Algebraic Geometry 13 (1) (2026) 1–24
doi:10.14231/AG-2026-001

A motivic change of variables formula for Artin stacks

Matthew Satriano and Jeremy Usatine

Abstract

Let $\mathcal{X} \to Y$ be a birational map from a smooth Artin stack to a (possibly singular) variety. We prove a change of variables formula that relates motivic integrals over arcs of Y to motivic integrals over arcs of $\mathcal{X}$. With a view toward the study of stringy Hodge numbers, this change of variables formula leads to a notion of crepantness for the map $\mathcal{X} \to Y$ that coincides with the usual notion in the special case where $\mathcal{X}$ is a scheme.

1. Introduction

Let Y be a variety. The theory of motivic integration, introduced by Kontsevich during a lecture at Orsay in 1995, defines a *motivic measure* on the *arc scheme* of Y that takes values in a certain completed Grothendieck ring of varieties. A key feature of this motivic measure is a *change of variables formula* that, given a birational modification $X \to Y$, relates the motivic measure for Y to the motivic measure for X; see, for example, [DL99, Loo02]. This change of variables formula is an essential ingredient in applications of motivic integration to many areas, including birational geometry, mirror symmetry, and the study of singularities, especially when one wants to understand invariants of a singular variety in terms of some resolution of singularities.

Numerous results across the last few decades have suggested that smooth stacks are a natural tool in studying singular varieties: This perspective is essential in recent work on functorial resolution of singularities [ATW24, MM19] and pervades (at the very least implicitly) work on the McKay correspondence since Reid's formulation (in lectures at the University of Utah and MSRI in 1992). In an example that is particularly relevant to this paper, Yasuda proved that if Y is proper and has Gorenstein quotient singularities and $\mathcal{X} \to Y$ is the canonical smooth Deligne–Mumford stack associated with Y, then the stringy Hodge numbers of Y coincide with the orbifold Hodge numbers of $\mathcal{X}$; see [Yas04]. This result was a consequence of Yasuda's motivic integration for Deligne–Mumford stacks and a change of variables formula [Yas04, Theorem 1.3] for the map $\mathcal{X} \to Y$, showing that $\mathcal{X}$ behaves like a *crepant* resolution of Y (see Section 1.3 below). In particular, this work showed that these stringy Hodge numbers are nonnegative, confirming a special case of a conjecture of Batyrev's [Bat98, Conjecture 3.10] and demonstrating the utility of a motivic integration and change of variables formula for Deligne–Mumford stacks (even when *a priori* one only cares about singular varieties).

A natural class of varieties with far worse than quotient singularities is those varieties that

Received 2 March 2022, accepted in final form 19 June 2024.
2020 Mathematics Subject Classification 14D23, 14E18, 14E16.
Keywords: motivic integration, stacks, McKay correspondence.

MS was partially supported by a Discovery Grant from the National Science and Engineering Research Council of Canada.

are the good moduli space Y, in the sense of [Alp13], of a smooth Artin stack $\mathcal{X}$. Such varieties naturally arise in the context of geometric invariant theory: They include all toric varieties as well as quotients of affine space by representations of linearly reductive groups. They have no obvious *crepant* resolution by a smooth Deligne–Mumford stack; however, the good moduli space map $\mathcal{X} \to Y$ is a natural candidate for a nice resolution of singularities by a smooth *Artin* stack. In [SU22], the authors introduced a conjectural (proven in the toric case) framework that uses motivic integration for the stack $\mathcal{X}$ to study the stringy Hodge numbers of the singular variety Y. This framework suggested that when the good moduli space map[1] $\mathcal{X} \to Y$ is birational and has exceptional locus with codimension at least 2, there should be a motivic change of variables formula making $\mathcal{X}$ behave like a crepant resolution of Y (see [SU22, Conjecture 1.1 and Theorem 1.7] for precise details). In particular, this illuminated that, purely from the perspective of studying singular varieties, there is need for

(1) a versatile motivic change of variables formula for maps $\mathcal{X} \to Y$ from smooth Artin stacks to singular varieties and
(2) a notion of *crepantness* for such maps $\mathcal{X} \to Y$.[2]

In this paper, we prove a motivic change of variables formula, Theorem 1.2, that applies in a high level of generality to birational maps from smooth Artin stacks to singular varieties, addressing point (1). We also discuss how the form this change of variables formula takes suggests a useful notion for crepantness, addressing point (2).

Remark 1.1. After this paper was originally written, we showed that this notion of crepantness coincides with a straightforward definition in terms of relative canonical divisors [SU24, Corollary 1.14]. Furthermore, we proved that any variety with log-terminal singularities admits a crepant resolution by a smooth Artin stack [SU24, Theorem 1.1].

1.1 Conventions

Throughout this paper, let k be an algebraically closed field of characteristic 0. For any stack $\mathcal{X}$ over k, we let $|\mathcal{X}|$ denote its associated topological space, and for any subset $\mathcal{C} \subset |\mathcal{X}|$ and field extension k' of k, we let $\mathcal{C}(k')$ (respectively, $\overline{\mathcal{C}}(k')$) denote the category of (respectively, set of isomorphism classes of) k'-valued points of $\mathcal{X}$ whose class in $|\mathcal{X}|$ is contained in $\mathcal{C}$.

For the benefit of the reader, we recall that the cotangent complex $L_{\mathcal{X}/\mathcal{Y}}$ for a morphism of stacks $f\colon \mathcal{X} \to \mathcal{Y}$ is concentrated in degrees less than or equal to 1. When $\mathcal{X}$ and $\mathcal{Y}$ are schemes (or more generally when f is representable), $L_{\mathcal{X}/\mathcal{Y}}$ is concentrated in degrees less than or equal to 0. When f is smooth, $L_{\mathcal{X}/\mathcal{Y}}$ is concentrated in degrees greater than or equal to 0.

1.2 Change of variables formula

The main result of this paper is a change of variables formula relating motivic integrals over arcs of a possibly singular variety Y to motivic integrals over arcs of a smooth Artin stack $\mathcal{X}$. A key input to this change of variables formula is a certain *height function* $\mathrm{ht}_E^{(0)}$ associated with an object E in the derived category of $\mathcal{X}$. The value of $\mathrm{ht}_E^{(0)}$ at an arc $\mathrm{Spec}(k'[\![t]\!]) \to \mathcal{X}$ is given by pulling back E to $\mathrm{Spec}(k'[\![t]\!])$ and taking the dimension (over k') of the resulting complex's 0th

[1]In fact, Theorem 1.2 does not assume that $\mathcal{X} \to Y$ is a good moduli space map. This extra level of generality is essential in light of [SU24].
[2]It is important to note that unlike in the case of Deligne–Mumford stacks, defining the canonical bundle for Artin stacks is a subtle issue, so there is no *a priori* obvious definition one can take for $\mathcal{X} \to Y$ to be crepant.

(hyper)cohomology; see Definition 3.1 for a precise definition of $\mathrm{ht}^{(0)}_E$. Since this hypercohomology group can be infinite-dimensional over k', the function $\mathrm{ht}^{(0)}_E$ takes values in $\mathbb{Z}_{\geqslant 0} \cup \{\infty\}$. We expect that these height functions will play an important role in motivic integration for Artin stacks, even beyond the change of variables formula, as Corollary 3.4 shows that they are natural functions to integrate.

We now state our change of variables formula. Our notions of cylinders, the arc stack $\mathscr{L}(\mathcal{X})$, the motivic measure $\mu_{\mathcal{X}}$, and constructible functions and their integrals are all straightforward generalizations of the corresponding notions in the case of schemes. See Section 2 or [SU22, Section 3] for precise definitions.

Theorem 1.2. *Let $\mathcal{X}$ be a smooth irreducible finite-type Artin stack over k with affine geometric stabilizers and separated diagonal, let Y be an irreducible finite-type scheme over k with $\dim Y = \dim \mathcal{X}$, let $\mathcal{X} \to Y$ be a morphism, and let $\mathcal{U}$ be an open substack of $\mathcal{X}$ such that $\mathcal{U} \hookrightarrow \mathcal{X} \to Y$ is an open immersion.*

(a) *Let $L_{\mathcal{X}/Y}$ and $L_{\mathcal{I}_\mathcal{X}/\mathcal{X}}$ be the relative cotangent complexes of $\mathcal{X} \to Y$ and the inertia map $\mathcal{I}_\mathcal{X} \to \mathcal{X}$, respectively, let $\sigma\colon \mathcal{X} \to \mathcal{I}_\mathcal{X}$ be the identity section of $\mathcal{I}_\mathcal{X} \to \mathcal{X}$, and set*
$$\mathrm{ht}_{\mathcal{X}/Y} = \mathrm{ht}^{(0)}_{L\sigma^* L_{\mathcal{I}_\mathcal{X}/\mathcal{X}}} - \mathrm{ht}^{(0)}_{L_{\mathcal{X}/Y}}\,.$$
For any cylinder $\mathcal{C} \subset |\mathscr{L}(\mathcal{X})| \setminus |\mathscr{L}(\mathcal{X} \setminus \mathcal{U})| \subset |\mathscr{L}(\mathcal{X})|$, the function $\mathrm{ht}_{\mathcal{X}/Y}\colon \mathcal{C} \to \mathbb{Z}$ is constructible.

