Algebraic Geometry 13 (1) (2026) 25–57
doi:10.14231/AG-2026-002

Generic curves and non-coprime Catalan numbers

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Abstract

We compute the Poincaré polynomials of the compactified Jacobians for plane curve singularities with Puiseux exponents $(nd, md, md+1)$, and relate them to the combinatorics of q,t-Catalan numbers in the non-coprime case. We also confirm a conjecture of Cherednik and Danilenko for such curves.

1. Introduction

In this paper, we study the topology of compactified Jacobians of plane curve singularities. We focus on the case where the curve is reduced and locally irreducible (or *unibranched*). It is known [AIK77, AK76] that in this case, the compactified Jacobian is irreducible as well. Throughout the paper, we work over $\mathbb{C}$. We explain where this assumption is used, in case our reader is curious about the extension of the result to other fields.

Compactified Jacobians play an important role in modern geometric representation theory. First, they are closely related to Hilbert schemes of points on singular curves, singular fibers in the Hitchin fibration and affine Springer fibers. In particular, counting points in the compactified Jacobians over finite fields is related to certain orbital integrals [Lau06, KT22, Tsa15]. Recent works [GK23, HKW23, GKO25] relate them to the representation theory of Coulomb branch algebras, defined by Braverman, Finkelberg and Nakajima [BFN18].

Second, a set of conjectures of the third author, Rasmussen and Shende [OS12, ORS18] relates the homology of compactified Jacobians to the *Khovanov–Rozansky homology* of the corresponding knots and links. In particular, the conjectures imply that the homology of the compactified Jacobian is expected to be determined by the topology of the link or, in the unibranched case, by the collection of Puiseux pairs of the singularity.

The progress in explicit computations of the homology of compactified Jacobians has been quite slow. For the quasi-homogeneous curves $C = \{x^m = y^n\}$ with $\mathrm{GCD}(m,n) = 1$, the homology was computed in many sources, starting with Lusztig and Smelt [LS91]. The key observation is that in this case, the compactified Jacobian $\overline{JC}$ admits a paving by affine cells. These cells and their dimensions have been given a number of combinatorial interpretations in [GM13, GM14, GMV16, Hik14], where they were related to q,t-Catalan combinatorics. In

Received 28 October 2022, accepted in final form 19 June 2024.
2020 Mathematics Subject Classification 14H20, 14H40, 05E14.
Keywords: compactified Jacobians, Dyck paths, q,t-Catalan numbers.

E. G. was partially supported by the NSF grant DMS-1760329, and by the NSF grant DMS-1928930 while E. G. was in residence at the Simons Laufer Mathematical Sciences Institute (previously known as MSRI) in Berkeley, California, during the Fall 2022 semester. M. M. was partially supported by the Simons Collaboration grant 524324. M. M. is also grateful to the MSRI for hospitality during the the Winter-Spring 2022 semester, where the project started. A. O. was supported by the NSF grants DMS-1760373 and DMS–2200798.

[OY16, OY17], the third author and Yun determined the ring structure on the homology in this case.

In a different direction, Piontkowski [Pio07] has computed the homology of compactified Jacobians for plane curve singularities with one Puiseux pair defined by the parametrization $(x, y) = (t^n, t^m + \cdots)$ with $\mathrm{GCD}(m, n) = 1$. He showed that $\overline{JC}$ again admits an affine paving, and the combinatorics of cells depends only on (m, n) and hence agrees with the quasi-homogeneous case $(x, y) = (t^n, t^m)$. Moreover, Piontkowski computed the cell decompositions for some singularities with two Puiseux exponents (see Section 6.2), where the combinatorics becomes rather subtle.

In our main theorem, we vastly generalize the results of Piontkowski and prove the following.

Theorem 1.1. *Suppose that* $\mathrm{GCD}(m, n) = 1$ *and* $d \geqslant 1$*, consider the plane curve singularity* C *defined by the parametrization*

$$(x(z), y(z)) = \left(t^{nd}, t^{md} + \lambda t^{md+1} + \cdots\right), \quad \lambda \neq 0\,. \tag{1.1}$$

Then:

(a) *The compactified Jacobian* $\overline{JC}$ *admits an affine paving where the cells are in bijection with Dyck paths* D *in an* $(nd) \times (md)$ *rectangle.*

(b) *The Poincaré polynomial of* $\overline{JC}$ *is given by*

$$P_{\overline{JC}}(T) = \sum_{D \in \mathrm{Dyck}(nd,md)} T^{2(\delta - \mathrm{dinv}(D))}\,,$$

where dinv *is a certain statistic on Dyck paths defined in Section 4 and* δ *is the number of boxes weakly under the diagonal in an* $(nd) \times (md)$ *rectangle.*

(c) *In particular, the Poincaré polynomial does not depend on* λ (*as long as it is non-zero*) *or on the higher-order terms in* (1.1).

We call the plane curve singularities (1.1) *generic* (or generic curves in short) since a generic plane curve singularity with first Puiseux pair (nd, md) has this form. In the notation of [Pio07], for $d > 1$ these are the plane curve singularities with Puiseux exponents $(nd, md, md + 1)$, while for $d = 1$ these have Puiseux exponents (n, m). We recall some basic facts about such singularities in Section 2.

The affine paving in the statement of Theorem 1.1 is obtained by intersecting the compactified Jacobian with the Schubert cell paving of the affine Grassmannian. Thus one can define a partial order on the strata such that the boundary of an affine cell lies in the union of the cells with smaller indices in this order. Hence we conclude the following.

Corollary 1.2. *For generic plane curve singularities, the cohomology* $H^*(\overline{JC})$ *is supported in even degrees, and the weight filtration on* $H^*(\overline{JC})$ *is pure in the sense of* [GKM06, GKM04].

Very recently, Kivinen and Tsai [KT22] used completely different methods (p-adic harmonic analysis) to count points in arbitrary compactified Jacobians over finite fields $\mathbb{F}_q$. They proved that the result is always a polynomial in q and hence recovers the weight polynomial of $\overline{JC}$ (see Theorem 5.3 for more details). Given the above purity result, for generic plane curve singularities, the Poincaré polynomial of $\overline{JC}$ agrees with the weight polynomial, and our result agrees with theirs.

It follows from Theorem 1.1 that the Euler characteristic of $\overline{JC}$ is given by the number of Dyck paths in an $(nd) \times (md)$ rectangle. For example, for the singularity $C = (t^4, t^6 + t^7)$, the

Euler characteristic $\chi(\overline{JC}) = 23$ is equal to the number of Dyck paths in a 4×6 rectangle (see Example 6.12), in agreement with [Pio07]. In general, the combinatorial results of [Ber21, Biz54] immediately imply the following.

Corollary 1.3. *Let us fix coprime m, n, and for each $d \geqslant 1$ let C_d be a generic plane curve singularity* (1.1). *Then*

$$1+\sum_{d=1}^{\infty} z^d \chi\left(\overline{JC_d}\right) = \exp\left(\sum_{d=1}^{\infty} \frac{z^d}{(m+n)d}\binom{md+nd}{md}\right).$$

Next, we address the conjectures of Cherednik, Danilenko and Philipp [CD16, CP18], which propose an expression for the Poincaré polynomials of compactified Jacobians in terms of certain matrix elements of certain operators in the double affine Hecke algebra; see Section 5 for more details. We are able to prove this conjecture for generic plane curve singularities.

Theorem 1.4. *Consider the two-variable polynomial*

$$C_{nd,md}(Q,T) = \sum_{D\in \mathrm{Dyck}(dn,dm)} Q^{\mathrm{area}(D)} T^{\mathrm{dinv}(D)}, \tag{1.2}$$

where area(D) *is the number of full boxes between a Dyck path D and the diagonal. It has the following properties:*

(a) *It is symmetric in Q and T; that is, $C_{nd,md}(Q,T) = C_{nd,md}(T,Q)$.*
(b) *At $Q = 1$, it specializes to the Poincaré polynomial of $\overline{JC}$ (up to a linear change of the variable).*
(c) *It is given by the matrix element $(\gamma_{n,m}(e_d)(1), e_{nd})$ of the elliptic Hall algebra operator $\gamma_{n,m}(e_d)$.*
(d) *It agrees with the Cherednik–Danilenko conjecture (Conjecture 5.2).*

Recently, the second author, Caprau, Gonzalez and Hogancamp were able to show that the polynomial $C_{nd,md}(Q,T)$ is the $a = 0$ specialization of the Poincaré polynomial of the reduced Khovanov–Rozansky homology of the $(d, mnd + 1)$-cable of the (n, m) torus knot [CGHM24]. Combined with Theorem 1.4, this confirms a conjecture of the third author, Rasmussen and Shende in the case of generic plane curve singularities.

In Section 5, we also give a different, manifestly symmetric in Q and T, explicit formula for $C_{nd,md}(Q,T)$. The fact that the two formulas agree is a special case of the compositional rational shuffle theorem, conjectured in [BGLX16] and proved in [Mel21]. In fact, parts (a) and (c) of Theorem 1.4 immediately follow from that theorem, and part (d) follows from part (c); see Section 5 for a detailed proof. Part (b) is a rephrasing of Theorem 1.1.

Theorem 1.4(a) allows us to give a simple formula for the Poincaré polynomial of the compactified Jacobian.

Corollary 1.5. *The Poincaré polynomial of $\overline{JC}$ equals*

$$T^{2\delta} C_{nd,md}\big(1, T^{-2}\big) = T^{2\delta} C_{nd,md}\big(T^{-2}, 1\big) = \sum_{D\in \mathrm{Dyck}(nd,md)} T^{2(\delta-\mathrm{area}(D))}.$$

1.1 Proof strategy and organization of the paper

Let us comment on the main ideas in the proof of Theorem 1.1. The ring of functions $\mathcal{O}_C = \mathbb{C}\big[\big[t^{nd}, t^{md} + \lambda t^{md+1} + \ldots\big]\big]$ on the plane curve singularity C has a natural valuation given by the order of a function in the parameter t. We consider the semigroup Γ of C consisting of all possible orders of functions; it is generated by the numbers $(nd, md, mnd+1)$.

By definition, the compactified Jacobian $\overline{JC}$ is the moduli space of torsion-free rank 1 $\mathcal{O}_C$-modules with appropriate framing or, equivalently, $\mathcal{O}_C$-submodules of $\mathbb{C}((t))$ with appropriate normalization; see Section 2.1. Clearly, the set of valuations for any such submodule M is a Γ-submodule Δ_M of $\mathbb{Z}$.

We stratify the compactified Jacobian by the possible combinatorial types of submodules $\Delta = \Delta_M$:

$$J_\Delta = \{M \subset \mathbb{C}((t)) \mid \mathcal{O}_C M \subset M,\, \Delta_M = \Delta\}\,.$$

Next, we introduce the key definition of an *admissible* Γ-submodule (Definition 3.13) generalizing the constructions of Piontkowski [Pio07] and Cherednik–Philipp [CP18]. In Lemma 3.16, we prove that if Δ is not admissible, then J_Δ is empty. The converse statement is significantly harder.

Theorem 1.6 [(Proposition 3.23)]. *Suppose that Δ is admissible. Then the corresponding stratum $J_\Delta \subset \overline{JC}$ is isomorphic to the affine space of dimension*

$$\dim J_\Delta = \sum_{i,j} (|\mathrm{Gaps}(a_{j,i})| - |\mathrm{Gaps}(a_{j,i} + md)|)\,,$$

where $\mathrm{Gaps}(x) = [x, +\infty) \setminus \Delta$ *and the* $a_{j,i}$ *are the* (nd)*-generators of* Δ.

Remark 1.7. Although both [Pio07] and [CP18] use the notions of admissible subsets, there is an important difference between their definitions. In [Pio07], as in this paper, the definition of admissible subsets is purely combinatorial. On the other hand, [CP18] gives a geometric definition by saying that Δ is admissible if J_Δ is non-empty. Lemma 3.16 and Theorem 1.6 show that the two definitions agree for $s = 1$. However, we explain in the next subsection that for $s > 1$ the combinatorial and geometric definitions of admissible subsets diverge, and we always use the combinatorial one in this paper.

To prove Theorem 1.6, we need to analyze the explicit equations defining J_Δ; this analysis occupies most of Section 3. The coordinates on J_Δ are indexed by $\mathrm{Gaps}(a_{j,i})$, and the equations are indexed by $\mathrm{Gaps}(a_{j,i}+md)$ for all j, i. We show in Proposition 3.23 that with an appropriately defined partial order on the coordinates, the equations can be written in such a way that they express certain coordinates in terms of smaller coordinates with respect to this order. This allows us to eliminate the variables one by one, until all the equations are exhausted.

Remark 1.8. This approach follows the ideas of [Pio07] and [CP18]. In the nutshell, in both settings of [Pio07] and [CP18], the linear parts of the equations are linearly independent, and the non-linear parts depend on smaller coordinates (with respect to the order); hence one can just keep track of the linear parts.

The main new conceptual obstacle that we encounter in the proof of Theorem 1.6 is that for sufficiently large m, n, d, the *linear parts of many equations become dependent* even if Δ is admissible; see Example 3.5 and Lemma 3.9. We develop combinatorial machinery to keep track of all such dependencies and introduce some novel changes of coordinates to resolve these. The order on the new variables is also changed. The change of variables heavily uses the fact that C is generic and $\lambda \neq 0$.

Theorem 1.6 immediately yields the Poincaré polynomial of $\overline{JC}$, described in terms of admissible subsets. Still, a significant combinatorial work is required to relate it to the Dyck paths that appear in Theorem 1.1. Following [GMV17], we introduce an equivalence relation on all (nd, md)-invariant subsets of $\mathbb{Z}_{\geqslant 0}$. The main result of [GMV17] establishes a bijection between the equivalence classes and Dyck paths in an $(nd) \times (md)$ rectangle. In this paper, we prove Theorem 4.10, which shows that in each equivalence class, there exists a unique admissible subset. By combining these results, we establish a bijection between the admissible subsets and the Dyck paths, and complete the proof of Theorem 1.1.

Finally, in Section 5, we recall some results on rational shuffle theorem, elliptic Hall algebra and double affine Hecke algebras (DAHA), and we prove Theorem 1.4.

1.2 Beyond generic curves

The last section of the paper is focused on more general plane curve singularities

$$(x(t), y(t)) = \left(t^{nd}, t^{md} + \lambda t^{md+s} + \cdots\right)$$

with two Puiseux pairs. We assume that $\mathrm{GCD}(m, n) = \mathrm{GCD}(d, s) = 1$ and $\lambda \neq 0$; the case $s = 1$ corresponds to the generic curves above.

The definition of admissible subsets has a natural generalization that we call s-admissible (see Definition 6.1). Combinatorially, we can parametrize the s-admissible subsets as follows.

Theorem 1.9. *There is a bijection between the s-admissible subsets and* cabled Dyck paths, *which are parametrized by the choice of a (d, s) Dyck path P with vertical runs $v_i(P)$ and a tuple of $(v_i(P)n, v_i(P)n)$ Dyck paths.*

See Definition 6.4 and Theorem 6.7 for more details. Note that for $s = 1$ there is only one (d, s) Dyck path with $v_1(P) = d$, and hence cabled Dyck paths are simply (dn, dm) Dyck paths as above.

