



Integrability of pushforward measures by analytic maps

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ABSTRACT

Given a map $\phi: X \rightarrow Y$ between F -analytic manifolds over a local field F of characteristic 0, we introduce an invariant $\epsilon_*(\phi)$ measuring the integrability of pushforwards of smooth compactly supported measures by ϕ , together with a local version $\epsilon_*(\phi; x)$ near $x \in X$.

These invariants are closely tied to the singularities of ϕ . When Y is one-dimensional, we provide an explicit formula for $\epsilon_*(\phi; x)$ and show that it is asymptotically equivalent to classical singularity invariants, in particular the F -log-canonical threshold $\text{lct}_F(\phi - \phi(x); x)$. In general, we prove that $\epsilon_*(\phi; x)$ is bounded below by the F -log-canonical threshold $\lambda = \text{lct}_F(\mathcal{J}_\phi; x)$ of the Jacobian ideal \mathcal{J}_ϕ , with equality when $\dim Y = \dim X$.

For polynomial maps $\varphi: X \rightarrow Y$ between smooth algebraic \mathbb{Q} -varieties, we show that the condition “ $\epsilon_*(\varphi_F) = \infty$ over a large family of local fields” is equivalent to φ being flat with semi-log-canonical fibers.

1. Introduction

The goal of this paper is to explore a singularity invariant $\epsilon_*(\phi)$ of a map ϕ between two manifolds over a local field. This invariant quantifies the integrability of pushforward measures by ϕ ; we define it in (1.2) below after introducing some notation that will also be used further on.

Throughout this paper, we fix a local field F of characteristic 0, that is, \mathbb{R} , \mathbb{C} or a finite extension of \mathbb{Q}_p . If X is an F -analytic manifold of dimension n , let $(U_\alpha \subset X, \psi_\alpha: U_\alpha \rightarrow F^n)_{\alpha \in \mathcal{A}}$ be an atlas. We denote by $C^\infty(X)$ the space of smooth functions on X , that is, functions $f: X \rightarrow \mathbb{C}$ such that $f \circ \psi_\alpha^{-1}|_{\psi_\alpha(U_\alpha)}$ is smooth for each $\alpha \in \mathcal{A}$, and by $C_c^\infty(X)$ the subspace of compactly supported smooth functions (if F is non-Archimedean, smooth means locally constant). We similarly write $\mathcal{M}^\infty(X)$ for the space of smooth measures on X , that is, measures such that

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each $(\psi_\alpha)_*(\mu|_{U_\alpha})$ has a smooth density with respect to the Haar measure. We use $\mathcal{M}_c^\infty(X)$ to denote the space of smooth compactly supported measures on X . For $1 \leq q \leq \infty$, consider the class $\mathcal{M}_{c,q}(X)$ of finite Radon measures μ on X that are compactly supported and such that, for any $\alpha \in \mathcal{A}$, the measure $(\psi_\alpha)_*(\mu|_{U_\alpha})$ is absolutely continuous with density in $L^q(F^n)$. All these classes do not depend on the choice of the atlas. For $\mu \in \mathcal{M}_{c,1}(X)$, we define

$$\epsilon_\star(\mu) := \sup\{\epsilon \geq 0 : \mu \in \mathcal{M}_{c,1+\epsilon}(X)\}.$$

Note that by Jensen's inequality, $\mu \in \mathcal{M}_{c,1+\epsilon}(X)$ for all $0 \leq \epsilon < \epsilon_\star(\mu)$.

Now let $\phi: X \rightarrow Y$ be an analytic map between F -analytic manifolds X and Y . If ϕ is *locally dominant*, that is, $\phi(U)$ contains a non-empty open set for each open subset $U \subseteq X$, then $\phi_*\mu \in \mathcal{M}_{c,1}(Y)$ whenever $\mu \in \mathcal{M}_{c,1}(X)$. We now set, for each $x \in X$,

$$\epsilon_\star(\phi; x) := \sup_{U \ni x} \inf_{\mu \in \mathcal{M}_c^\infty(U)} \epsilon_\star(\phi_*\mu) = \sup_{U \ni x} \inf_{\mu \in \mathcal{M}_{c,\infty}(U)} \epsilon_\star(\phi_*\mu), \quad (1.1)$$

where the supremum is over all open neighborhoods U of x . Finally, we can define

$$\epsilon_\star(\phi) := \inf_{\mu \in \mathcal{M}_c^\infty(X)} \epsilon_\star(\phi_*\mu) = \inf_{x \in X} \epsilon_\star(\phi; x). \quad (1.2)$$

Note that if there exists an open neighborhood U of x such that $\phi_*\mu$ lies in $\mathcal{M}_{c,q}(Y)$ for all $1 < q < \infty$ and all $\mu \in \mathcal{M}_c^\infty(U)$, then $\epsilon_\star(\phi; x) = \infty$. The best-case scenario in this setting is obtained when $\phi_*\mu \in \mathcal{M}_{c,\infty}(Y)$ for every $\mu \in \mathcal{M}_c^\infty(X)$. In this case, we say that ϕ is an L^∞ -map.

The main motivation for $\epsilon_\star(\phi; x)$ comes from singularity theory. In general, bad singularities of ϕ should manifest themselves in poor analytic behavior of the pushforward $\phi_*\mu$ of $\mu \in \mathcal{M}_c^\infty(X)$. This phenomenon has been extensively studied in the case $Y = F^m$, through the analysis of the Fourier transform $\mathcal{F}(\phi_*\mu)$, which takes the form of an *oscillatory integral* in the Archimedean case, and of an *exponential sum* in the p -adic case. When the dimension of the target space is equal to $m = 1$, the rate of decay of $\mathcal{F}(\phi_*\mu)$ is closely related to singularity invariants such as the log-canonical threshold (see for example [Igu78], [Den91, Corollary 1.4.5] and [CMN19, Theorem 1.5] for the non-Archimedean case, and [AGV88, Chapter 7] for the Archimedean case). In higher dimension, the connections are less explicit; however, milder singularities of ϕ still result in faster decay rates of $\mathcal{F}(\phi_*\mu)$. For more information on $\mathcal{F}(\phi_*\mu)$, we refer to Igusa's work [Igu78], the surveys of Denef, Meuser and León-Cardenal [Den91, Meu16, Leó22], as well as the book [AGV88, Parts II and III] by Arnold, Gusein-Zade and Varchenko, and also the discussion in Section 3.2 below.

The study of the integrability properties of $\phi_*\mu$ and their relation to the singularities of ϕ has received less treatment. The invariant $\epsilon_\star(\phi; x)$ is a natural step in this direction; it is more robust than Fourier-type invariants as it is also meaningful when Y is any smooth manifold, which is especially important for applications (see Section 1.1).

The invariant $\epsilon_\star(\phi; x)$ tends to be small whenever the singularities of ϕ are bad near x . When ϕ is a submersion, the pushforward $\phi_*\mu$ of any $\mu \in \mathcal{M}_c^\infty(X)$ is smooth and in particular lies in $\mathcal{M}_{c,\infty}(Y)$. Moreover, Aizenbud and Avni [AA16] have shown that for algebraic maps $\varphi: X \rightarrow Y$ between smooth algebraic \mathbb{Q} -varieties, the condition that the corresponding map $\varphi_{\mathbb{Q}_p}: X(\mathbb{Q}_p) \rightarrow Y(\mathbb{Q}_p)$ of \mathbb{Q}_p -analytic varieties is an L^∞ -map is equivalent to a certain mild singularity property, namely that φ is flat with fibers of rational singularities (abbreviated as (FRS), see Definition 1.8).

When analyzing $\epsilon_\star(\phi; x)$, one can further restrict the infima in (1.1)–(1.2) to the class of compactly supported measures μ that are constructible, in the sense of [CGH14, Section 3] and [CM11, Definition 1.2] (see also [LR97]). This class is preserved under pushforward by analytic

maps, and therefore $\phi_*\mu$ is constructible as well. Moreover, constructible measures admit a well-behaved structure theory and have tame analytic behavior (see for example [CL08, CL10, CM11, CM13, CGH14, CGH18]), and, in particular, it follows from [GH21, CM13] that $\epsilon_*(\phi_*\mu) > 0$. Positivity of $\epsilon_*(\phi_*\mu)$ in the real case can further be deduced from [RS88, Section 2].

Our goal in this paper is to explore in more detail the properties of $\epsilon_*(\phi; x)$, and in particular to obtain upper and lower bounds on $\epsilon_*(\phi; x)$ in terms of other singularity invariants that may be easier to compute, such as the log-canonical threshold of certain ideals. In Theorem 1.1 (proved in Section 4), we give a lower bound on $\epsilon_*(\phi; x)$. In particular, this provides a proof for the positivity of ϵ_* , without using the theory of motivic integration. In Theorem 1.5 (proved in Section 5), an upper bound on $\epsilon_*(\phi; x)$ is given in the complex case. In Theorem 1.2 (proved in Section 3), an explicit formula is given for $\epsilon_*(\phi; x)$ when Y is one-dimensional, over any local field. Finally, in Theorem 1.12 (proved in Section 6), we specialize to polynomial maps between smooth algebraic varieties, and we geometrically characterize the condition $\epsilon_*(\phi; x) = \infty$.

1.1 Application: Regularization by convolution

Apart from the geometric motivation discussed above, an additional source of motivation comes from the study of random walks on groups. Assume that G is an F -analytic group, and take a finite measure ν on G . Can one find a number $k \in \mathbb{N}$ such that the k th convolution power ν^{*k} lies in $\mathcal{M}_{c,\infty}(G)$, and if so, what is the smallest such number $k_*(\nu)$?

An important class of examples comes from the realm of word maps. Given a word w in r letters, by which we mean an element either of the free group F_r , or of the free Lie algebra \mathcal{L}_r , one can consider the corresponding word maps $w_G: G^r \rightarrow G$ or $w_{\mathfrak{g}}: \mathfrak{g}^r \rightarrow \mathfrak{g}$, where \mathfrak{g} is the Lie algebra of G . When G is a compact real or p -adic Lie group, w induces a natural measure $\nu := (w_G)_*\pi_G$, where π_G is the normalized Haar measure on G . Here, one may further ask what is the L^∞ -mixing time of the word measure ν , namely, how large should k be to ensure that $\|\nu^{*k} - \pi_G\|_\infty \ll 1$. Questions of this kind have been studied for example in [AA16, LST19, GH24, AG25, AGL24].

Similar problems have also appeared in a variety of other applications; we mention the work of Ricci and Stein on singular integrals on non-abelian groups (see the ICM survey of Stein [Ste87]). In the study of one-dimensional random operators, the regularity of the distribution of transfer matrices (lying in $SL(2, \mathbb{R})$, or, more generally, in $Sp(2W, \mathbb{R})$) plays a role in the work of Shubin–Vakilian–Wolff [SVW98] as well as in the recent work [GS22].

Motivated by the above examples, we focus on the following setting: G and X are analytic, and $\nu = \phi_*\mu$ is the pushforward of a measure $\mu \in \mathcal{M}_{c,\infty}(X)$ under a locally dominant analytic map $\phi: X \rightarrow G$. For $x \in X$, let

$$k_*(\phi; x) = \min_{U \ni x} \max_{\mu \in \mathcal{M}_{c,\infty}^\infty(U)} k_*(\phi_*\mu)$$

be the smallest k that works for any μ supported in a sufficiently small neighborhood U of x .

According to Young’s convolution inequality for locally compact groups (see [KR78, Corollary 2.3] and Remark 1.10),

$$\nu \in \mathcal{M}_{c,1+\epsilon}(G) \implies \nu^{*k} \in \mathcal{M}_{c,1+r}(G), \quad \text{where } r = \begin{cases} \frac{k\epsilon}{1 - (k-1)\epsilon} & \text{if } k < \frac{1+\epsilon}{\epsilon}, \\ \infty & \text{if } k \geq \frac{1+\epsilon}{\epsilon}. \end{cases}$$

This implies

$$k_{\star}(\phi; x) \leq \left\lfloor \frac{1 + \epsilon_{\star}(\phi; x)}{\epsilon_{\star}(\phi; x)} \right\rfloor + 1 < \infty. \quad (1.3)$$

As mentioned above, a more classical approach to bounding $k_{\star}(\phi; x)$ relies on the study of the decay of the Fourier transform of $\phi_{\star}\mu$. While the Fourier-analytic approach often provides sharper bounds (see [GH24, Proposition 5.7]), it is mainly applicable for abelian groups such as $G = F^n$, or mildly non-abelian groups such as the Heisenberg model. One can use non-commutative Fourier transforms to analyze compact Lie groups such as $\mathrm{SO}_n(\mathbb{R})$, but such representation-theoretic techniques are much less effective for compact p -adic groups and non-compact Lie groups. To treat the latter cases, one can use algebro-geometric techniques as in [AA16, GH24]; however, this method requires some assumptions on ϕ . Thus one can argue that the approach to regularization via (1.3) is currently the most efficient one for treating p -adic analytic groups and non-compact Lie groups such as $\mathrm{SL}_n(\mathbb{R})$.

1.2 Main results

1.2.1 *A lower bound on ϵ_{\star} .* While the mere positivity of ϵ_{\star} (and the mere finiteness of k_{\star}) are sufficient for some applications, other ones require explicit bounds. Our first result provides a bound in terms of an important exponent known as the log-canonical threshold; see for example [Mus12, Kol08] for $F = \mathbb{C}$. For an analytic map $\psi: X \rightarrow F$, define the *F-log-canonical threshold*

$$\mathrm{lct}_F(\psi; x) := \sup \left\{ s > 0 : \exists U \ni x \text{ s.t. } \forall \mu \in \mathcal{M}_{c,\infty}(U), \int_X |\psi(x)|_F^{-s} d\mu(x) < \infty \right\}, \quad (1.4)$$

where U runs over all open neighborhoods of x and $|\cdot|_F$ is the absolute value on F , normalized such that $\mu_F(aA) = |a|_F \cdot \mu_F(A)$ for all $a \in F^{\times}$ and $A \subseteq F$, and where μ_F is a Haar measure on F . In particular, $|\cdot|_{\mathbb{C}} = |\cdot|^2$ is the square of the usual absolute value $|\cdot|$ on \mathbb{C} . More generally, if J is a non-zero ideal of analytic functions generated by $\psi_1, \dots, \psi_{\ell}$, define

$$\mathrm{lct}_F(J; x) := \sup \left\{ s > 0 : \exists U \ni x \text{ s.t. } \forall \mu \in \mathcal{M}_{c,\infty}(U), \int_X \min_{1 \leq i \leq \ell} [|\psi_i(x)|_F^{-s}] d\mu(x) < \infty \right\}. \quad (1.5)$$

This definition does not depend on the choice of the generators, and thus it extends in a straightforward way to sheaves of ideals. Furthermore, the log-canonical threshold is always strictly positive (see Section 2).

Given a locally dominant analytic map $\phi: X \rightarrow Y$ between two F -analytic manifolds, we define the *Jacobian ideal sheaf* \mathcal{J}_{ϕ} as follows. If $X \subseteq F^n$ and $Y \subseteq F^m$ are open subsets, we define \mathcal{J}_{ϕ} to be the ideal in the algebra of analytic functions on X generated by the $m \times m$ minors of the differential $d_x(\phi)$ of ϕ . Note that if $\psi_1: X \rightarrow X' \subseteq F^n$ and $\psi_2: Y \rightarrow Y' \subseteq F^m$ are analytic diffeomorphisms, then $\psi_1^*(\mathcal{J}_{\psi_2 \circ \phi \circ \psi_1^{-1}}) = \mathcal{J}_{\phi}$. Hence, the definition of \mathcal{J}_{ϕ} can be generalized (or glued) to an ideal sheaf on X if X and Y are F -analytic manifolds.

We now describe the first main result, which provides a lower bound on $\epsilon_{\star}(\phi; x)$ in terms of the Jacobian ideal \mathcal{J}_{ϕ} of ϕ . The proof is given in Section 4.

THEOREM 1.1. *Let X and Y be analytic F -manifolds with $\dim X = n \geq \dim Y = m$, and let $\phi: X \rightarrow Y$ be a locally dominant analytic map. Then for every $x \in X$, we have*

$$\epsilon_{\star}(\phi; x) \geq \mathrm{lct}_F(\mathcal{J}_{\phi}; x); \quad (1.6)$$

if $m = n$, equality is achieved.

As we will see later in Remark 1.6(2), for $\dim X > \dim Y$, the inequality (1.6) may be strict. In the next paragraph, we discuss an additional case in which ϵ_\star can be computed explicitly: $\dim Y = 1$.

1.2.2 *A formula in the one-dimensional case and a reverse Young inequality.* When the target space Y is one-dimensional, Hironaka’s theorem on the embedded resolution of singularities [Hir64] provides a powerful tool to study the structural properties of algebraic and analytic maps. This theorem, as well as the asymptotic expansion of pushforward measures about a critical value of the map, allows us to obtain the following much more detailed results, the proofs of which are given in Section 3.

The first one is an exact formula relating ϵ_\star to the log-canonical threshold.

THEOREM 1.2. *Let X be an analytic F -manifold, and let $\phi: X \rightarrow F$ be a locally dominant analytic map. Then for each $x \in X$, we have*

$$\epsilon_\star(\phi; x) = \begin{cases} \infty & \text{if } \text{lct}_F(\phi - \phi(x); x) \geq 1, \\ \frac{\text{lct}_F(\phi - \phi(x); x)}{1 - \text{lct}_F(\phi - \phi(x); x)} & \text{if } \text{lct}_F(\phi - \phi(x); x) < 1. \end{cases} \quad (1.7)$$

By Theorem 1.2, by (1.3) and by a Thom–Sebastiani-type result for lct_F (Proposition 3.11(1)), one can further show

$$\left\lceil \frac{1}{\text{lct}_F(\phi_x; x)} \right\rceil \leq k_\star(\phi; x) \leq \left\lfloor \frac{1}{\text{lct}_F(\phi_x; x)} \right\rfloor + 1. \quad (1.8)$$

Consequently, we see that $\epsilon_\star(\phi; x)$, $\text{lct}_F(\phi_x; x)$ and $1/k_\star(\phi; x)$ are asymptotically equivalent as $\text{lct}_F(\phi_x; x) \rightarrow 0$. In Section 3.2, we will see that these invariants are also tightly related to an invariant δ_\star quantifying Fourier decay. We will further see in Section 3.4 that the close relation between all these quantities is a special feature of the one-dimensional case and does not generalize to higher dimensions.

We next provide a reverse Young result for pushforward measures by analytic maps. Recall that Young’s convolution inequality (see for example [Wei40, Section III.13, pp. 54–55]) implies that

$$\frac{\epsilon_1}{1 + \epsilon_1} + \frac{\epsilon_2}{1 + \epsilon_2} \geq \frac{\epsilon}{1 + \epsilon} \implies \mathcal{M}_{c, 1+\epsilon_1}(F) * \mathcal{M}_{c, 1+\epsilon_2}(F) \subseteq \mathcal{M}_{c, 1+\epsilon}(F). \quad (1.9)$$

Using the connection between $\epsilon_\star(\phi; x)$ and $k_\star(\phi; x)$ as well as the structure of pushforward measures, we show the following converse to (1.9).

THEOREM 1.3 (Reverse Young inequality). *Let $\nu_1, \nu_2 \in \mathcal{M}_{c, 1}(F)$ be pushforward measures of the form $\nu_j = (\phi_j)_* \mu_j$, where $\mu_j \in \mathcal{M}_c^\infty(F^{n_j})$ and $\phi_j: F^{n_j} \rightarrow F$ are analytic, locally dominant. If $\nu_1 * \nu_2 \in \mathcal{M}_{c, 1+\epsilon}(F)$ for some $\epsilon > 0$, then*

$$\frac{\epsilon_\star(\nu_1)}{1 + \epsilon_\star(\nu_1)} + \frac{\epsilon_\star(\nu_2)}{1 + \epsilon_\star(\nu_2)} > \frac{\epsilon}{1 + \epsilon}. \quad (1.10)$$

In particular, if ν_1 is equal to ν_2 , it lies in $\mathcal{M}_{c, 1+\epsilon/(2+\epsilon)}(F)$.

