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Rational approximation on cubic hypersurfaces

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Abstract

We compute the constant of approximation for an arbitrary rational point on an arbitrary smooth cubic hypersurface X over a number field k, provided that there is a k-rational line somewhere on X. In the process, we verify the Coba conjecture for X.

1. Introduction

Let X be a smooth projective variety defined over a number field k. In [McK07], the author makes a conjecture that the best rational approximations to a rational point P on a smooth projective variety can be found along some rational curve through P. Such a curve is called a curve of best approximation to P.

This is a little vague, so we must say what we mean by a good approximation. Generally speaking, an approximation to P is good (in this sense) if

(i) the approximating point is close to P, and

(ii) the approximating point has small height.

To define distance, we choose a place v of k and use the distance on X induced by v. That is, we embed X in projective space and restrict the standard v-adic distance there to X. (It will turn out that it does not matter which embedding we pick.) To define the height, we choose a divisor class L on X and choose some (multiplicative, relative to k) Weil height H_L associated with L. To balance these two factors, we work in aggregate over a sequence of rational points.

Let $\{x_i\}$ be a sequence of rational points that converges v-adically to P. The L-approximation constant of $\{x_i\}$ is $\alpha(\{x_i\}, P, L)$, a real number that we will define in the next section. Nevertheless, $\alpha(\{x_i\}, P, L)$ has the property that the larger it is, the worse the sequence $\{x_i\}$ v-adically L-approximates P. It might be thought of as the "cost" of approximating P with the $\{x_i\}$.

We define the L-approximation constant $\alpha(P, L)$ to be the infimum of $\alpha(\{x_i\}, P, L)$ over all nonconstant sequences $\{x_i\}$. Thus, the lower $\alpha(\{x_i\}, P, L)$ is, the better the sequence approximates P. The lower $\alpha(P, L)$ is, the better P can be approximated.

Conjecture 1.1 (Coba conjecture). *Let X be a smooth variety defined over a number field k and $P \in X$ a k-rational point. Let A be an ample line bundle on X. If $\alpha(P, A) < \infty$, then there*

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is a curve of best A-approximation to P. In other words, there is a curve C such that

$$\alpha(P, A|_C) = \alpha(P, A)\,,$$

where the first α is computed on C and the second is computed on X.

To paraphrase the conjecture still further: The best A-approximations to P approximate P just as well as the best A-approximations to P that happen to lie on C. This conjecture has been verified in a large number of cases, and Vojta's conjecture is known to imply it in a much larger set of cases. See [McK07] for a discussion of this.

In the work of Lehmann, McKinnon, and Satriano [LMS25], a framework is proposed to deduce Conjecture 1.1 from Vojta's conjecture for all smooth projective varieties. This framework uses the machinery of the minimal model program to reduce from a general smooth variety to a Mori fibre space (though not necessarily a smooth one). Thus, it is natural to look to Mori fibre spaces to verify Conjecture 1.1. In this paper, we consider cubic hypersurfaces. In dimension 3 or greater, these are all Mori fibre spaces when they are smooth.

In particular, we verify Conjecture 1.1 for smooth cubic hypersurfaces that contain a K-rational line, in Theorem 3.3. It is known by work of Dietmann and Wooley [DW03, Theorem 2] that any smooth cubic hypersurface of dimension at least 35 defined over a number field k contains a k-rational line. Work of Brandes and Dietmann [BD21, Theorem 1.1] improves this to dimension at least 29 when $k = \mathbb{Q}$, so it is likely that the hypothesis in Theorem 3.3 is satisfied in a large proportion of cases.

The structure of the paper is quite simple. Section 2 contains some tedious prerequisites like the definition of α. Section 3 first considers the cases of curves and surfaces separately in Theorem 3.1 and then proves the higher-dimensional case in Theorem 3.3.

2. Preliminaries

The purpose of this section is to make precise our terminology.

We start with a projective variety X defined over a number field k. With any line bundle L on X, we associate a multiplicative Weil height H_L. Details about how to do this can be found in a number of standard references, such as [Ser97, Section 1.1]. There is a small ambiguity in the choice of height, but for any two heights H_L and H'_L constructed in this way, the function H_L/H'_L is a bounded, positive function compactly supported away from zero. This ambiguity will not affect the definition of α, as we will shortly see below.

