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Kodaira dimension of moduli spaces of hyperkähler varieties

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Abstract

We study the Kodaira dimension of moduli spaces of polarized hyperkähler varieties deformation equivalent to the Hilbert scheme of points on a K3 surface or to O'Grady's ten-dimensional variety. This question was studied by Gritsenko–Hulek–Sankaran in the cases of $\mathrm{K3}^{[2]}$- and OG10-type when the divisibility of the polarization is 1. We generalize their results to higher dimension and divisibility. As a main result, for almost all dimensions $2n$, we provide a lower bound on the degree such that for all higher degrees, every component of the moduli space of polarized hyperkähler varieties of $\mathrm{K3}^{[n]}$-type is of general type.

1. Introduction

A hyperkähler manifold is a simply connected compact Kähler manifold with a unique nondegenerate 2-form (up to scaling). Hyperkähler manifolds are always even-dimensional with trivial canonical bundle. In this paper, we study moduli spaces of hyperkähler varieties, by which we mean hyperkähler manifolds that are projective.

Two-dimensional hyperkähler manifolds are always K3 surfaces. Mukai gave unirational parametrizations of the moduli spaces $\mathcal{F}_{2d}$ of polarized K3 surfaces of degree $2d$ for $d \leqslant 12$ and $d \in \{15, 17, 19\}$; see [Muk88, Muk92, Muk06, Muk10, Muk16]. Recently, the same was done for $d \in \{13, 21\}$; see [Nue17, FV18, FV21]. In contrast, Gritsenko–Hulek–Sankaran showed in [GHS07] that $\mathcal{F}_{2d}$ is of general type for $d > 61$ and $d \in \{46, 50, 54, 58, 60\}$.

For a hyperkähler manifold X, the group $H^2(X, \mathbb{Z})$ carries a quadratic form q_X, turning it into a lattice called the *Beauville–Bogomolov–Fujiki lattice*. In this paper, we focus on moduli spaces of hyperkähler manifolds coming from K3 surfaces: deformations of Hilbert schemes of n points on a K3 surface, called of $\mathrm{K3}^{[n]}$-type, and deformations of O'Grady's 10-dimensional examples, called of OG10-type. The corresponding moduli spaces $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ and $\mathcal{M}^{\gamma}_{\mathrm{OG10},2d}$ parametrize pairs (X, H), with X a projective hyperkähler variety of $\mathrm{K3}^{[n]}$-type or OG10-type, respectively,

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and H a primitive polarization on X so that $c_1(H)$ has degree $2d$ with respect to q_X and divisibility γ in $H^2(X,\mathbb{Z})$.

The study of the moduli spaces $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ and $\mathcal{M}^{\gamma}_{\mathrm{OG10},2d}$ has attracted much attention in the past decade. A fundamental question in hyperkähler geometry is whether one can effectively construct a general projective hyperkähler variety of a given deformation type. This is a question about the existence of unirational locally complete families. There are few known examples of such families; to our knowledge, there are no constructions outside $\mathrm{K3}^{[n]}$-type. For instance, such unirational parametrizations are known for $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ in the cases where (n,d,γ) equals $(2,3,2)$, see [BD85]; $(2,1,1)$, see [O'G06]; $(2,19,2)$, see [IR01, IR07] and also [Mon13, Proposition 1.4.1]; $(2,11,2)$, see [DV10]; $(4,1,2)$, see [LLSvS17]; $(3,2,2)$, see [IKKR19]; and additional higher-dimensional examples [BLM+21, PPZ23] (see (1.4) below).

The space $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ is not irreducible in general, but it is when $\gamma \in \{1,2\}$; see [Apo14a]. The space $\mathcal{M}^{\gamma}_{\mathrm{OG10},2d}$ is always irreducible [Ono22b, Ono22a]. Gritsenko–Hulek–Sankaran treat the "split" case $\gamma = 1$ in [GHS10, GHS11] (see also [Ma24]) and prove the following:

(i) The moduli space $\mathcal{M}^{1}_{\mathrm{K3}^{[2]},2d}$ is of general type if $d \geqslant 12$.

(ii) The moduli space $\mathcal{M}^{1}_{\mathrm{OG10},2d}$ is of general type for all $d \geqslant 3$ with $d \neq 2^m$ for $m \geqslant 0$.

Recently, similar results have been proven for moduli spaces of polarized hyperkähler manifolds deformation equivalent to a generalized Kummer fourfold; see [Daw25, Daw25].

1.1 Main results

We produce analogous general-type results for the higher-divisibility and higher-dimension cases of these moduli spaces. In particular, we give, for almost all dimensions, the first uniform bound (quadratic in γ and n) on the degree after which these moduli spaces are of general type (see Theorem 1.3 below). Moreover, in the more technical split case $\gamma = 1$, in which new subtleties arise, we give a list of $(n,d) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$, of positive density, for which the moduli space $\mathcal{M}^{1}_{K3^{[n]},2d}$ is of general type (see Theorem 1.6 below).

In the case $n = 2$, the remaining case (non-split) is $\gamma = 2$. In this case, $\mathcal{M}^{2}_{\mathrm{K3}^{[2]},2d}$ is non-empty if and only if $d \equiv -1 \mod 4$. Similarly, for OG10-type, the remaining case (non-split) is $\gamma = 3$, and $\mathcal{M}^{3}_{\mathrm{OG10},2d}$ is non-empty if and only if $d \equiv -3 \mod 9$.

Theorem 1.1. *Let $\mathcal{M}^{2}_{\mathrm{K3}^{[2]},2d}$ be the moduli space of primitively polarized hyperkähler varieties of $\mathrm{K3}^{[2]}$-type with non-split polarization of degree $2d = 8t-2$. Then for all $t \geqslant 12$ and $t = 10$, the moduli space $\mathcal{M}^{2}_{\mathrm{K3}^{[2]},2d}$ is of general type.*

Theorem 1.2. *Let $\mathcal{M}^{3}_{\mathrm{OG10},2d}$ be the moduli space of primitively polarized hyperkähler varieties of OG10-type with non-split polarization of degree $2d = 18t-6$. Then for all $t \geqslant 4$, the moduli space $\mathcal{M}^{3}_{\mathrm{OG10},2d}$ is of general type.*

We remark that the cases $t \in \{10,12\}$ in Theorem 1.1 and $t = 4$ in Theorem 1.2 were proved in [GHS13, Proposition 9.2] and [GHS11, Corollary 4.3].

The main results of our paper focus on the moduli spaces $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ in the case $n \geqslant 3$.

Theorem 1.3 (see Theorem 4.8). *Let (n,d,γ) be a triple such that the moduli space $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ is non-empty (see Proposition 3.1). We assume further that $n \geqslant 6$, $n \notin \{11,13\}$, and $\gamma \geqslant 3$. Then* *every component of* $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ *is of general type provided*

$$d \geqslant 6\gamma^2\big(n+3+\sqrt{2(n-1)}\big)^2\,,$$

except for one possible value of $d \geqslant 5\cdot 10^{10}$ *for each* n *that is odd or in the set* $\{10,12,52,\star/2+1\}$.

In the above theorem, $\star$ is a fixed integer greater than $5\cdot 10^{10}$ (see Section 2.6). This number $\star$ does not exist if the generalized Riemann hypothesis holds.

Remark 1.4. In Theorem 4.5, we give a much sharper lower bound on d after which a given connected component of $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ is of general type. However, computing this bound requires solving a linear Diophantine equation whose coefficients depend on the connected component, the value γ, and integers α_1, α_2, α_3 such that $\alpha_1^2+\alpha_2^2+\alpha_3^3 = 2(n-1)$. By contrast, Theorem 1.3 gives a uniform bound on d that does not require specifics about the numerics of n and γ.

Example 1.5. As an illustration of the difference between the bounds obtained from Theorem 4.5 versus Theorem 1.3, in the case of the moduli space $\mathcal{M}^{5}_{K3^{[26]},2d}$, Theorem 4.5 yields that both of its components are of general type provided $d \geqslant 225$ (see Example 4.6). By contrast, for the same moduli space, Theorem 1.3 yields the bound $d \geqslant 195169$.

We also consider $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ for $n \geqslant 3$ when $\gamma \in \{1,2\}$. These low divisibility cases come with several additional technical challenges. The hardest and perhaps most interesting case we tackle is the “split case” $\gamma = 1$. This case departs from previous literature: Modularity of the quasi-pullback is not guaranteed and vanishing of the resulting cusp form at the ramification of the modular projection cannot be proven via classification of root systems of small rank and discriminant. We obtain general-type results for a set of pairs (n,d) of density roughly $1/2$.

Theorem 1.6. *Suppose* $n \geqslant 3$, *and write* $n-1 = 4^c\cdot k$ *for* $c \geqslant 0$, $4 \nmid k$.

- (i) *If* k *is a square, then* $\mathcal{M}^{1}_{\mathrm{K3}^{[n]},2d}$ *is of general type for all* $d \geqslant 12$.
- (ii) *If* $k \equiv 1,2 \bmod 4$ *and* $k \notin \{1,2,5,6,10,13,25,37,58,85,130,\star\}$, *then* $\mathcal{M}^{1}_{\mathrm{K3}^{[n]},2d}$ *is of general type for all* d *of the following form:*
  - (a) $d = 4^e\cdot m$ *with* $e \geqslant 0$, $m \geqslant 3$, $m \not\equiv 0,4,7 \bmod 8$, *and* $m \notin \{5,10,13,25,37,58,85,130,\star\}$;
  - (b) $d = p\cdot r^2$, *where* p *is a prime congruent to* 3 *modulo* 4, *or any square* $d = r^2$ *when additionally* $k \neq 9$.

As an example, the lowest dimension for which one can rule out the existence of unirational locally complete families in degree 2 is $2n = 30$, and in degree 6, it is $2n = 20$.

Corollary 1.7. *There is no unirational locally complete family of polarized hyperkähler* 20*-folds of K3-type with split polarization of degree* 6 *or of* 30*-folds with split polarization of degree* 2.

The second case already follows from [GHS10] together with *strange duality*; see Proposition 3.9. In the case $\gamma = 2$, as a consequence of Theorem 1.1, we obtain the following general-type results for an infinite quadratic series of dimensions.

Theorem 1.8. *Suppose that* n *is an even integer such that* $n-1$ *is square. Then the moduli space* $\mathcal{M}^{2}_{\mathrm{K3}^{[n]},2d}$ *of primitively polarized hyperkähler varieties of degree* $2d = 8t-2$ *and divisibility* 2 *is of general type for* $t \geqslant 12$ *and* $t = 10$.

1.2 Relation to existing literature

To prove our main results, we reduce the question to the existence of a certain cusp form for an orthogonal modular variety. This strategy was introduced by Gritsenko–Hulek–Sankaran in [GHS07] and has been subsequently used in, among others, [GHS10, GHS11, TV19, FM21].

The moduli space $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ (respectively, $\mathcal{M}^{\gamma}_{\mathrm{OG10},2d}$) is a 20-dimensional (respectively, 21-dimensional) quasi-projective variety with finite quotient singularities; see [Vie95]. Its irreducible components Y admit algebraic open embeddings to orthogonal modular varieties

$$Y \longrightarrow \Omega(\Lambda_h)/\Gamma\,; \tag{1.1}$$

see [BB66, Ver13]. The target of this map is defined as follows. Let

$$\Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2(n-1)\rangle \quad (\text{respectively, } \Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus A_2(-1))$$

be the lattice isomorphic to $\big(H^2(X,\mathbb{Z}), q_X\big)$ and $\Lambda_h \subset \Lambda$ the lattice of signature $(2,20)$ (respectively $(2,21)$) isomorphic to the orthogonal complement of the first Chern class $h \in H^2(X,\mathbb{Z})$ of H. Then

$$\{[x] \in \mathbb{P}(\Lambda_h \otimes \mathbb{C}) \mid (x,x) = 0 \text{ and } (x,\overline{x}) > 0\}$$

is a type IV bounded Hermitian symmetric domain that has two isomorphic components, and the orthogonal group $O(\Lambda_h)$ acts on this domain. We fix one of the components $\Omega(\Lambda_h)$ and let $O^+(\Lambda_h)$ be the subgroup of $O(\Lambda_h)$ fixing $\Omega(\Lambda_h)$. The arithmetic group Γ, called the *monodromy group*, is a finite-index subgroup of $O^+(\Lambda_h)$.

In the case of the moduli spaces $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$, the monodromy group Γ was computed in [Mar11, Ono22b]; it is given by

$$\widehat{O}^+(\Lambda,h) = \big\{g \in O^+(\Lambda,h) \mid g|_{D(\Lambda)=\pm\mathrm{Id}}\big\}\,, \tag{1.2}$$

where $O^+(\Lambda,h) \subset O(\Lambda)$ is the stabilizer of $h \in \Lambda$ acting on Λ_h by restriction and $D(\Lambda)$ is the discriminant group of the lattice Λ. As an important ingredient in the proof of Theorem 1.3 (see Lemma 3.4 and Proposition 3.8), we show that if $n = 2$ or $\gamma \geqslant 3$, then this monodromy group Γ has the simpler description

$$\widehat{O}^+(\Lambda,h) = \widetilde{O}^+(\Lambda_h) := \{g \in O^+(\Lambda_h) \mid g|_{D(\Lambda_h)} = \mathrm{Id}\}. \tag{1.3}$$

By contrast, when $n \geqslant 3$ and $\gamma \in \{1,2\}$, the group $O^+(\Lambda,h)$ is an index 2 extension of $\widetilde{O}^+(\Lambda_h)$. This gives rise to several technical difficulties in the proof of Theorem 1.6.

Remarkably, Ma shows in [Ma18, Theorem 1.3] that there are only finitely many isomorphism classes of even lattices Λ_h of signature $(2,m)$ with $m \geqslant 9$ for which the modular variety $\Omega(\Lambda_h)/\widetilde{O}^+(\Lambda_h)$ is not of general type. Additionally, there are only finitely many lattices Λ_h (up to isomorphism) of signature $(2,m)$ with $m \geqslant 21$ or $m = 17$ such that $\Omega(\Lambda_h)/O^+(\Lambda_h)$ is not of general type [Ma18, Theorem 1.1].

One should note however that this does *not* imply that there are only finitely many choices of (n,d,γ) such that $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ is not of general type since infinitely many (n,d,γ) may yield isometric lattices Λ_h. In particular, $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ may not be of general type even when n or d are arbitrarily large. For instance, for any coprime integers (a,b), letting $n = a^2 - ab + b^2$, the moduli spaces

$$\begin{cases} \mathcal{M}^{(2/3)n}_{K3^{[n+1]},(2/3)n} & \text{if } 3|n\,, \\ \mathcal{M}^{2n}_{K3^{[n+1]},6n} & \text{otherwise} \end{cases} \tag{1.4}$$

are unirational, dominated by the moduli space of cubic fourfolds; see [BLM+21, Corollary 29.5]. Theorems 1.1 and 1.3 thus serve, once one has fixed n and γ, to give an explicit lower bound on d after which $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ is always of general type.

Note moreover that Ma's result does not apply in the cases $n \geqslant 3$ and $\gamma = 1, 2$ due to the failure of (1.3). The natural expectation that there are finitely many birational classes of moduli spaces $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ that are not of general type when $\gamma \in \{1, 2\}$ remains open. Theorems 1.6 and 1.8 are a contribution in this direction.

1.3 Ideas of the proof

The proofs of Theorems 1.1, 1.2, 1.3, and 1.6 use the strategy developed by Gritsenko–Hulek–Sankaran in [GHS07, GHS10, GHS11] involving modular forms of orthogonal type. For the relevant lattice Λ_h of signature $(2, m)$, the quotient $\mathcal{F}_{\Lambda_h} = \Omega(\Lambda_h)/\Gamma$ admits many projective toroidal compactifications with mild singularities; however, by [GHS07, Theorem 2], one can choose one $\overline{\mathcal{F}}_{\Lambda_h}$ with at worst canonical singularities. Taking a resolution of singularities of $\overline{\mathcal{F}}_{\Lambda_h}$ yields a smooth projective model $\overline{Y}$ of $\mathcal{F}_{\Lambda_h}$. Showing that the relevant moduli spaces are of general type thus amounts to showing that there is an abundance of holomorphic pluri-canonical forms on $\overline{Y}$.

This is accomplished using modular forms of orthogonal type and the so-called "low-weight cusp form trick" of Gritsenko–Hulek–Sankaran. A modular form of weight k and character $\chi\colon \Gamma \to \mathbb{C}^*$ is a holomorphic function $F\colon \Omega^\bullet(\Lambda_h) \to \mathbb{C}$ on the affine cone $\Omega^\bullet(\Lambda_h)$ of $\Omega(\Lambda_h)$ such that for all $Z \in \Omega^\bullet(\Lambda_h)$, $t \in \mathbb{C}^*$, and $g \in \Gamma$, we have

$$F(tZ) = t^{-k} F(Z) \quad \text{and} \quad F(gZ) = \chi(g) \cdot F(Z)\,.$$

These form a finite-dimensional vector space $\mathrm{Mod}_k(\Gamma, \chi)$. In the cases we consider, every modular form is holomorphic at the boundary. Those vanishing on the boundary are called *cusp forms* and form a subspace $S_k(\Gamma, \chi) \subset \mathrm{Mod}_k(\Gamma, \chi)$. By a classical result of Freitag [Fre83, Chapter 3], if F is a cusp form of weight $m\ell$ and character det vanishing on the ramification divisor of $\Omega(\Lambda_h) \to \mathcal{F}_{\Lambda_h}$, then the form $F \cdot dz^{\otimes \ell}$, where dz is a holomorphic volume form on $\Omega(\Lambda_h)$, descends to a global section of $\ell K_{\overline{Y}}$.

1.3.1 *Low-weight cusp form trick.* Gritsenko–Hulek–Sankaran's "low-weight cusp form trick" [GHS07] gives a method to produce such pluricanonical forms on $\overline{Y}$. To do this, by Freitag's result, we need to produce a fixed cusp form $F_a \in S_a(\Gamma, \det)$ with weight $a < m$ vanishing on the ramification divisor of $\Omega(\Lambda_h) \to \mathcal{F}_{\Lambda_h}$. As long as Γ has no irregular cusps [Ma24] (see Section 2.3), there is then an injection [GHS07, Theorem 1.1]

$$\mathrm{Mod}_{(m-a)\ell}(\Gamma, 1) \longrightarrow H^0\big(\overline{Y}, \ell K_{\overline{Y}}\big)\,, \quad \rho \longmapsto \rho \cdot F_a^\ell \cdot dz^{\otimes \ell}\,.$$

By Hirzebruch–Mumford proportionality [Mum77], the dimension of $\mathrm{Mod}_{(m-a)\ell}(\Gamma, 1)$ grows like ℓ^m, and from this it follows that $\overline{Y}$ is indeed of general type.

1.3.2 *Borcherds form and root counting.* In order to produce this low-weight cusp form F_a, one uses the Borcherds modular form $\Phi_{12} \in M_{12}\big(O^+(\mathrm{II}_{2,26}), \det\big)$, see [Bor95], where $\mathrm{II}_{2,26}$ is the unique even unimodular lattice of signature $(2, 26)$ given by

$$\mathrm{II}_{2,26} := U^{\oplus 2} \oplus E_8(-1)^{\oplus 3}\,.$$

If one can produce a primitive embedding of lattices

$$\Lambda_h \longhookrightarrow \mathrm{II}_{2,26} \tag{1.5}$$

with the right properties (see for example Propositions 3.10, 3.11, 5.3, 8.4), then the so-called "quasi-pullback" of Φ_{12} to the cone $\Omega^\bullet(\Lambda_h)$ yields the needed low-weight cusp form F_a. Thus the proofs of the main results boil down to producing embeddings $\Lambda_h \hookrightarrow \mathrm{II}_{2,26}$ with the desired properties. Modularity of the quasi-pullback is not automatic and strongly depends on the arithmetic group involved. The condition is automatically satisfied if Γ is the group $\widetilde{O}^+(\Lambda_h)$ defined in (1.3) but is much more delicate when Γ is larger. This is one source of difficulty for $\gamma \in \{1,2\}$. Moreover, in these cases, the embedding (1.5) has to be chosen so that isometries of Λ_h in the group Γ extend to isometries of $\mathrm{II}_{2,26}$. This is again automatic when $\Gamma = \widetilde{O}^+(\Lambda_h)$, but if Γ is larger, it is a more delicate endeavor.

Once we have an embedding where one can ensure modularity of the quasi-pullback, vanishing at the ramification divisor follows in the case $\gamma \geqslant 3$ by the classification of irreducible low-rank lattices of small discriminant [CS88]. In the case $\gamma \in \{1,2\}$, this problem is again more challenging, and we can solve it only by imposing additional constraints on both the dimension $2n$ and the degree $2d$.

2. Preliminaries

2.1 Lattices

Let L be a lattice, that is, a free abelian group of finite rank together with a symmetric non-degenerate bilinear pairing $(\cdot,\cdot)\colon L \times L \to \mathbb{Z}$. We denote by $O(L)$ the group of isometries of L and by $L(m)$, for an integer m, the lattice whose underlying abelian group is L and whose pairing is m times the pairing of L.

The dual lattice $L^\vee = \mathrm{Hom}(L,\mathbb{Z})$ can be embedded in $L \otimes \mathbb{Q}$ as those elements $x \in L \otimes \mathbb{Q}$ such that $(x,\ell) \in \mathbb{Z}$ for all $\ell \in L$, and the bilinear form is the restriction of the $\mathbb{Q}$-linear extension of $(\cdot,\cdot)$. The inclusion $L \subset L^\vee$ has finite index, and the quotient $D(L) = L^\vee/L$ is a finite abelian group, called the *discriminant group*. When L is even, that is, $(x,x) \in 2\mathbb{Z}$ for all $x \in L$, the discriminant group $D(L)$ comes endowed with a $\mathbb{Q}/2\mathbb{Z}$-valued quadratic form given by $x + L \mapsto (x,x) + 2\mathbb{Z}$. In this case, the natural projection induces a homomorphism

$$\pi\colon O(L) \longrightarrow O(D(L)) \tag{2.1}$$

that is surjective if L is indefinite, and $D(L)$ can be generated by at most $\mathrm{rk}(L) - 2$ elements. The following two groups play a prominent role:

$$\widetilde{O}(L) = \pi^{-1}(\mathrm{Id}) \quad \text{and} \quad \widehat{O}(L) = \pi^{-1}(\pm\mathrm{Id})\,.$$

The first one is called the *stable orthogonal group*. For $h \in L$, we denote by $O(L,h)$ the stabilizer of h in $O(L)$, and by $\widetilde{O}(L,h) \subset O(L,h)$ (respectively, $\widehat{O}(L,h)$) the subgroup of $\widetilde{O}(L)$ (respectively, $\widehat{O}(L)$) of elements fixing h.

Let $h \in L$; then $(h,L) \subset \mathbb{Z}$ is an ideal of the form $\gamma\mathbb{Z}$, where γ is a positive integer. We call γ the *divisibility* of h and write $\mathrm{div}(h) = \gamma$. Alternatively, γ can be defined as the positive integer making h/γ a primitive element in $L^\vee$. We denote by h_* the class of h/γ in $D(L)$.