(b) *Let $\mathcal{C} \subset |\mathscr{L}(\mathcal{X})| \setminus |\mathscr{L}(\mathcal{X} \setminus \mathcal{U})| \subset |\mathscr{L}(\mathcal{X})|$ and $D \subset \mathscr{L}(Y)$ be cylinders such that, for all field extensions k' of k, the map $\overline{\mathcal{C}}(k') \to D(k')$ is a bijection. Then*
$$\mu_Y(D) = \int_{\mathcal{C}} \mathbb{L}^{\mathrm{ht}_{\mathcal{X}/Y}} \mathrm{d}\mu_{\mathcal{X}}\,.$$

Remark 1.3. By Remark 3.5 below, $\mathrm{ht}^{(0)}_{L\sigma^* L_{\mathcal{I}_\mathcal{X}/\mathcal{X}}}$ and $\mathrm{ht}^{(0)}_{L_{\mathcal{X}/Y}}$ take only integer values on $\mathcal{C}$.

We see that the height functions $\mathrm{ht}^{(0)}_{L_{\mathcal{X}/Y}}$ and $\mathrm{ht}^{(0)}_{L\sigma^* L_{\mathcal{I}_\mathcal{X}/\mathcal{X}}}$ control the change of variables formula. Note that in the case where $\mathcal{X}$ is a scheme, the function $\mathrm{ht}^{(0)}_{L_{\mathcal{X}/Y}}$ coincides with the order function of the relative Jacobian ideal of $\mathcal{X} \to Y$; that is, it is the function controlling Kontsevich's original motivic change of variables formula. However, when $\mathcal{X}$ is a scheme (or even a Deligne–Mumford stack), $L\sigma^* L_{\mathcal{I}_\mathcal{X}/\mathcal{X}}$ is isomorphic to the 0 object, so the function $\mathrm{ht}^{(0)}_{L\sigma^* L_{\mathcal{I}_\mathcal{X}/\mathcal{X}}}$ is identically zero. Therefore, $\mathrm{ht}^{(0)}_{L\sigma^* L_{\mathcal{I}_\mathcal{X}/\mathcal{X}}}$ is a novel contribution that only appears when dealing with Artin stacks that are not Deligne–Mumford. Note that this function, which controls the stabilizer groups of the jet stacks of $\mathcal{X}$ (see Theorem 4.1), is nonzero in general (see, for example, [SU22, Section 5], or consider the stabilizers of jets in simple cases such as $\left[\mathbb{A}^2_k/\mathbb{G}_{m,k}\right]$, where $\mathbb{G}_{m,k}$ acts with weights 1 and -1).

Finally, we note that cylinders $\mathcal{C}$ and D satisfying the hypotheses of Theorem 1.2 seem to arise frequently. For example, the change of variables formula [SU22, Theorem 1.7] for toric Artin stacks is easily reduced to cylinders satisfying these hypotheses by [SU22, Theorem 4.9]. Another family of examples of such $\mathcal{C}$ and D is the subject of [SU22, Proposition 10.2]; these are examples involving quotients of affine space by certain representations of SL_2.

1.3 Crepantness and applications to stringy invariants

Now suppose that Y is a variety over k with log-terminal singularities, and for expositional simplicity, also assume that the canonical divisor K_Y is Cartier. Batyrev introduced a notion of *stringy Hodge numbers* for Y; see [Bat98]. If Y is proper and $X \to Y$ is a crepant resolution of Y (by a scheme X), the stringy Hodge numbers of Y coincide with the usual Hodge numbers of X. Refining the notion of stringy Hodge numbers, Denef and Loeser introduced the *Gorenstein measure* μ_Y^{Gor} on $\mathscr{L}(Y)$; see [DL02]. If $X \to Y$ is a crepant resolution of Y (by a scheme X), then μ_Y^{Gor} is essentially the usual motivic measure μ_X. A key insight of Yasuda's was that if Y has only quotient singularities and $\mathcal{X} \to Y$ is the canonical smooth Deligne–Mumford stack associated with Y, then μ_Y^{Gor} is essentially $\mu_{\mathcal{X}}$; that is, from this perspective, $\mathcal{X}$ behaves like a crepant resolution of Y.

Now suppose that $\mathcal{X}$ is an Artin stack over k, that $\mathcal{X} \to Y$ is a morphism, and that these satisfy the hypotheses of Theorem 1.2. Unlike when $\mathcal{X}$ is a scheme (or even a Deligne–Mumford stack), we are not aware of a general definition of relative canonical divisor $K_{\mathcal{X}/Y}$ that produces a nice definition for crepantness of $\mathcal{X} \to Y$. We therefore take an approach hinted at by the properties of μ_Y^{Gor} above: We propose that one should consider $\mathcal{X} \to Y$ to be crepant when μ_Y^{Gor} and $\mu_{\mathcal{X}}$ have a certain nice relationship; specifically, we propose that $\mathcal{X} \to Y$ should be considered crepant if whenever $\mathcal{C}$ and D satisfy the hypotheses in Theorem 1.2, we have $\mu_Y^{\mathrm{Gor}}(D) = \mu_{\mathcal{X}}(\mathcal{C})$. Theorem 1.2 tells us how to characterize this property in terms of the height functions $\mathrm{ht}^{(0)}_{L\sigma^* L_{\mathcal{I}_{\mathcal{X}}/\mathcal{X}}}$ and $\mathrm{ht}^{(0)}_{L_{\mathcal{X}/Y}}$, as we now make precise.

Let $\omega_Y = \iota_* \Omega^{\dim Y}_{Y_{\mathrm{sm}}}$, where $\iota\colon Y_{\mathrm{sm}} \to Y$ is the inclusion of the smooth locus, and let $\mathscr{J}_Y$ be the unique ideal sheaf on Y such that $\mathscr{J}_Y \omega_Y$ is the image of $\Omega_Y^{\dim Y} \to \omega_Y$. Recall that for any cylinder $D \subset \mathscr{L}(Y)$, we have

$$\mu_Y^{\mathrm{Gor}}(D) = \int_D \mathbb{L}^{\operatorname{ord} \mathscr{J}_Y} \mathrm{d}\mu_Y\,.$$

Definition 1.4. Let $\mathcal{X}$ be a smooth irreducible finite-type Artin stack over k with affine geometric stabilizers and separated diagonal, let Y be an integral finite-type separated scheme over k with log-terminal singularities and Cartier canonical divisor, assume $\dim Y = \dim \mathcal{X}$, and let $\pi\colon \mathcal{X} \to Y$ be a morphism. We say that π satisfies ($\triangle$) if there exists a nonempty open substack $\mathcal{U}$ of $\mathcal{X}$ such that $\mathcal{U} \hookrightarrow \mathcal{X} \to Y$ is an open immersion and

$$\operatorname{ord}_{\mathscr{J}_Y} \circ \mathscr{L}(\pi) + \mathrm{ht}^{(0)}_{L\sigma^* L_{\mathcal{I}_{\mathcal{X}}/\mathcal{X}}} - \mathrm{ht}^{(0)}_{L_{\mathcal{X}/Y}} = 0$$

on $|\mathscr{L}(\mathcal{X})| \setminus |\mathscr{L}(\mathcal{X} \setminus \mathcal{U})|$.

By [SU24, Remark 1.7 and Corollary 1.14], small resolutions satisfy ($\triangle$). For example, this includes the canonical Artin stack over a toric variety or the good moduli space map $[\mathbb{A}^{2r}/\mathrm{SL}_2] \to Y$, where $r \geqslant 3$, the group SL_2 acts on the affine space of $2\times r$ matrices $\mathbb{A}^{2r}$, and Y is the cone over the Grassmannian $\mathrm{Gr}(2,r)$; variants on the latter class of examples play a particularly important role in our aforementioned work [SU24], where we construct crepant resolutions of log-terminal varieties.

Remark 1.5. If $\mathcal{X} = X$ is a variety, ($\triangle$) coincides with the condition $K_{X/Y} = 0$. Indeed, when $\mathcal{X} = X$ is a variety, we have

$$\operatorname{ord}_{\mathscr{J}_Y} \circ \mathscr{L}(\pi) + \mathrm{ht}^{(0)}_{L\sigma^* L_{\mathcal{I}_{\mathcal{X}}/\mathcal{X}}} - \mathrm{ht}^{(0)}_{L_{\mathcal{X}/Y}} = \operatorname{ord}_{\mathscr{J}_Y} \circ \mathscr{L}(\pi) - \operatorname{ordjac}_\pi\,,$$

and the right-hand side vanishes if and only if $K_{X/Y} = 0$; see, for example, [CNS18, Chapter 7, Proposition 3.2.5]. Here, $\operatorname{ordjac}_\pi$ denotes the order of the Jacobian ideal sheaf of π.

An immediate consequence of Theorem 1.2 and Definition 1.4 is that if π satisfies ($\triangle$), then $\mu_Y^{\mathrm{Gor}}(D) = \mu_{\mathcal{X}}(\mathcal{C})$ for any $\mathcal{C}$, D satisfying the hypotheses of Theorem 1.2(b). We therefore propose ($\triangle$) as a new notion for crepantness. See also footnote 2.

Remark 1.6. Since [Yas04], Yasuda has further developed the theory of motivic integration for *Deligne–Mumford* stacks and obtained quite general change of variables formulas in that setting; see, for example, [Yas06, Yas24].

Also, Balwe introduced a motivic integration and change of variables formula for Artin n-stacks [Bal08, Theorem 7.2.5]. However, his hypothesis that the map be "0-truncated" prevents his formula from applying to our main case of interest here, namely maps from Artin stacks to varieties.