On the geometric side, we can still stratify the compactified Jacobian by the strata J_Δ parametrized by the (nd, md)-invariant subsets Δ. However, the situation for $s > 1$ is significantly more complicated than it is for $s = 1$. In general, J_Δ might be non-empty if Δ is not s-admissible, and it might not be isomorphic to an affine space if it is s-admissible. Nevertheless, we conjecture the following weaker statement.

Conjecture 1.10. *The Euler characteristic of the stratum J_Δ is given by*

$$\chi(J_\Delta) = \begin{cases} 1 & \text{if } \Delta \text{ is } s\text{-admissible}, \\ 0 & \text{if } \Delta \text{ is not } s\text{-admissible}. \end{cases}$$

All existing computations by Piontkowski [Pio07] and Cherednik–Philipp [CP18] are compatible with this conjecture. In particular, in examples one can find the strata J_Δ isomorphic to $\mathbb{C}^* \times \mathbb{C}^N$ (with Euler characteristic 0) and $\mathbb{C}^N \cup_{\mathbb{C}^{N-1}} \mathbb{C}^N$ (with Euler characteristic 1). Conjecture 1.10 and Theorem 1.9 immediately imply the following.

Corollary 1.11. *Under the assumption of Conjecture 1.10, the Euler characteristic of the compactified Jacobian equals the number of s-admissible subsets and the number of cabled Dyck paths with parameters (m, n), (d, s).*

In [KT22], Kivinen and Tsai proved unconditionally that $\chi(\overline{JC})$ equals the number of cabled Dyck paths.

We plan to study the geometry of strata J_Δ in more details and generalize the above results to singularities with more than two Puiseux pairs in future work. Another interesting future direction is understanding the Euler characteristics of Hilbert schemes of points on singular plane curves by similar methods, which would yield a new approach to the main conjecture of [OS12]. Finally, [CP18] introduced a "flagged" generalization of compactified Jacobians, and it would be interesting to see if these spaces associated with generic curves admit cell decompositions.

2. Background

2.1 Compactified Jacobians and semigroups

Let C be an irreducible (and reduced) plane curve singularity at $(0,0)$. We can parametrize C as $(x(t), y(t))$ and write the local ring of functions on C as $\mathcal{O}_C = \mathbb{C}[[x(t), y(t)]] \subset \mathbb{C}((t))$. Given a function $f(t) \in \mathbb{C}((t))$, we can write

$$f(t) = \alpha_k t^k + \alpha_{k+1} t^{k+1} + \cdots, \quad \alpha_k \neq 0,$$

and define the order of $f(t)$ by $\mathrm{Ord} f(t) = k$. It is well known that Ord is a valuation on $\mathcal{O}_C$, that is,

$$\mathrm{Ord}(f \cdot g) = \mathrm{Ord}(f) + \mathrm{Ord}(g), \quad \mathrm{Ord}(f+g) \geqslant \min(\mathrm{Ord}(f), \mathrm{Ord}(g)).$$

The compactified Jacobian $\overline{JC}$ is defined as the moduli space of rank 1 torsion-free sheaves on C or, equivalently, $\mathcal{O}_C$-submodules $M \subset t^{-N}\mathbb{C}[[t]]$ (for sufficiently large N) such that

$$\dim \frac{t^{-N}\mathbb{C}[[t]]}{M} = \dim \frac{t^{-N}\mathbb{C}[[t]]}{\mathcal{O}_C}. \tag{2.1}$$

Note that M is an $\mathcal{O}_C$-submodule if and only if $x(t)M \subset M$ and $y(t)M \subset M$. Given such a subspace M, we define

$$\Delta_M = \{\mathrm{Ord}\ f(t) \mid f(t) \in M\} \subset \mathbb{Z}.$$

In particular, for $M = \mathcal{O}_C$ we obtain the *semigroup* of C:

$$\Gamma = \Delta_{\mathcal{O}_C} = \{\mathrm{Ord}\ f(t) \mid f(t) \in \mathcal{O}_C\}.$$

If M is an $\mathcal{O}_C$-submodule, then Δ_M is a Γ-module: $\Delta_M + \Gamma \subset \Delta_M$. This motivates the following.

Definition 2.1. Given a subset $\Delta \subset \mathbb{Z}$, we define the stratum in the compactified Jacobian:

$$J_\Delta := \{M \subset \mathbb{C}((t)) \mid \mathcal{O}_C M \subset M,\ \Delta_M = \Delta\}.$$

Definition 2.2. A subset $\Delta \subset \mathbb{Z}$ is called *balanced* if $\Delta \subset [-N, +\infty)$ and $|[-N,+\infty) \setminus \Delta| = |[-N,+\infty) \setminus \Gamma|$ for sufficiently large N, and 0-*normalized* if $\min \Delta = 0$.

By definition, the subsets Δ_M corresponding to M satisfying (2.1) are balanced, but for many technical reasons, it is easier to work with 0-normalized subsets that correspond to $M \subset \mathbb{C}[[t]]$. The two classes of subsets are related by a shift, which corresponds to the multiplication of M by a power of t. Clearly, the shift does not change J_Δ up to isomorphism, so in the rest of the paper, we will abuse the notation and talk about the J_Δ for 0-normalized subsets Δ instead.

Clearly, J_Δ give a stratification of $\overline{JC}$, and J_Δ is empty if Δ is not Γ-invariant. As a warning to the reader, J_Δ could be empty even if Δ is Γ-invariant, as shown in the following example.

Example 2.3. Consider the singularity $(x(t), y(t)) = (t^4, t^6 + t^7)$. It is easy to see that $y^2(t) - x^3(t) = 2t^{13} + t^{14}$ has order 13, and one can check that the semigroup Γ in this case is generated by 4, 6 and 13. The subset

$$\Delta = \{0, 2, 4, 6, 8, 10, 12, 13, 14, \dots\}$$

is clearly Γ-invariant. Suppose that $M \subset J_\Delta$ and that $f_0 = 1 + at + \cdots$ and $f_2 = t^2 + bt^3 + \cdots$ are two elements of M of orders 0 and 2, respectively. Then

$$y(t)f_0 - x(t)f_2 = \big(t^6 + t^7\big)(1 + at + \cdots) - t^4\big(t^2 + bt^3 + \cdots\big) = (a + 1 - b)t^7 + \cdots\,,$$
$$x^2(t)f_0 - y(t)f_2 = t^8(1 + at + \cdots) - \big(t^6 + t^7\big)\big(t^2 + bt^3 + \cdots\big) = (a - b - 1)t^9 + \cdots\,.$$

Observe that $a + 1 - b$ and $a - b - 1$ cannot vanish at the same time, so M contains either an element of order 7 or an element of order 9, and we have a contradiction. Therefore, the corresponding stratum J_Δ is empty.

We will see that a necessary condition for J_Δ being non-empty is that Δ satisfies the so-called admissibility property, which we define below.

Remark 2.4. To replace the base field $\mathbb{C}$ with arbitrary field k, we need a parametrization $x(t)$, $y(t)$ of the germ of the curve singularity to be defined over k.

2.2 Generic curves

It is well known that any plane curve singularity C can be parametrized using a Puiseux expansion

$$x = t^{nd}\,, \quad y = t^{md} + \lambda t^{md+1} + \cdots\,, \quad \mathrm{GCD}(m, n) = 1\,.$$

It is convenient to assume that $n < m$, but sometimes we will not need this assumption. If $d = 1$, then C has one Puiseux pair (n, m), and its link is the (n, m) torus knot. In this paper, we will be mostly interested in the case $d > 1$.

Definition 2.5. Assume that $d > 1$. A plane curve singularity C is called *generic* (or a generic curve for short) if $\lambda \neq 0$.

It is easy to see that the definition of a generic curve is symmetric in n and m. Indeed, we can choose the new parameter

$$\widetilde{t} = \sqrt[md]{y(t)} = t\,\sqrt[md]{1 + \lambda t^{md+1} + \cdots} = t\left(1 + \frac{\lambda}{md}t + \cdots\right);$$

then

$$y = \widetilde{t}^{\,md}\,, \quad x = \widetilde{t}^{\,nd} - \frac{n\lambda}{md}\widetilde{t}^{\,nd+1} + \cdots\,.$$

For a generic plane curve singularity, the difference $y^n - x^m$ has order $nmd + 1$, and one can check that the semigroup Γ is generated by nd, md and $nmd + 1$.

If $n = 1$, then the singularity has one Puiseux pair $(d, md+1)$. Otherwise, it has two Puiseux pairs (nd, md) and $(d, md + 1)$, which completely determine the topological type of the corresponding knot as a $(d, mnd + 1)$-cable of the (n, m) torus knot.

Remark 2.6. The above argument requires the base field to be algebraically closed of characteristic 0. However, the symmetry is not used in the rest of the paper.

2.3 Invariant subsets

A subset $\Delta \subset \mathbb{Z}_{\geqslant 0}$ is called 0-*normalized* if $0 \in \Delta$. We will mostly consider 0-normalized subsets, as any subset of $\mathbb{Z}_{\geqslant 0}$ can be shifted to a unique 0-normalized one. We call Δ *cofinite* if $\mathbb{Z}_{\geqslant 0} \setminus \Delta$ is finite. For a cofinite subset Δ and $x \geqslant 0$, we write

$$\mathrm{Gaps}(x) = \mathrm{Gaps}_\Delta(x) := [x, +\infty) \setminus \Delta\,.$$

Definition 2.7. We call Δ an (nd)-*invariant subset* if $nd+\Delta\subset\Delta$. A number a is called an (nd)-*generator* of Δ if $a\in\Delta$ but $a-nd\notin\Delta$.

It is clear that for a cofinite (nd)-invariant subset Δ, there is exactly one (nd)-generator in each remainder modulo nd. We will group them according to their remainders modulo d so that the generators $a_{j,i}$ for $i=0,\ldots,n-1$ all have remainder j modulo d and different remainders modulo nd. We write

$$\Delta_j=\bigcup_{i=0,\ldots,n-1}(a_{j,i}+dn\mathbb{Z}_{\geqslant 0})\,,\quad \Delta=\bigcup_{j=0,\ldots,d}\Delta_j\,.$$

Lemma 2.8. *Suppose that Δ is a cofinite (nd)-invariant subset with (nd)-generators $a_{j,i}$ as above. Then Δ is (md)-invariant if and only if one can cyclically order the generators $a_{j,i}$ for each j such that*

$$a_{j,i}+md=a_{j,i+1}+\alpha_{j,i}nd\,,\quad \alpha_{j,i}\geqslant 0\,. \tag{2.2}$$

Proof. Since Δ is (nd)-invariant, it is (md)-invariant if and only if $a_{j,i}+md\in\Delta$. Since $a_{j,i}+md\equiv j \mod d$, this is equivalent to the existence of a generator $a_{j,i'}\equiv a_{j,i}+md \mod nd$ such that $a_{j,i'}\leqslant a_{j,i}+md$.

Observe that $(a_{j,i'}-j)/d\equiv(a_{j,i}-j)/d+m \mod n$. Since m and n are coprime, for a fixed j the permutation $a_{j,i}\to a_{j,i'}$ is a single cycle, and the result follows. □

For a cofinite (nd,md)-invariant (that is, both nd- and md-invariant) subset Δ, we will always assume that for each j the generators $a_{j,i}$ are cyclically ordered as above. Furthermore, we will consider the second index i modulo n, so $a_{j,i}=a_{j,i+n}$, and we will consider the first index j modulo m, so $a_{j,i}=a_{j+m,i}$.

We will call the integers $a_{j,i}+md$ the combinatorial syzygys of Δ. It is clear that in each remainder modulo d, there are n such syzygys, and the equations (2.2) form an n-cycle.

Example 2.9. In Example 2.3, we have $d=2$, $n=2$ and $m=3$. The subset

$$\Delta=\{0,2,4,6,8,10,12,13,14,\ldots\}$$

contains all integers starting from 12, so it is cofinite. The 4-generators are 0, 2, 13, 15, and the combinatorial syzygys are

$$0+6=2+4\,,\quad 2+6=0+2\cdot 4\,,\quad 13+6=15+4\,,\quad 15+6=13+2\cdot 4\,.$$

In the notation of Lemma 2.8, we can write

$$a_{0,0}=0\,,\quad a_{0,1}=2\,,\quad a_{1,0}=13\,,\quad a_{1,1}=15$$

and

$$\alpha_{0,0}=1\,,\quad \alpha_{0,1}=2\,,\quad \alpha_{1,0}=1\,,\quad \alpha_{1,1}=2\,.$$

Example 2.10. Suppose that $(nd,md)=(6,9)$, so $d=3$, $n=2$ and $m=3$. The subset

$$\Delta=\{0,3,6,7,9,10,12,13,15,16,17,18,19,20,\ldots\}$$

contains all integers starting from 15, so it is cofinite. It is not hard to see that it is $(6,9)$ invariant and its 6-generators are 0, 3, 7, 10, 17, 20. The combinatorial syzygys are

$$\begin{aligned}0+9&=3+6\,, & 3+9&=0+2\cdot 6\,, & 7+9&=10+6\,,\\ 10+9&=7+2\cdot 6\,, & 17+9&=20+6\,, & 20+9&=17+2\cdot 6\,.\end{aligned}$$

In the notation of Lemma 2.8, we can write

$$a_{0,0}=0\,,\quad a_{0,1}=3\,,\quad a_{1,0}=7\,,\quad a_{1,1}=10\,,\quad a_{2,0}=17\,,\quad a_{2,1}=20\,,$$
$$\alpha_{0,0}=1\,,\quad \alpha_{0,1}=2\,,\quad \alpha_{1,0}=1\,,\quad \alpha_{1,1}=2\,,\quad \alpha_{2,0}=1\,,\quad \alpha_{2,1}=2\,.$$

Note that the first index corresponds to the remainder of the generators modulo $d=3$.

LEMMA 2.11. *Assume that Δ is a cofinite (nd,md)-invariant subset. Suppose that $a_{j,i}+md=a_{j,i+1}+\alpha_{j,i}nd$ is a combinatorial syzygy and $a_{j,i}+md+x$ is a gap in Δ. Then both $a_{j,i}+x$ and $a_{j,i+1}+x$ are gaps as well.*

Proof. If $a_{j,i}+x\in\Delta$, then $a_{j,i}+x+md\in\Delta$. If $a_{j,i+1}+x\in\Delta$, then $a_{j,i+1}+x+\alpha_{j,i}nd\in\Delta$ and $a_{j,i+1}+x+\alpha_{j,i}nd=a_{j,i}+x+md$. We have a contradiction. □

3. Topology of compactified Jacobians

Throughout this section, we fix a generic plane curve singularity C with parametrization

$$(x(t),y(t))=\left(t^{nd},t^{md}+\lambda t^{md+1}+\cdots\right)$$

with $\lambda\neq 0$. We will use the notation $\mathcal{O}_C=\mathbb{C}[[x(t),y(t)]]$ for the ring of functions on C.

Remark 3.1. The existence of such a parametrization over an arbitrary field is not automatic. However, if we assume this specific parametrization for C, all the arguments in this section are valid over any field of characteristic larger than n (see Remarks 3.17 and 3.21 for more details).

