, m_n)$. Let $\chi_i := \chi^{p^{n-i}}$. If $1 \leqslant i \leqslant n$, then the conductor of χ_i is*

$$\iota_i := \sum_{x \in \mathbb{B}(\chi)} \iota_{x,i} = m_i + 1\,. \tag{3.8}$$

In particular, $\iota_i \geqslant p\iota_{i-1} - p + 1$, and if $\iota_i \equiv 1 \pmod{p}$, then $\iota_i = p\iota_{i-1} - p + 1$.

Proof. Item (i) is immediate from Proposition 3.20.

In the situation of part (ii), as χ_i also has good reduction, it has conductor $\mathfrak{C}_{\chi_i}(0,0,\overline{0}) = \sum_{x \in \mathbb{B}(\chi)} \iota_{x,i}$ by part (i). In addition, its reduction $\overline{\chi}_i \in \mathrm{H}^1_{p^i}(\kappa)$ is a well-defined character with upper ramification breaks $(m_1, \ldots, m_n)$. Hence, the Swan conductor of $\overline{\chi}_i$, which coincides with $\operatorname{sw}_{\chi_i}\big(0,0,\overline{0}\big)$ as discussed in Remark 3.11(i), is m_i; see [Ser79, Corollary 2 to Theorem 1]. Comparing with (3.7), we obtain (3.8). The remaining assertions follow from Remark 2.5. □

Remark 3.25. Corollary 3.24(i) indicates that, for a $\mathbb{Z}/p^n$-cover with étale reduction to have good reduction, it is necessary and sufficient that the conductor of the reduction is equal to the sum of the conductors of the generic branch points. It is thus equivalent to the different criterion (Proposition 2.25).

3.7 Characters of order p

Fix an admissible character $\chi \in \mathrm{H}^1_p(\mathbb{K})$ and $z \in D$. We will now provide an algorithm to calculate $\delta_\chi(r, z)$ and $\omega_\chi(r, z)$ for $r \in \mathbb{Q}_{\geqslant 0}$. We may assume $z = 0$. The proofs in this section can be easily translated to the case $z \neq 0$. The proposition below is the equal-characteristic analog of [OW14, Proposition 5.16].

Proposition 3.26. *Suppose $\chi = \mathfrak{K}_1(F)$, where $F \in \mathbb{K}^\times/\wp(\mathbb{K}^\times)$, and $r \in \mathbb{Q}_{\geqslant 0}$. Without loss of generality, one may assume $\nu_r(F) \leqslant 0$ and that χ is weakly unramified with respect to ν_r.*

(i) *We have*

$$\delta_\chi(r) = -\frac{1}{p}\max_a \nu_r(F + a^p - a)\,,$$

where a ranges over all elements of $\mathbb{K}$.

(ii) *The maximum of $\nu_r(F + a^p - a)$ in part (i) is achieved if and only if*

$$g := [F + a^p - a]_r \notin \kappa_r^p\,.$$

If this is the case and $\delta_\chi(r) > 0$, then $\omega_\chi(r) = dg$. If, instead, $\delta_\chi(r) = 0$, then $\overline{\chi}$ corresponds to the Artin–Schreier extension given by the equation $y^p - y = g$.

Proof. Part (i) is immediate from the definition of the depth for abelian characters.

Let us now prove part (ii). Suppose $g \notin \kappa_r^p$ and $\delta := -\nu_r(F + a^p - a)/p \geqslant 0$. Then

$$F + a^p - a = \pi^{-p\delta}u\,,$$

where $u \in \mathbb{K}$ satisfies $\nu_r(u) = 0$ and specializes to g. By [Dan20b, Proposition 3.16], we immediately obtain $\delta_\chi(r) = \delta$. If $\delta_\chi(r) > 0$, then it follows from the same result that $\omega_\chi(r) = dg$. The case $\delta_\chi(r) = 0$ is similar. ☐

Remark 3.27. Saïdi shows in [Saï07, Proposition 2.3.1] that a $\mathbb{Z}/p$-cover of a boundary is completely determined by its depth and its boundary Swan conductor. That result is motivated by Henrio [Hen00, Corollaire 1.8] for the case where R is of mixed characteristic. However, as in the case of mixed characteristic examined in [Bre09, Chapter 5], it is no longer true that a $\mathbb{Z}/p^n$-cover ($n > 1$) of a boundary $\operatorname{Spec} R[[X^{-1}]]\{X\}$ is determined by its Swan and boundary conductor.

3.8 Computing the slope of particular order p characters

This section is parallel to [OW14, § 5.5]. Let $\chi \in \mathrm{H}^1_p(\mathbb{K})$ be an admissible character of order p, hence giving rise to a branched cover of a disc $D[0, z]$, which we may associate with $\operatorname{Spec} R[[X]]$. As before, we may assume $z = 0$. Let $m > 1$ be a prime-to-p integer. We further assume that the following conditions hold:

(D1) The branch locus of χ is contained in the closed disc $D[pr_0]$ for some $r_0 > 0$, and $X = 0$ is one of the branch points.

(D2) For all $r \in (0, pr_0]$, the left derivative of δ_χ at r is less than or equal to m.

(D3) For all $r \in (0, pr_0]$, we have $\delta_\chi(r) > 0$ and $\delta_\chi(pr_0) = \delta$.

PROPOSITION 3.28. *Suppose that $\chi \in H^1_p(\mathbb{K})$ satisfies* (D1) *and* (D3)*. Then χ can be represented by the Artin–Schreier class of*

$$F = \sum_{i=0}^{\infty} a_i X^{-i} \in \mathbb{K}\,, \tag{3.9}$$

with $a_i \in R$ and $\nu(a_i) \geqslant p(r_0 i - \delta)$ for all i.

Proof. Condition (D1) tells us that the function F is holomorphic on the annulus $\{a \in K \mid 0 < \nu(A) < pr_0\}$. As there is also no pole at infinity, it follows from the classical theory of analytic functions that F is represented by a power series of the form (3.9) with

$$\liminf_{i\to\infty} \frac{\nu(a_i)}{i} \geqslant pr_0 \quad \text{or} \quad \liminf_{i\to\infty} \frac{\nu(a_i)}{pr_0 i} \geqslant 1\,. \tag{3.10}$$

Let us now consider the place pr_0. Replacing X with $X\pi^{-pr_0} =: X_{r_0}$ in (3.9), we obtain

$$F(X_{r_0}) = \sum_{i=0}^{\infty} \frac{a_i}{\pi^{pr_0 i} X^i_{r_0}}\,.$$

One can easily derive from (3.10) that there exists a positive integer M such that, for all $i \geqslant M$, $\nu(a_i) - pr_0 i > -p\delta = -p\delta(r_0)$. Thus, we may further assume, after replacing F by another function in the same Artin–Schreier class, that $a_j = 0$ for all $j < M$, $j \equiv 0 \pmod p$. It then follows from Proposition 3.26 and condition (D3) that

$$\nu(F(X_{r_0})) = \nu\left(\sum_{i=0}^{M} \frac{a_i}{\pi^{pr_0 i} X^i_{r_0}}\right) = \min_i(\nu(a_i) - pr_0 i) = -p\delta\,.$$

Hence, we have shown that $\nu(a_i) \geqslant p(r_0 i - \delta)$, completing the proof. □

Let us consider the function F in Proposition 3.28. We wish to find a polynomial a in T^{-1} such that $a^p - a$ approximates F well enough to use Proposition 3.26 simultaneously for all r in an interval $(0, s] \cap \mathbb{Q}$ for some $0 < s < pr_0$. We will then get explicit expressions for the slopes of δ_χ on the interval $[0, pr_0]$, which will be useful later in this article (Section 5.8).

For any $N \geqslant 1$, set

$$a := \sum_{j=1}^{N} b_j X^{-j} \in R[X]\,.$$

Here we consider the b_j as indeterminates for the moment. Write

$$F + a^p - a = \sum_{k=1}^{\infty} c_k X^{-k}\,,$$

where c_k is a polynomial in $b_1, \ldots, b_{\min(k,m)}$. Note that $c_k = a_k \in R$ for any $k > pN$.

LEMMA 3.29. *Assuming that condition* (D1) *holds, after replacing K by some finite extension, there exist $b_1, \ldots, b_N \in R$ such that*

(i) *$\nu(c_k) \geqslant p(r_0 k - \delta)$ for all k, and*

(ii) *$c_{kp} = 0$ for all $k \leqslant N$.*

Proof. In order to achieve item (ii), we solve the equations $c_{pN} = c_{p(N-1)} = \cdots = c_p = 0$

inductively. From top to bottom, they are

$$\begin{aligned} a_{pN} + b_N^p &= 0\,, \\ &\vdots \\ a_{p(N-i)} + b_{N-i}^p - b_{p(N-i)} &= 0\,, \\ &\vdots \\ a_p + b_1^p - b_p &= 0\,, \end{aligned} \tag{3.11}$$

where $b_{p(N-i)} = 0$ for $p(N-i) > N$. Moreover, one can easily check that b_{N-i} has valuation at least $p(r_0(N-i)-\delta)$. Hence, $\nu(c_{N-i})$ is at least $p(r_0(N-i)-\delta)$, proving item (i). ☐

Remark 3.30. The proof above also shows that there are only finitely many solutions for the b_j and that they vary analytically as the a_i do.

Proposition 3.31. *Assume that conditions* (D1), (D2), *and* (D3) *hold. Choose* $s \in (0, pr_0) \cap \mathbb{Q}$ *and* $N \in \mathbb{N}$ *such that*

$$pN \geqslant \frac{\delta}{r_0 - s}\,. \tag{3.12}$$

Let $b_1, \ldots, b_N$ *be as in Lemma* 3.29. *Define* $\lambda_m(\chi) \in [0, pr_0]$ *by*

$$\lambda_m(\chi) := \max(\{r \in (0, pr_0] \mid \mathrm{sw}_\chi(r,\infty) > -m\} \cup \{0\})\,.$$

Set

$$\mu_m(\chi) := \max\left(\left\{\frac{\nu(c_m) - \nu(c_k)}{m-k} \mid 1 \leqslant k < m\right\} \cup \{0\}\right).$$

Then the following hold:

(i) *For all* $r \in (0, s] \cap \mathbb{Q}$, *we have* $[F + a^p - a]_r \notin \kappa_r^p$. *Therefore,*

$$\delta_\chi(r) = -\frac{\nu_r(F + a^p - a)}{p} \quad \text{and} \quad \mathrm{sw}_\chi(r,\infty) = -\operatorname{ord}_\infty [F + a^p - a]_r\,.$$

(ii) *We have* $\lambda_m(\chi) < s \Leftrightarrow \mu_m(\chi) < s$.

(iii) *If* $\lambda_m(\chi) < s$, *then* $\lambda_m(\chi) = \mu_m(\chi)$.

Remark 3.32. Note that, if $\lambda_m(\chi) \neq r$, then Proposition 3.31 implies that $\lambda_m(\chi)$ is the largest value in the interval $(0, pr_0]$ where the piecewise-linear function δ_χ has a "kink."

Proof. Fix $r \in (0, s] \cap \mathbb{Q}$ and set $M := \operatorname{ord}_\infty [F + a^p - a]_r$. Applying Lemma 3.29, we obtain the following inequality:

$$\nu_r(F + a^p - a) = \nu(c_M) - prM \geqslant p(M(r_0 - r) - \delta) \geqslant p(M(r_0 - s) - \delta)\,. \tag{3.13}$$

On the other hand, as condition (D3) forces $\delta_\chi(r) > 0$, Proposition 3.26 shows that

$$\nu_r(F + a^p - a) < 0\,. \tag{3.14}$$

Hence, it must be true that $\delta/(r_0 - s) > M$. It then follows from (3.13), (3.14), and the choice of N that

$$M < \frac{\delta}{r_0 - s} \leqslant Np\,. \tag{3.15}$$

If M was divisible by p, then (3.15) and Lemma 3.29(ii) would show that $c_M = 0$, which contradicts the definition of M. Therefore, M is prime to p, and part (i) of the proposition follows from Proposition 3.26 and Remark 3.11.

The rest of the proof is exactly the same as that of [OW14, Proposition 5.19]. □

The proof of the following corollary is straightforward.

COROLLARY 3.33. *In the notation of Proposition 3.31, set*

$$\lambda_{m,l}(\chi) := \max(\lambda_m(\chi), l) \quad \textit{and} \quad \mu_{m,l}(\chi) := \max(\mu_m(\chi), l)\,.$$

Then Proposition 3.31 still holds when replacing $\lambda_m(\chi)$ (respectively, $\mu_m(\chi)$) by $\lambda_{m,l}(\chi)$ (respectively, $\mu_{m,l}(\chi)$).

3.9 Refined Swan conductor of $\mathbb{Z}/p^n$-extensions

In this section, we discuss a technique for calculating the refined Swan conductors of a fixed character $\chi := \mathfrak{K}_n(\underline{a})$, where $\underline{a} \in W_n(\mathbb{K})$. We mostly follow the settings from [Lea18, KLS19]. Different perspectives can be found in [Mat97, Bor04].

Write $\mathbb{K} = \kappa((\pi))$ for some $\pi \in \mathcal{O}_{\mathbb{K}}$, where κ is the residue of $\mathbb{K}$. Let $\{b_\lambda\}_{\lambda \in \Lambda}$ be a lift of a p-basis of κ to $\mathcal{O}_{\mathbb{K}}$. Then $\Omega^1_{\mathcal{O}_{\mathbb{K}}}(\log)$ is the $\mathcal{O}_{\mathbb{K}}$-module with basis $\{db_\lambda, d\log\pi \mid \lambda \in \Lambda\}$.

We first define valuations on $\Omega^1_{\mathbb{K}}$ and $W_n(\mathbb{K})$ as follows.

DEFINITION 3.34. If $\omega \in \Omega^1_{\mathbb{K}}$ and $\underline{a} = (a^1, \ldots, a^n) \in W_n(\mathbb{K})$, let

$$\nu^{\log}\omega = \sup\bigl\{i \mid \omega \in \pi^i \otimes_{\mathcal{O}_{\mathbb{K}}} \Omega^1_{\mathcal{O}_{\mathbb{K}}}(\log)\bigr\}\,,$$

and

$$\nu(\underline{a}) = \max_i\bigl\{-p^{n-1-i}\nu\bigl(a^i\bigr)\bigr\} = \min_i\bigl\{p^{n-1-i}\nu\bigl(a^i\bigr)\bigr\}\,. \tag{3.16}$$

If $\omega = f(x)dx \in \Omega^1_{\mathbb{K}}$, then we set $[\omega] = [f(x)]dx$.

These valuations define increasing filtrations of $\Omega^1_{\mathbb{K}}$ and $W_n(\mathbb{K})$ by the subgroups

$$F_s\Omega^1_{\mathbb{K}} = \bigl\{\omega \in \Omega^1_{\mathbb{K}} \mid \nu^{\log}\omega \geqslant -s\bigr\} \quad \text{and} \quad F_sW_n(\mathbb{K}) = \{\underline{a} \in W_n(\mathbb{K}) \mid \nu(\underline{a}) \geqslant -s\}\,, \tag{3.17}$$

respectively, where $n \in \mathbb{Z}_{\geqslant 0}$.

Remark 3.35. In [Kat89], Kato defined the filtration $F_s\mathrm{H}^1_{p^n}(\mathbb{K})$ as the image of $F_sW_n(\mathbb{K})$ under the ASW map (3.1). Note that, for $\chi = \mathfrak{R}(\underline{a}) \in \mathrm{H}^1_{p^n}(\mathbb{K})$, the Swan conductor $\mathrm{sw}(\chi)$ is defined to be the smallest s such that $\underline{a} \in F_s\mathrm{H}^1_{p^n}(\mathbb{K})$.

We will now define what it means for a Witt vector $\underline{a}$ in $W_n(\mathbb{K})$ to be "best" (which is a generalization of "reducible" in [Dan20b, Definition 3.14]).

DEFINITION 3.36. Let $\underline{a} \in W_n(\mathbb{K})$, and let s be the smallest non-negative integer such that $\underline{a} \in F_sW_n(\mathbb{K})$. We say that a is *best* if there is no $\underline{a}' \in W_n(\mathbb{K})$ that maps to the same element as $\underline{a}$ in $\mathrm{H}^1_{p^n}(\mathbb{K})$ such that $\underline{a}' \in F_{s'}W_n(\mathbb{K})$ for some non-negative integer $s' < s$.

When $\nu(\underline{a}) \geqslant 0$, the Witt vector $\underline{a}$ is clearly best. When $\nu(\underline{a}) < 0$, the Witt vector $\underline{a}$ is best if and only if there are no $\underline{a}', \underline{b} \in W_n(\mathbb{K})$ satisfying $\underline{a} = \underline{a}' + \wp(\underline{b})$ and $\nu(\underline{a}) < \nu(\underline{a}')$. When $n = 1$, the following result, which immediately follows from Proposition 3.26, characterizes when $a \in \mathbb{K}$ is best.

PROPOSITION 3.37. *A Witt vector $a \in \mathbb{K} \setminus \mathcal{O}_{\mathbb{K}}$ is best if and only if $a = \pi w$, where $\nu(\pi) < 0$ and $w \in \mathcal{O}_{\mathbb{K}}$ with $\overline{w} \in \kappa \setminus \kappa^p$.*

DEFINITION 3.38 ([Lea18, Definition 2.3]). We say that the ith position of a Witt vector $\underline{a} \in W_n(\mathbb{K})$ is *relevant* if $\nu(\underline{a}) = p^{n-1-i}\nu(a^i)$. Let $j = \min\{i \mid \nu(a) = p^{n-1-i}\nu(a^i)\}$. We call $n-j+1$ the *relevance length* of $\underline{a}$.

The following result gives us an explicit algorithm to calculate the refined Swan conductors of a character using its best Witt vector representation.

THEOREM 3.39 ([Lea18, Theorem 2.7]). *Let $\underline{a} \in W_n(\mathbb{K})$. The following conditions are equivalent:*

(i) *The Witt vector $\underline{a}$ is best.*

(ii) *There exists some relevant position i such that a^i is best in the sense of length* 1.

(iii) *We have $\nu(\underline{a}) = \nu^{\log}(d\underline{a})$, where $d\underline{a} := \sum_i (a^i)^{p^{n-i}-1} da^i$.*

In particular, when $\underline{a}$ is best, the differential conductor of the corresponding character can be calculated as follows.

PROPOSITION 3.40. *Suppose that $\underline{a} = (a^1, \ldots, a^n) \in W_n(\mathbb{K})$ is defined with relevance length $n-j+1$, and let $\chi := \mathfrak{K}_n(\underline{a}) \in \mathrm{H}^1_{p^n}(\mathbb{K})$. Then, $\mathrm{sw}(\chi) = \nu(d\underline{a})$, and if $\mathrm{sw}(\chi) > 0$, we have*

$$\mathrm{dsw}(\chi) = [d\underline{a}] = \sum_{i=j}^{n} \left[a^i\right]^{p^{n-i}-1} d\left[a^i\right].$$

Proof. In [Lea18, § 2], the author defines the map $\mathrm{rsw}\colon F_d\mathrm{H}^1_{p^n}(\mathbb{K}) \to F_d\Omega^1_{\mathbb{K}}/F_{\lfloor d/p \rfloor}\Omega^1_{\mathbb{K}}$ (which is the same as rsw in [KS19]) sending a best $\underline{a}$, which represents χ, to $d\underline{a}$. In [KS19, Theorem 1.5], Kato and Saito show that rsw in [Lea18] coincides with $\mathrm{rsw}^{\mathrm{ab}}$ defined in [Kat87a] (mentioned in (3.3)). See also [Kat89, Lemma 3.7] and the discussion following Proposition 6.8 of the same paper. In particular, in the notation above, we have

$$\mathrm{rsw}^{\mathrm{ab}}(\chi) = \pi^{-\nu(d\underline{a})} \otimes [d\underline{a}].$$

The rest then follows immediately from (3.3). □

Example 3.41. In this example, we calculate the refined Swan conductors of some restrictions of the deformation in Example 2.27. Recall that χ is defined by the ASW equation

$$\wp(Y_1, Y_2) = \left(\frac{1}{X^2\left(X - t^8\right)}, \frac{1}{X^3\left(X - t^8\right)^2\left(X - t^2\right)^2\left(X - t^2\left(1 + t^2\right)\right)^2}\right). \tag{3.18}$$

Over the subdisc $\mathcal{D}[2]$ associated with $\operatorname{Spec} R[[t^{-8}X]]$, replacing $t^{-8}X$ with X_4 in (3.18), we obtain

$$\left(\frac{1}{t^{24}X_4^2(X_4 - 1)}, \frac{1}{t^{48}X_4^3(X_4 - 1)^2\left(t^6X_4 - 1\right)^2\left(t^6X_4 - 1\left(1 + t^2\right)\right)^2}\right).$$

As $48 = 24 \cdot 2$ and

$$\frac{1}{x^2(x-1)}\frac{d}{dx}\left(\frac{1}{x^2(x-1)}\right) + \frac{d}{dx}\left(\frac{1}{x^3(x-1)^2}\right) = \frac{dx}{x^3(x-1)^3} \neq 0,$$

it follows from Proposition 3.40 that $\delta_{\chi_2}(4) = 24$ and

$$\omega_{\chi_2}(4) = \frac{dx}{x^3(x-1)^3},$$

which is not an exact differential form as in the $\mathbb{Z}/p$-covers case. Observe that $\mathcal{C}(\omega_{\chi_2}(4)) = \omega_{\chi_1}(4) = dx/x^2(x-1)^2$, where $\chi_1 = \chi^p$ and $\mathcal{C}$ is the Cartier operator of $\Omega^1_{k(x)}$. This phenomenon will be discussed in Section 3.9.1.

Iterating the above computation for $\mathcal{D}[1]$ (associated with $\operatorname{Spec} R[[t^{-2}X]]$), we get $\delta_{\chi_2}(1) = 9$ and $\omega_{\chi_2}(1) = dx/x^6(x-1)^4$, which is exact and thus satisfies Theorem 3.42(b).