Remark 1.7. Although we are not aware of any direct concrete relationship, this work shares a spiritual connection with the literature on variation of geometric invariant theory in the context of the D-equivalence conjecture, such as [BFK19, DS14, Hal15, HS20], in which derived categories of Artin stacks are used to produce derived equivalences between certain birational varieties. In a vague sense, this connection is analogous to the relationship between the motivic McKay correspondence and the derived McKay correspondence.

1.4 Recent progress

Using resolution of singularities to reduce to Theorem 1.2, we extended our change of variables formula to allow $\mathcal{X}$ to be singular in [SU23b, Theorem 7.2]. The key motivation was to apply such a formula to the case where $\mathcal{X}$ is a warping stack as introduced in [SU23a] and obtain a change of variables formula for our new theory of warped arcs.

It is reasonable to expect that a version of Theorem 1.2 also holds when Y is allowed to be a stack $\mathcal{Y}$, likely using the relative height function in [SU23b, Definition 6.2]. However, we note that the assumption that Y is a scheme is used in many places in the proofs below (just to name one example, the computation of $\ker f$ in the proof of Proposition 6.4 uses that L_Y is concentrated in nonpositive degree due to Y being a scheme). Furthermore, we expect that in the case of a stack $\mathcal{Y}$, more elaborate arguments than those below would be needed in order to account for the stabilizers of the $\mathscr{L}_n(\mathcal{Y})$. In fact, much of the technical difficulty below stems from handling the stabilizers of the $\mathscr{L}_n(\mathcal{X})$. Alternatively, maybe the case with a stack $\mathcal{Y}$ can somehow be reduced to Theorem 1.2. Either way, since our main motivation is using stacks to study singular varieties Y, we have not pursued such a generalization (other than the case where $\mathcal{X} \to \mathcal{Y}$ is a representable resolution of singularities [SU23b, Theorem 3.1] as an intermediate step in handling the case where $\mathcal{X}$ is singular).

2. Motivic integration for smooth Artin stacks

In this section, we set up the formalism and notation for a motivic integration for smooth Artin stacks. For any Artin stack $\mathcal{X}$ over k, we let $\mathscr{L}(\mathcal{X})$ denote the arc stack of $\mathcal{X}$, and for each $n \in \mathbb{Z}_{\geqslant 0}$, we let $\mathscr{L}_n(\mathcal{X})$ denote the nth jet stack of $\mathcal{X}$. In other words, $\mathscr{L}(\mathcal{X}) = \varprojlim_n \mathscr{L}_n(\mathcal{X})$, and $\mathscr{L}_n(\mathcal{X})$ is the Weil restriction of $\mathcal{X} \otimes_k k[t]/(t^{n+1})$ along the morphism $\mathrm{Spec}(k[t]/(t^{n+1})) \to \mathrm{Spec}(k)$. We will let $\theta_n \colon \mathscr{L}(\mathcal{X}) \to \mathscr{L}_n(\mathcal{X})$ and $\theta^n_m \colon \mathscr{L}_n(\mathcal{X}) \to \mathscr{L}_m(\mathcal{X})$ denote the truncation morphisms. See [SU22, Section 3] for some of the basic properties of $\mathscr{L}(\mathcal{X})$ and $\mathscr{L}_n(\mathcal{X})$. In particular, we recall that if $\mathcal{X}$ is of finite type over k, then so is each $\mathscr{L}_n(\mathcal{X})$, and if $\mathcal{X}$ has affine geometric stabilizers, so does each $\mathscr{L}_n(\mathcal{X})$. For us, the important subsets of $|\mathscr{L}(\mathcal{X})|$ will be the so-called cylinders, whose definition we now recall.

Definition 2.1. Let $\mathcal{X}$ be a finite type Artin stack over k. A subset of $|\mathscr{L}(\mathcal{X})|$ is called a *cylinder* if it is of the form $\theta_n^{-1}(\mathcal{C}_n)$ for some $n \in \mathbb{Z}_{\geqslant 0}$ and constructible subset $\mathcal{C}_n \subset |\mathscr{L}_n(\mathcal{X})|$.

Proposition 2.2. *Let $\mathcal{X}$ be a finite type Artin stack over k. Then any cover of $|\mathscr{L}(\mathcal{X})|$ by cylinders has a finite subcover.*

Proof. Let $V \to \mathcal{X}$ be a finite type smooth cover. Then infinitesimal lifting implies that $\mathscr{L}(V) \to |\mathscr{L}(\mathcal{X})|$ is surjective. It is easy to verify that the preimage in $\mathscr{L}(V)$ of a cylinder in $|\mathscr{L}(\mathcal{X})|$ is a cylinder, so the proposition reduces to the well-known case where $\mathcal{X}$ is a scheme; see, for example, [CNS18, Chapter 6, Example 4.1.3]. $\square$

For the remainder of this section, fix a smooth irreducible finite-type Artin stack $\mathcal{X}$ over k with affine geometric stabilizers. Recall that there is a motivic measure $\mu_{\mathcal{X}}$ that assigns to each cylinder in $|\mathscr{L}(\mathcal{X})|$ an element of a modified Grothendieck ring of varieties. First, we set some notation for that modified Grothendieck ring of varieties. We refer to [SU22, Section 2.2] for further details. Let $\widehat{\mathscr{M}_k}$ denote the ring obtained by starting with the Grothendieck ring of k-varieties, inverting the class of $\mathbb{A}^1_k$, and then completing with respect to the dimension filtration. Let $K_0(\mathbf{Stack}_k)$ denote the Grothendieck ring of stacks in the sense of [Eke25]. Since $K_0(\mathbf{Stack}_k)$ is an explicit localization of the Grothendieck ring of varieties over k, see [Eke25, Theorem 1.2], the universal property of localization gives a unique ring homomorphism $K_0(\mathbf{Stack}_k) \to \widehat{\mathscr{M}_k}$ taking each class in $K_0(\mathbf{Stack}_k)$ of any finite-type k-scheme to its class in $\widehat{\mathscr{M}_k}$. For any finite-type Artin stack $\mathcal{Y}$ over k with affine geometric stabilizers, let $\mathrm{e}(\mathcal{Y}) \in \widehat{\mathscr{M}_k}$ denote the image along $K_0(\mathbf{Stack}_k) \to \widehat{\mathscr{M}_k}$ of the class of $\mathcal{Y}$ in $K_0(\mathbf{Stack}_k)$, and for any constructible subset $\mathcal{D} \subset |\mathcal{Y}|$, let $\mathrm{e}(\mathcal{D}) \in \widehat{\mathscr{M}_k}$ denote the image along $K_0(\mathbf{Stack}_k) \to \widehat{\mathscr{M}_k}$ of the class of $\mathcal{D}$ in $K_0(\mathbf{Stack}_k)$. We will let $\mathbb{L}$ denote the class of $\mathbb{A}^1_k$ in $\widehat{\mathscr{M}_k}$. We now recall the definition of $\mu_{\mathcal{X}}$.

Definition 2.3. Let $\mathcal{C} \subset |\mathscr{L}(\mathcal{X})|$ be a cylinder. The *motivic measure* of $\mathcal{C}$ is

$$\mu_{\mathcal{X}}(\mathcal{C}) = \varinjlim_n \mathrm{e}(\theta_n(\mathcal{C}))\, \mathbb{L}^{-(n+1)\dim \mathcal{X}} \in \widehat{\mathscr{M}_k}.$$

Note that each $\theta_n(\mathcal{C}) \subset |\mathscr{L}_n(\mathcal{X})|$ is constructible and the above limit converges (in fact, the sequence stabilizes) by [SU22, Theorem 3.33].

Definition 2.4. Let $\mathcal{C} \subset |\mathscr{L}(\mathcal{X})|$ be a cylinder. A function $f\colon \mathcal{C} \to \mathbb{Z}$ is called *constructible* if for all $n \in \mathbb{Z}$, the set $f^{-1}(n) \subset |\mathscr{L}(\mathcal{X})|$ is a cylinder.

Remark 2.5. We use the terminology "constructible function" for consistency with [CNS18]. However, unlike in the case where $\mathcal{X}$ is a scheme, we do not know if cylinders coincide with constructible subsets of the topological space $|\mathscr{L}(\mathcal{X})|$ (using the general definition of constructible subset for not-necessarily noetherian spaces).

Remark 2.6. Let $\mathscr{I}$ be a quasi-coherent ideal sheaf on $\mathcal{X}$. We get a function $\mathrm{ord}_{\mathscr{I}}\colon |\mathscr{L}(\mathcal{X})| \to \mathbb{Z} \cup \{\infty\}$ in the usual manner: For any $\varphi\colon \mathrm{Spec}(k'[\![t]\!]) \to \mathcal{X}$, we set $\mathrm{ord}_{\mathscr{I}}(\varphi)$ to be such that $\mathscr{I} \cdot k'[\![t]\!] = (t^{\mathrm{ord}_{\mathscr{I}}(\varphi)})$, where (t^∞) means (0). Then just as in the case where $\mathcal{X}$ is a scheme, it is easy to verify that for any $n \in \mathbb{Z}$, the set $\mathrm{ord}_{\mathscr{I}}^{-1}(n)$ is a cylinder. Thus for any cylinder $\mathcal{C} \subset |\mathscr{L}(\mathcal{X})| \setminus |\mathscr{L}(\mathcal{Y})|$, where $\mathcal{Y}$ is the closed substack defined by $\mathscr{I}$, we have that $\mathrm{ord}_{\mathscr{I}}\colon \mathcal{C} \to \mathbb{Z}$ is constructible.