We also fix a cofinite (nd,md)-invariant subset $\Delta\subset\mathbb{Z}_{\geqslant 0}$ with (nd)-generators $a_{j,i}$ as in Section 2.3. We denote by $A=\{a_{j,i}\}$ the set of all (nd)-generators and follow the notation (2.2) (see Lemma 2.8) in ordering them.

The main goal of this section is to describe the stratum J_Δ in the compactified Jacobian $\overline{JC}$.

3.1 Equations for J_Δ

Consider an $\mathcal{O}_C$-module $M\in J_\Delta$. The following lemma is standard, but we include the proof for the reader's convenience.

LEMMA 3.2. *For all $k\in\Delta$ there exist a unique canonical representative*

$$f_k=t^k+\sum_{l\in\mathrm{Gaps}(k)}f_{k;l-k}t^l\in M\,.$$

The canonical generators form a topological basis in M.

Proof. Let us prove the existence first. For all $b\in\Delta$ we can choose some element $\widetilde{f}_b\in M$ of order b. Suppose that $\widetilde{f}_k=\sum_{i=0}^{\infty}\beta_i t^{k+i}$ and $\beta_0\neq 0$. By dividing by β_0, we can assume that $\beta_0=1$. If $\beta_i=0$ whenever $i>0$ and $k+i\in\Delta$, we are done since $\widetilde{f}_k$ is canonical. Otherwise, we consider a minimal $i>0$ such that $\beta_i\neq 0$ and $k+i\in\Delta$, and we replace $\widetilde{f}_k$ with $\widetilde{f}_k-\beta_i\widetilde{f}_{a+i}$. By repeating this process, we construct a sequence of elements of M that converges to a canonical element of order k in t-adic topology. Since M is closed in t-adic topology, we are done.

Now let us prove the uniqueness. Suppose that $f_k=t^k+\sum_{y\geqslant 0,k+y\notin\Delta}\alpha_y t^{k+y}$ and $f'_k=t^k+\sum_{y\geqslant 0,k+y\notin\Delta}\alpha'_y t^{k+y}$ are both contained in M. If $\alpha_y\neq\alpha'_y$ for some y, we get that $f_k-f'_k\neq 0$ and the order of $f_k-f'_k$ is a gap in Δ, so we have a contradiction. □

Note that the representatives $\{f_a \mid a \in A\}$ generate the submodule M over $\mathcal{O}_C$. Here $A = \{a_{j,i}\}$ is the set of all (nd)-generators of Δ as above, and the f_a are defined as in Lemma 3.2. Vice versa, consider a collection of polynomials

$$\Big\{g_a = t^a + \sum_{l \in \mathrm{Gaps}(a)} g_{a;l-a} t^l \mid a \in A\Big\}. \tag{3.1}$$

It will be convenient to consider the coefficients $g_{a;l-a}$ for $l \in \mathrm{Gaps}(a)$ as parameters. For $l \neq \mathrm{Gaps}(a)$ we use the conventions

$$g_{a;0} = 1\,, \quad g_{a;l-a} = 0 \quad \text{if } l \in \Delta \setminus \{a\}\,.$$

Let N be the $\mathcal{O}_C$-submodule generated by the collection (3.1), and let $\widetilde{N}$ be the $\mathbb{C}[[t^{dn}]]$-submodule generated by the same collection. Note that $\widetilde{N}$ has a topological basis

$$\{g_a t^{\alpha nd} \mid a \in A,\, \alpha \geqslant 0\}\,, \quad \mathrm{Ord}\big(g_a t^{\alpha nd}\big) = a + \alpha nd\,. \tag{3.2}$$

Note that by the definition of (nd)-generators (Definition 2.7), the orders of the polynomials in the basis (3.2) are pairwise distinct and the set of such orders coincides with Δ.

Then $N \in J_\Delta$ if and only if $N = \widetilde{N}$ or, equivalently, $y(t)g_a \in \widetilde{N}$ for all $a \in A$. For every $a \in A$ let s_{a+dm} be a polynomial in t whose coefficients are polynomials in $g_{a';r}$ for $a' \in A$, $r \in \mathbb{Z}_{>0}$ such that

$$y(t)g_a - s_{a+dm} \in \widetilde{N} \quad \text{and} \quad s_{a+dm} = \sum_{l \in \mathrm{Gaps}(a+dm)} s_{a+dm;l-dm-a} t^l\,. \tag{3.3}$$

Such an s_{a+dm} is unique: Given two distinct choices s_{a+dm}, s'_{a+dm}, the difference $s_{a+dm} - s'_{a+dm}$ is contained in $\widetilde{N}$, but its order is a gap of Δ, which is not possible.

Concretely, the polynomials s_{a+dm} can be defined by reducing $y(t)g_a$ modulo $\widetilde{N}$ using the basis (3.2), similarly to Lemma 3.2. For a recursive construction of s_{a+dm}, see formulas (3.8) and (3.9). The condition $y(t)g_a \in \widetilde{N}$ is equivalent to $s_{a+dm} = 0$ for all $a \in A$.

We can summarize the above discussion as follows.

Proposition 3.3. *The stratum J_Δ is isomorphic to a subset of the affine space* Gen $= \mathbb{C}^{G(\Delta)}$ *defined by $E(\Delta)$ equations, where*

$$G(\Delta) = \sum_{i,j} |\mathrm{Gaps}(a_{j,i})|\,, \quad E(\Delta) = \sum_{i,j} |\mathrm{Gaps}(a_{j,i} + dm)|\,. \tag{3.4}$$

In particular, the natural coordinates on Gen *are given by coefficients $g_{a_{j,i};x}$ for $a_{i,j} + x \in$* Gaps$(a_{j,i})$ *in the generators (3.1); there is a tuple of coordinates for each combinatorial generator $a_{i,j}$ of Δ. The equations are given by*

$$s_{a_{j,i}+dm;x}(g) = 0\,, \quad a_{j,i} + dm + x \notin \Delta\,.$$

There is a tuple of relations for each combinatorial syzygy of Δ as in Lemma 2.8.

Example 3.4. We can revisit Examples 2.3 and 2.9 with new notation. We have $g_0 = 1 + at + \cdots$ and $g_2 = t^2 + bt^3 + \cdots$, so in particular $g_{0;1} = a$ and $g_{2;1} = b$. Using the computations in Example 2.3, we observe that

$$s_{6;1} = a + 1 - b\,, \quad s_{8;1} = b + 1 - a\,.$$

In total, there are $\mathrm{Gaps}(0) + \mathrm{Gaps}(2) + \mathrm{Gaps}(13) + \mathrm{Gaps}(15) = 6 + 5 + 0 + 0 = 11$ coordinates and $\mathrm{Gaps}(6) + \mathrm{Gaps}(8) + \mathrm{Gaps}(19) + \mathrm{Gaps}(21) = 3 + 2 + 0 + 0 = 5$ equations, but the equations $s_{6;1} = s_{8;1} = 0$ are already contradictory.

Example 3.5. As in Example 2.10, consider $(nd, md) = (6, 9)$ and the module

$$\Delta = \{0, 3, 6, 7, 9, 10, 12, 13, 15, 16, 17, 18, 19, 20, \dots\}$$

with 6-generators 0, 3, 7, 10, 17, 20. We have

$$g_0 = 1 + a_1 t + a_2 t^2 + \cdots\,, \quad g_3 = t^3 + b_1 t^4 + b_2 t^5, \quad g_7 = t^7 + c_1 t^8 + \cdots\,, \quad g_{10} = t^{10} + d_1 t^{11} + \cdots\,.$$

Here $a_i = g_{0;i}$, $b_i = g_{3;i}$, $c_i = g_{7;i}$ and $d_i = g_{10;i}$ in the above notation. Note that $g_{17} = t^{17}$ and $g_{20} = t^{20}$ since $\mathrm{Gaps}(17) = \mathrm{Gaps}(20) = 0$. The only interesting syzygys are of degrees 9 and 12:

$$\begin{gathered}\left(t^9 + \lambda t^{10}\right) g_0 - t^6 g_3 - (a_1 + \lambda - b_1) g_{10} = (a_2 + \lambda a_1 - b_2 - (a_1 + \lambda - b_1) d_1) t^{11} + \cdots\,,\\ \left(t^9 + \lambda t^{10}\right) g_3 - t^{12} g_0 - (b_1 + \lambda - a_1) t^6 g_7 = (b_2 + \lambda b_1 - a_2 - (b_1 + \lambda - a_1) c_1) t^{14} + \cdots\,,\end{gathered}$$

so we get degree 2 equations

$$s_{9;2} = a_2 + \lambda a_1 - b_2 - (a_1 + \lambda - b_1) d_1 = 0\,, \quad s_{12;2} = b_2 + \lambda b_1 - a_2 - (b_1 + \lambda - a_1) c_1 = 0\,.$$

Note that the linear parts of these equations are $a_2 - b_2$ and $b_2 - a_2$, respectively, which are linearly dependent. If we add the equations, these cancel out, and we get

$$\lambda(a_1 + b_1) - (a_1 + \lambda - b_1) d_1 - (b_1 + \lambda - a_1) c_1 = 0\,.$$

For $\lambda = 0$ we get a non-linear equation $(a_1 - b_1)(c_1 - d_1) = 0$. For $\lambda \neq 0$, however, we can solve the equations as follows: First, fix arbitrary $b_1 - a_1 = \theta$; then

$$a_1 + b_1 = \frac{1}{\lambda}((\theta + \lambda) d_1 - (\lambda - \theta) c_1)\,,$$

and this determines a_1 and b_1 uniquely. Given a_1 and b_1, we can determine $a_2 - b_2$ as well. There is one more equation $s_{9;5} = 0$ of degree 5, which is easy to solve. Overall, we get an affine space of dimension

$$\mathrm{Gaps}(0) + \mathrm{Gaps}(3) + \mathrm{Gaps}(7) + \mathrm{Gaps}(10) - \mathrm{Gaps}(9) - \mathrm{Gaps}(12) = 7 + 5 + 3 + 2 - 2 - 1 = 14\,,$$

as predicted by [CP18, Section 4.4].

3.2 Admissibility

In this subsection, we discuss an admissibility condition on Δ generalizing Example 2.3. We show that if Δ is not admissible, then $J_\Delta = \emptyset$. In the next section, we describe J_Δ for admissible Δ.

We will consider (md, nd)-invariant subsets and say that such a subset B is supported in a remainder r if all elements of B are congruent to $r \mod d$.

Lemma 3.6. *Suppose that B, C are (md, nd)-invariant subsets supported in the same remainder $r \mod d$ and b_i, c_i are their (nd)-generators. Then $B \subset C$ if and only if $c_i - nd \notin B$ for all i.*

Proof. There exists a permutation σ such that $c_i \equiv b_{\sigma(i)} \mod nd$ for all j. We have

$$B \subset C \iff \forall i\ c_i \leqslant b_{\sigma(i)} \iff \forall i\ c_i - nd < b_{\sigma(i)} \iff \forall i\ c_i - nd \notin B\,. \qquad \square$$

Definition 3.7. Given an (nd, md)-invariant subset Δ and a remainder j modulo d, we call $x > 0$ *j-suspicious* if $a_{j,i} + md + x \notin \Delta$ for all i, and we call it *suspicious* if it is j-suspicious for at least one remainder j.

Example 3.8. In Example 2.9, one can check that $x = 1$ is 0-suspicious since $0 + 6 + 1 = 7$ and $2 + 6 + 1 = 9$ are both not in Δ. In Example 2.10, one can check that $x = 2$ is 0-suspicious since $0 + 9 + 2 = 11$ and $3 + 9 + 2 = 14$ are both not in Δ.

Note that in both examples, suspicious numbers correspond to dependencies between the linear parts of the equations. More generally, we have the following result.

Lemma 3.9. *A number x is j-suspicious if and only if the linear parts of the equations $s_{a_{j,i}+md;x}$ are linearly dependent for $i = 0, \dots, n-1$.*

Here by linear parts we mean linear terms in $g_{a;x}$ not containing λ or higher coefficients of $y(t)$.

Proof. Recall that by Lemma 2.8 we have $a_{j,i} + md = a_{j,i+1} + \alpha_{j,i}nd$. We can write

$$s_{a_{j,i}+md}(t) = y(t)g_{a_{j,i}}(t) - (t^{nd})^{\alpha_{j,i}} g_{a_{j,i+1}} - \cdots ,$$

where the three dots denote a certain correction term with non-linear coefficients; see (3.8) and (3.9) below. The coefficient at $t^{a_{j,i}+md+x}$ in $y(t)g_{a_{j,i}}(t)$ equals $g_{a_{j,i};x} + \lambda g_{a_{j,i};x-1} + \cdots$, while the coefficient at $(t^{nd})^{\alpha_{j,i}} g_{a_{j,i+1}}$ equals $g_{a_{j;i+1};x}$.

Therefore, the linear part of $s_{a_{j,i}+md;x}$ equals $g_{a_{j,i};x} - g_{a_{j,i+1};x}$, as long as $a_{j,i} + md + x \notin \Delta$ (otherwise, there is no equation), $a_{j,i} + x \notin \Delta$ and $a_{j,i+1} + x \notin \Delta$ (otherwise, the corresponding coordinates vanish). Note that it follows from Lemma 2.11 that $a_{j,i} + md + x \notin \Delta$ implies that $a_{j,i} + x \notin \Delta$ and $a_{j,i+1} + x \notin \Delta$. Now x is j-suspicious if and only if all such equations $s_{a_{j,i}+md;x}$ are present and their linear parts $g_{a_{j,i};x} - g_{a_{j;i+1};x}$ sum up to zero. Otherwise, these equations are linearly independent. □

Lemma 3.10. *A number x is j-suspicious if and only if $\Delta_{j+x} \subset \Delta_j + md + nd + x$.*

Proof. The subset $\Delta_j + md + nd + x$ is supported in remainder $j + x$ and has (nd)-generators $a_{j,i} + md + nd + x$. The result now follows from Lemma 3.6. □

Lemma 3.11. *Suppose that $\min \Delta_{j+x} \leqslant \min \Delta_j$. Then x is not j-suspicious.*

Proof. Set $b = \min \Delta_{j+x}$. We have $b \equiv j + x \mod d$, so $b - x \equiv j \mod d$, and there exists an i such that $b - x - md \equiv a_{j,i} \mod nd$. Now

$$a_{j,i} + md + x > a_{j,i} \geqslant \min \Delta_j \geqslant b\,, \quad a_{j,i} + md + x \equiv b \mod nd\,;$$

hence $a_{j,i} + md + x \in \Delta_{j+x}$. Therefore, x is not j-suspicious. □

Corollary 3.12. *For any 0-normalized Δ and $x \in \mathbb{Z}_{>0}$, the integer x is not $(d-x)$-suspicious.*

Proof. Note that by Lemma 3.11 if $d \mid x$, then x is not suspicious. Suppose that $d \nmid x$. Since Δ is 0-normalized, we have $\min \Delta_0 = 0 < \min \Delta_{d-x}$, and Lemma 3.11 applies. □

The following is a key definition of this section.

Definition 3.13. We call Δ *admissible* if 1 is not suspicious for Δ.