3.9.1 *Conditions on the refined Swan conductors of cyclic covers.* In order to answer induction-type questions like Proposition 2.35, one would be interested in learning what the n-level can be when the $(n-1)$-level is known. The next theorem, which is the equal-characteristic analog of [Wew14, Theorem 1.2], will do exactly that for our situation.

Let $\chi_n \in \mathrm{H}^1(K, \mathbb{Z}/p^n)$ be a radical character of order p^n, with $n \geqslant 1$. For $i = 1, \ldots, n$, we set

$$\chi_i := \chi_n^{p^{n-i}}, \quad \delta_i := \delta_{\chi_i}, \quad \omega_i := \omega_{\chi_i}.$$

The tuple $(\delta_i, \omega_i)_{i=1,\ldots,n}$ is called the *ramification datum* associated with χ_n. For $1 \leqslant j \leqslant n$, the pair (δ_j, ω_j) is called the *jth ramification datum* of χ_n.

THEOREM 3.42. *Let χ_n and $(\delta_i, \omega_i)_{i=1,\ldots,n}$ be as above. Let $\mathcal{C}\colon \Omega^1_\kappa \to \Omega^1_\kappa$ be the Cartier operator. Then for all $i = 1, \ldots, n$, the following hold:*

- (i) *We have $\mathcal{C}(\omega_1) = 0$.*
- (ii) *Suppose $i > 1$. Then $\delta_i \geqslant p\delta_{i-1}$. Moreover, we have the implications*
  - (a) *$\delta_i = p\delta_{i-1} \Rightarrow \mathcal{C}(\omega_i) = \omega_{i-1}$,*
  - (b) *$\delta_i > p\delta_{i-1} \Rightarrow \mathcal{C}(\omega_i) = 0$.*

For the definition of the Cartier operator, see [Wew14, § 3.2].

Proof. Part (i) follows immediately from the exactness of ω_1 asserted by Proposition 3.26.

Suppose that the length i Witt vector $\underline{a}_i = (a^1, \ldots, a^i)$ that defines χ_i is best with relevance length l. One may assume, after an ASW operation, that $\underline{a}_{i-1} = (a^1, \ldots, a^{i-1})$ is also best. First suppose $l = 1$. Then, it follows from the definition that

$$\delta_i = -\nu(a^i)/p > -\max_{j<i}\{p^{i-1-j}\nu(a^j)\}/p = p\delta_{i-1}.$$

In addition, the differential form $\omega_i = [d\underline{a}_i] = d[a^i]$ is exact, proving part (ii)(b).

Let us now consider the case where the relevance length satisfies $l > 1$. Thus, the relevance length of $\overline{a}_{i-1}$ is $l-1$. An easy computation shows

$$\mathcal{C}(\omega_i) = \mathcal{C}\Bigg(\sum_{m=i-l+1}^{i} [a^m]^{p^{n-m}-1} d[a^m]\Bigg) = \sum_{m=i-l+1}^{i-1} [a^m]^{p^{n-m-1}-1} d[a^m] = \omega_{i-1},$$

confirming part (ii)(a). □

DEFINITION 3.43. In the notation of Theorem 3.42(ii), we say that (δ_i, ω_i) *extends* $(\delta_{i-1}, \omega_{i-1})$.

DEFINITION 3.44. We call the equation $\mathcal{C}(y) = w$ for a given $w \in \Omega^1_\kappa$ the *Cartier operator equation*. Some solutions to this equation will be discussed in Section 6.1.

Remark 3.45. Theorem 3.42 can also be proved quite easily by adapting the computation from [Wew14] to the equal-characteristic case.

The following result says that the inverse of Theorem 3.42, that is, the equal-characteristic version of [Wew14, Theorem 4.6], also holds. The theorem itself is not employed in this paper. However, it contains the key idea of the main theorem's proof.

THEOREM 3.46. *Let $(\delta_i, \omega_i)_{i=1,\dots,n}$ be a tuple satisfying the conditions of Theorem 3.42. Then, after a finite extension of K, there exists a radical character χ of order p^n on $\mathbb{K}$ such that $(\delta_i, \omega_i)_i$ is the ramification datum associated with χ.*

Proof. We do induction on i. First suppose that we are given a dual (δ_1, ω_1), where $\delta_1 \in \mathbb{Q}_{\geqslant 0}$ and $\omega_1 \in \Omega^1_\kappa$ is such that $\mathcal{C}(\omega_1) = 0$. By the previous discussion, we may assume without loss of generality that $\omega_1 = dw$ for some $w \in \kappa \setminus \kappa^p$. Let W be a lift of w to $\mathbb{K}$. Then, it immediately follows from Proposition 3.26 that the character $\chi_u \in \mathrm{H}^1_p(\mathbb{K})$, where $u := \pi^{-p\delta_1} w$, satisfies $\delta_{\chi_u} = \delta_1$ and $\omega_{\chi_u} = \omega_1$, confirming the base case.

We complete the induction process in two steps corresponding to the following two lemmas.

LEMMA 3.47. *Let $\chi_{n-1} \in H^1_{p^{n-1}}(\mathbb{K})$ be a radical character of order p^{n-1}. Then there exists a character $\chi_{\min} \in H^1_{p^n}(\mathbb{K})$ extending χ_{n-1} and such that $\delta_{\chi_{\min}} = p\delta_{n-1}$.*

Proof. One may assume that χ_{n-1} is given by $\underline{f} = (f^1, \dots, f^{n-1}) \in W_{n-1}(\mathbb{K})$, where $\underline{f}$ is best (Definition 3.36). It follows from Theorem 3.39 that there exists some relevant position $0 \leqslant i \leqslant n-2$ such that f^i is best in the sense of length 1. Recall that this means that $\nu(f) = p^{n-2-i}\nu(f^i) = -\delta_{n-1}$ and $f^i = \pi^{-\nu(f^i)} w$, where $\overline{w} \in \kappa \setminus \kappa^p$. Consider the character $\chi_{\min}$ defined by $\underline{f'} = (f^1, \dots, f^{n-1}, 0) \in W_n(\mathbb{K})$. It is immediate from the definition that $\chi^p_{\min} = \chi_{n-1}$ and $\nu(\underline{f'}) = p^{n-i}\nu(f^i)$. Hence, i is also a relevant position of f'. Therefore, it again follows from Theorem 3.39 that f' is best. Thus, we have $\delta_{\chi_{\min}} = -p^{n-i-1}\nu(f^i) = p\delta_{n-1}$. ☐

LEMMA 3.48. *Let $\chi_{n-1} \in H^1_{p^{n-1}}(\mathbb{K})$ be a radical character of order p^{n-1} with $(n-1)$th ramification datum $(\delta_{n-1}, \omega_{n-1})$. Suppose that (δ_n, ω_n) extends $(\delta_{n-1}, \omega_{n-1})$ in the sense of Definition 3.43. Then there exists an extension χ_n of χ_{n-1} such that $\delta_{\chi_n} = \delta_n$ and $\omega_{\chi_n} = \omega_n$.*

Proof. Let $\chi_{\min}$ be the extension of χ_{n-1} with $\delta_{\chi_{\min}} = p\delta_{n-1}$, which exists by Lemma 3.47. Set $\omega_{\min} := \mathrm{dsw}_{\chi_{\min}}$. It follows from Theorem 3.42(ii)(a) that $\mathcal{C}(\omega_{\min}) = \omega_{n-1}$. If $\delta_n = p\delta_{n-1}$ and $\omega_n = \omega_{\min}$, then we are done. Otherwise, set $0 \neq \eta := \omega_n - \omega_{\min}$. Since $\mathcal{C}(\eta) = \omega_{n-1} - \omega_{n-1} = 0$, the differential form η is exact. We may thus write $\eta = du$ for some $u \in \kappa$. Just as in the base case, one shows that the character $\psi \in \mathrm{H}^1_p(\mathbb{K})$ defined by $\pi^{-p\delta_n} U$, where U is a lift of u to $\mathbb{K}$, has depth $\delta_\psi = \delta_n$ and differential Swan conductor $\mathrm{dsw}_\psi = \eta$. Set $\chi_n := \chi_{\min} \cdot \psi$. Lemma 3.12 asserts that $\delta_{\chi_n} = \delta_n$ and $\mathrm{dsw}_{\chi_n} = \omega_n$, as desired.

A similar process applies for the case $\delta_n > p\delta_{n-1}$. As before, there exists a $\psi \in \mathrm{H}^1_p(\mathbb{K})$ with $\delta_\psi = \delta$ and $\mathrm{dsw}(\psi) = \omega_n$. It, once more, follows from Lemma 3.12 that the character $\chi_n := \chi_{\min} \cdot \psi$ satisfies the conditions we are seeking. ☐

This completes the induction process and hence the proof of Theorem 3.46. ☐

Remark 3.49. In the situation of Proposition 2.35, one can apply the technique from the above proof to form a cyclic p^n-cover $Y_n \to C$ with étale reduction, extending $Y_{n-1} \to C$, whose completion of the reduction at $x = 0$ is birationally equivalent to $k[[y_n]]/k[[x]]$. However, it is not true in general that the cover $Y_n \to C$ has a good reduction as it would require the generic fiber to have the right ramification datum (Corollary 3.24), which is not easy to achieve using only the current method.

4. Hurwitz trees

4.1 Hurwitz trees and the deformation problem

Let $R = k[[t]]$ be a complete discrete valuation ring of equal characteristic, where k is an algebraically closed field of characteristic $p > 0$. In this section, we first introduce the notion of a Hurwitz tree for cyclic covers of a curve C that is étale outside an open disc $D \subset C^{\mathrm{an}}$. Then, we will describe how a $\mathbb{Z}/p^n$-cover gives rise to such a tree. Finally, we present an obstruction for the refined equal-characteristic deformation problem that is parallel to the obstruction for lifting given in [BW09].

The definition below is identical to [BW06, § 3.1]. We repeat it here for the convenience of the reader.

Definition 4.1. A *decorated tree* is given by the following data:

- a semi-stable curve $\overline{C}$ over k of genus 0,
- a family $(x_b)_{b\in B}$ of pairwise-distinct smooth k-rational points of $\overline{C}$, indexed by a finite non-empty set B,
- a distinguished smooth k-rational point $x_0 \in \overline{C}$, distinct from any of the point x_b.

We require that $\overline{C}$ is stably marked by the points $((x_b)_{b\in B}, x_0)$ in the sense of [Knu83].

The *combinatorial tree* underlying a decorated tree $\overline{C}$ is the graph $T = (V, E)$, defined as follows. The vertex set V of T is the set of irreducible components of $\overline{C}$, together with a distinguished element e_0. We write $\overline{C}_v$ for the component corresponding to a vertex $v \neq v_0$ and z_e for the singular point corresponding to an edge $e \neq e_0$. The singular point z_e associated with an edge e is adjacent to the vertices corresponding to the two components that intersect at z_e. The edge e_0 is adjacent to the root v_0 and the vertex v corresponding to the (unique) component $\overline{C}_v$ containing the distinguished point x_0. At any point on an edge e, we say that the direction away from the root is the *positive* one.

Note that, since $(\overline{C}, (x_b), x_0)$ is stably marked of genus 0, the components $\overline{C}_v$ have genus 0, too, and the graph T is a tree. Moreover, we have $|B| \geqslant 1$. For a vertex $v \in V$, we write $\overline{U}_v \subset \overline{C}_v$ for the complement in $\overline{C}_v$ of the set of singular and marked points.

Definition 4.2. Let $M = (e_{j,i})_{1\leqslant j\leqslant r}$ be an $r \times n$ matrix in $\Omega_{e_1,\dots,e_n}$ (see Definition 2.10). A $G = \mathbb{Z}/p^n$*-Hurwitz tree* $\mathcal{T}$ of type M is defined by the following data:

- a decorated tree $\overline{C} = (\overline{C}, (x_b), x_0)$ with underlying combinatorial tree $T = (V, E)$;
- for every $v \in V$, a rational $0 \leqslant \delta_v = \delta_{\mathcal{T}}(v)$, called the *depth* of v;
- for each $v \in V$ such that $\delta_v > 0$, a differential form $\omega_v = \omega_{\mathcal{T}}(v) \in \Omega^1_\kappa$, called the *differential conductor* at v;
- for each $v \in V$, a group $G_v \subseteq G$, called the *monodromy group* at v;
- for every $e \in E$, a positive rational number ϵ_e, called the *thickness* of e;
- for every $e \in E$, a positive integer d_e, called the *slope* on e;
- for every $b_j \in B = \{b_1, \dots, b_r\}$, the positive number $h_j = e_{j,n}$, called the *conductor* at b;
- for v_0 with $\delta_{v_0} = 0$, a *reduced* length n Witt vector $\underline{f} := (f^1, \dots, f^n) \in W_n(\kappa)$ with only pole 0, called the *degeneration* of the tree; the rational f^i is called the *ith degeneration* of $\mathcal{T}$; define
$$d := \max\{p^{n-l} \deg_{x^{-1}}(f^l) \mid l = 1, \dots, n\}.$$

These data are required to satisfy all of the following conditions:

(H1) Let $v \in V$. We have $\delta_v \neq 0$ if $v \neq v_0$.

(H2) For each $v \in V \setminus \{v_0\}$, the differential form ω_v does not have zeros or poles on $\overline{U}_v \subsetneq \overline{C}_v$.

(H3) For every edge $e \in E \setminus \{e_0\}$, we have the equality $-\operatorname{ord}_{z_e} \omega_{t(e)} - 1 = \operatorname{ord}_{z_e} \omega_{s(e)} + 1$.

(H4) For v_0, we have $d = \operatorname{ord}_{z_{e_0}} \omega_{t(e_0)} + 1$.

(H5) For every edge $e \in E$, we have

$$d_e = -\operatorname{ord}_{z_e} \omega_{t(e)} - 1 \underset{s(e) \neq v_0}{\overset{\text{(H2)}}{=}} \operatorname{ord}_{z_e} \omega_{s(e)} + 1\,.$$

(H6) For every edge $e \in E$, we have $\delta_{s(e)} + (p-1)\epsilon_e d_e = \delta_{t(e)}$.

(H7) For $b \in B$, let $\overline{C}_v$ be the component containing the point x_b. Then the differential ω_v has a pole at x_b of order h_b.

(H8) For each v, we have $G_{v'} \subseteq G_v$ for every successor vertex v' of v. Moreover, we have

$$\sum_{v \to v'} [G_v : G_{v'}] > 1\,,$$

except if $v = v_0$ is the root, in which case there exists exactly one successor v' and we have $G_v = G_{v'} = G$.

For each $v \in V \setminus \{v_0\}$, we call (δ_v, ω_v) the *degeneration type* of v. For each $e \in E$, we call $(\delta_{s(e)}, d_{s(e)})$ (respectively, $(\delta_{t(e)}, d_{t(e)})$) the *initial* (respectively, *final*) *degeneration type* of e. The positive integer $\mathfrak{C} := d + 1$ is called the *conductor* of the Hurwitz tree. The rational $\delta := \delta_{v_0}$ is the depth of $\mathcal{T}$. We define the *height* of the tree to be the maximal positive direction of edges from its root to its leaves. We call $\sum_{b \in B} h_b \cdot [B]$ the *branching divisor* of $\mathcal{T}$.

Remark 4.3. With the notation as above, fix $e \in E$ and let $\overline{C}_e \subseteq \overline{C}$ be the union of all components $\overline{C}_v$ corresponding to vertices v that are separated from the root v_0 by the edge e. Then

$$d_e = \sum_{\substack{b \in B \\ x_b \in \overline{C}_e}} h_b - 1 > 0\,.$$

In particular, we have $\mathfrak{C} = d + 1 = \sum_{b \in B} h_b$.

Remark 4.4. Depending on the study, one may like to add or remove some information from the Hurwitz tree in Definition 4.2. For instance, the monodromy groups can be omitted when $G = \mathbb{Z}/p$ as in [Dan20b] and [Hen00], or when $G = \mathbb{Z}/p \rtimes \mathbb{Z}/m$ as in [BW06]. In our situation, these data are useful for deducing the Hurwitz tree of a subcover as discussed in Section 4.4.2. When $G = \mathbb{Z}/p \rtimes_\psi \mathbb{Z}/m$, where ψ is non-trivial, one would like to add one extra piece of information coming from ψ to the Hurwitz tree of the $\mathbb{Z}/p$-subcover, as in [BW06, Definition 3.2].

4.2 Hurwitz trees arise from cyclic covers

Fix a cyclic group $G := \mathbb{Z}/p^n$. Let $R = k[[t]]$. Let K denote the fraction field of R. As usual, we may associate with $\mathcal{D} \subset C^{\text{an}}$ the spectrum of $R\{X\}$. Suppose that we are given an admissible G-character $\chi \in \mathrm{H}^1_{p^n}(\mathbb{K})$, which gives rise to an exponent p^n cover of the closed disc $\mathcal{D}$

$$\Phi\colon\ \operatorname{Spec} R\{X\} \longrightarrow \operatorname{Spec} R\{Z\}\,.$$

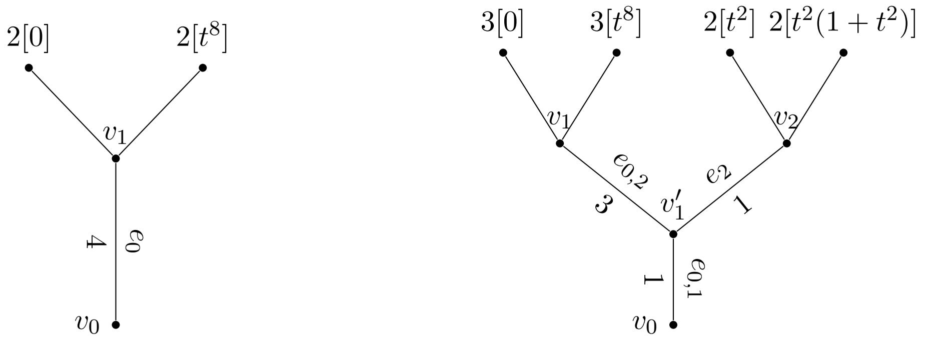
Let h be the conductor, and let δ be the depth of this cover. As discussed in Section 3.3, there exists a semi-stable model of C corresponding to the interior of $\mathcal{D}$ and the branch locus of Φ.

The dual graph of its special fiber $\overline{C}$ forms a decorated tree $T = (V, E)$. For each vertex v in T, we denote by $U_v \subset C^{\mathrm{an}}$ the affinoid subdomain with reduction $\overline{U}_v$, which can be thought of as a punctured disc. Following the exact procedure from [Dan20b, § 4.2], one can construct a Hurwitz tree $\mathcal{T}$ from the dual graph and the refined Swan conductors. More precisely, the data of $\mathcal{T}$ are as in Table 3. See also [BW06, BW09].

Data on $\mathcal{T}$	Degeneration data of the cover
The decorated tree	The branching geometry of Φ (Definition 3.6)
The depth of v	The depth of the restriction of Φ to U_v
The differential conductor at v	The differential conductor of the restriction of Φ to U_v
The thickness of an edge e	The thickness of the corresponding annulus divided by p
The conductor at a leaf b	The conductor of Φ at the branch point associated with b
The reduced degeneration at v_0	The reduced degeneration of Φ
The monodromy group at v	The largest inertia group of the leaves succeeding v

Table 3. Assigning a Hurwitz tree to an admissible cover

Example 4.5. Let us calculate the Hurwitz tree for the $\mathbb{Z}/4$-cover from Example 2.27. Figure 2a (respectively, 2b) is the one associated with the subcover χ_1 (respectively, χ_2), which we call $\mathcal{T}_1$ (respectively, $\mathcal{T}_2$). The monodromy group at each vertex of $\mathcal{T}_1$ is $\mathbb{Z}/2$. The reduction type at v_1 is $(12, dx/x^2(x-1)^2)$, and at v_0 it is $(0, 1/x^3)$.



(a) $\mathcal{T}_1$

(b) $\mathcal{T}_2$

Figure 2. The Hurwitz trees for Example 2.27

Regarding $\mathcal{T}_2$, it follows from the calculation in Example 3.41 that the degeneration types at v_1, v_1', v_2, and v_0 are, respectively, $(24, dx/x^3(x-1)^3)$, $(9, dx/x^6(x-1)^4)$, $(12, dx/x^2(x-1)^2)$, and $(0, (1/x^3, 1/x^9))$. At all vertices but v_2 and those of the two leaves $[t^2]$ and $[t^2(1+t^2)]$, the monodromy groups are $\mathbb{Z}/4$. The monodromy groups at those vertices are $\mathbb{Z}/2$.

Definition 4.6. Suppose that we are given a $\mathbb{Z}/p^{n-1}$-tree $\mathcal{T}_{n-1}$ and a $\mathbb{Z}/p^n$-tree $\mathcal{T}_n$. We say that $\mathcal{T}_n$ *extends* $\mathcal{T}_{n-1}$, denoted by $\mathcal{T}_{n-1} \prec \mathcal{T}_n$, if the following hold:

(i) The decorated tree of $\mathcal{T}_n$ is a refinement of that for $\mathcal{T}_{n-1}$.

(ii) At each vertex v of $\mathcal{T}_{n-1}$, the depth conductor and the differential conductor satisfy the conditions of Theorem 3.42.

(iii) At each vertex v (respectively, each leaf b) of $\mathcal{T}_{n-1}$, if the monodromy group is $\mathbb{Z}/p^i$ (with $i \leqslant n-1$), then the monodromy group at the corresponding vertex on $\mathcal{T}_n$ is $\mathbb{Z}/p^{i+1}$.

(iv) At each vertex v (respectively, each leaf b) of $\mathcal{T}_n$ that is not one of $\mathcal{T}_{n-1}$, the monodromy group is exactly $\mathbb{Z}/p$.

(v) Suppose $\delta_{\mathcal{T}_n}(v_0) = \delta_{\mathcal{T}_{n-1}}(v_0) = 0$. Then the reduced degeneration of $\mathcal{T}_n$ is a length n Witt vector $(f^1, \dots, f^{n-1}, f^n)$ such that $(f^1, \dots, f^{n-1})$ is the reduced degeneration of $\mathcal{T}_{n-1}$.