Remark 2.7. If $f, g\colon |\mathscr{L}(\mathcal{X})| \to \mathbb{Z}$ are constructible functions, then the function $f+g\colon |\mathscr{L}(\mathcal{X})| \to \mathbb{Z}$ is constructible by Proposition 2.2.

Definition 2.8. Let $\mathcal{C} \subset |\mathscr{L}(\mathcal{X})|$ be a cylinder, and let $f\colon \mathcal{C} \to \mathbb{Z}$ be a constructible function. We define the *integral* of $\mathbb{L}^f$ to be

$$\int_{\mathcal{C}} \mathbb{L}^f \mathrm{d}\mu_{\mathcal{X}} = \sum_{n \in \mathbb{Z}} \mu_{\mathcal{X}}(f^{-1}(n))\mathbb{L}^n ,$$

where we note that this sum is finite by Proposition 2.2.

3. Height functions for arcs and jets

In this section, fix a finite-type Artin stack $\mathcal{X}$ over k and an object $E \in D^-_{\mathrm{coh}}(\mathcal{X})$ in the derived category of $\mathcal{X}$.

Definition 3.1. For any $i \in \mathbb{Z}$, any field extension k' of k, and any arc $\varphi\colon \operatorname{Spec}(k'[\![t]\!]) \to \mathcal{X}$ (respectively, jet $\varphi_n\colon \operatorname{Spec}(k'[t]/(t^{n+1})) \to \mathcal{X}$), set

$$\operatorname{ht}^{(i)}_E(\varphi) = \dim_{k'} \mathbb{H}^i(L\varphi^* E) \quad \big(\text{respectively, } \operatorname{ht}^{(i)}_{n,E}(\varphi_n) = \dim_{k'} \mathbb{H}^i(L\varphi_n^* E)\big) .$$

The assignment $\varphi \mapsto \operatorname{ht}^{(i)}_E(\varphi)$ (respectively, $\varphi_n \mapsto \operatorname{ht}^{(i)}_{n,E}(\varphi_n)$) induces a function

$$\operatorname{ht}^{(i)}_E\colon |\mathscr{L}(\mathcal{X})| \longrightarrow \mathbb{Z}_{\geqslant 0} \cup \{\infty\} \quad \big(\text{respectively, } \operatorname{ht}^{(i)}_{n,E}\colon |\mathscr{L}_n(\mathcal{X})| \longrightarrow \mathbb{Z}_{\geqslant 0}\big)$$

that we refer to as the associated *height function*.

The main result of this section is the following theorem, which collects some important properties of these height functions.

Theorem 3.2. *Let $i \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geqslant 0}$.*

(a) *The function $\operatorname{ht}^{(i)}_{n,E}\colon |\mathscr{L}_n(\mathcal{X})| \to \mathbb{Z}_{\geqslant 0}$ is upper semi-continuous.*

(b) *For any $\varphi \in |\mathscr{L}(\mathcal{X})|$, we have*

$$\operatorname{ht}^{(i)}_E(\varphi) + \operatorname{ht}^{(i+1)}_E(\varphi) \geqslant \operatorname{ht}^{(i)}_{n,E}(\theta_n(\varphi)) ,$$

and if this inequality is strict, then

$$\operatorname{ht}^{(i)}_{n,E}(\theta_n(\varphi)) \geqslant n+1 \quad \text{or} \quad \operatorname{ht}^{(i+1)}_E(\varphi) = \infty .$$

In particular, if $n + 1 \geqslant \operatorname{ht}^{(i)}_E(\varphi) + \operatorname{ht}^{(i+1)}_E(\varphi)$, then

$$\operatorname{ht}^{(i)}_E(\varphi) + \operatorname{ht}^{(i+1)}_E(\varphi) = \operatorname{ht}^{(i)}_{n,E}(\theta_n(\varphi)) .$$

We prove Theorem 3.2 after a preliminary lemma.

Lemma 3.3. *Let k' be a field extension of k, let $R = k'[\![t]\!]$, and let M be a finitely generated R-module. Then for all $n \geqslant 0$, we have*

$$\dim_{k'}(M) \geqslant \dim_{k'}\big(M \otimes_R R/\big(t^{n+1}\big)\big) \geqslant \dim_{k'} \operatorname{Tor}^R_1\big(M, R/\big(t^{n+1}\big)\big) ,$$

and the second inequality is an equality if $\dim_{k'}(M) \neq \infty$. Furthermore, if

$$\begin{gathered}\dim_{k'}(M) > \dim_{k'}\big(M \otimes_R R/\big(t^{n+1}\big)\big) \\ \big(\text{respectively, } \dim_{k'}(M) > \dim_{k'} \operatorname{Tor}^R_1\big(M, R/\big(t^{n+1}\big)\big)\big) ,\end{gathered}$$

then

$$\dim_{k'}\big(M \otimes_R R/\big(t^{n+1}\big)\big) \geqslant n+1$$
$$\big(\textit{respectively,}\ \dim_{k'} \operatorname{Tor}_1^R\big(M, R/\big(t^{n+1}\big)\big) \geqslant n+1 \textit{ or } \dim_{k'} M = \infty\big)\,.$$

Proof. By the structure theorem for modules over a principal ideal domain (PID for short), $M \cong R^r \oplus \bigoplus_{i=1}^m R/(t^{e_i})$ for $e_i \geqslant 1$. Since $\operatorname{Tor}_1^R(R, R/(t^{n+1})) = 0$, we reduce immediately to the case where $M = R/(t^e)$ for some $e \geqslant 1$. Then from the short exact sequence

$$0 \longrightarrow R \xrightarrow{\cdot t^e} R \longrightarrow M \longrightarrow 0\,,$$

we obtain a long exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^R\big(M, R/\big(t^{n+1}\big)\big) \longrightarrow R/\big(t^{n+1}\big) \xrightarrow{\cdot t^e} R/\big(t^{n+1}\big) \longrightarrow M \otimes_R R/\big(t^{n+1}\big) \longrightarrow 0\,,$$

and so $\dim_k(M \otimes_R R/(t^{n+1})) = \dim_k \operatorname{Tor}_1^R(M, R/(t^{n+1}))$. Since $M \otimes_R R/(t^{n+1}) = R/(t^e, t^{n+1}) = R/(t^{\min(e,n+1)})$, we see that $\dim_{k'}(M) \geqslant \dim_{k'}(M \otimes_R R/(t^{n+1}))$, which finishes the proof of the first statement. Furthermore, if $e = \dim_{k'} M > \dim_{k'}(M \otimes_R R/(t^{n+1})) = \min(e, n+1)$, we have $\dim_{k'}(M \otimes_R R/(t^{n+1})) = \min(e, n+1) = n+1$, and the remainder of the lemma follows. □

Proof of Theorem 3.2. We first prove part (a). Let $V \to \mathscr{L}_n(\mathcal{X})$ be a finite-type smooth cover by a scheme, and consider the composition

$$\widetilde{\mathrm{ht}}\colon V \longrightarrow |\mathscr{L}_n(\mathcal{X})| \xrightarrow{\mathrm{ht}^{(i)}_{n,E}} \mathbb{Z}_{\geqslant 0}\,.$$

Because $V \to |\mathscr{L}_n(\mathcal{X})|$ is surjective and open, to show part (a) it is sufficient to prove that $\widetilde{\mathrm{ht}}$ is upper semi-continuous. The map $V \to \mathscr{L}_n(\mathcal{X})$ corresponds to a map

$$\Phi\colon V \times_k \operatorname{Spec}\big(k[t]/\big(t^{n+1}\big)\big) \longrightarrow \mathcal{X}\,.$$

Let $\pi\colon V \times_k \operatorname{Spec}(k[t]/(t^{n+1})) \to V$ be the projection, and for any $v \in V$, let Y_v be the fiber of π over v, and let $\iota_v\colon Y_v \to V \times_k \operatorname{Spec}(k[t]/(t^{n+1}))$ be the inclusion. Then for all $v \in V$,

$$\widetilde{\mathrm{ht}}(v) = \dim_{k(v)} \mathbb{H}^i(L\iota_v^* L\Phi^* E)\,,$$

where $k(v)$ is the residue field of v. Set $F = R\pi_* L\Phi^* E$. Because π is flat and proper, cohomology and base change (see, for example, [Sta21, Lemma 0CSC]) imply that F is pseudo-coherent (that is, $F \in D^-_{\mathrm{coh}}(V)$) and that for all $v \in V$, we have

$$\mathbb{H}^i(L\iota_v^* L\Phi^* E) \cong H^i\big(F \otimes^L_{\mathcal{O}_V} k(v)\big)\,.$$

Part (a) then follows from the fact that $v \mapsto \dim_{k(v)} H^i(F \otimes^L_{\mathcal{O}_V} k(v))$ is upper semi-continuous; see, for example, [Sta21, Lemma 0BDI].