The lattice E_8 plays a crucial role in our arguments. It can be realized as a sublattice of $\left(\frac{1}{2}\mathbb{Z}\right)^{\oplus 8}$

with the euclidean quadratic form via

$$E_8 = \left\{ (x_1, \ldots, x_8) \in \mathbb{Z}^{\oplus 8} \cup \left(\tfrac{1}{2} + \mathbb{Z}\right)^{\oplus 8} \,\middle|\, \sum_{i=1}^{8} x_i \equiv 0 \bmod 2 \right\}.$$

Elements in E_8 are points in $\mathbb{Q}^{\oplus 8}$ whose coordinates are either all integers or all half integers, and are such that the sum of the coordinates is an even integer. Recall that E_8 has 240 *roots*, elements r with $(r, r) = 2$. The 112 of these roots that have integer coordinates are of the form $\pm e_i \pm e_j$ for $i, j \in \{1, \ldots, 8\}$ and $i \neq j$, where $\{e_1, \ldots, e_8\}$ is the standard basis of $\mathbb{Z}^{\oplus 8}$. The remaining 128 roots have half-integer coordinates and are of the form $\frac{1}{2}\sum_{i=1}^{8} \pm e_i$, where the number of minus signs is even or, equivalently, the sum of all coordinates is even. Let us call the first set of roots *integral* and the second *fractional*. We view the lattice D_8 as a sublattice of E_8 via

$$D_8 = \left\{ (x_1, \ldots, x_8) \in \mathbb{Z}^{\oplus 8} \,\middle|\, \sum_{i=1}^{8} x_i \equiv 0 \bmod 2 \right\}.$$

Finally, recall that if L is a fixed even lattice, there is a natural one-to-one correspondence between finite-index even overlattices $L \subset L'$ and isotropic subgroups $H \subset D(L)$. Namely, given $L \subset L'$, there is a sequence of inclusions

$$L \subset L' \subset (L')^\vee \subset L^\vee, \tag{2.2}$$

and the isotropic subgroup associated with L' is given by $H = L'/L \subset L^\vee/L = D(L)$. Conversely, given an isotropic subgroup $H \subset D(L)$, the overlattice L' is given by $\pi^{-1}(H)$, where $\pi \colon L^\vee \to D(L)$ is the quotient map. The discriminant groups of L and L' are related by the following diagram:

$$\begin{array}{ccccc} H = L'/L & \hookrightarrow & H^\perp = (L')^\vee/L & \hookrightarrow & D(L) \\ & & \downarrow\!\!\!\downarrow & & \\ & & D(L') = H^\perp/H. & & \end{array} \tag{2.3}$$

An element $g \in O(L)$ can be extended to L' if and only if $\overline{g}(H) = H$, where $\overline{g} \in O(D(L))$ is the image of $g \in O(L)$ under the projection (2.1).

Recall that a lattice embedding $L \hookrightarrow L'$ is said to be *primitive* if the quotient L'/L is torsion-free. We now place ourselves in the following situation. Let Λ be an even lattice, $h \in \Lambda$ a primitive element of degree $2d$ and divisibility γ, and Λ_h the orthogonal complement of h in Λ. Then

$$D(\Lambda_h \oplus \langle h \rangle) = D(\Lambda_h) \oplus \mathbb{Z}/2d\mathbb{Z}, \tag{2.4}$$

where the last factor is the discriminant group of $\langle h \rangle$ generated by $(1/2d)h \mod \langle h \rangle$. Now Λ is a finite-index overlattice of $\Lambda_h \oplus \langle h \rangle$ that corresponds to the isotropic subgroup $H = \Lambda/(\Lambda_h \oplus \langle h \rangle)$ in (2.4). Moreover, since both inclusions $\Lambda_h \subset \Lambda$ and $\langle h \rangle \subset \Lambda$ are primitive, one checks that the projections

$$\pi_1 \colon H \longrightarrow D(\Lambda_h) \quad \text{and} \quad \pi_2 \colon H \longrightarrow D(\langle h \rangle) \tag{2.5}$$

are injective; see also [Nik80, Proposition 1.5.1]. That is, for any $x \in \pi_1(H)$, there exists a unique $y \in \pi_2(H)$ such that $x + y \in H$. The induced map $\pi_1(H) \to \pi_2(H)$ defines an automorphism of H, and if $g \in O(\Lambda, h)$, the restriction $g|_{\Lambda_h}$ acts as the identity on $\pi_1(H)$. Moreover, $\pi_1(H) \to \pi_2(H)$ respects the bilinear form up to a sign.

Lemma 2.1. *Let Λ be an even lattice and $h \in \Lambda$ primitive. View $\widetilde{O}(\Lambda, h)$ as a subgroup of $O(\Lambda_h)$ via the restriction map $O(\Lambda, h) \to O(\Lambda_h)$. Then $\widetilde{O}(\Lambda, h)$ contains $\widetilde{O}(\Lambda_h)$.*

Proof. Let $g \in \widetilde{O}(\Lambda_h)$, and consider the orthogonal transformation given by $g \oplus \mathrm{Id}$ on the lattice $\Lambda_h \oplus \langle h \rangle$. Then $g \oplus \mathrm{Id}$ acts as the identity on $D(\Lambda_h) \oplus D(\langle h \rangle)$. In particular, it fixes H, so $g \oplus \mathrm{Id}$ extends to an element of $O(\Lambda, h)$. Moreover, $\overline{g \oplus \mathrm{Id}}$ is the identity on $H^\perp$. From (2.3) it follows that the extension must act as the identity on $D(\Lambda) = H^\perp / H$. □

2.2 Modular forms

Let L be an even lattice of signature $(2, q)$ with $q \geqslant 3$. Recall that the symmetric domain

$$\left\{ [x] \in \mathbb{P}(L \otimes \mathbb{C}) \mid x^2 = 0 \text{ and } x \cdot \overline{x} > 0 \right\}$$

consists of two components exchanged by complex conjugation and is acted on by the arithmetic group $O(L)$. We fix one of the two connected components, denoted by $\Omega(L)$ and called the *period domain* for L. The index 2 subgroup of orientation-preserving isometries $O^+(L) \subset O(L)$ is the subgroup of $O(L)$ fixing $\Omega(L)$.

We denote by $\Omega^\bullet(L) \subset L \otimes \mathbb{C}$ the affine cone of $\Omega(L) \subset \mathbb{P}(L \otimes \mathbb{C})$ and let Γ be a finite-index subgroup of $O^+(L)$. A *modular form of weight k and character $\chi \colon \Gamma \to \mathbb{C}^*$* is a holomorphic function $F \colon \Omega^\bullet(L) \to \mathbb{C}$ such that for all $Z \in \Omega^\bullet(L)$, $t \in \mathbb{C}^*$, and $g \in \Gamma$, we have

$$F(tZ) = t^{-k} F(Z) \quad \text{and} \quad F(gZ) = \chi(g) \cdot F(Z)\,. \tag{2.6}$$

Modular forms of fixed weight and character form a finite-dimensional vector space $\mathrm{Mod}_k(\Gamma, \chi)$. When $q \geqslant 3$, every modular form is holomorphic at the boundary. Those vanishing on the boundary are called *cusp forms* and form a subspace denoted by $S_k(\Gamma, \chi) \subset \mathrm{Mod}_k(\Gamma, \chi)$. This subspace is fundamental in many ways. By a classical result of Freitag [Fre83, Chapter 3], cusp forms of weight $k = q$ and character det descend to global sections of the canonical divisor for any smooth model of the quotient. More precisely,

$$S_q\left(\Gamma, \det\right) \cong H^0\big(\overline{Y}, K_{\overline{Y}}\big)\,, \tag{2.7}$$

where $\overline{Y}$ is a smooth projective model of the quasi-projective variety $\Omega(L)/\Gamma$; see [BB66].

2.3 Irregular cusps and the low-weight cusp form trick

The period domain $\Omega(L)$ is an open subset of the isotropic quadric $\mathcal{Q} \subset \mathbb{P}(L \otimes \mathbb{C})$ defined by $(w, w) = 0$. It has two types of boundary components called *cusps*: 0-dimensional and 1-dimensional ones, corresponding to isotropic sublattices of L of rank 1 and 2, respectively.

The Fourier expansion of a modular form $F \in \mathrm{Mod}_k(\Gamma, \chi)$ at a 0-dimensional cusp is defined via Eichler transvections [Eic74, Section 3] (see also [GHS13, Section 8.3]). Recall that if I is an isotropic rank 1 sublattice of L and $U(I)_\mathbb{Q}$ is the unipotent part of the stabilizer of I in $O^+\left(L \otimes \mathbb{Q}\right)$, the group of translations defining the Fourier expansion of F at the cusp associated with I is $U(I)_\mathbb{Q} \cap \Gamma$, whereas the lattice of translations around the cusp I in the Γ-action is given by $U(I)_\mathbb{Q} \cap \langle \Gamma, -\mathrm{Id} \rangle$. When these two groups do not coincide, I is called an *irregular cusp* [Ma24]. Irregular 1-dimensional cusps are defined in a similar fashion, and their existence can be reduced to the 0-dimensional case.

Proposition 2.2 ([Ma24, Corollary 6.5]). *Let $\Gamma \subset O^+(L)$ be a finite-index subgroup. If $\Omega(L)$ has no irregular 0-dimensional cusps for Γ, then it has no irregular 1-dimensional cusps.*

After Freitag's result (2.7), the fundamental link between the existence of low-weight cusp forms and the Kodaira dimension of orthogonal Shimura varieties is the following theorem due to Gritsenko–Hulek–Sankaran, with a correction by Ma.

Theorem 2.3 ([GHS07, Theorem 1.1] and [Ma24, Theorem 1.2]). *Let L be a lattice of signature $(2,q)$ with $q \geqslant 9$ and $\Gamma \subset O^+(L)$ a finite-index subgroup with no irregular cusps. Then the modular variety $\Omega(L)/\Gamma$ is of general type if there exist a character χ and a cusp form F in $S_a(\Gamma,\chi)$ of weight $a < q$ that vanishes with order at least 1 at the ramification divisor of the projection $\Omega(L) \to \Omega(L)/\Gamma$. Moreover, if $S_q(\Gamma,\det) \neq 0$, then $\Omega(L)/\Gamma$ has non-negative Kodaira dimension.*

Remark 2.4. The condition of Γ not having irregular cusps is fairly mild, and for our purposes it will be satisfied with the possible exception of very few cases. This hypothesis can be dropped by requiring that F vanishes with certain order at the boundary of a toroidal compactification of the modular variety; see [Ma24, Theorem 1.2] for details.

An immediate observation is that if $-\mathrm{Id} \in \Gamma$, then Γ has no irregular cusps. We now explain a second criterion ensuring the non-existence of irregular cusps. Suppose that $I \subset L$ is an isotropic sublattice defining a 0-dimensional cusp. Consider the lattice

$$L(I) := \left(I^\perp/I\right) \otimes I\,,$$

and let $\Gamma(I)_\mathbb{Q}$ be the stabilizer of I in $O^+(L \otimes \mathbb{Q})$. For $m \otimes l \in L(I)_\mathbb{Q}$, the *Eichler transvection* $E_{m\otimes l} \in \Gamma(I)_\mathbb{Q}$ is defined by

$$E_{m\otimes l}(v) = v - (\widetilde{m},v)l + (l,v)\widetilde{m} - \tfrac{1}{2}(m,m)(l,v)l$$

for any $v \in L_\mathbb{Q}$, where $\widetilde{m} \in I^\perp_\mathbb{Q}$ is an arbitrary lift of $m \in \left(I^\perp/I\right)_\mathbb{Q}$; see [Eic74, Sca87]. The construction induces a canonical isomorphism

$$L(I)_\mathbb{Q} \longrightarrow U(I)_\mathbb{Q} \subset \Gamma(I)_\mathbb{Q}\,, \quad m \otimes l \longmapsto E_{m\otimes l}\,. \tag{2.8}$$

Proposition 2.5 ([Ma24, Proposition 3.1]). *The 0-dimensional cusp I is irregular for $\Gamma \subset O^+(L)$ if and only if $-\mathrm{Id} \notin \Gamma$ and $-E_w \in \Gamma \cap \Gamma(I)_\mathbb{Q}$ for some $w \in L(I)_\mathbb{Q}$.*

In Section 5, we use [Ma24, Proposition 1.1] together with Propositions 2.2 and 2.5 to rule out the existence of irregular cusps in the cases we consider.

2.4 The Borcherds modular form and quasi-pullback

Cusp forms of low weight are rare in nature. As in [GHS07, GHS10, GHS11, TV19], we produce the necessary cusp form to apply Theorem 2.3 by using the Borcherds form found in [Bor95]:

$$\Phi_{12} \in M_{12}\big(O^+(\mathrm{II}_{2,26}),\det\big)\,,$$

where $\mathrm{II}_{2,26}$ is the unique even unimodular lattice of signature $(2,26)$ given by

$$\mathrm{II}_{2,26} := U^{\oplus 2} \oplus E_8(-1)^{\oplus 3}\,. \tag{2.9}$$

We denote by $R(\mathrm{II}_{2,26})$ the set of (-2)-roots, and for a primitive embedding of lattices $L \hookrightarrow \mathrm{II}_{2,26}$, we denote by $R\big(L^\perp\big)$ the set of (-2)-roots in the orthogonal complement $L^\perp \subset \mathrm{II}_{2,26}$. Let $r \in R\big(L^\perp\big)$, and let $\sigma_r \in O^+(\mathrm{II}_{2,26})$ be the reflection with respect to r:

$$\sigma_r(v) = v - \frac{2(v,r)}{(r,r)}r\,. \tag{2.10}$$

The definition (2.6) of a modular form implies that for any Z in the σ_r-invariant hyperplane

$$H_r := r^{\perp} \cap \Omega^{\bullet}(\mathrm{II}_{2,26}) \subset \mathrm{II}_{2,26} \otimes \mathbb{C}\,,$$

one has $\Phi_{12}(Z) = -\Phi_{12}(Z)$. In particular, $\Phi_{12}(Z)$ vanishes along H_r. Furthermore, Φ_{12} vanishes only at the union of all H_r with $r \in R(\mathrm{II}_{2,26})$, and the vanishing multiplicity is 1; see [Bor95, Section 10, Example 2] and [BKPS98]. Note that $H_r = H_{-r}$ and the image of the induced embedding $\Omega^{\bullet}(L) \to \Omega^{\bullet}(\mathrm{II}_{2,26})$ lands inside H_r for any $\pm r \in R(L^{\perp})$. To get a non-zero modular form on $\Omega^{\bullet}(L)$ by means of restricting Φ_{12}, one has to divide by the corresponding linear factors (Z, r), one for each $\pm r \in R(L^{\perp})$.

We fix a choice of *positive roots*, that is, a subset $R(L^{\perp})_{>0} \subset R(L^{\perp})$ such that $R(L^{\perp})_{>0}$ and $-R(L^{\perp})_{>0}$ are disjoint and their union is all of $R(L^{\perp})$. We call $-R(L^{\perp})_{>0}$ the set of *negative roots* and denote it by $R(L^{\perp})_{<0}$. The function on $\Omega^{\bullet}(L)$ defined by

$$F(Z) = \left.\frac{\Phi_{12}(Z)}{\prod_{r\in R(L^{\perp})_{>0}} (Z, r)}\right|_{\Omega^{\bullet}(L)} \tag{2.11}$$

is called the *quasi-pullback* of Φ_{12} to $\Omega^{\bullet}(L)$. Observe that for $t \in \mathbb{C}^*$,

$$F(tZ) = t^{-(12+|R_{>0}|)} F(Z)\,. \tag{2.12}$$

On the other hand, the second condition of (2.6) is not always guaranteed for the full group $O^+(L)$ and a given character χ. Let us assume that $g \in O^+(L)$ is the restriction of $\widetilde{g} \in O^+(\mathrm{II}_{2,26})$. Then $\widetilde{g}|_{L^{\perp}}$ permutes the roots in $R(L^{\perp})$, and for $Z \in \Omega^{\bullet}(L)$,

$$\prod_{r\in R_{>0}} (g(Z), r) = \prod_{r\in R_{>0}} (\widetilde{g}(Z), r) = \prod_{r\in R_{>0}} \big(Z, \widetilde{g}^{-1}(r)\big) = (-1)^M \prod_{r\in R_{>0}} (Z, r)\,,$$

where M is the number of sign-changing roots via $\widetilde{g}$, that is,

$$M = \left|\widetilde{g}^{-1}(R_{>0}) \cap R_{<0}\right| = \left|\widetilde{g}(R_{>0}) \cap R_{<0}\right|. \tag{2.13}$$

Lemma 2.6. *The quasi-pullback F of Φ_{12} is modular with respect to $\chi \colon \Gamma \to \mathbb{C}^*$ for a finite-index subgroup $\Gamma \subset O^+(L)$ if for every $g \in \Gamma$, there exists an extension $\widetilde{g} \in O^+(\mathrm{II}_{2,26})$ such that*

$$\chi(g) = (-1)^M \cdot \det(\widetilde{g})\,,$$

where M is the number of sign-changing roots via $\widetilde{g}$ in $R(L^{\perp})$ defined in (2.13).

Proof. This follows immediately from the definitions:

$$F(g(Z)) = \frac{\det(\widetilde{g})\Phi_{12}(Z)}{(-1)^M \prod_{r\in R_{>0}} (Z, r)} = (-1)^M \det(\widetilde{g}) F(Z)\,. \qquad \square$$

As an immediate consequence, we have the following.

Corollary 2.7. *For any finite-index subgroup $\Gamma \subset \widetilde{O}^+(L)$, the quasi-pullback F is modular with respect to $\det \colon \Gamma \to \mathbb{C}^*$.*

Proof. Every element g in $\Gamma \subset \widetilde{O}^+(L)$ admits an extension $\widetilde{g} \in O^+(\mathrm{II}_{2,26})$ such that $\widetilde{g}$ restricts to the identity on $L^{\perp} \subset \mathrm{II}_{2,26}$; see [Nik80, Theorem 1.6.1 and Corollary 1.5.2] and also [Huy16, Chapter 14, Proposition 2.6]. In particular, $M = 0$ and $\det(g) = \det(\widetilde{g})$. $\square$

Provided that there are no irregular cusps for $\Gamma \subset O^+(L)$, in order to use Theorem 2.3, it is not enough to show that the quasi-pullback F of the Borcherds modular form Φ_{12} is modular and of low weight; one also has to ensure that it vanishes at all of the cusps and along the ramification divisor of the projection

$$\pi_L\colon \Omega(L) \longrightarrow \Omega(L)/\Gamma\,.$$

Let us place our attention on the latter. The ramification divisor of the projection π_L is given by the union of all reflective divisors

$$\mathrm{Ram}(\pi_L) = \bigcup_{\substack{r\in L \text{ primitive} \\ (r,r)<0 \\ \sigma_r \text{ or } -\sigma_r\in\Gamma}} \mathcal{D}_r\,, \tag{2.14}$$

where $\mathcal{D}_r = \{[Z]\in\Omega(\Lambda_h) \mid (Z,r)=0\}$ and σ_r is the reflection with respect to $r\in L$; see [GHS07, Corollary 2.13]. An element $r\in L$ of negative square $(r,r)<0$ such that σ_r or $-\sigma_r$ lies in $\Gamma\subset O^+(L)$ is called a *reflective element for* Γ. The condition $\pm\sigma_r\in\Gamma$ in general imposes strong restrictions on the numbers (r,r) and $\mathrm{div}(r)$. For instance, for σ_r to be integral, that is, an element in $O(L)$, one has to have

$$(r,r)\in\{\mathrm{div}(r), 2\mathrm{div}(r)\}\,. \tag{2.15}$$

An immediate consequence of the description of the ramification divisor in terms of reflective divisors is the following.

Lemma 2.8. *Let $G\in\mathrm{Mod}_k(\Gamma,\det)$, and assume $\mathrm{rk}(L)\equiv k \mod 2$. Then G vanishes at $\mathrm{Ram}(\pi_L)$.*

Proof. Let $[Z]\in\mathcal{D}_r$. If $\sigma_r\in\Gamma$, then the modularity of G for $\det\colon\Gamma\to\mathbb{C}^*$ and the fact that $\sigma_r(Z)=Z$ if $(Z,r)=0$ imply that we have $G(Z)=G(\sigma_r(Z))=-G(Z)$ and that G vanishes at $\mathcal{D}_r$. If $-\sigma_r\in\Gamma$, then the modularity implies

$$G(-Z)=G(-\sigma_r(Z))=(-1)^{\mathrm{rk}(L)+1}G(Z) \quad\text{and}\quad G(-Z)=(-1)^kG(Z)\,.$$

If $\mathrm{rk}(L)$ and k have the same parity, then $G(Z)=-G(Z)$ and G vanishes at $\mathcal{D}_r$. □

As $\mathrm{II}_{2,26}$ is unimodular, if σ_r or $-\sigma_r\in O^+(\mathrm{II}_{2,26})$, then $(r,r)=\pm2$. Moreover, the Borcherds form $\Phi_{12}\in M_{12}(\mathrm{II}_{2,26},\det)$ vanishes with order 1 at all reflective divisors associated with (-2)-roots $r\in\mathrm{II}_{2,26}$; see [Bor95, BKPS98] (see also [GHS07, Section 6]).

Proposition 2.9. *Let L be an even lattice of signature $(2,q)$ with $3\leqslant q\leqslant 26$ and $L\hookrightarrow\mathrm{II}_{2,26}$ a primitive embedding. Assume that the quasi-pullback F of the Borcherds form Φ_{12} to $\Omega(L)$ is modular with character $\det\colon\Gamma\to\mathbb{C}^*$. Let $L_r=r^\perp\subset L$ be the orthogonal complement of a reflective element $r\in L$, and consider the induced primitive embedding $L_r\hookrightarrow\mathrm{II}_{2,26}$. If for every reflective element $r\in L$, we have*

$$\big|R\big(L^\perp\big)\big| < \big|R\big(L_r^\perp\big)\big|\,,$$

then F vanishes along the ramification divisor of the modular projection $\pi_L\colon\Omega(L)\to\Omega(L)/\Gamma$.

Proof. The components of the ramification divisor (2.14) are given by reflective divisors

$$\mathcal{D}_r=\{[Z]\in\Omega(L)\mid (Z,r)=0\}\cong\Omega(L_r)\subset\Omega(L)\,, \tag{2.16}$$

where $(r,r)<0$ and σ_r or $-\sigma_r\in\Gamma$. If $\sigma_r\in\Gamma$, then the modularity of F with respect to det gives us the vanishing we want. So assume $-\sigma_r\in\Gamma$. Recall that for any primitive $(2,q)$-sublattice $S\subset\mathrm{II}_{2,26}$, the Borcherds form Φ_{12} vanishes at $\Omega(S)\subset\Omega(\mathrm{II}_{2,26})$ with order $\big|R\big(S^\perp\big)\big|/2$. In particular, the order of vanishing of Φ_{12} at $\mathcal{D}_r$ is $\left|R\left(L_r^\perp\right)\right|/2$, and by construction (2.11), the quasi-pullback F vanishes at $\mathcal{D}_r$ with order

$$\operatorname{ord}_{\mathcal{D}_r}(F) = \frac{\left|R\left(L_r^\perp\right)\right| - \left|R\left(L^\perp\right)\right|}{2} > 0\,. \qquad \square$$

2.5 Monodromy groups, moduli spaces, and components

Let X be a projective hyperkähler manifold X. The second cohomology group $H^2(X,\mathbb{Z})$ comes endowed with a lattice structure induced by a quadratic form q_X known as the *Beauville–Bogomolov–Fujiki form*. From now on, we focus on those X of OG10- or $\mathrm{K3}^{[n]}$-type. In these cases, the lattice $H^2(X,\mathbb{Z})$ is isomorphic to

$$\Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus L\,, \tag{2.17}$$

where $L = A_2(-1)$ if X is of OG10-type and $L = \langle -2(n-1)\rangle$ if X is of $\mathrm{K3}^{[n]}$-type. We call an isomorphism $\eta\colon H^2(X,\mathbb{Z}) \to \Lambda$ a *marking*. Let H be a polarization on X with first Chern class $h \in H^2(X,\mathbb{Z})$ that we assume to be primitive. The $O(\Lambda)$-orbit of $\eta(h)$ is called a *polarization type*; it is by definition independent of the marking. We denote the polarization type of h by $\mathfrak{h}$. There is a moduli space $\mathcal{M}_{\Lambda,\mathfrak{h}}$ parametrizing pairs (X,H), where X is a projective hyperkähler variety as above, H is a primitive polarization, $H^2(X,\mathbb{Z}) \cong \Lambda$, and the $O(\Lambda)$-orbit of $\eta(c_1(H))$ is $\mathfrak{h}$ for any marking η; see [Vie95]. The degree $2d$ and divisibility γ are constant for any polarization of a given polarization type, but there may be more than one polarization type for the same pair $(\deg,\operatorname{div}) = (2d,\gamma)$. We denote by $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ the union of all moduli spaces

$$\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d} = \bigcup_{\substack{\deg(\mathfrak{h})=2d\\ \operatorname{div}(\mathfrak{h})=\gamma}} \mathcal{M}_{\Lambda,\mathfrak{h}}\,, \tag{2.18}$$

where Λ is the $\mathrm{K3}^{[n]}$ lattice defined in (2.17). The definition of $\mathcal{M}^{\gamma}_{\mathrm{OG10},2d}$ is analogous. Note that for each polarization type $\mathfrak{h}$, the moduli space $\mathcal{M}_{\Lambda,\mathfrak{h}}$ may have several components. In the OG10 case, after fixing $2d$ and γ, there is at most one polarization type [Son23, Proposition 3.6], and by [Ono22b, Ono22a], the moduli spaces $\mathcal{M}^{\gamma}_{\mathrm{OG10},2d}$ are irreducible; see also [Son23, Proposition 3.4].