Next, we fix a place v of k. We need to define a v-adic distance function on X. As hinted at in the introduction, we do this by choosing an embedding of X in $\mathbb{P}^n$ and restrict the usual v-adic distance on $\mathbb{P}^n$ to X. The details of this are surprisingly unpleasant and are covered in [MR15, Section 2]. The important points are: (a) defining the v-adic distance can be done, and (b) any two distances constructed in this way are equivalent, in that their quotient is a bounded, positive function compactly supported away from zero. As with the heights, this will not affect the definition of α.

We now turn to α. We begin by defining the approximation constant for a single sequence.

Definition 2.1. Let $\{x_i\}$ be a sequence of k-rational points that converges v-adically to P. We define

$$A(\{x_i\}, L) = \{\gamma \in \mathbb{R} \mid \operatorname{dist}(P, x_i)^\gamma H_L(x_i) \text{ is bounded from above}\}\,.$$

The constant of L-approximation of $\{x_i\}$ to P is

$$\alpha(\{x_i\}, P, L) = \inf A(\{x_i\}, L)\,.$$

In particular, if $A(\{x_i\}, L) = \emptyset$, then $\alpha(\{x_i\}, P, L) = \infty$. We call $\alpha(\{x_i\}, P, L)$ the L-approximation constant of $\{x_i\}$.

The idea behind α is that it is the exponent you need on the distance $\operatorname{dist}(x_i, P)$ to make $\operatorname{dist}(x_i, P)^\alpha$ approximately equal to $H_L(x_i)$. The preceding definition is merely making that precise. Also, if we modify either the height or the distance by a bounded, positive function compactly supported away from zero, then the value of α is unchanged.

Next we must define α for a point, independently of the sequence. It is the infimum of all the approximation constants of sequences.

Definition 2.2. The L-approximation constant of P is defined to be

$$\alpha(P, L) = \inf_{\{x_i\}} \alpha(\{x_i\}, P, L)\,.$$

Next, we state a couple of results from [MR16] that will come in handy later. First, we state a result about how to compute α on a curve.

Theorem 2.3 ([MR16, Theorem 2.6]). *Let C be any curve birational over k to $\mathbb{P}^1_k$ and $\varphi\colon \mathbb{P}^1 \to C$ the normalization map. Then for any ample line bundle L on C and any $P \in C(\overline{\mathbb{Q}})$, we have the equality*

$$\alpha(P, L|_C) = \min_{q \in \varphi^{-1}(P)} d/r_q m_q\,,$$

where $d = \deg(L)$, m_q is the multiplicity of the branch of C through P corresponding to q, and

$$r_q = \begin{cases} 0 & \text{if } \kappa(q) \not\subseteq k_v\,, \\ 1 & \text{if } \kappa(q) = k\,, \\ 2 & \text{otherwise}\,, \end{cases}$$

where $\kappa(q)$ denotes the field of definition of q.

The other result is a Liouville-type result that provides lower bounds for α. This is a special case of [MR16, Theorem 3.3].

Theorem 2.4. *Let X be an algebraic variety defined over k and $P \in X(k)$ any k-rational point. Let $\pi\colon \tilde{X} \to X$ be the blowup of X at P with exceptional divisor E. Let L be a nef line bundle on X and $\gamma > 0$ a rational number such that $L_\gamma := \pi^*L - \gamma E$ is in the effective cone of $\tilde{X}$. Finally, let B' be the stable base locus of L_γ, and set $B = \pi(B')$.*

(i) *There is a positive real constant M such that for all $y \in X(k) - B(k)$, we have the lower bound $H_L(y)\operatorname{dist}_v(x, y)^\gamma \geqslant M$.*
(ii) *For any sequence $\{x_i\} \to P$ of k-rational points approximating P, if infinitely many points of $\{x_i\}$ are outside B, then $\alpha(\{x_i\}, P, L) \geqslant \gamma$.*
(iii) *If $\alpha(P, L) < \gamma$, then $P \in B$ and $\alpha(P, L) = \alpha(P, L|_B)$.*
(iv) *If $P \in B$ and $\alpha(P, L|_B) \geqslant \gamma$, then $\alpha(P, L) \geqslant \gamma$.*

In other words, Theorem 2.4 says that if L_γ is effective, then $\alpha(P, L) \geqslant \gamma$, except maybe for sequences that lie in the image of the stable base locus of L_γ.