We now prove part (b). Let $R = k'[\![t]\!]$, where k' is a field extension of k over which φ is defined. We have a spectral sequence

$$E_2^{pq} = \operatorname{Tor}^R_{-p}\big(L^q\varphi^* E, R/\big(t^{n+1}\big)\big) \Longrightarrow L^{p+q}\varphi_n^* E\,;$$

see, for example, [Huy06, Equation (3.10)]. Since R is a PID, all E_2^{pq} terms vanish except those for $p \in \{-1, 0\}$. Thus,

$$\begin{aligned}\mathrm{ht}^{(i)}_{n,E}(\theta_n(\varphi)) &= \dim_{k'} L^i\varphi_n^* E\\ &= \dim_{k'}\big(L^i\varphi^* E \otimes_R R/\big(t^{n+1}\big)\big) + \dim_{k'}\big(\operatorname{Tor}_1^R\big(L^{i+1}\varphi^* E, R/\big(t^{n+1}\big)\big)\big)\,,\end{aligned}$$

and we are done by Lemma 3.3. □

The following corollary and remark allow us to obtain constructible functions from our height functions.

COROLLARY 3.4. *Let $i \in \mathbb{Z}$, let $f\colon |\mathscr{L}(\mathcal{X})| \to \mathbb{Z} \cup \{\infty\}$ be the sum of height functions $\mathrm{ht}_E^{(i)} + \mathrm{ht}_E^{(i+1)}$, and let $\mathcal{C} \subset |\mathscr{L}(\mathcal{X})|$ be a cylinder such that $\mathcal{C} \cap (\mathrm{ht}_E^{(i+1)})^{-1}(\infty) = \emptyset$. Then for any $n \in \mathbb{Z}$, the set $f^{-1}(n) \cap \mathcal{C}$ is a cylinder.*

Proof. By Theorem 3.2(a), the set $(\mathrm{ht}_{n-1,E}^{(i)} \circ \theta_{n-1})^{-1}(\mathbb{Z}_{\geqslant n}) \subset |\mathscr{L}(\mathcal{X})|$ is a cylinder, so it is sufficient to prove that

$$f^{-1}(\mathbb{Z}_{\geqslant n}) \cap \mathcal{C} = \left(\mathrm{ht}_{n-1,E}^{(i)} \circ \theta_{n-1}\right)^{-1}(\mathbb{Z}_{\geqslant n}) \cap \mathcal{C}\,,$$

but this follows easily from Theorem 3.2(b). □

Remark 3.5. If $\mathcal{U}$ is an open substack of $\mathcal{X}$ such that E restricts to 0 on $\mathcal{U}$, then each $\mathrm{ht}_E^{(i)}$ is never ∞ on $|\mathscr{L}(\mathcal{X})| \setminus |\mathscr{L}(\mathcal{X} \setminus \mathcal{U})|$. Thus Corollary 3.4 implies that $\mathrm{ht}_E^{(i)} + \mathrm{ht}_E^{(i+1)}\colon \mathcal{C} \to \mathbb{Z}$ is a constructible function for any cylinder $\mathcal{C} \subset |\mathscr{L}(\mathcal{X})| \setminus |\mathscr{L}(\mathcal{X} \setminus \mathcal{U})|$.

We may now prove Theorem 1.2(a).

Proof of Theorem 1.2(a). In the notation of Theorem 1.2, the complexes $L_{\mathcal{X}/Y}$ and $L\sigma^* L_{\mathcal{I}_\mathcal{X}/\mathcal{X}}$ live in $D^-_{\mathrm{coh}}(\mathcal{X})$. By Remark 2.7, it suffices to show that $\mathrm{ht}^{(0)}_{L_{\mathcal{X}/Y}} + \mathrm{ht}^{(1)}_{L_{\mathcal{X}/Y}}$, $\mathrm{ht}^{(1)}_{L_{\mathcal{X}/Y}}$, and $\mathrm{ht}^{(0)}_{L\sigma^* L_{\mathcal{I}_\mathcal{X}/\mathcal{X}}}$ are constructible functions on $\mathcal{C}$ since then

$$\mathrm{ht}^{(0)}_{L\sigma^* L_{\mathcal{I}_\mathcal{X}/\mathcal{X}}} + \mathrm{ht}^{(1)}_{L_{\mathcal{X}/Y}} - \left(\mathrm{ht}^{(0)}_{L_{\mathcal{X}/Y}} + \mathrm{ht}^{(1)}_{L_{\mathcal{X}/Y}}\right)$$

is constructible.

First observe that the restriction of $L_{\mathcal{X}/Y}$ to $\mathcal{U}$ is given by $L_{\mathcal{U}/Y} = 0$ since $\mathcal{U} \to Y$ is an open immersion. Next, we have a Cartesian square

$$\begin{tikzcd}
\mathcal{U} \arrow[r] \arrow[d, "\sigma_\mathcal{U}"'] & \mathcal{X} \arrow[d, "\sigma"] \\
\mathcal{I}_\mathcal{U} \arrow[r] \arrow[d] & \mathcal{I}_\mathcal{X} \arrow[d] \\
\mathcal{U} \arrow[r] & \mathcal{X}\,,
\end{tikzcd}$$

from which we see that $L\sigma^* L_{\mathcal{I}_\mathcal{X}/\mathcal{X}}$ restricts over $\mathcal{U}$ to be $L\sigma_\mathcal{U}^* L_{\mathcal{I}_\mathcal{U}/\mathcal{U}}$. Since $\mathcal{U} \to Y$ is an open immersion, $\mathcal{U}$ is a scheme; hence $I_\mathcal{U} \to \mathcal{U}$ is an isomorphism. In particular, $L_{\mathcal{I}_\mathcal{U}/\mathcal{U}} = 0$. Thus, Remark 3.5 implies that $\mathrm{ht}^{(i)}_{L_{\mathcal{X}/Y}} + \mathrm{ht}^{(i+1)}_{L_{\mathcal{X}/Y}}$ and $\mathrm{ht}^{(i)}_{L\sigma^* L_{\mathcal{I}_\mathcal{X}/\mathcal{X}}} + \mathrm{ht}^{(i+1)}_{L\sigma^* L_{\mathcal{I}_\mathcal{X}/\mathcal{X}}}$ are constructible functions on $\mathcal{C}$. The result follows by taking $i \in \{0, 1\}$ after we prove that $\mathrm{ht}^{(2)}_{L_{\mathcal{X}/Y}}$ and $\mathrm{ht}^{(1)}_{L\sigma^* L_{\mathcal{I}_\mathcal{X}/\mathcal{X}}}$ vanish.

These vanishing results are general facts. Since the cotangent complex of any morphism of Artin stacks is concentrated in degree at most 1, we have $H^i L_{\mathcal{X}/Y} = 0$. Since $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is representable, $H^i L\sigma^* L_{\mathcal{I}_\mathcal{X}/\mathcal{X}} = 0$. From the definitions, we then see that $\mathrm{ht}^{(i)}_{L_{\mathcal{X}/Y}}$ is identically 0 for $i > 1$ and $\mathrm{ht}^{(i)}_{L\sigma^* L_{\mathcal{I}_\mathcal{X}/\mathcal{X}}}$ is identically 0 for $i > 0$. □

We end this section by proving the following lemma, which will allow us to relate our height functions to the various deformation spaces that will appear throughout the proof of Theorem 1.2(b).

Lemma 3.6. *Let $i \in \mathbb{Z}$, let $n \geqslant m \in \mathbb{Z}_{\geqslant 0}$, let k' be a field, let $\iota\colon \operatorname{Spec}(k'[t]/(t^m)) \to \operatorname{Spec}(k'[t]/(t^n))$ be the truncation map, and let F be a pseudo-coherent object in the derived category of $\operatorname{Spec}(k'[t]/(t^n))$. Then*

$$\dim_{k'} \operatorname{Ext}^i_{k'[t]/(t^n)}\big(F, (t^n)/(t^{n+m})\big) = \dim_{k'} \mathbb{H}^{-i}(L\iota^* F)\,.$$

Proof. Let $F^\bullet$ be a complex of projectives representing the object F. Then

$$\begin{aligned}\operatorname{Ext}^i_{k'[t]/(t^n)}\big(F,(t^n)/(t^{n+m})\big) &\cong H^i \operatorname{Hom}_{k'[t]/(t^n)}\big(F^\bullet, (t^n)/(t^{n+m})\big)\\ &\cong H^i \operatorname{Hom}_{k'[t]/(t^n)}\big(F^\bullet, k'[t]/(t^m)\big)\\ &\cong H^i \operatorname{Hom}_{k'[t]/(t^m)}\big(F^\bullet \otimes_{k'[t]/(t^n)} k'[t]/(t^m), k'[t]/(t^m)\big)\,,\end{aligned}$$

where the second isomorphism is due to the fact that $k'[t]/(t^m) \cong (t^n)/(t^{n+m})$ as $k'[t]/(t^n)$-modules. Baer's criterion and the fact that $k'[\![t]\!]$ is a principal ideal domain imply that $k'[t]/(t^m)$ is an injective $k'[t]/(t^m)$-module, so

$$\begin{aligned}&H^i \operatorname{Hom}_{k'[t]/(t^m)}\big(F^\bullet \otimes_{k'[t]/(t^n)} k'[t]/(t^m), k'[t]/(t^m)\big)\\ &\qquad\cong \operatorname{Hom}_{k'[t]/(t^m)}\big(H^{-i}\big(F^\bullet \otimes_{k'[t]/(t^n)} k'[t]/(t^m)\big), k'[t]/(t^m)\big)\\ &\qquad\cong H^{-i}\big(F^\bullet \otimes_{k'[t]/(t^n)} k'[t]/(t^m)\big)\\ &\qquad\cong L^{-i}\iota^* F\,,\end{aligned}$$

where the second isomorphism is due to the fact that $\operatorname{Hom}_{k'[t]/(t^m)}(M, k'[t]/(t^m)) \cong M$ for any finitely generated $k'[t]/(t^m)$-module M as a straightforward consequence of the structure theorem for finitely generated modules over $k'[\![t]\!]$. The lemma follows from the above isomorphisms and the fact that $\operatorname{Spec}(k'[t]/(t^m))$ is affine. □

4. Stabilizers of jet stacks

Throughout this section, let $\mathcal{X}$ be a finite-type Artin stack over k such that the inertia map $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is separated, and let $\sigma\colon \mathcal{X} \to \mathcal{I}_\mathcal{X}$ denote the identity section of $\mathcal{I}_\mathcal{X} \to \mathcal{X}$. We will use the fact that since $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is representable and separated, σ is a closed immersion.