Remark 1.4. Under the assumptions of Theorem 1.3, if $\nu_1 * \nu_2 \in \mathcal{M}_{c, \infty}(F)$, then (1.10) (applied with $\epsilon \rightarrow +\infty$) implies

$$\frac{\epsilon_\star(\nu_1)}{1 + \epsilon_\star(\nu_1)} + \frac{\epsilon_\star(\nu_2)}{1 + \epsilon_\star(\nu_2)} \geq 1. \quad (1.11)$$

In general, one cannot hope for a strict inequality in (1.11); indeed, taking $F = \mathbb{R}$ or $F = \mathbb{Q}_p$ for $p \equiv 3 \pmod{4}$, $\phi_1 = \phi_2 = x^2$ and $\mu_1 = \mu_2$ a uniform measure on some ball around 0, one has $\epsilon_*(\nu_1) = \epsilon_*(\nu_2) = 1$, so that the left-hand side of (1.11) equals 1. On the other hand, $\nu_1 * \nu_2 \in \mathcal{M}_{c,\infty}(F)$.

When $F = \mathbb{C}$, or more generally when the codimension of $(\phi_1 * \phi_2)^{-1}(0)$ in $F^{n_1+n_2}$ is one, we expect (1.11) to hold with a strict inequality.

1.2.3 Upper bounds on ϵ_* . For $n > m$, the lower bound (1.6) may in general not be an equality. However, the next result shows that when $F = \mathbb{C}$, the lower bound (1.6) is asymptotically sharp as $\text{lct}_{\mathbb{C}}(\mathcal{J}_\phi; x) \rightarrow 0$.

THEOREM 1.5. *Let X and Y be analytic \mathbb{C} -manifolds, and let $\phi: X \rightarrow Y$ be a locally dominant analytic map. Then, whenever $\text{lct}_{\mathbb{C}}(\mathcal{J}_\phi; x) < 1$, we have*

$$\epsilon_*(\phi; x) \leq \frac{\text{lct}_{\mathbb{C}}(\mathcal{J}_\phi; x)}{1 - \text{lct}_{\mathbb{C}}(\mathcal{J}_\phi; x)}. \quad (1.12)$$

Remark 1.6. (1) When $Y = F$, we have $\text{lct}_F(\phi - \phi(x); x) \leq \text{lct}_F(\mathcal{J}_\phi; x)$ (see Proposition 3.13), so (1.12) also holds for $F \neq \mathbb{C}$ whenever $\text{lct}_F(\mathcal{J}_\phi; x) < 1$. In particular, Theorem 1.2 implies Theorem 1.5 in the case $Y = \mathbb{C}$.

(2) The upper bound (1.12) is asymptotically tight, in the sense that the value of ϵ_* can be arbitrarily close to the upper bound (1.12), as seen from the following family of examples. Let $\phi := x_1^m x_2^m \cdots x_n^m$. Then $\nabla \phi = \langle x_1^{m-1} x_2^m \cdots x_n^m, \dots, x_1^m x_2^m \cdots x_n^{m-1} \rangle$, and thus by [How01, Main Theorem and Example 5], it follows that

$$\text{lct}_{\mathbb{C}}(\mathcal{J}_\phi; 0) = \frac{1}{m - 1/n},$$

so the upper bound in (1.12) becomes $\epsilon_*(\phi; 0) \leq 1/(m - 1 - 1/n)$, whereas the lower bound (1.6) is $\epsilon_*(\phi; 0) \geq 1/(m - 1/n)$. We see that the true value $\epsilon_*(\phi; 0) = 1/(m - 1)$ is closer to the upper bound than to the lower bound.

One may wonder whether Theorem 1.5 can be extended to $F \neq \mathbb{C}$. In the current proof, the volumes of balls in complex manifolds are bounded from below using Lelong's monotonicity theorem, and the latter fails for $F = \mathbb{R}$ and for any \mathbb{Q}_p . If $\varphi: X \rightarrow Y$ is a polynomial map between smooth varieties, defined over \mathbb{Q} , we expect the upper bound in Theorem 1.5 to hold for $\varphi_{\mathbb{Q}_p}: X(\mathbb{Q}_p) \rightarrow Y(\mathbb{Q}_p)$ for infinitely many primes p . This would follow from a positive answer to Question 1.14.

1.2.4 Applications to convolutions of algebraic morphisms. Throughout this and the next subsections, we assume that K is a number field. In [GH19, GH21] and [GH24], the first two authors have studied the following convolution operation in algebraic geometry.

DEFINITION 1.7. Let $\varphi: X \rightarrow G$ and $\psi: Y \rightarrow G$ be morphisms from algebraic K -varieties X and Y to an algebraic K -group G . We define their *convolution* by

$$\varphi * \psi: X \times Y \longrightarrow G, \quad (x, y) \longmapsto \varphi(x) \cdot_G \psi(y).$$

We denote by $\varphi^{*k}: X^k \rightarrow G$ the k th *self-convolution* of φ .

We restrict ourselves to the setting where X and Y are smooth algebraic K -varieties and G is a connected algebraic K -group. The main motto is that the algebraic convolution operation has

a smoothing effect on morphisms, similarly to the usual convolution operation in analysis (see [GH19, Proposition 1.3] and [GH21, Proposition 3.1]). For example, starting from any dominant map $\varphi: X \rightarrow G$, the k th self-convolution $\varphi^{*k}: X^k \rightarrow G$ is flat for every $k \geq \dim G$ (see [GH21, Theorem B]). To explain the connection to this work, we introduce the following property.

DEFINITION 1.8 ([AA16, Definition II]). (1) A K -variety Z has *rational singularities* if it is normal and for every resolution of singularities $\pi: \tilde{Z} \rightarrow Z$, the pushforward $\pi_*(\mathcal{O}_{\tilde{Z}})$ of the structure sheaf has no higher cohomologies.

(2) A morphism $\varphi: X \rightarrow Y$ between smooth K -varieties is called *flat with fibers of rational singularities*, abbreviated as (FRS), if it is flat and if every fiber of φ has rational singularities.

In [AA16, Theorem 3.4] (see Theorem 6.1 below), Aizenbud and Avni proved the following. A morphism $\varphi: X \rightarrow Y$ between smooth K -varieties is (FRS) if and only if for every non-Archimedean local field $F \supseteq K$, one has $(\varphi_F)_*\mu \in \mathcal{M}_{c,\infty}(Y(F))$ for every $\mu \in \mathcal{M}_c^\infty(X(F))$. A similar characterization can be given for $F = \mathbb{C}$; see Corollary 6.2.

This characterization of L^∞ -maps allows one to study random walks on analytic groups as in Section 1.1 in an algebro-geometric way, via the above algebraic convolution operation. Starting from a pushforward $\nu = (\varphi_F)_*\mu$ of $\mu \in \mathcal{M}_{c,\infty}(X(F))$ by an algebraic map $\varphi: X \rightarrow G$, instead of showing that $\nu^{*k} \in \mathcal{M}_{c,\infty}(G(F))$, it is enough to show that $\varphi^{*k}: X^k \rightarrow G$ is an (FRS) morphism. This method was used in [AA16, GH24] to study word maps. Moreover, in [GH19, GH21] it was shown that any locally dominant morphism $\varphi: X \rightarrow G$ becomes (FRS) after sufficiently many self-convolutions.

Using (1.3), Theorem 1.1 and Corollary 6.2, explicit bounds can be given on the required number of self-convolutions, in terms of the Jacobian ideal of φ .

COROLLARY 1.9. *Let X be a smooth K -variety, let G be a connected K -algebraic group, and let $\varphi: X \rightarrow G$ be a locally dominant morphism. Then the morphism φ^{*k} is (FRS) for any $k \geq \lfloor (1 + \text{lct}_{\mathbb{C}}(\mathcal{J}_\varphi)) / \text{lct}_{\mathbb{C}}(\mathcal{J}_\varphi) \rfloor + 1$.*

Remark 1.10. In the setting of locally compact groups, Young's convolution inequality is commonly stated under the assumption that the group G is unimodular. In [KR78, Lemma 2.1 and Corollary 2.3], a version for non-unimodular groups is given; if G is a locally compact group, with modular character $\Delta: G \rightarrow \mathbb{R}_{>0}$, and if $1 \leq p, q, r \leq \infty$ satisfy $1/p + 1/q = 1 + 1/r$, then we have $\|\mu * (\Delta^{1-1/p}\nu')\|_r \leq \|\mu\|_p \|\nu'\|_q$ whenever $\mu \in \mathcal{M}_{c,p}(G)$ and $\nu' \in \mathcal{M}_{c,q}(G)$. However, since the modular character $\Delta: G \rightarrow \mathbb{R}_{>0}$ is a continuous homomorphism, it is bounded on the compact support of ν' . Hence, for every $\mu \in \mathcal{M}_{c,p}(G)$ and $\nu \in \mathcal{M}_{c,q}(G)$, we deduce that

$$\|\mu * \nu\|_r = \|\mu * (\Delta^{1-\frac{1}{p}}(\Delta^{\frac{1}{p}-1} \cdot \nu))\|_r \leq \|\mu\|_p \cdot \|\Delta^{\frac{1}{p}-1} \cdot \nu\|_q \leq C \cdot \|\mu\|_p \cdot \|\nu\|_q$$

for some constant C depending on G and ν . In particular, $\mu * \nu \in \mathcal{M}_{c,r}(G)$.

1.2.5 *An algebraic characterization of $\epsilon_\star = \infty$.* Let $\varphi: X \rightarrow Y$ be a morphism between smooth K -varieties. We would like to characterize the condition that $\epsilon_\star(\varphi_F) = \infty$ for all F in certain families of local fields, in terms of the singularities of φ . The singularity properties we consider play a central role in birational geometry (see [Kol97]).

Let X be a normal K -variety, and let $\omega \in \Omega^{\text{top}}(X_{\text{sm}})$ be a rational top form on the smooth locus X_{sm} of X . The zeros and poles of ω give rise to a divisor $\text{div}(\omega)$ on X . Let $\pi: \tilde{X} \rightarrow X$ be a resolution of singularities, namely, a proper morphism from a smooth variety \tilde{X} that is an isomorphism over X_{sm} . Then $\pi^*\omega$ defines a unique rational top form on \tilde{X} . Moreover, when X

is nice enough (for example, if X is a local complete intersection), $\operatorname{div}(\omega)$ is \mathbb{Q} -Cartier, and we can define its pullback $\pi^* \operatorname{div}(\omega)$. The divisor $K_{\tilde{X}/X} := \operatorname{div}(\pi^*\omega) - \pi^* \operatorname{div}(\omega)$ on \tilde{X} is called the *relative canonical divisor*, and one can verify that it does not depend on the choice of ω . The divisor $K_{\tilde{X}/X}$ can be written as $K_{\tilde{X}/X} = \sum_{i=1}^M a_i E_i$, for some prime divisors E_i and $a_i \in \mathbb{Q}$. We say that X has *canonical singularities* (respectively, *log-canonical singularities*) if $a_i \geq 0$ (respectively, $a_i \geq -1$) for all $1 \leq i \leq M$. When X is a local complete intersection (for example a fiber of a flat morphism between smooth schemes), canonical singularities are equivalent to rational singularities. Let us give an example.

Example 1.11. Let $X \subseteq \mathbb{A}_{\mathbb{C}}^n$ be the variety defined by $\sum_{i=1}^n x_i^{d_i} = 0$, with $n \geq 3$. Then X has canonical singularities if and only if $\sum_{i=1}^n 1/d_i > 1$, and log-canonical singularities if and only if $\sum_{i=1}^n 1/d_i \geq 1$.

As seen from Example 1.11, log-canonical singularities are very close to being canonical, so one could suspect that being flat with fibers of log-canonical singularities is equivalent to $\epsilon_*(\varphi_{\mathbb{Q}_p}) = \infty$, that is, to being almost in L^∞ . Unfortunately, the normality hypothesis required for log-canonical singularities turns out to be too strong. For example, the map $\varphi(x, y) = xy$ satisfies $\epsilon_*(\varphi_{\mathbb{Q}_p}) = \infty$ for all p , but the fiber over 0 is not normal (see Section 6, after Corollary 6.2). This technical issue can be resolved by considering the slightly weaker notion of *semi-log-canonical singularities*, which is an analogue of log-canonical singularities for demi-normal schemes (see [KS88, Section 4]). Indeed, the variety $\{xy = 0\}$ is demi-normal and has semi-log-canonical singularities. We can now state the main result of this section.

THEOREM 1.12. *Let $\varphi: X \rightarrow Y$ be a map between smooth K -varieties. Then the following are equivalent:*

- (1) *The map φ is flat with fibers of semi-log-canonical singularities.*
- (2) *For every local field F containing K , we have $\epsilon_*(\varphi_F) = \infty$; that is, for every $\mu \in \mathcal{M}_c^\infty(X(F))$, the measure $\varphi_*\mu$ lies in $\mathcal{M}_{c,q}(Y(F))$ for all $1 < q < \infty$.*
- (3) *For every large enough prime p such that $\mathbb{Q}_p \supseteq K$, we have $\epsilon_*(\varphi_{\mathbb{Q}_p}) = \infty$.*
- (4) *We have $\epsilon_*(\varphi_{\mathbb{C}}) = \infty$.*

We prove Theorem 1.12 by showing the implications (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1) and (1) \Rightarrow (2) \Rightarrow (4) \Rightarrow (1). The implications (2) \Rightarrow (3) and (2) \Rightarrow (4) are immediate. In the proof of (3) \Rightarrow (1) and (4) \Rightarrow (1), we reduce to the case $Y = \mathbb{A}^n$ and show that $\varphi: X \rightarrow Y$ satisfies that $\varphi * \psi$ is (FRS) for every dominant map $\psi: Y' \rightarrow Y$. By analyzing the jets of φ and using a jet-scheme interpretation of semi-log-canonical singularities (Lemma 6.5), we deduce item (1). The proof also uses the Archimedean counterpart of [AA16, Theorem 3.4], which is stated in Corollary 6.2.

In the proof of (1) \Rightarrow (2), we reduce to showing item (2) for constructible measures and utilize their structure theory; namely, we use [CM11, CM12] for the Archimedean case, and [CGH18] for the non-Archimedean case.

1.3 Future directions and further applications

1.3.1 ϵ_* as an invariant of singularities. Similarly to the analytic definition of $\epsilon_*(\phi)$ in (1.2), one can also define an algebro-geometric invariant. Let Loc_0 be the collection of all non-Archimedean local fields F of characteristic 0.

DEFINITION 1.13. Let $\varphi: X \rightarrow Y$ be a morphism between smooth \mathbb{Q} -varieties. We define $\epsilon_\star(\varphi) := \min_{F \in \text{Loc}_0} \epsilon_\star(\varphi_F)$ and $\epsilon_\star(\varphi; x) = \sup_{U \ni x} \epsilon_\star(\varphi|_U)$, where U varies over all Zariski open neighborhoods of $x \in X(\overline{\mathbb{Q}})$.

It is a consequence of Theorem 1.1 that $\epsilon_\star(\varphi) > 0$ for any φ , and it essentially follows from [GH21] that $\epsilon_\star(\varphi) \in \mathbb{Q}$. We further expect $\epsilon_\star(\varphi; x)$ to have a purely algebro-geometric formula, and to have a good behavior in families, which means the following. Suppose that $\varphi: \tilde{X} \rightarrow \tilde{Y}$ is a morphism over $\mathbb{A}_{\mathbb{C}}^1$, where $\pi_1: \tilde{X} \rightarrow \mathbb{A}_{\mathbb{C}}^1$ and $\pi_2: \tilde{Y} \rightarrow \mathbb{A}_{\mathbb{C}}^1$ are smooth morphisms. This gives a family $\{\varphi_t: \tilde{X}_t \rightarrow \tilde{Y}_t\}_{t \in \mathbb{A}_{\mathbb{C}}^1}$ of morphisms between smooth varieties. It follows from [Var83] that the function $x \mapsto \text{lct}(\varphi_{\pi_1(x)}; x)$ is lower semicontinuous. By Theorem 1.2, the function $x \mapsto \epsilon_\star(\varphi_{\pi_1(x)}; x)$ is lower semicontinuous as well if $\pi_2: \tilde{Y} \rightarrow \mathbb{A}^1$ has fibers of dimension one. We expect $x \mapsto \epsilon_\star(\varphi_{\pi_1(x)}; x)$ to be lower semicontinuous in general. We further expect the following question to have a positive answer.

Question 1.14. Let $\varphi: X \rightarrow Y$ be as in Definition 1.13. Is it true that $\epsilon_\star(\varphi_{\mathbb{C}}) = \epsilon_\star(\varphi)$?

Given a \mathbb{Q} -morphism $\varphi: X \rightarrow Y$ as above, one may further wonder whether the quantity $\epsilon_\star(\varphi_{\mathbb{C}}; x) = \sup_{U \ni x} \inf_{\mu \in \mathcal{M}_{c, \infty}(U)} \epsilon_\star((\varphi_{\mathbb{C}})_* \mu)$ in (1.1) stays the same when the supremum is taken over all Zariski open neighborhoods U of x instead of analytic ones.

1.3.2 ϵ_\star as an invariant of words. As discussed in Section 1.1, one particularly interesting potential application is to the study of word maps on semisimple algebraic groups. In [GH24, Theorem A], it was shown that Lie algebra word maps $w_{\mathfrak{g}}: \mathfrak{g}^r \rightarrow \mathfrak{g}$, where \mathfrak{g} is a simple Lie algebra, become (FRS) after $O(\deg(w)^4)$ self-convolutions, where $\deg(w)$ is the degree of w .

Question 1.15. Can we find $\alpha, C > 0$ such that for any $w \in F_r$ of length $\ell(w)$ and every simple algebraic group G , the word map $w_G^{*C\ell(w)^\alpha}$ is (FRS)?

A potential way to tackle Question 1.15 is by studying $\epsilon_\star(w_G)$ in the sense of Definition 1.13. For each $w \in F_r$ (respectively, $w \in \mathcal{L}_r$), we define $\epsilon_\star(w) := \inf_G \epsilon_\star(w_G)$ (respectively, $\epsilon_\star(w) := \inf_{\mathfrak{g}} \epsilon_\star(w_{\mathfrak{g}})$), where G runs over all simple, simply connected algebraic groups and $\mathfrak{g} = \text{Lie}(G)$. We can now ask the following.

Question 1.16. Can we find, for each $l \in \mathbb{N}$, a constant $\epsilon(l) > 0$ such that the following hold:

- (1) For every $w \in F_r$ of length $\ell(w) = l$, we have $\epsilon_\star(w) \geq \epsilon(l)$?
- (2) For every $w \in \mathcal{L}_r$ of degree $\deg(w) = l$, we have $\epsilon_\star(w) \geq \epsilon(l)$?

1.3.3 ϵ_\star as an invariant of representations. In a recent work [GGH24] of the first two authors with Julia Gordon, we apply the results and point of view of this paper to the realm of representation theory, and we define and study a new invariant of representations of reductive groups G over local fields. Harish-Chandra's regularity theorem says that every character $\Theta(\pi)$ of an irreducible representation π of G is given by a locally L^1 -function. Since characters are of motivic nature, a variant of [GH21, Theorem F] suggests that they should in fact be locally in $L^{1+\epsilon}$ for some $\epsilon > 0$. This gives rise to a new invariant $\epsilon_\star(\pi)$, which is not equivalent to previously known invariants of representations, such as the Gelfand–Kirillov dimension (see for example [Vog78]). We use a geometric construction and Theorem 1.1 to provide a formula for $\epsilon_\star(\pi)$ in terms of the nilpotent orbits appearing in the local character expansion of π (see [GGH24, Theorems A and D]).