We say that $\mathcal{T}_n$ extends a $\mathbb{Z}/p^i$-tree $\mathcal{T}_i$ if there exist a sequence of consecutive $n-i+1$ extending trees $\mathcal{T}_i \prec \mathcal{T}_{i+1} \prec \cdots \prec \mathcal{T}_{n-1} \prec \mathcal{T}_n$.

Definition 4.7. Suppose that r is a rational place on an edge e of a Hurwitz tree $\mathcal{T}$. Set

$$\mathfrak{C}_{\mathcal{T}}(e) := \sum_{b \in \mathbb{B}_{\mathcal{T}}(e)} h_b \,,$$

which is equal to $d_e + 1$ by Remark 4.3. We say that a $\mathbb{Z}/p^n$-Hurwitz tree $\mathcal{T}$ is *étale* if $\delta_{\mathcal{T}} = 0$ and call it *radical* otherwise. We define, for an edge e and $r \in [s(e), t(e)] \cap \mathbb{Q}$, the depth of $\mathcal{T}$ at the place r of e as follows:

$$\delta_{\mathcal{T}}(r, e) := \delta_{\mathcal{T}}(s(e)) + d_e(r - s(e)) = \delta_{\mathcal{T}}(t(e)) - d_e(t(e) - r) \,.$$

The existence of a Hurwitz tree $\mathcal{T}$ with particular conditions is necessary for a cyclic extension to deform non-trivially.

Corollary 4.8. *Suppose that $\mathcal{T}$ is an étale $\mathbb{Z}/p^n$-tree that extends $\mathcal{T}_{n-1}$, and $\chi \in H^1_{p^n}(\mathbb{K})$ is a character extends χ_{n-1}. Suppose, moreover, that the tree associated with χ has the same shape as $\mathcal{T}$ and the conductors of its branch points agree with those of $\mathcal{T}$. Then, on each edge e of $\mathcal{T}$,*

$$\delta_{\chi}(r, z_e) \geqslant \delta_{\mathcal{T}}(r, e)$$

holds for all $r \in [s(e), t(e)] \cap \mathbb{Q}_{\geqslant 0}$. Equality holds everywhere if χ is étale (Definition 3.7).

Proof. The result follows immediately from Remark 3.23. □

Theorem 4.9. *The $\mathbb{Z}/p^n$-cover $\phi_n \colon Y_n \to C$ satisfying conditions (i) and (ii) in Proposition 2.35 is a deformation of $k[[y_n]]/k[[x]]$ over R if it gives rise to an étale tree $\mathcal{T}_n$ and its reduction is in the same ASW class as an étale tree that defines $k[[y_n]]/k[[x]]$.*

Proof. The assumption about the degeneration of $\mathcal{T}_n$ implies that the cover has étale reduction and the special fiber is birationally equivalent to $k[[y_n]]/k[[x]]$. In addition, suppose that the nth conductor of $k[[y_n]]/k[[x]]$ is ι_n, which is equal to $d+1$ by definition. Then, by Remark 4.3, the sum of the conductors of the branch points of ϕ_n is also ι_n. The theorem follows immediately from Corollaries 4.8 and 3.24(i). □

Suppose that we are given a $\mathbb{Z}/p^n$-extension ϕ_n of $k[[x]]$ that is defined by $\underline{f} := (f^1, \dots, f^n) \in W_n(k[[x]])$, and a $\mathbb{Z}/p^{n-1}$-deformation χ_{n-1} of its subcover. Theorem 4.9 implies that one can extend χ_{n-1} to a deformation of ϕ_n only if the Hurwitz tree it gives rise to can be extended. We call that obligation the *Hurwitz tree obstruction* for the deforming in towers problem.

When $n = 1$, it is known that every $\mathbb{Z}/p$-tree arises from some $\mathbb{Z}/p$-cover of the rigid disc.

Theorem 4.10 ([Dan20b, Theorem 1.1]). *Suppose that $\mathcal{T}$ is an étale $\mathbb{Z}/p$-tree. Then there exists a $\mathbb{Z}/p$-cover whose associated tree coincides with $\mathcal{T}$.*

Remark 4.11. The proof of Theorem 4.10 utilizes the formal patching technique for $G \cong \mathbb{Z}/p$-torsors. Recall that a $\mathbb{Z}/p$-cover of the boundary of a disc is determined by its depth and its boundary Swan conductor, as mentioned in Remark 3.27. Hence, one can "glue" the covers of the discs and the annuli that partition a unit disc along their boundaries in a G-equivariant way to get a $\mathbb{Z}/p$-cover of the whole disc. That is the technique first used by Henrio in [Hen00] to construct a $\mathbb{Z}/p$-lift from a Hurwitz tree in mixed characteristic. Unfortunately, we cannot apply the same technique for $\mathbb{Z}/p^n$-covers (where $n > 1$), since it is no longer true that the depth and the boundary conductor determine the action on the boundary, as discussed in the same remark. We can, however, generalize that approach to an arbitrary $(\mathbb{Z}/p \rtimes_{\psi} \mathbb{Z}/m)$-cover by adding information coming from ψ to the Hurwitz tree of the $\mathbb{Z}/p$-subcover, as explained in Remark 4.4. We thus expect an analog of Theorem 4.10 for $(\mathbb{Z}/p \rtimes_{\psi} \mathbb{Z}/m)$-covers, which is also parallel to the main result of [BW06].

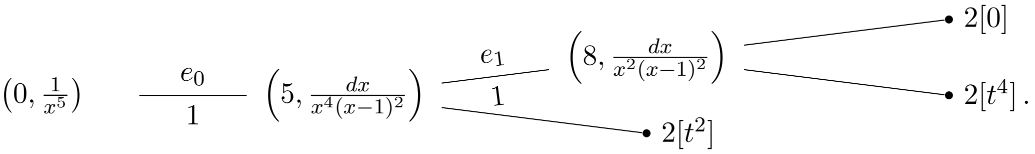
Definition 4.12. Suppose that $\mathcal{T}_n$ is a $\mathbb{Z}/p^n$-Hurwitz tree and $v \neq v_0$ is one of its vertices with m branches branching out $e_1, \ldots, e_m$, that is, $v = s(e_1) = \cdots = s(e_m)$. Suppose, moreover, that the differential conductor at v is of the form

$$\omega_{\mathcal{T}_n}(v) = \frac{c_v dx}{\prod_{i=1}^{m}(x-a_i)^{h_i}} = \left(\sum_{i=1}^{m} \sum_{j=1}^{h_i} \frac{a_{i,j}}{(x-a_i)^j} \right) dx =: \sum_{i=1}^{m} \omega_i \,,$$

where $a_i = [z_{e_i}]_v$. We call ω_i the e_i-part of $\omega_{\mathcal{T}_n}(v)$.

The example below shows how one can derive the relative positions of unknown branch points from the conditions for Hurwitz trees and their extensions.

Example 4.13. Suppose $p = 2$ and that $\phi_2 \in \mathrm{H}^1_4(k(x))$ is defined by $(1/x^5, 0)$, hence has branching datum $[6, 11]$, and Φ_1 is a deformation of $\phi_1 := \phi_2^2$ of type $[6] \to [2,2,2]^\top$ over $R := k[[t]]$, whose branch points are $(0, t^2, t^4)$. Then the tree $\mathcal{T}_1$ associated with Φ_1 has the following form:



The existence of such a deformation is asserted by Theorem 4.10. Suppose in addition that there is a deformation Φ_2 of ϕ_2 extending Φ_1 of type

$$M := \begin{bmatrix} 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 2 \end{bmatrix}^\top . \tag{4.1}$$

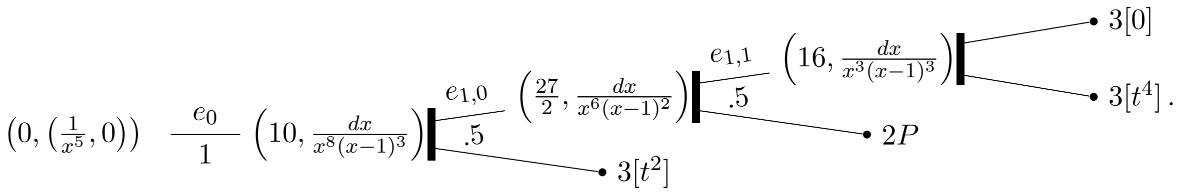
Denote by $\mathcal{T}_2$ the corresponding tree. Then it must be true that $\omega_{\mathcal{T}(2)}(s(e_0))$ is equal to one of

$$\text{(i)}\ \frac{a dx}{x^6(x-1)^3(x-b)^2}\,,\quad \text{(ii)}\ \frac{c dx}{x^6(x-1)^5}\,,\quad \text{(iii)}\ \frac{e dx}{x^8(x-1)^3}\,,\quad \text{(iv)}\ \frac{f dx}{x^6(x-1)^3}$$

for some $a, b, c, e, f \neq 0$ and $b \neq 1$. In all cases, $\mathcal{C}(\omega_{\mathcal{T}(2)}(s(e_0))) \neq 0$. Theorem 3.42(ii)(a) then forces $\delta_{\mathcal{T}_2}(s(e_0)) = 10$ and, respectively,

$$\text{(i)}\ d_{e_0} = 10,\ d_{e_1} = 5\,, \quad \text{(ii)}\ d_{e_0} = 10,\ d_{e_1} = 5\,, \quad \text{(iii)}\ d_{e_0} = 10\,, \quad \text{(iv)}\ d_{e_1} = 5\,.$$

A similar reasoning shows that $\delta_{\mathcal{T}_2}(s(e_1)) = 16$. If Φ_2 has no branch point in the interior of the annulus associated with e_1, then $d_{e_1} = 6$. That contradicts the information we got for cases (i), (ii), and (iv). Finally, a straightforward calculation shows that the branch point of ramification breaks $(0,1)$, which we call P, has valuation 3, and the tree $\mathcal{T}_2$ should be of the following form:



4.2.1 *Deforming Artin–Schreier–Witt covers with no essential part.* Recall from Section 2.3.5 that every one-point $\mathbb{Z}/p^n$-cover with an essential component can deform into a cyclic cover whose branching datum has no essential component. Using the Hurwitz tree technique, we demonstrate that further non-trivial deformations of these covers are not possible. This result plays a crucial role in determining the geometry of the moduli space of cyclic covers, as detailed in [DH24, Theorem 4.16].

PROPOSITION 4.14. *A one-point $\mathbb{Z}/p^n$-cover ϕ_n whose branching datum has no essential part cannot be non-trivially deformed.*

Proof. Suppose that ϕ_n has branching datum $[e_1, \ldots, e_n]$, where $e_1 \neq 0$. Furthermore, suppose that ϕ_n deforms non-trivially to Φ_n. Without loss of generality, one may assume that Φ_n has type

$$M = [e_1, e_2, \ldots, e_n] \longrightarrow \begin{bmatrix} e_{1,1} & e_{1,2} & \ldots & e_{1,n} \\ e_{2,1} & e_{2,2} & \ldots & e_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{r,1} & e_{r,2} & \ldots & e_{r,n} \end{bmatrix} = N \,, \tag{4.2}$$

where $e_{1,1} \neq 0$. Assume $e_{j,1} \neq 0$ for some $1 < j \leqslant n$. Let B_i be the branch point of Φ_n associated with the ith row of N. Then there exists a non-trivial Artin–Schreier deformation $\Phi_1 := (\Phi_n)^{p^{n-1}}$ of type $[e_1] \to [e_{1,1}, \ldots, e_{r,1}]^\top$. Let $\mathcal{T}_1$ be the associated Hurwitz tree. Then Theorem 3.42(i) asserts the existence of exact differential forms at certain leaves of $\mathcal{T}_1$, of the form

$$\omega = \frac{c\,dx}{\prod_{i \in I}(x - a_i)^{e_{i,1}}} \in \Omega^1_\kappa \,, \tag{4.3}$$

where $c \neq 0$, $a_i \neq a_j$ for $i \neq j$, and $I \subseteq \{1, \ldots, r\}$. However, as $e_1 < p$ due to the assumption of no essential part, we have $\sum_{i \in I} e_{i,1} \leqslant e_1 < p + |I|$. Therefore, ω cannot exist, as shown in [Dan20b, Proposition 6.4].

By induction, we may rewrite (4.2) as

$$M = [e_1, e_2, \ldots, e_n] \longrightarrow \begin{bmatrix} e_1 & e_2 & \ldots & e_{n-1} & e_{1,n} \\ 0 & 0 & \ldots & 0 & e_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & e_{r,n} \end{bmatrix} = N \,. \tag{4.4}$$

Also, since M lacks an essential part, we have $pe_{n-1} \geqslant e_n = \sum_{i=1}^{r} e_{i,n}$. Therefore, we have the condition

$$pe_{n-1} - p + 1 \leqslant e_{1,n} < pe_{n-1}\,,$$

where the first inequality follows from the condition on the upper jump.

Suppose $e_{1,n} = pe_{n-1} - p + 1 \equiv 1 \pmod{p}$. Let $\mathcal{T}_n$ (respectively, $\mathcal{T}_{n-1}$) represent the Hurwitz tree associated with Φ_n (respectively, $\Phi_{n-1} := (\Phi_n)^p$). Then, for each edge e between the root and the leaf associated with B_1 in $\mathcal{T}_n$, its slope, denoted by $d_e(\mathcal{T}_n)$, can be expressed as

$$d_e(\mathcal{T}_n) = \sum_{b \in B(\mathcal{T}_n),\, x_b \in \overline{C}_e} h_b - 1 > e_{1,n} - 1 \geqslant p(e_{n-1} - 1)\,,$$

given that $x_{B_1} \in \overline{C}_e$ and $|\overline{C}_e| > 1$. On the other hand, the slope of the associated edge in $\mathcal{T}_{n-1}$ is precisely $e_{n-1} - 1$. As a consequence, the depth at the vertex v_1 preceding the leaf associated with B_1 in $\mathcal{T}_n$ is strictly greater than p times the corresponding depth in $\mathcal{T}_{n-1}$. Therefore, in accordance with Theorem 3.42(ii)(b), the differential form at the vertex v_1 within $\mathcal{T}_n$ is exact. This, however, leads to a contradiction, considering that $e_{1,n} \equiv 1 \pmod{p}$.

Now suppose $e_{1,n} = pe_{n-1} - p + a + 1$, where $0 < a < p$. In this case, the Hurwitz tree $\mathcal{T}_n$ is étale of type $[e_{1,n}, \ldots, e_{r,n}]$ with all the differential forms being exact. However, one can demonstrate that such a tree does not exist, using an argument similar to that employed for Artin–Schreier deformations. □

4.3 Reduction type

Suppose that we are given a cyclic character $\chi_n \in \mathrm{H}^1_{p^n}(\mathbb{K})$ of order p^n that is defined by a length n Witt vector $\underline{F}_n := (F^1, \ldots, F^n) \in W_n(\mathbb{K})$. Suppose, moreover, that $\delta_{\chi_n}(0) = 0$. Then the reduction $\overline{\chi}_n \in \mathrm{H}^1_{p^n}(\kappa)$ of χ_n is well defined. One may further assume that $\overline{\chi}_n$ is given by a length n Witt vector

$$\underline{f}_n := \left(f^1, f^2, \ldots, f^n\right) \in W_n(\kappa)\,.$$

We call $\underline{f}_n$ a *reduction type* of χ_n. If $\underline{f}_n$ is reduced, we say that it is the *reduced reduction type* of χ_n and f^i is its *ith degeneration*. It is clear that the character $\chi_i := (\chi_n)^{p^{n-i}} \in \mathrm{H}^1_{p^i}(\kappa)$ has $\underline{f}_i = (f^1, \ldots, f^i)$ as reduction type.

PROPOSITION 4.15. *Suppose that we are given a character $\overline{\chi}_n \in H^1_{p^n}(\kappa)$ of order p^n defined by a length n Witt vector $\underline{f}_n = (f^1, f^2, \ldots, f^n) \in W_n(\kappa)$, a complete discrete valuation ring R that is finite over $k[[t]]$, and a deformation χ_{n-i} of the subcharacter $\overline{\chi}_{n-i} := \overline{\chi}_n^{p^i}$ over R given by*

$$\wp(\underline{Y}_{n-i}) = \left(F^1, \ldots, F^{n-i}\right).$$

Suppose, moreover, that χ_n is a character given by

$$\wp(\underline{Y}_n) = \left(F^1, \ldots, F^{n-1}, F^n\right)$$

and gives rise to an étale tree $\mathcal{T}_n$ that extends $\mathcal{T}_{n-i}$ (in the sense of Definition 4.6). Then χ_n extends χ_{n-i} if it has reduction type $\underline{f}_n$.

Proof. It is clear from the definition that χ_n extends χ_{n-i}. Since the Hurwitz tree $\mathcal{T}_n$ to which it gives rise is étale, χ_n has good reduction by Theorem 4.9. Moreover, having reduction type $\underline{f}_n$ guarantees that the special fiber of χ_n is birationally equivalent to $\overline{\chi}_n$. It then follows from the definition that χ_n is a deformation of $\overline{\chi}_n$ extending χ_{n-i}. □

Remark 4.16. Proposition 4.15 implies that one can extend a $\mathbb{Z}/p^{n-1}$-deformation by finding a rational function F^n that gives the right degeneration data at the root of the associated tree $\mathcal{T}_n$ and the right branching datum to the generic fiber. In general, it is not easy to calculate the refined Swan conductor of a character, let alone control it to obtain the wanted reduction. To get around it, we start our construction of F^n from the disc corresponding to a "final vertex" (see induction step *Ind* 1.) of the extending Hurwitz tree $\mathcal{T}_n$, where the degeneration is easy to manage. We then continuously modify F^n until we get to the boundary of the unit disc $\operatorname{Spec} R[[X]]$. More details will be given in Section 5.4. This strategy again resembles one used by Obus and Wewers in [OW14] to prove Oort's conjecture.

4.4 The compatibility of the differential conductors

Definition 4.17. Suppose that $\mathcal{T}$ is a $\mathbb{Z}/p^n$-Hurwitz tree that arises from $\chi \in \mathrm{H}^1_{p^n}(\mathbb{K})$, which has conductor ι_n and good reduction. Suppose, moreover, that v is a vertex (which can be the root) of the étale tree $\mathcal{T}$ initiating m edges $e_1, \ldots, e_m$ and has a differential conductor (or nth degeneration type) of the form

$$\omega_{\mathcal{T}}(v) = \frac{c_v dx}{\prod_{i=1}^m (x-a_i)^{h_i}} = \sum_{i=1}^m \sum_{j=1}^{h_i} \frac{c_{v,j} dx}{(x-a_i)^j} \quad \left(\text{or } \sum_{j=1}^{l} \frac{d_j}{x^j}, \text{ where } d_l \neq 0, p \nmid l\right),$$

where $c_{v,h_i} \neq 0$ for $i = 1, \ldots, m$ and $a_i = [z_{e_i}]_v$. Then we call c_v (or $-ld_l$) the *constant coefficient at* v and c_{v,h_i} the *constant coefficient of its* e_i *part.*

Suppose that e is an edge in $\mathcal{T}$ and $r \in (s(e), t(e)) \cap \mathbb{Q}$. Then it follows from the discussion in Section 3.6 that the differential conductor at r of χ is a finite sum of the form

$$\omega_\chi(r) = \sum_{j \geqslant l} \frac{c_j dx}{(x-[z_e]_r)^j}$$

for some $l \in \mathbb{N}$ where $c_l \neq 0$.

Definition 4.18. With the notation above, we say that $\omega_{\mathcal{T}}$ has coefficient c_l at r.

The following result shows that the constant coefficients of the differential conductors along the Hurwitz tree constructed in Section 4.2 are "compatible" in some senses.

Theorem 4.19. *Suppose that $\mathcal{T}$ is a tree that arises from a cyclic cover with good reduction. Let e be an edge in $\mathcal{T}$. If $\delta_{\mathcal{T}}(s(e)) > 0$, then the following hold:*

(i) *Suppose that $s(e) < r < t(e)$ is a rational place on e. Then the constant coefficient at r is equal to that at $t(e)$.*

(ii) *The constant coefficient at $t(e)$ is equal to that of the e-part at $s(e)$.*

Suppose that $\mathcal{T}$ is an étale tree and the nth level reduced degeneration at its root is a polynomial in x^{-1} of degree l with coefficient $-ld_l$. If $l < p\iota_{n-1} - p$, then we only need the differential conductors starting from v_1 to be compatible. Otherwise, we have $\iota_n = l+1$, and the differential conductor at v_1 has the same coefficient as that at v_0.

We first prove the result above for the situation of Section 3.8. The rest of the proof will be given Section 5.3, where we have extra tools to generalize Proposition 4.20.

Proposition 4.20. *Suppose that $\chi \in H^1_{p^n}(\mathbb{K})$ satisfies conditions* (D1), (D2), *and* (D3) *of Section* 3.8 *with respect to a rational place r_0. Suppose that at r_0, the differential conductor $\omega_\chi(r_0)$*

has constant coefficient c and order of infinity $\iota - 2 > 0$. Then there exists an $r \in [0, r_0) \cap \mathbb{Q}$ such that, for all $s \in (r, r_0) \cap \mathbb{Q}$, we have

$$\delta_\chi(s) = \delta_\chi(r_0) - (\iota - 1)(r_0 - s) \quad \textit{and} \quad \omega_\chi(s) = \frac{c\,dx}{x^\iota}\,.$$

In particular, for all $r' \in [0, r_0) \cap \mathbb{Q}$ such that $\delta_\chi(r') = \delta_\chi(r_0) - (\iota - 1)(r_0 - r')$, we have

$$\omega_\chi(r') = \frac{c\,dx}{x^\iota} + \sum_{i=1}^{\iota-1} \frac{c_i\,dx}{x^i} =: \frac{c\,dx}{x^\iota} + \omega'_\chi(r')\,,$$

where $\omega'_\chi(r')$ is exact, that is, $c_i = 0$ for all $i \equiv 1 \pmod p$.