Theorem 4.1. *Let $\mathscr{J}$ denote the ideal sheaf on $\mathcal{I}_\mathcal{X}$ defining the closed substack $\mathcal{X} \xrightarrow{\sigma} \mathcal{I}_\mathcal{X}$.*

(a) *If $\mathcal{U}$ is an open substack of $\mathcal{X}$ that is isomorphic to an algebraic space, then there exists a quasi-coherent ideal sheaf $\mathscr{I}$ on $\mathcal{X}$ such that*
  - *$\mathscr{I}\mathcal{O}_{\mathcal{I}_\mathcal{X}} \cdot \mathscr{J} = 0$ and*
  - *the closed substack of $\mathcal{X}$ defined by $\mathscr{I}$ has underlying topological space equal to $|\mathcal{X}| \setminus |\mathcal{U}|$.*

(b) *Let $\mathscr{I}$ be a quasi-coherent ideal sheaf on $\mathcal{X}$ such that $\mathscr{I}\mathcal{O}_{\mathcal{I}_\mathcal{X}} \cdot \mathscr{J} = 0$, let k' be a field extension of k, and let $\varphi \in \mathscr{L}(\mathcal{X})(k')$. Then for all*

$$n \geqslant \max\big(\operatorname{ord}_\mathscr{I}(\varphi), 2\max\big(\operatorname{ord}_\mathscr{I}(\varphi), \operatorname{ht}^{(0)}_{L\sigma^* L_{\mathcal{I}/\mathcal{X}}}(\varphi)\big) - 1\big)\,,$$

the stabilizer of $\theta_n(\varphi) \in \mathscr{L}_n(\mathcal{X})(k')$ is isomorphic as an algebraic group to $\mathbb{G}_{a,k'}^{\operatorname{ht}^{(0)}_{L\sigma^* L_{\mathcal{I}_\mathcal{X}/\mathcal{X}}}(\varphi)}$.

We begin with a lemma that will be used to prove Theorem 4.1(a).

Lemma 4.2. *Let $\mathcal{Y}$ be a noetherian Artin stack, and let $\mathcal{Z}, \mathcal{W}$ be closed substacks of $\mathcal{Y}$. Let $\mathscr{I}$ and $\mathscr{J}$ be the ideal sheaves on $\mathcal{Y}$ defining $\mathcal{Z}$ and $\mathcal{W}$, respectively, and let $\mathcal{U} \hookrightarrow \mathcal{Y}$ be the open substack such that $|\mathcal{U}| = |\mathcal{Y}| \setminus |\mathcal{Z}|$. If $\mathcal{W} \times_\mathcal{Y} \mathcal{U} \to \mathcal{U}$ is an isomorphism, then there exists some $r \in \mathbb{Z}_{>0}$ such that $\mathscr{I}^r \cdot \mathscr{J} = 0$.*

Proof. For any $r \in \mathbb{Z}_{>0}$, let $\mathcal{V}_r \hookrightarrow \mathcal{Y}$ denote the closed substack defined by $\mathscr{I}^r \cdot \mathscr{J}$. It is sufficient to prove that there exists some $r \in \mathbb{Z}_{>0}$ such that $\mathcal{V}_r \hookrightarrow \mathcal{Y}$ is an isomorphism. Let $\operatorname{Spec}(A) \to \mathcal{Y}$ be a smooth cover. Then it is sufficient to show that there exists some $r \in \mathbb{Z}_{>0}$ such that $\mathcal{V}_r \times_{\mathcal{Y}} \operatorname{Spec}(A) \hookrightarrow \operatorname{Spec}(A)$ is an isomorphism. Let $I, J \subset A$ be the global sections of the ideal sheaves $\mathscr{I}\mathcal{O}_{\operatorname{Spec}(A)}$ and $\mathscr{J}\mathcal{O}_{\operatorname{Spec}(A)}$, respectively. Because $\mathcal{V}_r \times_{\mathcal{Y}} \operatorname{Spec}(A) \hookrightarrow \operatorname{Spec}(A)$ is the closed subscheme defined by $I^r \cdot J$, it is sufficient to prove that there exists some $r \in \mathbb{Z}_{>0}$ such that $I^r \cdot J = 0$. Because A is noetherian, it is sufficient to show that for any $f \in I$, every element of J is f-torsion.

Let $f \in I$, and let A_f be the ring obtained from A by inverting f. Then the open subscheme $\operatorname{Spec}(A_f) \hookrightarrow \operatorname{Spec}(A)$ is contained in the open subscheme $\mathcal{U} \times_{\mathcal{Y}} \operatorname{Spec}(A) \hookrightarrow \operatorname{Spec}(A)$. Set $B = A_f \otimes_A (A/J)$. Then $\operatorname{Spec}(B) \to \operatorname{Spec}(A_f)$ is the base change of $\mathcal{W} \to \mathcal{Y}$ along the composition $\operatorname{Spec}(A_f) \hookrightarrow \mathcal{U} \times_{\mathcal{Y}} \operatorname{Spec}(A) \to \mathcal{U} \hookrightarrow \mathcal{Y}$, so $A_f \to B$ is an isomorphism. Therefore,

$$J = \ker(A \to A/J) \subset \ker(A \to B) = \ker(A \to A_f)\,,$$

so every element of J is f-torsion. Recall that at the end of the previous paragraph, we showed that this is sufficient to complete the proof. $\square$

We now prove Theorem 4.1(a).

Proof of Theorem 4.1(a). Let $\mathscr{I}'$ be the ideal sheaf on $\mathcal{X}$ that defines the reduced substack with underlying topological space equal to $|\mathcal{X}| \setminus |\mathcal{U}|$, and let $\mathcal{Z} \hookrightarrow \mathcal{I}_{\mathcal{X}}$ be the closed substack defined by $\mathscr{I}'\mathcal{O}_{\mathcal{I}_{\mathcal{X}}}$. Then $|\mathcal{I}_{\mathcal{U}}| = |\mathcal{I}_{\mathcal{X}}| \setminus |\mathcal{Z}|$. Let $\mathcal{U} \to \mathcal{I}_{\mathcal{U}} \to \mathcal{U}$ be the base change of $\mathcal{X} \xrightarrow{\sigma} \mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ along $\mathcal{U} \hookrightarrow \mathcal{X}$. Then $\mathcal{U} \to \mathcal{I}_{\mathcal{U}}$ is a section of the isomorphism $\mathcal{I}_{\mathcal{U}} \xrightarrow{\sim} \mathcal{U}$ (this is an isomorphism because $\mathcal{U}$ is an algebraic space), so $\mathcal{U} \to \mathcal{I}_{\mathcal{U}}$ is an isomorphism. Thus $\mathcal{X} \xrightarrow{\sigma} \mathcal{I}_{\mathcal{X}}$ is a closed immersion such that $\mathcal{X} \times_{\mathcal{I}_{\mathcal{X}}} \mathcal{I}_{\mathcal{U}} \to \mathcal{I}_{\mathcal{U}}$ is an isomorphism. Therefore, Lemma 4.2 implies that there exists some $r \in \mathbb{Z}_{>0}$ such that $(\mathscr{I}'\mathcal{O}_{\mathcal{I}_{\mathcal{X}}})^r \cdot \mathscr{J} = 0$. Setting $\mathscr{I} = (\mathscr{I}')^r$, we get $\mathscr{I}\mathcal{O}_{\mathcal{I}_{\mathcal{X}}} \cdot \mathscr{J} = 0$, and the closed substack of $\mathcal{X}$ defined by $\mathscr{I}$ has underlying topological space equal to $|\mathcal{X}| \setminus |\mathcal{U}|$. $\square$

The next lemmas will be used in proving Theorem 4.1(b).