Let $h \in H^2(X,\mathbb{Z})$ be the first Chern class of a primitive polarization on X of degree given by $q_X(h) = 2d$. Recall the following classical result [GHS11, Lemma 3.3] known as *Eichler's criterion.*

Theorem 2.10. *Let L be an even lattice containing two copies of the hyperbolic lattice U. Two primitive elements h, h' are in the same $\widetilde{O}(L)$-orbit if and only if they have the same square $(h,h) = (h',h')$ and the same class $h_* = h'_*$ in $D(L)$.*

In particular, two primitive elements $h, h' \in \Lambda$ with the same degree and discriminant class define the same polarization type. In light of the criterion, if X is of $\mathrm{K3}^{[n]}$-type, we can always assume $h = \gamma(e+tf) - a\ell$ for appropriate t and a, where $\{e,f\}$ is the standard basis of a copy of U, ℓ is the generator of the last factor $\langle -2(n-1)\rangle$, and γ is the divisibility of h. Similarly, if X is of OG10-type, we can choose h of the form $h = \gamma(e+tf) + v$ with $v = 0$ or v a primitive element in $A_2(-1)$. We will make these choices explicit later on.

Let (X, H) be a primitively polarized hyperkähler variety of $\mathrm{K3}^{[n]}$- or OG10-type. After fixing a marking η for (X, H), the group of monodromy operators acting on Λ fixing h is denoted by $\mathrm{Mon}^2(\Lambda, h) = \eta \circ \mathrm{Mon}^2(X, h) \circ \eta^{-1}$. In both cases, we have

$$\mathrm{Mon}^2(X, h) = \widehat{O}^+(\Lambda, h)\,; \tag{2.19}$$

see [Mar11, Ono22b]. In particular, the definition does not depend on the choice of (X, H) or η.

Note that $D(\Lambda) = \mathbb{Z}/3\mathbb{Z}$ and $\widehat{O}^+(\Lambda) = O^+(\Lambda)$ in the OG10 case. The Torelli theorem for primitively polarized hyperkähler varieties of $\mathrm{K3}^{[n]}$- or OG10-type with polarization of fixed divisibility γ and degree $2d$ reads as follows.

Theorem 2.11 ([Ver13], [Mar11, Theorem 8.4]). *Let Y be an irreducible component of $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ or $\mathcal{M}^{\gamma}_{\mathrm{OG10},2d}$. Then there exists an algebraic open embedding*

$$Y \longrightarrow \Omega(\Lambda_h)/\widehat{O}^+(\Lambda, h)\,, \tag{2.20}$$

where Λ_h is the orthogonal complement of h in Λ and $\widehat{O}^+(\Lambda, h)$ acts on $\Omega(\Lambda_h)$ via the restriction map $\widehat{O}^+(\Lambda, h) \longrightarrow O^+(\Lambda_h)$.

In particular, (2.20) is a birational map, and the Kodaira dimension of every component Y of the moduli space $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ or $\mathcal{M}^{\gamma}_{\mathrm{OG10},2d}$ is given by the Kodaira dimension of the modular variety $\Omega(\Lambda_h)/\widehat{O}^+(\Lambda, h)$, where

$$\Lambda = \begin{cases} U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2(n-1)\rangle & \text{for } \mathrm{K3}^{[n]}\text{-type}\,, \\ U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus A_2(-1) & \text{for OG10-type}\,. \end{cases} \tag{2.21}$$

The map (2.20) depends not only on the polarization type, but also on the monodromy orbit of $h \in \Lambda$. The number of connected components of $\mathcal{M}_{\Lambda,\mathfrak{h}}$ is equal to the number of $\mathrm{Mon}(\Lambda)$-orbits in $\mathfrak{h}$. The number of such orbits has been computed in the $\mathrm{K3}^{[n]}$ case in [Apo14a] (see also [Son23, Proposition 3.4]). We recall the Mukai lattice

$$\widetilde{\Lambda} = U^{\oplus 4} \oplus E_8(-1)^{\oplus 2}\,. \tag{2.22}$$

For (X, H) fixed, there is a canonical choice [Mar11, Corollary 9.5] of primitive embedding $i_X \colon H^2(X, \mathbb{Z}) \hookrightarrow \widetilde{\Lambda}$ up to the action of $O(\widetilde{\Lambda})$. The number of connected components of $\mathcal{M}_{\Lambda,\mathfrak{h}}$ is the number of isometries of a rank 2 lattice T fixing $i_X(h)$, where T is the saturation in $\widetilde{\Lambda}$ of $i_X(h) \oplus i_X\big(H^2(X, \mathbb{Z})\big)^{\perp}$. This has discriminant $4d(n-1)/\gamma^2$, and $h^{\perp T} = \langle \ell \rangle$ with $(\ell, \ell) = -2(n-1)$. Moreover, the isomorphism class of the modular variety in the target of (2.20) does not depend on the monodromy orbit, but only on the polarization type. This in particular means that any two components of $\mathcal{M}_{\Lambda,\mathfrak{h}}$ are birational.

2.6 Sums of squares

The representability of a positive integer by a quadratic form is a very classical problem and essential in many of our arguments. It was already known to Gauss that a positive integer n can be expressed as a sum of three squares if and only if n is not of the form $4^k \cdot u$ with $u \equiv -1 \bmod 8$; see [Iwa97, Chapter 11]. In this paper, we make use of several finer representability results due to Halter-Koch [Hal82].

Theorem 2.12 ([Hal82, Korollar 1]). *Let n be a positive integer such that $n \not\equiv 0, 4, 7 \bmod 8$. Then n can be expressed as a sum of three positive coprime squares if and only if*

$$n \notin \{1, 2, 5, 10, 13, 25, 37, 58, 85, 130, \star\}\,,$$

where $\star$ *is a number that is at least* $5 \cdot 10^{10}$ *whose existence is unknown. Moreover,* n *can be expressed as a sum of three pairwise distinct coprime squares if and only if*

$$n \notin \left\{ \begin{matrix} 1, 2, 3, 6, 9, 11, 18, 19, 22, 27, 33, 43, 51, \\ 57, 67, 99, 102, 123, 163, 177, 187, 267, 627, \star \end{matrix} \right\}.$$

An immediate consequence of this theorem that will be used later is the following.

Corollary 2.13. *Let* n *be a positive even integer. Then either* n *or* $n-2$ *can be expressed as the sum of three pairwise distinct coprime squares, with the exception of*

$$n \in \{2, 4, 6, 8, 18, 20, 22, 24, 102, 104, \star, \star + 2\}.$$

As for quadratic forms of rank 4, every positive integer can be expressed as the sum of four squares. In the case of primitive representations of odd numbers, we have the following.

Theorem 2.14 ([Hal82, Satz 3]). *An odd number* $n \in \mathbb{N}$ *can be expressed as the sum of four pairwise distinct positive coprime squares if and only if*

$$n \notin \left\{ \begin{matrix} 1, \dots, 37, 41, 43, 45, 47, 49, 55, 59, 61, \\ 67, 69, 73, 77, 83, 89, 97, 101, 103, 115, 157 \end{matrix} \right\}.$$

Checking by hand the n listed as exceptions in the above theorem, one concludes as follows.

Corollary 2.15. *Every odd number* n *can be expressed as a sum of four positive coprime squares, with the exception of* $n \in \{1, 3, 5, 9, 11, 17, 29, 41\}$. *The numbers* 9, 11, 17, 29, 41 *can be expressed as sums of three positive coprime squares.*

Combining Theorem 2.12 and Corollary 2.15 then yields the following.

Corollary 2.16. *Any integer* $n \geqslant 3$ *with* $n \neq 5$ *such that* $n \not\equiv 0 \mod 4$ *can be written as a sum of four coprime squares at most one of which is zero.*

3. Monodromy, period domains, and ramification for K3$^{[n]}$-type

Let X be a hyperkähler variety of K3$^{[n]}$-type. Recall that the Beauville–Bogomolov–Fujiki lattice $H^2(X, \mathbb{Z})$ is isometric to

$$\Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \mathbb{Z}\ell,$$

where $(\ell, \ell) = -2(n-1)$, and for a fixed primitive element of positive square $h \in \Lambda$, the monodromy group acting on $\Omega(\Lambda_h)$ is given by the restriction to Λ_h of

$$\mathrm{Mon}^2(\Lambda, h) = \widehat{O}^+(\Lambda, h).$$

Recall moreover that $\mathrm{div}(h) = \gamma$ divides $\gcd(2d, 2(n-1))$.

Proposition 3.1. *The moduli space* $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ *is non-empty if and only if there exists an* $a \in \mathbb{Z}$ *coprime to* γ *such that*

$$\frac{2d}{\gamma} \equiv -\frac{2(n-1)}{\gamma} a^2 \pmod{2\gamma}. \tag{3.1}$$

In particular, once n *and* γ *are fixed, the degree* $2d$ *of any class* h *in a polarization type* $\mathfrak{h}$ *for which the corresponding moduli space* $\mathcal{M}_{\mathfrak{h}}$ *is non-empty must satisfy*

$$d = \gamma^2 t - (n-1)a^2 \tag{3.2}$$

for some (t, a) *with* a *coprime with* γ.

Remark 3.2. Once a triple (n, γ, a) is fixed, d and t determine each other. In view of our lattice computations, it will be easier to keep track of t instead of d.

Proof of Proposition 3.1. First, assume $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d} \neq \emptyset$. Suppose $(X, H) \in \mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ and that $\eta\colon H^2(\Lambda, \mathbb{Z}) \to \Lambda$ is a marking, and let $h = \eta(c_1(H))$. Write $h = \alpha x - a\ell$, with x primitive in $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$. Unimodularity implies $\mathrm{div}(x) = 1$ and $(\ell, \Lambda) = 2(n-1)\mathbb{Z}$. We are under the assumption

$$\mathrm{div}(h) = \gamma = \gcd(\alpha, 2a(n-1)) = \gcd(\alpha, 2(n-1))\,,$$

the last inequality following from the fact that h is primitive and thus α and a are coprime. Take t, d', k, α' such that $2t = (x, x)$, $2d = \gamma d'$, $2(n-1) = \gamma k$, and $\alpha = \gamma\alpha'$. By hypothesis, we have $2d = (\alpha x - a\ell)^2 = 2t\alpha^2 - 2(n-1)a^2$, so $d' \equiv -ka^2 \pmod{2\gamma}$.

Conversely, let $a \in \mathbb{Z}$ be an integer coprime to γ and satisfying equation (3.1). Then for some integer t, one has $2d = 2t\gamma^2 - 2(n-1)a^2$. Note that $h = \gamma(e + tf) - a\ell$ is a primitive element of divisibility γ and degree $2d$. Let

$$\Omega_{\mathrm{marked}} = \big\{[\omega] \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid (w, w) = 0 \text{ and } (\omega, \overline{\omega}) > 0\big\}$$

be the period domain for marked hyperkähler varieties. By choosing $[\omega] \in \Omega_{\mathrm{marked}}$ very general in the hyperplane $\mathbb{P}\big(h^{\perp} \otimes \mathbb{C}\big) \cap \Omega_{\mathrm{marked}}$, one concludes that the only integral point in $\omega^{\perp}$ is h. Then, the surjectivity of the period map for marked hyperkähler varieties [Huy99] implies that there is a marked hyperkähler variety (X, η) such that $\eta(\mathrm{Pic}(X)) = h\mathbb{Z}$. Since $h^2 > 0$, up to a sign, the class $H = \eta^{-1}(h) \in \mathrm{Pic}(X)$ is ample of degree $2d$. In particular, $(X, H) \in \mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$. □

Remark 3.3. When $\gamma = 1$, we can fix $a = 0$ so that $t = d$. In this case, the moduli space $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ is never empty. This is the only divisibility with this property. When $\gamma = 2$, we can take $a = 1$, and then $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ is non-empty if and only if $d \equiv -(n-1) \mod 4$, that is, $d = 4t - (n-1)$ for some $t > (n-1)/4$.

Lemma 3.4. *For all $h \in \Lambda$ primitive of positive square and divisibility γ, there exist a, t, with a coprime to γ, such that $\gamma(e + tf) - a\ell$ lies in the $\widetilde{O}(\Lambda)$-orbit of h, where $\{e, f\}$ are the standard generators of the first copy of U in Λ.*

Proof. Let $(h, h) = 2d$ and $h = \alpha x - a\ell$ as in the proof of Proposition 3.1. In particular, γ divides α and a is coprime to γ. Write $h' = \gamma(e + tf) - a\ell$, where $t = (d + (n-1)a^2)/\gamma^2$. Then $(h', h') = (h, h)$, and in $D(\Lambda)$, we have

$$h_* = \frac{h}{\gamma} \equiv \frac{-a\ell}{\gamma} = -\frac{2a(n-1)}{\gamma}\ell_* = h'_*\,.$$

It follows from Theorem 2.10 that h and h' are in the same $\widetilde{O}(\Lambda)$-orbit. □

Proposition 3.5. *The connected components of $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ are in one-to-one correspondence with the $a \in \big\{0, \ldots, \lfloor \gamma/2 \rfloor\big\}$ coprime with γ satisfying (3.1) (note that $a = 0$ if and only if $\gamma = 1$).*

Proof. The moduli space $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ is a disjoint union of moduli spaces with fixed polarization type $\mathfrak{h}$ as in (2.18), each of which has a connected component for every $\widehat{O}(\Lambda)$-orbit in $\mathfrak{h}$; see [Son23, Proposition 3.4]. Hence, the connected components of $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ are in one-to-one correspondence with the $\widehat{O}(\Lambda)$-orbits of all primitive $h \in \Lambda$ with square $2d$ and divisibility γ. Since the map (2.1) is surjective, the proof of Lemma 3.4 shows that classes $h = \gamma(e+tf) - a\ell$ and $h' = \gamma(e + t'f) - a'\ell$

with the same square $2d$ are in the same $\widehat{O}(\Lambda)$-orbit if and only if $h_* = -a(2(n-1)/\gamma)\ell_*$ equals $\pm h'_* = \pm(-a'(2(n-1)/\gamma)\ell_*)$ in $D(\Lambda)$. This holds if and only if $a \equiv \pm a' \mod \gamma$. □

We will denote the connected component corresponding to a by $\mathcal{M}^{\gamma,a}_{\mathrm{K3}^{[n]},2d}$.

Remark 3.6. Using [Son23, Lemma 3.2], one sees that in Proposition 3.5, two such integers a give rise to the same polarization type $\mathfrak{h}$ if and only if the corresponding discriminant classes h_* and h'_* are related via $O(D(\Lambda))$.

By Lemma 3.4, the orthogonal complement of h in Λ is given by

$$\Lambda_h = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus Q_h(-1)\,,$$

where $Q_h(-1)$ is generated by

$$z_1 = \frac{2a(n-1)}{\gamma} f - \ell \quad \text{and} \quad z_2 = e - tf\,. \tag{3.3}$$

In particular, the Gram matrix of Q_h is

$$Q_h = \begin{pmatrix} 2(n-1) & -\frac{2a(n-1)}{\gamma} \\ -\frac{2a(n-1)}{\gamma} & 2t \end{pmatrix}. \tag{3.4}$$

Moreover, by equation (3.2) we have

$$|D(\Lambda_h)| = |D(Q_h)| = \frac{4d(n-1)}{\gamma^2}\,. \tag{3.5}$$

Recall, see (2.19), that the monodromy group in the polarized $\mathrm{K3}^{[n]}$ case is given by $\widehat{O}^+(\Lambda, h)$.

LEMMA 3.7. *Let $n \geqslant 2$ and $h \in \Lambda$ be a primitive element of positive square. Then the index of $\widetilde{O}(\Lambda, h) \subset \widehat{O}(\Lambda, h)$ is given by*

$$\left[\widehat{O}(\Lambda, h) : \widetilde{O}(\Lambda, h)\right] = \begin{cases} 1 & \text{if } n = 2 \text{ or } \operatorname{div}(h) \geqslant 3\,, \\ 2 & \text{otherwise}\,. \end{cases} \tag{3.6}$$

The same holds for $\left[\widehat{O}^+(\Lambda, h) : \widetilde{O}^+(\Lambda, h)\right]$. Moreover, when $\gamma \in \{1, 2\}$ and $n \geqslant 3$, the group $\widehat{O}^+(\Lambda, h)$ is generated by $\widetilde{O}^+(\Lambda_h)$ and the restriction to Λ_h of the reflection $\sigma_{z_1} \in O(\Lambda)$ with respect to z_1 defined in equation (3.3).

Proof. Note that

$$\left[\widehat{O}(\Lambda) : \widetilde{O}(\Lambda)\right] = |\{\pm \mathrm{Id}_{D(\Lambda)}\}| \leqslant 2 \tag{3.7}$$

and equality holds when $n \geqslant 3$. In particular,

$$\left[\widehat{O}(\Lambda, h) : \widetilde{O}(\Lambda, h)\right] \leqslant 2\,. \tag{3.8}$$

By Lemma 3.4, we can assume $h = \gamma(e + tf) - a\ell$. Suppose that there exists a $g \in O(\Lambda, h)$ such that $\overline{g} = -\mathrm{Id}_{D(\Lambda)}$. Write

$$g(x) = u + r\ell \quad \text{and} \quad g(\ell) = u' + s\ell$$

for some integers r, s and $u, u' \in \ell^\perp$. The condition

$$h = g(h) = \gamma u + au' + (\gamma r + as)\ell$$

implies that $a(s-1) = -\gamma r$. Since γ and a are coprime, it follows that γ divides $s - 1$. Now $s\ell_* = \overline{g}(\ell_*) = -\ell_*$ implies that $s \equiv -1 \mod 2(n-1)$, so γ must divide $s - 1 = 2(n-1)k - 2$ for some $k \in \mathbb{Z}$. Since γ divides $2(n-1)$, it follows that γ divides 2. This shows that $\widehat{O}(\Lambda, h) = \widetilde{O}(\Lambda, h)$ when $\gamma \geqslant 3$, proving the first statement of the lemma. To finish, we assume that γ is 1 or 2. By (3.8), it is enough to exhibit an element in $\widehat{O}(\Lambda, h)$ not acting as the identity on $D(\Lambda)$. Let $\sigma_{z_1} \colon \Lambda \otimes \mathbb{Q} \to \Lambda \otimes \mathbb{Q}$ be the reflection (see (2.10)) with respect to z_1. Concretely, σ_{z_1} is given by

$$e \longmapsto e + \left(\frac{2}{\gamma}\right)^2 a^2(n-1)f - \frac{2}{\gamma}a\ell\,, \quad f \longmapsto f\,, \quad \ell \longmapsto \frac{4}{\gamma}a(n-1)f - \ell \tag{3.9}$$

and is the identity on $(U \oplus \langle \ell \rangle)^\perp \otimes \mathbb{Q}$. One checks [Mar11, Proposition 9.12] that $\sigma_{z_1} \in \widehat{O}(\Lambda, h)$ and $\overline{\sigma}_{z_1}(\ell_*) = -\ell_*$ if and only if $\gamma \in \{1, 2\}$. As σ_{z_1} is a reflection with respect to a negative-square primitive element, it preserves orientation; see [Mar11, Section 9]. □

Lemma 3.7 and the following generalization of [GHS10, Proposition 3.12(i)] will yield the modularity of the quasi-pullback when $n = 2$ or $\gamma \geqslant 3$ (see Corollary 2.7).

Proposition 3.8. *In the above setting, the restriction map $O(\Lambda, h) \to O(\Lambda_h)$ induces an isomorphism $\widetilde{O}(\Lambda, h) \cong \widetilde{O}(\Lambda_h)$.*

Proof. First observe that $D(\Lambda_h) = D(Q_h(-1))$. Consider the following elements in $Q_h(-1) \otimes \mathbb{Q}$:

$$u = \frac{\gamma}{2d}(az_1 + \gamma z_2) \quad \text{and} \quad v = \frac{\gamma^2}{2d(n-1)}\left(tz_1 + a\frac{n-1}{\gamma}z_2\right).$$

From (3.2) and (3.4) one checks that

$$(u, z_1) = (v, z_2) = 0 \quad \text{and} \quad (v, z_1) = (u, z_2) = -1\,.$$

In particular, $\{u, v\}$ forms a basis of $Q_h(-1)^\vee$, and $\{u_*, v_*\}$ generates the discriminant group $D(Q_h(-1))$.

Now, since the inclusion $\widetilde{O}(\Lambda_h) \subset \widetilde{O}(\Lambda, h)$ holds by Lemma 2.1, for the proposition we just need to show the reverse inclusion. So, let $g \in \widetilde{O}(\Lambda, h)$, acting on Λ_h via restriction. We have to show that $g|_{\Lambda_h}$ is in $\widetilde{O}(\Lambda_h)$, meaning that

$$g(u) \equiv u \quad \text{and} \quad g(v) \equiv v \bmod \Lambda_h\,. \tag{3.10}$$

From (3.2) it follows that $u, v \in \Lambda \otimes \mathbb{Q}$ can be expressed as

$$u = \frac{\gamma}{2d}h - f \quad \text{and} \quad v = \frac{a}{2d}h - \frac{1}{2(n-1)}\ell\,. \tag{3.11}$$

Since $g(h) = h$ and $\overline{g}(\ell_*) = \ell_*$ in $D(\Lambda)$, one has $g(v) = v + w$ with $w \in \Lambda$. Moreover, we have

$$0 = (h, v) = (h, g(v)) = (h, w)\,.$$

Thus, $w \in \Lambda_h$ and $g(v) \equiv v \mod \Lambda_h$. For the first congruence in (3.10), observe that we have $[\Lambda : \Lambda_h \oplus \langle h \rangle] = 2d/\gamma$; see [Huy16, Chapter 14, equation (0.2)] and (3.5). Moreover, f generates the group $H = \Lambda/(\Lambda_h \oplus \langle h \rangle)$. Indeed,

$$\frac{2d}{\gamma}f = -az_1 - \gamma z_2 + h \in \Lambda_h \oplus \langle h \rangle$$

is primitive in $\Lambda_h \oplus \langle h \rangle$. In particular, the order of the image $\overline{f}$ of f in the quotient H is exactly $2d/\gamma$. Recall, see (2.5), that $\pi_1 \colon H \to D(\Lambda_h)$ and $\pi_2 \colon H \to D(\langle h \rangle)$ are injective and every element in H can be written uniquely as a sum of an element in $\pi_1(H)$ and one in $\pi_2(H)$. From the first equation in (3.11), one observes that $\overline{f} = -u + \gamma h_*$, where

$$\pi_1(\overline{f}) = -u \in D(\Lambda_h) \quad \text{and} \quad \pi_2(\overline{f}) = \gamma h_* \in D\left(\langle h \rangle\right).$$

In particular, $\pi_1(H)$ is generated by u. Finally, by [Nik80, Corollary 1.5.2] (see also [GHS10, Lemma 3.2]), the image $\overline{g}$ of g in $O\left(D(\Lambda_h)\right)$ acts as the identity on $\pi_1(H) = \langle u\rangle$. This shows the first congruence in (3.10); thus we have the inclusion $\widetilde{O}(\Lambda, h) \subset \widetilde{O}(\Lambda_h)$. □

We next explain an interesting phenomenon known as *strange duality*. It was first observed by Apostolov in [Apo14b, Proposition 3.2] in the case $\gamma \in \{1,2\}$ and $d = 1$, and then generalized to $\gamma \in \{1,2\}$ and $d > 1$ by Song; see [Deb22, Remark 3.24]. As a consequence of Lemma 3.7 and Proposition 3.8, one can generalize strange duality to all triples (n, d, γ) such that the corresponding moduli space $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ is non-empty. For completeness, we prove the result for all $\gamma \geqslant 1$. In Section 6, we will make use of strange duality to propagate general-type results.