1.4 Conventions

- (1) By \mathbb{N} , we mean the set $\{0, 1, 2, \dots\}$.
- (2) We use K (respectively, F) to denote number (respectively, local) fields and \mathcal{O}_K (respectively, \mathcal{O}_F) for their rings of integers.
- (3) Given an algebraic map $\varphi: X \rightarrow Y$ between smooth K -varieties and a local field F , we denote by $\varphi_F: X(F) \rightarrow Y(F)$ the corresponding F -analytic map. We sometimes write φ instead of φ_F if the local field F is clear from the setting.
- (4) If $f, g: X \rightarrow \mathbb{R}$ are two functions, we write $f \lesssim g$ if there exists a number $C > 0$, possibly depending on X , f and g , such that $f \leq Cg$. We write $f \sim g$ if $g \lesssim f \lesssim g$.
- (5) We write $d_x\phi$ for the differential of an analytic map $\phi: X \rightarrow Y$ at $x \in X$, and we denote by $\text{Jac}_x\phi := \det(d_x\phi)$ the Jacobian of ϕ if $\dim X = \dim Y$. The generalization of the Jacobian to the case of unequal dimensions is defined in Section 1.2.
- (6) Throughout the paper, we write $|\cdot|_F$ for the absolute value on F , normalized so that $\mu_F(aA) = |a|_F \cdot \mu_F(A)$ for all $a \in F^\times$ and $A \subseteq F$, where μ_F is a Haar measure on F . Note that $|\cdot|_{\mathbb{C}}$ is the square of the usual absolute value on \mathbb{C} .
- (7) We write H_n for the n -dimensional Hausdorff measure. We recall that given a metric space X and a subset $Z \subseteq X$, we define $H_n(Z) := \sup_{\delta \rightarrow 0^+} H_n^\delta(Z)$, where

$$H_n^\delta(Z) = \inf \left\{ \sum_{i=1}^{\infty} \alpha(n) (\text{diam}(U_i))^n : \bigcup_{i=1}^{\infty} U_i \supseteq Z, \text{diam}(U_i) < \delta \right\},$$

where $\text{diam}(A)$ denotes the diameter of a set $A \subseteq X$ and $\alpha(n) := (\pi/4)^{n/2} / \Gamma(n/2 + 1)$. The normalization constant $\alpha(n)$ is chosen such that H_n coincides with the Lebesgue measure in the case $X = \mathbb{R}^n$.

- (8) All the measures we consider are non-negative unless stated otherwise.

2. Preliminaries: Embedded resolution of singularities

Let F be a local field of characteristic 0. We use the following analytic version of Hironaka's theorem [Hir64] on embedded resolution of singularities. The map $\pi: \tilde{X} \rightarrow U$ below is called a *log-principalization* (or *uniformization*) of J .

THEOREM 2.1 (see [VZ08, Theorem 2.3], [DvdD88, Theorem 2.2], [BM89] and [Wło09]). *Let $U \subseteq F^n$ be an open subset, and let $f_1, \dots, f_r: U \rightarrow F$ be F -analytic maps generating a non-zero ideal J in the algebra of F -analytic functions on U . Then there exist an F -analytic manifold \tilde{X} , a proper F -analytic map $\pi: \tilde{X} \rightarrow U$ and a collection of closed submanifolds $\{E_i\}_{i \in T}$ of \tilde{X} of codimension one, equipped with pairs of non-negative integers $\{(a_i, b_i)\}_{i \in T}$, such that the following hold:*

- (1) *The map π is locally a composition of a finite number of blow-ups at closed submanifolds and is an isomorphism over the complement of the common zero set $V(J)$ of J in U .*
- (2) *For every $c \in \tilde{X}$, there are local coordinates $(\tilde{x}_1, \dots, \tilde{x}_n)$ in a neighborhood $V \ni c$ such that each E_i containing c is given by the equation $\tilde{x}_i = 0$. Moreover, if without loss of generality E_1, \dots, E_m contain c , then there exists an F -analytic unit $v: V \rightarrow F$ such that the pullback of J is the principal ideal*

$$\pi^* J = \langle \tilde{x}_1^{a_1} \cdots \tilde{x}_m^{a_m} \rangle \tag{2.1}$$

and such that the Jacobian $\text{Jac}_{\tilde{x}}(\pi) = \det d_{\tilde{x}}\pi$ of π is given by

$$\text{Jac}_{\tilde{x}}(\pi) = v(\tilde{x}) \cdot \tilde{x}_1^{b_1} \cdots \tilde{x}_m^{b_m}. \quad (2.2)$$

Remark 2.2. (1) Condition (2.1) means that for each $f_i: U \rightarrow F$, one can write $f_i \circ \pi(\tilde{x}) = u_i(\tilde{x})\tilde{x}_1^{a_1} \cdots \tilde{x}_m^{a_m}$ for some analytic functions u_i , and $u_j(0) \neq 0$ for at least one $j \in \{1, \dots, r\}$. Note that a_1, \dots, a_m are the same for all the f_i .

(2) If $r = 1$, so that $J = \langle f \rangle$, we may further assume that $f \circ \pi(\tilde{x}_1, \dots, \tilde{x}_n) = C \cdot \tilde{x}_1^{a_1} \cdots \tilde{x}_m^{a_m}$ locally on each chart, for some constant $C \neq 0$. Indeed, $u(0) \neq 0$. If $u(0)$ is an a_1 th power in F , then the same holds for $u(\tilde{x})$ in a small neighborhood of 0. In this case, we may apply the change of coordinates

$$(\tilde{x}_1, \dots, \tilde{x}_n) \mapsto (u(\tilde{x})^{-\frac{1}{a_1}} \tilde{x}_1, \dots, \tilde{x}_n).$$

If $u(0)$ is not an a_1 th power, we may multiply it by a number $b \in F^\times$ such that $u(\tilde{x})b$ is an a_1 th power, and apply a similar change of coordinates.

The next lemma follows directly by changing coordinates using a log-principalization of J and computing the integral $\int \min_{1 \leq i \leq l} [|f_i(x)|^{-s}] d\mu(s)$ with respect to the new coordinates.

LEMMA 2.3 (cf. [Mus12, Theorem 1.2] and [VZ08, Theorem 2.7]). *Let $J = \langle f_1, \dots, f_r \rangle$ be an ideal of F -analytic functions on $U \subseteq F^n$. Let $\pi: \tilde{X} \rightarrow U$ be a log-principalization of J , with data $\{E_i\}_{i \in T}$ and $\{(a_i, b_i)\}_{i \in T}$ as in Theorem 2.1. Then the F -log-canonical threshold of J at $x \in U$ is equal to*

$$\text{lct}_F(J; x) = \min_{i: x \in \pi(E_i)} \frac{b_i + 1}{a_i}. \quad (2.3)$$

Remark 2.4. Note that while the data $\{E_i\}_{i \in T}$ and $\{(a_i, b_i)\}_{i \in T}$ depends on the log-principalization π , Definition 1.5 is intrinsic; thus the right-hand side of (2.3) does not depend on π .

3. The one-dimensional case

3.1 A formula for ϵ_\star

In this section, we provide a formula for ϵ_\star in the one-dimensional case (Theorem 1.2). The formula will be phrased in terms of the F -log-canonical threshold, where F is any local field of characteristic 0.

When F is non-Archimedean, we denote by \mathcal{O}_F its ring of integers, by k_F its residue field and by q_F the number of elements in k_F . Write $\varpi_F \in \mathcal{O}_F$ for a fixed uniformizer (that is, a generator of the maximal ideal of the ring of integers) of F , and let $|\cdot|_F$ be as in Section 1.4, so that $|\varpi_F|_F = q_F^{-1}$. We write μ_F for the Haar measure on F , normalized such that $\mu_F(\mathcal{O}_F) = 1$ when F is non-Archimedean, and such that μ_F is the Lebesgue measure when F is Archimedean. We write $\mu_{\mathcal{O}_F} := \mu_F|_{\mathcal{O}_F}$. We write dx instead of μ_F when we integrate a function $g(x)$ with respect to μ_F . We denote by $\|(a_1, \dots, a_n)\|_F := \max_i |a_i|_F$ the maximum norm on F^n .

For an analytic map $\phi: X \rightarrow F$ and $x \in X$, we set $\phi_x(z) := \phi(z) - \phi(x)$.

To prove Theorem 1.2, we reduce to the monomial case using Hironaka's resolution of singularities (Section 2) and prove the monomial case in Lemma 3.2. However, we first note that the upper bound in (1.7) can be proved by elementary arguments, as follows.

LEMMA 3.1. *Let X be an analytic F -manifold, and let $\phi: X \rightarrow F$ be a locally dominant analytic map. Then for every $x \in X$ with $\text{lct}_F(\phi_x; x) < 1$, one has*

$$\epsilon_\star(\phi; x) \leq \frac{\text{lct}_F(\phi_x; x)}{1 - \text{lct}_F(\phi_x; x)}.$$

Proof. We need to show

$$1 - \frac{1}{1 + \epsilon_\star(\phi; x)} \leq \text{lct}_F(\phi_x; x).$$

Let $\epsilon < \epsilon_\star(\phi; x)$. Then there exists a neighborhood U of x such that $\phi_\star\sigma \in L^{1+\epsilon}$ for every σ in $\mathcal{M}_{c,\infty}(U)$. Write $\phi_\star\sigma = g(t)\mu_F$. Let $B(a, \delta) := \{t : |t - a|_F \leq \delta\}$, and note that $\mu_F(B(a, \delta)) \sim \delta$. By Jensen's inequality, we have

$$(\phi_\star\sigma)(B(a, \delta)) = \frac{1}{\delta} \int_{B(a, \delta)} \delta g(t) dt \leq \left(\frac{1}{\delta} \int_{B(a, \delta)} (\delta g(t))^{1+\epsilon} dt \right)^{\frac{1}{1+\epsilon}} \lesssim \delta^{1 - \frac{1}{1+\epsilon}};$$

that is, we have the distributional estimate $\sigma(\{z : |\phi(z) - \phi(x)|_F \leq \delta\}) \lesssim \delta^{1-1/(1+\epsilon)}$. Using Fubini's theorem, we obtain

$$\begin{aligned} \int_U |\phi_x(z)|_F^{-s} d\sigma &= \int_U \left(\int_0^\infty 1_{\{(t,z): |\phi_x(z)|_F^{-s} \geq t\}}(t, z) dt \right) d\sigma \\ &= \int_0^\infty \sigma\{z : |\phi(z) - \phi(x)|_F^{-s} \geq t\} dt \\ &= \int_0^\infty \sigma\{z : |\phi(z) - \phi(x)|_F \leq t^{-\frac{1}{s}}\} dt \\ &\lesssim 1 + \int_1^\infty t^{-\frac{1}{s}(1 - \frac{1}{1+\epsilon})} dt < \infty \end{aligned}$$

whenever $s < 1 - 1/(1 + \epsilon)$. This implies that $\text{lct}_F(\phi_x; x) \geq 1 - 1/(1 + \epsilon_\star(\phi; x))$. \square

We now return to the main narrative. Specializing to the setting of Hironaka's theorem, we consider the monomial case.

LEMMA 3.2. *Let $f: F^n \rightarrow F$ be a monomial map $f(x_1, \dots, x_n) = x_1^{a_1} \cdots x_n^{a_n}$, let $g: F^n \rightarrow \mathbb{R}_{\geq 0}$ be a continuous function, and let $\mu = g(x)|x_1|_F^{b_1} \cdots |x_n|_F^{b_n} \mu_F^n$ for $a_1, \dots, a_n \in \mathbb{Z}_{\geq 1}$ and $b_1, \dots, b_n \in \mathbb{Z}_{\geq 0}$. Then*

$$\epsilon_\star(f_\star\mu) \begin{cases} = \infty & \text{if } \min_i \frac{b_i + 1}{a_i} \geq 1, \\ \geq \frac{\min_i (b_i + 1)/a_i}{1 - \min_i (b_i + 1)/a_i} & \text{otherwise.} \end{cases}$$

Furthermore, if $g(0) \neq 0$, the second bound is in fact an equality.

Proof. Without loss of generality, we may assume $\min_i (b_i + 1)/a_i = (b_n + 1)/a_n$. We first consider the special case where $(b_n + 1)/a_n$ is the unique minimum. We write f as a composition $f = \text{pr} \circ \psi$, where $\psi: F^n \rightarrow F^n$ is given by $\psi(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, x_1^{a_1} \cdots x_n^{a_n})$ and pr is the projection to the last coordinate. Write $x = (\bar{x}, x_n)$, where $\bar{x} = (x_1, \dots, x_{n-1})$, and similarly $y = (\bar{y}, y_n)$. Note that $\text{Jac}_{(x_1, \dots, x_n)}(\psi) = a_n x_1^{a_1} \cdots x_{n-1}^{a_{n-1}} x_n^{a_n-1}$ and that if $\psi(\bar{x}, x_n) = (\bar{y}, y_n)$, then $\bar{x} = \bar{y}$ and $x_n^{a_n} = y_n y_1^{-a_1} \cdots y_{n-1}^{-a_{n-1}}$. Hence the Radon–Nikodym density of $\psi_\star\mu$ is equal to

$$\frac{d(\psi_\star\mu)}{(\mu_F^n)}(\bar{y}, y_n) = \sum_{t: \psi(\bar{y}, t) = y} \frac{\left(\prod_{j=1}^{n-1} |y_j|_F^{b_j} \right) |t|_F^{b_n} g(\bar{y}, t)}{|\text{Jac}_{(y_1, \dots, y_{n-1}, t)}(\psi)|_F}$$

$$\begin{aligned}
 &= \sum_{t:\psi(\bar{y},t)=y} g(\bar{y},t) |a_n|_F^{-1} \cdot \left(\prod_{j=1}^{n-1} |y_j|_F^{b_j - a_j} \right) |t|_F^{b_n - a_n + 1} \\
 &= \left(\sum_{t:\psi(\bar{y},t)=y} g(\bar{y},t) \right) \cdot |a_n|_F^{-1} \left(\prod_{j=1}^{n-1} |y_j|_F^{b_j - a_j - \frac{(b_n - a_n + 1)a_j}{a_n}} \right) |y_n|_F^{\frac{b_n - a_n + 1}{a_n}}.
 \end{aligned}$$

Since $(b_n + 1)/a_n < (b_j + 1)/a_j$, for every $1 \leq j \leq n - 1$, we get

$$a_n(b_j - a_j + 1) > (b_n - a_n + 1)a_j,$$

and hence

$$b_j - a_j - \frac{(b_n - a_n + 1)a_j}{a_n} > -1.$$

In particular, integrating over the first $n - 1$ coordinates, we get $d(f_*\mu)/d\mu_F(s) \lesssim |s|_F^{(b_n - a_n + 1)/a_n}$. If $(b_n + 1)/a_n \geq 1$, then $\epsilon_*(f_*\mu) = \infty$ as required. Thus we may assume $(b_n + 1)/a_n < 1$. Then $f_*\mu \in \mathcal{M}_{c,1+\epsilon}(F)$ whenever $((b_n - a_n + 1)/a_n)(1 + \epsilon) > -1$, that is, whenever

$$\epsilon < \frac{b_n + 1}{a_n - b_n - 1} = \frac{(b_n + 1)/a_n}{1 - (b_n + 1)/a_n} = \frac{\min_i (b_i + 1)/a_i}{1 - \min_i (b_i + 1)/a_i}.$$

If $g(0) \neq 0$, we also have $d(f_*\mu)/d\mu_F(s) \gtrsim |s|_F^{(b_n - a_n + 1)/a_n}$, whence the inequality in the statement of the lemma is in fact an equality.

We are left to deal with the case where $\min_i (b_i + 1)/a_i$ is not uniquely achieved. For the lower bound, take

$$\tilde{\mu} = |x_1|_F^{b_1} \cdots |x_{n-1}|_F^{b_{n-1}} |x_n|_F^{b_n - \delta} g(x) \mu_F^n$$

for an arbitrarily small $\delta > 0$. Since we have $\tilde{\mu} \geq \mu$ inside a small neighborhood of 0, we deduce that $f_*\mu \in \mathcal{M}_{c,1+\epsilon}(F)$ for $\epsilon < ((b_n - \delta + 1)/a_n)/((1 - (b_n - \delta + 1)/a_n))$. Since δ can be taken arbitrarily small, we are done. Similarly, for the upper bound, we take

$$\tilde{\mu} = |x_1|_F^{b_1 + \delta} \cdots |x_{n-1}|_F^{b_{n-1} + \delta} |x_n|_F^{b_n} g(x) \mu_F^n \leq \mu$$

with $g(0) \neq 0$ and deduce that $\epsilon_*(f_*\mu) \leq ((b_n + 1)/a_n)/(1 - (b_n + 1)/a_n)$. \square

Proof of Theorem 1.2. Let $\phi: X \rightarrow F$ be a locally dominant analytic map, and let $x_0 \in X$. Replacing ϕ with $\phi_{x_0} = \phi - \phi(x_0)$, we may assume $\phi(x_0) = 0$. We may further assume that $X \subseteq F^n$ is open and apply Theorem 2.1 to get a log-principalization $\pi: \tilde{X} \rightarrow X$ such that, locally on a chart around a point in $\pi^{-1}(x_0)$, we have $\phi_{x_0} \circ \pi(\tilde{x}_1, \dots, \tilde{x}_n) = C\tilde{x}_1^{a_1} \cdots \tilde{x}_n^{a_n}$ and

$$\text{Jac}_{\tilde{x}}(\pi) = v(\tilde{x}) \cdot \tilde{x}_1^{b_1} \cdots \tilde{x}_n^{b_n}.$$

Let $\sigma \in \mathcal{M}_{c,\infty}(X)$ with $\sigma = g(x)\mu_F^n$ and $g > 0$ in a neighborhood of x_0 . Then $\sigma = \pi_*\mu$, where

$$\mu = (g \circ \pi)(\tilde{x}_1, \dots, \tilde{x}_n) |v(\tilde{x})|_F |\tilde{x}_1|_F^{b_1} \cdots |\tilde{x}_n|_F^{b_n} \mu_F^n.$$

Since $\phi_*\sigma = (\phi \circ \pi)_*\mu$, the theorem now follows from Lemmas 3.2 and 2.3. \square

3.2 Relation to Fourier decay and other invariants

Let $\phi: X \rightarrow Y$ be an F -analytic map between F -analytic manifolds X and Y . We have seen that each of the exponents $\epsilon_*(\phi; x)$ and $\text{let}_F(\phi; x)$ provides a different quantification for the singularities of ϕ near $x \in X$. When $Y = F^m$, one can further consider other invariants involving the Fourier transform of pushforward measures.

In Section 1.1, we have defined $k_*(\phi; x)$ as the minimal number of self-convolutions after which the pushforward densities of smooth measures supported near x become bounded. Note that by the Plancherel theorem, for each $\nu \in \mathcal{M}_{c,1}(F^m)$, we have $\nu^{*k} \in \mathcal{M}_{c,2}(F^m)$ if and only if $\mathcal{F}(\nu) \in L^{2k}(F^m)$, whence

$$\nu^{*k} \in \mathcal{M}_{c,\infty}(F^m) \implies \mathcal{F}(\nu) \in L^{2k}(F^m) \implies \nu^{*2k} \in \mathcal{M}_{c,\infty}(F^m).$$

Thus, the exponent $k_*(\phi; x)$ is, in general, roughly comparable to the L^p -class of $\mathcal{F}(\phi_*\mu)$ rather than to the L^p -class of the pushforward measure $\phi_*\mu$ itself. Instead of the L^p -class, we now focus on an invariant quantifying the Fourier decay of $\phi_*\mu$ on the power-law scale:

$$\delta_*(\phi; x) := \sup_{U \ni x} \inf_{\mu \in \mathcal{M}_{c,\infty}(U)} \delta_*(\phi_*\mu),$$

where

$$\delta_*(\nu) := \sup\{\delta \geq 0 : |\mathcal{F}(\nu)(y)| \lesssim \|y\|_F^{-\delta}\}. \quad (3.1)$$

The study of the invariant $\delta_*(\mu)$, and variations of it, goes back at least to the 1920s, when the classical van der Corput lemma was introduced, relating lower bounds on the derivative of a smooth function $f: \mathbb{R} \rightarrow \mathbb{R}$ to bounds as in (3.1); see [Ste93, Proposition 2] and [CCW99]. This invariant was also studied extensively in Igusa's work [Igu78] in the case $\dim Y = 1$; it is much less understood in high dimensions.