The idea is to use Theorem 2.4 to show that most points do not approximate P very well and then Theorem 2.3 to show that the curve C we will find actually *does* approximate P well.

3. Main theorem

Let k be a number field, and let v be a place of k. Let $X \subset \mathbb{P}^n$ be a smooth cubic hypersurface defined over k, and let L be the hyperplane class on X.

If the dimension of X is 1, then of course there cannot be any line on X, k-rational or not, so the claim is vacuous. Nevertheless, it is classical[1] that $\alpha(P, L) = \infty$ for every point P of X, and Conjecture 1.1 is vacuously true for X.

If the dimension of X is 2, then the question of computing $\alpha(P, L)$ is treated in [MR16]. In [MR16], the Picard group of the surface is assumed to be entirely defined over k, but the argument for computing $\alpha(P, L)$ for the anticanonical class – that is, the class that embeds X as a cubic hypersurface – depends only on the existence of a k-rational conic through P, which is equivalent to the existence of a k-rational line on X ... unless P is the intersection of two conjugate irrational lines, in which case the argument breaks down. We remedy this failing with the following theorem, which also relaxes the smoothness hypothesis on the hypersurface.

Theorem 3.1. *Let $X \subset \mathbb{P}^3$ be a cubic surface (irreducible, but possibly singular) defined over k. Let $P \in X(k)$ be any k-rational smooth point, and let L be the hyperplane class. Assume that there is a line T on X defined over k, and let S_P be the intersection of X with the tangent plane at P. Assume further that the singular locus of X is supported on T. Then*

$$\alpha(P, L) = \begin{cases} 1 & \text{if } P \text{ lies on a } k\text{-rational line}\,, \\ 2 & \text{if } P \text{ is an isolated point in } S_P(k) \\ & \text{or if } S_P \text{ has two } k\text{-rational tangent lines at } P\,, \\ 3/2 & \text{otherwise}\,. \end{cases}$$

In all cases, there is a curve of best approximation to P.

Proof. Let $\pi \colon Y \to X$ be the blowup of X at P, with exceptional divisor E. Since $\pi^*L - 2E$ is effective with stable base locus S_P, Theorem 3.3 of [MR16] immediately shows that $\alpha_L(P) \geqslant 2$, except possibly for sequences contained in S_P.

Now, S_P is a plane curve of degree 3 with a singularity at P, so we can compute $\alpha(P, L|_{S_P})$ using Theorem 2.3. If P is v-adically isolated on S_P, then of course $\alpha(P, L|_{S_P}) = \infty$, and so S_P is no threat to the computation of α.

Otherwise, we can assume that P is the v-adic limit of k-rational points on S_P. Theorem 2.3 confidently lists the possibilities here:

- If S_P is reducible or nonreduced, then it contains at least one k-rational line and therefore $\alpha(P, L|_{S_P}) = 1$.
- If S_P is irreducible and there are two tangent lines to S_P at P, both defined over k, then $\alpha(P, L|_{S_P}) = 3$.
- If S_P is irreducible and there are two tangent lines to S_P at P, neither defined over k, then $\alpha(P, L|_{S_P}) = 3/2$. (Remember that P is not v-adically isolated on S_P, so the tangent lines are defined over the completion k_v.)
- If S_P is irreducible and there is only one tangent line to S_P at P, then $\alpha(P, L|_{S_P}) = 3/2$.

If you match up this list with the statement of the theorem, you find that the values of $\alpha(P, L|_{S_P})$ must equal the values of $\alpha(P, L)$ when they are less than 2, and we already knew

[1] See for example the second theorem of [Ser97, Section 7.3].

$\alpha(P, L) \geqslant 2$ in all the other cases. It therefore only remains to show $\alpha(P, L) \leqslant 2$. In other words, we need to find a curve C such that $\alpha(P, L|_C) \leqslant 2$.