Proof. We first assume $G \cong \mathbb{Z}/p$. Set $Y := X/t^{pr_0}$ and $\delta := \delta_\chi(r_0)$. By Proposition 3.26, we may assume that the restriction of χ to $D[pr_0]$ is represented by an Artin–Schreier class with representative $f_\chi = t^{-p\delta} F(Y)$, where the reduction (modulo t) of F is $\overline{F} := f \notin \kappa^p$. In addition, condition (D1) allows us to write

$$df = \left(\sum_{j \geqslant \iota} \frac{\overline{b}_j}{x^j} \right) dx \in \Omega^*_\kappa\,,$$

where $\overline{b}_j \in k$ and $\overline{b}_\iota = c$. Set $I := \{i \in \mathbb{N} \mid \overline{b}_i \neq 0\}$. It then follows from the discussion in Section 3.8 that

$$f_\chi = \sum_{i=0}^{\infty} a_i Y^{-i} = \frac{1}{t^{p\delta}} \left(\sum_{j \geqslant \iota - 1} \frac{d_j}{Y^j} + \sum_{l \notin I} \frac{e_l}{Y^l} \right), \tag{4.5}$$

where $d_j \in R$, $\nu(d_j) = 0$, and $\nu(e_l) > 0$. In addition, using the assumption about the constant coefficient and Proposition 3.26, we learn that $c = (1 - \iota)\overline{d}_{\iota-1}$. Let $Z := Y t^{ps}$. We thus have

$$f_\chi = \sum_{i=0}^{\infty} a_i t^{psi} Z^{-i} = \frac{1}{t^{p\delta}} \left(\sum_{j \geqslant \iota - 1} \frac{d_j t^{psj}}{Z^j} + \sum_{l \notin I} \frac{e_l t^{psl}}{Z^l} \right). \tag{4.6}$$

Consequently, there exists a $v \in (0, r_0) \cap \mathbb{Q}$ such that, for all $w \in [v, r_0) \cap \mathbb{Q}$, the value $\nu(d_{\iota-1}) + p(r_0 - w)(\iota - 1)$ is strictly smaller than the valuations of other terms. The first part then immediately follows from Proposition 3.26. Let $r' \geqslant v$ be the largest kink of δ_χ on $(0, r_0]$. Applying the process from Section 3.8 for some $r \in (0, r') \cap \mathbb{Q}$, one may replace f_χ of (4.5) by another function in the same Artin–Schreier class, which we may assume to have the form of (4.6), such that Proposition 3.26 applies immediately for any rational place in $(0, v)$. In addition, that procedure does not change the $\iota - 1$ coefficient of f_χ and, for $s := r_0 - r$, we have $\nu\big(d_{\iota-1} t^{ps(\iota-1)}\big) \leqslant \nu\big(e_l t^{psl}\big)$ only when $l < \iota - 1$ and $l \not\equiv 0 \pmod p$. The rest once more follows from an application of Proposition 3.26 to the place r'.

We now consider the case $G \cong \mathbb{Z}/p^n$ ($n > 1$). Suppose that the length n Witt vector $\underline{f} = (f^1, \ldots, f^{n-1}, f^n)$ that represent χ is best at r_0 and $n - j + 1$ is its relevance length (see Definition 3.36). Recall from Proposition 3.40 that its differential Swan conductor at r_0 is

$$\operatorname{dsw}_\chi(r_0) = \sum_{i \geqslant j}^{n} [f^i]_{r_0}^{p^{n-i}-1} d[f^i]_{r_0} = \left[\sum_{i=1}^{n} (f^i)^{p^{n-i}-1} df^i \right]_{r_0}.$$

The first assertion follows from the application of the base case's argument to $\sum_{i=1}^{n} (f^i)^{p^{n-i}-1} df^i$. Let us consider the case $\delta_\chi(r_0) > p\delta_{\chi^p}(r_0)$. Then the relevance length of $\underline{f}$ (see Definition 3.38) is 1 at r_0, and $\operatorname{dsw}_\chi(r_0) = d[f^n]_{r_0}$. One may then prove the second assertion for this case using the same reasoning as for the case $n = 1$, where f^n plays the role of f_χ. Now suppose that $\delta_\chi(r_0) = p\delta_{\chi^p}(r_0)$, that is, the relevance length of $\underline{f}$ is greater than 1. Then, as before, there exists a smallest $r' \in [0, r_0) \cap \mathbb{Q}$ such that $\delta_\chi(s)$ is linear on $[r', r_0)$ with slope $\iota - 1$. In addition, for all $s \in [r, r_0) \cap \mathbb{Q}$, it must be true that

$$\omega_\chi(s) = \frac{c_s dx}{x^\iota} \quad \text{and} \quad \omega_\chi(r') = \frac{c_{r'} dx}{x^\iota} + \sum_{i=1}^{\iota-1} \frac{c_{r',i} dx}{x^i}\,.$$

Note that $c_s = c$ for $s \in (r, r_0) \cap \mathbb{Q}$. Let us consider two separate cases:

- If $\delta_\chi(s') > p\delta_{\chi^p}(s')$ for $s' \in (r', r_0)$, then $\delta_\chi(s) > p\delta_{\chi^p}(s)$ for all $s \in [r', r_0) \cap \mathbb{Q}$. Therefore, it must be true that $\underline{f}$ is best of relevance length 1 on the interval (r', r_0). We are in a situation similar to one where $\delta_\chi(r_0) > p\delta_{\chi^p}(r_0)$, where s' plays the role of r_0.
- If $\delta_\chi(s') = p\delta_{\chi^p}(s')$ for $s' \in (r, r_0)$, then, by Theorem 3.42, it must be true that $\delta_\chi(s) = p\delta_{\chi^p}(s)$ for all $s \in [r, r_0]$. Note that, if $\delta_{\chi^p}(-)$ concaves down at a place s on $(0, r_0)$, then $\operatorname{ord}_\infty \omega_{\chi^p}(s) + 1 > -\operatorname{ord}_0 \omega_{\chi^p}(s) - 1$ (see Proposition 3.15), which implies that $\omega_{\chi^p}(s)$ has a pole outside 0, contradicting Corollary 3.22. Therefore, if $\omega_{\chi^p}(s') = mdx/x^l$, then $\omega_{\chi^p}(s) = mdx/x^l$ for all $s \in (r', r_0) \cap \mathbb{Q}$ by induction, so $\omega_\chi(s) = m^p dx/x^{pl+1}$. The rest easily follows.

That completes the proof of the proposition. ☐

Definition 4.21. With the above notation, we say that the conductor at $t(e)$ is *compatible* with that at $s(e)$ if the constant coefficient at $t(e)$ is equal to that of the e-part at $s(e)$. Suppose $v \prec v'$; then we say that v' is compatible with v if, given any pair of adjacent vertices $v \prec v_1 \prec v_2 \prec v'$, the vertex v_1 is compatible with v_2.

4.4.1 *Partition of a tree by its edges and vertices.*

Suppose that we are given a Hurwitz tree $\mathcal{T}$ and that e is one of its edges. We define $\mathcal{T}(e)$ to be a (non-Hurwitz) subtree of $\mathcal{T}$ whose

- edges, vertices, and information on them coincide with those succeeding $s(e)$,
- differential datum at $s(e)$ is equal to the e-part of $\omega_{\mathcal{T}}(s(e))$.

These trees will play a critical role in the extension of a Hurwitz tree in Section 4.5 and in the "partition process" Section 5.3. More precisely, Section 5 will construct a cover that gives rise to a whole tree inductively from its subtree. Note that $\mathcal{T}(e)$ satisfies all of the Hurwitz tree's axioms but ones that involve the root and the trunk.

4.4.2 *Deriving a $\mathbb{Z}/p^{n-1}$-tree from a $\mathbb{Z}/p^n$-tree.*

Suppose that χ_n is a $\mathbb{Z}/p^n$-character that gives rise to a Hurwitz tree $\mathcal{T}_n$. By following the process below, one can derive the tree $\mathcal{T}_{n-1}$ for level $n-1$ just from the level n tree and the branching datum of the generic fiber, as follows:

(i) Erase the $\mathbb{Z}/p$-leaves and the edges e of $\mathcal{T}_n$ such that the monodromy group at $t(e)$ is $\mathbb{Z}/p$.
(ii) Set the conductors at the leaves according to the $(n-1)$-branching datum of the generic fiber.
(iii) If the monodromy group at a vertex v is $G_{\mathcal{T}_n}(v) = \mathbb{Z}/p^m$, where $m > 1$, then set $G_{\mathcal{T}_{n-1}}(v) = \mathbb{Z}/p^{m-1}$.
(iv) Starting from the root, assign the degeneration data at the succeeding vertices inductively so that the differential conductors are compatible and satisfy the conditions of Theorem 3.42.

One can thus easily derive the degeneration data of the lower (than $n-1$) levels from the highest one by reiterating the process above. The readers can see this from the examples in Section 4.5.

4.5 The vanishing of the differential Hurwitz tree obstruction

In the proposition below, we discuss the vanishing of the Hurwitz tree obstruction for the refined deformation problem. In addition, those extending Hurwitz trees are specially designed to be used as the "skeletons" to build the lifts in Section 5. The proof is postponed to Section 6.2 as they are quite technical. However, we will provide some generalized-enough examples and discuss some of their features to help the readers follow Section 5.

Proposition 4.22. *Suppose that $k[[y_n]]/k[[x]]$ is cyclic Galois of order p^n with conductor ι_n. Suppose, moreover, that $\phi_{n-1}\colon Y_{n-1} \to C$ is an admissible deformation of the $\mathbb{Z}/p^{n-1}$-subextension $k[[y_{n-1}]]/k[[x]]$ over a finite extension R of $k[[t]]$, and $\mathcal{T}_{n-1}$ is its associated Hurwitz tree. Then there exists a good $\mathbb{Z}/p^n$-Hurwitz tree $\mathcal{T}_n$ of conductor ι_n that extends $\mathcal{T}_{n-1}$.*

Next, we will give three examples of extending a $\mathbb{Z}/p$-tree of conductor ι_1 to certain $\mathbb{Z}/p^2$-trees, one for a minimal second upper jump (that is, $\iota_2 = p\iota_1 - p$) and two for a non-minimal second upper jump (that is, $\iota_2 = p\iota_1 - p + a_2$, where $p \nmid a_2$).

Example 4.23. Suppose that $\overline{\chi}$ is a character in $\mathrm{H}^1_{7^2}(\kappa)$ that is given by the equation

$$\wp(y_1, y_2) = \left(\frac{1}{x^{29}}, \frac{2}{x^{135}} + \frac{1}{x}\right).$$

One can think of $\overline{\chi}$ as a $\mathbb{Z}/49$-extension of the complete discrete valuation ring $k[[x]]$. By Theorem 2.4, the extension $\overline{\chi}$ has upper jumps $u_1 = 29$ and $u_2 = \max\{29 \cdot 7, 135\} = 203$. This means that the second jump is minimal. The $\mathbb{Z}/7$-subextension $\overline{\chi}_1$ of $\overline{\chi}$ is given by

$$y_1^7 - y_1 = \frac{1}{x^{29}}\,.$$

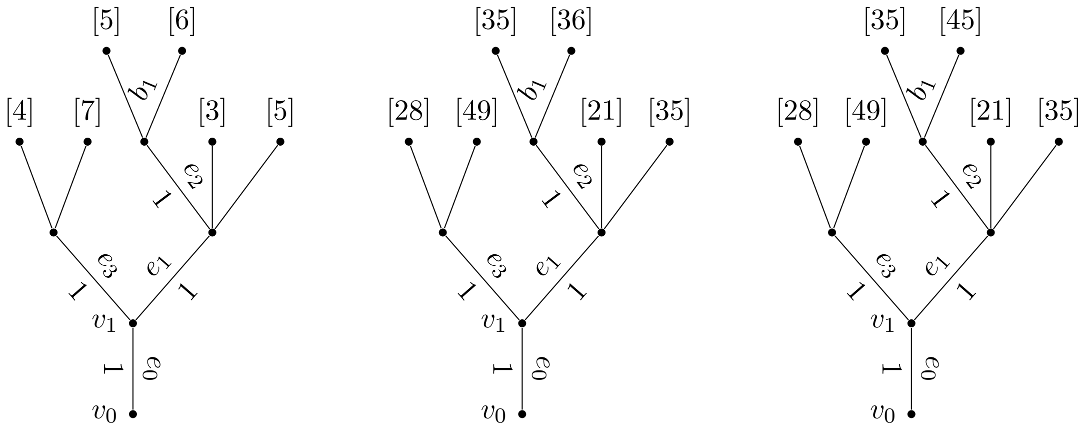
Consider the $\mathbb{Z}/7$-Hurwitz tree $\mathcal{T}_1$ in Figure 3a, whose degeneration data are on Table 4. The number a can be either $2 + 5\sqrt{6}$ or $2 - 5\sqrt{6}$. Note that these choices of a make the differential $dx/(x^{11}(x-1)^5(x-a)^3)$ exact. Also note that the constant coefficient $-1/a^3$ of the differential conductor at $t(e_2)$ is necessary for it to be compatible with one at $t(e_1)$. Recall that one can construct from $\mathcal{T}_1$ a $\mathbb{Z}/7$-character χ_1 of type $[5, 6, 5, 3, 4, 7]^\top$ that deforms $\overline{\chi}_1$ (see Theorem 4.10). The tree $\mathcal{T}_2^{\min}$ in Figure 3b with degeneration data in the third column of Table 4 and with monodromy group $\mathbb{Z}/49$ at each vertex is a $\mathbb{Z}/49$-tree that extends $\mathcal{T}_1$ and has conductor 204. One can show that $\delta_{\mathcal{T}_2}(v) = 7\delta_{\mathcal{T}_1}(v)$ and $\mathcal{C}(\omega_{\mathcal{T}_2^{\min}}(v)) = \omega_{\mathcal{T}_1}(v)$ for each $v \in V_{\mathcal{T}_2^{\min}} \setminus \{v_0\}$. It follows from the data in Table 4 that $\mathcal{T}_2^{\min}$ can represent a deformation of type

$$\begin{bmatrix} 30 & 204 \end{bmatrix} \longrightarrow \begin{bmatrix} 5 & 6 & 5 & 3 & 4 & 7 \\ 35 & 36 & 35 & 21 & 28 & 49 \end{bmatrix}^\top .$$

Example 4.24. If we replace the Witt vector that defined $\overline{\chi}$ in the previous example by

$$\wp(y_1, y_2) = \left(\frac{1}{x^{29}}, \frac{3}{x^{212}} + \frac{1}{x^3}\right),$$

then the upper jump of the new character is $(30, 213)$. Suppose that the deformation at the $\mathbb{Z}/7$-level is the same. Hence, its corresponding tree $\mathcal{T}_1$ is also identical to one in the previous example.



(a) $\mathcal{T}_1$ (b) $\mathcal{T}_2^{\mathrm{min}}$ (c) $\mathcal{T}_2^{9}$

Figure 3. Extending trees

Vertices	$\mathcal{T}_1$	$\mathcal{T}_2^{\mathrm{min}}$	$\mathcal{T}_2^{9}$
v_0	$\left(0, \frac{1}{x^{29}}\right)$	$\left(0, \left(\frac{1}{x^{29}}, \frac{2}{x^{135}} + \frac{1}{x}\right)\right)$	$\left(0, \left(\frac{1}{x^{29}}, \frac{3}{x^{212}} + \frac{1}{x^3}\right)\right)$
v_1	$\left(29, \frac{dx}{x^{19}(x-1)^{11}}\right)$	$\left(203, \frac{dx}{x^{127}(x-1)^{77}}\right)$	$\left(212, \frac{dx}{x^{136}(x-1)^{77}}\right)$
$t(e_3)$	$\left(39, \frac{dx}{x^7(x-1)^4}\right)$	$\left(280, \frac{dx}{x^{49}(x-1)^{28}}\right)$	$\left(288, \frac{dx}{x^{49}(x-1)^{28}}\right)$
$t(e_1)$	$\left(47, \frac{-dx}{x^{11}(x-1)^5(x-a)^3}\right)$	$\left(329, \frac{-dx}{x^{71}(x-1)^{35}(x-a)^{21}}\right)$	$\left(347, \frac{-dx}{x^{80}(x-1)^{35}(x-a)^{21}}\right)$
$t(e_2)$	$\left(57, \frac{-dx}{a^3x^5(x-1)^6}\right)$	$\left(399, \frac{-dx}{a^{21}x^{35}(x-1)^{36}}\right)$	$\left(426, \frac{-dx}{a^{21}x^{35}(x-1)^{52}}\right)$

Table 4. Degeneration data of the trees in Figure 3

Figure 3c is a tree $\mathcal{T}_2^9$ that extends $\mathcal{T}_1$ and has conductor 213. Note that $\delta_{\mathcal{T}_2}(v) > 7\delta_{\mathcal{T}_1}(v)$ and $\mathcal{C}(\omega_{\mathcal{T}_2^9}(v)) = 0$ for each $v \in V_{\mathcal{T}_2^9} \setminus \{v_0\}$.

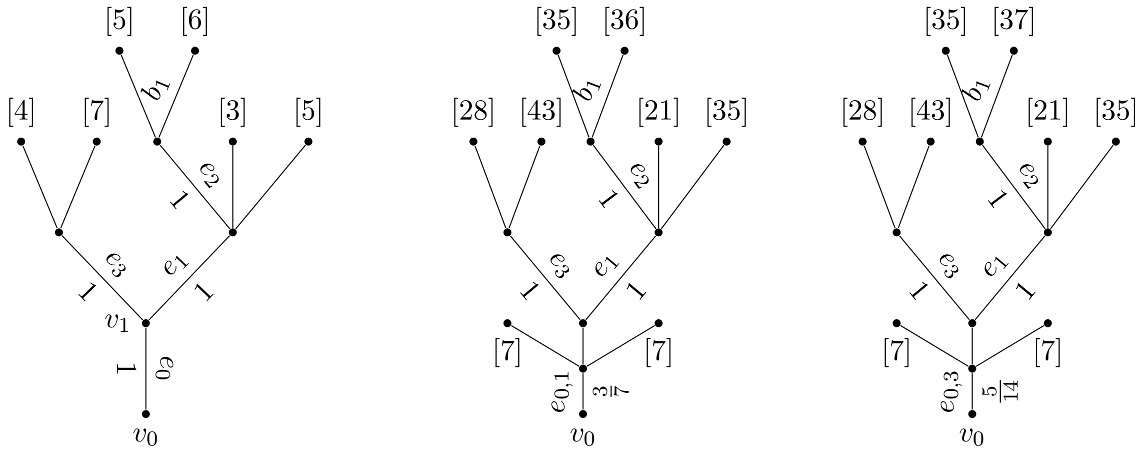
Note that the extension tree in Example 4.24 is not unique. The following example illustrates an alternative construction where the extending tree $\mathcal{T}_2^{\mathrm{ness}}$ has no essential jumps from level $n-1$ to level n (*ness* stands for non-essential). The general construction will be given in Proposition 6.5.

Example 4.25. Suppose that the character $\overline{\chi}$ from the previous example corresponds to

$$\wp(y_1, y_2) = \left(\frac{1}{x^{29}}, \frac{2}{x^{211}} + \frac{1}{x}\right) \quad \left(\text{respectively, } \left(\frac{1}{x^{29}}, \frac{3}{x^{212}} + \frac{1}{x^3}\right)\right),$$

and the $\mathbb{Z}/7$-deformation is the same. Then the diagram in Figure 4b (respectively, Figure 4c) gives an extending tree of $\mathcal{T}_1$ with the right reduction type. Its degeneration data are in Table 5. The monodromy groups at all vertices but the ends of the two leaves of conductors 7 are isomorphic to $\mathbb{Z}/49$. The monodromy groups at those two vertices are $\mathbb{Z}/7$. Observe that, in both

Figure 4b and 4c, we have $\delta_{\mathcal{T}_2}(v_1) = 7\delta_{\mathcal{T}_1}(v_1)$.



(a) $\mathcal{T}_1$ (b) $\mathcal{T}_2^{8\,\text{ness}}$ (c) $\mathcal{T}_2^{9\,\text{ness}}$

Figure 4. Extending trees with no essential jumps from level 1 to 2

Vertices	$\mathcal{T}_1$	$\mathcal{T}_2^{8\,\text{ness}}$	$\mathcal{T}_2^{9\,\text{ness}}$
v_0	$\left(0, \frac{1}{x^{29}}\right)$	$\left(0, \left(\frac{1}{x^{29}}, \frac{2}{x^{211}} + \frac{1}{x}\right)\right)$	$\left(0, \left(\frac{1}{x^{29}}, \frac{3}{x^{212}} + \frac{1}{x^3}\right)\right)$
$t(e_{0,1})$	$\left(\frac{89}{7}, \frac{dx}{x^{30}}\right)$	$\left(\frac{633}{7}, \frac{5dx}{x^{198}(x-1)^7(x-5)^7}\right)$	
$t(e_{0,3})$	$\left(\frac{145}{14}, \frac{dx}{x^{30}}\right)$		$\left(\frac{530}{7}, \frac{dx}{x^{199}(x-2)^7(x-4)^7}\right)$
v_1	$\left(29, \frac{dx}{x^{19}(x-1)^{11}}\right)$	$\left(203, \frac{dx}{x^{127}(x-1)^{71}}\right)$	$\left(203, \frac{dx}{x^{128}(x-1)^{71}}\right)$
$t(e_3)$	$\left(39, \frac{dx}{x^7(x-1)^4}\right)$	$\left(273, \frac{dx}{x^{43}(x-1)^{28}}\right)$	$\left(273, \frac{dx}{x^{43}(x-1)^{28}}\right)$
$t(e_1)$	$\left(47, \frac{-dx}{x^{11}(x-1)^5(x-a)^3}\right)$	$\left(329, \frac{-dx}{x^{71}(x-1)^{35}(x-a)^{21}}\right)$	$\left(330, \frac{-dx}{x^{72}(x-1)^{35}(x-a)^{21}}\right)$
$t(e_2)$	$\left(57, \frac{-dx}{a^3x^5(x-1)^6}\right)$	$\left(399, \frac{-dx}{a^{21}x^{35}(x-1)^{36}}\right)$	$\left(401, \frac{-dx}{a^{21}x^{35}(x-1)^{37}}\right)$

Table 5. Degeneration data of the trees in Figure 4

Remark 4.26. Suppose that r is an arbitrary rational point on the tree $\mathcal{T}_2$. Then one can observe from the previous examples that there are at most two succeeding leaves whose conductors are not divisible by p. This also holds for all the trees $\mathcal{T}_n$ that we will construct to prove Proposition 4.22. This fact is essential for the proof of Lemma 5.15, which, in turn, is a key ingredient of the proof of Proposition 2.35.