Lemma 3.14. *Suppose that Δ is admissible and x is j-suspicious. Then there exists an i such that $a_{j,i} + x - 1 \notin \Delta$.*

Proof. Assume that $a_{j,i} + x - 1 \in \Delta$ for all i; then $\Delta_j + x - 1 \subset \Delta_{j+x-1}$ and $\Delta_j + md + nd + x \subset \Delta_{j+x-1} + md + nd + 1$. Since x is j-suspicious, we have the inclusions $\Delta_{j+x} \subset \Delta_j + md + nd + x \subset \Delta_{j+x-1} + md + nd + 1$. Therefore, 1 is suspicious for the remainder $j + x - 1$, and we have a contradiction. □

Lemma 3.15. *If Δ is admissible, then Δ is $(mnd + 1)$-invariant.*

Proof. Suppose that $a \in \Delta$ has remainder $j \mod d$. Since Δ is admissible, there exists an integer i such that $a_{j,i}+md+1 \in \Delta$. There exists some $0 \leqslant s \leqslant n-1$ such that $a+mds \equiv a_{j,i} \mod nd$. Since Δ is (md)-invariant, we get $a+mds = a_{j,i}+\alpha nd$ for some $\alpha \geqslant 0$. Now

$$\begin{aligned} a+mdn+1 &= (a+md(s+1)+1)+md(n-1-s) \\ &= (a_{j,i}+md+1)+\alpha nd+md(n-1-s) \in \Delta\,. \end{aligned}$$

□

Lemma 3.16. *Suppose that $J_\Delta \neq \emptyset$. Then Δ is admissible.*

Proof. We follow the strategy of Lemma 3.9. Suppose that Δ is not admissible. Then 1 is suspicious for Δ, and there exists some j such that $a_{j,i}+md+1 \notin \Delta$ for all i. This implies $a_{j,i}+1 \notin \Delta$ for all i, so we have canonical generators $g_{a_{j,i}} = t^{a_{j,i}}+g_{a_{j,i};1}t^{a_{j,i}+1}+\cdots$. Recall that by Lemma 2.8, we get $a_{j,i}+md = a_{j,i+1}+\alpha_{j,i}nd$ and

$$\begin{aligned} t^{\alpha_{j,i}nd}g_{a_{j,i+1}} - y(t)g_{a_{j,i}} &= t^{\alpha_{j,i}nd}g_{a_{j,i+1}} - \left(t^{md}+\lambda t^{md+1}+\cdots\right)g_{a_{j,i}} \\ &= (g_{a_{j,i+1};1}-g_{a_{j,i};1}-\lambda)t^{a_{j,i}+md+1}+\cdots\,; \end{aligned}$$

hence $g_{a_{j,i+1};1}-g_{a_{j,i};1}-\lambda = 0$ for all i. By adding all these together, we get $n\lambda = 0$, so we have a contradiction. □

Remark 3.17. The very last part of the argument for Lemma 3.16 relies on the fact that we work in characteristic 0. More generally, the argument is valid if we assume that the characteristic of the base field is coprime with n.

See Example 2.3 for an illustration of this lemma.

Example 3.18. If $d=1$, then all (n,m)-invariant modules are admissible. Indeed, if Δ is (n,m)-invariant with n-generators $a_1 < \cdots < a_n$, then $[a_n,+\infty) \subset \Delta$. In particular, for any $x>0$ one has $a_n+m+x \in \Delta$, so x is not suspicious.

Example 3.19. Suppose that $n=1$ and Δ is (d,md)-invariant (that is, d-invariant). Suppose that the $a_{j,0}$ are d-generators of Δ. Then 1 is j-suspicious if and only if $a_{j,0}+md+1 \notin \Delta$, so Δ is admissible if and only if $a_{j,0}+md+1 \in \Delta$ for all j, which is equivalent to Δ being $(md+1)$-invariant.

3.3 Paving by affine spaces

Recall that Gen has coordinates $g_{a_{j,i};x}$, where $a_{j,i}+x \notin \Delta$. Sometimes we will use the notation $g_{a_{j,i};x}$ for all x, assuming

$$g_{a_{j,i};x} = 0 \quad \text{if } a_{j,i}+x \in \Delta\,. \tag{3.5}$$

It is also convenient to consider Gen as a graded vector space

$$\mathrm{Gen} = \bigoplus_{x=1}^{\infty} \mathrm{Gen}_x\,,$$

where Gen_x is spanned by the $g_{a_{j,i};x}$ with a fixed x.

There are coordinates on Gen that are most suitable to study equations s_k in the case where Δ is admissible. Let us define

$$g^-_{j,i;x} = g_{a_{j,i};x} - g_{a_{j,i+1};x}\,, \quad i=0,\ldots,n-1\,.$$

These functions are linearly dependent, for example $\sum_i g^-_{j,i;x} = 0$. We choose a subset of these as follows:

(1) If x is j-suspicious, we define $I(j;x)=\{0,\dots,n-2\}$.

(2) If x is not j-suspicious, we can define $I(j;x)$ to be a set of i such that $a_{j,i}+dm+x\notin\Delta$.

Lemma 3.20. *The functions $g^-_{j,i;x}$, $i\in I(j;x)$ are linearly independent.*

Proof. Since Δ is (dm)-invariant, the condition $a_{j,i}+dm+x\notin\Delta$ implies $a_{j,i}+x\notin\Delta$. We consider two cases:

(1) If x is j-suspicious, then by definition, $a_{j,i}+md+x\notin\Delta$ and hence $a_{j,i}+x\notin\Delta$ for all i. In particular, (3.5) does not apply, and all $g_{a_{j,i};x}$ are linearly independent coordinates on Gen. Therefore, the $g^-_{j,i;x}$ for $i\in\{0,\dots,n-2\}$ are linearly independent as well.

(2) Suppose that x is not j-suspicious. Then for some s we have $a_{j,s}+dm+x\in\Delta$ and $s\notin I(j;x)$; therefore, $I(j;x)$ is a proper subset of $\{0,\dots,n-1\}$. Furthermore, for $i\in I(j;x)$ condition (3.5) does not apply, so the $g_{a_{j,i};x}$ for $i\in I(j;x)$ form a linearly independent subset of coordinates on Gen.

Assume that there is a non-trivial linear relation between the $g^-_{j,i';x}=g_{a_{j,i'};x}-g_{a_{j,i'+1};x}$ that contains $g^-_{j,i;x}$ for some $i\in I(j;x)$. Since $g_{a_{j,i};x}$ is a coordinate on Gen that cancels out, the relation must contain $g^-_{j,i-1;x}$ as well, so $i-1\in I(j;x)$. By induction, we conclude that $i\in I(j;x)$ for all i, and we have a contradiction. Therefore, the $g^-_{j,i;x}$ for $i\in I(j;x)$ are linearly independent. □

Finally, if there exists at least one integer i such that $a_{j,i}+x\notin\Delta$, then we set

$$g^+_{j;x}=\sum_{i:a_{j,i}+x\notin\Delta} g_{a_{j,i};x}\,.$$

In particular, if $x+1$ is suspicious, then Lemma 3.14 guarantees that the sum is not empty.

Thus if $I(j;x)\neq\emptyset$, then the $\{g^-_{j,i;x},g^+_{j;x}\}$ for $i\in I(j;x)$ are linearly independent linear coordinates on the space of generators Gen_x. For each j and x such that $I(j;x)\neq\emptyset$, let us fix a subset $\bar I(j;x)$ such that $\{g^-_{j,i;x},g^+_{j;x},g_{a_{j,i'};x}\}$ for $i\in I(j;x)$ and $i'\in\bar I(j;x)$ is a basis of linear coordinates on Gen_x. For $I(j;x)=\emptyset$ set $\bar I(j;x)=\{0,1,\dots,n-1\}$.

Remark 3.21. To show that $\{g^-_{j,i;x},g^+_{j;x},g_{a_{j,i'};x}\}$ for $i\in I(j;x)$ and $i'\in\bar I(j;x)$ is a basis of linear coordinates, we used that we work over $\mathbb{C}$. The argument also works for a base field of characteristic larger than n. The rest of the arguments in this section are also valid under this assumption.

We will need to briefly use the higher coefficients in the expansion of $y(t)$, so let

$$y(t)=t^{md}+\lambda t^{md+1}+\lambda_2t^{md+2}+\lambda_3t^{md+3}+\cdots\,.$$

It will be convenient to think of the coefficients $\lambda,\lambda_2,\dots$ as parameters.

As we will see later, the coordinates $g_{a_{j,i'};x}$ for $i'\in\bar I(j;x)$ are not constrained by the equations for J_Δ. Thus let us set notation for the polynomial ring of these free variables and of λ:

$$R_{\mathrm{free}}=\mathbb{C}[\lambda,\lambda^{-1},\lambda_2,\lambda_3,\dots][g_{a_{j,i'};x}]_{i'\in\bar I(j;x)}.$$

On the rest of the coordinates $g^*_{*;*}$, we introduce a partial order generated as follows. Allowing $*$ to take any independent values,

$$g^-_{*,*;x}<g^+_{*;x}<g^-_{*,*,x+1}\,,$$

the coordinates $g^-_{*,*;x}$ are ordered in any arbitrary way, and

$$g^+_{j+1;x}<g^+_{j;x}$$

(cyclic notation modulo d) for $j \neq d-x-1$.

The partial order is compatible with the grading on the algebra $\mathbb{C}[\mathrm{Gen} \times \mathbb{C}_\lambda]$ defined by

$$\deg_+(g_{a;x}) = x\,, \quad \deg_+(\lambda) = 1\,, \quad \deg_+(\lambda_i) = i\,.$$

Let us analyze graded properties of the equations for J_Δ. We fix $\deg_g$ for the usual degree

$$\deg_g(g_{a;x}) = 1\,, \quad \deg_g(\lambda) = \deg_g(\lambda_i) = 0\,.$$

LEMMA 3.22. *For x such that $a_{ji} + dm + x \notin \Delta$, the polynomial $s_{a_{j,i}+dm;x}$ is homogeneous with respect to the grading $\deg_+$, and*

$$\deg_+(s_{a_{j,i}+dm;x}) = x\,.$$

Furthermore, for any j, i, x

$$s_{a_{j,i}+dm;x} = g^-_{j,i;x} + \text{polynomial in variables} < g^-_{j,i;x} \text{ with coefficients in } R_{\mathrm{free}}\,, \tag{3.6}$$

and if x is j-suspicious, then

$$\sum_{i=0}^{n-1} s_{a_{j,i}+dm;x} = \lambda g^+_{j;x-1} + \text{polynomial in variables} < g^+_{j;x-1} \text{ with coefficients in } R_{\mathrm{free}}\,. \tag{3.7}$$

Proof. The expansion of $s_{a_{j,i}+dm}(t)$ is computed by an iterated process. We define a sequence of power series

$$s^{(k)}_{a_{j,i}+dm} = \sum_{\ell=0}^{\infty} s^{(k)}_{a_{j,i}+dm;l} t^{a_{j,i}+dm+l}$$

such that $s_{a_{j,i}+dm} = s^{(\infty)}_{a_{j,i}+dm}$. Here the seed of the process is defined by

$$s^{(0)}_{a_{j,i}+dm} = g_{a_{j,i}} y(t) - t^{dn\alpha_{j,i}} g_{a_{j,i+1}}\,, \tag{3.8}$$

where the $\alpha_{j,i}$ are defined by (2.2). The step of the process is defined as follows. We set

$$s^{(k+1)}_{a_{j,i}+dm} = s^{(k)}_{a_{j,i}+dm} - \sum_{x>0: a_{j,i}+dm+x\in\Delta} s^{(k)}_{a_{j,i}+dm;x} g_{a_{j+x,i'}} t^{ndu}\,, \tag{3.9}$$

where for every summand, i' and u are chosen so that $a_{j,i} + dm + x = a_{j+x,i'} + ndu$. Note that at each step, the term with the smallest x such that $s^{(k)}_{a_{j,i}+dm;x} \neq 0$ and $a_{j,i} + dm + x \in \Delta$ cancels out. Therefore, the process converges. Since $s^{(0)}_{a_{j,i}+dm;x}$ is homogeneous of degree x with respect to $\deg_+$ and the iteration process preserves homogeneity, the first statement follows.

One immediately concludes by induction that for every $k \in \mathbb{Z}_{\geqslant 0}$ and every $x \in \mathbb{Z}_{\geqslant 0}$ such that $a_{j,i} + dm + x \in \Delta$, one has $\deg_g(s^{(k)}_{a_{j,i}+dm;x}) > k$. Furthermore, any monomial with $\deg_+ = x$ and $\deg_g > 2$ can only depend on $g_{a_{*,*};z}$ with $z < x-1$. Therefore, for $k > 0$ and any x, one has

$$s^{(k)}_{a_{j,i}+dm;x} = s^{(k+1)}_{a_{j,i}+dm;x} + \cdots\,, \tag{3.10}$$

where the three dots represent a series depending only on $g_{a_{*,*};z}$ with $z < x-1$. In particular,

$$s_{a_{j,i}+dm;x} = s^{(1)}_{a_{j,i}+dm;x} + \cdots\,. \tag{3.11}$$

Here and below we use the same convention about the three dots as in (3.10).

Furthermore, from (3.8) we get

$$s^{(0)}_{a_{j,i}+dm;x} = g_{a_{j,i};x} - g_{a_{j,i+1};x} + \lambda g_{a_{j,i};x-1} + \cdots = g^-_{j,i,x} + \lambda g_{a_{j,i};x-1} + \cdots\,.$$

Using (3.9), we get

$$
\begin{aligned}
s^{(1)}_{a_{j,i}+dm;x} &= s^{(0)}_{a_{j,i}+dm;x} - \sum_{0<y<x:a_{j,i}+dm+y\in\Delta} s^{(0)}_{a_{j,i}+dm;y} g_{a_{j+y,i'};x-y} \\
&= g^{-}_{j,i,x} + \lambda g_{a_{j,i};x-1} + \sum_{0<y<x:a_{j,i}+dm+y\in\Delta} (g^{-}_{j,i;y} + \dots) g_{a_{j+y,i'};x-y} + \cdots . \qquad (3.12)
\end{aligned}
$$

Combining (3.11) and (3.12), we immediately get equation (3.6). To get equation (3.7), we take the sum

$$
\begin{aligned}
\sum_{i=0}^{n-1} s_{a_{j,i}+dm;x} &= \sum_{i=0}^{n-1} s^{(1)}_{a_{j,i}+dm;x} + \cdots \\
&= \lambda g^{+}_{j;x-1} + \sum_{i=0}^{n-1} \sum_{0<y<x:a_{j,i}+dm+y\in\Delta} (g^{-}_{j,i;y} + \cdots) g_{a_{j+y,i'};x-y} + \cdots .
\end{aligned}
$$

Note that only terms with $y = 1$ and $y = x-1$ contribute; other terms depend on $g_{a_{j,i};z}$ with $z < x-1$ only. Furthermore, for $y = x-1$ we have $g^{-}_{j,*;x-1} < g^{+}_{j,x-1}$. For $y = 1$ the variable $g_{a_{j+1,i'};x-1}$ is a linear combination of $g^{-}_{j+1,*;x-1}$ and $g^{+}_{j+1;x-1}$ (modulo variables from R_{free}). Since x is j-suspicious, according to Corollary 3.12 we have $j \neq d - x = d - (x-1) - 1$, and therefore $g^{+}_{j+1;x-1} < g^{+}_{j;x-1}$. Hence we obtain equation (3.7). □

PROPOSITION 3.23. *If Δ is admissible, then*

$$
J_\Delta = \mathbb{C}^{\dim(\Delta)}, \quad \dim(\Delta) = G(\Delta) - E(\Delta),
$$

where $G(\Delta)$ and $E(\Delta)$ are given by the equations (3.4).