In most cases, one can use the conic obtained by intersecting the plane through P and T with X. Unfortunately, that "conic" might actually be the union of two irrational lines, in which case it does not actually give an upper bound for $\alpha(P, L)$. So we use a different argument. First, if X is singular along all of T, then the hyperplane through P and T is the union of T with a *line*, so $\alpha(P, L|_{S_P}) = 1$. Thus, in what follows, we will assume that X has only isolated singularities, all supported on T by assumption.

Let $f\colon \tilde{X} \to X$ be a minimal desingularization of X, let $-K = f^*\mathcal{O}(1)$ be the anticanonical class on $\tilde{X}$, and let $\tilde{T}$ be the class on $\tilde{X}$ of the strict transform of the k-rational line T. It is classical (see for example [BW79]) that the singularities of X are canonical, so f is a crepant resolution of X: we have $K_{\tilde{X}} = f^*K_X$. This means that since $T\cdot(-K_X) = 1$, we get $\tilde{T}\cdot(-K_{\tilde{X}}) = 1$ (curves contracted by f intersect $K_{\tilde{X}}$ trivially), and so by adjunction we have $\tilde{T}^2 = -1$.

Let A be the class $\tilde{T} - K$; then $\deg_{-K}(A) = A \cdot (-K) = 4$ and $A^2 = 4$. Note that A is nef because $-K$ is ample and $\tilde{T} \cdot A = 0$, and therefore big because $A^2 = 4$. We now compute $h^0(A)$. The class A is $\pi^*(-K')$, where $\pi\colon X \to Y$ is the blowing down of $\tilde{T}$ and $-K'$ is the anticanonical class of Y. By [Bea96, Proposition IV.9], we have $h^0(-K') = 5$, and so $h^0(A) = 5$ as well.

There is therefore a pencil $\mathcal{F} \cong \mathbb{P}^1$ of curves in $|A|$ such that each curve in $\mathcal{F}$ has a singularity at P. Moreover, adjunction tells us that a smooth curve in $|A|$ has genus 1, so a general curve in $\mathcal{F}$ has geometric genus zero with a double point at P. (Bertini's theorem tells us that all but finitely many elements of $\mathcal{F}$ are irreducible.)

Let $\mathcal{H}$ be the projective line parametrizing tangent directions at P in the Zariski tangent space at P. Define $\Psi\colon \mathcal{F} \to \mathrm{Sym}^2(\mathcal{H})$ by $\Psi(C) = \{t_1, t_2\}$, where t_1 and t_2 represent the two tangent directions to C at P. No two curves in $\mathcal{F}$ can have the same tangent line at P because $A^2 = 4$, and so the morphism $\Psi\colon \mathcal{F} \to \mathrm{Sym}^2(\mathcal{H})$ is injective.

We will find a curve in $\mathcal{F}$ with irrational tangent lines that are defined over k_v. Theorem 2.3 will tell us that this curve C will be the one we are looking for, with $\alpha(P, L|_C) = \deg C/2 = 4/2 = 2$.

If $B \subset \mathrm{Sym}^2(\mathcal{H})$ is the image of Ψ, then its preimage $\tilde{B} \subset \mathcal{H} \times \mathcal{H}$ is in the class $(1, 1)$, and therefore induces an involution on $\mathcal{H}$ whose quotient is a surjective double cover $q\colon \mathcal{H} \to \mathcal{F}$. Roughly speaking, q maps (a pair of tangent directions) to (the curve in $\mathcal{F}$ that they come from). All we need now is a k-rational element $C \in \mathcal{F}$ such that $q^{-1}(C)$ is the union of two distinct irrational points that are defined over the completion k_v. In fact, we will show that there are infinitely many of these. To do this, it is enough to prove the following lemma, which is probably not new.

Lemma 3.2. *Let $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism of degree 2, defined over a number field k. For any place v of k, there are infinitely many points $P \in \mathbb{P}^1$ such that $f^{-1}(P)$ consists of two points that are defined over k_v but not k.*

Proof. Let $Q \in \mathbb{P}^1(k)$ be a rational point where f is not ramified. Then there is an open v-adic neighbourhood U of Q such that f is unramified at R for all $R \in U$.