Proposition 3.9 (Strange duality). *Let (n, d, γ) be a triple such that $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ is non-empty (see Proposition 3.1). There is a natural bijection between the connected components of $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ and those of $\mathcal{M}^{\gamma}_{K3^{[d+1]},2(n-1)}$ such that each component of $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ is birational to the corresponding component of $\mathcal{M}^{\gamma}_{K3^{[d+1]},2(n-1)}$.*

Proof. To prove the non-emptiness of $\mathcal{M}^{\gamma}_{K3^{[d+1]},2(n-1)}$, consider an $a \in \mathbb{Z}$ such that (3.1) holds. Since $\gcd(a, \gamma) = 1$, there exist $z, a' \in \mathbb{Z}$ such that $aa' = 1 + z\gamma$. Then (3.2) yields

$$\frac{2(n-1)}{\gamma} = 2\gamma\left(t(a')^2 - \frac{2(n-1)}{\gamma}(2z + z^2\gamma)\right) - \frac{2d}{\gamma}(a')^2 .$$

Let $t' = t(a')^2 - (2(n-1)/\gamma)(2z+z^2\gamma)$; then (3.2) holds for a', t', $d' = n-1$ and $n' = d+1$. In fact, there is a unique $a' \in \left\{0, \ldots, \lfloor\gamma/2\rfloor\right\}$ such that $\pm a'a \equiv 1 \mod \gamma$. Hence by Proposition 3.5, the map sending $\mathcal{M}^{\gamma,a}_{\mathrm{K3}^{[n]},2d}$ to $\mathcal{M}^{\gamma,a'}_{K3^{[d+1]},2(n-1)}$ is a bijection between the sets of components.

Let $h \in \Lambda$ be as in Lemma 3.4. Likewise, let $\Lambda' = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \mathbb{Z}\ell'$ with $(\ell', \ell') = -2(n'-1) = -2d$, and let $h' = \gamma(e + t'f) - a'\ell' \in \Lambda'$. To prove that the two corresponding components of the moduli spaces are birational, by Theorem 2.11 it suffices to provide an isometry $\Lambda_h \to (\Lambda')_{h'}$ that identifies the projectivization $P\widehat{O}^+(\Lambda, h)$ with $P\widehat{O}^+(\Lambda', h')$. Let Q_h be as in (3.4), with generators $\{z_1, z_2\}$. Analogously, define $Q_{h'}$ so that the orthogonal complement of h' in Λ is given by $\Lambda_{h'} = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus Q_{h'}(-1)$, so $Q_{h'}$ has generators $\{z'_1, z'_2\}$ with respect to which the Gram matrix is

$$Q_{h'} = \begin{pmatrix} 2d & -\frac{2a'd}{\gamma} \\ -\frac{2a'd}{\gamma} & 2t' \end{pmatrix}.$$

One checks that the map $Q_h \to Q_{h'}$ sending z_1 to $a'z'_1 + \gamma z'_2$ and z_2 to $-(zz'_1 + az'_2)$ is an isomorphism of lattices. The induced isometry $\alpha\colon \Lambda_h \to (\Lambda')_{h'}$ identifies $\widetilde{O}(\Lambda_h)$ with $\widetilde{O}((\Lambda')_{h'})$, which completes the proof when $\gamma \geqslant 3$.

For $\gamma \in \{1,2\}$, first assume $n > 2$ and $d > 1$. For $\gamma = 1$, we have $(a, a', z) = (0, 0, 1)$; for $\gamma = 2$, we have $(a, a', z) = (1, 1, 0)$. In both cases, $\alpha(z_1) \in Q_{h'}$ is orthogonal to z'_1; hence, we have $\sigma_{z'_1} = -\sigma_{\alpha(z_1)}$. It follows from Lemma 3.7 that α identifies $P\widehat{O}^+(\Lambda, h)$ with $P\widehat{O}^+(\Lambda', h')$. Finally, for $\gamma \in \{1,2\}$ and $n = 2$ (or, analogously, $d = 1$), we have $\widehat{O}^+(\Lambda, h) = \widetilde{O}^+(\Lambda_h)$ and $\widehat{O}^+(\Lambda', h')/\widetilde{O}^+((\Lambda')_{h'}) = \{\pm \mathrm{Id}\}$, so again, α identifies $P\widehat{O}^+(\Lambda, h)$ with $P\widehat{O}^+(\Lambda', h')$. □

Given a primitive embedding $\Lambda_h \hookrightarrow \mathrm{II}_{2,26}$, the quasi-pullback F of the Borcherds form Φ_{12} vanishes along the ramification divisor of the modular projection $\Omega(\Lambda_h) \to \Omega(\Lambda_h)/\widetilde{O}^+(\Lambda_h)$ if for every reflective element $r \in \Lambda_h$ for which $-\sigma_r \in \widetilde{O}^+(\Lambda_h)$, there is a root in $\mathrm{II}_{2,26}$ orthogonal to $(\Lambda_h)_r$ but not to all of Λ_h; see Proposition 2.9.

Proposition 3.10. *Let $Q_h \hookrightarrow E_8$ be a primitive embedding such that the number of 2-roots in the orthogonal complement is bounded by $\left|R\left(Q_h^\perp\right)\right| \leqslant 54$. Then the quasi-pullback F of the Borcherds form Φ_{12} to $\Omega(\Lambda_h)$ with respect to the induced embedding $\Omega(\Lambda_h) \hookrightarrow \Omega(\mathrm{II}_{2,26})$ vanishes along the ramification divisor of the modular projection $\Omega(\Lambda_h) \to \Omega(\Lambda_h)/\widetilde{O}^+(\Lambda_h)$.*

Proof. Let $r \in \Lambda_h$ be a reflective element such that σ_r or $-\sigma_r$ lies in $\widetilde{O}^+(\Lambda_h)$. If $\sigma_r \in \widetilde{O}^+(\Lambda_h)$, the modularity of F (see Corollary 2.7) implies that $F(Z) = -F(Z)$ for all $Z \in \Omega^\bullet(\Lambda_h)$ such that $[Z] \in \mathcal{D}_r$. In particular, F vanishes along $\mathcal{D}_r$. We now assume $-\sigma_r \in \widetilde{O}(\Lambda_h)$. Then by [GHS07, Proposition 3.2], one must have

$$D(\Lambda_h) \cong D(Q_h) \cong (\mathbb{Z}/2\mathbb{Z})^m \times \mathbb{Z}/D\mathbb{Z}\,,$$

where $m = 0$ or $m = 1$ depending on whether $D(Q_h)$ is cyclic or not and

$$D = \frac{\operatorname{disc}(\Lambda_h)}{2^m} = \frac{\operatorname{disc}(Q_h)}{2^m} = \frac{4d(n-1)}{2^m\gamma^2}\,.$$

Moreover, the possibilities for $(r,r) < 0$ and $\operatorname{div}(r)$ are

(i) $(r,r) = -2D$ and $\operatorname{div}(r) = D$,

(ii) $(r,r) = -D$ and $\operatorname{div}(r) = D$, or

(iii) $(r,r) = -D$ and $\operatorname{div}(r) = D/2$.

Therefore,

$$\operatorname{disc}((\Lambda_h)_r) = \left|\frac{(r,r)\cdot\operatorname{disc}(\Lambda_h)}{\operatorname{div}(r)^2}\right| \in \{2,4,8\}\,;$$

see [GHS13, Lemma 7.2]. Since $\mathrm{II}_{2,26}$ is unimodular, the orthogonal complement $(\Lambda_h)_r^\perp$ of $(\Lambda_h)_r$ in $\mathrm{II}_{2,26}$ has rank 7 and discriminant 2, 4, or 8. By [CS88, Table 1], it contains one of the root systems E_6, E_7, D_6, D_7, or A_7. In particular, $|R((\Lambda_h)_r^\perp)| \geqslant |R(A_7)| = 56$, and F must vanish with order at least 1 along $\mathcal{D}_r$; see Proposition 2.9. □

Finally, our main proposition before going into lattice embeddings and root counting is the following.

Proposition 3.11. *Let n, d, γ, and a be positive integers satisfying the hypothesis of Proposition 3.1. Assume further that*

$$\frac{d(n-1)}{\gamma^2} > 4\,.$$

If there exists a primitive embedding $Q_h \subset E_8$ such that the number of roots $\left|R\left(Q_h^\perp\right)\right|$ in $Q_h^\perp \subset E_8$ is at least 2 and at most 14 (respectively, 16), then the modular variety

$$\Omega(\Lambda_h)/\widetilde{O}^+(\Lambda_h)$$

is of general type (respectively, non-negative Kodaira dimension). In particular, if $n = 2$ or $\gamma \geqslant 3$ and such an embedding exists, then the component of the moduli space $\mathcal{M}^\gamma_{\mathrm{K3}^{[n]},2d}$ corresponding to a (see Proposition 3.5) is of general type (respectively, non-negative Kodaira dimension).

Proof. Let $\Omega(\Lambda_h) \subset \Omega(I_{2,26})$ be the induced embedding of period domains and F the corresponding quasi-pullback of the Borcherds form $\Phi_{12} \in M_{12}(O^+(\mathrm{II}_{2,26}), \det)$. Then Corollary 2.7 implies that $F \in \operatorname{Mod}_k\left(\widetilde{O}^+(\Lambda_h), \det\right)$, with weight $k = 12 + |R(Q_h)|/2$; see (2.12). Moreover, F is a cusp form that vanishes along the ramification divisor of the projection $\Omega(\Lambda_h) \to \Omega(\Lambda_h)/\widetilde{O}^+(\Lambda_h)$; see

Proposition 3.10 and [GHS13, Corollary 8.12]. Recall, see (3.5), that the discriminant group of Λ_h has order

$$\operatorname{disc}(\Lambda_h) = \frac{4d(n-1)}{\gamma^2}$$

and the minimal number of generators is at most two. From [Ma24, Proposition 4.4], one concludes that if $4d(n-1)/\gamma^2 > 16$, then $\widetilde{O}^+(\Lambda_h)$ has no irregular cusps. Lemma 3.7 and Proposition 3.8 imply $\widehat{O}^+(\Lambda, h) \cong \widetilde{O}^+(\Lambda_h)$. The proposition now follows from Theorems 2.3 and 2.11. $\square$

4. Kodaira dimension for K3$^{[n]}$-type with divisibility at least 3

4.1 Rank 2 primitive embeddings and root counting

We start with a numerical lemma.

Lemma 4.1. *Let $C \in \mathbb{Z}$ and m_1, m_2, m_3 be pairwise distinct non-negative integers. There is at most one choice of signs in front of m_1, m_2, m_3 with an even number of $+$ signs such that*

$$C = \pm m_1 \pm m_2 \pm m_3\,.$$

The same holds for an odd number of $+$ signs.

Proof. Suppose that there are two choices of signs both with an even number of $+$ signs such that $C = \pm m_1 \pm m_2 \pm m_3$. Then the two equations must differ by two signs. Hence subtracting the two equations and dividing by ± 2 yields an equation of the form

$$0 = m_i \pm m_j$$

for some $1 \leqslant i < j \leqslant 3$. However, this is impossible since m_i and m_j are assumed non-negative and distinct. The argument when the number of $+$ signs is odd is analogous. $\square$

A similar analysis shows the following.

Lemma 4.2. *Let C, m_1, m_2, m_3 be non-negative integers with $C \geqslant 1$ and $m_1 > m_2 > m_3 \geqslant 0$. Then there are at most three choices of signs such that*

$$m_1 \pm m_2 \pm m_3 \pm C \pm C = 0\,. \tag{4.1}$$

We now place ourselves in the following general situation. Let Q be a rank 2 even positive-definite lattice having Gram matrix

$$\begin{pmatrix} M & N \\ N & P \end{pmatrix} \tag{4.2}$$

for some basis $\{z_1, z_2\}$. In the interest of applying Proposition 3.11, our goal will be to show that under certain assumptions on M, N, and P, one can primitively embed Q in E_8 in such a way that the number of roots in E_8 orthogonal to Q is between 2 and 14. We start with a lemma. We view E_8 as a sublattice of $\left(\frac{1}{2}\mathbb{Z}\right)^{\oplus 8}$; see Section 2.1.

Lemma 4.3. *Let $x = \alpha_1 e_1 + \cdots + \alpha_8 e_8$ be an integral vector in the lattice E_8, that is, with $\alpha_1, \ldots, \alpha_8 \in \mathbb{Z}$. Assume further that x is primitive in $\mathbb{Z}^8$, that is, $\alpha_1, \ldots, \alpha_8$ are coprime. If $x = mv$ for some integer m and $v \in E_8 \setminus D_8$, then α_i is odd for all $i \in \{1, \ldots, 8\}$.*

Proof. Since $v \notin D_8$, we have $v \in \frac{1}{2}(e_1 + \cdots + e_8) + D_8$. Writing $v = \frac{1}{2}(e_1 + \cdots + e_8) + \sum_{i=1}^8 y_i e_i$ with $\sum_{i=1}^8 y_i e_i \in D_8$, we have

$$x = \sum_{i=1}^{8} \left(\frac{m}{2} + m y_i\right) e_i\,.$$

Since $\alpha_1,\dots,\alpha_8\in\mathbb{Z}$, it follows that m is even. However, since $v\notin D_8$ and $x\in\mathbb{Z}^8$ is primitive, we must have that 4 does not divide m. Hence for each $i\in\{1,\dots,8\}$, we have $\alpha_i=m/2+my_i$, where $m/2$ is odd and my_i is even. □

In Proposition 4.4 below, we consider solutions (x_1,x_2,x_3) to an integral linear Diophantine equation of the form

$$\alpha_1X_1+\alpha_2X_2+\alpha_3X_3=K\,, \tag{4.3}$$

where α_1, α_2, α_3 are fixed coprime integers exactly one of which is even. Further, we require all the x_i to be odd when K is even, and exactly one x_i to be even when K is odd. In both cases, solutions exist. If K is odd, if we put $X_i=2Y_i+1$, then the odd solutions that we want correspond to solutions of

$$\alpha_1Y_1+\alpha_2Y_2+\alpha_3Y_3=\tfrac{1}{2}(K-(\alpha_1+\alpha_2+\alpha_3))\,,$$

which exist because the α_i are coprime. If K is even, we can reduce to the odd case: We may assume without loss of generality that α_1 is odd; then if we put $X_1=Y_1+1$, $X_2=Y_2$, and $X_3=Y_3$, the solutions that we want correspond to odd solutions of

$$\alpha_1Y_1+\alpha_2Y_2+\alpha_3Y_3=K-\alpha_1\,.$$

The following proposition constructs the embeddings of the lattice (4.2) into E_8 needed to apply Proposition 3.11 by making essential use of Theorem 2.12. The given embeddings will depend on the residues of M and P modulo 4.

Proposition 4.4. *Let Q be an even lattice of rank 2 with basis and Gram matrix given by (4.2). Assume further that $M>8$ and $M\notin\{20,24\}$. We fix*

$$\Omega=\begin{cases}0 & \text{if } M\equiv 2 \bmod 4,\ M\notin\{18,22,102,\star\}\,,\\ 1 & \text{if } M\equiv 0 \bmod 4,\ M\notin\{104,\star+2\}\,,\\ 2 & \text{if } M\in\{18,22,102,\star\}\,,\\ 3 & \text{if } M\in\{104,\star+2\}\end{cases}$$

and write $M-2\Omega^2$ as a sum of three pairwise distinct coprime squares (see Theorem 2.12):

$$M-2\Omega^2=\alpha_1^2+\alpha_2^2+\alpha_3^2$$

with $\alpha_1>\alpha_2>\alpha_3\geqslant 0$. Let (x_1,x_2,x_3) be a solution to the equation

$$\alpha_1X_1+\alpha_2X_2+\alpha_3X_3=N-2\Omega\Theta\,,$$

where $\Theta=0$ if $4|P$ and $\Theta=1$ otherwise, and we require all x_i to be odd when N is even and exactly one x_i to be even when N is odd. When we have $\alpha_3=0$, we set $x_3=1$. Let $S=P-\left(x_1^2+x_2^2+x_3^2\right)-2\Theta^2$. If $S>5$ and

$$S\notin\begin{cases}\{8\} & \text{if } \Omega=\Theta=0\,,\\ \{6\} & \text{if } \Omega=0,\Theta=1\,,\\ \{6,9,18,22,33,57,102,177,\star\} & \text{otherwise}\,,\end{cases}$$

there is a primitive embedding $Q\hookrightarrow E_8$ such that the number of roots in the orthogonal complement $Q^\perp$ satisfies

$$2\leqslant|R(Q)|\leqslant 14\,.$$

Proof. Begin by noting that $S = P - (x_1^2 + x_2^2 + x_3^3) - 2\Theta^2$ must satisfy $S \equiv 1, 2, 5, 6 \mod 8$. We separate the proof into several cases depending on the values of S. We are under the assumption $S > 5$.

Case 1. Assume

$$S \notin \begin{cases} \{10, 13, 25, 37, 58, 85, 130, \star\} & \text{if } \Omega = \Theta = 0\,, \\ \{6, 9, 18, 22, 33, 57, 102, 177, \star\} & \text{otherwise}\,. \end{cases}$$

Let $x_6 \geqslant x_7 \geqslant x_8$ be non-negative integers such that

$$S = x_6^2 + x_7^2 + x_8^2\,,$$

where if $\Omega = \Theta = 0$, we ask that x_6, x_7, x_8 are positive and coprime, and if Θ, Ω are not both zero, then we ask the x_i to be pairwise distinct and coprime; see Theorem 2.12. Consider the embedding $Q \hookrightarrow E_8$ given by $z_1 \mapsto v_1$, $z_2 \mapsto v_2$, where

$$v_1 = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \Omega e_4 + \Omega e_5\,,$$
$$v_2 = x_1 e_1 + x_2 e_2 + x_3 e_3 + \Theta e_4 + \Theta e_5 + x_6 e_6 + x_7 e_7 + x_8 e_8\,.$$

Note that $(v_1, v_1) = M$, $(v_1, v_2) = N$, and $(v_2, v_2) = P$. Moreover, as M and P are even, the sums of the coefficients of each of v_1 and v_2 are even; hence v_1 and v_2 are primitive elements in $D_8 \subset \mathbb{Z}^8$. In order to check the primitivity of the embedding, assume that there exist coprime integers r, s such that

$$rv_1 + sv_2 = mv$$

is a multiple of an element $v \in E_8$. If $v \in D_8$, then m divides sx_6, sx_7, and sx_8. Since x_6, x_7, x_8 are coprime, it follows that m divides s. Similarly, m must divide $r\alpha_i + sx_i$ for $i \in \{1, 2, 3\}$, and as α_1, α_2, α_3 are coprime, it follows that m divides r. But r and s are assumed to be coprime; thus $m = \pm 1$. If $v \in E_8 \setminus D_8$, since neither M nor S is congruent to 3 modulo 8, we know that at least one α_i and one of x_6, x_7, x_8 must be even. Primitivity then follows from Lemma 4.3.

We now count the roots in $Q^\perp$. If $\Omega = \Theta = 0$, one observes that the integral roots are

(i) $\pm e_4 \pm e_5$,
(ii) $\pm(e_3 - e_j)$ for $j \in \{6, 7, 8\}$ if $\alpha_3 = 0$ and $1 = x_j$,
(iii) $\pm(e_i - e_j)$ for $6 \leqslant i < j \leqslant 8$ if $x_i = x_j$.

Note that since $S \neq 3$, there are at most 4 roots of type (ii), and since x_6, x_7, x_8 are coprime, there are at most 2 roots of type (iii). Hence there are between 4 and 10 integral roots in $Q^\perp \subset E_8$. Moreover, if $\alpha_3 \neq 0$, there are at most 6 integral roots.

If Ω and Θ are not both 0, the integral roots in $Q^\perp$ are

(i) $\pm(e_4 - e_5)$,
(ii) $\pm(e_k - e_i)$ for $k \in \{1, 2, 3\}$ and $i \in \{4, 5\}$ if $\alpha_k = \Omega$ and $x_k = \Theta$,
(iii) $\pm(e_3 - e_j)$ for $j \in \{6, 7, 8\}$ if $\alpha_3 = 0$ and $x_j = 1$,
(iv) $\pm(e_i - e_j)$ for $i = 4, 5$ and $j \in \{6, 7, 8\}$ if $\Omega = 0$ and $\Theta = x_j$.

Note that since α_1, α_2, α_3 are distinct, there are at most 4 roots of type (ii). Since x_6, x_7, x_8 are distinct, there are at most 2 roots of type (iii) and 4 of type (iv). Therefore, in this case there are between 2 and 12 integral roots in $Q^\perp$. Moreover, if $\alpha_3 \neq 0$, the number of integral roots is at most 6 (since in this case (ii) and (iv) cannot happen simultaneously), and if $\Omega \neq 0$, the number of integral roots is at most 8.

Next, we count the fractional roots in $Q^\perp$. Suppose that

$$w = \tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 \pm e_7 \pm e_8) \tag{4.4}$$

is a fractional root in $Q^\perp$. So, the number of + signs in (4.4) is even. Since $(w, v_1) = 0$, we must have a choice of signs such that

$$\alpha_1 \pm \alpha_2 \pm \alpha_3 \pm \Omega \pm \Omega = 0\,. \tag{4.5}$$

Let C be the sum $x_1 \pm x_2 \pm x_3 \pm \Theta \pm \Theta$ for the same choice of signs as in (4.5). Then as $(w, v_2) = 0$, we have

$$C = \pm x_6 \pm x_7 \pm x_8\,, \tag{4.6}$$

where the numbers of + signs in (4.6) and (4.5) have opposite parities. If $\Omega = \Theta = 0$, then x_6, x_7, x_8 are all positive and not necessarily distinct. Begin by noting that since $\alpha_1 > \alpha_2 > \alpha_3$ and $\Omega = 0$, the only way for an equation of the form (4.5) to hold is if $\alpha_1 = \alpha_2 + \alpha_3$, so the first three signs in w are $\pm(e_1 - e_2 - e_3)$. Moreover, since α_1, α_2, α_3 are distinct, this can only happen if $\alpha_3 \neq 0$. Hence when $\Omega = \Theta = 0$, if there are fractional roots, there are at most 6 integral roots.

Note moreover that in the case $\Omega = \Theta = 0$, if there are two choices of signs differing by one sign such that (4.6) holds, then subtracting the two equations will yield $0 = \pm 2x_i$ for some $i \in \{6, 7, 8\}$. But this is impossible since x_6, x_7, x_8 are assumed to be non-zero.

If there are two choices of signs differing by two signs such that (4.6) holds, then adding the two equations yields $2x_i = \pm 2C$ for some $i \in \{6, 7, 8\}$, and subtracting the two equations yields $2x_j \pm 2x_k = 0$ for $j, k \in \{6, 7, 8\}$ and $j, k \neq i$. Since x_6, x_7, and x_8 are coprime and positive, we then know that x_j and x_k cannot also be equal to $\pm C$ and moreover $C \neq 0$. It follows that the choices of signs such that $C = \pm x_6 \pm x_7 \pm x_8$ are precisely $C = C \pm (x - x)$. Hence possible fractional roots in this case are of the form

$$w = \pm\tfrac{1}{2}(e_1 - e_2 - e_3 \pm e_4 \pm e_5 - (e_6 \pm (e_7 - e_8)))\,,$$

where the number of + signs is even. Possible fractional roots in this case satisfy

$$\begin{aligned}\pm w &= \tfrac{1}{2}(e_1 - e_2 - e_3 \pm e_4 \pm e_5 - (e_6 \pm (e_7 - e_8)))\\ &= \tfrac{1}{2}(e_1 - e_2 - e_3 \pm (e_4 + e_5) - e_6 \pm (e_7 - e_8))\,,\end{aligned}$$

as the number of + signs on the right-hand side must be even. In particular, there are at most 8 fractional roots.

The only remaining way to have two choices of signs such that equation (4.6) holds is if the two choices differ by three signs. In this case $C = 0$, so by the previous analysis, we cannot have any additional choices of signs such that $C = \pm x_6 \pm x_7 \pm x_8$. These two choices will each correspond to at most 4 fractional roots. We have shown that when $\Omega = \Theta = 0$, there are at most 6 integral and 8 fractional roots; in particular,

$$2 \leqslant |R(Q)| \leqslant 14\,.$$

Now consider the case where Ω and Θ are not both 0. Then x_6, x_7, x_8 are pairwise distinct, and by Lemma 4.1, there is at most one choice of signs such that (4.6) holds. It follows that for each choice of signs such that (4.5) holds, there are at most 2 fractional roots in $Q^\perp$ (of the form $\pm w$). However, by Lemma 4.2, there are at most three choices of signs such that equation (4.5) holds. Hence there are at most 6 fractional roots in $Q^\perp$.