LEMMA 4.3. *Let k' be a field, let $n \in \mathbb{Z}_{\geqslant 0}$, let A be a flat $k'[t]/(t^{n+1})$ algebra, and let $a \in A$. If $m \leqslant n$ is such that $t^m a = 0$, then $a \in (t^{n-m+1})A$.*

Proof. Consider the exact sequence

$$0 \longrightarrow \big(t^{n-m+1}\big)/\big(t^{n+1}\big) \longrightarrow k'[t]/\big(t^{n+1}\big) \xrightarrow{\cdot t^m} k'[t]/\big(t^{n+1}\big)\,.$$

Tensoring with the flat algebra A, we get the sequence

$$0 \longrightarrow \big(t^{n-m+1}\big)/\big(t^{n+1}\big) \otimes_{k'[t]/(t^{n+1})} A \longrightarrow A \xrightarrow{\cdot t^m} A\,,$$

whose exactness implies the lemma. $\square$

LEMMA 4.4. *Let k' be a field, let Y be a scheme over $k'[\![t]\!]$, let $W \hookrightarrow Y$ be a closed subscheme, let $n \in \mathbb{Z}_{\geqslant 0}$, let Z be a flat scheme over $k'[t]/(t^{n+1})$, and let $Z \to Y$ be a morphism over $k'[\![t]\!]$. Let $\mathscr{J}$ be the ideal sheaf on Y that defines W. If $m \leqslant n$ is such that $(t^m)\mathcal{O}_Y \cdot \mathscr{J} = 0$, then $Z \times_{\operatorname{Spec}(k'[t]/(t^{n+1}))} \operatorname{Spec}(k'[t]/(t^{n-m+1})) \to Z \to Y$ factors through $W \hookrightarrow Y$.*

Proof. It is sufficient to prove that if $\operatorname{Spec}(B) \subset Y$ is an affine open and $\operatorname{Spec}(A) \subset Z$ is an affine open mapped into $\operatorname{Spec}(B)$ by $Z \to Y$, then $\operatorname{Spec}(A/(t^{n-m+1})) \to \operatorname{Spec}(A) \to \operatorname{Spec}(B)$ factors through $W \times_Y \operatorname{Spec}(B) \hookrightarrow \operatorname{Spec}(B)$.

Let $\operatorname{Spec}(B) \subset Y$ be an affine open, let $\operatorname{Spec}(A) \subset Z$ be an affine open mapped into $\operatorname{Spec}(B)$ by $Z \to Y$, and let $J \subset B$ be the set of global sections of $\mathscr{J}\mathcal{O}_{\operatorname{Spec}(B)}$, so the closed immersion $W \times_Y \operatorname{Spec}(B) \hookrightarrow \operatorname{Spec}(B)$ is defined by the ideal J, and $(t^m)J = 0$. Let $b \in J$, and let $a \in A$ be the image of b under $B \to A$. Then $t^m a = 0$, so by Lemma 4.3, we have $a \in (t^{n-m+1})A$. Thus $B \to A/(t^{n-m+1})$ factors through $B \to B/J$, and we are done. $\square$

For the remainder of this section, let $\mathscr{J}$ denote the ideal sheaf on $\mathcal{I}_{\mathcal{X}}$ defining the closed substack $\mathcal{X} \xrightarrow{\sigma} \mathcal{I}_{\mathcal{X}}$. We will eventually prove Theorem 4.1(b) by expressing the relevant stabilizers as certain deformation spaces.

The following proposition is key to setting up the appropriate deformation problem. We prove that for any arc $\varphi\colon \operatorname{Spec}(k'[[t]]) \to \mathcal{X}$ and any A-valued automorphism g_n of the truncation φ_n, after further truncation by a controlled amount, g_n reduces to the identity automorphism. As a result, automorphisms of φ_n come from deformations of the identity automorphism and can therefore be understood in terms of $L\sigma^* L_{\mathcal{I}_{\mathcal{X}}/\mathcal{X}}$.

Proposition 4.5. *Let k' be a field extension of k, let $\varphi\colon \operatorname{Spec}(k'[\![t]\!]) \to \mathcal{X}$ be a morphism over k, let $n \in \mathbb{Z}_{\geqslant 0}$, let A be a k'-algebra, and let $g_n\colon \operatorname{Spec}(A[t]/(t^{n+1})) \to \mathcal{I}_{\mathcal{X}}$ make the diagram*

$$\begin{tikzcd} \operatorname{Spec}\big(A[t]/\big(t^{n+1}\big)\big) \arrow[r, "g_n"] \arrow[d] & \mathcal{I}_{\mathcal{X}} \arrow[d] \\ \operatorname{Spec}(k'[\![t]\!]) \arrow[r, "\varphi"] & \mathcal{X} \end{tikzcd}$$

2-commute. If $\mathscr{I}$ is an ideal sheaf on $\mathcal{X}$ such that $\mathscr{I}\mathcal{O}_{\mathcal{I}_{\mathcal{X}}} \cdot \mathscr{J} = 0$ and $n \geqslant \operatorname{ord}_{\mathscr{I}}(\varphi)$, then the following diagram 2-commutes:

$$\begin{tikzcd} & \operatorname{Spec}(k'[\![t]\!]) \arrow[r, "\varphi"] & \mathcal{X} \arrow[d, "\sigma"] \\ \operatorname{Spec}\big(A[t]/\big(t^{n-\operatorname{ord}_{\mathscr{I}}(\varphi)+1}\big)\big) \arrow[ur] \arrow[r] & \operatorname{Spec}\big(A[t]/\big(t^{n+1}\big)\big) \arrow[r, "g_n"] & \mathcal{I}_{\mathcal{X}}\,. \end{tikzcd}$$

Proof. Let $\operatorname{Spec}(k'[\![t]\!]) \xrightarrow{s} G \to \operatorname{Spec}(k'[\![t]\!])$ be the base change of $\mathcal{X} \xrightarrow{\sigma} \mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ along $\varphi\colon \operatorname{Spec}(k'[\![t]\!]) \to \mathcal{X}$. By choosing a 2-isomorphism that makes

$$\begin{tikzcd} \operatorname{Spec}\big(A[t]/\big(t^{n+1}\big)\big) \arrow[r, "g_n"] \arrow[d] & \mathcal{I}_{\mathcal{X}} \arrow[d] \\ \operatorname{Spec}(k'[\![t]\!]) \arrow[r, "\varphi"] & \mathcal{X} \end{tikzcd}$$

2-commute, we obtain a map $h_n\colon \operatorname{Spec}(A[t]/(t^{n+1})) \to G$ making

$$\begin{tikzcd} \operatorname{Spec}\big(A[t]/\big(t^{n+1}\big)\big) \arrow[dr, "h_n"] \arrow[drr, bend left, "g_n"] \arrow[ddr, bend right] & & \\ & G \arrow[r] \arrow[d] & \mathcal{I}_{\mathcal{X}} \\ & \operatorname{Spec}(k'[\![t]\!]) & \end{tikzcd}$$

2-commute. Set $m = \operatorname{ord}_{\mathscr{I}}(\varphi)$. Let $h_{n-m}\colon \operatorname{Spec}(A[t]/(t^{n-m+1})) \to G$ be the composition of $\operatorname{Spec}(A[t]/(t^{n-m+1})) \hookrightarrow \operatorname{Spec}(A[t]/(t^{n+1}))$ with $h_n\colon \operatorname{Spec}(A[t]/(t^{n+1})) \to G$. It is sufficient to prove that the map h_{n-m} factors through the closed immersion $\operatorname{Spec}(k'[\![t]\!]) \xrightarrow{s} G$.

The closed immersion $\operatorname{Spec}(k'[\![t]\!]) \xrightarrow{s} G$ is defined by the ideal sheaf $\mathscr{J}\mathcal{O}_G$, and

$$(t^m)\mathcal{O}_G \cdot \mathscr{J}\mathcal{O}_G = (\mathscr{I}\mathcal{O}_{\operatorname{Spec}(k'[\![t]\!])})\mathcal{O}_G \cdot \mathscr{J}\mathcal{O}_G = (\mathscr{I}\mathcal{O}_{\mathcal{I}_\mathcal{X}} \cdot \mathscr{J})\mathcal{O}_G = 0\,.$$

Therefore, Lemma 4.4 implies that h_{n-m} factors through the closed immersion $\operatorname{Spec}(k'[\![t]\!]) \xrightarrow{s} G$, as desired. $\square$

Before completing the proof of Theorem 4.1(b), we need the following lemma. We expect this to be well known, but we include a proof for lack of a suitable reference.

Lemma 4.6. *Let S be a scheme and G a (not-necessarily flat) group algebraic space over S with multiplication map $m\colon G \times_S G \to G$ and identity section $e\colon S \to G$. Then for all $\mathcal{O}_S$-modules $\mathcal{F}$, the map*

$$\operatorname{Hom}\big(e^*\Omega^1_{G/S}, \mathcal{F}\big) \times \operatorname{Hom}\big(e^*\Omega^1_{G/S}, \mathcal{F}\big) \longrightarrow \operatorname{Hom}\big(e^*\Omega^1_{G/S}, \mathcal{F}\big)$$

induced by m is given by addition of functions $(\varphi_1, \varphi_2) \mapsto \varphi_1 + \varphi_2$.

Proof. We have an isomorphism

$$(e \times e)^*\Omega^1_{(G\times_S G)/S} \simeq e^*\Omega^1_{G/S} \oplus e^*\Omega^1_{G/S}\,,$$

where the two projection maps $\pi_j\colon (e\times e)^*\Omega^1_{(G\times_S G)/S} \to e^*\Omega^1_{G/S}$ can be described as follows. Let $i_1, i_2\colon G \to G\times_S G$ be given by $i_1(g) = (g, e)$ and $i_2(g) = (e, g)$. Since $e \times e = i_j \circ e$ for $j \in \{1, 2\}$, applying e^* to the morphisms $i_j^*\Omega^1_{G/S} \to \Omega^1_{(G\times_S G)/S}$ yields two maps $(e\times e)^*\Omega^1_{(G\times_S G)/S} \to e^*\Omega^1_{G/S}$, which are precisely the maps π_j.