Remark 3.3. Note that $\epsilon_*(\phi; x)$ and $\text{lct}_F(\phi_x; x)$ are preserved under analytic changes of coordinates around x . On the other hand, $\delta_*(\phi; x)$ and $k_*(\phi; x)$ might depend on the choice of coordinate system. For example, the map $\phi(x, y) = (x, x^2(1 + y^{1000}))$ satisfies $\epsilon_*(\phi) = 1/999$ (by Theorem 1.1), while $\delta_*(\phi) = 1/2$ and $k_*(\phi) \leq 4$. Applying the change of coordinates $\psi(x, y) = (x, y - x^2)$, we get $\psi \circ \phi(x, y) = (x, x^2 y^{1000})$ and still have $\epsilon_*(\psi \circ \phi) = 1/999$, while $\delta_*(\psi \circ \phi) = 1/1000$ and $k_*(\psi \circ \phi) \geq 1000$.

We next discuss the relations between the different exponents. In the one-dimensional case, it turns out that all of the exponents above are essentially equivalent (whenever $\text{lct}_F(\phi; x) \leq 1$). In order to explain this, we need to discuss the structure of pushforward measures by analytic maps.

3.2.1 Asymptotic expansions of pushforward measures and their Fourier transform. Let F be a local field and $f: U \rightarrow F$ be a locally dominant analytic map with $U \subseteq F^n$ an open set. Let $\mu \in \mathcal{M}_c^\infty(U)$, and consider the pushforward measure $f_*\mu$. Fix a non-trivial additive character Ψ of F . We may identify F and F^\vee by $t \leftrightarrow \Psi_t$, where $\Psi_t(y) = \Psi(ty)$. The Fourier transform $\mathcal{F}(f_*\mu)$ can then be written as

$$\mathcal{F}(f_*\mu)(t) = \int_F \Psi(ty) df_*\mu(y) = \int_U \Psi(t \cdot f(x)) d\mu(x).$$

Given μ and f , one can further define *Igusa's local zeta function*

$$Z_{\mu,f}(s) := \int_U |f(x)|_F^s d\mu(x), \quad s \in \mathbb{C}, \text{Re}(s) > 0.$$

Igusa's local zeta function admits a meromorphic continuation to the complex plane (see [BG69, Ati70, Ber72] for the Archimedean case, and [Igu74, Igu78] for the non-Archimedean case). The poles of $Z_{\mu,f}(s)$ (as well as certain twisted versions of it), and the Laurent expansions around them, control the asymptotic expansions for $f_*\mu$ as $|y|_F \rightarrow 0$ and for $\mathcal{F}(f_*\mu)$ as $|t|_F \rightarrow \infty$, via the theory of Mellin transform (see [Igu78, Theorems 4.2, 4.3 and 5.3]). We next describe the asymptotic expansions of $f_*\mu$ and $\mathcal{F}(f_*\mu)$.

DEFINITION 3.4 (Asymptotic expansion, see [Igu78, Section I.2]). Let x_∞ be 0 or ∞ . A sequence $\{\varphi_k\}_{k \in \mathbb{N}}$ of complex-valued functions on an open subset U of $F \in \{\mathbb{R}, \mathbb{C}\}$, with $\varphi_k(x) \neq 0$ in a punctured neighborhood of x_∞ , is said to constitute an *asymptotic scale* if for every k ,

$$\lim_{x \rightarrow x_\infty} \frac{\varphi_{k+1}(x)}{\varphi_k(x)} = 0.$$

A function $f: U \rightarrow \mathbb{C}$ is said to have an *asymptotic expansion* at x_∞ if there exists a sequence $\{a_n\}_{n \in \mathbb{N}}$ of complex numbers such that for every $k \geq 0$, there exists a $C > 0$ such that for all x close enough to x_∞ , we have

$$\left| f(x) - \sum_{i=0}^k a_i \varphi_i(x) \right| \leq C \cdot \varphi_{k+1}(x).$$

In this case, we write

$$f(x) \approx \sum_{k=0}^{\infty} a_k \varphi_k(x) \quad \text{as } x \rightarrow x_\infty. \quad (3.2)$$

Example 3.5. Given a monotone increasing sequence $\Lambda = (-1 < \lambda_0 < \lambda_1 < \dots < \lambda_n < \dots)$ of real numbers, with no finite accumulation points, and given a sequence $\{m_n\}_{n \in \mathbb{N}}$ of positive integers, set $\varphi_0, \varphi_1, \varphi_2, \dots$ to be the sequence

$$x^{\lambda_0} \log(x)^{m_0-1}, x^{\lambda_0} \log(x)^{m_0-2}, \dots, x^{\lambda_0}, x^{\lambda_1} \log(x)^{m_1-1}, \dots, x^{\lambda_1}, \dots$$

for $x > 0$. Then $\{\varphi_k\}_{k \in \mathbb{N}}$ is an asymptotic scale at $x_\infty = 0$.

We now describe the asymptotic expansions of pushforward measures and their Fourier transforms. We fix a local field F of characteristic 0 and an analytic F -manifold X . If F is non-Archimedean, we further fix a uniformizer $\varpi_F \in \mathcal{O}_F$. We set $F_1^\times := \{x \in F : |x|_F = 1\}$ and denote by $\text{ac}: F^\times \rightarrow F_1^\times$ the angular component map

$$\text{ac}(t) = \begin{cases} t \varpi_F^{-\text{val}(t)} & \text{if } F \text{ is non-Archimedean,} \\ \frac{t}{|t|} & \text{if } F \text{ is Archimedean.} \end{cases}$$

THEOREM 3.6 ([Jea70, Mal74, Igu78], see also [VZ17, Section 4] and [Den91, Theorem 1.3.2 and Corollary 1.4.5]). *Let $f: X \rightarrow F$ be a locally dominant analytic map, let $\mu \in \mathcal{M}_c^\infty(X)$, and write $f_*\mu = g(y)d\mu_F$. Suppose that 0 is the only critical value of f . Then there exist*

- a sequence $\Lambda = \{\lambda_k\}_{k \geq 0}$, consisting of strictly increasing positive real numbers with $\lim_{k \rightarrow \infty} \lambda_k = \infty$ if $F \in \{\mathbb{R}, \mathbb{C}\}$, or a finite set of complex numbers λ_k with $\text{Re } \lambda_k > 0$ and $\text{Im } \lambda_k \in (2\pi/N \ln(q_F))\mathbb{N}$ for some $N \in \mathbb{N}$ if F is non-Archimedean;
- a sequence $\{m_k\}_{k \geq 0}$ of positive integers;
- smooth functions $a_{k,m,\mu}, b_{k,m,\mu}$ on F_1^\times

such that the following hold:

- (1) For $F \in \{\mathbb{R}, \mathbb{C}\}$, the function $g(y)$ admits an asymptotic expansion at 0 of the form

$$g(y) \approx \sum_{k \geq 0} \sum_{m=1}^{m_k} a_{k,m,\mu}(\text{ac}(y)) \cdot |y|_F^{\lambda_k-1} (\log |y|_F)^{m-1} \quad \text{as } |y|_F \rightarrow 0, \quad (3.3)$$

and $\mathcal{F}(f_*\mu)$ admits an asymptotic expansion of the form

$$\mathcal{F}(f_*\mu)(t) \approx \sum_{k \geq 0} \sum_{m=1}^{m_k} b_{k,m,\mu}(\text{ac}(t)) \cdot |t|_F^{-\lambda_k} (\log |t|_F)^{m-1} \quad \text{as } |t|_F \rightarrow \infty. \quad (3.4)$$

- (2) For F non-Archimedean, $g(y)$ and $\mathcal{F}(f_*\mu)$ admit expansions as in (3.3) and (3.4), where “ \approx ” is replaced with “=” and both sums are finite.
- (3) For each λ_k , the functions $(b_{k,m,\mu})_{m=1}^{m_k}$ are determined by the functions $(a_{k,m,\mu})_{m=1}^{m_k}$. If $\text{Re } \lambda_k < 1$, the map taking the latter to the former is one-to-one, and, moreover, the leading function $b_{k,m_k,\mu}$ is not identically zero (provided that m_k is defined so that $a_{k,m_k,\mu}$ is not identically 0).

Remark 3.7. (1) The asymptotic expansion in Theorem 3.6(1) is also *termwise differentiable*, and *uniform* in the angular component; we refer to [Igu78, Sections 3 and 4, pp. 19-24] for the precise meaning of those notions.

(2) The maps $(a_{k,m,\mu})_{m=1}^{m_k} \mapsto (b_{k,m,\mu})_{m=1}^{m_k}$ in Theorem 3.6(3) can be explicitly described; see [VZ17, Proposition 4.6] and [Igu78, Section 2.2].

(3) Note that Igusa’s theory (and in particular Theorem 3.6) was originally developed for polynomial maps but works for analytic maps as well. Indeed, the proof uses resolution of singularities to reduce to the case of pushforwards of measures with monomial density by monomial maps. The same reduction can be made for analytic maps via an analytic version of resolution of singularities (as stated in Theorem 2.1). For a generalization of Theorem 3.6 to the case of meromorphic maps, see [VZ17, Section 5].

We record the following immediate corollary of equation (3.3) in Theorem 3.6.

COROLLARY 3.8. *Let $f: X \rightarrow F$ be a locally dominant analytic map, let $\mu \in \mathcal{M}_c^\infty(X)$, and suppose $\epsilon_*(f_*\mu) < \infty$. Then $f_*\mu \notin \mathcal{M}_{c,1+\epsilon_*(f_*\mu)}(F)$; that is, the supremum in the definition of ϵ_* is not achieved.*

Theorem 3.6 also implies the following corollary, relating ϵ_* to δ_* .

COROLLARY 3.9. *Let $\phi: X \rightarrow F$ be a locally dominant analytic map such that 0 is the only critical value. Then for each $\mu \in \mathcal{M}_c^\infty(X)$, we have*

$$\epsilon_*(\phi_*\mu) = \begin{cases} \infty & \text{if } \delta_*(\phi_*\mu) \geq 1, \\ \frac{\delta_*(\phi_*\mu)}{1 - \delta_*(\phi_*\mu)} & \text{if } \delta_*(\phi_*\mu) < 1. \end{cases} \quad (3.5)$$

Proof. Write $\phi_*\mu = g(y)d\mu_F$. If $F = \mathbb{R}$, then by Theorem 3.6, the function $g(y)$ can be expanded near $y = 0$, so that the leading term is $|y|_F^{\lambda_{k_0}-1} (\log |y|_F)^{m_{k_0}-1}$ for some $\lambda_{k_0} \in \mathbb{R}_{>0}$ and $m_{k_0} \in \mathbb{N}$. Similarly, $\mathcal{F}(\phi_*\mu)$ can be expanded near $t = \infty$. If $\lambda_{k_0} < 1$, then by item (3) of Theorem 3.6, the leading term of $\mathcal{F}(\phi_*\mu)$ is $|y|_F^{-\lambda_{k_0}} (\log |y|_F)^{m_{k_0}-1}$, and thus $\delta_*(\phi_*\mu) = \lambda_{k_0}$. If $\lambda_{k_0} = 1$, then item (3) implies that $\delta_*(\phi_*\mu) \geq 1$.

Since 0 is the only critical value of ϕ , the function $g(y)$ is bounded outside any neighborhood of 0, so $g(y)^{1+\epsilon}$ is integrable if and only if it is integrable in a small ball B around 0, and this holds if and only if $(\lambda_{k_0} - 1)(1 + \epsilon) > -1$, that is, either $\lambda_{k_0} \geq 1$ and then $\epsilon_*(\phi_*\mu) = \infty$, or $\epsilon < \lambda_{k_0}/(1 - \lambda_{k_0}) = \delta_*(\phi_*\mu)/(1 - \delta_*(\phi_*\mu))$.

The case where F is non-Archimedean should be done with care since there might be multiple terms in (3.3) with the same real part, and some cancellations may occur (see Example 3.10

below). Let $\lambda := \min_k \operatorname{Re} \lambda_k$ and suppose $\lambda < 1$. Then by Theorem 3.6, the function $g(y)$ can be written as $g(y) = g_{\text{lead}}(y) + g_{\text{error}}(y)$, where

$$g_{\text{lead}}(y) := \sum_m \sum_{j=0}^{N-1} \tilde{a}_{j,m,\mu}(\operatorname{ac}(y)) \cdot |y|_F^{\lambda + \frac{2\pi ij}{N \ln(q_F)} - 1} (\log |y|_F)^{m-1},$$

$\tilde{a}_{j,m,\mu}(\operatorname{ac}(y))$ is equal to $a_{k_j,m,\mu}(\operatorname{ac}(y))$ if $\lambda_{k_j} = \lambda + 2\pi ij/(N \ln(q_F)) \in \Lambda$ and 0 otherwise and, furthermore,

$$g_{\text{error}}(y) := \sum_{k: \operatorname{Re} \lambda_k > \lambda} \sum_{m=1}^{m_k} a_{k,m,\mu}(\operatorname{ac}(y)) \cdot |y|_F^{\lambda_k - 1} (\log |y|_F)^{m-1}.$$

We first show that there is an arithmetic progression $I_{a,N,l} := \{a + bN\}_{l < b \in \mathbb{N}}$, for some $a, l \in \mathbb{N}$ and $y_0 \in F_1^\times$, such that

$$|g_{\text{lead}}(y)| > C_F |y|_F^{\lambda-1} \quad (3.6)$$

for all $y \in S_{a,N,l,d,y_0} := \{z \in \mathcal{O}_F : \operatorname{val}(\operatorname{ac}(z) - y_0) \geq d, \operatorname{val}(z) \in I_{a,N,l}\}$ for some constant C_F independent of y . It is enough to show (3.6) for the terms in $g_{\text{lead}}(y)$ where $m = m_0$ is maximal such that there is a $0 \leq j \leq N-1$ such that $\tilde{a}_{j,m_0,\mu}(\operatorname{ac}(y))$ is not identically zero. Note that

$$|y|_F^{\frac{2\pi ij}{N \ln(q_F)}} = q_F^{-\operatorname{val}(y) \frac{2\pi ij}{N \ln(q_F)}} = e^{-\ln(q_F) \operatorname{val}(y) \frac{2\pi ij}{N \ln(q_F)}} = e^{-\operatorname{val}(y) \frac{2\pi ij}{N}}.$$

Let $y_0 \in F_1^\times$ be such that $\tilde{a}_{j,m_0,\mu}(y_0) \neq 0$ for some $0 \leq j \leq N-1$. Choose $d \in \mathbb{N}$ such that each of $\tilde{a}_{0,m_0,\mu}(z), \dots, \tilde{a}_{N-1,m_0,\mu}(z)$ is constant on the ball $\operatorname{val}(z - y_0) \geq d$. Note that the functions $f_0, \dots, f_{N-1}: \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}$, $f_j(t) = e^{-t(2\pi ij)/N}$ are the irreducible characters of $\mathbb{Z}/N\mathbb{Z}$ and hence are linearly independent. In particular, there exists an $a \in \mathbb{N}$ such that

$$\left| \sum_{j=0}^{N-1} \tilde{a}_{j,m_0,\mu}(y_0) e^{-t \frac{2\pi ij}{N}} \right| = C_F > 0$$

for all $t \in I_{a,N,0}$. Taking $y \in S_{a,N,l,d,y_0}$ for $l \gg 1$, we deduce (3.6) as required.

Finally, by Theorem 3.6(3) and by an argument similar to the one above, it follows that $\delta_\star(\phi_*\mu) = \lambda$. Hence, (3.6) implies that $\int_{S_{a,N,l,d,y_0}} g(y)^{1+\epsilon} d\mu_F$ diverges for every $\epsilon \geq \lambda/(1-\lambda) = \delta_\star(\phi_*\mu)/(1-\delta_\star(\phi_*\mu))$. On the other hand, a similar argument as in the case $F = \mathbb{R}$ shows that $\int_F g(y)^{1+\epsilon} d\mu_F$ converges for $\epsilon < \lambda/(1-\lambda)$, so the corollary follows. \square

In the proof of Corollary 3.9, we have seen that there might be some cancellations between the terms in (3.3) with the same real part, and that these cancellations are insignificant for infinitely many values of $\operatorname{val}(y)$. Here is a simple example that illustrates this phenomenon.

Example 3.10. Let $d \in \mathbb{N}$ and let $p > d$ be a prime. Let $\phi: \mathbb{Q}_p \rightarrow \mathbb{Q}_p$ be the map $\phi(x) = x^d$. Write $\phi_*\mu_{\mathbb{Z}_p} = g(y)\mu_{\mathbb{Z}_p}$. Then for almost all $y \in \mathbb{Z}_p$, we have

$$g(y) = \#\phi^{-1}(y) \cdot |y|_p^{-1+\frac{1}{d}} = a(\operatorname{ac}(y)) 1_{\{z \in \mathbb{Z}_p: d|\operatorname{val}(z)\}}(y) \cdot |y|_p^{-1+\frac{1}{d}},$$

where $a(z) := \#\{x \in \mathbb{Z}_p : x^d = z\}$. Note that by Schur orthogonality, $\sum_{j=0}^{d-1} e^{-t(2\pi ij)/d}$ equals d if $d|t$ and equals 0 if $d \nmid t \in \mathbb{N}$. In particular,

$$1_{\{z \in \mathbb{Z}_p: d|\operatorname{val}(z)\}}(y) = \frac{1}{d} \sum_{j=0}^{d-1} e^{-\operatorname{val}(y) \frac{2\pi ij}{d}} = \frac{1}{d} \sum_{j=0}^{d-1} |y|_p^{\frac{2\pi ij}{d \ln(q_F)}},$$

and therefore the expansion of $g(y)$ as in (3.3) is

$$g(y) = \sum_{j=0}^{d-1} \frac{a(\text{ac}(y))}{d} |y|_p^{-1 + \frac{1}{d} + \frac{2\pi i j}{d \ln(q_F)}}.$$

3.2.2 Relations between the invariants. We now show that $\epsilon_*(\phi; x)$, $\delta_*(\phi; x)$ and $k_*(\phi; x)$ are essentially determined by the log-canonical threshold $\text{lct}_F(\phi_x; x)$ whenever $\text{lct}_F(\phi; x) \leq 1$. In particular, we show that

$$\epsilon_*(\phi; x), \delta_*(\phi; x), \frac{1}{k_*(\phi; x)} \quad \text{and} \quad \text{lct}_F(\phi_x; x)$$

are asymptotically equivalent as $\text{lct}_F(\phi_x; x) \rightarrow 0$.

Theorem 1.2 already shows that $\epsilon_*(\phi; x) = \text{lct}_F(\phi_x; x)/(1 - \text{lct}_F(\phi_x; x))$ if $\text{lct}_F(\phi_x; x) < 1$. We further have the following.

PROPOSITION 3.11. *Let $\phi: X \rightarrow F$ be a dominant F -analytic map. Then the following hold:*

(1) *If $\text{lct}_F(\phi_x; x) < 1$, then $\delta_*(\phi; x) = \text{lct}_F(\phi_x; x)$. In particular,*

$$\text{lct}_F(\phi_x * \phi_y; (x, y)) \begin{cases} = \text{lct}_F(\phi_x; x) + \text{lct}_F(\phi_y; y) & \text{if } \text{lct}_F(\phi_x; x) + \text{lct}_F(\phi_y; y) < 1, \\ \geq 1 & \text{otherwise.} \end{cases}$$

(2) *We have*

$$\left\lceil \frac{1}{\text{lct}_F(\phi_x; x)} \right\rceil \leq k_*(\phi; x) \leq \left\lfloor \frac{1}{\text{lct}_F(\phi_x; x)} \right\rfloor + 1.$$

Proof. Let us first prove item (1). Let $x \in X$. Replacing ϕ with ϕ_x , we may assume $\phi(x) = 0$. We may choose an open neighborhood U such that 0 is the only critical value of $\phi|_U$, that \bar{U} is compact and that for each $\mu \in \mathcal{M}_c^\infty(U)$ and each $s < \text{lct}_F(\phi; x)$, one has $\int |\phi(x)|_F^{-s} d\mu(s) < \infty$. Taking any μ that does not vanish at x , by Theorem 1.2 and Lemma 3.2, we get

$$\epsilon_*(\phi_*\mu) = \frac{\text{lct}_F(\phi_x; x)}{1 - \text{lct}_F(\phi_x; x)} = \epsilon_*(\phi; x) < \infty.$$

Corollary 3.9 implies that

$$\delta_*(\phi_*\mu) = \frac{\epsilon_*(\phi_*\mu)}{1 + \epsilon_*(\phi_*\mu)} = \frac{\epsilon_*(\phi; x)}{1 + \epsilon_*(\phi; x)} = \text{lct}_F(\phi_x; x).$$

Since the equalities above hold for μ of arbitrarily small support around x , we get $\delta_*(\phi; x) = \delta_*(\phi_*\mu) = \text{lct}_F(\phi_x; x)$, as required. Since Fourier transform translates convolution into product, we have

$$\delta_*(\phi * \phi; (x, y)) = \delta_*(\phi; x) + \delta_*(\phi; y),$$

which implies the second part of item (1) (see also [Den91, Section 5.1]).