Remark 4.27. In the tree $\mathcal{T}_2^{\text{min}}$ of Figure 3b, there is one leaf b_1 with conductor $l_{1,2} = 7l_{1,1} - 7 + 1$. The other leaves b_i have conductor $l_{i,2} = 7l_{i,1}$. Furthermore, at each vertex v adjacent to the edges e_2, e_1, and e_0, we have $\delta_{\mathcal{T}_2}(v) = 7\delta_{\mathcal{T}_1}(v)$. We say that the tree $\mathcal{T}_2^{\text{min}}$ extends $\mathcal{T}_1$ *minimally.*

Note that $\mathcal{T}_2^{\min}(e_1)$ (respectively, $\mathcal{T}_2^{\min}(e_2)$) also extends $\mathcal{T}_1(e_1)$ (respectively, $\mathcal{T}_1(e_2)$) minimally. In addition, the tree $\mathcal{T}_2^9(e_1)$ of Figure 3c has one leaf b_1 with conductor $l_{1,2} = 7l_{1,1} - 7 + 9 + 1$, and the other leaves b_i have conductor $l_{i,2} = 7l_{i,1}$. We say that $\mathcal{T}_2^9$ extends $\mathcal{T}_1$ 9-*additively*. Similarly, the tree $\mathcal{T}_2^{\min}(e_3)$ extends $\mathcal{T}_1(e_3)$ $(7-1)$-additively.

At the vertex v_1 of the $\mathbb{Z}/49$-trees in Figure 4b and 4c, all but two subtrees starting from it extend the corresponding subtrees of $\mathcal{T}_1$ $(7-1)$-additively. The tree $\mathcal{T}_2^{9\,\mathrm{ness}}(e_1)$ extends $\mathcal{T}_1(e_1)$ 1-additively, and $\mathcal{T}_2^{8\,\mathrm{ness}}(e_3)$ and $\mathcal{T}_2^{9\,\mathrm{ness}}(e_3)$) extend $\mathcal{T}_1(e_3)$ minimally. Formal definitions of the above conventions will be given in Section 6.2.

5. Proof of Proposition 2.35

In this section, we give the final step (Step 4) of the proof of Proposition 2.35, hence of Theorem 2.34, hence of Theorem 1.2. From now on, we set $\mathbb{K} := K(X)$ and denote by $\kappa := k(x)$ its residue field.

The inverse process of Section 4.2, constructing a cover from a Hurwitz tree, was utilized by Henrio in [Hen00] and by Bouw and Wewers in [BW06] to solve the lifting problem for $\mathbb{Z}/p$-covers and $(\mathbb{Z}/p \rtimes \mathbb{Z}/m)$-covers, respectively. However, the main technique in these papers is the gluing method, which no longer works in our situation (as discussed in Remarks 3.27 and 4.11). We instead generalize Obus and Wewers' approach in [OW14].

5.1 Geometric setup

In the rest of the paper, we set $C \cong \mathbb{P}^1_K = \operatorname{Proj} K[X,V]$. Recall that $\mathcal{D}$ is the closed unit disc inside $\left(\mathbb{A}^1_K\right)^{\mathrm{an}} \subset C^{\mathrm{an}}$ centered at $X = 0$. We are given a character $\overline{\chi}_n \in \mathrm{H}^1_{p^n}(\kappa)$ corresponding to a cyclic cover of $\mathcal{D}$ over k of order exactly p^n, branching only at 0 with upper ramification breaks $(m_1, \ldots, m_{n-1}, m_n)$. Set $(\iota_1, \ldots, \iota_n) := (m_1+1, \ldots, m_n+1)$. For $i = 1, \ldots, n-1$, we set $\overline{\chi}_i := (\overline{\chi}_n)^{p^{n-i}} \in \mathrm{H}^1_{p^i}(\kappa)$. Suppose that there is a character $\chi_{n-1} \in \mathrm{H}^1_{p^{n-1}}(\mathbb{K})$ deforming $\overline{\chi}_{n-1}$ (in the sense of Definition 3.9). In order to prove the refined local deformation problem, we must show that there exists a character $\chi_n \in \mathrm{H}^1_{p^n}(\mathbb{K})$ that smoothly deforms $\overline{\chi}_n$ and satisfies $\chi_n^p = \chi_{n-1}$.

We may assume that χ_{n-1} corresponds to an extension of $\mathbb{K}$ given by a length $n-1$ Witt vector $\underline{G}_{n-1} = (G^1, \ldots, G^{n-1})$. Then any $\chi_n \in \mathrm{H}^1_{p^n}(\mathbb{K})$ such that $\chi_n^p = \chi_{n-1}$ is given by a length n Witt vector $\underline{G}_n = (G^1, \ldots, G^{n-1}, G^n) \in W_n(\mathbb{K})$. After a finite extension of K, one may assume that $\underline{G}_{n-1}$ is best (see Definition 3.36). Recall that, by Proposition 4.22, one can construct a Hurwitz tree $\mathcal{T}_n$ with conductor $\iota_n = m_n + 1$ that extends $\mathcal{T}_{n-1}$ (see Definition 4.6). Suppose that the type of such a $\mathcal{T}_n$ is $(\iota_{n,1}, \ldots, \iota_{n,r})$, where $\sum_i \iota_{n,i} = \iota_n$. We would then like G^n to be of the form

$$G^n = \sum_{j=1}^{r} \sum_{i=1}^{\iota_{n,j}-1} \frac{a_{j,i}}{(X-b_j)^i} \in \mathbb{K},$$

where $a_{i,j} \in k((t))$, the b_i are pairwise-distinct K-points of the open unit disc D, and the geometry of the poles (see Definition 3.6) of $\underline{G}_n$ agrees with that of $\mathcal{T}_n$. We say that the polynomial G^n *gives rise* to the character χ_n (from χ_{n-1}).

5.2 The base case

The case $n = 1$ is trivial as one can always deform an Artin–Schreier cover trivially over $k[[t]]$ (using the same Artin–Schreier equation that defines the special fiber to represent the generic fiber). Recall that Theorem 4.10 asserts that the existence of a deformation of Artin–Schreier covers with a given type is fully determined by the presence of $\mathbb{Z}/p$-differential Hurwitz trees of the same type. One may utilize this fact to study the deformation ring $R_{\mathbb{Z}/p}$ (see Section 2.3.2).

5.3 Partition of the extending Hurwitz tree

In this section, we discuss a strategy to translate our situation to one that can be solved using the tools from [OW14]. Suppose that χ_{n-1} branches at s points $b_1, \dots, b_s$, where $b_i \in \mathfrak{m}R$. We first show that its corresponding Witt vector $\underline{G}_{n-1} = (G^1, \dots, G^{n-1})$ can be partitioned with respect to a rational place on the associated Hurwitz tree.

Proposition 5.1. *With the notation as in the geometric setup, suppose that A and B partition the branch locus $\mathbb{B}(\chi_{n-1}) = \{b_1, \dots, b_s\}$. Then there exist two length $n-1$ Witt vectors*

$$\left(G_A^1, \dots, G_A^{n-1}\right) \quad \text{and} \quad \left(G_B^1, \dots, G_B^{n-1}\right)$$

such that the former vector only has poles in A, the latter one only has poles in B, and

$$\left(G^1, \dots, G^{n-1}\right) = \left(G_A^1, \dots, G_A^{n-1}\right) + \left(G_B^1, \dots, G_B^{n-1}\right).$$

Proof. We give a proof by induction on n. Let us first assume $n = 2$. We can then write G^1 as the sum of G_A^1 and G_B^1 simply by splitting its partial-fraction decomposition. When $n > 2$, we have

$$\begin{aligned}\left(G^1, G^2, \dots, G^{n-1}\right) - \left(G_A^1, 0, \dots, 0\right) &= \left(G_B^1, G_1^2, \dots, G_1^{n-1}\right) \\ &= \left(G_B^1, 0, \dots, 0\right) + \left(0, G_1^2, \dots, G_1^{n-1}\right).\end{aligned}$$

The second equation comes from the "decomposition" property of Witt vector [Lor08, § 26, (32)]. The second term is $\mathbb{V}(G_1^2, \dots, G_1^{n-1})$, where $\mathbb{V}$ is the Verschiebung operation for Witt vectors. By induction, we may write $\left(G_1^2, \dots, G_1^{n-1}\right) = \left(G_{1,A}^2, \dots, G_{1,A}^{n-1}\right) + \left(G_{1,B}^2, \dots, G_{1,B}^{n-1}\right)$, using the obvious notation. Therefore,

$$\left(G^1, \dots, G^{n-1}\right) = \left(G_A^1, G_{1,A}^2, \dots, G_{1,A}^{n-1}\right) + \left(G_B^1, G_{1,B}^2, \dots, G_{1,B}^{n-1}\right),$$

as we want, completing the proof. □

Definition 5.2. Let s be a rational place on an edge e of a Hurwitz tree $\mathcal{T}_{n-1}$, and let $\overline{z} \in k$ or $\overline{z} = \infty$. We define $\underline{G}_{n-1,s,\overline{z}}$ to be a partition of $\underline{G}_{n-1}$ whose poles specify to $\overline{z}$ on the canonical reduction of $\mathcal{D}[s, z_e]$. For any $s(e) < r < t(e)$, set

$$\underline{G}_{n-1,r} := \underline{G}_{n-1,r,0} \quad \text{and} \quad \underline{G}_{n-1,s(e)} = \underline{G}_{n-1,t(e)} := \underline{G}_{n-1,r}.$$

If $v = t(e)$ is a vertex, we set $\underline{G}_v := \underline{G}_{t(e)}$. Define

$$\chi_{n-1,s} := \mathfrak{K}_{n-1}(\underline{G}_{n-1,s}) \quad \text{and} \quad \chi_{n-1,s,\infty} := \mathfrak{K}_{n-1}(\underline{G}_{n-1,s,\infty}).$$

Remark 5.3. Observe that $\underline{G}_{n-1,s_1} = \underline{G}_{n-1,s_2}$ for all $s(e) < s_1 \leqslant s_2 < t(e)$. In addition, if $s(e) < s < t(e)$, then $\underline{G}_{n-1,t(e)} = \underline{G}_{n-1,s} + \underline{G}_{n-1,s,\infty}$. Finally, if $r = t(e) = s(e_1) = \cdots = s(e_l)$, then $\underline{G}_{n-1,t(e)} = \sum_{j=1}^{l} \underline{G}_{n-1,s(e_j)}$.

The following result is simple but will turn out to be critical for our approach.

Proposition 5.4. *Suppose that we are given a character $\chi_{n-1} \in H^1_{p^{n-1}}(\mathbb{K})$, which is defined by $\underline{G}_{n-1} := (G^1, \dots, G^{n-1}) \in W_{n-1}(\mathbb{K})$, has good reduction, and gives rise to a $\mathbb{Z}/p^{n-1}$-Hurwitz tree $\mathcal{T}_{n-1}$. Then the following hold:*

(i) *Suppose* $s(e) < s < t(e) \in \mathbb{Q}_{\geqslant 0}$*, where* e *is an edge of* $\mathcal{T}_{n-1}$*, and* $\underline{G}_{n-1} = \underline{G}_{n-1,s} + \underline{G}_{n-1,s,\infty}$ *as in Proposition 5.1. Then*

$$\delta_{\chi_{n-1}}(s) = \delta_{\chi_{n-1,s}}(s) \quad \textit{and} \quad \omega_{\chi_{n-1}}(s) = \omega_{\chi_{n-1,s}}(s)\,.$$

(ii) *Suppose* $s = t(e) = s(e_1) = \cdots = s(e_l)$*. Then*

$$\delta_{\chi_{n-1}}(s) = \delta_{\chi_{n-1,s(e_i)}}(s) \quad \textit{and} \quad \omega_{\chi_{n-1}}(s) = \sum_{i=1}^{l} \omega_{\chi_{n-1,s(e_i)}}(s).$$

In particular, the constant coefficient (*see Section 4.4*) *of the* e_i*-part of* $\omega_{\chi_{n-1}}(s)$ *is equal to the constant coefficient of* $\omega_{\chi_{n-1,s(e_i)}}(s)$*.*

Proof. Suppose $\delta_{\chi_{n-1}}(r) \neq \delta_{\chi_{n-1,s}}(r)$ at a rational place $s(e) < s \leqslant r < t(e)$ in e. Since χ_{n-1} has good reduction, it follows from Proposition 3.20 and Remark 3.23 that $\delta_{\chi_{n-1}}(r)$ is the minimal depth possible at r with the given positions and conductors of the branch points whose valuations (with respect to z_e) are greater than r. Note that the restrictions of $\chi_{n-1,s}$ and χ_n to $D[pr, z_e]$ have the same branching data. Hence $\delta_{\chi_{n-1}}(r) < \delta_{\chi_{n-1,s}}(r)$. Therefore, it must be true that $\delta_{\chi_{n-1,s}}(r) = \delta_{\chi_{n-1,s,\infty}}(r)$, else $\delta_{\chi_{n-1}}(r) = \max(\delta_{\chi_{n-1,s}}(r), \delta_{\chi_{n-1,s,\infty}}(r))$ by Proposition 3.12, which contradicts the previous sentence. On the other hand, by applying Proposition 3.20, we obtain

$$\operatorname{ord}_0(\omega_{\chi_{n-1,s,\infty}}(w)) \geqslant 0 \quad \text{and} \quad \operatorname{ord}_\infty(\omega_{\chi_{n-1,s}}(w)) \geqslant 0$$

for any $w \in \mathbb{Q} \cap [s(e), t(e)]$. This means that, in the direction from $s(e)$ to $t(e)$ in e, the depth $\delta_{\chi_{n-1,s}}$ is strictly increasing, and $\delta_{\chi_{n-1,s,\infty}}$ is strictly decreasing (recall Corollary 3.16, which says that the differential determines the rate of change of the depth). Thus, it must be true that $\delta_{\chi_{n-1,s,\infty}}(w) \leqslant \delta_{\chi_{n-1,s}}(w)$, and equality can only happen when $w = s(e)$. That gives a contradiction. The claim about the differential conductor then follows immediately from Lemma 3.12.

The rest is straightforward, following a similar line of reasoning. ☐

Remark 5.5. Suppose that $\chi_{n-1} := \mathfrak{K}_{n-1}(\underline{G}_{n-1})$ is a $\mathbb{Z}/p^{n-1}$-cover that has $\mathcal{T}_{n-1}$ as the corresponding tree, and $\mathcal{T}_n$ is a $\mathbb{Z}/p$-extension of $\mathcal{T}_{n-1}$. Consider a vertex $v = t(e)$ on $\mathcal{T}$ with m branches $e_1, e_2, \ldots, e_m$ in the directions away from the root. As before. we can partition $\underline{G}_{n-1}$ into a sum

$$\underline{G}_{n-1} = \underline{G}_{n-1,v,\infty} + \sum_{i=1}^{m} \underline{G}_{n-1,v,i}\,.$$

Each $\underline{G}_{n-1,v,i}$ here is a length $n-1$ Witt vector whose roots are the leaves of $\mathcal{T}_{n-1}$ succeeding e_i. The poles of $\underline{G}_{n-1,v,\infty}$ are the leaves of $\mathcal{T}_{n-1}$ that are not leaves of one of the $\mathcal{T}_{n-1}(e_i)$. Set $\chi_{n-1,v,i} := \mathfrak{K}_{n-1}(\underline{G}_{n-1,v,i}) \in \mathrm{H}^1_{p^{n-1}}(\mathbb{K})$. For any rational place r on $\mathcal{T}_n(e_i)$, we have $\delta_{\chi_{n-1,v,i}}(r) = \delta_{\chi_{n-1}}(r)$ by Proposition 5.4. Hence, we obtain full information about the depth and the differential conductor of $\chi_{n-1,v,i}$. Moreover, all of the branch points of $\chi_{n-1,v,i}$ lie in the closed subdisc $\mathcal{D}[pv, z_e]$. This is similar to the situation considered in [OW14], thus allows us to apply many technique from that paper. More precisely, we will extend the $\chi_{n-1,v,i}$ one at a time. Their sum then gives an extension of $\chi_{n-1,v}$ with tree $\mathcal{T}_n(v)$.

Definition 5.6. Suppose that we are given a tree $\mathcal{T}_n$ that extends $\mathcal{T}_{n-1}$, and r is a rational place on an edge e of $\mathcal{T}_n$. When $s(e) < r \leqslant t(e)$, we define $\mathcal{H}_r$ to be the collection of all $H \in \mathbb{K}$ that give rise to $\mathfrak{K}_1(H) \in \mathrm{H}^1_p(\mathbb{K})$ whose branching divisor is everywhere smaller than or equal to that of $\mathcal{T}_n(s(e)) = \mathcal{T}_n(r)$. We define $\mathcal{H}_{s(e)} = \mathcal{H}_r$ for a rational $s(e) < r < t(e)$. Hence $\mathcal{H}_r = \mathcal{H}_s$ for all $s(e) \leqslant s \leqslant r \leqslant t(e)$. As before, if $v = t(e)$ is a vertex, then we set $\mathcal{H}_v := \mathcal{H}_{t(e)}$.

COROLLARY 5.7. *With the notation as in Definition 5.6, a polynomial $H \in \mathcal{H}_{s(e_0)}$ extends χ_{n-1} to a character χ_n that deforms $\overline{\chi}_n$ if and only if its reduction type coincides with the reduced representation of $\overline{\chi}_n$.*

Proof. “⇐” Since $H \in \mathcal{H}_{s(e_0)}$ and $\underline{G}_{n-1}$ is best (as we assume in the geometric setup), it follows from Theorem 2.4 that $\mathfrak{C}(\chi_n, e_0, 0) \leqslant \iota_n$. Moreover, as the different of the generic fiber is at least that of the special fiber, by Proposition 2.25, the assumption on the reduction type of χ_n implies that the conductor of the special fiber is ι_n, enforcing the equality. That information, combining with Proposition 4.15, proves the direction “⇐”.

“⇒” This direction is immediate from the definition. □

DEFINITION 5.8. Suppose given a $\mathbb{Z}/p^n$-tree $\mathcal{T}_n$ that extends $\mathcal{T}_{n-1}$ and a vertex v of $\mathcal{T}_n$. We define $\mathcal{G}_v \subseteq \mathcal{H}_v$ to be the collection of $G \in \mathbb{K}$ such that the branching datum of the character $\chi_{n,G} \in \mathrm{H}^1_{p^n}(\mathbb{K})$ to which it gives rise fits into $\mathcal{T}_n(v)$, and such that

$$\delta_{\chi_{n,G}}(v) = \delta_{\mathcal{T}_n}(v)\,.$$

We denote by $\mathcal{G}_v^* \subseteq \mathcal{G}_v$ the collection of G such that $\omega_{\chi_{n,G}}(v)$ is a non-zero multiple of $\omega_{\mathcal{T}_n}(v)$.

DEFINITION 5.9. Suppose that we are given a rational $G \in \mathcal{G}_v$, and let v' be a vertex succeeding v in $\mathcal{T}_{n-1}$. Then, as before, one may write $G = G_{v'} + G_{v',\infty}$. We set $G|_{v'} := G_{v'}$.

PROPOSITION 5.10. *Suppose that $\mathcal{T}_n$ is a good $\mathbb{Z}/p^n$-Hurwitz tree that extends $\mathcal{T}_{n-1}$, and let $G \in \mathcal{G}_v$ for some vertex v of $\mathcal{T}_n$. Let χ be the extension of χ_{n-1} by G. Then*

$$\delta_\chi(r) = \delta_{\mathcal{T}_n}(r)$$

for every rational place $r > v$. In particular, at every vertex v' succeeding v, we have $G|_{v'} \in \mathcal{G}_{v'}$.

Proof. Suppose $\delta_\chi(r) \neq \delta_{\mathcal{T}_n}(r)$ for some $r > v$. Then, it follows from Corollary 4.8 that $\delta_\chi(r) > \delta_{\mathcal{T}_n}(r)$. On the other hand, the right derivative of δ_χ at any rational place r is $-\operatorname{ord}_0(\omega_\chi(r)) - 1$ by Corollary 3.16, which is at most $\mathfrak{C}(\chi, r, \overline{0}) - 1$ by Proposition 3.20. Hence, as $\delta_\chi(v) = \delta_{\mathcal{T}_n}(v)$, it must be true that $\delta_\chi(r) \leqslant \delta_{\mathcal{T}_n}(r)$, which negates the assumption. The last assertion follows easily from Proposition 5.4(ii). □

DEFINITION 5.11. At a vertex $v = t(e)$ with r direct successor edges $e_1, e_2, \ldots, e_l$, we define

$$\sum_{i=1}^{r} \mathcal{G}_{s(e_i)} := \left\{ \sum_{i=1}^{l} G_{s(e_i)} \mid G_{s(e_i)} \in \mathcal{G}_{s(e_i)} \right\}.$$

COROLLARY 5.12. *With the notation as in Definition 5.11, we have $\mathcal{G}_v = \mathcal{G}_{t(e)} = \sum_{i=1}^{l} \mathcal{G}_{s(e_i)}$.*

Proof. The inclusion $\mathcal{G}_{t(e)} \supseteq \sum_{i=1}^{l} \mathcal{G}_{s(e_i)}$ is immediate from the definition and Lemma 3.12(ii). Suppose $G \in \mathcal{G}_{t(e)}$. Then, as discussed in Remark 5.5, we may write $G = \sum_{i=1}^{r} G_{s(e_i)}$, where $G_{s(e_i)} \in \mathcal{H}_{s(e_i)}$. Finally, as $\delta_{\chi_{s(e_i)}}(r) = \delta_\chi(r) = \delta_{\mathcal{T}_n}(r)$ for all $s(e) < r$ by Proposition 5.10, Proposition 3.15 asserts that $\delta_{\chi_{s(e_i)}}(s(e_i)) = \delta_\chi(s(e_i)) = \delta_{\mathcal{T}_n}(s(e_i))$. Therefore, $G_{s(e_i)} \in \mathcal{G}_{s(e_i)}$. □

Note that we also have $\mathcal{H}_{t(e)} = \sum_{i=1}^{l} \mathcal{H}_{s(e_i)}$.