The set $V = f(U)$ is therefore an open v-adic neighbourhood of the k-rational point $f(Q)$ of $\mathbb{P}^1$, with the property that for every point $P \in V$, the set $f^{-1}(P)$ consists of two points defined over k_v. The set V contains a positive proportion of the k-rational points of $\mathbb{P}^1$ by height, but the Hilbert irreducibility theorem (see for example [Ser97, Section 9]) implies that the proportion of k-rational points of $\mathbb{P}^1$ by height whose preimage is two k-rational points is zero. Therefore, there are infinitely many points of V whose preimage is defined over k_v but not k. □

Proof of Theorem 3.1, continued. We therefore have a curve C that is defined over k, irreducible, of genus zero, singular at P, and has tangent directions at P that are not defined over k but are defined over the completion k_v. (Indeed, we have infinitely many such curves.) By [MR16, Theorem 2.6], the curve C contains a sequence with approximation constant $\deg(C)/2 = 2$ and is therefore the curve we seek.

Thus, there is always a curve of best approximation to P, as desired, and the value of α is as advertised. □

Having treated the curve and surface cases, we will henceforth assume that the dimension of X is at least 3.

THEOREM 3.3. *Let $n \geqslant 4$ be an integer, $X \subset \mathbb{P}^n$ a smooth cubic hypersurface defined over k, $P \in X(k)$ any k-rational point, and let L be the hyperplane class. Assume that there is a line on X defined over k, and let S_P be the intersection of X with the tangent hyperplane at P. Then*

$$\alpha_L(P) = \begin{cases} 1 & \text{if } P \text{ lies on a } k\text{-rational line}\,, \\ 2 & \text{if } P \text{ is an isolated point in } S_P(k)\,, \\ 3/2 & \text{otherwise}\,. \end{cases}$$

In all cases, there is a curve of best approximation to P.

Remark 3.4. Note that the three cases are mutually exclusive, as any k-rational line through P must lie on S_P.

Proof of Theorem 3.3. First, notice that by Theorem 2.3, if P lies on a k-rational line, then $\alpha_L(P) = 1$, and the line is a curve of best approximation. Thus, we assume for the rest of the proof that P does not lie on a k-rational line.

Let $\pi\colon Y \to X$ be the blowup of X at P, with exceptional divisor E. Then $\pi^*L - 2E$ is effective (the tangent hyperplane has multiplicity 2 at P), with stable base locus contained in S_P. By Theorem 2.4, this means that $\alpha(P, L) \geqslant 2$ unless there is a sequence contained in S_P whose approximation constant is less than 2.

We first consider the case where P is an isolated rational point on S_P. We just showed that $\alpha(P, L) \geqslant 2$. All that is left is to show that $\alpha(P, L) \leqslant 2$.

Let H be any 3-dimensional linear space that intersects the tangent space at P properly and transversely, contains P and the line ℓ, and is defined over k. Let $V = H \cap X$ be the intersection of H and X. By Bertini's theorem, we may choose H so that V is a cubic surface in H that is defined over k and smooth at P, and whose singularities are supported on ℓ. Thus, by Theorem 3.1, we deduce that $\alpha(P, L) \leqslant \alpha(P, L|_V) \leqslant 2$.

We may therefore assume that P is not an isolated rational point on S_P and restrict our attention to S_P. We will show that $\alpha(P, L|_{S_P}) = 3/2$ and find a curve of best approximation to P with respect to $L|_{S_P}$. (Remember that we assumed that P does not lie on a k-rational line!)

Let $\phi\colon S_P \dashrightarrow \mathbb{P}^{n-2}$ be the linear projection from the point P, considered inside the tangent hyperplane $T_P \cong \mathbb{P}^{n-1}$. The fibres of ϕ (over $\bar{k}$) are either single points or lines. Therefore, if ϕ is not dominant, then all the fibres are lines and S_P is a cone. Since P is not an isolated rational point on S_P, there is another rational point on S_P, and therefore a k-rational line containing P, which contradicts our earlier assumption. Thus, we may assume that ϕ is dominant. In this case, we may change coordinates in T_P (which does not change α) so that $P = [0:0:\ldots:1]$ and the defining equation for S_P is

$$f(x_1, \ldots, x_{n-1}) + x_n g(x_1, \ldots, x_{n-1}) = 0\,,$$

where f is a homogeneous cubic and g is a nonzero homogeneous quadric. The rational map

$$\psi(\mathbf{x}) = [x_1 g(x_1, \ldots, x_{n-1}) : \cdots : x_{n-1} g(x_1, \ldots, x_{n-1}) : -f(x_1, \ldots, x_{n-1})]$$

is inverse to ϕ, defined everywhere except $f = g = 0$. Thus, S_P is the blowup of $\mathbb{P}^{n-2}$ along the scheme $f = g = 0$, with the strict transform of $g = 0$ contracted.