We now turn to item (2). Set $k_0 := \lceil 1/\text{lct}_F(\phi_x; x) \rceil$. By item (1), for a positive integer $k < k_0$, we get that $\text{lct}_F(\phi_x^{*k}; (x, \dots, x)) < 1$, so that $\epsilon_*(\phi_x^{*k}; (x, \dots, x)) < \infty$. This implies the lower bound $k_*(\phi; x) \geq k_0$. The upper bound follows from (1.3) and Theorem 1.2. \square

Remark 3.12. Item (1) of Proposition 3.11 can be seen as an F -analytic interpretation of a theorem by Thom–Sebastiani [ST71] (see for example [MSS20] for the case $F = \mathbb{C}$).

The combination of Corollaries 3.8 and 3.9 implies Theorem 1.3, as follows.

Proof of Theorem 1.3. Let $\mu_j \in \mathcal{M}_c^\infty(F^{n_j})$, and suppose that

$$\nu_1 * \nu_2 \in \mathcal{M}_{1+\epsilon}(F), \quad \epsilon > 0, \quad (3.7)$$

where $\nu_j = (\phi_j)_* \mu_j$. Assume toward a contradiction that

$$\frac{\epsilon_*(\nu_1)}{1 + \epsilon_*(\nu_1)} + \frac{\epsilon_*(\nu_2)}{1 + \epsilon_*(\nu_2)} \leq \frac{\epsilon}{1 + \epsilon}.$$

For each $j \in \{1, 2\}$, choose a finite cover $\bigcup_i U_{i,j}$ of $\text{Supp}(\mu_j)$ by open balls in F^{n_j} such that, for each i , the restriction $\phi_j|_{U_{i,j}}$ has at most one critical value $z_{i,j}$ of ϕ_j . We can write $\mu_j = \sum_i \mu_{i,j}$ with $\mu_{i,j}$ supported inside $U_{i,j}$. Taking $\phi_{i,j} = \phi_j|_{U_{i,j}} - z_{i,j}$, we can find i_1 and i_2 such that

$$\epsilon_*(\nu_j) = \epsilon_*((\phi_{i_j,j})_*(\mu_{i_j,j})), \quad j = 1, 2.$$

Now, Corollary 3.9 implies that

$$\delta_*((\phi_{i_j,j})_*(\mu_{i_j,j})) = \frac{\epsilon_*((\phi_{i_j,j})_*(\mu_{i_j,j}))}{1 + \epsilon_*((\phi_{i_j,j})_*(\mu_{i_j,j}))},$$

whence

$$\delta_*((\phi_{i_1,1})_*(\mu_{i_1,1}) * (\phi_{i_2,2})_*(\mu_{i_2,2})) \leq \frac{\epsilon}{1 + \epsilon}.$$

Using Corollary 3.9 once again, we obtain that

$$\epsilon_*(\nu_1 * \nu_2) \leq \epsilon_*((\phi_{i_1,1})_*(\mu_{i_1,1}) * (\phi_{i_2,2})_*(\mu_{i_2,2})) \leq \epsilon.$$

In view of Corollary 3.8, this contradicts (3.7). \square

3.3 Consistency of the various bounds in the one-dimensional case

The following proposition, which can be seen as a variant of the Łojasiewicz gradient inequality (see for example [Loj65, Section 18, Proposition 1], [BM88, Proposition 6.8]), shows that the formula for $\epsilon_*(\phi; x)$ given in Theorem 1.2 in the one-dimensional case is consistent with the lower and upper bounds in Theorems 1.1 and 1.5. The proof of Proposition 3.13 is similar to that of [Fee19, Theorem 1], but applied to any local field.

Note that for $\phi: F^n \rightarrow F$, we have $\mathcal{J}_\phi = \langle \partial\phi/\partial x_1, \dots, \partial\phi/\partial x_n \rangle$, which we denote by $\langle \nabla\phi \rangle$.

PROPOSITION 3.13. *Let $\phi: F^n \rightarrow F$ be an analytic map. Then for every $x \in F^n$, we have*

$$\frac{\text{lct}_F(\langle \nabla\phi \rangle; x)}{\max(0, 1 - \text{lct}_F(\langle \nabla\phi \rangle; x))} \geq \frac{\text{lct}_F(\phi_x; x)}{\max(0, 1 - \text{lct}_F(\phi_x; x))} \geq \text{lct}_F(\langle \nabla\phi \rangle; x) \geq \text{lct}_F(\phi_x; x).$$

Proof. The middle inequality follows from Theorems 1.1 and 1.2, and the left inequality follows from the right inequality, so it is left to show that $\text{lct}_F(\langle \nabla\phi \rangle; x_0) \geq \text{lct}_F(\phi_{x_0}; x_0)$ for a fixed $x_0 \in F^n$.

Recall that $\|(a_1, \dots, a_n)\|_F := \max_i |a_i|_F$ is the maximum norm on F^n . We would like to relate $\int_B \|\nabla\phi\|_F^{-s} dx$ and $\int_B |\phi_{x_0}|_F^{-s} dx$, where B is a small ball around x_0 , for $s > 0$ small enough. Let $\pi: \tilde{X} \rightarrow F^n$ be a resolution of singularities of ϕ_{x_0} . Then we have

$$\int_B \|\nabla\phi\|_F^{-s} dx = \int_{\pi^{-1}(B)} \|\nabla\phi|_{\pi(\tilde{x})}\|_F^{-s} |\text{Jac}_{\tilde{x}}(\pi)|_F d\tilde{x}. \quad (3.8)$$

Since $\pi^{-1}(B)$ is compact, by working locally over finitely many pieces, and using Theorem 2.1, we may replace $\pi^{-1}(B)$ with a compact neighborhood $L \subseteq F^n$ of 0, assume that $\pi(0) = x_0$, and further assume that for $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)$, we have

$$\tilde{\phi}(\tilde{x}) := \phi \circ \pi(\tilde{x}) = C\tilde{x}_1^{a_1} \dots \tilde{x}_n^{a_n} \quad \text{and} \quad \text{Jac}_{\tilde{x}}(\pi) = v(\tilde{x})\tilde{x}_1^{b_1} \dots \tilde{x}_n^{b_n} \quad (3.9)$$

for some analytic unit $v(\tilde{x})$ and a constant $C \in F$. Note that on L , all of the entries of $d_{\tilde{x}}\pi$ are smaller, in absolute value, than some constant \tilde{C} , so that the operator norm $\|d_{\tilde{x}}\pi\|_{\text{op}} := \sup_{v \in F^n: \|v\|_F=1} \|d_{\tilde{x}}\pi \cdot v\|_F$ of $d_{\tilde{x}}\pi$ is bounded by a constant. Since $\nabla\tilde{\phi}|_{\tilde{x}} = \nabla\phi|_{\pi(\tilde{x})} \cdot d_{\tilde{x}}\pi$, for $s > 0$, we get

$$\begin{aligned} \int_L \|\nabla\phi|_{\pi(\tilde{x})}\|_F^{-s} |\text{Jac}_{\tilde{x}}(\pi)|_F d\tilde{x} &\leq \int_L \|d_{\tilde{x}}\pi\|_{\text{op}}^s \|\nabla\tilde{\phi}|_{\tilde{x}}\|_F^{-s} |\text{Jac}_{\tilde{x}}(\pi)|_F d\tilde{x} \\ &\lesssim \int_L \|\nabla\tilde{\phi}|_{\tilde{x}}\|_F^{-s} |\text{Jac}_{\tilde{x}}(\pi)|_F d\tilde{x} \\ &\lesssim \int_L (\max_j |\tilde{x}_j|_F^{a_1} \cdots |\tilde{x}_j|_F^{a_{j-1}} \cdots |\tilde{x}_n|_F^{a_n})^{-s} |\text{Jac}_{\tilde{x}}(\pi)|_F d\tilde{x} \\ &\lesssim \int_L (|\tilde{x}_1|_F^{a_1} \cdots |\tilde{x}_n|_F^{a_n})^{-s} |\text{Jac}_{\tilde{x}}(\pi)|_F d\tilde{x} \lesssim \int_{\pi(L)} |\phi_{x_0}|_F^{-s} dx, \end{aligned}$$

which concludes the proof. \square

Remark 3.14. Here is an alternative approach to Proposition 3.13, as suggested by the referee. For simplicity suppose $F = \mathbb{C}$. We may assume that $x = 0 \in \mathbb{C}^n$. Recall from [HS06, Definition 1.1.1] that the integral closure \bar{I} of an ideal I in a commutative ring R is the set of elements $r \in R$ for which there are $a_j \in I^j$ such that $\sum_{j=0}^n a_j r^{n-j} = 0$ for some $n \in \mathbb{N}$. We now take $R = \mathbb{C}\{x_1, \dots, x_n\}$ to be the ring of convergent power series, $I := \langle \nabla\phi \rangle$ and $J := \langle \phi_0 \rangle$. By [HS06, Corollary 7.1.4], we have $J \subseteq \bar{I}$ and in particular $\text{lct}_{\mathbb{C}}(J; 0) \leq \text{lct}_{\mathbb{C}}(\bar{I}; 0)$ (see [Mus12, Property 1.15]). However, by [Mus12, Property 1.12] (see also [dFM09, Proposition 2.4]), we have $\text{lct}_{\mathbb{C}}(I; 0) = \text{lct}_{\mathbb{C}}(\bar{I}; 0)$, which implies that $\text{lct}_{\mathbb{C}}(J; 0) \leq \text{lct}_{\mathbb{C}}(I; 0)$, as required. This argument likely generalizes to any local field F of characteristic 0, that is, the full generality of Proposition 3.13.

3.4 Some examples in higher dimension

We next discuss the higher-dimensional case (that is, $\dim Y > 1$). Here we will see that the connection between the four invariants $\epsilon_*(\phi; x)$, $\delta_*(\phi; x)$, $k_*(\phi; x)$ and $\text{lct}_F(\phi_x; x)$ is not as tight as in the one-dimensional case. We first provide a simpler description of $\delta_*(\phi; x)$.

LEMMA 3.15. *Let F be a local field of characteristic 0 and $\phi: F^n \rightarrow F^m$ be an analytic map. Then*

$$\delta_*(\phi; x) = \inf_{\ell} \{\delta_*(\ell \circ \phi; x)\},$$

where ℓ runs over all non-zero linear functionals $\ell: F^m \rightarrow F$.

Proof. For each $\mu \in \mathcal{M}_c^\infty(F^n)$, we have

$$|\mathcal{F}(\phi_*\mu)(z)| \lesssim \|z\|_F^{-\delta} \iff |\mathcal{F}(\phi_*\mu)(ta_1, \dots, ta_m)| \lesssim |t|_F^{-\delta}$$

for each $a = (a_1, \dots, a_m)$ with $\|a\|_F = 1$ and $t \in F$. Setting $\ell_a(y_1, \dots, y_m) = \sum_{i=1}^m a_i y_i$, we have

$$\mathcal{F}(\phi_*\mu)(ta_1, \dots, ta_m) = \int_{F^m} \Psi(t\ell_a(y)) \cdot d\phi_*\mu = \int_F \Psi(tz) \cdot d(\ell_a \circ \phi)_*\mu = \mathcal{F}((\ell_a \circ \phi)_*\mu)(t),$$

which concludes the proof of the lemma. \square

The following examples demonstrate that the connection between the four invariants can get loose as the dimension of Y grows.

Example 3.16. Consider the map $\phi: \mathbb{C}^m \rightarrow \mathbb{C}^m$ defined by $\phi(x_1, \dots, x_m) = (x_1^d, x_1^d x_2, \dots, x_1^d x_m)$. Then

- (1) $\epsilon_\star(\phi) = 1/(dm - 1)$;
- (2) $\text{lct}_{\mathbb{C}}(J) = 1/d$, where $J = \langle x_1^d, x_1^d x_2, \dots, x_1^d x_m \rangle = \langle x_1^d \rangle$;
- (3) $dm \leq k_\star(\phi) \leq dm + 1$;
- (4) $\delta_\star(\phi) = 1/d$.

Item (1) follows from Theorem 1.1. Young’s inequality (see (1.3)) shows that $k_\star(\phi) \leq dm + 1$; on the other hand, we have $\text{lct}_{\mathbb{C}}(J) = 1/d$, which implies that at least dm convolutions are needed to obtain the (FRS) property (see [GH24, Lemmas 3.23 and 3.26]); hence $k_\star(\phi) \geq dm$. Note that for any $a_1, \dots, a_m \in \mathbb{C}$ with $\|a\|_{\mathbb{C}} = 1$, we have $\text{lct}_{\mathbb{C}}(a_1 x_1^d + a_2 x_1^d x_2 + \dots + a_m x_1^d x_m) = 1/d$, so $\delta_\star(\phi) = 1/d$, by Lemma 3.15.

Remark 3.17. One can replace ϕ in Example 3.16 with

$$\phi(x_1, \dots, x_m) = (x_1, x_1^d x_2, \dots, x_1^d x_m).$$

Here, we get $\epsilon_\star(\phi) = 1/(dm - d)$, which is similar to Example 3.16 when m is large. On the other hand, the log-canonical threshold of $J = \langle x_1, x_1^d x_2, \dots, x_1^d x_m \rangle = \langle x_1 \rangle$ is $\text{lct}_{\mathbb{C}}(J) = 1$, which is much larger than in Example 3.16.

The following example shows that a reverse Young inequality (Theorem 1.3) does not hold in dimension $m \geq 2$.

Example 3.18. Consider $\phi(x, y) = (x, x^2(1 + y^{1000}))$. Then $\epsilon_\star(\phi) = 1/999$, while $\delta_\star(\phi) = 1/2$, and consequently $k_\star(\phi) \leq 4$.

4. Lower bound: Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We start with the equidimensional case, which we restate as Proposition 4.1 below.

Recall that given a locally dominant analytic map $\phi: X \rightarrow Y$ between two F -analytic manifolds, the Jacobian ideal sheaf \mathcal{J}_ϕ is defined locally as the ideal in the algebra of analytic functions on X generated by the $m \times m$ minors of the differential $d_x(\phi)$ of ϕ .

PROPOSITION 4.1. *Let X and Y be F -analytic manifolds, $\dim X = \dim Y = n$, and let $\phi: X \rightarrow Y$ be a locally dominant analytic map. Then for any $x_0 \in X$,*

$$\epsilon_\star(\phi; x_0) = \text{lct}_F(\mathcal{J}_\phi; x_0). \tag{4.1}$$

Proof. We may assume that X and Y are compact balls in F^n and $x_0 = 0$. Since $\dim X = \dim Y = n$, we have $\mathcal{J}_\phi = \langle \text{Jac}_x(\phi) \rangle$, where $\text{Jac}_x(\phi) = \det(d_x \phi)$. Since ϕ is analytic and X is compact, there exist an open dense set $U \subseteq X$ and an $M \in \mathbb{N}$ such that $\#\{\phi^{-1}\phi(x)\} \leq M$ and $\text{Jac}_x(\phi) \neq 0$ for every $x \in U$. We choose a disjoint cover $\bigcup_{i \in \mathbb{N}} U_i$ of U by locally closed subsets¹ U_i such that $\phi|_{U_i}$ is a diffeomorphism. Write $\mu := \mu_F^n|_X$ and $\mu_i := \mu_F^n|_{U_i}$, and note that $\phi_*\mu = g(y) \cdot \mu_F^n$ and $\phi_*\mu_i = g_i(y) \cdot \mu_F^n$, where

$$g(y) = \sum_{x \in \phi^{-1}(y)} |\text{Jac}_x(\phi)|_F^{-1} \quad \text{and} \quad g_i(y) = |\text{Jac}_{\phi^{-1}(y)}(\phi)|_F^{-1}.$$

¹Recall that a set is locally closed if it is the difference of two open sets.

We have

$$\int_Y g(y)^{1+s} dy = \int_{\phi(U)} g(y)^s d\phi_*\mu = \int_U g \circ \phi(x)^s d\mu \geq \int_U \frac{1}{|\text{Jac}_x(\phi)|_F^s} d\mu. \quad (4.2)$$

On the other hand, since $\#\{i \in I : y \in \phi(U_i)\} \leq M$ for each $y \in Y$, using Jensen's inequality, we have

$$\begin{aligned} \int_Y g(y)^{1+s} dy &\leq \int_Y \left(\sum_{i \in I: y \in \phi(U_i)} g_i(y) \right)^{1+s} dy \leq M^s \int_Y \sum_{i \in I: y \in \phi(U_i)} g_i(y)^{1+s} dy \\ &= M^s \sum_{i \in I} \int_Y g_i(y)^{1+s} dy = M^s \sum_{i \in I} \int_{U_i} \frac{1}{|\text{Jac}_x(\phi)|_F^s} d\mu = M^s \int_U \frac{1}{|\text{Jac}_x(\phi)|_F^s} d\mu, \end{aligned} \quad (4.3)$$

which implies the proposition. \square

Proof of Theorem 1.1. Let $\phi: X \rightarrow Y$ be a locally dominant analytic map, and let $x_0 \in X$. Since the claim is local, we may assume that $X \subseteq F^n$ is an open subset and $Y = F^m$ with $n \geq m$. Let B be a small ball around x_0 . For each subset $I \subseteq \{1, \dots, n\}$ of size m , let M_I be the corresponding $m \times m$ minor of $d_x\phi$, and set

$$V_I := \left\{ x \in B : \max_{I' \subseteq \{1, \dots, n\}} |M_{I'}(x)|_F = |M_I(x)|_F \right\}.$$

If $V_I \neq \emptyset$, the map $\phi_I(x) := (\phi(x), x_{j_1}, \dots, x_{j_{n-m}})$ is locally dominant, where $\text{Jac}_x(\phi_I) = M_I(x)$ and $\{j_1, \dots, j_{n-m}\} = \{1, \dots, n\} \setminus I$. For each V_I of positive measure, let $\mu_B := \mu_F^n|_B$ and $\mu_I := \mu_F^n|_{B \cap V_I}$, and write

$$\phi_*\mu_B = g(y) \cdot \mu_F^m, \quad \phi_*\mu_I = g_I(y) \cdot \mu_F^m \quad \text{and} \quad (\phi_I)_*\mu_I = \tilde{g}_I(z) \cdot \mu_F^n.$$

Let \tilde{B} be a large ball in F^{n-m} that contains the projection of $\phi_I(B \cap V_I)$ from F^n to the last $n-m$ coordinates F^{n-m} . Since $\phi = \pi_I \circ \phi_I$, where $\pi_I: F^n \rightarrow F^m$ is a projection to the first m coordinates, we have

$$g_I(y) = \int_{F^{n-m}} \tilde{g}_I(y, z_{m+1}, \dots, z_n) dz_{m+1} \cdots dz_n = \int_{\tilde{B}} \tilde{g}_I(y, z_{m+1}, \dots, z_n) dz_{m+1} \cdots dz_n.$$

By Jensen's inequality, we have

$$\begin{aligned} \int_{F^m} g_I(y)^{1+s} dy &= \int_{F^m} \mu_F^{n-m}(\tilde{B})^{1+s} \left(\frac{1}{\mu_F^{n-m}(\tilde{B})} \int_{\tilde{B}} \tilde{g}_I(y, z_{m+1}, \dots, z_n) dz_{m+1} \cdots dz_n \right)^{1+s} dy \\ &\leq \mu_F^{n-m}(\tilde{B})^s \cdot \int_{F^m \times \tilde{B}} \tilde{g}_I(y, z_{m+1}, \dots, z_n)^{1+s} dy dz_{m+1} \cdots dz_n \lesssim \int_{F^n} \tilde{g}_I(z)^{1+s} dz \end{aligned}$$

for every $s > 0$. Since $\bigcup_I V_I$ is of full measure in B , by Proposition 4.1 and (1.5), we have

$$\begin{aligned} \int_{F^m} g(y)^{1+s} dy &\lesssim \sum_I \int_{F^m} g_I(y)^{1+s} dy \lesssim \sum_I \int_{F^n} \tilde{g}_I(z)^{1+s} dz \\ &\lesssim \sum_I \int_{V_I} |M_I(x)|_F^{-s} dx \lesssim \int_B \min_{I \subseteq \{1, \dots, n\}} [|M_I(x)|_F^{-s}] dx < \infty \end{aligned}$$

for every $s < \text{lct}_F(\mathcal{J}_\phi; x_0)$, as required. \square

5. Proof of Theorem 1.5: An upper bound over \mathbb{C}

In this section, we prove Theorem 1.5. We use the following easy consequence of the coarea formula (see [Fed69]).