Note moreover that when $\Omega \neq 0$, there are at most 8 integral roots in $Q^\perp$, so the total number of roots in $Q^\perp$ is between 2 and 14. If $\Omega = 0$ and $\Theta \neq 0$, then as $\alpha_1 > \alpha_2 > \alpha_3$, the only way to have equation (4.5) hold is if $\alpha_1 = \alpha_2 + \alpha_3$, meaning in particular that $\alpha_3 \neq 0$. It follows that if $\Omega = 0$, $\Theta \neq 0$, and there are fractional roots, then there are most 6 integral roots. Hence the total number of roots is between 2 and 12.

Case 2. Assume $\Omega = 0$ and

$$S \in \begin{cases} \{13, 25, 37, 58, 85, 130, \star\} & \text{if } \Theta = 0\,, \\ \{33, 57, 102, 177, \star\} & \text{if } \Theta = 1\,. \end{cases}$$

We fix $\Xi = 2$ if $\Theta = 0$ and $\Xi = 3$ if $\Theta = 1$. Observe that $R := S + 2\Theta - 2\Xi^2$ can be written as a sum of three different coprime squares, $R = y_6^2 + y_7^2 + y_8^2$, where without loss of generality $y_6 > y_7 > y_8$. Note, moreover, that $R \equiv S \mod 8$; thus in particular, R is not congruent to 3 modulo 8. Hence one of y_6, y_7, y_8 must be even. In this case, consider the embedding $Q^\perp \subset E_8$ given by $z_1 \mapsto v_1$ and $z_2 \mapsto v_2$, where

$$\begin{aligned} v_1 &= \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3\,, \\ v_2 &= x_1 e_1 + x_2 e_2 + x_3 e_3 + \Xi e_4 + \Xi e_5 + y_6 e_6 + y_7 e_7 + y_8 e_8\,. \end{aligned}$$

As with the previous case, note that the v_i are in D_8. From

$$\gcd(\alpha_1, \alpha_2, \alpha_3) = \gcd(y_6, y_7, y_8) = 1$$

together with the fact that one of the α_j and one of the y_j must be even, we see that the embedding is primitive; see Lemma 4.3.

The integral roots for the above embedding are the same as those listed for the previous embedding in the case where Ω and Θ are not both 0. Thus there are between 2 and 12 integral roots in $Q^\perp$, with at most 6 integral roots when $\alpha_3 \neq 0$. The analysis of the fractional roots is also the same as in the previous case where Ω and Θ are not both 0. In particular, because $\Omega = 0$, in order to have fractional roots, we must have $\alpha_3 \neq 0$, and there are at most 6 fractional roots. Since in this case, there are at most 6 integral roots, it follows that the total number of roots is between 2 and 12.

For the next cases, we only give the embedding. The primitivity and root counting arguments are analogous to the first two cases.

Case 3. Assume $\Omega = 0$, $\Theta = 0$, and $S = 10$. In this case, the embedding $Q^\perp \subset E_8$ is given by

$$\begin{aligned} z_1 &\longmapsto v_1 = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3\,, \\ z_2 &\longmapsto v_2 = x_1 e_1 + x_2 e_2 + x_3 e_3 + e_4 + e_5 + 2e_6 + 2e_7\,. \end{aligned}$$

Case 4. Assume $\Omega = 0$, $\Theta = 1$, and $S = 9$. In this case, the embedding $Q^\perp \subset E_8$ is given by

$$\begin{aligned} z_1 &\longmapsto v_1 = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3\,, \\ z_2 &\longmapsto v_2 = x_1 e_1 + x_2 e_2 + x_3 e_3 + 3e_6 + e_7 + e_8\,. \end{aligned}$$

Case 5. Assume $\Omega = 0$, $\Theta = 1$, and $S = 18$. In this case, the embedding $Q^\perp \subset E_8$ is given by

$$\begin{aligned} z_1 &\longmapsto v_1 = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3\,, \\ z_2 &\longmapsto v_2 = x_1 e_1 + x_2 e_2 + x_3 e_3 + e_4 + e_5 + 3e_6 + 3e_7\,. \end{aligned}$$

Case 6. Assume $\Omega = 0$, $\Theta = 1$, and $S = 22$. In this case, the embedding $Q^\perp \subset E_8$ is given by

$$\begin{aligned} z_1 &\longmapsto v_1 = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3\,, \\ z_2 &\longmapsto v_2 = x_1 e_1 + x_2 e_2 + x_3 e_3 + e_4 + e_5 + 3e_6 + 3e_7 + 2e_8\,. \end{aligned}$$

We have exhausted all cases of the proposition. □

4.2 General-type results for divisibility at least 3

If $\gamma \geqslant 3$, then combining Propositions 3.11 and 4.4 yields the following result.

Theorem 4.5. *Let n, d, γ, a be positive integers satisfying the hypothesis of Proposition 3.1 such that $n \geqslant 6$, $n \notin \{11, 13\}$, $\gamma \geqslant 3$, and $d(n-1)/\gamma^2 > 4$. Let $M = 2(n-1)$, $N = -a(2(n-1)/\gamma)$, and $P = 2t$. Then, in the notation of Proposition 4.4, if $S > 5$ and*

$$S \notin \begin{cases} \{8\} & \text{if } \Omega = \Theta = 0\,, \\ \{6\} & \text{if } \Omega = 0, \Theta = 1\,, \\ \{6, 9, 18, 22, 33, 57, 102, 177, \star\} & \text{otherwise}\,, \end{cases}$$

the component of $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ corresponding to a (see Proposition 3.5) is of general type.

We spell out an example below to show that the above result can give different bounds on d for different components of the moduli space $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$.

Example 4.6. Consider the moduli space $\mathcal{M}^{5}_{K3^{[26]},2d}$ given by taking $n = 26$ and $\gamma = 5$. Then the number M in Proposition 4.4 is $2(n-1) = 50$, and $\Omega = 0$. Note that there are exactly two ways to write $2(n-1) = 50$ as a sum of three distinct coprime squares:

$$50 = 7^2 + 1^2 + 0^2 = 5^2 + 4^2 + 3^2\,.$$

We can take $(\alpha_1, \alpha_2, \alpha_3)$ to be $(7, 1, 0)$ or $(5, 4, 3)$. The numbers N and P are, respectively, $-10a$ and $2t = 2d/25 + 2a^2$, where $a \in \{1, 2\}$ (see (3.2)). If we consider the component $\mathcal{M}^{5,1}_{K3^{[26]},2d}$ of $\mathcal{M}^{5}_{K3^{[26]},2d}$ corresponding to $a = 1$, we then consider odd solutions to the equations, for the two possibilities of $(\alpha_1, \alpha_2, \alpha_3)$, respectively

$$7x_1 + x_2 = -10 \quad \text{and} \quad 5x_1 + 4x_2 + 3x_3 = -10\,.$$

The minimal-norm solutions are then respectively

$$(x_1, x_2, x_3) = (-1, -3, 1) \quad \text{and} \quad (x_1, x_2, x_3) = (1, -3, -1)\,,$$

both of which have norm 11. The number Θ is 0 when $2t \equiv 0 \mod 4$ and is 1 otherwise, and we find $S = 2\big(t - \Theta^2\big) - 11$. The bounds given in Proposition 4.4 imply, via Proposition 3.11, that $\mathcal{M}^{5,1}_{K3^{[26]},2d}$ is of general type when $t \geqslant 10$, that is, $d \geqslant 225$. By contrast, if we consider the component $\mathcal{M}^{5,2}_{K3^{[26]},2d}$, we take odd solutions to the equations, respectively

$$7x_1 + x_2 = -20 \quad \text{and} \quad 5x_1 + 4x_2 + 3x_3 = -20\,.$$

The minimal-norm solutions are then respectively

$$(x_1, x_2, x_3) = (-3, -1, 1) \quad \text{and} \quad (x_1, x_2, x_3) = (-1, -3, -1)\,,$$

both of which again have norm 11. Hence, $\mathcal{M}^{5,2}_{K3^{[26]},2d}$ is of general type when $t \geqslant 10$, that is, $d \geqslant 150$.

The above illustrates for instance that the moduli space $\mathcal{M}^{5}_{K3^{[26]},300}$ has one connected component (corresponding to $a = 2$) that our results show is of general type and one connected component (corresponding to $a = 1$) for which our results do not yield a statement about the Kodaira dimension.

Given particular (n, γ, a), Theorem 4.5 allows one to compute a lower bound on d after which the given connected component of $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ is of general type. However, since obtaining this bound requires computing α_1, α_2, α_3 as well as x_1, x_2, x_3, Theorem 4.5 is not so useful for understanding uniformly when $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ is of general type. For this, we will compute a much coarser bound obtained by using the following.

Lemma 4.7 ([BFRT89, Main Theorem]). *Let $\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_N X_N = b$ be a linear equation with integer coefficients. If it admits an integral solution, then it has a solution $(x_1, \ldots, x_N)$ such that $\max|x_i| \leqslant \max\{|a_1|, \ldots, |a_N|, |b|\}$.*

Theorem 4.8. *Let (n, d, γ) be a triple such that the moduli space $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ is non-empty (see Proposition 3.1). We assume further that $n \geqslant 6$, $n \notin \{11, 13\}$, and $\gamma \geqslant 3$. Then every component of $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$ is of general type provided*

$$d \geqslant 6\gamma^2\big(n + 3 + \sqrt{2(n-1)}\big)^2\,,$$

except for one possible value of $d \geqslant 5\cdot 10^{10}$ for each n that is odd or in the set $\{10, 12, 52, \star/2+1\}$.

Proof. Let Q_h be the lattice

$$Q_h = \begin{pmatrix} 2(n-1) & -a\frac{2(n-1)}{\gamma} \\ -a\frac{2(n-1)}{\gamma} & 2t \end{pmatrix},$$

where a and t satisfy

$$\frac{2d}{\gamma} \equiv -\frac{2(n-1)}{\gamma}a^2 \mod 2\gamma \quad \text{and} \quad d = \gamma^2 t - (n-1)a^2\,.$$

Observe that if $d \geqslant 6\gamma^2\big(n + 3 + \sqrt{2(n-1)}\big)^2$, then certainly $d(n-1)/\gamma^2 > 4$ and so by Proposition 3.11, we need only verify that the hypotheses of Proposition 4.4 are satisfied.

Let $M = 2(n-1)$, $N = -a(2(n-1)/\gamma)$, and $P = 2t$. Then fix Ω, α_1, α_2, α_3, (x_1, x_2, x_3), Θ, and S as in the statement of Proposition 4.4. Then since $M - 2\Omega^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2$, for all $i \in \{1, 2, 3\}$, we have

$$0 \leqslant \alpha_i < \sqrt{M - 2\Omega^2}\,. \tag{4.7}$$

Additionally, the fact that $0 < a < \gamma$ implies

$$0 < -N < M\,. \tag{4.8}$$

Let us first consider the case N even. Recall that odd solutions (x_1, x_2, x_3) to the Diophantine equation

$$\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 = N - 2\Omega\Theta$$

are (by setting $X_i = 2Y_i + 1$) equivalent to integral solutions of

$$\alpha_1 Y_1 + \alpha_2 Y_2 + \alpha_3 Y_3 = \tfrac{1}{2}(N - 2\Omega\Theta - (\alpha_1 + \alpha_2 + \alpha_3))\,. \tag{4.9}$$

Lemma 4.7 implies the existence of a solution (y_1, y_2, y_3) to (4.9) satisfying

$$\max|y_i| \leqslant \max\left\{\alpha_1, \alpha_2, \alpha_3, \tfrac{1}{2}(-N + 2\Theta\Omega + \alpha_1 + \alpha_2 + \alpha_3)\right\}.$$

Also note that

$$\alpha_1 + \alpha_2 + \alpha_3 \leqslant \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2} = \sqrt{M - 2\Omega^2}. \tag{4.10}$$

Equations (4.8) and (4.10) give us

$$\tfrac{1}{2}(-N + 2\Omega\Theta + (\alpha_1 + \alpha_2 + \alpha_3)) < \tfrac{1}{2}\big(M + 2\Omega\Theta + \sqrt{M - 2\Omega^2}\big)\,.$$

Moreover, note that for all integers $M > 0$, we have

$$0 < \sqrt{M - 2\Omega^2} < \tfrac{1}{2}\big(M + 2\Omega\Theta + \sqrt{M - 2\Omega^2}\big)\,.$$

Therefore, when N is even,

$$\max|y_i| < \tfrac{1}{2}\big(M + 2\Omega\Theta + \sqrt{M - 2\Omega^2}\big) \quad \text{and} \quad \max|x_i| < M + 2\Omega\Theta + 1 + \sqrt{M - 2\Omega^2}\,.$$

We now turn to the case where N is odd. As discussed before Proposition 4.4, we may produce a solution (x_1, x_2, x_3) with exactly one x_i even to the equation

$$\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 = N - 2\Omega\Theta\,,$$

where α_i is odd. By setting $X_i = 2(Y_i+1)$ and $X_j = 2Y_j+1$ for $i \neq j$, such a solution (x_1, x_2, x_3) can be obtained from an integral solution (y_1, y_2, y_3) to

$$\alpha_1 Y_1 + \alpha_2 Y_2 + \alpha_3 Y_3 = \tfrac{1}{2}(N - 2\Omega\Theta - \alpha_i - (\alpha_1 + \alpha_2 + \alpha_3))\,.$$

Lemma 4.7 implies that such a solution (y_1, y_2, y_3) satisfies

$$\max|y_i| \leqslant \max\big\{\alpha_1, \alpha_2, \alpha_3, \tfrac{1}{2}(-N + 2\Omega\Theta + \alpha_i + (\alpha_1 + \alpha_2 + \alpha_3))\big\}\,.$$

Hence using the bounds (4.7), (4.8), and (4.10), when N is odd, we get

$$\max|y_i| < \tfrac{1}{2}\big(M + 2\Omega\Theta + 2\sqrt{M - 2\Omega^2}\big) \quad \text{and} \quad \max|x_i| < M + 2\Omega\Theta + 2 + 2\sqrt{M - 2\Omega^2}.$$

Comparing the bounds on $|x_i|$ in the N even and N odd cases, we see that for any N,

$$x_1^2 + x_2^2 + x_3^2 < 3\big(M + 2\Omega\Theta + 2 + 2\sqrt{M - 2\Omega^2}\big)^2\,. \tag{4.11}$$

It follows that

$$S > P - 3\big(M + 2\Omega\Theta + 2 + 2\sqrt{M - 2\Omega^2}\big)^2 - 2\Theta^2\,. \tag{4.12}$$

However, the assumption $d \geqslant 6\gamma^2\big(n + 3 + \sqrt{2(n-1)}\big)^2$ can be rewritten as

$$P\gamma^2 - Ma^2 \geqslant 3\gamma^2\big(M + 8 + \sqrt{M}\big)^2\,.$$

In particular, we have

$$P \geqslant 3\big(M + 8 + \sqrt{M}\big)^2\,. \tag{4.13}$$

Now a verification shows that for all $M \geqslant 10$, $\Omega \in \{0, 1, 2, 3\}$, and $\Theta \in \{0, 1\}$, we have

$$3\big(M + 8 + \sqrt{M}\big)^2 - 3\big(M + 2\Omega\Theta + 2 + 2\sqrt{M - 2\Omega^2}\big)^2 - 2\Theta^2 > 600\,.$$

In particular, (4.12) implies that $S > 600$, which by Proposition 4.4 finishes the proof. □

5. Monodromy and pullbacks for K3$^{[n]}$-type with low divisibility

In this section, we treat the case $\gamma \in \{1, 2\}$. The main difficulty lies in the fact that the monodromy group $\widehat{O}^+(\Lambda, h)$ and the stable orthogonal group $\widetilde{O}^+(\Lambda_h)$ are no longer isomorphic. For a primitive embedding $\Lambda_h \hookrightarrow \mathrm{II}_{2,26}$, the quasi-pullback $F(Z)$ defined in (2.11) is not necessarily modular for $\widehat{O}^+(\Lambda, h)$, so we have to choose the embedding carefully to ensure that $\widehat{O}^+(\Lambda, h)$ satisfies the hypothesis of Lemma 2.6. Recall that the restriction $O(\Lambda, h) \to O(\Lambda_h)$ induces an isomorphism $\widetilde{O}^+(\Lambda_h) \cong \widetilde{O}^+(\Lambda, h)$; see Proposition 3.8. When $\gamma \in \{1, 2\}$ and $n \geqslant 3$, the group $\widehat{O}^+(\Lambda, h)$ is generated (see Lemma 3.7) by $\widetilde{O}^+(\Lambda_h)$ and $\sigma_v \in O(\Lambda)$, where

$$v = \frac{2a(n-1)}{\gamma} f - \ell \in \Lambda\,.$$

In order to treat the case $\gamma \in \{1,2\}$, we need to ensure that the reflection $\sigma_v \in \widehat{O}^+(\Lambda, h)$ of Lemma 3.7 extends to $\mathrm{II}_{2,26}$ after choosing our embedding $Q_h \hookrightarrow E_8$. Let $\sigma \in O(E_8)$ be an involution on E_8 and V_+, V_- the invariant and anti-invariant lattices, respectively. Note that both V_- and V_+ are primitive, $V_- = (V_+)^\perp$, and the sublattice $V_- \oplus V_+ \subset E_8$ has finite index. Moreover, the unimodularity of E_8 implies that there is an isometry

$$\phi\colon D(V_+) \longrightarrow D(V_-)$$

(up to a sign) and the finite-index extension $V_- \oplus V_+ \subset E_8$ corresponds (see Section 2.1) to the isotropic subgroup

$$H_\phi = \{(x, \phi(x)) \in D(V_+ \oplus V_-) \mid x \in D(V_+)\} \subset D(V_+ \oplus V_-)\,.$$

Moreover, an isometry of the form

$$g = g_- \oplus g_+ \in O(V_- \oplus V_+)$$

extends to an isometry on E_8 if $\overline{g}$ fixes H_ϕ, which is equivalent to

$$\phi \circ \overline{g_+} = \overline{g_-} \circ \phi\,; \tag{5.1}$$

see [Nik80, Corollary 1.5.2]. Recall that a lattice L is called *n-elementary* if $D(L)$ is isomorphic to a direct sum of cyclic groups of order n. An immediate consequence is then the following.

LEMMA 5.1. *Let $Q \subset E_8$ be a primitive sublattice and $\sigma_r \in O(Q)$ a reflection with respect to an element $r \in Q$. Let M be the orthogonal complement of r in Q. The following are equivalent:*

(i) *There exists a 2-elementary primitive sublattice $V_+ \subset E_8$ such that $M \subset V_+$ and $r \in V_- = (V_+)^\perp$.*

(ii) *The map σ_r extends to an involution on E_8 acting as* Id *on V_+ and as* $-$Id *on V_-.*

Proof. Assume that statement (i) holds. Then $g = -\mathrm{Id} \oplus \mathrm{Id} \in O(V_- \oplus V_+)$ extends the reflection $\sigma_r \in O(Q)$. Moreover, g extends to an element of $O(E_8)$ since V_- is 2-elementary, $-\mathrm{Id}$ acts as the identity on $D(V_-) \cong D(V_+)$, and (5.1) is immediately satisfied. The other implication is [GHS07, Lemma 3.5]. ☐

LEMMA 5.2. *Let $Q \subset E_8$ be a primitive embedding and $\sigma_r \in O(Q)$ a reflection with respect to a primitive element $r \in Q$ that extends to an involution $\widetilde{\sigma}_r \in O(E_8)$. As before, we write V_+ and V_- for the invariant and anti-invariant sublattices of $\widetilde{\sigma}_r$. Denote by $R\big(r^\perp \cap V_-\big)$ the set of roots in $r^\perp \cap V_-$. Let $L = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus Q(-1)$, and consider the induced primitive embedding $L \hookrightarrow \mathrm{II}_{2,26}$. If*

$$\mathrm{rk}(V_-) + \tfrac{1}{2}\big|R\big(r^\perp \cap V_-\big)\big| \equiv 1 \mod 2\,, \tag{5.2}$$

then the quasi-pullback of $\Phi_{12} \in M_{12}(O^+(\mathrm{II}_{2,26}), \det)$ to $\Omega^\bullet(L)$ is modular with respect to $\det\colon \Gamma \to \mathbb{C}^*$ for the subgroup $\Gamma \subset O^+(L)$ generated by $\widetilde{O}^+(L)$ and σ_r.

Proof. Let $R_{>0} \cup R_{<0}$ be a sign-partition of the set of roots $R\big(Q^\perp\big)$ in E_8 orthogonal to Q (see Section 2.4). Let M be the number of roots that change sign under $\widetilde{\sigma}_r$; see (2.13). By Lemma 2.6 and Corollary 2.7, it is enough to check that $\det(\sigma_r) = (-1)^M \cdot \det(\widetilde{\sigma}_r)$ or, equivalently,

$$M + \mathrm{rk}(V_-) \equiv 1 \mod 2\,.$$

Note that for any root $\eta \in R_{>0}$, if $\eta \in V_+$, then $\widetilde{\sigma}_r(\eta) = \eta \in R_{>0}$, and if $\eta \in V_-$, then $\widetilde{\sigma}_r(\eta) = -\eta \in R_{<0}$. If $\eta \notin V_- \cup V_+$, then $\widetilde{\sigma}_r(\eta) \neq \pm\eta$. Let $\{\eta_1, \ldots, \eta_k\}$ be all roots in $R_{>0}$ not in $V_- \cup V_+$. Then $\widetilde{\sigma}_r$ induces an injection

$$\{\eta_1, \ldots, \eta_k\} \longrightarrow \{\eta_1, \ldots, \eta_k, -\eta_1, \ldots, -\eta_k\}$$

with $\widetilde{\sigma}_r(\eta_i) \neq \pm\eta_i$. One immediately sees that if $\widetilde{\sigma}_r(\eta_i) = -\eta_j$, then $\widetilde{\sigma}_r(\eta_j) = -\eta_i$; thus

$$\big|\{\widetilde{\sigma}_r(\eta_1), \ldots, \widetilde{\sigma}_r(\eta_k)\} \cap \{-\eta_1, \ldots, -\eta_k\}\big| \equiv 0 \mod 2$$

and $M \equiv |R_{>0} \cap V_-| \mod 2$. Finally, we have $Q \cap V_- = \langle r \rangle$ and

$$|R_{>0} \cap V_-| = \tfrac{1}{2} R\big(Q^\perp \cap V_-\big) = \tfrac{1}{2} R\big(r^\perp \cap V_-\big)\,. \qquad \square$$

We have to choose our embedding $Q_h \hookrightarrow E_8$ not only to have a controlled number of roots in the orthogonal complement, but also in such a way that the reflection σ_v defined in Lemma 3.7 can be extended to an involution $\widetilde{\sigma}_v \in O(E_8)$ satisfying (5.2). Note that if $\gamma \in \{1, 2\}$, by Proposition 3.5, we may choose

$$a = \begin{cases} 0 & \text{when } \gamma = 1\,, \\ 1 & \text{when } \gamma = 2\,. \end{cases} \tag{5.3}$$

In particular, by Lemma 3.4, the element $h \in \Lambda$ can be chosen as

$$h = \begin{cases} e + df & \text{when } \gamma = 1\,, \\ 2(e + tf) - \ell & \text{when } \gamma = 2\,, \end{cases} \tag{5.4}$$

where $d = 4t - (n-1)$ when $\gamma = 2$; see Proposition 3.1.

Recall that $Q_h(-1) = \langle h \rangle^\perp \subset U \oplus \langle \ell \rangle$ is generated by $\{(2a(n-1)/\gamma)f - \ell, e - tf\}$, and

$$Q_h = \begin{pmatrix} 2(n-1) & 0 \\ 0 & 2d \end{pmatrix} \text{ if } \gamma = 1\,, \quad Q_h = \begin{pmatrix} 2(n-1) & -(n-1) \\ -(n-1) & 2t \end{pmatrix} \text{ if } \gamma = 2\,. \tag{5.5}$$

Summarizing the discussion above, and adding the conditions for modularity of the quasi-pullback in Lemma 2.6, the vanishing at the ramification in Proposition 2.9, and a numerical condition ensuring that $\widehat{O}^+(\Lambda, h)$ has no irregular cusps, we have the following.