Proof. Recall that J_Δ is defined by the equations $s_{a_{j,i}+dm;x} = 0$ for j, i, x such that $a_{j,i}+dm+x$ is not in Δ. We can modify this system of equations as follows. Whenever x is j-suspicious, replace $s_{a_{j,n-1}+dm;x} = 0$ by $\sum_{i=0}^{n-1} s_{a_{j,i}+dm;x} = 0$. Clearly, the new system of equations is equivalent to the old one. Furthermore, according to Lemma 3.22, the new system of equations expresses some of the elements of a basis of the space Gen in terms of the smaller variables with respect to the order $<$. Therefore, one can use the equations to eliminate these variables one by one. Since $\dim \mathrm{Gen} = G(\Delta)$ and there are $E(\Delta)$ equations, we obtain the required result. □

4. Combinatorics

4.1 More on invariant subsets

Let Δ be an (nd, md)-invariant subset. We call b an *(md)-cogenerator* for Δ if $b \notin \Delta$ but $b + md \in \Delta$.

LEMMA 4.1. *Let Δ be an admissible cofinite (nd, md)-invariant subset. The dimension of J_Δ equals the number of pairs (a, b) such that a is an (nd)-generator of Δ, b is an (md)-cogenerator and $a < b$.*

Proof. By Proposition 3.23, the dimension of J_Δ equals $\sum_{a\in A}(|\mathrm{Gaps}(a)| - |\mathrm{Gaps}(a+md)|)$, where the sum is over (nd)-generators. The difference $|\mathrm{Gaps}(a)| - |\mathrm{Gaps}(a+md)|$ equals the number of (md)-cogenerators greater than a. □

4.2 Relatively prime case

Let Θ be an (n, m)-invariant subset in $\mathbb{Z}$. As before, n, m are relatively prime.

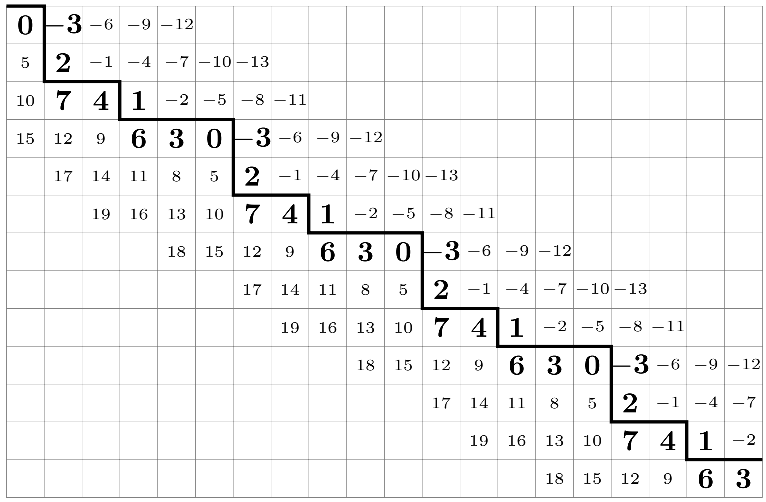

Figure 1. Periodic lattice path corresponding to the $(3,5)$-invariant subset $\Theta = \{0, 3, 4, 5, \dots\}$. The 5-generators of Θ are $\{0, 3, 4, 6, 7\}$, and the 3-cogenerators are $\{-3, 1, 2\}$ (both in bold).

Definition 4.2. The *skeleton* S of Θ is the union of the n-generators and m-cogenerators of Θ.

An important way of visualizing (n,m)-invariant subsets is by using the *periodic lattice paths* as follows. Consider a 2-dimensional square lattice with the boxes labeled by integers via the linear function $l(x,y) = -nx - my$ (here (x,y) are the coordinates of the, say, northeast corner of the square box). Since n and m are relatively prime, every integer appears exactly once in each horizontal strip of width n (and in each vertical strip of width m). Furthermore, the labeling is periodic with period $(m,-n)$. Given an (n,m)-invariant subset $\Theta \subset \mathbb{Z}$, the corresponding periodic lattice path $P(\Theta)$ is the boundary that separates the set of boxes labeled by integers from Θ from the set of boxes labeled by integers from the complement $\mathbb{Z} \setminus \Theta$. Note that the n-generators of Θ are exactly the labels in the boxes just south of the path, and the m-cogenerators of Θ are exactly the labels in the boxes just to the east of the path. In particular, a number k belongs to the skeleton of an (n,m)-invariant subset if and only if the northwest corners of the boxes labeled by k lie on the corresponding periodic path. We immediately get the following.

Lemma 4.3. *Skeletons of two (n,m)-invariant subsets intersect if and only if the corresponding periodic lattice paths intersect.*

See Figure 1 for an example of a periodic path corresponding to a $(3,5)$-invariant subset.

4.3 Equivalence classes of (dn, dm)-invariant subsets

Let us recall the definitions of the equivalence classes of invariant subsets from [GMV17]. Let $\Theta = \{\Theta_0^0, \dots, \Theta_l^0\}$ be a collection of 0-normalized (n,m)-invariant subsets. For every $(x_1, \dots, x_l)$ in $\mathbb{R}^l_{\geqslant 0}$ consider

$$\Delta = \Delta(x_1, \dots, x_l) := \bigcup_{k=0}^{l} \left(d\Theta_k^0 + x_k\right),$$

where $x_0 = 0$. If all shift parameters $x_0, \dots, x_l$ are integers with different remainders modulo d, then Δ is a 0-normalized (nd, md)-invariant subset. Furthermore, if in addition $l = d-1$, then Δ is cofinite.

Remark 4.4. In [GMV17], only cofinite invariant subsets were considered (that is, $l = d-1$). However, many results generalize to non-cofinite invariant subsets without any change.

For each Θ_k^0 let S_k^0 be its skeleton. Consider the space $\mathbb{R}^l_{\geqslant 0}$ of all possible shifts $x_1, \dots, x_l$. Consider the subset $\Sigma_\Theta \subset \mathbb{R}^l_{\geqslant 0}$ consisting of all shifts $x_1, \dots, x_l$ for which there exist i and j such that

$$dS_i^0 + x_i \cap dS_j^0 + x_j \neq \emptyset\,.$$

Clearly, Σ_Θ is a hyperplane arrangement. We say that two (nd, md)-invariant subsets are equivalent if they can be obtained as $\Delta(x_1, \dots, x_l)$ and $\Delta(y_1, \dots, y_l)$ for the same collection Θ and the shifts $(x_1, \dots, x_l)$ and $(y_1, \dots, y_l)$ belong to the same connected component of the complement to Σ_Θ. One can show (in fact, it will follow from the construction below) that every connected component of the complement to Σ_Θ contains at least one point corresponding to a (dn, dm)-invariant subset. We call the connected components of the complement to a hyperplane arrangement the *regions* of the arrangement. See Figure 2 for an example of a hyperplane arrangement Σ_Θ.

To summarize, in order to get all the equivalence classes of (dn, dm)-invariant subsets that have non-trivial intersections with exactly $(l+1)$ congruence classes modulo d, one should consider all possible $(l+1)$-tuples of 0-normalized (n, m)-invariant subsets $\Theta_0^0, \dots, \Theta_l^0$, and for each such $(l+1)$-tuple consider the set of regions of Σ_Θ in the space of shifts. Furthermore, one should consider these regions up to symmetry: If two of the subsets are equal, $\Theta_i^0 = \Theta_j^0$, then switching the corresponding shift coordinates x_i and x_j interchanges the connected components corresponding to the same equivalence class of (dn, dm)-invariant subsets.

It can be observed (see [GMV17, Lemma 3.16]) that for any region R of Σ_Θ, its closure $\overline{R}$ contains the minimal point $(m_1, \dots, m_l) \in \overline{R}$ such that $m_i = \min\{x_i \mid (x_1, \dots, x_l) \in \overline{R}\}$ for all i. Furthermore, it is easy to see that $m_1, \dots, m_l \in d\mathbb{Z}_{\geqslant 0}$. For every $k \in \{0, \dots, l\}$ set $\Theta_k := \Theta_k^0 + m_k/d$ and $S_k := S_k^0 + m_k/d$ so that S_k is the skeleton of Θ_k. Here $m_0 = 0$.

LEMMA 4.5. *One has $S_i \cap S_j = \emptyset$ if and only if either $\Theta_i \subset \Theta_j + n + m$ or $\Theta_j \subset \Theta_i + n + m$.*

Proof. According to Lemma 4.3, the equality $S_i \cap S_j = \emptyset$ is equivalent to the fact that the corresponding periodic lattice paths do not intersect, which implies that one of them sits weakly below the other one shifted one south and one to the west $(+n+m)$, which is equivalent to $\Theta_i \subset \Theta_j + n + m$ or $\Theta_j \subset \Theta_i + n + m$. □

Note that the hyperplanes of Σ_Θ that pass through a point $(m_1, \dots, m_l)$ are of the form $x_i - x_j = m_i - m_j$ with one such hyperplane for each pair $i > j$ such that the corresponding skeletons intersect: $S_i \cap S_j \neq \emptyset$ (here j might be zero, in which case $m_0 = x_0 = 0$). The point $(m_1, \dots, m_l)$ might be the minimal point of closures of multiple regions of Σ_Θ. In order to pick one such region R, for each hyperplane $x_i - x_j = m_i - m_j$ of Σ_Θ one should pick on which side of that hyperplane the region lies. This information can be encoded into a digraph G with vertices $\Theta_0, \dots, \Theta_l$ and edges connecting subsets Θ_i and Θ_j with intersecting skeletons, going in the direction of Θ_i whenever $R \subset \{x_i - m_i > x_j - m_j\}$. Note that G is acyclic and defines a partial order on Θ. The minimality of the point $(m_1, \dots, m_l)$ also implies that Θ_0 is the unique source.

DEFINITION 4.6. We call the graph G described above the *intersection graph* of the skeletons.

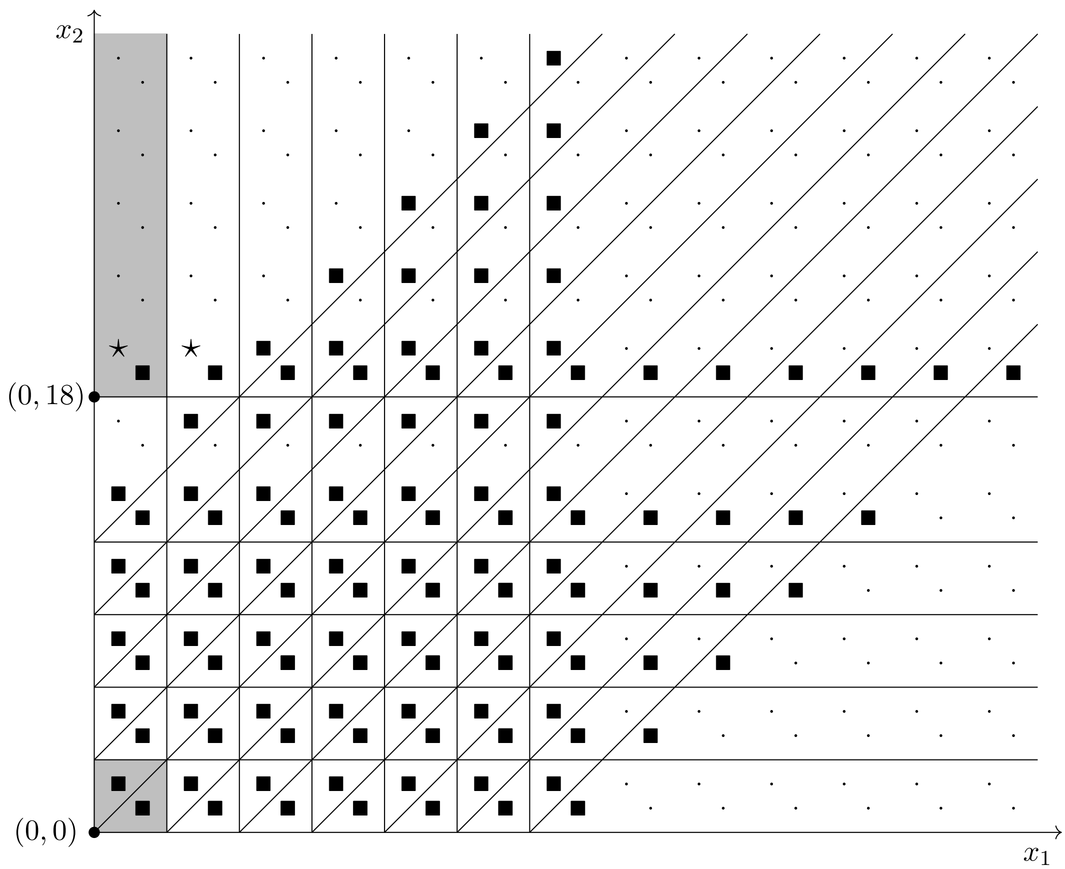

FIGURE 2. The hyperplane arrangement Σ_Θ for the collection of $(3, 2)$-invariant subsets from Examples 4.7, 4.8 and 4.11. The lattice of dots are the integer shifts with different remainders modulo 3. They correspond to the $(9, 6)$-invariant subsets. The squares correspond to the admissible subsets, and the stars correspond to the minimal representatives that are not admissible. The three shaded regions are the regions considered in examples, and the bold black dots are the minimal shifts on the boundaries of those regions.

Let $\sigma = (\sigma_0, \dots, \sigma_l) \in S_{l+1}$ be a permutation such that $\Theta_{\sigma_0}, \dots, \Theta_{\sigma_l}$ are weakly increasing in the partial order defined by the acyclic orientation on G (in particular, $\sigma_0 = 0$). Then

$$\Delta(m_0 + \sigma_0, m_1 + \sigma_1, \dots, m_l + \sigma_l) = \bigcup d\Theta_k + \sigma_k$$

is a (dn, dm)-invariant subset such that the corresponding shift belongs to the region R. In particular, it follows that R corresponds to a non-empty equivalence class. The subsets constructed as above are called the *minimal representatives* of the class corresponding to R. Note that there might be more than one minimal representative: They are parametrized by the linearizations of the partial order on Θ given by the acyclic orientation on G.

It also follows that if $(l_1, \dots, l_l) \in d\mathbb{Z}^l_{\geqslant 0} \subset \mathbb{R}^l_{\geqslant 0}$ is a shift such that the corresponding intersection graph of skeletons is connected, then this shift is the minimal point of at least one of the regions. The regions with that minimal point are then parametrized by the acyclic orientations of the graph such that Θ_0 is the unique source.