LEMMA 5.1. *Let $B \subseteq \mathbb{C}^n$ be a compact ball, and let $\phi: B \rightarrow \mathbb{C}^m$ be a dominant analytic map. Let $\mu_B := \mu_{\mathbb{C}}^n|_B$, and write $\phi_*\mu_B = g(y) \cdot \mu_{\mathbb{C}}^m$. Then*

$$g(y) = \int_{\phi^{-1}(y)} \frac{dH_{2(n-m)}(x)}{\det((d_x\phi)^*(d_x\phi))},$$

where the integral is taken with respect to the $2(n-m)$ -dimensional Hausdorff measure (recall Section 1.4(7)).

Let $\phi: X \rightarrow Y$ be a locally dominant analytic map between complex analytic manifolds, and let $x_0 \in X$. Since the claim is local, we may assume that $Y = \mathbb{C}^m$ and $X = B$ is a ball in \mathbb{C}^n . To show (1.12), it is enough to bound $\epsilon_*(\phi_*\mu_B)$, where B is of arbitrarily small radius around x_0 . Set $G(x) := \det((d_x\phi)^*(d_x\phi))$. By Lemma 5.1, we have

$$g(y) = \int_{\phi^{-1}(y)} \frac{dH_{2(m-n)}(x)}{G(x)},$$

whence

$$\int_Y g(y)^{1+\epsilon} dy = \int_X \left[\int_{\phi^{-1}(\phi(x))} \frac{dH_{2(m-n)}(x')}{G(x')} \right]^\epsilon dx. \quad (5.1)$$

We apply Theorem 2.1 to the ideal \mathcal{J}_ϕ . Let $\pi: \tilde{X} \rightarrow B$ be the corresponding resolution. Without loss of generality, we can assume that $\tilde{X} \subseteq \mathbb{C}^m$ is an open subset, so that

$$\det d_{\tilde{x}}\pi = v(\tilde{x})\tilde{x}_1^{b_1} \cdots \tilde{x}_n^{b_n}, \quad v(0) \neq 0,$$

and each of the $m \times m$ minors M_I of $d_{\tilde{x}}\phi$ satisfies

$$M_I(\pi(\tilde{x})) = u_I(\tilde{x})\tilde{x}_1^{a_1} \cdots \tilde{x}_n^{a_n},$$

where at least one of the functions u_I does not vanish at 0.

We first perform the change of variables $x = \pi(\tilde{x})$ in the external integral in (5.1), yielding

$$\int_Y g(y)^{1+\epsilon} dy = \int_{\tilde{X}} |\det d_{\tilde{x}}\pi|_{\mathbb{C}} \left[\int_{\phi^{-1}(\phi(\pi(\tilde{x})))} \frac{dH_{2(m-n)}(x')}{G(x')} \right]^\epsilon d\tilde{x},$$

and then perform the change of variables $x' = \pi(t)$ in the internal integral. Since the map π is continuously differentiable and B is compact, the product of the singular values of the restriction of $d_t\pi$ to any subspace is bounded from below in absolute value by a number times the absolute value of the determinant of $d_t\pi$. Thus,

$$\int_Y g(y)^{1+\epsilon} dy \gtrsim \int_{\tilde{X}} |\det d_{\tilde{x}}\pi|_{\mathbb{C}} \left[\int_{\tilde{\phi}^{-1}(\tilde{\phi}(\tilde{x}))} \frac{|\det d_t\pi|_{\mathbb{C}} dH_{2(m-n)}(t)}{G(\pi(t))} \right]^\epsilon d\tilde{x}, \quad (5.2)$$

where $\tilde{\phi} := \phi \circ \pi$. By the Cauchy–Binet formula, the right-hand side is equal to

$$\int_{\tilde{X}} |\det d_{\tilde{x}}\pi|_{\mathbb{C}} \left[\int_{\tilde{\phi}^{-1}(\tilde{\phi}(\tilde{x}))} \frac{|v(t)|_{\mathbb{C}} |t_1|_{\mathbb{C}}^{b_1} \cdots |t_n|_{\mathbb{C}}^{b_n} dH_{2(m-n)}(t)}{\sum_I |u_I(t)|_{\mathbb{C}} |t_1|_{\mathbb{C}}^{a_1} \cdots |t_n|_{\mathbb{C}}^{a_n}} \right]^\epsilon d\tilde{x}.$$

We bound the internal integral from below as follows. For each $\tilde{x} \in \tilde{X}$, set

$$Q(\tilde{x}) := \left\{ t \in \tilde{X} : \sum_{j=1}^n \frac{|t_j - \tilde{x}_j|_{\mathbb{C}}}{|\tilde{x}_j|_{\mathbb{C}}} \leq \frac{1}{4} \right\}.$$

Then

$$\begin{aligned} & \int_{\tilde{\phi}^{-1}(\tilde{\phi}(\tilde{x}))} \frac{|v(t)|_{\mathbb{C}} |t_1|_{\mathbb{C}}^{b_1} \cdots |t_n|_{\mathbb{C}}^{b_n} dH_{2(m-n)}(t)}{\sum_I |u_I(t)|_{\mathbb{C}} |t_1|_{\mathbb{C}}^{a_1} \cdots |t_n|_{\mathbb{C}}^{a_n}} \\ & \gtrsim \int_{\tilde{\phi}^{-1}(\tilde{\phi}(\tilde{x})) \cap Q(\tilde{x})} |t_1|_{\mathbb{C}}^{b_1 - a_1} \cdots |t_n|_{\mathbb{C}}^{b_n - a_n} dH_{2(m-n)}(t) \\ & \gtrsim |\tilde{x}_1|_{\mathbb{C}}^{b_1 - a_1} \cdots |\tilde{x}_n|_{\mathbb{C}}^{b_n - a_n} H_{2(n-m)}(\tilde{\phi}^{-1}(\tilde{\phi}(\tilde{x})) \cap Q(\tilde{x})). \end{aligned} \quad (5.3)$$

CLAIM 5.2. For any $\tilde{x} \in \tilde{X}$, we have

$$H_{2(n-m)}(\tilde{\phi}^{-1}(\tilde{\phi}(\tilde{x})) \cap Q(\tilde{x})) \gtrsim \min_{|I|=n-m} \prod_{i \in I} |\tilde{x}_i|_{\mathbb{C}}.$$

Proof of Claim 5.2. Let $T: \tilde{X} \rightarrow \tilde{X}$ be the affine map given by $T(\tilde{x}')_j = \tilde{x}_j(1 + \tilde{x}'_j)$. Then $Q(\tilde{x}) = TB_{1/2}$, where $B_{1/2}$ is a ball of radius $1/2$ centered at the origin. By a theorem of Lelong [Lel57] (see also [Thi67, LG86]), for any analytic set M of pure dimension $n - m$ and any $r > 0$, one has

$$\frac{H_{2(n-m)}(M \cap B_r)}{H_{2(n-m)}(M_0 \cap B_r)} \geq \lim_{\rho \rightarrow +0} \frac{H_{2(n-m)}(M \cap B_\rho)}{H_{2(n-m)}(M_0 \cap B_\rho)},$$

where M_0 is a linear subspace of the same dimension $n - m$. The limit on the right-hand side is the *Lelong number* of M , which is the algebraic multiplicity of M at \tilde{x} . It is strictly positive; thus

$$H_{2(n-m)}(M \cap B_r) \gtrsim r^{2(n-m)}.$$

Applying this estimate to $M = T^{-1}(\tilde{\phi}^{-1}(\tilde{\phi}(\tilde{x})))$ and observing that

$$H_{2(n-m)}(TA) \geq \min_{|I|=n-m} \prod_{i \in I} |\tilde{x}_i|_{\mathbb{C}} \cdot H_{2(n-m)}(A)$$

for any Borel set $A \subset \tilde{X}$, we obtain the claimed assertion. \square

We now proceed with the proof of the theorem. Using Claim 5.2, we deduce

$$\int_{\tilde{\phi}^{-1}(\tilde{\phi}(\tilde{x}))} \frac{|\det d_t \pi|_{\mathbb{C}} dH_{2(m-n)}(t)}{G(\pi(t))} \gtrsim \prod_{i=1}^n |\tilde{x}_i|_{\mathbb{C}}^{b_i - a_i + 1},$$

whence

$$\int_Y g(y)^{1+\epsilon} dy \gtrsim \int_{\tilde{X}} \prod_{i=1}^n |\tilde{x}_i|_{\mathbb{C}}^{b_i + \epsilon(b_i - a_i + 1)} d\tilde{x}.$$

This integral diverges whenever there is an index i such that $b_i + \epsilon(b_i - a_i + 1) \leq -1$, that is,

$$\epsilon_{\star}(\phi; x) \leq \min_i \left\{ \frac{b_i + 1}{a_i - b_i - 1} : b_i + 1 < a_i \right\}.$$

On the other hand, we have

$$\text{lct}_{\mathbb{C}}(\mathcal{J}_{\phi}; x_0) = \min_i \left\{ \frac{b_i + 1}{a_i} \right\},$$

and under the assumption that this quantity is strictly less than 1, we have an index i such that

$$\text{lct}_{\mathbb{C}}(\mathcal{J}_{\phi}; x_0) = \frac{b_i + 1}{a_i} < 1.$$

For this index,

$$\frac{b_i + 1}{a_i - b_i - 1} = \frac{\text{lct}_{\mathbb{C}}(\mathcal{J}_{\phi}; x_0)}{1 - \text{lct}_{\mathbb{C}}(\mathcal{J}_{\phi}; x_0)}.$$

This concludes the proof of Theorem 1.5.

6. Geometric characterization of $\epsilon_{\star} = \infty$

From now on, let K be a number field, and let \mathcal{O}_K be its ring of integers. Let Loc_0 be the collection of all non-Archimedean local fields F that contain K . We use the notation $\text{Loc}_{0, \gg}$ for the collection of $F \in \text{Loc}_0$ with large enough residual characteristic, depending on some given data. Throughout this section, we denote by $B(y, r)$ a ball of radius r centered at $y \in Y(F)$.

In this section, we focus on algebraic morphisms $\varphi: X \rightarrow Y$ between algebraic K -varieties. We would like to characterize morphisms φ where $\epsilon_{\star}(\varphi_F) = \infty$ for certain collections of local fields F , in terms of the singularities of φ . In order to effectively do this, it is necessary to consider an ‘‘algebraically closed’’ collection of local fields, such as the following:

- $\{\mathbb{C}\}$,
- $\{F\}_{F \in \text{Loc}_{0, \gg}}$ or $\{F\}_{F \in \text{Loc}_0}$.

Aizenbud and Avni have shown the following characterization of the (FRS) property.

THEOREM 6.1 ([AA16, Theorem 3.4]). *Let $\varphi: X \rightarrow Y$ be a map between smooth K -varieties. Then φ is (FRS) if and only if for each $F \in \text{Loc}_0$ and every $\mu \in \mathcal{M}_{\mathbb{C}}^{\infty}(X(F))$, one has $(\varphi_F)_{\star}(\mu) \in \mathcal{M}_{c, \infty}(Y(F))$.*

The Archimedean counterpart of this theorem was studied in [Rei18], where it was shown that given an (FRS) morphism φ , the maps $\varphi_{\mathbb{R}}$ and $\varphi_{\mathbb{C}}$ are L^{∞} -morphisms. For the other direction, the non-Archimedean proof of [AA16, Theorem 3.4] (see [AA16, Section 3.7]) can be easily adapted to the complex case, with less complications due to the fact that \mathbb{C} is algebraically closed. We arrive at the following characterization of L^{∞} -morphisms.

COROLLARY 6.2. *Let $\varphi: X \rightarrow Y$ be a map between smooth K -varieties. Then the following are equivalent:*

- (1) *The map φ is (FRS).*
- (2) *For every local field F containing K , the map φ_F is an L^{∞} -morphism.*
- (3) *For each $F \in \text{Loc}_{0, \gg}$, the map φ_F is an L^{∞} -map.*
- (4) *The map $\varphi_{\mathbb{C}}$ is an L^{∞} -map.*

Our goal is to characterize the weaker property that $\epsilon_{\star}(\varphi_F) = \infty$ over $F = \mathbb{C}$, or over all $F \in \text{Loc}_0$ (Theorem 1.12, restated below as Theorem 6.6). Let us first present an example

showing that the (FRS) condition is too strong for this purpose. Let $\varphi: \mathbb{A}_{\mathbb{Q}}^2 \rightarrow \mathbb{A}_{\mathbb{Q}}^1$ be the map $\varphi(x, y) = xy$. Then

$$\begin{aligned} \varphi_{\mathbb{Q}_p}^{-1}(B(0, p^{-k})) \cap \mathbb{Z}_p^2 &= \{(x, y) \in \mathbb{Z}_p^2 : \text{val}(x) + \text{val}(y) \geq k\} \\ &= \bigcup_{r \geq k, 0 \leq l \leq r} \{(x, y) \in \mathbb{Z}_p^2 : \text{val}(x) = l, \text{val}(y) = r - l\}. \end{aligned}$$

In particular, we have

$$\begin{aligned} \frac{\varphi_{\mathbb{Q}_p*}(\mu_{\mathbb{Z}_p}^2)(B(0, p^{-k}))}{\mu_{\mathcal{O}_F}(B(0, p^{-k}))} &= p^k \cdot \sum_{r=k}^{\infty} \sum_{l=0}^r \left(\frac{p-1}{p}\right)^2 p^{-l} p^{-(r-l)} \\ &= p^k \cdot \left(\frac{p-1}{p}\right)^2 \sum_{r=k}^{\infty} (r+1) p^{-r} \geq \left(\frac{p-1}{p}\right)^2 (k+1). \end{aligned}$$

Therefore, the measure $\varphi_{\mathbb{Q}_p*}(\mu_{\mathbb{Z}_p}^2)$ does not have bounded density. On the other hand, since $\text{lct}_{\mathbb{Q}_p}(\varphi_{\mathbb{Q}_p}; 0) = 1$, by considering the asymptotic expansion of $\varphi_{\mathbb{Q}_p*}(\mu_{\mathbb{Z}_p}^2)$ as in Theorem 3.6, one sees that

- (1) $\varphi_{\mathbb{Q}_p*}(\mu_{\mathbb{Z}_p}^2)$ explodes logarithmically around 0; that is, the density of $\varphi_{\mathbb{Q}_p*}(\mu_{\mathbb{Z}_p}^2)$ behaves like $\text{val}(t)$ around 0;
- (2) $\epsilon_{\star}(\varphi_{\mathbb{Q}_p}) = \infty$ for every prime p .

By Corollary 6.2, the map φ cannot be (FRS), and indeed $\{xy = 0\}$ is not normal, so in particular it does not have rational singularities.

In order to prove Theorem 1.12, we recall the notion of jet schemes. Let $X \subseteq \mathbb{A}_K^n$ be an affine K -scheme whose coordinate ring is

$$K[x_1, \dots, x_n]/(f_1, \dots, f_k).$$

Then the m th jet scheme $J_m(X)$ of X is the affine scheme with the following coordinate ring:

$$K[x_1, \dots, x_n, x_1^{(1)}, \dots, x_n^{(1)}, \dots, x_1^{(m)}, \dots, x_n^{(m)}]/(\{f_j^{(u)}\}_{j=1, u=1}^{k, m}),$$

where $f_i^{(u)}$ is the u th formal derivative of f_i .

Let $\varphi: \mathbb{A}^{n_1} \rightarrow \mathbb{A}^{n_2}$ be a morphism between affine spaces. Then the m th jet morphism $J_m(\varphi): \mathbb{A}^{n_1(m+1)} \rightarrow \mathbb{A}^{n_2(m+1)}$ of φ is given by formally deriving φ ; explicitly, we have $J_m(\varphi) = (\varphi, \varphi^{(1)}, \dots, \varphi^{(m)})$. Similarly, the m th jet $J_m(\varphi)$ of a morphism $\varphi: X \rightarrow Y$ of affine K -schemes is given by the formal derivative of φ . Both $J_m(X)$ and $J_m(\varphi)$ can be generalized to arbitrary K -schemes and K -morphisms (see [CNS18, Chapter 3] and [EM09] for more details).

Given a subscheme $Z \subseteq X$ of a smooth variety X , with Z defined by an ideal J , we denote by $\text{lct}(X, Z) := \text{lct}(J)$ the log-canonical threshold of the pair (X, Z) . Mustařă showed that the log-canonical threshold $\text{lct}(X, Z)$ can be characterized in terms of the growth rate of the dimensions of the jet schemes of Z .

THEOREM 6.3 ([Mus02, Corollary 0.2], [CNS18, Corollary 7.2.4.2]). *Let X be a smooth, geometrically irreducible K -variety, and let $Z \subsetneq X$ be a closed subscheme. Then*

$$\text{lct}(X, Z) = \dim X - \sup_{m \geq 0} \frac{\dim J_m(Z)}{m+1}.$$

Furthermore, the supremum is achieved for m divisible enough. In particular, $\text{lct}(X, Z)$ is a rational number.

Note that $\text{lct}(X, Z)$ depends on Z and $\dim X$, and neither on the ambient space X nor on the embedding of Z in X .

We now introduce the following definitions from [GH24]. For a morphism $\varphi: X \rightarrow Y$ between schemes, we denote by $X_{y,\varphi}$ the scheme-theoretic fiber of φ over $y \in Y$.

DEFINITION 6.4. Let $\varphi: X \rightarrow Y$ be a morphism of smooth, geometrically irreducible K -varieties, and let $\epsilon > 0$. Then

- (1) φ is called ϵ -flat if for every $x \in X$, we have $\dim X_{\varphi(x),\varphi} \leq \dim X - \epsilon \dim Y$;
- (2) φ is called ϵ -jet flat if $J_m(\varphi): J_m(X) \rightarrow J_m(Y)$ is ϵ -flat for every $m \in \mathbb{N}$;
- (3) φ is called jet-flat if it is 1-jet flat.

In particular, note that φ is flat if and only if it is 1-flat.

Note that by Theorem 6.3, a morphism φ is ϵ -jet-flat if and only if $\text{lct}(X, X_{\varphi(x),\varphi}) \geq \epsilon \dim Y$ for all $x \in X$. We will need the following lemma to give a jet scheme interpretation to rational and semi-log-canonical singularities (from Theorem 6.3).

LEMMA 6.5. Let $\varphi: X \rightarrow Y$ be a morphism of smooth K -varieties. Then

- (1) φ is (FRS) if and only if $J_m(\varphi)$ is flat with locally integral fibers for every $m \geq 0$ (in particular, φ is jet-flat);
- (2) φ is jet-flat if and only if φ is flat with fibers of semi-log-canonical singularities.