Proposition 5.3. *Let Q_h be the lattice defined in (5.5), $Q_h \hookrightarrow E_8$ a primitive embedding, and $\Lambda_h \hookrightarrow \mathrm{II}_{2,26}$ the induced embedding. Assume further that the following hold:*

(i) *There exists a primitive 2-elementary sublattice $V_+ \subset E_8$ such that*

$$z_1 = \frac{2a(n-1)}{\gamma} f - \ell \in V_- := (V_+)^\perp \subset E_8 \quad \text{and} \quad (z_1)^\perp \cap Q_h \subset V_+\,.$$

(ii) *The number of roots $\big|R\big(Q_h^\perp\big)\big|$ in the orthogonal complement $Q_h^\perp \subset E_8$ is at least 2 and at most 14 (respectively, 16).*

(iii) *The rank of V_- and half the number of roots orthogonal to Q_h in V_- have opposite parities:*

$$\mathrm{rk}(V_-) + \tfrac{1}{2}\big|R\big((z_1)^\perp \cap V_-\big)\big| \equiv 1 \mod 2\,.$$

(iv) *One of the following holds:*

  (a) *The number $|R(Q_h^\perp)|/2$ is even.*

  (b) *For any $r \in \Lambda_h \subset \mathrm{II}_{2,26}$ such that $-\sigma_r \in \widehat{O}^+(\Lambda, h)$, one has*

$$\big|R\big(\Lambda_h^\perp\big)\big| < \big|R\big((\Lambda_h)_r^\perp\big)\big|\,,$$

  where $(\Lambda_h)_r$ is the orthogonal complement of r in Λ_h and the orthogonal complements in the inequality are taken in $\mathrm{II}_{2,26}$.

(v) *We have $2d/\gamma \notin \{1, 2, 4, 8\}$.*

Then the modular variety $\Omega(\Lambda_h)/\widehat{O}^+(\Lambda, h)$ is of general type (respectively, has non-negative Kodaira dimension).

Proof. By Lemmas 5.2 and 3.7, the quasi-pullback F of the Borcherds form Φ_{12} to $\Omega^\bullet(\Lambda_h)$ is modular with character $\det\colon \widehat{O}^+(\Lambda,h) \to \mathbb{C}^*$ and weight $12 + |R(Q_h^\perp)|/2$; see (2.12). Moreover, since $2 \leqslant \left|R\left(Q_h^\perp\right)\right| \leqslant 16$, the form F is a cusp form [GHS13, Corollary 8.12]. We claim that F vanishes along the ramification divisor of the modular projection

$$\pi\colon \Omega(\Lambda_h) \longrightarrow \Omega(\Lambda_h)/\widehat{O}^+(\Lambda,h)\,.$$

Indeed, if the embedding $Q_h \hookrightarrow E_8$ satisfies assumption (iv)(a), this follows from Proposition 2.9. If the embedding satisfies assumption (iv)(b), it follows from Lemma 2.8, where we use that $\mathrm{rk}(\Lambda_h) = 22$. The proposition now follows from Theorem 2.3 once we have shown that $\Omega(\Lambda_h)$ has no irregular cusps for $\widehat{O}^+(\Lambda,h)$. By Proposition 2.2, it is enough to show this for 0-dimensional cusps. Let $I \subset \Lambda_h$ be a rank 1 primitive isotropic sublattice corresponding to a 0-dimensional cusp of $\Omega(\Lambda_h)$. Assume that I is irregular for $\widehat{O}^+(\Lambda,h)$; then by Proposition 2.5 (see also the proof of [Ma24, Proposition 4.11]), we have $-\mathrm{Id} \notin \widehat{O}^+(\Lambda,h)$ and $-E_w \in \Gamma(I)_\mathbb{Q} \cap \widehat{O}^+(\Lambda,h)$ for some $w \in L(I)_\mathbb{Q}$. Since I has rank 1, the vector w can be written as $m \otimes l$ with $m \in \left(I^\perp/I\right)_\mathbb{Q}$ and $l \in I_\mathbb{Q}$. Then by Lemma 3.7,

$$E_{2m\otimes l} = (-E_{m\otimes l}) \circ (-E_{m\otimes l}) \in \Gamma(I)_\mathbb{Q} \cap \widetilde{O}^+(\Lambda_h)\,.$$

The isomorphism (2.8) when restricted to $\widetilde{O}^+(\Lambda_h)$ induces an isomorphism $L(I) \cong U(I) \cap \widetilde{O}^+(\Lambda_h)$; see [Ma24, Lemma 4.1]. In particular, $2m \otimes l \in L(I)$. We choose a lift $\widetilde{m} \in \frac{1}{2}I^\perp$ of m; then for any $u \in (\Lambda_h)_\mathbb{Q}$, we have

$$-E_{m\otimes l}(u) = -u + (\widetilde{m},u)l - (l,u)\widetilde{m} + \tfrac{1}{2}(m,m)(l,u)l\,. \tag{5.6}$$

Moreover, any $g \in \widehat{O}^+(\Lambda,h)$ fixes the class $\overline{u} \in D(\Lambda_h)$ of any $u \in (z_1)^\perp \subset (\Lambda_h)^\vee$ (see Lemma 3.7). In particular, for any such u, we have $-E_{m\otimes l}(u) \in u + \Lambda_h$. From (5.6), since $(2\widetilde{m},u)$, (l,u), and $(2m,m)$ are integers, it follows that

$$2u \in \tfrac{1}{4}\Lambda_h\,,$$

so $8u \in \Lambda_h$ and $\mathrm{ord}(\overline{u})$ divides 8. Take $u = (\gamma/2d)\,(az_1 + \gamma z_2)$ as in the proof of Proposition 3.8. Indeed, then $(u,z_1) = 0$ and thus $\sigma_{z_1}(u) = u$. The order of the class $\overline{u}$ in $D(\Lambda_h)$ is $2d/\gamma$. In particular, we have $2d/\gamma \in \{1,2,4,8\}$. □

Recall that for $\gamma \in \{1,2\}$, the moduli space $\mathcal{M}^\gamma_{\mathrm{K3}^{[n]},2d}$ is connected when non-empty; see Proposition 3.5. As a corollary, we have the following.

Corollary 5.4. *Let (n,d,γ) be a triple such that $\gamma \in \{1,2\}$ and $\mathcal{M}^\gamma_{\mathrm{K3}^{[n]},2d}$ is non-empty. Assume that there exists an embedding $Q_h \hookrightarrow E_8$ satisfying the hypothesis of Proposition 5.3, with at most 14 (respectively, 16) roots in assumption (ii). Then $\mathcal{M}^\gamma_{\mathrm{K3}^{[n]},2d}$ is of maximal (respectively, non-negative) Kodaira dimension.*

Proof. This follows from Proposition 5.3 together with Theorem 2.3. □

6. Kodaira dimension for K3$^{[n]}$-type with divisibility 1

From now on we concentrate on the split case $\gamma = 1$. Recall that in this case, the moduli space $\mathcal{M}^1_{\mathrm{K3}^{[n]},2d}$ is irreducible; see Proposition 3.5. Note that in this case $a = 0$, and the element z_2 in Proposition 5.3 is given by

$$z_2 = \frac{2a(n-1)}{\gamma}f - \ell = -\ell\,.$$

Proposition 6.1. *Consider the two primitive 2-elementary orthogonal sublattices of E_8 given by*

$$V_- := \langle e_1 - e_5, e_2 - e_6, e_3 - e_7, e_4 - e_8 \rangle \cong A_1^{\oplus 4}\,,$$
$$V_+ := \langle e_1 + e_5, e_2 + e_6, e_3 + e_7, e_4 + e_8 \rangle \cong A_1^{\oplus 4}\,.$$

Let Q_h be as in (5.5), which in the case $\gamma = 1$ has basis

$$z_1 = \frac{2a(n-1)}{\gamma} f - \ell = -\ell \quad \text{and} \quad z_2 = e - tf = e - df\,. \tag{6.1}$$

Assume further that $n-1 > 6$, $n-1 \equiv 1, 2 \mod 4$, and $n - 1 \notin \{10, 13, 25, 37, 58, 85, 130, \star\}$. Then for any $d \geqslant 3$ with $d \not\equiv 0 \mod 4$, there is a primitive embedding $Q_h \hookrightarrow E_8$ sending z_1 to V_- and z_2 to V_+ such that the following hold:

(i) *The number $\frac{1}{2}\big|R\big((z_1)^\perp \cap V_-\big)\big|$ is odd.*
(ii) *The number of roots $\big|R\big(Q_h^\perp\big)\big|$ in the orthogonal complement $Q_h^\perp \subset E_8$ satisfies*
$$2 \leqslant \big|R\big(Q_h^\perp\big)\big| \leqslant 14\,.$$
(iii) *If, moreover, $d \not\equiv 7 \mod 8$ and $d \notin \{5, 10, 13, 25, 37, 58, 85, 130, \star\}$, then the primitive embedding $Q_h \hookrightarrow E_8$ may be chosen so that $\big|R\big(Q_h^\perp\big)\big| \equiv 0 \mod 4$.*

Proof. By Theorem 2.12, we may write $n - 1 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2$, where α_1, α_2, α_3 are coprime and positive. Moreover, since $n - 1 \not\equiv 3 \mod 4$, we may assume without loss of generality that α_1 is even and α_2 is odd. Similarly, by Corollary 2.16, we may write

$$d = x_1^2 + x_2^2 + x_3^2 + x_4^2\,, \tag{6.2}$$

with $x_1 \geqslant 0$ and $x_i > 0$ for $i \in \{2, 3, 4\}$ (except in the case $d = 5$, where $x_1 = x_2 = 0$). Since we know $d \not\equiv 0 \mod 4$, we may without loss of generality take x_1 to be even and x_4 odd. In the case where $d \not\equiv 7 \mod 8$ and $d \notin \{10, 13, 25, 37, 58, 85, 130, \star\}$, we let $x_1 = 0$ (which is possible by Theorem 2.12).

Consider the embedding $Q_h \hookrightarrow E_8$ given by

$$\begin{aligned} z_1 &\longmapsto v_1 = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 - \alpha_1 e_5 - \alpha_2 e_6 - \alpha_3 e_7\,, \\ z_2 &\longmapsto v_2 = x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_1 e_5 + x_2 e_6 + x_3 e_7 + x_4 e_8\,. \end{aligned} \tag{6.3}$$

Using Lemma 4.3 together with our assumptions on the x_i and α_i, one checks that the embedding (6.3) is primitive. Moreover, the given embedding has the property that $v_1 \in V_-$ and $v_2 \in V_+$. The intersection $v_1^\perp \cap V_-$ contains the single root $e_4 - e_8$. In particular, the number of roots in $v_1^\perp \cap V_-$ is odd.

Now we count roots in $Q_h^\perp \subset E_8$. We first assume $d \notin \{5, 6\}$. Then, in particular, $x_i > 0$ for $i \in \{2, 3, 4\}$. The integral roots are then

(i) $\pm(e_4 - e_8)$,
(ii) $\pm(e_1 + e_5)$ if $x_1 = 0$,
(iii) $\pm(e_i - e_j)$ and $\pm(e_{i+4} - e_{j+4})$ for $i, j \in \{1, 2, 3\}$ if $\alpha_i = \alpha_j$ and $x_i = x_j$.

Since α_1, α_2, α_3 are coprime and thus in particular cannot all coincide, there are either 0 or 4 roots of type (iii). Hence the total number of integral roots is either 2 or 6 if $x_1 \neq 0$ and is either 4 or 8 if $x_1 = 0$.

Suppose that w is a fractional root in $Q_h^\perp$. Then w is of the form

$$w = \tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 \pm e_7 \pm e_8)\,, \tag{6.4}$$

where the number of + signs is even. Hence, the number of indices $i \in \{1,2,3,4\}$ such that e_i and e_{i+4} have the same sign in (6.4) must be even.

Case 1. There is no $i \in \{1,2,3\}$ such that e_i and e_{i+4} have the same sign in (6.4). The equation $(w, v_1) = 0$ implies that there is a choice of signs such that

$$\pm 2\alpha_1 \pm 2\alpha_2 \pm 2\alpha_3 = 0\,.$$

Hence the α_i must satisfy a relation of the form $\alpha_\ell = \alpha_j + \alpha_k$. But α_1, α_2, α_3 are positive and coprime, and $n-1 > 6$, so if such a relation holds, no two α_i can be equal. So if there are any fractional roots of this type, there must be exactly 4, given by

$$\pm\tfrac{1}{2}((e_\ell - e_{\ell+4}) - (e_j - e_{j+4}) - (e_k - e_{k+4}) \pm (e_4 - e_8))\,.$$

Case 2. There is exactly one index $i \in \{1,2,3\}$ such that e_i and e_{i+4} have the same sign in (6.4). Then e_4 and e_8 must have the same sign in (6.4). Moreover, $(w, v_2) = 0$ implies that there is a choice of sign such that

$$\pm 2x_i \pm 2x_4 = 0\,.$$

Since x_i is non-negative and x_4 is positive, it follows that $x_i = x_4$. Additionally, $(w, v_1) = 0$ implies that there is a choice of sign such that

$$\pm 2\alpha_j \pm 2\alpha_k = 0\,,$$

where $j, k \neq i$, and so $\alpha_j = \alpha_k$. Thus, since α_1, α_2, α_3 cannot all be equal, if there are any fractional roots of this type, there must be exactly 4, given by

$$\tfrac{1}{2}\left(\pm(e_i + e_{i+4} - e_4 - e_8) \pm (e_j - e_{j+4} - e_k + e_{k+4})\right).$$

Case 3. There are exactly two indices $i \in \{1,2,3\}$ such that e_i and e_{i+4} have the same sign in (6.4). In this case, the equation $(w, v_1) = 0$ implies that there is a j with $2\alpha_j = 0$. This contradicts our assumptions on α_j.

Case 4. All indices $i \in \{1,2,3\}$ are so that e_i and e_{i+4} have the same sign in (6.4). In this case, e_4 and e_8 must have the same sign in (6.4). Since $(w, v_2) = 0$, there is a choice of sign such that

$$\pm x_1 \pm x_2 \pm x_3 \pm x_4 = 0\,. \tag{6.5}$$

Note that each choice of signs such that (6.5) holds corresponds to exactly 1 fractional root in $Q_h^\perp$. The number of choices of signs such that (6.5) holds is twice the number of choices of signs such that

$$x_1 = \pm x_2 \pm x_3 \pm x_4\,. \tag{6.6}$$

Note that if $x_1 = 0$ and there is a choice of signs such that (6.6) holds, then the opposite choice of signs also makes (6.6) hold. In general, if there are two choices of signs such that (6.6) holds, the choices of signs differ by either one, two, or three signs, where the latter only occurs if $x_1 = 0$. If two such equations differ by one sign, subtracting the equations yields $x_i = 0$ for some $i \neq 1$, which is not possible. If two such equations differ by two signs, we have relations $x_1 = x_i$ for some $i \in \{1,2,3\}$ (in particular, $x_1 \neq 0$) and $x_j = x_k$ for $j, k \neq i$. Hence there are at most two choices of signs such that (6.6) holds, and if $x_1 = 0$, there are exactly two such choices. It follows that there are at most 4 fractional roots of this type and that if $x_1 = 0$ and there are any fractional roots of this type, there must be exactly 4.

This gives us the root count for $d \notin \{5,6\}$. Indeed, if α_1, α_2, α_3 are all different, then there are at most 4 integral roots, 4 fractional roots from Case 1, and 4 fractional roots from Case 3.

Hence there are between 2 and 12 roots in $Q_h^\perp$. If moreover $x_1 = 0$, then there either 8 or 12 roots in $Q_h^\perp$.

If two of the α_i are equal and $x_1 \neq 0$, then there are at most 6 integral roots, no fractional roots from Case 1, 4 fractional roots from Case 2, and 4 fractional roots from Case 4. Hence there are between 2 and 14 roots.

Finally, assume that two of the α_i are equal and $x_1 = 0$. If x_2, x_3, x_4 are all different, then the only roots are 4 integral roots and either 0 or 4 fractional roots from Case 4. If two of x_2, x_3, x_4 are equal, then the only roots are either 4 or 8 integral roots and either 0 or 4 fractional roots from Case 2 (Case 4 cannot occur because x_2, x_3, x_4 are coprime and $d \neq 6$). Hence there are 4, 8, or 12 roots in $Q_h^\perp$. This finishes the count for $d \notin \{5,6\}$.

When $d = 5$, we take $(x_1, x_2, x_3, x_4) = (0, 0, 2, 1)$. Since $\alpha_1 \neq \alpha_2$, there are 6 integral roots: $\pm(e_4-e_8)$, $\pm(e_1+e_5)$, $\pm(e_2+e_6)$. Moreover, if w is a fractional root in $Q_h^\perp$, then $(w, v_2) = 0$ implies that the signs of e_3 and e_4 are opposite to the signs of e_7 and e_8 in (6.4). Then $\alpha_\ell = \alpha_j + \alpha_k$, in which case there are 4 fractional roots (as in Case 1). Therefore, there are between 6 and 10 roots in $Q_h^\perp$ in this case.

When $d = 6$, we take $(x_1, x_2, x_3, x_4) = (0, 2, 1, 1)$. Then there are 4 integral roots. Moreover, there are either 0 or 4 fractional roots in Case 1, no fractional roots in Case 2 (since $\alpha_1 \neq \alpha_2$), and either 0 or 4 fractional roots in Case 4. Hence there either 4, 8, or 12 roots in $Q_h^\perp$. This finishes the proof for $d \geqslant 3$. ☐

In light of Proposition 5.3, the only missing ingredient in the proof of Theorem 1.6 is the vanishing of the quasi-pullback of the Borcherds form along the ramification divisor of the modular projection $\pi\colon \Omega(\Lambda_h) \to \Omega(\Lambda_h)/\widehat{O}^+(\Lambda, h)$. When d satisfies the conditions of part (iii) of Proposition 6.1 above, this follows from Lemma 2.8: Note that the weight of the pullback of the Borcherds form to $\Omega(\Lambda_h)$ is $12 + \frac{1}{2}\left|R\left(Q_h^\perp\right)\right|$ by (2.12). For other d, the vanishing along the ramification divisor is the content of Proposition 6.4. In order to prove it, we have to put constraints on the possible $r \in \Lambda$ such that $\pm\sigma_r$ lies inside the modular group $\widehat{O}^+(\Lambda, h)$. Recall that we are under the assumption that Q_h is of the form (5.5) with basis (6.1). We keep the same basis for $Q_h(-1)$. Recall that

$$D(\Lambda_h(-1)) \cong D(Q_h) = \left\langle \frac{z_1}{2(n-1)} \right\rangle \oplus \left\langle \frac{z_2}{2d} \right\rangle \cong \mathbb{Z}/2(n-1)\mathbb{Z} \oplus \mathbb{Z}/2d\mathbb{Z}\,.$$

As usual, we regard $\widehat{O}^+(\Lambda, h)$ as a subgroup of $O(\Lambda_h)$ via restriction. Elements $g \in \widehat{O}^+(\Lambda, h)$ act on $D(Q_h)$ as $\pm\mathrm{Id} \oplus \mathrm{Id}$.

Lemma 6.2. *Let $r \in \Lambda_h$ be a reflective element such that $-\sigma_r \in \widehat{O}^+(\Lambda, h)$ and $-\sigma_r \notin \widetilde{O}^+(\Lambda_h)$. Then $r^2 = -2d$, and $\operatorname{div}(r)$ is either d or $2d$.*

Proof. Recall, see (2.15), that if $-\sigma_r \in \widehat{O}^+(\Lambda, h)$, then

$$(r, r) \in \{\operatorname{div}(r), 2\operatorname{div}(r)\}\,. \tag{6.7}$$

Let $r = Au + Bz_1 + Cz_2$ with $u \in U^{\oplus 2} \oplus E_8^{\oplus 2}$ primitive in the unimodular part of $\Lambda_h(-1)$ and A, B, C coprime integers. Write $(r, r) = 2m$. Unimodularity and (6.7) imply that m divides A. Furthermore, since

$$-\sigma_r\left(\frac{z_1}{2(n-1)}\right) \equiv -\frac{z_1}{2(n-1)} \quad \text{and} \quad -\sigma_r\left(\frac{z_2}{2d}\right) \equiv \frac{z_2}{2d} \mod \Lambda_h(-1)\,,$$

one has

$$\frac{B}{m}r \equiv 0 \quad \text{and} \quad \frac{C}{m}r \equiv \frac{1}{d}z_2 \mod \Lambda_h(-1)\,.$$

As r is primitive, it follows that m divides B and dC. Since $\gcd(A,B,C) = 1$, the integer m cannot divide C. Comparing orders in the second equation together with our primitivity assumption on r gives us that $m = d$. ☐

Lemma 6.3. *Assume that both d and $n-1$ are not divisible by 4. Then for any reflective element $r \in \Lambda_h$ such that $-\sigma_r \in \widehat{O}^+(\Lambda, h)$, there exists an isometry $g \in O(\Lambda_h)$ such that*

$$g(r) = \begin{cases} de + z_2 & \text{when } \operatorname{div}(r) = d\,, \\ z_2 & \text{when } \operatorname{div}(r) = 2d\,, \end{cases} \tag{6.8}$$

and $-\sigma_{g(r)}$ acts as $-\mathrm{Id} \oplus \mathrm{Id}$ on $D(\Lambda_h)$. In particular, $-\sigma_{g(r)} \in \widehat{O}^+(\Lambda, h)$. Moreover, if d is an odd prime and $(n-1)d \equiv 2, 3 \mod 4$, then we can take $g \in \widehat{O}^+(\Lambda, h)$.

Proof. As before, let $r = Au + Bz_1 + Cz_2 \in \Lambda_h(-1)$ with $\gcd(A,B,C) = 1$. Since $2d = (r,r) = A^2(u,u) - B^2 \cdot 2(n-1) - C^2 \cdot 2d$, see Lemma 6.2, and d divides A and B, then d must divide $C^2 - 1$. Consider the map $\phi\colon D(\Lambda_h(-1)) \to D(\Lambda_h(-1))$, given by

$$(z_1)_* \longmapsto (z_1)_* \quad \text{and} \quad (z_2)_* \longmapsto \begin{cases} C(z_2)_* & \text{if } C \text{ is odd}\,, \\ (C+d)(z_2)_* & \text{if } C \text{ is even}\,. \end{cases} \tag{6.9}$$

One checks that this map is a group isomorphism, and $d \not\equiv 0 \mod 4$ implies that it preserves the quadratic form, that is, $\phi \in O(D(\Lambda_h(-1)))$. Now if $\operatorname{div}(r) = d$ and we write $r = A'du + B'dz_1 + Cz_2$, then

$$r_* = \frac{r}{\operatorname{div}(r)} = A'u + B'z_1 + \frac{C}{d}z_2 \equiv 2C\frac{z_2}{2d} = 2C(z_2)_* \mod \Lambda_h(-1)\,.$$

Note that via (6.9), the element $2C(z_2)_*$ is in the same $O(D(\Lambda_h(-1)))$-orbit as $2(z_2)_*$ if C is odd, and as $2(d+1)(z_2)_* = 2(z_2)_*$ if C is even. Since

$$2\frac{z_2}{2d} \equiv \frac{de + z_2}{d} = \frac{de + z_2}{\operatorname{div}(de + z_2)} \mod \Lambda_h(-1)$$

and the projection map $O(\Lambda_h(-1)) \to O(D(\Lambda_h(-1)))$ is surjective, Eichler's criterion (Theorem 2.10) gives us the lemma when $\operatorname{div}(r) = d$. The case $\operatorname{div}(r) = 2d$ and B even is treated analogously. If $\operatorname{div}(r) = 2$ and B is odd, we have $C^2 - 1 + (n-1)d \equiv 0 \mod 4d$. Under the additional assumption $n - 1 \not\equiv 0 \mod 4$, it follows that C is even. Using the map $\phi \in O(D(\Lambda_h(-1)))$ that sends $(z_1)_*$ to $n(z_1)_* + d(z_2)_*$ and $(z_2)_*$ to $(n-1)(z_1)_* + C(z_2)_*$, one shows that r is equivalent to z_2 under $O(\Lambda_h(-1))$.