Proof of Theorem 4.19. We now have the necessary tools to complete the proof of the compatibility theorem. Recall that the situation of Section 3.8 has been solved by Proposition 4.20. Let us now consider an arbitrary rational place $r \in (s(e), t(e)]$. We may again write $\underline{G} = \underline{G}_r + \underline{G}_{r,\infty}$. Applying Proposition 5.4 with $\chi_r := \mathfrak{K}(\underline{G}_r)$, one obtains

$$\delta_\chi(r) = \delta_{\chi_r}(r) \quad \text{and} \quad \omega_\chi(r) = \omega_{\chi_r}(r)\,.$$

Hence, as the differential conductor of χ_r follows the compatibility rules, so does that of χ. ☐

Remark 5.13. Most of the results in Sections 4 and 5 until this point easily translate to the case where R is of mixed characteristic. The most important among them is Proposition 5.4(i). The only problematic part is Proposition 5.4(ii), hence Theorem 4.19 is also problematic since we do not yet have an exact analog of Theorem 3.39 in mixed characteristic.

5.4 The induction process

The following is the outline of the induction process to construct a character χ_n that deforms $\overline{\chi}_n$, extends χ_{n-1}, and gives rise to $\mathcal{T}_n$. One may assume that the length $n-1$ Witt vector that defines χ_{n-1} is best. We would like to find a rational function $G^n \in \mathbb{K}$ that extends χ_{n-1} to a deformation χ_n of $\overline{\chi}_n$. Below is the induction (Ind) process.

Ind 1. We start at a vertex $v = t(e)$ of $\mathcal{T}_n$ that only precedes leaves. We call v a *final vertex* of $\mathcal{T}_n$. In Section 5.6.1, we consider the affinoid $\mathcal{G}_{t(e)}$. It is a collection of $G^n_{t(e)} \in \mathbb{K}$ that gives rise to a cover $\chi_{n,t(e)}$ with desired depth and (generic) branching datum at $t(e)$. We then show that the subset $\mathcal{G}^*_{t(e)} \subsetneq \mathcal{G}_{t(e)}$, whose elements give rise to the characters with the right differential conductor at $t(e)$, is non-empty. Recall that either $\omega_{\mathcal{T}_n}(t(e))$ is exact, or $\mathcal{C}(\omega_{\mathcal{T}_n}(t(e))) = \omega_{\chi_{n-1}}(t(e))$ (see Theorem 3.42). The case where $\omega_{\mathcal{T}_n}(v)$ is exact is considered in Section 5.6.2; the non-exact case is examined in Section 5.6.3.

Ind 2. This step is carried out in Sections 5.7 and 5.8. Let e be an edge of $\mathcal{T}_n$ that is the direct predecessor of a final vertex. Then $\underline{G}_{n-1,t(e)} = \underline{G}_{n-1,s(e)}$, and there are no leaves in $\mathcal{T}_n$ with valuation (with respect to z_e) greater than $s(e)$ besides one at $t(e)$. In order for the depth of $\chi_{n,s(e)}$ at $s(e)$ to be equal to $\delta_{\mathcal{T}_n}(s(e))$, we continuously control its differential conductor along e by modifying $G^n_{t(e)}$. The key ingredient is Proposition 5.18, which says that one can always replace $G^n_{t(e)}$ with another element of $\mathcal{G}_{t(e)}$ that "fits" better into $\mathcal{T}_n$. Furthermore, we prove that the collection of $G^n_{t(e)}$ whose depth at $s(e)$ is $\delta_{\mathcal{T}_n}(s(e))$, that is, the quasi-compact quasi-separated (qcqs) set $\mathcal{G}_{s(e)}$, is a non-empty subset of $\mathcal{G}^*_{t(e)}$ (see Proposition 5.20). This shows that one can secure the wanted ramification datum over the edge e, completing the base case.

Ind 3. Suppose that e is an edge of $\mathcal{T}_n$ where $t(e)$ is *not* final, and $e_1, \dots, e_l$ are the direct successors of $t(e)$. As in Remark 5.5, we may write $\underline{G}_{n-1}$ as

$$\underline{G}_{n-1} = \sum_{i=1}^{l} \underline{G}_{n-1,s(e_i)} + \underline{G}_{n-1,t(e),\infty}\,.$$

By induction, for each i, there exists a qcqs $\mathcal{G}_{s(e_i)}$ such that for each $G^n_{s(e_i)} \in \mathcal{G}_{s(e_i)}$ and for each $i = 1, \dots, l$, the following hold:

(a) The branching datum of $\chi_{n,s(e_i)} = \mathfrak{K}_n\big(\underline{G}_{n-1,s(e_i)}, G^n_{s(e_i)}\big)$ fits into $\mathcal{T}_n(e_i)$.

(b) The depth of $\chi_{n,s(e_i)}$ at $t(e)$ is equal to $\delta_{\mathcal{T}_n}\big(t(e)\big)$.

Moreover, it follows from Corollary 5.12 that $\mathcal{G}_{t(e)} = \sum_{i=1}^{l} \mathcal{G}_{s(e_i)}$. Hence, $\mathcal{G}_{t(e)}$ is also qcqs. We call $t(e)$ an *exact vertex* if $\delta_{\mathcal{T}_n}(t(e)) > p\delta_{\mathcal{T}_{n-1}}(t(e))$. Otherwise, $\delta_{\mathcal{T}_n}(t(e)) = p\delta_{\mathcal{T}_{n-1}}(t(e))$, and we call $t(e)$ a *Cartier vertex*. The controlling of exact vertices (respectively, Cartier vertices) to achieve $\omega_{\mathcal{T}_n}(t(e)) = \omega_{\chi_{n,t(e)}}(t(e))$ is similar to step *Ind* 2.. We again obtain a non-empty open subset $\mathcal{G}^*_{t(e)} \subset \mathcal{G}_{t(e)}$ whose elements gives the desired differential conductor at $t(e)$.

Ind 4. Let e be the edge from the previous step. We can simply repeat the process from step *Ind* 2. to achieve the right degeneration data over e. That completes the inductive step.

Ind 5. Iterating the previous two steps, we acquire a non-empty qcqs $\mathcal{G}_{s(e_0)}$ (recall that e_0 is the trunk of $\mathcal{T}_n$). Finally, Section 5.10 shows that $\mathcal{G}_{s(e_0)}$ contains a non-empty subset $\mathcal{G}^*_{s(e_0)}$ whose elements give rise to characters with desired reduction type. That result, together with Corollary 5.7, completes the proof of the main theorem.

5.5 Some natural degree p characters

DEFINITION 5.14. Suppose that $\mathcal{T}_n$ is a tree that extends $\mathcal{T}_{n-1}$ in Proposition 4.22, and r is a rational place on an edge e of $\mathcal{T}_n$. We may assume $z_e = 0$. Suppose, moreover, that $b_1, \ldots, b_m$ are the successor leaves of e on $\mathcal{T}_n$ and l_j is the conductor at b_j, and set $l := \sum_{j=1}^m l_j$. Define $\mathcal{W}_r$ to be the collection of exact differentials forms

$$\omega_r = \sum_{i=1}^{l} \frac{c_i dx}{x^i}\,, \tag{5.1}$$

where $c_i \in k$. Note that, as ω_r is exact, the coefficients c_{qp-1} are 0 for all $q \in \mathbb{Z}$.

LEMMA 5.15. *Suppose that $\mathcal{T}_n$ is a tree extending $\mathcal{T}_{n-1}$ in Proposition 4.22, and r is a rational place on an edge e of $\mathcal{T}_n$. Suppose, moreover, that we are given a $\delta \in \mathbb{Q}_{>0}$ and an exact differential form ω in $\mathcal{W}_r$. Then there exists an $H_r \in \mathcal{H}_r$ such that, for $\psi := \mathfrak{K}_1(H_r)$, we have $\delta_\psi(r) = \delta$ and $\omega_\psi(r) = \omega$.*

For the definition of $\mathcal{H}_r$, see Definition 5.6.

Proof. Suppose that ω has the form shown in (5.1). Suppose that, for each i, there exists a rational function $H_{r,i} \in \mathcal{H}_r$ such that, for $\psi_i := \mathfrak{K}_1(H_{r,i})$, we have $\delta_{\psi_i}(r) = \delta$ and $\omega_{\psi_i}(r) = c_i dx/x^i$. Then, by Lemma 3.12(ii), the character $\psi := \sum_{i=1}^r \psi_i$ works. Therefore, we may further assume

$$\omega = \frac{c_i dx}{x^i}\,,$$

where $c_i \in k$, $p \nmid (i-1)$, and $i \leqslant l$. By the construction of $\mathcal{T}_n$, all but at most two of the l_i are divisible by p (this is mentioned in Remark 4.26). Without loss of generality, we may assume that they are l_1 and l_2. We will treat each case separately.

First suppose $p \mid l_2$. This is the situation on most of the places of $\mathcal{T}_2^{\min}$ in Example 4.23. Consider the rational function

$$H := \frac{x^{l-i}}{t^{p(\delta - r(i-1))}(x-b_1)^{l_1 - 1}\prod_{j=2}^m (x-b_j)^{l_j}} \in \mathbb{K}\,.$$

Then it is straightforward to check that $H \in \mathcal{H}_r$, $[H]_r = 1/x_r^{i-1} \notin (k(x_r))^p$, and $\nu_r(H) = -p\delta$. Hence, by Proposition 3.26, it is exactly what we are seeking.

Suppose $p \nmid l_1$ and $p \nmid l_2$, for instance, where r is a place on the edge $e_{0,1}$ of $\mathcal{T}_2^{8\,\mathrm{ness}}$ in Example 4.25 and $\mathcal{T}_n = \mathcal{T}_2^{8\,\mathrm{ness}}(r)$. We first regard the case $l_1 = m_1 p + 1$ and $l_2 = m_2 p + 1$. Then, as $\omega \in \mathcal{W}_r$ and $l = pq + 2$, we may assume $i \leqslant l - 2$. One can show that

$$H := \frac{x^{l-1-i}}{t^{p(\delta - r(i-1))}(x-b_1)^{mp_1}(x-b_2)^{mp_2}\prod_{j=3}^m (x-b_j)^{l_j}}$$

works as before.

Finally, let us consider the case $l_2 = m_2p + 1$ and $1 < \bar{l}_1 < p$ as on the edge $e_{0,1}$ of $\mathcal{T}_2^{9\,\text{ness}}$ in Example 4.25. Then, similarly to the other cases, the rational function that does the job is

$$H := \frac{x^{l-1-i}}{t^{p(\delta - r(i-1))}(x-b_1)^{l_1-1}(x-b_2)^{m_2p}\prod_{j=3}^{m}(x-b_j)^{l_j}} \,.$$

We hence have solutions for all the cases. ☐

Corollary 5.16. *Suppose that ι_n is the conductor of $\mathcal{T}_n$ and $g = \sum_{j=1}^{m_n} c_j/x^j$ is a polynomial in $k[x^{-1}]$ that consists of only terms of prime-to-p degree. Then, after a possible finite extension of $\mathbb{K}$, there exists a $G'_{s(e_0)} \in \mathcal{H}_{s(e_0)}$ such that the character $\psi := \mathfrak{K}_1(G'_{s(e_0)})$ satisfies*

- *$\delta_\psi(0) = 0$, and*
- *its reduction $\overline{\psi}$, considered as a character in $H^1_p(\kappa)$, corresponds to the extension $y^p - y = g$.*

Proof. The proof is almost identical to that of Lemma 5.15. In particular, c_j/x^j in this case plays the role of $-jc_j dx/x^{j+1}$ in the proof of that lemma. ☐

5.6 Controlling a final vertex

Suppose that $v = t(e)$ is a final vertex of $\mathcal{T}_n$ with differential conductor

$$\omega_{\mathcal{T}_n}(v) = \frac{c\,dx}{\prod_{j=1}^{r}(x - [b_j]_v)^{l_j}} \,,$$

where l_j is the nth conductor of the branch point associated with the leaf b_j on $\mathcal{T}_n$. It follows from the rules asserted in Theorem 3.42 that either $\mathcal{C}(\omega_{\mathcal{T}_n}(v)) = \omega_{\mathcal{T}_{n-1}}(v) = \omega_{\chi_{n-1}}(v)$, or $\mathcal{C}(\omega_{\mathcal{T}_n}(v)) = 0$. We discuss the former case in Section 5.6.2 and the latter one in Section 5.6.3.

Recall from Section 5.3 that χ_{n-1} can be represented by a best length $n-1$ Witt vector $(G^1, \ldots, G^{n-1}) = \left(G^1_v, \ldots, G^{n-1}_v\right) + \left(G^1_\infty, \ldots, G^{n-1}_\infty\right)$. The discussion in Remark 5.5 allows us to reduce our study to the case where $\underline{G}_{n-1}$ is equal to $\left(G^1_v, \ldots, G^{n-1}_v\right)$ and is also best. We first set $G^n = 0$. Then, it follows from Theorems 3.39 and 3.42 that the character χ_n it gives rise to satisfies

$$\delta_{\chi_n}(v) = p\delta_{\mathcal{T}_{n-1}}(v) \quad \text{and} \quad \mathcal{C}(\omega_{\chi_n}(v)) = \omega_{\mathcal{T}_{n-1}}(v) \,.$$

Furthermore, Theorem 2.4 shows that the nth conductor of χ_n at b_j is $\iota_{j,n} = p\iota_{j,n-1}$ for all j. Therefore, $0 \in \mathcal{H}_v$ and $\delta_{\chi_n}(v) \leqslant \delta_{\mathcal{T}_n}(v)$. In the rest of this subsection, we show that one can replace 0 with some $G^n \in \mathcal{H}_v$ such that

$$\delta_{\chi_n}(v) = \delta_{\mathcal{T}_n}(v) \quad \text{and} \quad \omega_{\chi_n}(v) = \omega_{\mathcal{T}_n}(v) \,.$$

5.6.1 *The final qcqs.* Suppose that there are m leaves $b_1, \ldots, b_m$, with respective nth conductors $l_1, \ldots, l_m$, that are attached to v. As discussed in Remark 4.26, one may assume that p divides l_i for all i but maybe l_1 and l_2. Consider the subset $\mathcal{H}'_v$ of $\mathcal{H}_v$ that consists of the elements of the form

$$G = \frac{1}{t^{p\delta_{\mathcal{T}_n}(v)}}\left(\sum_{j=1}^{m}\sum_{i=1}^{l_j-1}\frac{a_{j,i}}{(x_v - (b_j)_v)^i}\right), \tag{5.2}$$

where $a_{j,i} \in k((t))$, $\min_{i \not\equiv 0 \pmod p}(\nu(a_{j,i})) = 0$, and $a_{j,i} = 0$ for $i \equiv 0 \pmod p$ for each j. Note that $\mathcal{H}'_v$ can be identified with an affinoid subset of $\mathbb{A}_{\mathbb{K}}^{\sum_{j=1}^{m}(l_j-1)}$ via the $a_{j,i}$. We first show that this collection is exactly the $\mathcal{G}_v$ defined in Definition 5.8.

Proposition 5.17. *The subset $\mathcal{H}'_v$ coincides with $\mathcal{G}_v$. In particular, $\mathcal{G}_v$ is an affinoid* (*hence qcqs*).

Proof. Suppose $G \in \mathcal{H}'_v$. It is immediate from (5.2) that $[G]_v \in \kappa_v \setminus \kappa_v^p$. Hence, we have $\delta_G(v) = \delta_{\mathcal{T}_n}(v)$ by Proposition 3.26. Therefore, $G \in \mathcal{G}_v$; hence $\mathcal{H}'_v \subseteq \mathcal{G}_v$. Conversely, each $H \in \mathcal{G}_v$ is in the same Artin–Schreier class as one of the form

$$\tilde{H} = \frac{1}{t^{p\delta_{\mathcal{T}_n(v)}}}(H')_v = \frac{1}{t^{p\delta_{\mathcal{T}_n(v)}}}\left(\sum_{j=1}^{m}\sum_{i=1}^{l_j-1}\left(\frac{b_{j,i}}{(x-b_j)^i}\right)\right)_v ,$$

where $[H']_v \notin \kappa^p$, again by Proposition 3.26. It is straightforward to realize that $\tilde{H}$ must be of the form (5.2). □

5.6.2 *The exact case.* First suppose that $\omega_{\mathcal{T}_n}(v)$ is exact. Then, by the conditions on an extending tree (Definition 4.6), we have

$$\delta_{\mathcal{T}_n}(v) > p\delta_{\mathcal{T}_{n-1}}(v) \quad \text{and} \quad \omega_{\mathcal{T}_n}(v) = \frac{cdx}{\prod_{j=1}^{m}(x-[b_j]_v)^{l_j}} =: d\overline{h}_v$$

for some $\overline{h}_v \in k(x) \setminus k(x)^p$. We may further assume

$$\overline{h}_v = \sum_{j=1}^{m}\sum_{i=1}^{l_j-1} \frac{b_{j,i}}{(x-[b_j]_v)^i} ,$$

where $b_{j,i} \in k$ and $b_{j,l_j-1} \neq 0$ for each j. One can consider $\overline{h}_v$, after replacing x with x_v, as an element of $k[[t]](x_v)$. Hence, the rational function

$$H_v := \frac{1}{t^{p\delta_{\mathcal{T}_n}(v)}}\left(\sum_{j=1}^{m}\sum_{i=1}^{l_j-1} \frac{b_{j,i}}{(x_v-(b_j)_v)^i}\right) \in \mathbb{K}$$

is an element of $\mathcal{H}_v$ such that, for $\psi := \mathfrak{K}_1(H_v)$, we have $\delta_\psi(v) = \delta_{\mathcal{T}_n}(v)$ and $\omega_\psi(v) = \omega_{\mathcal{T}_n}(v)$. It follows from Lemma 3.12i that adding H_v to $G^n = 0$ makes $\delta_{\chi_n} = \delta_{\mathcal{T}_n}(v)$ and $\omega_{\chi_n} = \omega_{\mathcal{T}_n}(v)$. We thus achieve the desired branching datum at v. Define

$$\mathcal{G}^*_{t(e)} := \left\{G \in \mathcal{G}_{t(e)} \mid \omega_{\chi_n}(v) = \omega_{\mathcal{T}_n}(v)\right\} ,$$

which is non-empty as the above construction suggests.

5.6.3 *The non-exact case.* Let us now consider the case $\mathcal{C}(\omega_{\mathcal{T}_n}(v)) = \omega_{\mathcal{T}_{n-1}}(v) = \omega_{\chi_{n-1}}(v)$. Recall that we start with $G^n = 0$. Then, as $\underline{G}_{n-1}$ is best, it follows from Theorem 3.39 that

$$\delta_{\chi_n}(v) = p\delta_{\chi_{n-1}}(v) = \delta_{\mathcal{T}_n}(v) .$$

Thus, $\mathcal{C}(\omega_{\mathcal{T}_n}(v) - \omega_{\chi_n}(v)) = \omega_{\mathcal{T}_{n-1}}(v) - \omega_{\mathcal{T}_{n-1}}(v) = 0$, that is, the difference between the two differential forms is exact. Therefore, we may assume

$$\omega_{\mathcal{T}_n}(v) - \omega_{\chi_n}(v) = d\overline{h}_v$$

for some $\overline{h}_v \in k(x) \setminus k(x)^p$. In addition, Theorem 2.4 again tells us that the nth conductor at the branch point b_j is $\iota_{j,n} = p\iota_{j,n-1} \leqslant l_j$ for $1 \leqslant j \leqslant m$. It then follows from Proposition 3.20 that

$\mathrm{ord}_{[b_j]_v}(d\overline{h}_v) \geqslant -l_j$ for all j. One may thus assume that

$$\overline{h}_v = \sum_{j=1}^{m} \sum_{i=1}^{l_j-1} \frac{b_{j,i}}{(x-[b_j]^i)} \,.$$

Hence, $d\overline{h}_v \in \mathcal{W}_v$. By the same argument as before, there exists an $H_v \in \mathcal{H}_v$ such that, for $\psi := \mathfrak{K}_1(H_v)$, we have $\delta_\psi(v) = \delta_{\mathcal{T}_n}(v)$ and $\omega_\psi(v) = d\overline{h}_v$. Again, replacing G^n by $G^n + H_v$ results in $\omega_{\chi_n}(v) = \omega_{\mathcal{T}_n}(v)$. Thus, the set $\mathcal{G}^*_{t(e)}$ defined in the previous subsection is not empty, completing step *Ind* 1..

5.7 Controlling an edge

Suppose that e is an edge where $t(e)$ is final. We again may assume $z_e = 0$. Recall that $\mathcal{G}^*_{t(e)}$, which is the collection of $G \in \mathbb{K}$ that give rise to a character χ_n satisfying

$$\delta_{\chi_n}(t(e)) = \delta_{\mathcal{T}_n}(t(e)) \quad \text{and} \quad \omega_{\chi_n}(t(e)) = \omega_{\mathcal{T}_n}(t(e))\,,$$

is non-empty by step *Ind* 1.. Suppose, moreover, that the order of ∞ of $\omega_{\mathcal{T}_n}(t(e))$ is $l-2$. Then, by Corollary 3.16, the left derivative of δ_{χ_n} at $t(e)$ is $l-1$. Therefore, as δ_{χ_n} is piecewise linear by Proposition 3.15, there exists a rational $s(e) \leqslant \lambda < t(e)$ such that

$$\delta_{\chi_n}(s) = \delta_{\mathcal{T}_n}(t(e)) - (l-1)(t(e)-s) = \delta_{\mathcal{T}_n}(s)$$

for all $s \in [\lambda, t(e)]$. Let $\lambda_e(G)$ be the minimal value of λ with this property. This means that $\lambda_e(G)$ is the largest "kink" of the function δ_{χ_n} on $(s(e), t(e))$ (or is $s(e)$ if δ_{χ_n} is linear on $[s(e), t(e))$). Our goal is proving that there exists a $G \in \mathcal{G}^*_{t(e)}$ satisfying $\lambda_e(G) = s(e)$. First, we will show that one can always reduce the kink on e, that is, make $\lambda_e(G)$ strictly smaller (if it is greater than $s(e)$) by replacing G with another element of $\mathcal{G}^*_{t(e)}$.