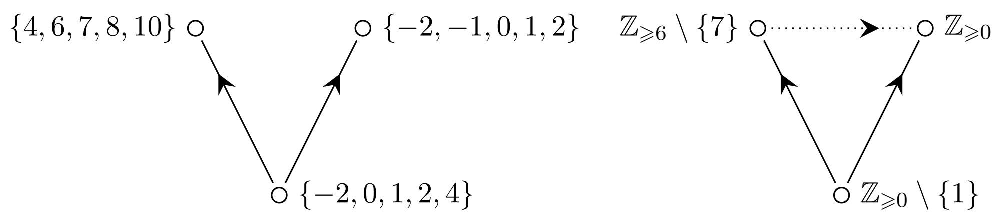

Figure 3. On the left: the digraph G corresponding to the minimal point on the boundary of a region with vertices labeled by the skeletons. On the right: the corresponding bicolored digraph with vertices labeled by the $(2,3)$-invariant subsets.

Example 4.7. Let $(n,m)=(3,2)$ and $d=3$. Also let

$$\Theta_0^0=\Theta_2^0=\{0,2,3,\dots\}=\mathbb{Z}_{\geqslant 0}\setminus\{1\}\,,\quad \Theta_1^0=\{0,1,2,\dots\}=\mathbb{Z}_{\geqslant 0}$$

so that

$$S_0^0=S_2^0=\{-2,0,1,2,4\}\,,\quad S_1^0=\{-2,-1,0,1,2\}\,.$$

In Figure 2, we present the corresponding hyperplane arrangement Σ_Θ.

For the zero shift $(m_1,m_2)=(0,0)$, all three skeletons pairwise intersect. Therefore, the intersection graph is complete and, in particular, connected. There are two ways to orient the graph so that S_0^0 is the unique source: The edges connecting S_0^0 to S_1^0 and S_2^0 have to be oriented pointing away from S_0^0, but the edge connecting S_1^0 and S_2^0 can be oriented either way. Therefore, this is the minimal point on the boundary of two regions. Both orientations determine linear orders; therefore, the corresponding regions contain unique minimal representatives:

$$\begin{aligned}
&3\Theta_0^0\sqcup\left(3\Theta_1^0+2\right)\sqcup\left(3\Theta_2^0+1\right)=\{0,1,2,5,6,\dots\}=\mathbb{Z}_{\geqslant 0}\setminus\{3,4\}\quad\text{for } S_2^0\to S_1^0\,,\\
&3\Theta_0^0\sqcup\left(3\Theta_1^0+1\right)\sqcup\left(3\Theta_2^0+2\right)=\{0,1,2,4,6,\dots\}=\mathbb{Z}_{\geqslant 0}\setminus\{3,5\}\quad\text{for } S_1^0\to S_2^0\,.
\end{aligned}$$

(See the two shaded triangular regions adjacent to the origin in Figure 2.)

Example 4.8. Now consider the shift $(m_1,m_2)=(0,3\times 6)=(0,18)$. One gets

$$S_0=S_0^0=\{-2,0,1,2,4\}\,,\quad S_1=S_1^0=\{-2,-1,0,1,2\}\,,\quad S_2=S_2^0+6=\{4,6,7,8,10\}\,.$$

Note that $S_1\cap S_2=\emptyset$ while both S_1 and S_2 have non-trivial intersections with S_0. The corresponding graph has a unique orientation such that S_0 is the unique source. (See Figure 3 on the left for the graph and the shaded region adjacent to the point $(0,18)$ in Figure 2 for the corresponding region.)

The permutation σ in this case can be either $(0,1,2)$ or $(0,2,1)$, giving rise to two minimal representatives:

$$\begin{aligned}
&3\Theta_0^0\sqcup\left(3\Theta_1^0+1\right)\sqcup\left(3\Theta_2^0+18+2\right)=\mathbb{Z}_{\geqslant 0}\setminus\{2,3,5,8,11,14,17,23\}\quad\text{for } \sigma=(0,1,2)\,,\\
&3\Theta_0^0\sqcup\left(3\Theta_1^0+2\right)\sqcup\left(3\Theta_2^0+18+1\right)=\mathbb{Z}_{\geqslant 0}\setminus\{1,3,4,7,10,13,16,22\}\quad\text{for } \sigma=(0,2,1)\,.
\end{aligned}$$

(See the stars and the squares in the shaded rectangular region in Figure 2.)

4.4 Admissible representatives

Let Δ be a 0-normalized (dn, dm)-invariant subset such that

$$\begin{aligned}\Delta \cap (d\mathbb{Z} + r) &\neq \emptyset\,, \quad r \in \{0, 1, \ldots, l\}\,,\\ \Delta \cap (d\mathbb{Z} + r) &= \emptyset\,, \quad r \in \{l+1, l+1, \ldots, d-1\}\,.\end{aligned}$$

In the spirit of the above notation, let

$$\Delta = \bigcup_k d\Theta_k^0 + x_k + k\,,$$

where $(x_1, \ldots, x_l) \in d\mathbb{Z}^l_{\geqslant 0} \subset \mathbb{R}^l_{\geqslant 0}$, $x_0 = 0$ and $\Theta_0^0, \ldots, \Theta_l^0$ are 0-normalized (n, m)-invariant subsets. Set $\Theta_k := \Theta_k^0 + x_k/d$ for all $k \in \{0, \ldots, l\}$.

One can generalize the notion of an admissible subset (see Definition 3.13 and Lemma 3.10) to the case of non-cofinite (nd, md)-invariant subsets of the type described above in a straightforward way. We get that Δ is admissible if for every $k \in \{0, \ldots, l-1\}$ one has

$$\begin{aligned}&\Delta \cap (d\mathbb{Z} + k + 1) \not\subseteq \Delta \cap (d\mathbb{Z} + k) + dn + dm + 1\,,\\ \iff\ & d\Theta_{k+1} + k + 1 \not\subseteq d\Theta_k + k + dn + dm + 1\,,\\ \iff\ & \Theta_{k+1} \not\subseteq \Theta_k + n + m\,.\end{aligned}$$

In view of Lemma 4.5, this is also equivalent to saying that Δ is admissible if and only if for every $k \in \{0, \ldots, l-1\}$ either $S_k \cap S_{k+1} \neq \emptyset$ or $\Theta_k \subset \Theta_{k+1} + n + m$, where $S_0, \ldots, S_l$ are the skeletons of $\Theta_0, \ldots, \Theta_l$.

Lemma 4.9. *If Δ is admissible, then it is a minimal representative of an equivalence class.*

Proof. According to the constructions above, all we need to check is that $(x_1, \ldots, x_l)$ is the minimal point on the boundary of a region, which is equivalent to showing that the intersection graph of the skeletons of $\Theta_0, \ldots, \Theta_l$ is connected.

Indeed, suppose that it is not connected. Let k be the minimal number such that Θ_k is not in the same connected component as Θ_0. Note that $\Theta_0 \not\subseteq \Theta_k + n + m$ because $0 \in \Theta_0$ and $\min(\Theta_k + n + m) \geqslant n + m > 0$. Furthermore, $S_0 \cap S_k = \emptyset$ because Θ_k is not connected to Θ_0 in the intersection graph. Therefore, according to Lemma 4.5, we have $\Theta_k \subset \Theta_0 + n + m$.

Now suppose that Θ_i and Θ_j are connected by an edge in the intersection graph, while Θ_k is not connected to either of them. That is equivalent to saying that the periodic lattice paths corresponding to Θ_i and Θ_j intersect, while the periodic path corresponding to Θ_k does not intersect either of them. It then follows that either the periodic path corresponding to Θ_k is strictly below both periodic paths corresponding to Θ_i and Θ_j, or it is strictly above both of them. In other words, if $\Theta_k \subset \Theta_i + n + m$, then also $\Theta_k \subset \Theta_j + n + m$.

Since Θ_0 and Θ_{k-1} are in the same connected component of the intersection graph and $\Theta_k \subset \Theta_0 + n + m$, applying the argument from the previous paragraph several times, we obtain that $\Theta_k \subset \Theta_{k-1} + n + m$, which means that Δ is not admissible. We have a contradiction. □

Theorem 4.10. *Every equivalence class contains a unique admissible representative.*

Proof. According to Lemma 4.9, it is enough to show that for every equivalence class, exactly one of the minimal representatives is admissible. We now follow the notation from the previous section. We get the intersection digraph of the skeletons of $\Theta_0, \ldots, \Theta_{d-1}$, which is acyclic and contains the unique source Θ_0. Consider a minimal representative

$$\Delta(m_0 + \sigma_0, m_1 + \sigma_1, \ldots, m_l + \sigma_l) = \bigcup d\Theta_k + \sigma_k\,.$$

(Here $\sigma \in S_{l+1}$ is such that $\Theta_{\sigma_0}, \dots, \Theta_{\sigma_l}$ is weakly increasing in the order defined by the orientation.) This minimal representative is admissible if and only if whenever $S_{\sigma_k} \cap S_{\sigma_{k+1}} = \emptyset$, one has $\Theta_{\sigma_k} \subset \Theta_{\sigma_{k+1}} + n + m$.

Let us enhance the digraph G as follows.[1] Let us mark the edges of the original digraph solid, and whenever two vertices Θ_i and Θ_j are not connected by a solid edge, connect them by a dotted edge oriented in such a way that if $\Theta_i \subset \Theta_j + n + m$, then $\Theta_i \to \Theta_j$. We obtain a complete graph with edges marked in two ways: solid and dotted, and oriented in such a way that

(1) both solid and dotted subgraphs are acyclic;
(2) the dotted subgraph is transitive; that is, if one has dotted edges $\Theta_i \to \Theta_j$ and $\Theta_j \to \Theta_k$, then one also has a dotted edge $\Theta_i \to \Theta_k$.

The proof of the theorem is reduced to the following claim: Such a graph contains a unique oriented path passing through every vertex exactly once (possibly using both types of edges) and such that it is weakly monotone with respect to the partial order defined by the solid edges.

Existence. We construct the path step by step. Suppose that $\Theta_{\sigma_0} \to \Theta_{\sigma_1} \to \cdots \to \Theta_{\sigma_k}$ is an initial segment of the path already constructed, and suppose that it satisfies the required conditions; that is, all the consecutive edges are correctly oriented, the path is weakly monotone with respect to the solid partial order, and there are no solid edges going from one of the unused vertices to one of the used vertices (otherwise, it would not be possible to extend the path to that vertex because of the monotonicity condition). Consider all sources of the restriction of the solid graph to the unused vertices. Note that it is a non-empty set unless $k = l$, in which case we are done. These vertices can only be connected by dotted edges. In particular, they are totally ordered with respect to the dotted orientation. Let σ_{k+1} be such that $\Theta_{\sigma_{k+1}}$ is the smallest of these vertices with respect to the dotted order. The solid monotonicity condition for this extension of the path is satisfied by construction. All we need to check is that there cannot be a dotted arrow $\Theta_{\sigma_{k+1}} \to \Theta_{\sigma_k}$. Indeed, if that would be the case, then $\Theta_{\sigma_{k+1}}$ would have been added before Θ_{σ_k} according to our algorithm. We repeat this step until we obtain a path going through every vertex.

Uniqueness. Suppose that we have a path $\Theta_{\omega_0} \to \Theta_{\omega_1} \to \cdots \to \Theta_{\omega_l}$ satisfying the required conditions and different from the one constructed using the algorithm above. That means that there exist two indices $1 \leqslant i < j \leqslant l$ such that Θ_{ω_j} should have been added instead of Θ_{ω_i} according to the algorithm. In particular, it implies that Θ_{ω_j} cannot be connected to any of the vertices $\Theta_{\omega_i}, \Theta_{\omega_{i+1}}, \dots, \Theta_{\omega_{j-1}}$ by solid edges in either direction, and we have a dotted edge $\Theta_{\sigma_j} \to \Theta_{\sigma_i}$. We claim that all edges connecting Θ_{σ_j} with the vertices $\Theta_{\omega_i}, \Theta_{\omega_{i+1}}, \dots, \Theta_{\omega_{j-1}}$ are oriented *from* Θ_{σ_j}. Indeed, otherwise, let Θ_{σ_k} for $i < k < j$ be the first vertex such that we have a dotted edge $\Theta_{\sigma_k} \to \Theta_{\sigma_j}$. Then we also have a dotted edge $\Theta_{\sigma_j} \to \Theta_{\sigma_{k-1}}$, and by the transitivity of the dotted edges, we get a dotted edge $\Theta_{\sigma_k} \to \Theta_{\sigma_{k-1}}$, which contradicts the conditions on the path. □

EXAMPLE 4.11. Continuing Example 4.8, note that

$$\Theta_1 = \Theta_1^0 = \mathbb{Z}_{\geqslant 0}\,, \quad \Theta_2 = \Theta_2^0 + 6 = \{6, 8, 9, \dots\} = \mathbb{Z}_{\geqslant 6} \setminus \{7\}\,.$$

Therefore, one gets $\Theta_2 \subset \Theta_1 + n + m = \Theta_1 + 5$, so the dotted edge points from Θ_2 to Θ_1 (see Figure 3 on the right). One concludes that the minimal representative corresponding

[1] Solid and dotted edges in Figure 4 correspond to the colors blue and green in the arXiv version.

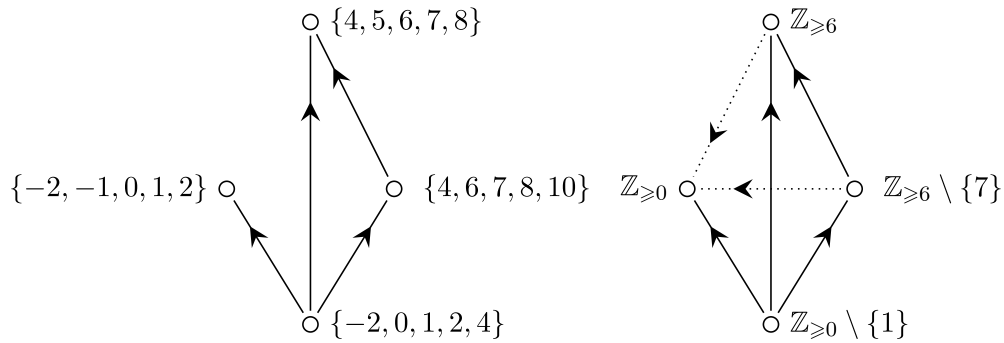

Figure 4. On the left: the digraph with vertices labeled by the skeletons of the $(3,2)$-invariant subsets. On the right: the corresponding bicolored graph with vertices labeled by the $(2,3)$-invariant subsets.

to $\sigma=(0,2,1)$ is admissible, while the minimal representative corresponding to $\sigma=(0,1,2)$ is not (see the stars versus squares in the shaded rectangular region in Figure 2).

Example 4.12. In Figure 4 on the left, we see an example of a digraph corresponding to the minimal point on the boundary of a region for $(n,m)=(2,3)$ and $d=4$ from [GMV17]. The vertices are labeled by the skeletons of the corresponding $(2,3)$-invariant subsets. Note that there are three distinct linearizations of the partial order defined by the orientation of the graph, each corresponding to a minimal representative of the corresponding equivalence class.