If d is an odd prime, then $c^2 - 1 \equiv 0 \mod d$ implies $c \equiv \pm 1 \mod d$. Hence when $\operatorname{div}(r) = d$, the class $r_* = 2p(z_2)_* \in D(\Lambda_h(-1))$ is equivalent to $\pm 2(z_2)_*$, and the statement follows. When $\operatorname{div}(r) = 2d$ and B is even, we have $c^2 - 1 \equiv 0 \mod 4d$ and therefore $c \equiv \pm 1 \mod 2d$. The class $r_* = p(z_2)_*$ is equivalent to $\pm(z_2)_*$, and the statement follows. Finally, when $\operatorname{div}(r) = 2d$ and B is odd, we have $c^2 - 1 + (n-1)d \equiv 0 \mod 4d$, which cannot happen when $(n-1)d \equiv 2, 3 \mod 4$. ☐

Proposition 6.4. *Suppose $\gamma = 1$ and that d is prime, that either $n - 1 \equiv 1$ and $d \equiv 3 \bmod 4$ or $n - 1 \equiv 2$ and $d \equiv 1 \bmod 4$, and that $n - 1 > 6$ and $n - 1 \notin \{10, 13, 25, 37, 58, 85, 130, \star\}$. In particular, assertions (i) and (ii) of Proposition 6.1 hold. Let $Q_h \hookrightarrow E_8$ be the primitive* *embedding defined in* (6.3)*, and consider the induced embedding* $\Lambda_h \hookrightarrow \mathrm{II}_{2,26}$*. Then for any* $r \in \Lambda_h$ *such that* $-\sigma_r \in \widehat{O}^+(\Lambda, h)$*, one has*

$$\left|R\big(\Lambda_h^\perp\big)\right| < \left|R\big((\Lambda_h)_r^\perp\big)\right| .$$

In particular, the quasi-pullback F *of the Borcherds form* Φ_{12} *to* $\Omega^\bullet(\Lambda_h)$ *is modular with respect to* $\det\colon \widehat{O}^+(\Lambda, h) \to \mathbb{C}^*$*, and it vanishes along the ramification divisor of the modular projection* $\pi\colon \Omega(\Lambda_h) \to \Omega(\Lambda_h)/\widehat{O}^+(\Lambda, h)$*.*

Proof. If $-\sigma_r \in \widetilde{O}^+(\Lambda_h)$, the statement follows from Proposition 3.10. If $-\sigma_r$ is in $\widehat{O}^+(\Lambda, h)$ and not in $\widetilde{O}^+(\Lambda_h)$, then in light of Lemma 6.3, we may assume $r = de + z_2$ or $r = z_2$. First assume $r = z_2$; then

$$(\Lambda_h)_r^\perp = (\mathbb{Z}z_1)^\perp \subset E_8(-1)\,,$$

and under the embedding (6.3) restricted to $\langle z_1\rangle$, our counting argument in Proposition 6.1 leads to more than 14 roots only orthogonal to z_1. Indeed, recall that by assumption if $d > 1$, then $x_1, \dots, x_3$ in (6.2) are not all equal to zero. If $x_i \neq 0$ with $i \in \{1, 2, 3\}$, then $e_i + e_{i+4}$ is orthogonal to z_1 but not to z_2, that is,

$$e_i + e_{i+4} \in R\big((\Lambda_h)_r^\perp\big) \setminus R\big(\Lambda_h^\perp\big)\,.$$

Now assume $r = de + z_2$. Then

$$(\Lambda_h)_r = U \oplus E_8(-1)^{\oplus 2} \oplus \langle e, z_1, 2f - z_2\rangle\,.$$

Note that $(e_i, e_{i+4}) = -4x_i$. In particular, there is always a root $s \in E_8(-1)$ such that $(s, z_1) = 0$ and $(s, z_2) = 2k$ for some non-zero integer k. Consider the root given by $t = ke + s \in \mathrm{II}_{2,26}$. Then one immediately checks that t is orthogonal to e, z_1, and $2f - z_2$, but not orthogonal to z_2. In particular, $t \in R\big((\Lambda_h)_r^\perp\big) \setminus R\big(\Lambda_h^\perp\big)$. □

Summarizing, we have proved the following.

Theorem 6.5. *Let* n, d *be two positive integers such that*

(i) $n - 1 > 6$,

(ii) $n - 1 \equiv 1, 2 \mod 4$,

(iii) $n - 1 \notin \{10, 13, 25, 37, 58, 85, 130, \star\}$, *and*

(iv) d *satisfies one of the following:*

 (a) $d \geqslant 3$, $d \not\equiv 0, 4, 7 \mod 8$, *and* $d \notin \{5, 10, 13, 25, 37, 58, 85, 130, \star\}$;

 (b) d *is an odd prime and* $d \equiv 3 \mod 4$.

Then $\mathcal{M}^1_{\mathrm{K3}^{[n]},2d}$ *is of general type.*

Proof. Note that under our assumptions, $d \geqslant 3$ and $(n-1)d \equiv 2, 3 \mod 4$. If assumption (iv)(a) holds, the statement follows from Proposition 6.1 and Corollary 5.4. If instead assumption (iv)(b) holds, we additionally use Proposition 6.4, where we note that under our assumptions, $d \geqslant 3$ and $(n-1)d \equiv 2, 3 \mod 4$. □

Similarly to the strange duality phenomenon, one can construct various generically finite rational maps between moduli spaces $\mathcal{M}^\gamma_{\mathrm{K3}^{[n]}2d}$ using lattice morphisms. This has already been done by O'Grady [O'G89, Appendix, p. 163] (see also [Kon93, Kon99]) in the surface case; our argument is similar. We restrict to the cases $\gamma \in \{1, 2\}$. The following are examples in a much larger class of generically finite maps that one can construct between the moduli spaces $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[n]},2d}$. We will use them to prove Theorems 1.6 and 1.8.

Lemma 6.6. *Let d, r, n be positive integers with $n \geqslant 2$.*

(i) *The moduli spaces $\mathcal{M}^1_{\mathrm{K3}^{[n]},2dr^2}$ and $\mathcal{M}^1_{K3^{[(n-1)r^2+1]},2d}$ admit dominant rational maps to $\mathcal{M}^1_{\mathrm{K3}^{[n]},2d}$.*

(ii) *Assume $n-1=k^2$ with k odd. The space $\mathcal{M}^2_{K3^{[n]},2d}$ is non-empty if and only if $\mathcal{M}^2_{\mathrm{K3}^{[2]},2d}$ is, and the former admits a dominant rational map to the latter.*

Proof. Consider the elements $h_1 = e + (dr^2)f$ and $h_2 = e + df$ in the lattices

$$\Lambda_1 := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2(n-1)\rangle \quad \text{and} \quad \Lambda_2 := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \big\langle -2\big((n-1)r^2\big)\big\rangle\,,$$

respectively. Let $h = e + df$ in the standard lattice Λ (see equation (2.17)). Then

$$(\Lambda_i)_{h_i} = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus Q_i(-1)$$

with

$$Q_1 = \langle x_1, y_1\rangle = \begin{pmatrix} 2(n-1) & 0 \\ 0 & 2dr^2 \end{pmatrix} \quad \text{and} \quad Q_2 = \langle x_2, y_2\rangle = \begin{pmatrix} 2(n-2)r^2 & 0 \\ 0 & 2d \end{pmatrix}.$$

As usual $\Lambda_h = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus Q_h(-1)$, with Q_h as in (5.5) with basis $\{z_1, z_2\}$. Consider the lattice morphism $Q_1 \to Q_h$ given by $x_1 \mapsto z_1$ and $y_1 \mapsto rz_2$, and the induced map $\psi_1\colon \Lambda_{1,h_1} \to \Lambda_h$. Via ψ_1, the group $\widetilde{O}^+((\Lambda_1)_{h_1})$ is identified with a finite-index subgroup of $\widetilde{O}^+(\Lambda_h)$; see [GHS13, Lemma 7.1]. The reflection $\sigma_{x_1} \in \widehat{O}^+(\Lambda_1, h_1)$, which generates the quotient $\widehat{O}^+(\Lambda_1, h_1)/\widetilde{O}^+((\Lambda_1)_{h_1})$, is the restriction to $\psi_1(\Lambda_{1,h_1})$ of $\sigma_{z_1} \in \widehat{O}^+(\Lambda, h)$. This shows that ψ_1 identifies $\widehat{O}(\Lambda_1, h_1)$ with a finite-index subgroup of $\widehat{O}(\Lambda, h)$. Therefore, ψ_1 induces a finite morphism $\Omega((\Lambda_1)_{h_1})/\widehat{O}^+(\Lambda_1, h_1) \to \Omega(\Lambda_h)/\widehat{O}^+(\Lambda_h)$ and hence a dominant rational map $\mathcal{M}^1_{\mathrm{K3}^{[n]},2dr^2} \dashrightarrow \mathcal{M}^1_{\mathrm{K3}^{[n]},2d}$. The argument for $\mathcal{M}^1_{K3^{[(n-1)r^2+1]},2d}$ is analogous. In the case $\gamma = 2$, by (3.1) both moduli spaces $\mathcal{M}^2_{\mathrm{K3}^{[n]},2d}$ and $\mathcal{M}^2_{\mathrm{K3}^{[2]},2d}$ are non-empty if and only if $d \equiv -1 \pmod 4$ since k is odd. Denote by Λ' and Q' the corresponding lattices for $(n,d,\gamma,a) = \big(k^2+1, d, 2, 1\big)$ with basis $\{z_1', z_2'\}$ for Q', and by Λ, Q the lattices for $(n,d,\gamma,a) = (2,d,2,k)$, with $\{z_1, z_2\}$ the standard basis of Q. Then the lattice morphism inducing the finite map on period domains is given by $z_1' \mapsto kz_1$ and $z_2' \mapsto z_2$. One checks that this map descends to the quotient. ☐

Proof of Theorem 1.6. The first statement follows from Lemma 6.6 as $\mathcal{M}^1_{\mathrm{K3}^{[2]},2d}$ is of general type for all $d \geqslant 12$; see [GHS10]. For statement (ii)(a), we may assume $e = 0$ by Lemma 6.6. The statement then follows from Theorem 6.5. Similarly, for statement (ii)(b), we may assume $r = 1$ by Lemma 6.6. Then for $p = 1$, the theorem follows from strange duality, Proposition 3.9, since our assumptions on k imply $n - 1 \geqslant 12$. For p prime with $p \equiv 3 \mod 4$, the statement follows from Theorem 6.5. ☐

7. Non-split fourfolds of K3$^{[2]}$-type and the K3$^{[n]}$ divisibility 2 case

In this section, we establish general-type results in the 4-dimensional case and then use Lemma 6.6 to generalize to higher dimensions. Recall that the moduli space $\mathcal{M}^2_{\mathrm{K3}^{[2]},2d}$ is irreducible when non-empty; see [Apo14a]. Let Λ be as in (2.21) of K3$^{[2]}$-type. The discriminant group $D(\Lambda)$ is cyclic of order 2; thus $\gamma \in \{1, 2\}$, and when $\gamma = 2$, the moduli space $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[2]},2d}$ is non-empty if and only if $d \equiv -1 \mod 4$. Note that $\mathrm{Id} = -\mathrm{Id}$ in $D(\Lambda)$; in particular,

$$\widetilde{O}(\Lambda, h) = \widehat{O}(\Lambda, h)\,.$$

We fix $h \in \Lambda$ of degree $2d = 8t - 2$ and divisibility $\mathrm{div}(h) = \gamma = 2$. In light of Theorem 2.10, we may assume $h = 2(e + tf) - \ell$, where as usual $\{e, f\}$ are the standard generators of the first copy of U and $\langle \ell \rangle = \langle -2 \rangle$ a generator of the last factor of Λ in (2.21); see Lemma 3.4. Moreover, by [GHS10, Proposition 3.12], the natural inclusion in Lemma 2.1 is an equality

$$\widetilde{O}^{+}(\Lambda_h) = \widetilde{O}^{+}(\Lambda, h)\,. \tag{7.1}$$

Let $\Lambda_h = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus Q_t(-1)$, where

$$Q_t = \begin{pmatrix} 2 & -1 \\ -1 & 2t \end{pmatrix}. \tag{7.2}$$

As a consequence of (7.1), we have the following special case of Proposition 3.11.

Proposition 7.1. *Let $d = 4t - 1 > 16$, and assume that there exists a primitive embedding $Q_t \subset E_8$ such that the number of roots $\left|R(Q_t^{\perp})\right|$ is at least 2 and at most 14 (respectively, 16). Then the modular variety $\Omega(\Lambda_h)/\widetilde{O}^{+}(\Lambda_h)$ is of general type (respectively, non-negative Kodaira dimension). In particular, if such embedding exists, the moduli space $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[2]},2d}$ is of general type (respectively, non-negative Kodaira dimension) for $d = 4t - 1$ and $\gamma = 2$.*

Now we can prove our theorem about the moduli space $\mathcal{M}^{2}_{\mathrm{K3}^{[2]},2d}$.

Proof of Theorem 1.1. The cases $t = 10$ and $t = 12$ are treated in [GHS13, Proposition 9.2], so we assume $t \geqslant 13$. Let x_1 be the unique positive integer such that

$$x_1^2 + (x_1 + 1)^2 + 6 < 2t < (x_1 + 1)^2 + (x_1 + 2)^2 + 6\,. \tag{7.3}$$

Note that $x_1^2 + (x_1 + 1)^2 + 6$ and $(x_1 + 1)^2 + (x_1 + 2)^2 + 6$ are odd, and the above inequalities are strict. Now consider the odd number $R = 2t - (x_1^2 + (x_1 + 1)^2)$. From (7.3) we have

$$6 < R < (x_1 + 1)^2 + (x_1 + 2)^2 + 6 - \left(x_1^2 + (x_1 + 1)^2\right) = 4x_1 + 10\,. \tag{7.4}$$

By Corollary 2.15, we may write $R = x_5^2 + x_6^2 + x_7^2 + x_8^2$, where $x_5 \geqslant x_6 \geqslant x_7 \geqslant x_8$ are coprime integers with x_8 positive except in the cases where $R \in \{9, 11, 17, 29, 41\}$, where we take $x_8 = 0$ and x_7 positive. Consider the embedding $Q_t \hookrightarrow E_8$ given by $z_1 \mapsto v_1, z_2 \mapsto v_2$, where $\{z_1, z_2\}$ is the basis of (7.2) and

$$v_1 = e_1 + e_2\,, \quad v_2 = x_1 e_1 - (x_1 + 1)e_2 + x_5 e_5 + x_6 e_6 + x_7 e_7 + x_8 e_8\,. \tag{7.5}$$

Note that $(v_1, v_1) = 2$, $(v_2, v_2) = 2t$, and $(v_1, v_2) = -1$. As in the proof of Proposition 4.4, the primitivity of the embedding follows from Lemma 4.3. Note that the integral roots of E_8 in the orthogonal complement $Q_t^{\perp}$ are

(i) $\pm e_3 \pm e_4$,
(ii) $\pm e_i \pm e_8$ for $i \in \{3, 4\}$ if $x_8 = 0$,
(iii) $\pm(e_j - e_k)$ for $j, k \in \{5, 6, 7, 8\}$ if $x_j = x_k$.

Observe that if $x_8 \neq 0$, there are at most 6 roots of type (iii), and so there are between 4 and 10 total integral roots in $Q^{\perp}$. If $x_8 = 0$, then there are at most 2 roots of type (iii), and so there are between 12 and 14 total integral roots.

We now we count fractional roots of E_8 in the orthogonal complement $Q_t^\perp$. Suppose that w is such a fractional root. Since $(w, v_1) = (w, v_2) = 0$, we must have a choice of signs such that

$$2x_1 + 1 = \pm x_5 \pm x_6 \pm x_7 \pm x_8 . \tag{7.6}$$

Since $2x_1 + 1$, x_5, x_6, x_7, x_8 are all non-negative, it follows that

$$2x_1 + 1 \leqslant x_5 + \cdots + x_8 . \tag{7.7}$$

The above inequality together with (7.4) imply

$$\sum_{i=5}^{8}(x_i - 1)^2 = R - 2\sum_{i=5}^{8} x_i + 4 \leqslant 4x_1 + 9 - 2(2x_1 + 1) + 4 = 11 .$$

Then, either $x_5 = 4$, in which case $x_6, x_7 \leqslant 2$ and $x_8 \leqslant 1$, or $x_5, x_6 \leqslant 3$ and $x_7, x_8 \leqslant 2$. Since $\sum_{i=5}^{8} x_i$ is odd, we must have $\sum_{i=5}^{8} x_i \leqslant 9$ and $x_1 \leqslant 4$. In particular, if $t \geqslant 34$, there are no fractional roots in $Q_t^\perp \subset E_8$ and

$$4 \leqslant \left|R\left(Q_t^\perp\right)\right| \leqslant 14 .$$

It remains to count fractional roots when $13 \leqslant t \leqslant 33$. This is done case by case; the full list of lattice embeddings together with the count of fractional roots for each embedding is in the appendix. □

Finally, Lemma 6.6 together with Theorem 1.1 give us Theorem 1.8.

Proof of Theorem 1.8. Write $n - 1 = k^2$ for some odd integer k. By Proposition 3.1, we then have that $\mathcal{M}^2_{\mathrm{K3}^{[n]},2d}$ is non-empty precisely when $d \equiv -1 \mod 4$ (as in the $n = 2$ case). Since $\gamma = 2$, we know $(k, \gamma) = 1$. Hence by Lemma 6.6, we have that $\mathcal{M}^2_{\mathrm{K3}^{[n]},2d}$ dominates $\mathcal{M}^2_{\mathrm{K3}^{[2]},2d}$. The result then follows from Theorem 1.1. □

8. Non-split hyperkähler tenfolds of OG10-type

Let X be a hyperkähler variety of OG10-type. Recall that the Beauville–Bogomolov–Fujiki lattice $(H^2(X,\mathbb{Z}), q_X)$ is isometric to

$$\Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus A_2(-1) ,$$

and after fixing a marking, by [Ono22b], one has $\mathrm{Mon}^2(\Lambda) = \widehat{O}^+(\Lambda) = O^+(\Lambda)$.

As a direct consequence, we obtain (see also [Ono22a, Section 1]) the following.

Proposition 8.1 ([Ono22b, Ono22a, Son23]). *The moduli space $\mathcal{M}^\gamma_{\mathrm{OG10},2d}$ is irreducible and non-empty when $\gamma = 1$ and when $\gamma = 3$ and $2d \equiv 12 \mod 18$.*

Note that the discriminant group $D(\Lambda)$ is cyclic of order 3; hence $\gamma \in \{1, 3\}$. This leaves two possibilities for γ. The first one, known as the *split case*, is $\gamma = 1$. In this case, h can be chosen (up to the action of $\widetilde{O}(\Lambda)$) as $h = e + df \in U$, and the orthogonal complement of h in Λ is of the form

$$\Lambda_h = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus A_2(-1) \oplus \langle -2d \rangle , \tag{8.1}$$

where the last factor is generated by $e - df$. This case was treated in [GHS11]. The second possibility is $\gamma = 3$, known as the *non-split case*. In this case, $\mathcal{M}^\gamma_{\mathrm{OG10},2d}$ is non-empty if and only if $2d \equiv 12 \mod 18$; see [GHS11, Lemma 3.4]. Equivalently, $2d = 18t - 6$, and we can choose

$$h = 3(e + tf) + (2a_1 + a_2) ,$$

where again $\{e, f\}$ are standard generators of U and $\{a_1, a_2\}$ are standard generators of $A_2(-1)$. In this case,

$$\Lambda_h = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus Q_t(-1)\,, \tag{8.2}$$

where $Q_t(-1)$ is given by

$$Q_t(-1) = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & -1 \\ 0 & -1 & -2t \end{pmatrix} \tag{8.3}$$

with generators $\{a_2, e + a_1, e - tf\} \subset U \oplus A_2(-1)$. Moreover, the discriminant group $D(\Lambda_h)$ is cyclic of order $2d/3$; see [GHS11, Theorem 3.1].

Lemma 8.2. *In the above setting, the restriction map $\widehat{O}(\Lambda, h) \to O(\Lambda_h)$ induces an isomorphism*

$$\widehat{O}^+(\Lambda, h) \cong \widetilde{O}^+(\Lambda_h)\,, \tag{8.4}$$

and the arithmetic group $\widetilde{O}^+(\Lambda_h)$ has no irregular cusps.

Proof. The isomorphism (8.4) is [GHS11, Theorem 3.1]. Moreover, $D(\Lambda_h)$ is cyclic of order $2d/3$, and by [Ma24, Proposition 1.1], the group $\widetilde{O}^+(\Lambda_h)$ has irregular cusps only if $D(\Lambda_h) \cong \mathbb{Z}/8\mathbb{Z}$, but this contradicts the assumption $2d \equiv 12 \mod 18$. □

The following lemma follows immediately from [GHS13, Proposition 8.13] together with the proof of [GHS07, Theorem 4.2].

Lemma 8.3 ([GHS07, GHS13]). *Assume that the exists a primitive embedding $Q_t \hookrightarrow E_8$ such that $2 \leqslant \left|R\big(Q_t^{\perp}\big)\right| \leqslant 16$, and let $\Lambda_h \hookrightarrow \mathrm{II}_{2,26}$ be the induced lattice embedding. Then the quasi-pullback F of the Borcherds form Φ_{12} to $\Omega^\bullet(\Lambda_h)$ is a cusp form of weight $12 + \left|R\big(Q_t^{\perp}\big)\right|/2$ and character $\det\colon \widetilde{O}^+(\Lambda_h) \to \mathbb{C}^*$ that vanishes along the ramification divisor of the modular projection $\Omega(\Lambda_h) \to \Omega(\Lambda_h)/\widetilde{O}^+(\Lambda_h)$.*

Proof. Since $D(\Lambda_h)$ is cyclic, isotropic subgroups are cyclic as well, and the proof of [GHS07, Theorem 6.2(ii)] applies in this case; see [GHS07, Theorem 4.2]. The vanishing at the ramification divisor is [GHS13, Proposition 8.13]. □

Proposition 8.4. *If there exists a primitive embedding $Q_t \hookrightarrow E_8$ such that the number of roots $\left|R(Q_t^{\perp})\right|$ in the orthogonal complement $Q_t^{\perp} \subset E_8$ is at least 2 and at most 16, then the non-split moduli space $\mathcal{M}^{\gamma}_{\mathrm{OG10},2d}$ is of general type, where $\gamma = 3$ and $2d = 18t - 6$.*

Proof. The proposition follows from Theorems 2.3 and 2.11 together with Lemma 8.3 for the modular variety $\Omega(\Lambda_h)/\widehat{O}^+(\Lambda, h) \cong \Omega(\Lambda_h)/\widetilde{O}^+(\Lambda_h)$. □

Now we can prove our main result regarding the non-split moduli space $\mathcal{M}^3_{\mathrm{OG10},2d}$. The proof is very similar to Theorem 1.1, so we omit details.

Proof of Theorem 1.2. The case of $t = 4$ was proved in [GHS11, Corollary 4.3]. For the moment, we will assume $t \geqslant 7$ and then tackle the cases $t \in \{5, 6\}$ subsequently. Let x_1 be the unique non-negative even integer such that

$$2x_1^2 + (x_1 + 1)^2 + 12 < 2t < 2(x_1 + 2)^2 + (x_1 + 3)^2 + 12\,, \tag{8.5}$$

and consider the odd number $R = 2t - \big(2x_1^2 + (x_1 + 1)^2\big)$. Then,

$$12 < R < 12x_1 + 18\,. \tag{8.6}$$

Recall [Hal82, Satz 1b] that any positive integer $n \equiv 5 \mod 8$ is a sum of three pairwise distinct coprime squares. In particular, we can assume $\star \not\equiv 5 \mod 8$. Let

$$\Theta = \begin{cases} 0 & \text{if } R \not\equiv 7 \text{ mod } 8 \text{ and } R \notin \{19, 27, 33, 43, 51, 57, 67, 99, 123, 163, 177, 187, 267, 627, \star\}\,, \\ 1 & \text{if } R \equiv 7 \text{ mod } 8 \text{ or } R \in \{19, 27, 43, 51, 67, 99, 123, 163, 187, 267, 627\}\,, \\ 2 & \text{if } R \in \{33, 57, 177, \star\}\,, \end{cases}$$

and consider the integer

$$S = 2t - \left(2x_1^2 + (x_1+1)^2\right) - 2\Theta^2\,.$$

Observe that S can be written as a sum of three distinct coprime squares: $S = x_6^2 + x_7^2 + x_8^2$, with $x_6 > x_7 > x_8 \geqslant 0$. Our embedding $Q_t \hookrightarrow E_8$ is given by $z_1 \mapsto v_i$, where $\{z_1, z_2, z_3\}$ is a basis for Q_t, with matrix given by minus the matrix defined in (8.3), and

$$v_1 = e_2 - e_1\,,\quad v_2 = e_3 - e_2\,,\quad v_3 = x_1e_1 + x_1e_2 + (x_1+1)e_3 + \Theta e_4 + \Theta e_5 + x_6e_6 + x_7e_7 + x_8e_8\,. \quad (8.7)$$

As in the proof of Proposition 4.4 and Theorem 1.1, primitivity follows from parity considerations together with Lemma 4.3.