Proposition 5.18. *Suppose $G \in \mathcal{G}^*_{t(e)}$ with $\lambda_e(G) > s(e)$. Then there exists a $G' \in \mathcal{G}^*_{t(e)}$ with $\lambda_e(G') < \lambda_e(G)$.*

Proof. Suppose that $\lambda := \lambda_e(G)$ is the largest kink of χ_n, which is the extension of χ_{n-1} by G, on e. We know from Proposition 3.20 that $\mathrm{ord}_\infty(\omega_{\chi_n}(\lambda))$ is equal to the left derivative of δ_{χ_n} plus 1, which is at most $l-2$. Moreover, it follows from the same corollary that

$$\mathrm{ord}_0(\omega_{\chi_n}(\lambda)) = -l \quad \text{and} \quad \mathrm{ord}_{\overline{z}}(\omega_{\chi_n}(\lambda)) \geqslant 0\,, \quad \forall\, \overline{z} \neq 0, \infty\,.$$

Let us first consider the case $\delta_{\chi_n}(\lambda) > p\delta_{\chi_{n-1}}(\lambda)$. Then $\omega_{\chi_n}(s)$ is exact, and $l-1$ is prime to p. Therefore, we may assume that $\omega_{\chi_n}(\lambda)$ is of the form

$$\omega_{\chi_n}(\lambda) = \frac{cdx}{x^l} + d\overline{F}\,,$$

where $0 \neq d\overline{F} \in \mathcal{W}_\lambda$. Applying Lemma 5.15, we obtain a $F \in \mathcal{H}_\lambda$ such that, for $\psi := \mathfrak{K}_1(F)$,

$$\delta_\psi(\lambda) = \delta_{\chi_n}(\lambda) \quad \text{and} \quad \omega_\psi(\lambda) = -d\overline{F}\,.$$

One may check that $G' := G + F$ lies in $\mathcal{G}_{t(e)}$. Finally, it follows from Lemma 3.12(ii) that replacing G by G' changes χ_n as follows:

$$\delta_{\chi_n}(\lambda) = \delta_{\mathcal{T}_n}(\lambda) \quad \text{and} \quad \omega_{\chi_n}(\lambda) = \frac{cdx}{x^l}\,.$$

Thus, χ_n has no kink at λ, whence $\lambda_e(G') < \lambda_e(G)$.

Let us now consider the case $\delta_{\chi_n}(\lambda) = p\delta_{\chi_{n-1}}(\lambda)$. The same line of reasoning as in Section 5.6.3 gives $\omega_{\chi_n}(\lambda) - \omega_{\mathcal{T}_n(\lambda)} = d\overline{F}$, where $\overline{F}$ is as in the previous case. Repeating the above process, we obtain a $G' \in \mathcal{G}^*_{t(e)}$ that satisfies $\lambda_e(G') < \lambda$. □

Remark 5.19. If $\lambda_e(G')$ is once more strictly greater than $s(e)$, we can repeat the procedure from the above proof to obtain an element of $\mathcal{G}^*_{t(e)}$ with strictly smaller kink (on e). Thus, if we can show that λ_e achieves a minimum value, then that value must be equal to $s(e)$. That is the goal of the next section.

5.8 The minimal depth Swan conductor

This section adapts [OW14, § 6.4]. We would like to show that the function $\lambda_e \colon \mathcal{G}_{t(e)} \to \mathbb{Q}_{\geqslant 0}$ defined in Section 5.7 takes the value $s(e)$ for some $G_{t(e)} \in \mathcal{G}^*_{t(e)}$. As discussed in Remark 5.19, it suffices to prove the following.

Proposition 5.20. *The function $G_{t(e)} \mapsto \lambda_e(G_{t(e)})$ attains a minimal value on $\mathcal{G}_{t(e)}$ (that is, $\mathcal{G}_{s(e)}$ is non-empty). Moreover, $\mathcal{G}_{s(e)}$ can be identified with a qcqs set.*

5.8.1 *A lemma from rigid analysis.* The following lemma will be a crucial ingredient in the proof of Proposition 5.20.

Lemma 5.21 (cf. [OW14, Lemma 6.16]). *Let X be qcqs over K and $f^1, \dots, f^n \in A$ be analytic functions on X. Then the function*

$$\phi \colon X \longrightarrow \mathbb{R}\,, \quad x \longmapsto \max_{1 \leqslant i \leqslant n} \sqrt[i]{|f^i(x)|}$$

takes a minimal value on X. Equivalently, the function

$$x \longmapsto \tau(x) := \min_{1 \leqslant i \leqslant n} \frac{\nu(f^i(x))}{i}$$

takes a maximal value on X. Furthermore, the subset of X on which the minimum (maximum) is attained is qcqs.

Proof. The proof adapts [OW14, Lemma 6.16] and [OW16, Lemma 4.19]. It suffices to show that, if the inequality

$$\gamma := \sup_{a \in X} \left(\min_{1 \leqslant i \leqslant n} \frac{\nu(f^i(a))}{i} \right) \geqslant 0$$

holds, then γ is achieved on a qcqs subset of X. We may assume that X is an affinoid. We set $g_i(x) := \nu(f^i(x))/i$ for $1 \leqslant i \leqslant n$. For each i, let $X_i \subseteq X$ be the rational subdomain where g_i is minimal among all the g_j. Then the restriction of τ to X_i is simply equal to g_i, which attains its maximum on X_i by the maximum modulus principle. Furthermore, the subspace of X_i where this maximum is attained is a Weierstrass domain in X_i, hence qcqs. Thus, the subspace of X where γ is attained is a union of finitely many qcqs spaces, whence also qcqs, completing the proof. □

5.8.2 *A quasi-compact quasi-separated set contained in $\mathcal{G}_{t(e)}$.* Consider an edge e of the tree $\mathcal{T}_n$. Assume for a moment that we have the desired refined Swan conductor at every $r \geqslant t(e)$. Suppose, moreover, that there are m edges $e_1, e_2, \dots, e_m$ where $s(e_i) = t(e)$ for each i. Set I_{e_j} to be an indexing set of leaves succeeding e_j. It follows from the previous constructions that an element $G_{t(e)} \in \mathcal{G}_{t(e)}$ can be represented by

$$G_{t(e)} = \sum_{j=1}^{m} G_{s(e_j)} = \sum_{j=1}^{m}\left(\sum_{h\in\bigcup I_{e_j}}\left(\sum_{i=1}^{l_h-1}\frac{a_{h,i}}{(x-b_h)^i}\right)\right), \tag{5.3}$$

where $G_{s(e_j)} \in \mathcal{G}_{s(e_j)}$ and $a_{h,i} = 0$ for $i \equiv 0 \pmod p$. Set $m_{t(e)} := \sum_{j=1}^{m}\sum_{h\in\bigcup I_{e_j}}(l_h - 1 - \lfloor l_h/p \rfloor)$. As before, we can think of $\mathcal{G}_{t(e)}$ as a subset of an affine $m_{t(e)}$-space over K via the coordinates $a_{h,i}$ and of $G_{t(e)}$ as one of its K-rational point. Moreover, by the induction hypothesis in Section 5.4, the subset $\mathcal{G}_{s(e_j)}$ is qcqs for $j = 1, \dots, l$. Hence, $\mathcal{G}_{t(e)} = \sum_{j=1}^{m}\mathcal{G}_{s(e_j)}$ is also a qcqs subset of $(\mathbb{A}_K^{m_{t(e)}})^{\mathrm{an}}$.

It is a consequence of Section 5.9 that $\varnothing \neq \mathcal{G}^*_{t(e)} \subsetneq \mathcal{G}_{t(e)}$, where

$$\mathcal{G}^*_{t(e)} = \{G \in \mathcal{G}_{t(e)} \mid \omega_{\chi_n(G)}(t(e)) = \omega_{\mathcal{T}_n}(t(e))\}\,.$$

As a rigid analytic space, $\mathcal{G}^*_{t(e)}$ is an open subset of $\mathcal{G}_{t(e)}$. The goal of this section is to show that $\lambda_e(\mathcal{G}_{t(e)})$ takes the minimal value on $\mathcal{G}_{t(e)}$ and that the points where that minimum is achieved form a qcqs subset of $\mathcal{G}^*_{t(e)}$.

5.8.3 *Analysis.* Let $\phi_{n-1}\colon Y_{n-1} \to X$ be the Galois cover corresponding to the character χ_{n-1}. By our induction hypothesis, it has good reduction and is totally ramified above $x = z_e$, which we may assume to be 0. As usual, $D \subset X^{\mathrm{an}}$ is an open unit disc centered at 0. It follows that the rigid analytic subspace $C := \phi_{n-1}^{-1}(D) \subseteq Y_{n-1}$ is another unit disc and contains the unique point $y_{n-1} \in Y_{n-1}$ above $x = 0$. We choose a parameter $\widetilde{x}$ for the disc C such that $\widetilde{x}(y_{n-1}) = 0$. Then $x = \widetilde{x}^{p^{n-1}}u(\widetilde{x})$, with $u(\widetilde{x}) \in R[[\widetilde{x}]]^\times$. One sees that for $r > 0$, the inverse image of the closed disc $D[pr] \subset D$ defined by the condition $\nu(x) \geqslant r$ is the closed disc $C[\widetilde{r}]$ defined by $\nu(\widetilde{x}) \geqslant \widetilde{r} := p^{-n+1}r$. Set $\widetilde{r}_{n-1} := p^{-n+1}r_{n-1}$. Let $\mathbb{K}_{n-1}$ denote the function field of Y_{n-1}.

Let us fix a character $\chi \in \mathrm{H}^1_{p^n}(\mathbb{K})$ that arises from χ_{n-1} by a rational $G \in \mathcal{G}_{t(e)}$. Let $\widetilde{\chi} := \chi|_{\mathbb{K}_{n-1}} \in \mathrm{H}^1_p(\mathbb{K}_{n-1})$ denote the restriction of χ to $\mathbb{K}_{n-1}$. If χ corresponds to a cover $Y_n \to X$, then $\widetilde{\chi}$ corresponds to the cover $Y_n \to Y_{n-1}$. In analogy to $\lambda_e(\chi)$, we define the function $\lambda_e(\widetilde{\chi})\colon \mathrm{H}^1_p(\mathbb{K}_{n-1}) \to \mathbb{Q}$ as follows:

$$\lambda_e(\widetilde{\chi}) = \begin{cases} \min\{\widetilde{s(e)} \leqslant \widetilde{r} < \widetilde{t(e)} \mid \delta_{\widetilde{\chi}} \text{ linear on } [\widetilde{s(e)}, \widetilde{t(e)}]\}\,, & \chi \in \mathcal{G}^*_{t(e)}\,, \\ \widetilde{t(e)}\,, & \chi \in \mathcal{G}_{t(e)} \setminus \mathcal{G}^*_{t(e)}\,. \end{cases}$$

Let m_i (respectively, m) be the left derivative (on e) of δ_{χ_i} (respectively, δ_χ) at $t(e)$. The following result suggests that one may apply the tools from Section 3.8 to determine the kinks of χ on e.

Lemma 5.22. (i) *For $\chi \in \mathcal{G}^*_{t(e)}$, we have $\lambda_e(\widetilde{\chi}) = p^{-n+1}\lambda_e(\chi)$.*

(ii) *Let $\widetilde{m} = p^{n-1}m - \sum_{i=1}^{n-1} m_i(p-1)p^{i-1}$. Then $p \nmid \widetilde{m}$ and the character $\widetilde{\chi} \in H^1_p(\mathbb{K}_{n-1})$ satisfies conditions* (D1), (D2), *and* (D3) *of Section* 3.8 *with respect to $\widetilde{m}$, the open disc $C \in Y^{\mathrm{an}}_{n-1}$, and the family of subdisc $C[\widetilde{r}]$ for $\widetilde{r} \in [\widetilde{s(e)}, \widetilde{t(e)}]$.*

(iii) *If $\lambda_{\widetilde{m},\widetilde{s(e)}}(\widetilde{\chi})$ is defined as in Corollary* 3.33 *and we set r_0 in Proposition* 3.31 *to be equal to $\widetilde{t(e)}$, then $\lambda_e(\widetilde{\chi}) = \lambda_{\widetilde{m},\widetilde{s(e)}}(\widetilde{\chi})$ for all $\chi \in \mathcal{G}_{t(e)}$.*

Proof. For $r > s(e)$, we systematically use the notation $\widetilde{r} := p^{-n+1}r$. Let the valuation $\nu_{\widetilde{r}}$ of $\mathbb{K}_{n-1}$, which corresponds to the Gauss valuation on $C[\widetilde{r}]$ (see Section 3.2), be the unique extension of ν_r. By [Wew14, § 7.1], we have the equation

$$\delta_{\widetilde{\chi}}(\widetilde{r}) = \psi_{\mathbb{K}_{n-1}/\mathbb{K}}(\delta_\chi(r)) = \delta_\chi(r) - \left(\delta_1(r)\frac{p-1}{p^{n-1}} + \cdots + \delta_{n-1}(r)\frac{p-1}{p}\right), \tag{5.4}$$

where ψ is the inverse Herbrand function [Ser79, § IV.3]. Since all the characters $\chi_i := \chi^{p^{n-i}}$ $(1 \leqslant i < n)$ have good reduction by the hypothesis and their branch points are all contained in $\mathcal{D}[t(e)]$, each $\delta_i := \delta_{\chi_i}(1 \leqslant i < n)$ is linear of slope m_i on the interval $[s(e), t(e)]$. Therefore, the left and the right derivatives of $\delta_i(r)$ are equal to m_i for all $r \in (s(e), t(e))$. Let c be the left slope of δ_χ at r. Then the left slope of $\delta_{\widetilde{\chi}}$ at $\widetilde{r}$ is equal to

$$p^{n-1}c - \sum_{i=1}^{n-1} m_i(p-1)p^{i-1} = p^{n-1}c + \widetilde{m} - p^{n-1}m\,.$$

Part (i) then follows immediately. Part (ii) follows from the fact that $c \leqslant m$. Part (iii) also follows from this property, along with Proposition 3.15 and the fact that $\mathrm{sw}_\chi(t(e), \infty) = m$ if and only if $\chi \in \mathcal{G}^*_{t(e)}$. ☐

One may assume that the character $\widetilde{\chi}$ is the Artin–Schreier class of the rational function

$$\widetilde{G} := F^n(y_1, \ldots, y_{n-1}) + G^n \in \mathbb{K}^\times_{n-1}\,,$$

where F^n is a polynomial over $\mathbb{F}_p$ in $n-1$ variables as in [Obu12, Proposition 6.5]. We thus may write $\widetilde{G}$ as a power series in the parameters $\widetilde{x}$ as follows:

$$\widetilde{G} = \sum_{l=1}^{\infty} a_l \widetilde{x}^{-l}\,.$$

Since $G^1, \ldots, G^{n-1}$ are fixed, $\widetilde{G}$ is determined by the choice of G^n. So, we may think of the a_l as analytic functions on the space $\mathcal{G}_{t(e)}$. In fact, a_l is a polynomial in the coordinate $a_{k,i}$ (as in (5.3)) with coefficients in R.

5.8.4 We continue with the process of proving Proposition 5.20. Suppose that G_0 is an arbitrary element of $\mathcal{G}^*_{t(e)}$ and χ_0 is the character it gives rise to. Then it is immediate that $\lambda_e(G_0) < t(e)$. We therefore may choose a rational number $s \in (\lambda(G_0), t(e))$. With the notation from Section 5.8.3, recall that $\widetilde{\chi}$ is the restriction of χ to the function field $\mathbb{K}_{n-1}$ of Y_{n-1}. It is an Artin–Schreier cover defined by a rational function $\widetilde{G} \in \mathbb{K}^\times_{n-1}$. By Lemma 5.22, we have

$$\lambda(\widetilde{G}) < \widetilde{s} := p^{1-n}s < \widetilde{t(e)}\,.$$

Set $\widetilde{\delta} := \delta_{\widetilde{\chi}}(\widetilde{t(e)})$. Let N be an integer such that

$$Np \geqslant \frac{\widetilde{\delta}}{\widetilde{t(e)} - \widetilde{s}}\,.$$

We hence arrive at the situation of (3.12). The following is parallel to [OW14, Lemma 6.8].

Lemma 5.23. *There exist a finite cover $\mathcal{G}'_{t(e)} \to \mathcal{G}_{t(e)}$ and analytic functions $b_1, \ldots, b_N$ on $\mathcal{G}'_{t(e)}$ with the following property: Set*

$$d := \sum_{j=0}^{N} b_j \widetilde{x}_1^{-j}$$

and write

$$F + d^p - d = \sum_{l=1}^{\infty} c_l \widetilde{x}_1^{-l} \, ,$$

where the c_l are now analytic functions on $\mathcal{G}'_{t(e)}$. Then

(i) *for all $l \geqslant 1$ and all points $y \in \mathcal{G}'_{t(e)}$, we have $\nu(c_l(y)) \geqslant p\big(\widetilde{t(e)}l - \widetilde{\delta}\big)$;*

(ii) *we have $c_{pl} = 0$ for $l \leqslant N$.*

Proof. The lemma follows immediately from Proposition 3.31 and Remark 3.32. □

Proof of Proposition 5.20. We can now complete the proof of Proposition 5.20. Let $\widetilde{m}$ be as in part (ii) of Lemma 5.22. Define the function $\mu_{\widetilde{m},\widetilde{s(e)}} \colon \mathcal{G}'_{t(e)} \to \mathbb{R}$ as follows:

$$\mu_{\widetilde{m},\widetilde{s(e)}}(x) := \max\left(\left\{ \frac{\nu(c_{\widetilde{m}}(x)) - \nu(c_l(x))}{\widetilde{m} - l} \mid 1 \leqslant l < \widetilde{m} \right\} \cup \left\{ \widetilde{s(e)} \right\} \right).$$

Let $\chi \in \mathcal{G}_{t(e)}$, write $\widetilde{\chi} := \chi|_{\mathbb{K}_{n-1}}$ for its restriction to the function field of Y_{n-1}, and let x be an arbitrary point of $\mathcal{G}'_{t(e)}$ over χ. Thanks to Lemma 5.23, we can apply Proposition 3.31 to compare $\mu_{\widetilde{m},\widetilde{s(e)}}(x)$ to $\lambda_{\widetilde{m},\widetilde{s(e)}}(\widetilde{\chi})$, which, in turn, is equal to $\lambda_e(\widetilde{\chi})$ by Lemma 5.22(iii). We thus conclude that $\mu_{\widetilde{m},\widetilde{s(e)}}(x) < \widetilde{s}$ if and only if $\lambda_e(\widetilde{\chi}) < \widetilde{s}$. Moreover, if this is the case, then we have $\mu_{\widetilde{m},\widetilde{s(e)}}(x) = \lambda_e(\widetilde{\chi})$. In any case, by Lemma 5.22(i), we have $\lambda_e(\chi) = p^{n-1}\lambda_e(\widetilde{\chi})$ when $\chi \in \mathcal{G}^*_{t(e)}$.

The rest of the proof is identical to one in [OW14, § 6.4.5]. Let $G_0 \in \mathcal{G}^*_{t(e)}$ be the rational function at the beginning of Section 5.8.4. Let $G'_0 \in \mathcal{G}'_{t(e)}$ be a point above G_0. Because $\lambda_e(G_0) < s$, we have $\mu_{\widetilde{m},\widetilde{s(e)}}(G'_0) < \widetilde{s}$, as discussed above.

It follows from Lemma 5.21 that the function $\mu_{\widetilde{m},\widetilde{s(e)}}$ takes a minimum on $\mathcal{G}'_{t(e)}$. Let $w \in \mathcal{G}'_{t(e)}$ be a point where this minimum is achieved, and let $W \in \mathcal{G}_{t(e)}$ be its image under $\mathcal{G}'_{t(e)} \to \mathcal{G}_{t(e)}$. We have $\mu_{\widetilde{m},\widetilde{s(e)}}(w) \leqslant \mu_{\widetilde{m},\widetilde{s(e)}}(G'_0) < \widetilde{s}$. Since

$$\lambda_{\widetilde{m},\widetilde{s(e)}}(\widetilde{W}) = \mu_{\widetilde{m},\widetilde{s(e)}}(w) < \widetilde{s} < \widetilde{t(e)} \, ,$$

we see that $W \in \mathcal{G}^*_{t(e)}$. Applying the above arguments a second time, we conclude that $\lambda_e(W) = p^{n-1}\mu_{\widetilde{m},\widetilde{s(e)}}(w)$, and this is actually the minimal value of the function $\lambda_e \colon \mathcal{G}^*_{t(e)} \to \mathbb{R}$. Moreover, the subset on which the minimum is attained is qcqs by the Lemma 5.21. This completes the proof of Proposition 5.20. □

5.9 Controlling a non-final vertex

Suppose that $t(e) = s(e_1) = \cdots = s(e_m)$ is a non-final vertex of $\mathcal{T}_n$. By the induction process, we obtain a collection of qcqs sets $\mathcal{G}_{s(e_1)}, \ldots, \mathcal{G}_{s(e_m)}$. Recall that each $G_i \in \mathcal{G}_{s(e_1)}$ has (generic) branching datum that fits into $\mathcal{T}_n(e_i)$ and satisfies, for χ_{G_i} the extension of χ_{n-1,e_i} by G_i, the equation $\delta_{\chi_{G_i}}(s(e_i)) = \delta_{\mathcal{T}_n}(s(e_i))$. Suppose, moreover, that all the branch points of χ_{G_i} specialize to $\overline{a}_i \in k$ at $t(e)$. Then the $\overline{a}_i$ are distinct, and the differential conductor $\omega_{\chi_{G_i}}(t(e))$ is of the form

$$\omega_{\chi_{G_i}}(t(e)) = \frac{f^i(x)dx}{(x - \overline{a}_i)^{l_i}} \, ,$$

where $f^i(x) \in k[x]$ and $l_i := \sum_{b \in \mathbb{B}(\mathcal{T}_n(e_i))} h_b$. Set $G := \sum_{i=1}^m G_i$, and let χ_G be the character that arises from $\chi_{n-1,e}$ by G. By Lemma 3.12(ii), we have

$$\omega_{\chi_G}(t(e)) = \sum_{i=1}^{m} \omega_{\chi_{G_i}}(t(e)) \quad \text{and} \quad \delta_{\chi_G}(t(e)) = \delta_{\chi_{G_i}}(t(e)) , \quad \forall i = 1, \ldots, m .$$

Hence, the collection of such G, which we call $\mathcal{G}_{t(e)}$, is equal to $\sum_{i=1}^m \mathcal{G}_{s(e_i)}$ and thus is qcqs. Its elements give rise to characters with the desired depth and the geometry of the branch locus at $t(e)$. Suppose that the wanted differential form at $t(e)$ is

$$\omega_{\mathcal{T}_n}(t(e)) = \frac{c dx}{\prod_{i=1}^m (x - \overline{a}_i)^{l_i}} \quad (c \neq 0) .$$

Recall that $\mathcal{C}(\omega_{\mathcal{T}_n}(t(e))) = \omega_{\mathcal{T}_{n-1}}(t(e))$ when $\delta_{\mathcal{T}_n}(t(e)) = p\delta_{\mathcal{T}_{n-1}}(t(e))$, and $\mathcal{C}(\omega_{\mathcal{T}_n}(t(e))) = 0$ otherwise. In either case, as we achieve the right depth Swan conductor at $t(e)$, we may assume that ω_{χ_G} has the form

$$\omega_{\chi_G}(t(e)) = \frac{c dx}{\prod_{i=1}^m (x - \overline{a}_i)^{l_i}} + \sum_{i=1}^{m} \omega'_{\chi_G}(s(e_i)) ,$$

where $\omega'_{\chi_G}(s(e_i)) \in \mathcal{W}_{s(e_i)}$ is exact. Now, by repeating the process in Section 5.6 for each i, we replace G with another element of $\mathcal{G}_{t(e)}$ where $\omega'_{\chi_G}(s(e_i)) = 0$ for all i. This shows that $\mathcal{G}^*_{t(e)}$ is non-empty.