Proof. Item (1) is proved in [GH24, Corollary 3.12] and essentially follows from a characterization of rational singularities by Musta \check{a} [Mus01]. For the proof of item (2), note that by [GH24, Corollary 2.7], the morphism φ is jet-flat if and only if $J_m(\varphi)$ is flat over $Y \subseteq J_m(Y)$. Since a fiber of a morphism between smooth varieties is flat if and only if its fibers are local complete intersections, the latter condition is equivalent to the condition that for every $x \in X$ and every $m \in \mathbb{N}$, the scheme $J_m(X_{\varphi(x),\varphi})$ is a local complete intersection. By [Ish18, Corollary 10.2.9] and [EI15, Corollary 3.17], this is equivalent to the condition that $X_{\varphi(x),\varphi}$ has semi-log-canonical singularities for every $x \in X$. \square

6.1 Proof of Theorem 1.12

We are now in a position to prove Theorem 1.12. Let us recall its formulation, slightly restated using Lemma 6.5.

THEOREM 6.6. Let $\varphi: X \rightarrow Y$ be a map between smooth K -varieties. Then the following are equivalent:

- (1) The map φ is jet-flat.
- (2) For every local field F containing K , we have $\epsilon_*(\varphi_F) = \infty$.
- (3) For every $F \in \text{Loc}_{0,\gg}$, we have $\epsilon_*(\varphi_F) = \infty$.
- (4) We have $\epsilon_*(\varphi_{\mathbb{C}}) = \infty$.

The proof of Theorem 6.6 is done by showing the implications (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1) and (1) \Rightarrow (2) \Rightarrow (4) \Rightarrow (1). The implications (2) \Rightarrow (3) and (2) \Rightarrow (4) are immediate. We first prove (3) \Rightarrow (1) and (4) \Rightarrow (1). Then we will prove (1) \Rightarrow (2) in the non-Archimedean case in Section 6.2, and in the Archimedean case in Section 6.3.

PROPOSITION 6.7. *Let $\varphi: X \rightarrow Y$ be a map between smooth K -varieties. Assume that either $\epsilon_*(\varphi_{\mathbb{C}}) = \infty$ or $\epsilon_*(\varphi_F) = \infty$ for all $F \in \text{Loc}_{0, \gg}$. Then φ is jet-flat.*

Proof. Working locally, and composing φ with an étale map $\Phi: Y \rightarrow \mathbb{A}^m$, we may assume that $Y = \mathbb{A}^m$. Let $\psi: \mathbb{A}^m \rightarrow \mathbb{A}^m$ be any dominant morphism, and let $\mu_1 \in \mathcal{M}_c^\infty(X(F))$ and $\mu_2 \in \mathcal{M}_c^\infty(F^m)$. By Theorem 1.1, for all $F \in \text{Loc}_{\gg} \cup \{\mathbb{C}\}$, we have $\psi_*(\mu_2) \in L^{1+\epsilon}$ for some $\epsilon > 0$. Taking q large enough, by Young's convolution inequality, one has

$$(\varphi * \psi)_*(\mu_1 \times \mu_2) = \varphi_*(\mu_1) * \psi_*(\mu_2) \subseteq L^q * L^{1+\epsilon} \subseteq L^\infty,$$

where $\varphi * \psi$ is as in Definition 1.7. By Corollary 6.2, the convolution $\varphi * \psi$ is (FRS). We now claim that since φ is a morphism whose convolution with any dominant morphism produces an (FRS) morphism, φ must be jet-flat.

Indeed, assume that it is not the case. Then by Theorem 6.3 and Definition 6.4, there exist $y \in \overline{K}^m$ and $N > 0$ such that the scheme-theoretic fiber $X_{y, \varphi}$ of φ over y satisfies

$$\text{lct}(X, X_{y, \varphi}) = \dim X - \sup_{k \geq 0} \frac{\dim J_k X_{y, \varphi}}{k+1} \leq m \left(1 - \frac{2}{N}\right).$$

Moreover, this supremum is achieved for k divisible enough. Thus the map

$$J_k(\varphi): J_k(X) \longrightarrow J_k(Y)$$

is not $(1 - 1/N)$ -flat for k divisible enough. On the other hand, the map

$$\psi_N(y_1, \dots, y_m) = (y_1^{2N}, \dots, y_m^{2N})$$

satisfies that $J_l(\psi_N)$ is not $1/N$ -flat for divisible enough l (since $\text{lct}(y_i^{2N}) = 1/2N$). Thus we may find a $k_0 \in \mathbb{N}$ such that $J_{k_0}(\varphi)$ is not $(1 - 1/N)$ -flat and $J_{k_0}(\psi_N)$ is not $1/N$ -flat. But then $J_{k_0}(\varphi * \psi) = J_{k_0}(\varphi) * J_{k_0}(\psi)$ is not flat (see [GH24, Lemma 3.26]), which gives a contradiction by Lemma 6.5, as $\varphi * \psi$ is (FRS). \square

6.2 Proof of (1) \Rightarrow (2) in the non-Archimedean case

We first prove the following variant of [CGH23, Theorem 4.12].

PROPOSITION 6.8 ([CGH23, Theorem 4.12]). *Let $\varphi: X \rightarrow Y$ be a jet-flat map between smooth K -varieties. Then there exists an $M \in \mathbb{N}$ such that for each $F \in \text{Loc}_0$, each $\mu \in \mathcal{M}_c^\infty(X(F))$ and each non-vanishing $\tau \in \mathcal{M}^\infty(Y(F))$, one can find a $C_{F, \mu, \tau} > 0$ such that for each $y \in Y(F)$ and $k \in \mathbb{N}$, one has*

$$G_{F, \mu}(y, k) := \frac{(\varphi_* \mu)(B(y, q_F^{-k}))}{\tau(B(y, q_F^{-k}))} \leq C_{F, \mu, \tau} k^M.$$

Proof. We may assume that $Y = \mathbb{A}_K^m$ and that $\tau = \mu_F^m$. We may further assume that X is affine and thus embeds in \mathbb{A}_K^n . Let $\tilde{\mu}$ be the canonical measure on $X(F)$ (see [Ser81, Section 3.3] and also [CCL12, Section 1.2]). It is enough to consider measures $\mu \in \mathcal{M}_c^\infty(X(F))$ that are of the form $\mu_l := \tilde{\mu}|_{B(0, q_F^l) \cap X(F)}$.

Write $g_F(l, y)$ for the density of $\varphi_* \mu_l$ with respect to μ_F^m , and set $G_F(y, l, k) := G_{F, \mu_l}(y, k)$. Then the collections $\{g_F(l, y)\}_{F \in \text{Loc}_0}$ and $\{G_F(y, l, k)\}_{F \in \text{Loc}_0}$ are both motivic functions, in the sense of [CGH18, Section 1.2]. By [CGH18, Theorem 2.1.3], there exists a motivic function $H(l, k) = \{H_F: \mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{R}\}_{F \in \text{Loc}_0}$ that approximates the supremum of $G(y, l, k)$; that is,

$$\frac{1}{C_F} H_F(l, k) \leq \sup_{y \in Y(F)} G_F(y, l, k) \leq H_F(l, k) \tag{6.1}$$

for all $F \in \text{Loc}_0$ and $(l, k) \in \mathbb{Z} \times \mathbb{N}$, where C_F depends only on the local field.²

Since H is motivic, using [CGH18, Proposition 1.4.2], for each $F \in \text{Loc}_0$, we may divide $\mathbb{Z} \times \mathbb{N}$ into a finite disjoint union $\mathbb{Z} \times \mathbb{N} = \bigsqcup_{A_F \in \mathcal{A}} A_F$, with $|\mathcal{A}| < \infty$ independent of F , such that on each part $A_F \subseteq \mathbb{Z} \times \mathbb{N}$, the following hold. There exist finitely many $a_i \in \mathbb{N}$ and $b_i \in \mathbb{Q}$ independent of F , a finite set Λ_F (of size depending on F) and a finite partition of A_F into subsets $A_F = \bigsqcup_{\xi \in \Lambda_F} A_{F,\xi}$, such that for all $(k, l) \in A_{F,\xi}$,

$$H_F(l, k) = \sum_{i=1}^L c_i(\xi, l, F) \cdot k^{a_i} q_F^{b_i k}$$

for some constants $c_i(\xi, l, F)$ depending on l, F and ξ . Moreover, for fixed $F \in \text{Loc}_0$ and $l \in \mathbb{N}$, the set $A_{F,\xi,l} := \{k \in \mathbb{N} : (k, l) \in A_{F,\xi}\}$ is either finite or a fixed congruence class modulo some $e \in \mathbb{Z}_{\geq 1}$.

To prove the proposition, it is enough to show that for each $F \in \text{Loc}_0$ and $l \in \mathbb{Z}$, we have $H_F(l, k) \leq C_{F,l} k^M$ on each $A_{F,\xi,l}$ for some constant $C_{F,l}$ depending on F and l . It is enough to prove this for $A_{F,\xi,l}$ infinite, as otherwise we have

$$H_F(l, k) \leq C_{F,l} := \sum_{k \in A_{F,\xi,l}} \sum_{i=1}^L |c_i(\xi, l, F)| \cdot k^{a_i} q_F^{b_i k}.$$

Now suppose that $A_{F,\xi,l}$ is infinite. By rearranging the constants $c_i(\xi, l, F)$, we may assume that the pairs (a_i, b_i) are disjoint and that $(b_i, a_i) > (b_{i+1}, a_{i+1})$ in lexicographic order, that is, either $b_i > b_{i+1}$ or $b_i = b_{i+1}$ and $a_i > a_{i+1}$. Note that if $b_1 \leq 0$, then we are done since for each $F \in \text{Loc}_0$ and each $k \in A_{F,\xi,l}$, we have

$$\sup_{y \in Y(F)} G_F(y, l, k) \leq H_F(l, k) \leq \left(\sum_{i=1}^L |c_i(\xi, l, F)| \right) k^M, \quad (6.2)$$

where $M = \max\{a_i\}$. Assume toward a contradiction that $b_1 > 0$ and $c_1(\xi, l, F) \neq 0$. Then for all large enough $k \in A_{F,\xi,l}$, one has

$$\sup_{y \in Y(F)} G_F(y, l, k) \geq C_F^{-1} H_F(l, k) \geq q_F^{\frac{1}{2} b_1 k}. \quad (6.3)$$

Now let $\psi_R: \mathbb{A}_K^m \rightarrow \mathbb{A}_K^m$ be the map $\psi_R(x_1, \dots, x_m) = (x_1^R, \dots, x_m^R)$ for $R := \lceil 4m/b_1 \rceil$. Then by [GH24, Corollary 3.18], the convolution $\varphi * \psi_R: X \times \mathbb{A}_K^m \rightarrow \mathbb{A}_K^m$ is (FRS). By Corollary 6.2, we have

$$(\varphi * \psi_R)_*(\mu_l \times \mu_{\mathcal{O}_F}^m) \in \mathcal{M}_{c,\infty}(F^m). \quad (6.4)$$

On the other hand, note that

$$(\varphi * \psi_R)^{-1}(B(y, q_F^{-k})) \supseteq \varphi^{-1}(B(y, q_F^{-k})) \times \psi_R^{-1}(B(0, q_F^{-k})).$$

Further note that

$$\psi_R^{-1}(B(0, q_F^{-k})) = B\left(0, q_F^{-\lceil \frac{k}{R} \rceil}\right).$$

²In the statement of [CGH18, Theorem 2.1.3], the approximation (6.1) is stated for $|G_F(y, l, k)|_{\mathbb{C}}$ instead of $G_F(y, l, k)$. Since $G_F(y, l, k)$ is a non-negative real-valued motivic function, their argument yields the current statement as well (see the first four lines of the proof on p. 146).

Thus, we have

$$\begin{aligned}
 \frac{(\varphi * \psi_R)_*(\mu_l \times \mu_{\mathcal{O}_F}^m)(B(y, q_F^{-k}))}{\mu_F^m(B(y, q_F^{-k}))} &= \frac{\mu_l \times \mu_{\mathcal{O}_F}^m((\varphi * \psi_R)^{-1}(B(y, q_F^{-k})))}{\mu_F^m(B(y, q_F^{-k}))} \\
 &\geq \frac{\mu_l(\varphi^{-1}(B(y, q_F^{-k})))}{\mu_F^m(B(y, q_F^{-k}))} \cdot \mu_{\mathcal{O}_F}^m(\psi_R^{-1}(B(0, q_F^{-k}))) \\
 &= \frac{\varphi_* \mu_l(B(y, q_F^{-k}))}{\mu_F^m(B(y, q_F^{-k}))} \cdot \mu_{\mathcal{O}_F}^m\left(B\left(0, q_F^{-\lceil \frac{k}{R} \rceil}\right)\right) \\
 &\geq G_F(y, l, k) \cdot q_F^{-\lceil \frac{k}{R} \rceil m}.
 \end{aligned}$$

By (6.3), for each large enough k , we may find a $y_0 \in Y(F)$ such that

$$\frac{(\varphi * \psi_R)_*(\mu_l \times \mu_{\mathcal{O}_F}^m)(B(y_0, q_F^{-k}))}{\mu_F^m(B(y_0, q_F^{-k}))} \geq q_F^{-\lceil \frac{k}{R} \rceil m} q_F^{\frac{1}{2} b_1 k} \geq q_F^{\frac{b_1}{8} k},$$

which contradicts (6.4). Thus $b_1 \leq 0$ and we are done by (6.2). \square

We are now ready to prove the implication (1) \Rightarrow (2) of Theorem 6.6.

Proof of (1) \Rightarrow (2) of Theorem 6.6, non-Archimedean case. Let $\varphi: X \rightarrow Y$ be a jet-flat morphism. We may assume $Y = \mathbb{A}_K^m$. Let $\mu \in \mathcal{M}_c^\infty(X(F))$, and write g_F for the density of $\varphi_* \mu$ with respect to μ_F^m . Let $Y^{\text{sm}, \varphi}$ be the set of $y \in Y$ such that φ is smooth over y . For every $F \in \text{Loc}_0$, the map φ_F is smooth over $Y^{\text{sm}, \varphi}(F)$, and therefore $g_F(y)$ is locally constant on $Y^{\text{sm}, \varphi}(F)$.

By [CGH18, Corollary 1.4.3], the constancy radius of $g_F(y)$ can be taken to be definable; that is, there exists a definable function $\alpha: Y^{\text{sm}, \varphi} \rightarrow \mathbb{N}$ such that $g_F(y)$ is constant around every ball $B(y, q_F^{-\alpha_F(y)})$. In particular, for every $y \in Y^{\text{sm}, \varphi}(F)$, we have $g_F(y) = G_{F, \mu}(y, \alpha_F(y))$. In addition, by Proposition 6.8, we have $G_{F, \mu}(y, k) \leq C_{F, \mu} k^M$ for $F \in \text{Loc}_0$. We arrive at the following:

$$\begin{aligned}
 \int_{F^m} |g_F(y)|^s dy &= \int_{Y^{\text{sm}, \varphi}(F)} |G_{F, \mu}(y, \alpha_F(y))|^s dy \leq C_{F, \mu}^s \int_{F^m} \alpha_F(y)^{Ms} dy \\
 &= C_{F, \mu}^s \sum_{t \in \mathbb{N}} t^{Ms} \cdot \mu_F^m(\{y \in F^m : \alpha_F(y) = t\}) \\
 &\leq C'(F) + C \sum_{t \in \mathbb{N}} t^{Ms} q_F^{-\lambda t} < \infty.
 \end{aligned}$$

The last inequality follows by [CGH18, Theorem 3.1.1] as $\lim_{t \rightarrow \infty} \mu_F^m(\{y \in F^m : \alpha_F(y) = t\}) = 0$ and thus $\mu_F^m(\{y \in F^m : \alpha_F(y) = t\}) \leq q_F^{-\lambda t}$ for some $\lambda > 0$ and every large enough t . \square

In [CGH23, Theorem 4.12], Cluckers and the first two authors showed that if $\varphi: X \rightarrow Y$ is a jet-flat morphism that is defined over \mathbb{Z} , and one chooses $\mu = \mu_{X(\mathbb{Z}_p)}$ and $\tau = \mu_{Y(\mathbb{Z}_p)}$ to be the canonical measures on $X(\mathbb{Z}_p)$ and $Y(\mathbb{Z}_p)$ (see [CGH23, Lemma 4.2]), then the constant $C_{\mathbb{Q}_p, \mu, \tau}$ in Proposition 6.8 can be taken to be independent of p (that is, $C_{\mathbb{Q}_p, \mu, \tau} = C$). Theorem 4.12 of [CGH23], together with the ideas of the proof of the implication (1) \Rightarrow (2) of Theorem 6.6, allows us to give bounds on the L^s norms of $\varphi_* \mu_{X(\mathbb{Z}_p)} / \mu_{Y(\mathbb{Z}_p)}$ that are independent of p .

PROPOSITION 6.9. *Let $\varphi: X \rightarrow Y$ be a dominant morphism between finite type \mathbb{Z} -schemes X and Y , with $X_{\mathbb{Q}}$ and $Y_{\mathbb{Q}}$ smooth and geometrically irreducible. For any prime p , let g_p be the density of $\varphi_* \mu_{X(\mathbb{Z}_p)}$ with respect to $\mu_{Y(\mathbb{Z}_p)}$. Then the following are equivalent:*

- (1) The map $\varphi_{\mathbb{Q}}: X_{\mathbb{Q}} \rightarrow Y_{\mathbb{Q}}$ is jet-flat.
 (2) For every $s > 1$, there exists a constant $C(s) > 0$ such that for every prime p , we have

$$\|g_p\|_s := \int_{Y(\mathbb{Z}_p)} |g_p(y)|^s d\mu_{Y(\mathbb{Z}_p)} < C(s).$$

Proof. The implication (2) \Rightarrow (1) follows from Proposition 6.7. By the implication (1) \Rightarrow (2) of Theorem 6.6, it is enough to prove (1) \Rightarrow (2) for $p \gg_s 1$. Suppose that $\varphi_{\mathbb{Q}}$ is jet-flat. We may assume that $Y = \mathbb{A}^m$. Indeed, working locally, since $Y_{\mathbb{Q}}$ is smooth, we may assume that there exists a morphism $\psi: Y \rightarrow \mathbb{A}^m$ such that $\psi_{\mathbb{Q}}$ is an étale map. If \tilde{g}_p is the density of $\psi_*\varphi_*\mu_{X(\mathbb{Z}_p)}$ with respect to μ_p^m and N is an upper bound on the size of the geometric fibers of ψ , then

$$\left| \frac{\psi_*\mu_{Y(\mathbb{Z}_p)}}{\mu_p^m} \right| < N$$

for p large enough. In particular, we have

$$\begin{aligned} \int_{Y(\mathbb{Z}_p)} |g_p(y)|^s d\mu_{Y(\mathbb{Z}_p)} &\leq \int_{Y(\mathbb{Z}_p)} |\tilde{g}_p \circ \psi(y)|^s d\mu_{Y(\mathbb{Z}_p)} = \int_{\mathbb{Z}_p^m} |\tilde{g}_p(\tilde{y})|^s d(\psi_*\mu_{Y(\mathbb{Z}_p)}) \\ &\leq N \int_{\mathbb{Z}_p^m} |\tilde{g}_p(\tilde{y})|^s d\mu_p^m. \end{aligned}$$

As in the proof of the implication (1) \Rightarrow (2) of Theorem 6.6 above, by [CGH18, Corollary 1.4.3], there exists a definable function $\alpha: (\mathbb{A}^m)^{\text{sm},\varphi} \rightarrow \mathbb{N}$ such that $g_p(y) = G_p(y, \alpha_{\mathbb{Q}_p}(y))$, where $G_p(y, k) := G_{\mathbb{Q}_p, \mu_{X(\mathbb{Z}_p)}}(y, k)$. Hence, applying [CGH23, Theorem 4.12], we can find $C, M \in \mathbb{N}$ such that for $p \gg 1$,

$$\begin{aligned} \int_{\mathbb{Z}_p^m} |g_p(y)|^s dy &= \int_{(\mathbb{A}^m)^{\text{sm},\varphi}(\mathbb{Q}_p) \cap \mathbb{Z}_p^m} |G_p(y, \alpha_{\mathbb{Q}_p}(y))|^s dy \leq C^s \int_{(\mathbb{A}^m)^{\text{sm},\varphi}(\mathbb{Q}_p) \cap \mathbb{Z}_p^m} \alpha_{\mathbb{Q}_p}(y)^{Ms} dy \\ &= C^s \sum_{t \in \mathbb{N}} t^{Ms} \cdot \mu_p^m(\{y \in \mathbb{Z}_p^m : \alpha_{\mathbb{Q}_p}(y) = t\}). \end{aligned}$$

From [CGH18, Theorem 3.1.1], it follows that there exist $L \in \mathbb{N}$ and $\lambda > 0$ such that we have $\mu_p^m(\{y \in \mathbb{Z}_p^m : \alpha_{\mathbb{Q}_p}(y) = t\}) \leq p^{-\lambda t}$ for every $t > L$ and every prime p . We therefore get the desired claim as

$$\int_{\mathbb{Z}_p^m} |g_p(y)|^s dy \leq C^s \sum_{t=0}^L t^{Ms} + C^s \sum_{t>L} t^{Ms} p^{-\lambda t} < C^s \left((L+1) \cdot L^{Ms} + \sum_{t>L} t^{Ms} 2^{-\lambda t} \right) < C(s). \quad \square$$

6.3 Proof of (1) \Rightarrow (2) in the Archimedean case

In this subsection, we prove (1) \Rightarrow (2) in the cases $F = \mathbb{R}$ and $F = \mathbb{C}$. Let $\varphi: X \rightarrow Y$ be a jet-flat morphism between smooth algebraic varieties defined over F . Using restriction of scalars, we may assume that $F = \mathbb{R}$. We would like to show that for each $\mu \in \mathcal{M}_c^\infty(X(\mathbb{R}))$, we have $\varphi_*\mu \in \mathcal{M}^q(Y(\mathbb{R}))$ for all $1 \leq q < \infty$. We start with the following proposition.