One observes that integral roots in $Q_t^\perp$ are given by

(i) $\pm(e_4 - e_5)$,

(ii) $\pm(e_4 + e_5)$ if $\Theta = 0$,

(iii) $\pm e_4 \pm e_8$ and $\pm e_5 \pm e_8$ if $\Theta = e_8 = 0$,

(iv) $\pm(e_4 - e_i)$ and $\pm(e_5 - e_i)$ for $i \in \{6, 7, 8\}$ if $\Theta = x_i \neq 0$.

Hence if $\Theta \neq 0$, there are between 2 and 6 integral roots. If $\Theta = 0$, there are either 4 or 12 integral roots (with 12 occurring only if $x_8 = 0$).

Suppose that w is a fractional root in $Q_t^\perp$. Since $(w, v_i) = 0$ for $i \in \{1, 2, 3\}$, we must have a choice of signs such that

$$3x_1 + 1 = \pm\Theta \pm \Theta \pm x_6 \pm x_7 \pm x_8\,, \quad (8.8)$$

where the number of + signs on the right-hand side is even. Since $3x_1 + 1$, x_5, x_6, x_7, x_8 are all non-negative,

$$3x_1 + 1 \leqslant 2\Theta + \sum_{i=6}^{8} x_i\,. \quad (8.9)$$

From (8.6) and (8.9), we have

$$\sum_{i=6}^{8}(x_i - 2)^2 = S - 4\sum_{i=6}^{8} x_i + 12 \leqslant 12x_1 + 17 - 2\Theta^2 - 4(3x_1 + 1 - 2\Theta) + 12$$
$$= 25 + 2\Theta(4 - \Theta)\,,$$

where $2\Theta(4-\Theta) \in \{0, 6, 8\}$. In particular, if $\Theta = 0$, then $\sum_{i=6}^{8}(x_i - 2)^2 \leqslant 25$ and $\sum_{i=6}^{8} x_i \leqslant 13$. And if $\Theta \in \{1, 2\}$, then $\sum_{i=6}^{8} x_i \leqslant 15$. From (8.9) it follows that either $x_1 < 6$ and $\Theta < 2$, or $x_1 < 8$ and $\Theta = 2$. In other words, if $x_1 \geqslant 6$, there cannot be fractional roots in $Q_t^\perp$. Moreover, when $x_1 = 6$, it follows from (8.6) that the only values of R for which $\Theta = 2$ are 33 and 57. In these cases, taking, respectively, $(x_6, x_7, x_8) = (4, 3, 0)$ and $(x_6, x_7, x_8) = (6, 3, 2)$, the bound (8.9) is not satisfied; that is, there cannot be fractional roots in $Q_t^\perp$ in these cases either. We have shown the desired result for all $t \geqslant 67$. The embeddings and root counts in the cases $4 \leqslant t \leqslant 66$ are listed in the appendix. This shows that for all $t \geqslant 4$, there is a primitive embedding $Q_t \hookrightarrow E_8$ such that $2 \leqslant \left|R(Q_t^\perp)\right| \leqslant 16$, and Proposition 8.4 gives us the result. $\square$

Appendix. Lattice embeddings for low degree

Here we complete the proofs of Theorems 1.1 and 1.2 by listing all the lattice embeddings for low values of t.

Non-split hyperkähler varieties of K3-type and low degree

We list the corresponding embeddings for $13 \leqslant t \leqslant 33$. The embedding is always the one given by (7.5), except when $t = 21$ (see below). For $t \in \{19, 20, 22, 31, 32, 33\}$, the parameters are as follows:

t	x_1	R	(x_5, x_6, x_7, x_8)	Fractional roots
19	3	13	(2,2,2,1)	4
20	3	15	(3,2,1,1)	4
22	3	19	(4,1,1,1)	4
31	4	21	(3,2,2,2)	4
32	4	23	(3,3,2,1)	4
33	4	25	(4,2,2,1)	4

In these cases, one shows that there are 4 fractional roots in $Q_t^\perp$ given by $\pm\frac{1}{2}(e_1 - e_2 \pm (e_4 - e_5) - (e_5 + e_6 + e_7 + e_8)$. Since $x_8 \neq 0$, there are between 4 and 10 integral roots, so $8 \leqslant \left|R\left(Q_t^\perp\right)\right| \leqslant 14$. When $24 \leqslant t \leqslant 30$, we take $x_1 = 4$; then we have $7 \leqslant R \leqslant 19$. Arguing as in the case $t \geqslant 34$, one rules out the possibility of fractional roots in $Q_t^\perp$. For the list

t	x_1	R	(x_5, x_6, x_7, x_8)	Fractional roots
15	2	17	(3,2,2,0)	0
16	3	7	(2,1,1,1)	0
17	3	9	(2,2,1,0)	0
18	3	11	(3,1,1,0)	0
23	3	21	(3,2,2,2)	0

one checks that there are no fractional roots. For the embeddings

t	x_1	R	(x_5, x_6, x_7, x_8)	Fractional roots
13	2	13	(2,2,2,1)	4
14	2	15	(3,2,1,1)	8

there are 10 integral roots and 4 fractional roots in $Q_t^\perp$, respectively 6 integral roots and 8 fractional roots given by

$$\pm\tfrac{1}{2}\left(e_1 - e_2 \pm (e_3 + e_4) - e_5 - e_6 - e_7 + e_8\right)$$
$$\pm\tfrac{1}{2}\left(e_1 - e_2 \pm (e_3 + e_4) - e_5 - e_6 \pm (e_7 - e_8)\right),$$

respectively. Finally, when $t = 21$, the embedding given in (7.5) yields too many roots in $Q_t^\perp$. However, a method described in [GHS13, Section 9] called the "$(\frac{1}{2}++)$-algorithm" works in this case. The embedding $Q_t \hookrightarrow E_8$ is given by $z_1 \mapsto v_1$, $z_2 \mapsto v_2$, with

$$v_1 = -e_1 + e_2\,, \quad v_2 = \tfrac{1}{2}(e_1 - e_2 + 11e_3 + 5e_4 + 3e_5 + 3e_6 + e_7 + e_8)\,.$$

One checks that the embedding is primitive and there are 6 integral roots in $Q_t^\perp$ given by $\pm(e_1 + e_2), \pm(e_5 - e_6), \pm(e_7 - e_8)$ and 8 fractional roots given by

$$\pm\tfrac{1}{2}\left(e_1 + e_2 \pm (e_3 - e_4 - e_5 - e_6 \pm (e_7 - e_8))\right).$$

Hence there are a total of 14 roots in $Q_t^\perp$. We have treated all cases $t \geqslant 13$. Proposition 7.1 together with [GHS13, Proposition 9.2] for $t \in \{10, 12\}$ give the theorem.

Non-split hyperkähler varieties of OG10-type and low degree

Here we complete the proof of Theorem 1.2 by listing the remaining data t, R, and (x_6, x_7, x_8) defining the embedding $Q_t^\perp \hookrightarrow E_8$ given by (8.7) where the number of roots satisfies the inequalities $2 \leqslant \left|R(Q_t^\perp)\right| \leqslant 16$. Recall that the number of integral roots for $t \geqslant 7$ is between 2 and 6 when $\Theta \neq 0$ and between 4 and 12 when $\Theta = 0$, with 12 only occurring when $x_8 = 0$. For $54 \leqslant t \leqslant 66$, the parameters are as follows:

t	x_1	R	Θ	(x_6, x_7, x_8)	Fractional roots
54	4	51	0	(6,3,2)	0
55	4	53	0	(7,2,0)	0
56	4	55	1	(7,2,0)	0
57	4	57	2	(6,3,2)	0
58	4	59	0	(7,3,1)	0
59	4	61	0	(6,5,0)	0
60	4	63	1	(6,5,0)	0

t	x_1	R	Θ	(x_6, x_7, x_8)	Fractional roots
61	4	65	0	(7,4,0)	0
62	4	67	1	(7,4,0)	0
63	4	69	0	(7,4,2)	4
64	4	71	1	(7,4,2)	4
65	4	73	0	(8,3,0)	0
66	4	75	0	(7,5,1)	4

The fractional roots in the cases $t = 63$, $t = 64$, and $t = 66$ are

$$\pm\tfrac{1}{2}(e_1 + e_2 + e_3 \pm (e_4 - e_5) - e_6 - e_7 - e_8)\,.$$

If $35 \leqslant t \leqslant 53$ and $t \notin \{45, 50, 52\}$, then $13 \leqslant R \leqslant 49$ and (as in the high-degree case) if one assumes that there is a fractional root, equations (8.6) and (8.9) lead to a contradiction. In these cases, there are no fractional roots. For $t \in \{45, 50, 52\}$, the parameters are as follows:

t	x_1	R	Θ	(x_6, x_7, x_8)	Fractional roots
45	4	33	2	(4,3,0)	0
50	4	43	1	(6,2,1)	0
52	4	47	1	(5,4,2)	0

For the range $15 \leqslant t \leqslant 34$ and $t \notin \{20, 21, 27, 28, 34\}$, the embedding parameters are as follows:

t	x_1	R	Θ	(x_6, x_7, x_8)	Fractional roots
15	2	13	0	(3,2,0)	0
16	2	15	1	(3,2,0)	2
17	2	17	0	(4,1,0)	0
18	2	19	1	(4,1,0)	2
19	2	21	0	(4,2,1)	4
22	2	27	1	(4,3,0)	4
23	2	29	0	(4,3,2)	0
24	2	31	1	(5,2,0)	4

t	x_1	R	Θ	(x_6, x_7, x_8)	Fractional roots
25	2	33	2	(4,3,0)	4
26	2	35	0	(5,3,1)	4
29	2	41	0	(5,4,0)	0
30	2	43	1	(5,4,0)	2
31	2	45	0	(5,4,2)	4
32	2	47	1	(5,4,2)	4
33	2	49	0	(6,3,2)	4

In the cases $t \in \{20, 21, 27, 34\}$, we use the embedding given by (8.7) with parameters as follows:

t	x_1	Θ	(x_6, x_7, x_8)	Integral roots	Fractional roots
20	0	3	(2,1,0)	2	2
21	0	0	(6,2,1)	4	4
27	2	4	(2,1,0)	2	2
34	2	5	(1,0,0)	6	0

When $t = 28$, one considers the embedding given by $z_i \mapsto v_i$, where $v_1 = e_2 - e_1$, $v_2 = e_3 - e_2$, and

$$v_3 = 2e_1 + 2e_2 + 3e_3 + 5e_4 + 3e_5 + 2e_6 + e_7 .$$

One can check that the embedding is primitive with no integral roots and 2 fractional roots in the orthogonal complement. The embeddings in the cases $8 \leqslant t \leqslant 14$ with $t \notin \{12, 13\}$ are as in (8.7) with the following parameters:

t	x_1	R	Θ	(x_6, x_7, x_8)	fractional roots
8	0	15	1	(3,2,0)	12
9	0	17	0	(4,1,0)	0
10	0	19	1	(4,1,0)	2
11	0	21	0	(4,2,1)	4
14	0	27	1	(4,3,0)	12

Finally, the cases $t \in \{5, 6, 7, 12, 13\}$ are given by $z_i \mapsto v_i$, where $v_1 = e_2 - e_1$, $v_2 = e_3 - e_2$ and

$$v_3 = a_1 e_1 + \cdots + a_8 e_8$$

with $(a_1, \ldots, a_8)$ as follows:

t	$(a_1, \ldots, a_8)$	Integral roots	Fractional roots
5	(1,1,2,0,1,1,1,1)	12	0
6	(1,1,2,0,0,2,1,1)	6	0
7	(1,1,2,2,1,1,1,1)	12	0
12	(1,1,2,0,0,4,1,0)	12	0
13	(1,1,2,1,1,4,1,0)	6	2

With these, we have covered all cases $5 \leqslant t \leqslant 66$, finishing the proof of Theorem 1.2.

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References

Apo14a A. Apostolov, *Moduli spaces of polarized irreducible symplectic manifolds are not necessarily connected*, Ann. Inst. Fourier (Grenoble) **64** (2014), no. 1, 189–202; doi:10.5802/aif.2844.

Apo14b ———, *On irreducible symplectic varieties of $K3^{[n]}$-type*, Ph.D. thesis, Universität Hannover, 2014, available at https://repo.uni-hannover.de/handle/123456789/827; doi:10.15488/8218.

BB66 W. L. Baily, Jr. and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. (2) **84** (1966), 442–528; doi:10.2307/1970457.

BLM+21 A. Bayer, M. Lahoz, E. Macrì, H. Nuer, A. Perry, and P. Stellari, *Stability conditions in families*, Publ. Math. Inst. Hautes Études Sci. **133** (2021), 157–325; doi:10.1007/s10240-021-00124-6.

BD85 A. Beauville and R. Donagi, *La variété des droites d'une hypersurface cubique de dimension* 4, C. R. Acad. Sci. Paris Sér. I Math. **301** (1985), no. 14, 703–706.

Bor95 R. E. Borcherds, *Automorphic forms on* $O_{s+2,2}(\mathbb{R})$ *and infinite products*, Invent. Math. **120** (1995), no. 1, 161–213; doi:10.1007/BF01241126.

BKPS98 R. E. Borcherds, L. Katzarkov, T. Pantev, and N. I. Shepherd-Barron, *Families of $K3$ surfaces*, J. Algebraic Geom. **7** (1998), no. 1, 183–193.

BFRT89 I. Borosh, M. Flahive, D. Rubin, and B. Treybig, *A sharp bound for solutions of linear Diophantine equations*, Proc. Amer. Math. Soc. **105** (1989), no. 4, 844–846; doi:10.2307/2047041.

CS88 J. H. Conway and N. J. A. Sloane, *Low-dimensional lattices. I. Quadratic forms of small determinant*, Proc. Roy. Soc. London Ser. A **418** (1988), no. 1854, 17–41; doi:10.1098/rspa.1988.0072.

Daw25 M. Dawes, *General type results for moduli of deformation generalised Kummer varieties*, Math. Nachr. **298** (2025), no. 10, 3376–3393, doi:10.1002/mana.70043.

Daw25 ———, *On the Kodaira dimension of the moduli of deformation generalised Kummer varieties*, Algebr. Geom. **12** (2025), no. 5, 601–660, doi:10.14231/AG-2025-017.

Deb22 O. Debarre, *Hyper-Kähler manifolds*, Milan J. Math. **90** (2022), no. 2, 305–387; doi:10.1007/s00032-022-00366-x.

DV10 O. Debarre and C. Voisin, *Hyper-Kähler fourfolds and Grassmann geometry*, J. reine angew. Math. **649** (2010), 63–87; doi:10.1515/CRELLE.2010.089.

Eic74 M. Eichler, *Quadratische Formen und orthogonale Gruppen*, Grundlehren math. Wiss., vol. 63 (Springer-Verlag, Berlin-New York, 1974); doi:10.1007/978-3-642-80764-0.

FV18 G. Farkas and A. Verra, *The universal $K3$ surface of genus* 14 *via cubic fourfolds*, J. Math. Pures Appl. (9) **111** (2018), 1–20; doi:10.1016/j.matpur.2017.07.014.

FV21 ———, *The unirationality of the moduli space of $K3$ surfaces of genus* 22, Math. Ann. **380** (2021), no. 3-4, 953–973; doi:10.1007/s00208-020-02036-y.

FM21 M. Fortuna and G. Mezzedimi, *The Kodaira dimension of some moduli spaces of elliptic $K3$ surfaces*, J. Lond. Math. Soc. (2) **104** (2021), no. 1, 269–294; doi:10.1112/jlms.12430.

Fre83 E. Freitag, *Siegelsche Modulfunktionen*, Grundlehren math. Wiss., vol. 254 (Springer-Verlag, Berlin, 1983); doi:10.1007/978-3-642-68649-8.

GHS07 V. A. Gritsenko, K. Hulek, and G. K. Sankaran, *The Kodaira dimension of the moduli of $K3$ surfaces*, Invent. Math. **169** (2007), no. 3, 519–567; doi:10.1007/s00222-007-0054-1.

GHS10 ———, *Moduli spaces of irreducible symplectic manifolds*, Compos. Math. **146** (2010), no. 2, 404–434; doi:10.1112/S0010437X0900445X.

GHS11 ———, *Moduli spaces of polarized symplectic O'Grady varieties and Borcherds products*, J. Differential Geom. **88** (2011), no. 1, 61–85; doi:10.4310/jdg/1317758869.

GHS13 ———, *Moduli of K3 surfaces and irreducible symplectic manifolds*, in *Handbook of moduli. Vol. I*, Adv. Lect. Math. (ALM), vol. 24 (Int. Press, Somerville, MA, 2013), 459–526.

Hal82 F. Halter-Koch, *Darstellung natürlicher Zahlen als Summe von Quadraten*, Acta Arith. **42** (1982), no. 1, 11–20; doi:10.4064/aa-42-1-11-20.

Huy99 D. Huybrechts, *Compact hyperkähler manifolds: basic results*, Invent. Math. **135** (1999), no. 1, 63–113; doi:10.1007/s002220050280.

Huy16 ———, *Lectures on K3 surfaces*, Cambridge Stud. Adv. Math., vol. 158 (Cambridge Univ. Press, Cambridge, 2016); doi:10.1017/CBO9781316594193.

IKKR19 A. Iliev, G. Kapustka, M. Kapustka, and K. Ranestad, *EPW cubes*, J. reine angew. Math. **748** (2019), 241–268; doi:10.1515/crelle-2016-0044.

IR01 A. Iliev and K. Ranestad, *K3 surfaces of genus 8 and varieties of sums of powers of cubic fourfolds*, Trans. Amer. Math. Soc. **353** (2001), no. 4, 1455–1468; doi:10.1090/S0002-9947-00-02629-5.

IR07 ———, *Addendum to K3 surfaces of genus 8 and varieties of sums of powers of cubic fourfolds*, C. R. Acad. Bulgare Sci. **60** (2007), no. 12, 1265–1270.

Iwa97 H. Iwaniec, *Topics in classical automorphic forms*, Grad. Stud. Math., vol. 17 (Amer. Math. Soc., Providence, RI, 1997); doi:10.1090/gsm/017.

Kon93 S. Kondō, *On the Kodaira dimension of the moduli space of K3 surfaces*, Compos. Math. **89** (1993), no. 3, 251–299.

Kon99 ———, *On the Kodaira dimension of the moduli space of K3 surfaces. II*, Compos. Math. **116** (1999), no. 2, 111–117; doi:10.1023/A:1000675831026.

LLSvS17 C. Lehn, M. Lehn, C. Sorger, and D. van Straten, *Twisted cubics on cubic fourfolds*, J. reine angew. Math. **731** (2017), 87–128; doi:10.1515/crelle-2014-0144.

Ma18 S. Ma, *On the Kodaira dimension of orthogonal modular varieties*, Invent. Math. **212** (2018), no. 3, 859–911; doi:10.1007/s00222-017-0781-x.

Ma24 ———, *Irregular cusps of orthogonal modular varieties*, Canad. J. Math. **77** (2024), no. 3, 739–769; doi:10.4153/S0008414X24000129.

Mar11 E. Markman, *A survey of Torelli and monodromy results for holomorphic-symplectic varieties*, in *Complex and differential geometry*, Springer Proc. Math., vol. 8 (Springer, Heidelberg, 2011), 257–322; doi:10.1007/978-3-642-20300-8_15.

Mon13 G. Mongardi, *Automorphisms of hyperkähler manifolds*, Ph.D. thesis, Università Roma Tre, 2013, arXiv:1303.4670.

Muk88 S. Mukai, *Curves, K3 surfaces and Fano 3-folds of genus* $\leqslant 10$, in *Algebraic geometry and commutative algebra, Vol. I* (Kinokuniya, Tokyo, 1988), 357–377; doi:10.1016/B978-0-12-348031-6.50026-7.

Muk92 ———, *Polarized K3 surfaces of genus 18 and 20*, Complex projective geometry (Trieste, 1989/Bergen, 1989), London Math. Soc. Lecture Note Ser., vol. 179 (Cambridge Univ. Press, Cambridge, 1992), 264–276; doi:10.1017/CBO9780511662652.019.

Muk06 ———, *Polarized K3 surfaces of genus thirteen*, in *Moduli spaces and arithmetic geometry*, Adv. Stud. Pure Math., vol. 45 (Math. Soc. Japan, Tokyo, 2006), 315–326; doi:10.2969/aspm/04510315.

Muk10 ———, *Curves and symmetric spaces, II*, Ann. of Math. (2) **172** (2010), no. 3, 1539–1558; doi:10.4007/annals.2010.172.1539.

Muk16 ———, *K3 surfaces of genus sixteen*, Minimal models and extremal rays (Kyoto, 2011), Adv. Stud. Pure Math., vol. 70 (Math. Soc. Japan, Tokyo, 2016), 379–396; doi:10.2969/aspm/07010379.

Mum77 D. Mumford, *Hirzebruch's proportionality theorem in the noncompact case*, Invent. Math. **42** (1977), 239–272; doi:10.1007/BF01389790.

Nik80 V. V. Nikulin, *Integral symmetric bilinear forms and some of their applications*, Math. USSR Izv. **14** (1980), 103–167; doi:10.1070/IM1980v014n01ABEH001060.

Nue17 H. Nuer, *Unirationality of moduli spaces of special cubic fourfolds and K3 surfaces*, Algebr. Geom. **4** (2017), no. 3, 281–289; doi:10.14231/AG-2017-015.

O'G89 K. G. O'Grady, *On the Kodaira dimension of moduli spaces of abelian surfaces*, Compos. Math. **72** (1989), no. 2, 121–163, http://www.numdam.org/item?id=CM_1989__72_2_121_0.

O'G06 ———, *Irreducible symplectic 4-folds and Eisenbud–Popescu–Walter sextics*, Duke Math. J. **134** (2006), no. 1, 99–137; doi:10.1215/S0012-7094-06-13413-0.

Ono22a C. Onorati, *Connected components of moduli spaces of irreducible holomorphic symplectic manifolds of Kummer type*, Rend. Mat. Appl. (7) **43** (2022), no. 4, 251–266.

Ono22b ———, *On the monodromy group of desingularised moduli spaces of sheaves on K3 surfaces*, J. Algebraic Geom. **31** (2022), 425–465; doi:10.1090/jag/802.

PPZ23 A. Perry, L. Pertusi, and X. Zhao, *Stability conditions and moduli spaces for Kuznetsov components of Gushel–Mukai varieties*, Geom. Topol. **26** (2023), no. 7, 3055–3121; doi:10.2140/gt.2022.26.3055.

Sca87 F. Scattone, *On the compactification of moduli spaces for algebraic $K3$ surfaces*, Mem. Amer. Math. Soc. **70** (1987), no. 374; doi:10.1090/memo/0374.

Son23 J. Song, *On the image of the period map for polarized hyperkähler manifolds*, Int. Math. Res. Not. IMRN (2023), no. 13, 11404–11431; doi:10.1093/imrn/rnac155.

TV19 S. Tanimoto and A. Várilly-Alvarado, *Kodaira dimension of moduli of special cubic fourfolds*, J. reine angew. Math. **752** (2019), 265–300; doi:10.1515/crelle-2016-0053.

Ver13 M. Verbitsky, *Mapping class group and a global Torelli theorem for hyperkähler manifolds*, Duke Math. J. **162** (2013), no. 15, 2929–2986; doi:10.1215/00127094-2382680.

Vie95 E. Viehweg, *Quasi-projective moduli for polarized manifolds*, Ergeb. Math. Grenzgeb. (3), vol. 30 (Springer-Verlag, Berlin, 1995); doi:10.1007/978-3-642-79745-3.

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