On the right in Figure 4, we see the corresponding bicolored digraph with vertices labeled by the $(2,3)$-invariant subsets. Indeed, one has $\mathbb{Z}_{\geqslant 6}\subset\mathbb{Z}_{\geqslant 0}+5$ and $\mathbb{Z}_{\geqslant 6}\setminus\{7\}\subset\mathbb{Z}_{\geqslant 0}+5$, which determine the orientations of the dotted edges. It follows that only one of the three linearizations corresponds to an admissible subset:

$$\begin{aligned}\Delta &= [4(\mathbb{Z}_{\geqslant 0}\setminus\{1\})]\sqcup[4(\mathbb{Z}_{\geqslant 6}\setminus\{7\})+1]\sqcup[4\mathbb{Z}_{\geqslant 6}+2]\sqcup[4\mathbb{Z}_{\geqslant 0}+3]\\ &= \mathbb{Z}_{\geqslant 0}\setminus\{1,2,4,5,6,9,10,13,14,17,18,21,22,29\}\,.\end{aligned}$$

Let $\mathrm{Inv}(nd,md)^{l+1}$ be the set of 0-normalized (nd,md)-invariant subsets such that

$$\begin{aligned}&\Delta\cap(d\mathbb{Z}+r)\neq\emptyset\,, \quad r\in\{0,1,\dots,l\}\,,\\ &\Delta\cap(d\mathbb{Z}+r)=\emptyset\,, \quad r\in\{l+1,l+1,\dots,d-1\}\end{aligned}$$

as above. The following theorem is the main result of [GMV17].

Theorem 4.13 ([GMV17]). *There exists a bijection*

$$\mathcal{D}\colon \mathrm{Inv}(nd,md)^{l+1}/{\sim}\longrightarrow \mathrm{Dyck}(n(l+1),m(l+1))\,,$$

where $\sim$ *is the equivalence relation defined above and* $\mathrm{Dyck}(n(l+1),m(l+1))$ *is the set of* $(n(l+1),m(l+1))$ *Dyck paths. In addition, in the cofinite case* $l=d-1$*, one gets* $\dim\Delta=\delta-\mathrm{dinv}(\mathcal{D}(\Delta))$ *for all* $\Delta\in\mathrm{Inv}(nd,md):=\mathrm{Inv}(nd,md)^d$.

Remark 4.14. In [GMV17], only the cofinite case $l=d-1$ is considered, but the construction of the bijection $\mathcal{D}$ generalizes to the non-cofinite case $l<d-1$ without any change.

4.5 Proof of Theorem 1.1

We combine all of the above results to prove Theorem 1.1. The compactified Jacobian $\overline{JC}$ is stratified into locally closed subsets J_Δ, and by Proposition 3.23, they are isomorphic to affine spaces of dimension $\dim(\Delta)$ if Δ is admissible and empty otherwise. Therefore, the Poincaré polynomial has the form

$$P(T) = \sum_{\Delta \text{ admissible}} T^{2\dim(\Delta)}.$$

Next, we consider the infinite set $\mathrm{Inv}(nd, md)$ of all (nd, md)-invariant subsets in $\mathbb{Z}_{\geqslant 0}$ and the equivalence relation on it. By Theorem 4.10, in each equivalence class, there is a unique admissible representative. Furthermore, by Lemma 4.1, the "combinatorial dimension" $\dim(\Delta)$ depends only on the order of generators and cogenerators, and hence is constant on each equivalence class. Therefore, we can write

$$P(T) = \sum_{\Delta \in \mathrm{Inv}(nd,md)/\sim} T^{2\dim(\Delta)}.$$

Finally, by Theorem 4.13, there is a bijection between the equivalence classes and Dyck paths in an $(nd) \times (md)$ rectangle, and the statistic $\dim(\Delta)$ on the former corresponds to the statistic $\mathrm{codinv} = \delta - \mathrm{dinv}$ on the latter, so

$$P(T) = \sum_{D \in \mathrm{Dyck}(nd,md)} T^{2(\delta - \mathrm{dinv}(d))}.$$

5. Rational shuffle theorem and generic curves

5.1 Elliptic Hall algebra

We briefly recall some notation for the elliptic Hall algebra [BS12]; we refer the reader to [BGLX16, GN15, Mel21, Neg14] for more precise statements and details.

Let $\Lambda = \Lambda_{Q,T}$ be the ring of symmetric functions in infinitely many variables with coefficients in $\mathbb{Q}(Q, T)$. The elliptic Hall algebra $\mathcal{E}$ acts on Λ and has the following properties:

(1) The multiplication operators by power sum symmetric functions p_k correspond (up to a scalar) to the generators $P_{k,0}$ of $\mathcal{E}$.

(2) By expanding an arbitrary symmetric function f in p_k and replacing each p_k by $P_{k,0}$, one associates with f an operator in $\mathcal{E}$.

(3) The universal cover of the group $\mathrm{SL}(2,\mathbb{Z})$ acts on $\mathcal{E}$ by automorphisms. For coprime m and n, we denote by $\gamma_{n,m}$ an element of $\mathrm{SL}(2,\mathbb{Z})$ such that $\gamma_{n,m}(1,0) = (n,m)$ and write $P_{kn,km} = \gamma_{n,m}(P_{k,0})$. Note that the choice of $\gamma_{n,m}$ is not unique, but different choices lead to the same operators, up to a scalar. The algebra $\mathcal{E}$ is generated by the $P_{kn,km}$ for all possible (kn, km).

(4) The algebra $\mathcal{E}$ is graded, and the grading is compatible with the grading on Λ. If $A \in \mathcal{E}$ is a degree d operator, then it shifts the degrees on Λ by d; in particular, $A(1)$ is a symmetric function of degree d. The generator $P_{kn,km}$ has degree kn.

(5) For $n = 0$ the generators $P_{0,m}$ have degree 0 and correspond to (generalized) Macdonald operators on Λ that have modified Macdonald polynomials as eigenvectors.

(6) One has $P_{kn,km+kn} = \nabla P_{kn,km} \nabla^{-1}$, where ∇ is a certain degree-preserving operator on symmetric functions that is diagonal in the modified Macdonald basis.

We also note that $\mathcal{E}$ and its representation Λ can be thought of as limits of spherical double affine Hecke algebras and their representations in symmetric polynomials [SV13]. We refer to [GN15] for the precise normalization conventions for this limit, which we implicitly use here. In particular, the Cherednik–Danilenko conjecture is formulated in terms of DAHA, but we reformulate it here in terms of the elliptic Hall algebra.

5.2 Rational shuffle theorem

We can use the above notation to explore a special case of the compositional rational shuffle theorem conjectured in [BGLX16] and proved in [Mel21]; see also [BHM$^+$23].

Let

$$S_i = S_i(nd, md) = \left\lceil \frac{im}{n} \right\rceil - \left\lceil \frac{(i-1)m}{n} \right\rceil, \quad 1 \leqslant i \leqslant nd\,.$$

Theorem 5.1. *Consider the operator $\gamma_{n,m}(e_d)$ in $\mathcal{E}$ and the symmetric function $\gamma_{n,m}(e_d)(1)$. We have*

$$\begin{aligned} C_{md,nd}(Q,T) &= (\gamma_{n,m}(e_d)(1), e_{nd}) \\ &= \sum_{\mathbf{T}\in \mathrm{SYT}(nd)} \frac{z_1^{S_1}\cdots z_{nd}^{S_{nd}}}{\prod_{i=2}^{nd}(1-z_i^{-1})(1-QTz_{i-1}/z_i)} \prod_{i<j} \frac{(1-z_i/z_j)(1-QTz_i/z_j)}{(1-Qz_i/z_j)(1-Tz_i/z_j)}\,, \end{aligned}$$

where $C_{md,nd}(Q,T)$ is defined by (1.2) and z_i is the (Q,T)-content of a box labeled by i in the standard Young tableau (SYT) $\mathbf{T}$ of size nd.

Proof. The identity $C_{md,nd}(Q,T) = (\gamma_{n,m}(e_d)(1), e_{nd})$ follows from [BGLX16, Conjecture 3.1],[2] which is a special case of the compositional rational shuffle conjecture [BGLX16, Conjecture 3.3] proved by Mellit in [Mel21]. The explicit formula for $\gamma_{n,m}(e_d)(1)$ follows from [Neg14, Proposition 6.13]. □

5.3 Cherednik–Danilenko conjecture

We can now explain the connection between our results and a conjecture of Cherednik and Danilenko [CD16]. The latter is phrased for arbitrary algebraic plane curve singularities with Puiseux expansion

$$y = b_1 x^{m_1/r_1} + b_2 x^{m_2/r_1 r_2} + b_3 x^{m_3/r_1 r_2 r_3} + \cdots\,, \quad b_i \neq 0\,.$$

Here we assume that $\mathrm{GCD}(m_i, r_1\cdots r_i) = 1$. The exponents are related to characteristic pairs (r_i, s_i) by the equations $m_1 = s_1$, $m_i = s_i + r_i m_{i-1}$ $(i > 1)$.

Given a sequence of characteristic pairs $(r_1, s_1), \ldots, (r_\ell, s_\ell)$, the authors of [CD16] define a sequence of symmetric functions as follows. Start from a degree 1 polynomial $f_{\ell+1} := p_1 \in \Lambda$, view it as a multiplication operator in $\mathcal{E}$, and apply the automorphism γ_{r_ℓ,s_ℓ}. The resulting operator $\gamma_{r_\ell,s_\ell}(f_{\ell+1}) = P_{r_\ell,s_\ell}$ has degree r_ℓ, and we can consider the symmetric polynomial

$$f_\ell = \gamma_{r_\ell,s_\ell}(f_{\ell+1})(1) = P_{r_\ell,s_\ell}(1)$$

of degree r_ℓ. Similarly, one can define an operator $\gamma_{r_{\ell-1},s_{\ell-1}}(f_\ell)$ and a symmetric function $f_{\ell-1} = \gamma_{r_{\ell-1},s_{\ell-1}}(f_\ell)(1)$ of degree $r_\ell r_{\ell-1}$ and so on.

Conjecture 5.2 ([CD16]). *Let f_1 be the symmetric function of degree $r_1\cdots r_\ell$ obtained by the above procedure. The specialization of $(f_1, e_{r_1\cdots r_\ell})$ at $T = 1$ is well defined and agrees with the*

[2] Note that [BGLX16] use a slightly different normalization of operators, which leads to an additional sign.

Poincaré polynomial of the compactified Jacobian of a plane curve singularity with characteristic pairs $(r_1, s_1), \ldots, (r_\ell, s_\ell)$.

In the recent paper [KT22], Kivinen and Tsai proved the following.

Theorem 5.3 ([KT22]). *The specialization of* $(f_1, e_{r_1\cdots r_\ell})$ *at* $T = 1$ *from Conjecture 5.2 is a polynomial in* Q *that agrees with the point count of the compactified Jacobian over a finite field* $\mathbb{F}_Q$.

Remark 5.4. A priori, $(f_1, e_{r_1\cdots r_\ell})$ is a rational function in Q and T. In the course of the proof of Theorem 5.3, Kivinen and Tsai proved using the compositional rational shuffle theorem that its specialization at $T = 1$ is indeed well defined.

In the case of generic curves (1.1), we have

$$y = x^{m/n} + \lambda x^{(md+1)/nd} + \cdots, \quad m_1 = m, \quad r_1 = n, \quad m_2 = md + 1, \quad r_2 = d.$$

This means that $s_1 = m$ and $s_2 = 1$. To follow the above procedure, we first need to compute the operator $\gamma_{r_2,s_2}(p_1)$ and the corresponding symmetric function f_2. By [GN15, Corollary 6.5], we have

$$f_2 = \gamma_{d,1}(p_1)(1) = P_{d,1}(1) = e_d.$$

Therefore, the next symmetric function is

$$f_1 = \gamma_{n,m}(e_d)(1),$$

and $(f_1, e_{nd}) = C_{nd,md}(Q, T)$ by Theorem 5.1.

We conclude that for generic curves, Conjecture 5.2 is true and follows from Theorems 1.1 and 5.1.

Remark 5.5. Strictly speaking, we need to consider the special cases $d = 1$ and $n = 1$ where the singularity has only one Puiseux pair separately. For $d = 1$ we have $e_1 = p_1$, so $f_1 = \gamma_{n,m}(p_1)(1)$ as expected.

For $n = 1$ we have one Puiseux pair $(d, md + 1)$. Since $\nabla(1) = 1$, we have

$$f_1 = \gamma_{1,m}(e_d)(1) = \nabla^m(e_d \cdot \nabla^{-m}(1)) = \nabla^m e_d = \nabla^m(P_{d,1}(1)) = P_{d,md+1}(1).$$

6. Curves with two Puiseux pairs

6.1 Cabled Dyck paths and s-admissible subsets

In this section, we consider arbitrary plane curve singularities with two Puiseux pairs:

$$(x(t), y(t)) = \left(t^{nd}, t^{md} + \lambda t^{md+s} + \cdots\right), \quad \lambda \neq 0,$$

where $\mathrm{GCD}(m, n) = 1$ and $\mathrm{GCD}(d, s) = 1$. As above, any $\mathcal{O}_C$-module corresponds to a (dn, dm)-invariant subset $\Delta \subset \mathbb{Z}_{\geqslant 0}$. Let J_Δ denote the corresponding stratum in the compactified Jacobian.

Definition 6.1. An (nd, md)-invariant subset Δ is called s-admissible if s is not suspicious for Δ.

For $s = 1$ this agrees with Definition 3.13. We conjecture the following weaker version of Theorem 1.1.

Conjecture 6.2. *The Euler characteristic of the stratum J_Δ is given by*

$$\chi(J_\Delta) = \begin{cases} 1 & \text{if } \Delta \text{ is } s\text{-admissible}\,, \\ 0 & \text{if } \Delta \text{ is not } s\text{-admissible}\,. \end{cases}$$

For $s = 1$ the conjecture clearly follows from Theorem 1.1. Piontkowski [Pio07] constructed cell decompositions for several cases with $s > 1$ (see Theorem 6.10 below for details), and the conjecture follows from his results in these cases.

Remark 6.3. Unlike in the case $s = 1$, the geometry of the strata J_Δ might be more complicated. In particular, [CP18, Appendix A] lists some non-affine strata for the singularity $(t^6, t^9 + t^{13})$ that corresponds to $(m, n) = (2, 3)$, $(d, s) = (3, 4)$. Some of the strata J_Δ are isomorphic to a union of two affine spaces glued along a codimension 1 subspace—these have Euler characteristic 1 and correspond to 4-admissible Δ. Other strata are isomorphic to $\mathbb{C}^* \times \mathbb{C}^N$—these have Euler characteristic 0 and correspond to Δ that are not 4-admissible.

In particular, Conjecture 6.2 would imply that the Euler characteristic of the compactified Jacobian equals the number of s-admissible subsets. To understand this number, we need the following combinatorial definition.

Definition 6.4. A cabled Dyck path with parameters (n, m), (d, s) is the following collection of data:

- an (d, s) Dyck path P called the *pattern*;
- the numbers $v_1(P), \ldots, v_k(P)$ parametrizing the lengths of vertical runs in P, so that
$$v_1(P) + \cdots + v_k(P) = d\,;$$
- an arbitrary tuple of $(v_i(P)n, v_i(P)m)$ Dyck paths for $1 \leqslant i \leqslant k$.