PROPOSITION 6.10. *Let $\varphi: X \rightarrow Y$ be a jet-flat map between smooth \mathbb{R} -varieties. Then for every $\mu \in \mathcal{M}_c^\infty(X(\mathbb{R}))$, every non-vanishing $\tau \in \mathcal{M}^\infty(Y(\mathbb{R}))$ and every $p \in \mathbb{N}$, one can find $C_{\mu,\tau,p} > 0$ and $M_{\mu,p} > 0$ such that for each $y \in Y(\mathbb{R})$ and $0 < r < 1/2$, one has*

$$\int_{Y(\mathbb{R})} \left(\frac{(\varphi_*\mu)(B(y, r))}{\tau(B(y, r))} \right)^p d\tau(y) < C_{\mu,\tau,p} |\log(r)|^{M_{\mu,p}}.$$

Remark 6.11. Proposition 6.10 is weaker than its non-Archimedean counterpart, Proposition 6.8. The main obstacle is that we do not know of an Archimedean analogue to [CGH18, Theorem 2.1.3], that is, whether one can approximate the supremum of constructible functions by constructible functions as in (6.1). It is conjectured by Raf Cluckers that such a statement should be true under suitable assumptions (see [AM23, Conjecture 6.9] and that paper’s sequel, due to appear in *Dissert. Math.*). We thank the anonymous referee for bringing this to our attention. We further believe that the current proposition should hold for $M_{\mu,p} = C \cdot p$ for a sufficiently large C independent of μ .

Analogously to the non-Archimedean case, we introduce the following notion of constructible functions.

DEFINITION 6.12 ([CM11, Section 1.1], see also [LR97]). (1) A *restricted analytic function* is a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ such that $f|_{[-1,1]^n}$ is analytic and $f|_{\mathbb{R}^n \setminus [-1,1]^n} = 0$.

(2) A subset $A \subseteq \mathbb{R}^n$ is *subanalytic* if it is definable in \mathbb{R}_{an} , the extension of the ordered real field by all restricted analytic functions. A function $f: A \rightarrow \mathbb{R}$ is *subanalytic* if its graph $\Gamma_f \subseteq \mathbb{R}^n \times \mathbb{R}$ is subanalytic.

(3) A function $h: A \rightarrow \mathbb{R}$ is called *constructible* if there exist subanalytic functions $f_i: A \rightarrow \mathbb{R}$ and $f_{ij}: A \rightarrow \mathbb{R}_{>0}$ such that

$$h(x) = \sum_{i=1}^N f_i(x) \cdot \prod_{j=1}^{N_i} \log(f_{ij}(x)). \quad (6.5)$$

We denote the class of constructible functions on A by $\mathcal{C}(A)$.

(4) Given an analytic manifold Z , a measure $\mu \in \mathcal{M}(Z)$ is called *constructible* if locally it is of the form $f \cdot |\omega|$, where ω is a regular top form and f is constructible. We denote the class of constructible measures by $\mathcal{CM}(Z)$. Similarly, we write $\mathcal{CM}_{c,q}(Z)$, $\mathcal{CM}^\infty(Z)$ and $\mathcal{CM}_c^\infty(Z)$.

Note that $X(\mathbb{R})$ is defined by polynomials, so it is definable in the real field, and in particular subanalytic. Since $\mathcal{C}(X(\mathbb{R}))$ contains indicators of balls, we may assume that $\mu \in \mathcal{CM}_{c,\infty}(X(\mathbb{R}))$ when proving Proposition 6.10 and Theorem 6.6.

Proof of Proposition 6.10. We may assume that $Y = \mathbb{A}_{\mathbb{R}}^m$, $\tau = \mu_{\mathbb{R}}^m$ and $\mu \in \mathcal{CM}_{c,\infty}(X(\mathbb{R}))$. Write $g(y)$ for the density of $\varphi_*\mu$ with respect to $\mu_{\mathbb{R}}^m$. Set

$$G(y, r) = \frac{(\phi_*\mu)(B(y, r))}{r^m} = \frac{1}{r^m} \int_{B(y, r)} g(y') dy'. \quad (6.6)$$

For each $p \in \mathbb{N}$, let

$$G_p(r) := \int_{\mathbb{R}^m} G(y, r)^p dy.$$

By [CM11, Theorem 1.3], the functions $G: \mathbb{R}^m \times \mathbb{R}_{>0} \rightarrow \mathbb{R}$ and $G_p(r): \mathbb{R}_{>0} \rightarrow \mathbb{R}$ are constructible. Writing G_p as in (6.5), and using a preparation theorem for constructible functions [CM12, Corollary 3.5], we see that there exist $\delta > 0$ and $\theta \in \mathbb{R}$ such that $\theta = 0$ or $\theta \notin [0, \delta]$, and for each $r \in (0, \delta)$, one can write

$$G_p(r) = \sum_{i=1}^M d_i \cdot S_i(r) \cdot |r - \theta|^{\alpha_i} \log(|r - \theta|)^{l_i}, \quad (6.7)$$

where $d_i \in \mathbb{R}$, $l_i, M \in \mathbb{N}$, $\alpha_i \in \mathbb{Q}$, and where the S_i are certain subanalytic units, called *strong functions* (see [CM11, Definition 2.3]). We may assume that $\theta = 0$ as otherwise, $G_p(r)$ is bounded on $(0, \delta)$ and we are done. In addition, [CM12, Corollary 3.5] also ensures that for each $1 \leq i \leq M$, either $S_i(r) = 1$ for each $r \in (0, \delta)$, or $\alpha_i > -1$. This additional property is achieved by writing each strong function $S_i(r)$ as a converging infinite sum $S_i(r) = \sum_{j=0}^{\infty} c_{ij} \cdot r^{j/p}$ for some $p \in \mathbb{Q}_{\geq 0}$, and splitting it into a finite sum $\sum_{j=0}^s c_{ij} \cdot r^{j/p}$ and an infinite sum $\tilde{S}_i(r) := \sum_{j>s} c_{ij} \cdot r^{j/p}$. By taking s large enough and rearranging the terms in (6.7), Cluckers and Miller ensured that $\alpha_i > -1$ whenever $S_i(r) \neq 1$. Following the same argument, and taking s even larger, one can guarantee that $\alpha_i > N$ for some fixed $N \in \mathbb{N}$, as large as we wish. Hence, we may assume that $G_p(r)$ has the following form:

$$G_p(r) = \sum_{i=1}^{M'} d_i r^{\alpha_i} \log(r)^{l_i} + \sum_{i=M'+1}^M d_i S_i(r) r^{\alpha_i} \log(r)^{l_i}, \quad (6.8)$$

where $\alpha_i > N$ for $M' + 1 \leq i \leq M$ and $N \in \mathbb{N}$ as large as we like. For $1 \leq i \leq M'$, we may further assume that the (α_i, l_i) are mutually different and lexicographically ordered, that is, either $\alpha_i < \alpha_{i+1}$, or $\alpha_i = \alpha_{i+1}$ and $l_i > l_{i+1}$. In particular, by taking δ small enough, for $0 < r < \delta$, we have

$$\frac{1}{2} d_1 r^{\alpha_1} |\log(r)|^{l_1} < |G_p(r)| < 2 d_1 r^{\alpha_1} |\log(r)|^{l_1}. \quad (6.9)$$

We claim that $\alpha_1 \geq 0$. Assume not; then for r small enough, we have

$$\|G(\cdot, r)\|_{\infty} \geq \|G(\cdot, r)\|_p = G_p(r)^{\frac{1}{p}} \gtrsim r^{\frac{\alpha_1}{p}} |\log(r)|^{\frac{l_1}{p}} \gtrsim r^{\frac{\alpha_1}{p}}. \quad (6.10)$$

We now use an argument analogous to the one in Proposition 6.8. Take $\psi_R: \mathbb{A}^m \rightarrow \mathbb{A}^m$ to be the map

$$\psi_R(x_1, \dots, x_m) = (x_1^R, \dots, x_m^R)$$

for $R := \lceil 2mp/|\alpha_1| \rceil$, and let $\eta \in C_c^{\infty}(\mathbb{R}^m)$ be a bump function that is equal to 1 on the unit ball in \mathbb{R}^m . Then $\varphi * \psi_R: X \times \mathbb{A}^m \rightarrow \mathbb{A}^m$ is (FRS), and thus by Corollary 6.2,

$$(\varphi * \psi_R)_*(\mu \times \eta) \in \mathcal{M}_{c,\infty}(\mathbb{R}^m). \quad (6.11)$$

Repeating precisely the same argument as in Proposition 6.8 and using (6.10), we may find $\{y_r\}_r$ such that

$$\frac{(\varphi * \psi_R)_*(\mu \times \eta)(B(y_r, 2r))}{\mu_{\mathbb{R}}^m(B(y_r, 2r))} \gtrsim G(y_r, r) \cdot \psi_{R*} \eta(B(0, r)) \gtrsim G(y_r, r) r^{\frac{m}{R}} \gtrsim r^{\frac{\alpha_1}{4p}},$$

which leads to a contradiction. Hence $\alpha_1 \geq 0$, so on $(0, \delta)$ for δ small enough, we have

$$G_p(r) \lesssim |\log(r)|^{l_1}.$$

For $r > \delta$, we have $G(y, r) \lesssim \delta^{-m}$ and thus $G_p(r) \lesssim \delta^{-pm} < \infty$. This concludes the proof. \square

In order to prove Theorem 6.6, we need to control the oscillations of constructible functions.

DEFINITION 6.13. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a subanalytic function. Define $\alpha_f: \mathbb{R}^n \times \mathbb{R}_{>0} \rightarrow \mathbb{R}_{\geq 0}$ by

$$\alpha_f(y, r) := \min(1, \sup\{t \in \mathbb{R}_{\geq 0} : \forall y' \in B(y, t), |f(y) - f(y')| < r\}). \quad (6.12)$$

Note that α_f is subanalytic, and for any $r > 0$, we have $\alpha_f(y, r) > 0$ for almost every y . The next lemma extends this construction to the ring of constructible functions.

LEMMA 6.14. *Let $g \in \mathcal{C}(\mathbb{R}^n)$. Then there exists a subanalytic function $\alpha_g: \mathbb{R}^n \times \mathbb{R}_{>0} \rightarrow \mathbb{R}_{\geq 0}$ such that for any $r > 0$, we have $\alpha_g(y, r) > 0$ for almost all $y \in \mathbb{R}^n$ and*

$$|g(y) - g(y')| \leq r \quad \text{for all } y' \in B(y, \alpha_g(y, r)). \quad (6.13)$$

Proof. Write $g = \sum_{i=1}^M g_i$ for $g_i \in \mathcal{C}(\mathbb{R}^n)$, and suppose that we already constructed α_{g_i} for each i . Then we may set $\alpha_g(y, r) = \min_i \alpha_{g_i}(y, r/M)$. Hence, by (6.5), we may assume that

$$g(y) = f(y) \cdot \prod_{j=1}^N \log(f_j(y)) \quad (6.14)$$

for some subanalytic $f, f_1, \dots, f_N \in \mathcal{C}(\mathbb{R}^n)$. By setting $\alpha_g(y, r) = \alpha_g(y, 1)$ for $r > 1$, we may assume $r \leq 1$.

There is an open subanalytic subset $U \subseteq \mathbb{R}^n$ with complement U^c of measure 0 such that $f|_U$ and all $f_j|_U$ are continuous (see for example [DvdD88, Theorem 3.2.11]). Set

$$M(y) := \max_j (\max\{|f(y)| + 1, 2f_j(y), 2/f_j(y)\})$$

and $h(y) := M(y)^{-N}/2N$. Let α_f and α_{f_j} be as in (6.12), and set

$$\alpha_g(y, r) = \frac{1}{2} \min\{\alpha_f(y, rh(y)), \alpha_{f_1}(y, rf_1(y)h(y)), \dots, \alpha_{f_N}(y, rf_N(y)h(y))\}.$$

Note that $h, \alpha_{f_i}, \alpha_{f_{ij}}$ are subanalytic, and thus α_g also is subanalytic. Moreover, by the continuity of $f|_U$ and $f_j|_U$, we get that $\alpha_f(y, \cdot), \alpha_{f_j}(y, \cdot) > 0$ and thus also $\alpha_g(y, \cdot) > 0$ for all $y \in U$. Note that for any real numbers $a_1, \dots, a_N, b_1, \dots, b_N \in [-L, L]$, we have

$$\begin{aligned} \left| \prod_{i=1}^N a_i - \prod_{i=1}^N b_i \right| &= \left| \sum_{j=1}^N \left(\prod_{i=1}^{N-j+1} a_i \prod_{i=N-j+2}^N b_i - \prod_{i=1}^{N-j} a_i \prod_{i=N-j+1}^N b_i \right) \right| \\ &\leq \sum_{j=1}^N \left| \prod_{i=1}^{N-j} a_i \prod_{i=N-j+2}^N b_i \right| |a_{N-j+1} - b_{N-j+1}| \\ &\leq L^{N-1} \sum_{j=1}^N |a_{N-j+1} - b_{N-j+1}|. \end{aligned} \quad (6.15)$$

By (6.12), for every $y' \in B(y, \alpha_g(y, r))$ and $r < 1$, we have $|f(y') - f(y)| \leq rh(y) \leq 1/2$ and

$$|f(y')| \leq |f(y') - f(y)| + |f(y)| \leq M(y).$$

Similarly, we have

$$|\log(f_j(y')) - \log(f_j(y))| \leq \log(1 + rh(y)) \leq rh(y) \leq \frac{1}{2}$$

and

$$|\log(f_j(y'))| \leq \frac{1}{2} + |\log(f_j(y))| \leq \frac{1}{2} + \max\left(f_j(y), \frac{1}{f_j(y)}\right) \leq M(y).$$

By (6.15), for every $y' \in B(y, \alpha_g(y, r))$, we have

$$\begin{aligned} |g(y) - g(y')| &= \left| f(y) \cdot \prod_{j=1}^N \log(f_j(y)) - f(y') \cdot \prod_{j=1}^N \log(f_j(y')) \right| \\ &\leq M(y)^N \left(|f(y) - f(y')| + \sum_{j=1}^N \left| \log \left(\frac{f_j(y)}{f_j(y')} \right) \right| \right) \leq rh(y)(N+1)M(y)^N \leq r. \quad \square \end{aligned}$$

We can now finish the proof of Theorem 6.6.

Proof of (1) \Rightarrow (2) of Theorem 6.6, Archimedean case. Let $\mu \in \mathcal{CM}_{c,\infty}(X(\mathbb{R}))$, and denote by $g(y) \in \mathcal{C}(\mathbb{R}^m)$ the density of $\varphi_*\mu$ with respect to $\mu_{\mathbb{R}}^m$. Let α_g be as in Lemma 6.14, and set

$$S(2) := \left\{ y \in \mathbb{R}^m : \frac{1}{2} \leq \alpha_g \left(y, \frac{1}{2} \right) \text{ and } g(y) \neq 0 \right\}.$$

Then, for each $r \in \mathbb{R}_{\geq 3}$, define the following subanalytic set:

$$S(r) := \left\{ y \in \mathbb{R}^m : \frac{1}{r} \leq \alpha_g \left(y, \frac{1}{2} \right) < \frac{1}{r-1} \text{ and } g(y) \neq 0 \right\}.$$

We fix $L \in \mathbb{N}$ large enough. Setting

$$G(y, r) = \frac{(\phi_*\mu)(B(y, r))}{r^m} = \frac{1}{r^m} \int_{B(y, r)} g(y') dy' \quad (6.16)$$

and using Lemma 6.14, Proposition 6.10 and Hölder's inequality, we have

$$\begin{aligned} \int_{S(r)} g(y)^p dy &\leq \mu_{\mathbb{R}}^m(S(r)) + \int_{S(r) \cap \{g(y) > 1\}} g(y)^p dy \lesssim 1 + \int_{\mathbb{R}^m} 1_{S(r)}(y) \cdot G \left(y, \frac{1}{r} \right)^p dy \\ &\lesssim 1 + \mu_{\mathbb{R}}^m(S(r))^{\frac{1}{1+1/L}} \left(\int_{\mathbb{R}^m} G \left(y, \frac{1}{r} \right)^{(L+1)p} dy \right)^{\frac{1}{L+1}} \\ &\lesssim 1 + \mu_{\mathbb{R}}^m(S(r))^{\frac{1}{1+1/L}} \left| \log \left(\frac{1}{r} \right) \right|^{\frac{M_{\mu, (L+1)p}}{L+1}} \\ &= 1 + \mu_{\mathbb{R}}^m(S(r))^{\frac{1}{1+1/L}} |\log(r)|^{\frac{M_{\mu, (L+1)p}}{L+1}}. \end{aligned} \quad (6.17)$$

Since $\mu_{\mathbb{R}}^m(S(r))$ is a constructible function, using a similar argument as in the Proposition 6.10, and writing $\mu_{\mathbb{R}}^m(S(1/r))$ as in (6.8) and (6.9), we get $\mu_{\mathbb{R}}^m(S(r)) \sim r^\beta |\log(r)|^\gamma$ as $r \rightarrow \infty$ for $\beta \in \mathbb{Q}$ and $\gamma \in \mathbb{N}$. But since $\sum_{r=2}^{\infty} \mu_{\mathbb{R}}^m(S(r)) < \infty$, we must have $\beta < -1$. In particular, taking L large enough, we get that $\mu_{\mathbb{R}}^m(S(r))^{1/(1+1/L)} \sim r^{-1-\delta} |\log(r)|^{\gamma'}$ for some $\delta > 0$. Thus, the following holds for any $\gamma'' \in \mathbb{R}$:

$$\sum_{r=2}^{\infty} \mu_{\mathbb{R}}^m(S(r))^{\frac{1}{1+1/L}} |\log(r)|^{\gamma''} < \infty. \quad (6.18)$$

By (6.17) and (6.18), we get

$$\int_{\mathbb{R}^m} g(y)^p dy = \sum_{r=2}^{\infty} \int_{S(r)} g(y)^p dy \lesssim 1 + \sum_{r=3}^{\infty} \mu_{\mathbb{R}}^m(S(r))^{\frac{1}{1+1/L}} |\log(r)|^{\frac{M_{\mu, (L+1)p}}{L+1}} < \infty. \quad \square$$

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