5.10 Controlling the root

By the previous steps of the induction process, we get to the point where $\mathcal{G}_{s(e_0)} \neq \varnothing$. This means that there exists a $G_{\min} \in \mathcal{G}_{s(e_0)}$ that gives rise to a character $\chi_{\min}$ whose branching datum fits into $\mathcal{T}_n$ and whose depth is zero at $s(e_0)$; that is, $\chi_{\min}$ has good reduction.

Recall that $\overline{\chi}_n$ is our original character. Assume that the ramification breaks of the one-point cover corresponding to $\overline{\chi}_n$ are $(m_1, \ldots, m_n)$, and that it is represented (upon completion at $x = 0$) by a Witt vector $\underline{g}_n := (g^1, \ldots, g^n) \in W_n(k(x))$. As discussed in Section 2.4, one may regard $\overline{\chi}_n$ as a one-point cover of $\mathbb{P}^1_k$. Therefore, we further assume that each g^i is a polynomial in $k[x^{-1}]$, all of whose terms have prime-to-p degree. On the other hand, it follows from the previous constructions that $\overline{\chi}_{\min}$ corresponds to a Witt vector $\underline{g}_{\min} = (g^1, \ldots, g^{n-1}, g_{\min})$, where $g_{\min} \in k[x^{-1}]$ has degree less than (only when $m_n = pm_{n-1}$) or equal to m_n and consists of only terms of prime-to-p degree. Subtracting Witt vectors yields

$$\underline{g}_n - \underline{g}_{\min} = \big(0, \ldots, 0, g^n - g_{\min}\big) .$$

We define $g := g^n - g_{\min}$. Recall that Corollary 5.7 asserts that $\chi_{\min}$ deforms $\overline{\chi}_n$ if and only if $g = 0$.

PROPOSITION 5.24. *There is a character χ_n that is a deformation of $\overline{\chi}_n$.*

Proof. Applying Corollary 5.16 for g, we obtain an $H \in \mathcal{H}_{s(e_0)}$ such that $\psi := \mathfrak{K}_1(H) \in \mathrm{H}^1_p(\mathbb{K})$ satisfies $\delta_\psi(0) = 0$ and $\overline{\psi}$ is given by $y^p - y = g$. Hence, if $\psi' := \mathfrak{K}_n((0, \ldots, 0, H)) \in \mathrm{H}^1_{p^n}(\mathbb{K})$, then we also have $\delta_{\psi'}(0) = 0$ and the reduction $\overline{\psi}'$ corresponds to the same extension, which is represented by the Witt vector $(0, \ldots, g) \in W_n(\kappa)$.

On the other hand, recall that $\chi_{\min}$ corresponds to the length n Witt vector

$$\big(G^1, G^2, \ldots, G^{n-1}, G_{\min}\big) \in W_n(\mathbb{K})/\wp(W_n(\mathbb{K})) .$$

Therefore, replacing $G_{\min}$ by $G^n := G_{\min} + H$ equates to multiplying ψ' to $\chi_{\min}$. By Lemma 3.12, the result is an étale character $\chi_n \in \mathrm{H}^1_{p^n}(\mathbb{K})$ with reduction $\overline{\chi}_{\min} \cdot \overline{\psi}'$, which, in turn, is defined by the Witt vector $(g^1, \ldots, g^{n-1}, g_{\min} + g = g^n)$. Finally, since $G^n \in \mathcal{G}_{s(e_0)}$ as $H \in \mathcal{H}_{s(e_0)}$, it follows from Corollary 5.7 that χ_n is a lift of $\overline{\chi}_n$. $\square$

This completes the proof of Theorem 1.2.

Remark 5.25. It follows from the construction and the compatibility of the differential conductors that the $\mathbb{Z}/p^n$-tree arising from χ_n coincides with the tree $\mathcal{T}_n$.

6. Proofs of technical results

6.1 A solution to the Cartier operator equation

Proposition 6.1. *Suppose that we are given a differential form*

$$\omega = \frac{c\,dx}{\prod_{i=1}^{r}(x-d_i)^{\iota_i}},$$

where c and the d_i are in k, $d_i \neq d_j$ for $i \neq j$, and $r \geqslant 2$, as well as a fixed integer $1 \leqslant a \leqslant p-1$. Set $\iota_1' := p\iota_1 - p + a + 1$, $\iota_2' := p\iota_2 - p + 1$, and $\iota_i' := p\iota_i$ for each $i \neq 1, 2$. Then the differential form

$$\omega' = \frac{c^p\,dx}{(-d_1-d_2)^{p-a-1}\prod_{i=1}^{r}(x-d_i)^{\iota_i'}}$$

satisfies $\mathcal{C}(\omega') = \omega$.

Proof. The differential form ω' above can be rewritten as

$$\omega' = \frac{c^p(x-d_1)^{p-a-1}dx}{(x-d_2)(-d_1-d_2)^{p-a-1}(x-d_2)^{p(\iota_2-1)}(x-d_1)^{p\iota_1}\prod_{i=3}^{r}(x-d_i)^{p\iota_i}}.$$

The proposition then follows from the fact that $\mathcal{C}(u^p\omega) = u\mathcal{C}(\omega)$ for all $u \in K$. $\square$

6.2 Construction of the extending Hurwitz tree

We dedicate this section to the proof of Proposition 4.22. In each proof, we will only demonstrate the construction of the extending Hurwitz trees. The readers can easily check that these trees satisfy the axioms in Definition 4.2 together with the conditions for an extension asserted by Theorems 3.42 and 4.19.

Recall the setup of Proposition 4.22. We are given a $\mathbb{Z}/p^{n-1}$-tree $\mathcal{T}_{n-1}$ and an nth degeneration datum $(\delta_n, \omega_n) \in \mathbb{Q}_{>0} \times \Omega^1_\kappa$ (or $(0, f_n)$, where $f_n \in \kappa$) at its root. The goal is to construct a $\mathbb{Z}/p^n$-tree $\mathcal{T}_n$ that extends $\mathcal{T}_{n-1}$ and whose degeneration datum at its root is the same as the given one. Note that the original result only requires $\delta_n = 0$, but we will prove the general case in order to utilize an induction technique that we will soon see.

Suppose that $\mathcal{T}_{n-1}$ is a $\mathbb{Z}/p^{n-1}$-tree that has m leaves $b_1, \ldots, b_m$ ($m \geqslant 2$) and that the conductor at b_i is $\iota_{i,n-1}$. As usual, v_0 and e_0 denote the root and the trunk of $\mathcal{T}_{n-1}$. Set $v_1 := t(e_0)$. Suppose, moreover, that the differential conductors (the $(n-1)$th degenerations) at v_1 and v_0

of $\mathcal{T}_{n-1}$ are

$$\begin{aligned} \omega_{\mathcal{T}_{n-1}}(v_1) &= \frac{c_{v_1}dx}{\prod_{i=1}^{r}(x-[b_i]_{v_1})^{\iota_{i,n-1}}}\,, \\ \omega_{\mathcal{T}_{n-1}}(v_0) &= \frac{c_{v_0}dx}{x^{\iota_{n-1}}} + \sum_{j<\iota_{n-1}} \frac{c_j dx}{x^j} \quad \left(\text{or } g^{n-1} = \sum_{j=1}^{l} \frac{d_j}{x^j}\right), \end{aligned} \tag{6.1}$$

respectively, where $l < \iota_{n-1}$. We may write $\omega_{\mathcal{T}_{n-1}}(v_0)$ in (6.1) as $f(x)dx/x^{\iota_{n-1}}$. Then, when $\delta_{\mathcal{T}_{n-1}}(v_0) > 0$, as the two forms are compatible (see Section 4.4), we have the relation $c_{v_0} = c_{v_1} =: c$. When $\delta_{\mathcal{T}_{n-1}}(v_0) = 0$ and $\iota_{n-1} = p\iota_{n-2} - p + 1$, where ι_{n-2} is the conductor of the $\mathbb{Z}/p^{n-2}$-subtree, as discussed in Theorem 4.19, we do not have to worry about the compatibility at the root. Otherwise, the same theorem asserts that $l = \iota_{n-1} - 1$ and $c_{v_1} = -ld_l$.

Suppose, moreover, that we are given a level n depth $\delta_{\mathcal{T}_n}(v_0) \geqslant p\delta_{\mathcal{T}_{n-1}}(v_0)$ and a level n differential conductor (or degeneration) at the root as follows:

$$\omega_{\mathcal{T}_n}(v_0) = \frac{C_{v_0}dx}{x^{\iota_n}} + \sum_{j<\iota_n} \frac{C_j dx}{x^j} \quad \left(\text{or } g^{n} = \sum_{j=1}^{L} \frac{D_j}{x^j}\right), \tag{6.2}$$

where $L < \iota_n$. When $\delta_{\mathcal{T}_n}(v_0) > 0$, it is required that $\mathcal{C}(\omega_{\mathcal{T}_n}(v_0)) = \omega_{\mathcal{T}_{n-1}}(v_0)$ if $\delta_{\mathcal{T}_n}(v_0) = p\delta_{\mathcal{T}_{n-1}}(v_0)$, or that $\omega_{\mathcal{T}_n}(v_0)$ is exact if $\delta_{\mathcal{T}_n}(v_0) > p\delta_{\mathcal{T}_{n-1}}(v_0)$. If $\delta_{\mathcal{T}_n}(v_0) = p\delta_{\mathcal{T}_{n-1}}(v_0) = 0$, then $L = \iota_n - 1$ if $\iota_n > \iota_{n-1} - p + 1$.

Just as the main theme of the whole paper, we first construct the extensions of certain subtrees of the given tree $\mathcal{T}_{n-1}$.

DEFINITION 6.2. With the notation as above, let $\mathcal{T}_n$ be an extension of $\mathcal{T}_{n-1}$ with the same underlying tree and such that the depth at the root of $\mathcal{T}_n$ is p *times* that of $\mathcal{T}_{n-1}$. Let $a_n \in \mathbb{Z}_{>0}$ and $a_n =: a'_n + pu_n$ with $1 \leqslant a'_n < p$. We say that

- $\mathcal{T}_n$ is an *additive a_n-extension* of $\mathcal{T}_{n-1}$ if, at all but one leaf (which we may assume to be b_1), we have $\iota_{i,n} = p\iota_{i,n-1}$, hence $\iota_{1,n} = p\iota_{1,n-1} - p + 1 + a_n$;
- $\mathcal{T}_n$ is a *minimum extension* of $\mathcal{T}_{n-1}$ if $\iota_{1,n} = p\iota_{1,n-1} - p + 1$ and $\iota_{i,n} = p\iota_{i,n-1}$ for $i \neq 2$.

We first show that one can always construct an additive extension for the tree $\mathcal{T}_{n-1}$. This is the extension of $\mathcal{T}_1(e_3)$ in Example 4.23.

PROPOSITION 6.3. *With the above settings, there exists a Hurwitz tree $\mathcal{T}_n$ that extends $\mathcal{T}_{n-1}$ a_n-additively. In particular, Proposition 4.22 holds when $\iota_n = p\iota_{n-1} - p + 1 + a_n$.*

Proof. First, we set the decorated tree of $\mathcal{T}_n$ and the thickness of the edges to be those of $\mathcal{T}_{n-1}$. We then assign to b_1 the conductor $\iota_{1,n} = p\iota_{i,n-1} - p + a_n + 1$, to each leaf b_i $(i \neq 1)$ a conductor $\iota_{i,n} = p\iota_{i,n-1}$, to each edge e a slope $d_e := \sum_{h \succ e} \iota_{h,n} - 1$, and to the root v_0 the depth $p\delta_{\mathcal{T}_{n-1}}(v_0)$. At each vertex or leaf of $\mathcal{T}_n$ where the corresponding one in $\mathcal{T}_{n-1}$ has monodromy group $\mathbb{Z}/p^i$, we equip it with the monodromy group $\mathbb{Z}/p^{i+1}$. We then assign the depth at each vertex $v \neq v_0$ inductively on the positive direction starting from v_0 so that condition (H5) is satisfied. Then it is straightforward to check that $\delta_{\mathcal{T}_n}(v) > p\delta_{\mathcal{T}_{n-1}}(v)$ at all vertices $v \neq v_0$. Finally, we equip each vertex v away from the root with an *exact* differential form

$$\omega_{\mathcal{T}_n}(v) = \frac{c_v dx}{\prod_{b \succ v}(x-[b]_v)^{\iota_{b,n}}}\,, \tag{6.3}$$

where $c_v \in k^{\times}$ is determined inductively from the root so that the differential conductors at all vertices are compatible. More precisely, if e is an edge of $\mathcal{T}_{n-1}$, then we set the coefficient of $\omega_{\mathcal{T}_n}(t(e))$ to be the e-part of $\omega_{\mathcal{T}_n}(s(e))$ (respectively, the coefficient of the nth degeneration) when $\delta_{\mathcal{T}_n}(s(e)) > 0$ (respectively, when $\delta_{\mathcal{T}_n}(s(e)) = 0$). ☐

We then construct a minimum extension. That is the extension of $\mathcal{T}_1$ or $\mathcal{T}_1(e_1)$ in Example 4.23.

Proposition 6.4. *With the above settings, there exists a Hurwitz tree $\mathcal{T}_n$ that minimally extends $\mathcal{T}_{n-1}$. In particular, Proposition 4.22 holds for the case $\iota_n = p\iota_{n-1} - p + 1$.*

Proof. First suppose that the height of the tree $\mathcal{T}_{n-1}$ is 1. We assign the conductor $\iota_{1,n} := p\iota_{1,n-1} - p + 1$ to b_1 and $\iota_{i,n} := p\iota_{i,n-1}$ to b_i when $i \neq 1$, making the sum of all conductors equal to ι_n. We then set the depth at v_1 to be $p\delta_{\mathcal{T}_{n-1}}(v_1)$, the slope of e_0 to be $\iota_n - 1$, and the differential conductor at v_1 to be

$$\omega_{\mathcal{T}_n}(v_1) = \frac{c^p dx}{\prod_{i=1}^{r}(x - [b_i]_{v_1})^{\iota_{i,n}}},$$

which satisfies $\mathcal{C}(\omega_{\mathcal{T}_n}(v_1)) = \omega_{\mathcal{T}_{n-1}}(v_1)$, as shown in Proposition 6.1. As before, the monodromy group of a vertex (or leaf) is cyclic of order p times that of the corresponding one in $\mathcal{T}_{n-1}$. The hypothesis $\iota_n = p\iota_{n-1} - p + 1$ also implies that the differential conductor (respectively, the nth degeneration) is of the form

$$\omega_{\mathcal{T}_n(x)}(v_0) = \frac{c^p dx}{x^{\iota_n}} + \sum_{j \leqslant \iota_{n-1}} \frac{c_j^p dx}{x^{pj-p+1}} + dg(x) \quad (\text{respectively, } g^n(x)), \tag{6.4}$$

where $g(x)$ (respectively, $g^n(x)$) is a polynomial in x^{-1} of degree at most $\iota_n - 2$. This makes v_0 compatible with v_1 in $\mathcal{T}_n$ when $\delta_{\mathcal{T}_n}(v_0) > 0$, completing the base case. Recall that, in our minimal jump assumption and when $\delta_{\mathcal{T}_n}(v_0) = 0$, we do not need the degenerations at v_0 and v_1 to be compatible.

Let us now consider the case where the height of $\mathcal{T}_{n-1}$ is greater than 1. Suppose that there are m edges $e_1, \ldots, e_m$ ($m \geqslant 2$) starting from v_1. We then first assign degeneration data at v_0 and v_1 identical to that in the base case. Note that the Hurwitz tree $\mathcal{T}_{n-1}(e_i)$ has height at most $h(\mathcal{T}_{n-1}) - 1$. We thus can extend $\mathcal{T}_{n-1}(e_1)$ to a minimum tree by induction, and $\mathcal{T}_{n-1}(e_i)$ for $i \neq 1$ to an additive tree as in Proposition 6.3, all with depth $\delta_{\mathcal{T}_n}(v_1)$. The conductor of $\mathcal{T}_n(e_1)$ (respectively, $\mathcal{T}_n(e_i)$) is $\iota_{n,e_1} := p\mathfrak{C}_{\mathcal{T}_{n-1}}(e_1) - p + 1$ (respectively, $\iota_{n,e_i} := p\mathfrak{C}_{\mathcal{T}_{n-1}}(e_i)$). Hence, the sum of the conductors is ι_n. That completes the construction of the tree $\mathcal{T}_n$. ☐

We have just proved Proposition 4.22. One can also do the following alternative construction, which is utilized in Example 4.25.

Proposition 6.5. *There exists a $\mathbb{Z}/p^n$-tree $\mathcal{T}_n$ that extends $\mathcal{T}_{n-1}$ with no essential jumps.*

Proof. Suppose that there are m edges $e_1, \ldots, e_m$ succeeding v_1, with conductors $\mathfrak{C}_1, \ldots, \mathfrak{C}_m$, respectively, on $\mathcal{T}_{n-1}$. Hence $\sum_{i=1}^{m} \mathfrak{C}_i = \iota_{n-1}$. Set $\mathfrak{C}'_1 := p\mathfrak{C}_1 - p + a'_n$, $\mathfrak{C}'_2 := p\mathfrak{C}_2 - p + 1$, and $\mathfrak{C}'_i := p\mathfrak{C}_i$. We first assign $p\delta_{\mathcal{T}_{n-1}}(v_1)$ as the depth at v_1 on $\mathcal{T}_n$, and the non-exact differential form

$$\omega_{\mathcal{T}_n}(v_1) = \frac{C_{v_1} dx}{\prod_{i=1}^{s}(x - [b_i]_{v_1})^{\mathfrak{C}'_i}},$$

that satisfies $\mathcal{C}(\omega_{\mathcal{T}_n}(v_1)) = \omega_{\mathcal{T}_{n-1}}(v_1)$ (which exists by Proposition 6.1) as the differential conductor at v_1. We then extend $\mathcal{T}_{n-1}(e_1)$ to an additive a'_n-tree, $\mathcal{T}_{n-1}(e_2)$ to a minimum tree, and each $\mathcal{T}_{n-1}(e_i)$ where $i \neq 1, 2$ to an additive-$(p-1)$-tree, all with depth $p\delta_{\mathcal{T}_{n-1}}(v_1)$, and so that they are compatible with the differential conductor $\omega_{\mathcal{T}_n}(v_1)$. This makes the current sum of conductors equal to $\mathfrak{C}(v_1) := p\iota_{n-1} - 2p + a'_n + 1$, which is strictly smaller than $p\iota_{n-1} - p + 1$ and not congruent to 1 modulo p. Let us now consider the trunk e_0 of $\mathcal{T}_{n-1}$. A simple calculation shows that one can break down the edge e_0 into two rational ones, called $e_{0,1}$ and $e_{0,2}$, so that $(\mathfrak{C}(v_1) - 1)\epsilon_{e_{0,2}} + (\iota_n - 1)\epsilon_{e_{0,1}} = p\delta_{\mathcal{T}_{n-1}}$. We then assign to $t(e_{0,1})$ on $\mathcal{T}_n$ the exact differential

$$\omega_{\mathcal{T}_n}(t(e_{0,1})) = \frac{C_{v_0}dx}{x^{\mathfrak{C}(v_1)}(x-a)^{p(u_n+1)}},$$

where $(-a)^{p(u_n+1)} = C_{v_0}/C_{v_1}$. The constant coefficient at $t(e_{0,1})$ is C_{v_0}, while the constant coefficient of its $e_{0,2}$-part is $C_{v_0}/(-a)^{p(u_n+1)} = C_{v_1}$. Hence $t(e_{0,1})$ is compatible with both v_0 and v_1. We then glue a leaf of conductor $p(u_n+1)$ and monodromy group $\mathbb{Z}/p$ to $t(e_{0,1})$. Finally, we set the remaining degeneration data at $e_{0,1}$, $e_{0,2}$, and v_0 in the obvious manner, completing the proof. □

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