Clearly, the total number of cabled Dyck paths equals

$$c_{(m,n),(d,s)} = \sum_{P \in \mathrm{Dyck}(d,s)} c_{v_1(P)n, v_1(P)m} \cdots c_{v_k(P)n, v_k(P)m}\,,$$

where $c_{a,b} = C_{a,b}(1, 1)$ is the number of Dyck paths in an $a \times b$ rectangle.

Example 6.5. For $s = 1$ there is only one (d, s) Dyck path P with $v_1(P) = d$, so

$$c_{(n,m),(d,1)} = c_{dn,dm}\,.$$

For $d = 2$ there is one pattern with $v_1(P) = 2$, and there are $\frac{1}{2}(s - 1)$ patterns with $v_1(P) = v_2(P) = 1$. Therefore,

$$c_{(n,m),(2,s)} = c_{2n,2m} + \frac{s-1}{2} c_{n,m}^2\,.$$

Remark 6.6. One can interpret the data of a cabled Dyck path geometrically as follows. First, draw the pattern P in the $d \times s$ rectangle. Then, apply an affine transformation in $\mathrm{SL}(2, \mathbb{Z})$ that sends the vector $(0, 1)$ to $(-m, n)$; it will transform a Dyck path to a sawtooth-like shape where the vertical steps v_i in P become diagonals in some $(v_i m) \times (v_i n)$ rectangles, shown dashed in Figure 5. Now $(v_i m, v_i n)$ Dyck paths can be inscribed in $(v_i m) \times (v_i n)$ rectangles below these diagonals (that is, contained in gray areas in Figure 5). For example, one can consider the gray paths in Figure 5.

Note that the gray areas are overlapping, so the $(v_i m, v_i n)$ Dyck paths might intersect in a complicated way.

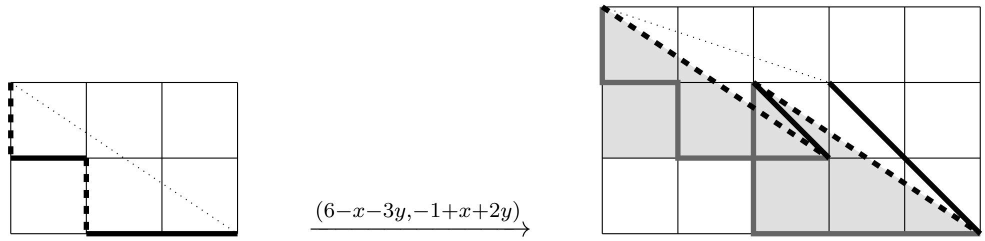

FIGURE 5. Cabled Dyck path for the singularity (t^4, t^6+t^9): Here $(d,s)=(n,m)=(2,3)$. The pattern P (left) in the $d\times s$ rectangle has $v_1=v_2=1$ and $h_1=1$, $h_2=2$. On the right, we give the same pattern after affine transformation (dashed and solid black) and a pair of (v_im, v_in) Dyck paths (gray).

THEOREM 6.7. *There is a bijection between the sets of s-admissible (dn,dm)-invariant subsets and cabled Dyck paths with parameters (n,m), (d,s).*

Proof. Let $\Delta=\bigcup_{j=0}^d \Delta_j$ be a (dn,dm)-invariant subset.

We consider another subset

$$\overline{\Delta}=\bigcup_{j=0}^{d}\overline{\Delta}_j\,,\quad \overline{\Delta}_j=\Delta_{sj}-(s-1)j\,.$$

Note that $\overline{\Delta}$ might have negative elements. Since Δ_{sj} is supported in remainder $sj \mod d$, the set $\overline{\Delta}_j$ is supported in remainder $j \mod d$.

We claim that Δ is s-admissible if and only if 1 is not j-suspicious for $\overline{\Delta}$ for $0\leqslant j\leqslant d-2$. Indeed, for $0\leqslant j\leqslant d-2$ it is clear from Lemma 3.10 that s is (sj)-suspicious for Δ if and only if

$$\Delta_{sj+s}\subset\Delta_{sj}+md+nd+s \iff \overline{\Delta}_{j+1}\subset\overline{\Delta}_j+1\,,$$

that is, 1 is j-suspicious for $\overline{\Delta}$. Since Δ is 0-normalized, we have $\min\Delta_0=0$ and $\min\Delta_{s(d-1)}>0$, so by Lemma 3.11, the integer s is not $s(d-1)$-suspicious.

Next, we define the cabled Dyck path corresponding to $\overline{\Delta}$. Note that $\overline{\Delta}_0=\Delta_0$, so $\min\overline{\Delta}_0=0$. Suppose that for $j=0,\ldots,t_1-1$ we have $\min\overline{\Delta}_j\geqslant 0$ but $\min\overline{\Delta}_{t_1}<0$; then we define $v_1(P)=t_1$. By Theorem 4.10, the collection of subsets $\overline{\Delta}_0,\ldots,\overline{\Delta}_{t_1-1}$ defines a unique 1-admissible collection of (m,n) Dyck paths in the corresponding equivalence class, and hence by Theorem 4.13 a unique (t_1m,t_1n) Dyck path. Note that by Lemma 3.11, the number 1 is not t_1-suspicious for $\overline{\Delta}$.

Next, suppose that for $j=t_1,\ldots,t_2-1$ we have $\min\overline{\Delta}_j\geqslant\min\overline{\Delta}_{t_1}$ but $\min\overline{\Delta}_{t_2}<\min\overline{\Delta}_{t_1}$; then we define $v_2(P)=t_2-t_1$. As above, $\overline{\Delta}_{t_1},\ldots,\overline{\Delta}_{t_2-1}$ define a unique $((t_2-t_1)m,(t_2-t_1)n)$ Dyck path, and so on.

This defines a collection of vertical runs $v_i(P)$ with $\sum v_i(P)=d$ and a collection of Dyck paths in the $(v_i(P)m)\times(v_i(P)n)$ rectangles. To complete the bijection, we need to define the horizontal steps of the pattern P. Consider a sequence of integers q_i defined by the equation

$$\min\overline{\Delta}_{t_i}=t_i-q_id\,; \tag{6.1}$$

then by construction, we have

$$t_i-q_id<t_{i-1}-q_{i-1}d\,,\quad (q_i-q_{i-1})d>t_i-t_{i-1}>0\,,$$

hence $q_i>q_{i-1}$, and we can define $h_i(P)=q_i-q_{i-1}>0$. On the other hand,

$$t_i-q_id=\min\Delta_{st_i}-st_i+t_i\geqslant t_i-st_i\,,$$

so

$$(h_1(P) + \cdots + h_i(P))d = q_i d \leqslant st_i = s(v_1(P) + \cdots + v_i(P))$$

and a lattice path P with vertical steps v_i and horizontal steps h_i stays below the diagonal.

Conversely, assume that we are given a cabled Dyck path with pattern P and a collection of $(v_i(P)m, v_i(P)n)$ Dyck paths. Using the vertical and horizontal steps of P, we can reconstruct t_i and q_i as above and set up $t_0 = q_0 = 0$. By Theorems 4.13 and 4.10, we can translate each $(v_i(P)m, v_i(P)n)$ Dyck path into a unique 1-admissible collection of (dn, dm) invariant subsets $\overline{\Delta}_{t_{i-1}}, \ldots, \overline{\Delta}_{t_i-1}$ normalized by (6.1). This determines $\overline{\Delta}$ completely.

Finally, since $\mathrm{GCD}(d, s) = 1$, the values of $sj \mod d$ run over all remainders modulo d once, and we can uniquely define Δ_j by

$$\Delta_{sj} = \overline{\Delta}_j + sj - j\,.$$

By reversing the argument above, we see that Δ is s-admissible. Assume that $t_i \leqslant j < t_{i+1}$; then

$$\min \Delta_{sj} = \min \overline{\Delta}_j + (s-1)j \geqslant \min \overline{\Delta}_{t_i} + (s-1)t_i = t_i - q_i d + (s-1)t_i = st_i - q_i d > 0\,,$$

and Δ is 0-normalized. □

EXAMPLE 6.8. Let us reconstruct the $(4, 6)$-invariant 3-admissible subset Δ corresponding to the data of cabled Dyck paths in Figure 5 (see also Remark 6.6). We have $(d, s) = (n, m) = (2, 3)$ and $v_1 = v_2 = 1$ and $h_1 = 1$, $h_2 = 2$. This means that $t_1 = 1$, $q_1 = 1$ and

$$\min \overline{\Delta}_0 = 0\,, \quad \min \overline{\Delta}_1 = -1\,.$$

Furthermore, it is easy to see that $\overline{\Delta}_0$ (corresponding to the top gray path in Figure 5) up to a shift coincides with $2\mathbb{Z}_{\geqslant 0} = \{0, 2, 4, 6, \ldots\}$, while $\overline{\Delta}_1$ (corresponding to the bottom gray path in Figure 5) up to a shift coincides with $2(\mathbb{Z}_{\geqslant 0} \setminus \{2\} = \{0, 4, 6, \ldots\}$. Therefore,

$$\overline{\Delta}_0 = \{0, 2, 4, 6, \ldots\}\,, \quad \overline{\Delta}_1 = \{-1, 3, 5, \ldots\}\,.$$

Now $\Delta_0 = \overline{\Delta}_0 = \{0, 2, 4, 6, \ldots\}$, $\Delta_1 = \overline{\Delta}_1 + 3 - 1 = \{1, 5, 7, \ldots\}$ and

$$\Delta = \{0, 1, 2, 4, 5, 6, \ldots\} = \mathbb{Z}_{\geqslant 0} \setminus \{3\}\,.$$

LEMMA 6.9. *The specialization of Conjecture 5.2 at $Q = T = 1$ for singularities of the form*

$$(x(t), y(t)) = \left(t^{nd}, t^{md} + \lambda t^{md+s} + \cdots\right), \quad \lambda \neq 0\,,$$

agrees with the number of cabled Dyck paths with parameters (n, m), (d, s).

Proof. We follow the notation of Section 5.3. There are two characteristic pairs (n, m) and (d, s). We start from a symmetric function $f_2 = P_{d,s}(1)$ that specializes to

$$f_2|_{Q=T=1} = \sum_P e_{v_1(P)} \cdots e_{v_k(P)}\,.$$

At $Q = T = 1$ the operators $\gamma_{n,m}(f)$ are simply multiplication operators on Λ (see for example [KT22, Neg24]), so

$$\gamma_{n,m}(fg)_{Q=T=1}(1) = \gamma_{m,n}(f)_{Q=T=1}(1) \cdot \gamma_{m,n}(g)_{Q=T=1}(1)\,.$$

Let $F_{nd,md} = \gamma_{n,m}(e_d)(1)|_{Q=T=1}$; then

$$\begin{aligned} f_1|_{Q=T=1} = \gamma_{n,m}(f_2)(1)|_{Q=T=1} &= \sum_P \left[\gamma_{n,m}(e_{v_1(P)}) \cdots \gamma_{n,m}(e_{v_k(P)})(1)\right]|_{Q=T=1} \\ &= \sum_P F_{v_1(P)n, v_1(P)m} \cdots F_{v_k(P)n, v_k(P)m}\,. \end{aligned}$$

Finally, by Theorem 5.1, we have $(F_{nd,md}, e_{nd}) = C_{nd,md}(1,1) = c_{nd,md}$, and the result follows. See [KT22] for related computations (at $Q = 1$) and more details. □

6.2 Piontkowski's results

For the reader's convenience, we summarize the main results of [Pio07] for singularities with two Puiseux pairs using the terminology of this paper.

Theorem 6.10 ([Pio07]). *Suppose that $d = 2$, $(n, m) = (2, q)$, $(3, 4)$ or $(3, 5)$ and s is an arbitrary odd number.*

(a) *If Δ is not s-admissible, then J_Δ is empty.*

(b) *If Δ is s-admissible, then J_Δ is isomorphic to an affine space (and the formula for its dimension is given in [Pio07]).*

(c) *The Euler characteristic of $\overline{JC}$ is equal to the number of s-admissible subsets, which is given by the following table:*

$(2,q)$	$\frac{(q+1)(q^2+5q+3)}{12} + \frac{(q+1)^2}{8}(2q+s)$
$(3,4)$	$\frac{229}{2} + \frac{25}{2}(8+s)$
$(3,5)$	$\frac{511}{2} + \frac{49}{2}(10+s)$

Observe that Theorem 6.10(c) can be compactly written as follows.

Proposition 6.11. *Let C be a plane curve singularity with Puiseux exponents $(2n, 2m, 2m+s)$, where $(n, m) = (2, q)$, $(3, 4)$ or $(3, 5)$. Then*

$$\chi(\overline{JC}) = c_{2n,2m} + (c_{n,m})^2 \cdot \frac{s-1}{2} = c_{(n,m),(2,s)}\,.$$

For $s = 1$ we recover $\chi(\overline{JC}) = c_{2n,2m}$, in agreement with Theorem 1.1.

Proof. We have $c_{2,q} = \frac{1}{2}(q+1)$, $c_{3,4} = 5$ and $c_{3,5} = 7$. Using for example the formula in Corollary 1.3, we compute

$$c_{4,2q} = \frac{(2q+3)!}{4!(2q)!} + \frac{1}{2}\left(\frac{q+1}{2}\right)^2 = \frac{(q+1)(2q+1)(2q+3)}{12} + \frac{(q+1)^2}{8}\,,$$
$$c_{6,8} = \frac{1}{14}\binom{14}{6} + \frac{5^2}{2} = \frac{429}{2} + \frac{25}{2} = 227\,,$$
$$c_{6,10} = \frac{1}{16}\binom{16}{6} + \frac{7^2}{2} = \frac{1001}{2} + \frac{49}{2} = 525\,.$$

Now the result is easy to check using the above table. □

Example 6.12. The simplest singularity with two Puiseux pairs is $(t^4, t^6 + t^7)$. The Dyck paths in a 4×6 rectangle can be thought of as subdiagrams of $\lambda_{4,6} = (4, 3, 1)$ that can be listed as follows:

$\lvert D\rvert$	D
0	$(0,0,0)$
1	$(1,0,0)$
2	$(2,0,0)$, $(1,1,0)$
3	$(3,0,0)$, $(2,1,0)$, $(1,1,1)$
4	$(3,1,0)$, $(2,2,0)$, $(2,1,1)$, $(4,0,0)$
5	$(3,2,0)$, $(3,1,1)$, $(2,2,1)$, $(4,1,0)$
6	$(3,2,1)$, $(3,3,0)$, $(4,2,0)$, $(4,1,1)$
7	$(3,3,1)$, $(4,3,0)$, $(4,2,1)$
8	$(4,3,1)$

The generating function equals

$$P_{4,6}(T)=\sum_D T^{|D|}=1+T+2T^2+3T^3+4T^4+4T^5+4T^6+3T^7+T^8\,,$$

which agrees with the Poincaré polynomial of $\overline{JC}$ computed in [Pio07], as in Corollary 1.5. The Euler characteristic equals

$$23=\frac{1}{10}\binom{10}{4}+\frac{1}{2}\left(\frac{1}{5}\binom{5}{2}\right)^2=21+\frac{2^2}{2}\,.$$

Acknowledgements

We thank Francois Bergeron, Oscar Kivinen, Anton Mellit and Monica Vazirani for useful discussions.

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