More precisely, let $Z \subset T_P \times \mathbb{P}^{n-2}$ be the closure of the graph of ϕ. If π_i is the projection of Z onto the ith factor, then

- $\pi_2 \colon Z \to \mathbb{P}^{n-2}$ is the blowup along $f = g = 0$,
- $\pi_1 \colon Z \to S_P$ is an isomorphism away from the strict transform C of $g = 0$ from π_2 and contracts C to the point P.

Now let $\{y_i\} \subset S_P$ be any sequence of rational points converging to P. We will show that $\alpha(\{y_i\}, P, L) \geqslant 3/2$. The idea is to relate the line bundle L to the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^{n-2}$ because we know all about α on projective space already.

First, note that all of the y_i lie in the subset of S_P on which ϕ is an isomorphism since none of them is P and none of the lines through P are k-rational. That means that there is a unique sequence $\{z_i\}$ of rational points on Z such that $\pi_1(z_i) = y_i \in S_P$.

Now let us pull back both L and $H = \mathcal{O}_{\mathbb{P}^{n-2}}(1)$ to Z and compare them there. Let $L' = L|_{S_P}$, a divisor class on S_P, and let $B = \pi_1^* L'$, a divisor class on Z. Let $H = \mathcal{O}_{\mathbb{P}^{n-2}}(1)$, a divisor class on $\mathbb{P}^{n-2}$. Then $B = \pi_2^* H + C$ and $C = 2\pi_2^* H - D$ for some effective divisor D. (Recall that C is the strict transform of $g = 0$.) Therefore, $2B - 3C = 2\pi_2^* H - C = D$, which is effective.

Therefore, up to multiplication by bounded functions compactly supported away from zero, for any $\gamma < 3/2$, we have

$$\begin{aligned} H_{L'}(y_i)\,\mathrm{dist}(P, y_i)^\gamma &= H_B(z_i)\,\mathrm{dist}(P, y_i)^\gamma \\ &\geqslant H_C(z_i)^{3/2}\,\mathrm{dist}(P, y_i)^\gamma \\ &\geqslant \mathrm{dist}(P, y_i)^{-3/2}\,\mathrm{dist}(P, y_i)^\gamma \\ &\to \infty \quad \text{as } i \to \infty\,. \end{aligned}$$

We conclude that $\alpha_{L'}(\{y_i\}, P) \geqslant 3/2$ and therefore $\alpha_{L'}(P) \geqslant 3/2$, as desired.

All that is left is to show that $\alpha_{L'}(P) \leqslant 3/2$, which we can do by exhibiting an approximating sequence that achieves $\alpha = 3/2$. We can do that by finding a curve through P of L'-degree 3 and a suitable singularity at P.

Since P is not an isolated rational point on S_P, it follows that the quadric $g = 0$ must have points defined over the completion k_v of k at the place v. Let Q be an irrational point of $g = 0$ (which we may choose to be away from the locus $f = 0$), defined over a quadratic extension of k that is contained in k_v, and let T be the line joining Q to its conjugate.

The line T is defined over k, so let T' be the closure of the curve $\psi(T)$ in S_P. The map ψ is an isomorphism away from $g = 0$, which is contracts to P, so T' is just T with the point Q glued to its conjugate. We have

$$L' \cdot T' = (\pi_1^* L') \cdot (\pi_2^* T) = B \cdot \pi_2^* T = (\pi_2^* H + C) \cdot (\pi_2^* T) = 3\,,$$

so T' is a cubic curve T' with a node at P, with tangent lines corresponding to Q and its conjugate. Since the tangent lines to T' at P are defined over k_v but not over k, by Theorem 2.3, there is a sequence on T' with $\alpha = 3/2$, and T' is a curve of best approximation to P. $\square$

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