Next, m yields a map $\mu\colon m^*\Omega^1_{G/S} \to \Omega^1_{(G\times_S G)/S}$. Since $m \circ i_j = \mathrm{id}$, we have $\pi_j \circ (e\times e)^*\mu = \mathrm{id}$. In other words, the map

$$e^*\Omega^1_{G/S} \xrightarrow{(e\times e)^*\mu} (e\times e)^*\Omega^1_{(G\times_S G)/S} \simeq e^*\Omega^1_{G/S} \oplus e^*\Omega^1_{G/S}$$

induced by m, is given by $\omega \mapsto (\omega, \omega)$, from which the lemma follows. $\square$

We end this section by proving Theorem 4.1(b).

Proof of Theorem 4.1(b). Set $r = \max(\operatorname{ord}_{\mathscr{I}}(\varphi), \operatorname{ht}^{(0)}_{L\sigma^* L_{\mathcal{I}/\mathcal{X}}}(\varphi))$. We have $n \geqslant \operatorname{ord}_{\mathscr{I}}(\varphi)$ and $r \geqslant \operatorname{ord}_{\mathscr{I}}(\varphi)$, so Proposition 4.5 implies that for every k'-algebra A, the A-valued points of the stabilizer of $\theta_n(\varphi) \in \mathscr{L}_n(\mathcal{X})(k')$ are given by deformations $\operatorname{Spec}(A[t]/(t^{n+1})) \to \mathcal{I}_\mathcal{X}$ (up to 2-isomorphism) over $\mathcal{X}$ of the composition

$$\operatorname{Spec}\big(A[t]/\big(t^{n-r+1}\big)\big) \longrightarrow \operatorname{Spec}\big(k'[t]/\big(t^{n-r+1}\big)\big) \xrightarrow{\varphi_{n-r}} \mathcal{X} \xrightarrow{\sigma} \mathcal{I}_\mathcal{X}\,,$$

where φ_{n-r} is the jet associated with $\theta_{n-r}(\varphi) \in \mathscr{L}_{n-r}(\mathcal{X})(k')$. Because $n \geqslant 2r-1$, the closed immersion $\operatorname{Spec}(A[t]/(t^{n-r+1})) \hookrightarrow \operatorname{Spec}(A[t]/(t^{n+1}))$ is defined by a square zero ideal. Therefore, the A-valued points of the stabilizer of $\theta_n(\varphi)$ are naturally in bijection with

$$\operatorname{Ext}^0\big(L\varphi_{n-r}^* L\sigma^* L_{\mathcal{I}_\mathcal{X}/\mathcal{X}}, \big(t^{n+1}\big)/\big(t^{n-r+1}\big)\big) \otimes_{k'} A\,.$$

This bijection is compatible with the group structures by Lemma 4.6, so the stabilizer of $\theta_n(\varphi)$ is isomorphic as an algebraic group to

$$\mathbb{G}_{a,k'}^{\dim_{k'} \operatorname{Ext}^0(L\varphi_{n-r}^* L\sigma^* L_{\mathcal{I}_\mathcal{X}/\mathcal{X}}, (t^{n+1})/(t^{n-r+1}))}\,.$$

Therefore, it is now sufficient to prove that

$$\dim_{k'} \operatorname{Ext}^0\big(L\varphi_{n-r}^* L\sigma^* L_{\mathcal{I}_{\mathcal{X}}/\mathcal{X}}, (t^{n+1})/(t^{n-r+1})\big) = \operatorname{ht}^{(0)}_{L\sigma^* L_{\mathcal{I}_{\mathcal{X}}/\mathcal{X}}}(\varphi)\,.$$

Because $n+1 \geqslant r$, Lemma 3.6 implies that

$$\dim_{k'} \operatorname{Ext}^0\big(L\varphi_{n-r}^* L\sigma^* L_{\mathcal{I}_{\mathcal{X}}/\mathcal{X}}, (t^{n+1})/(t^{n-r+1})\big) = \operatorname{ht}^{(0)}_{r-1, L\sigma^* L_{\mathcal{I}_{\mathcal{X}}/\mathcal{X}}}(\varphi)\,.$$

Note that since $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is representable, $\operatorname{ht}^{(1)}_{L\sigma^* L_{\mathcal{I}_{\mathcal{X}}/\mathcal{X}}}(\varphi) = 0$. Thus because $r \geqslant \operatorname{ht}^{(0)}_{L\sigma^* L_{\mathcal{I}_{\mathcal{X}}/\mathcal{X}}}(\varphi)$, Theorem 3.2(b) implies that $\operatorname{ht}^{(0)}_{r-1, L\sigma^* L_{\mathcal{I}_{\mathcal{X}}/\mathcal{X}}}(\varphi) = \operatorname{ht}^{(0)}_{L\sigma^* L_{\mathcal{I}_{\mathcal{X}}/\mathcal{X}}}(\varphi)$, completing the proof. $\square$

5. A technical theorem

The main result in this section is Theorem 5.1, which is a key technical lemma in the proof of Theorem 1.2(b).

Theorem 5.1. *Let $\mathcal{X}$ be a smooth irreducible finite-type Artin stack over k, let Y be an irreducible finite-type scheme over k, let $\pi\colon \mathcal{X} \to Y$ be a morphism, let $\mathcal{U}$ be an open substack of $\mathcal{X}$ such that $\mathcal{U} \to \mathcal{X} \to Y$ is an open immersion, and let $\mathcal{C} \subset |\mathscr{L}(\mathcal{X})| \setminus |\mathscr{L}(\mathcal{X} \setminus \mathcal{U})| \subset \mathscr{L}(\mathcal{X})$ be a cylinder. Let $h, e' \in \mathbb{Z}_{\geqslant 0}$, let $\mathscr{J}_Y$ be the $(\dim Y)$th Fitting ideal of $\Omega^1_{Y/k}$, and assume that $\operatorname{ht}^{(0)}_{L_{\mathcal{X}/Y}}$ and $\operatorname{ord}_{\mathscr{J}_Y} \circ \mathscr{L}(\pi)$ are at most h and e', respectively, on $\mathcal{C}$. Let $\ell \in \mathbb{Z}_{\geqslant 0}$ be such that $\mathcal{C}$ is the preimage along θ_ℓ of a constructible subset of $\mathscr{L}_\ell(\mathcal{X})$. Assume that the map $\overline{\mathcal{C}}(k') \to \mathscr{L}(Y)(k')$ is injective for all field extensions k' of k.*

Let k' be a field extension of k, let $n \in \mathbb{Z}_{\geqslant 0}$, let $\varphi, \varphi' \in \mathcal{C}(k')$, and assume $\theta_n(\mathscr{L}(\pi)(\varphi)) = \theta_n(\mathscr{L}(\pi)(\varphi'))$. If $n \geqslant \max(\ell+h, e')$, then $\theta_{n-e}(\varphi) \cong \theta_{n-e}(\varphi')$ for any $e \in \mathbb{Z}_{\geqslant 0}$ satisfying $n \geqslant e \geqslant h$.

The proof of Theorem 5.1 will occupy the entirety of this section. We claim that it suffices to prove the following.

Proposition 5.2. *We use the notation and hypotheses of the first paragraph of Theorem 5.1 and also assume $n \geqslant e \geqslant h$. Let k' be a field extension of k, let $n \in \mathbb{Z}_{\geqslant 0}$, let $\varphi \in \mathcal{C}(k')$, let $\alpha = \pi \circ \varphi$, and let $\mathfrak{m} = (t) \subset k'[\![t]\!]$. Suppose $n \geqslant \max(h, e')$. Then if $\Phi \in \operatorname{Hom}(\alpha^*\Omega^1_{Y/k}, \mathfrak{m}^{n+1}/\mathfrak{m}^{2(n+1)})$ and $\phi \in \operatorname{Hom}(\alpha^*\Omega^1_{Y/k}, \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2})$ is the induced map, then ϕ lifts to an element of the group $\operatorname{Ext}^0(L\varphi^* L_{\mathcal{X}/k}, \mathfrak{m}^{n+1-e}/\mathfrak{m}^{n+2})$.*

Let us show how the above result implies Theorem 5.1.

Proof of Theorem 5.1 assuming Proposition 5.2. It is enough to show $\theta_{n-h}(\varphi) \cong \theta_{n-h}(\varphi')$ since then $\theta_{n-e}(\varphi) \cong \theta_{n-e}(\varphi')$ for all $n \geqslant e \geqslant h$. Let $\alpha = \pi \circ \varphi$ and $\alpha' = \pi \circ \varphi'$. Set $\varphi^{(0)} := \varphi$ and $\alpha^{(0)} := \alpha$. We will prove the result by inductively constructing a sequence of arcs $\{\varphi^{(i)}\}_i$ such that if $\alpha^{(i)} := \pi \circ \varphi^{(i)}$, then

$$\theta_{n+i-h}\big(\varphi^{(i+1)}\big) \simeq \theta_{n+i-h}\big(\varphi^{(i)}\big) \quad \text{and} \quad \theta_{n+i}\big(\alpha^{(i)}\big) = \theta_{n+i}(\alpha')\,.$$

Let us show how the result follows assuming the existence of the above sequence $\{\varphi^{(i)}\}_i$. Consider $\{\theta_{n+i-h}(\varphi^{(i)})\}_i$, which is an inverse system of jets, so there exists a unique arc $\varphi^{(\infty)}$ such that $\theta_{n+i-h}(\varphi^{(\infty)}) \simeq \theta_{n+i-h}(\varphi^{(i)})$ for all i. Let $\alpha^{(\infty)} := \pi\varphi^{(\infty)}$. We prove that $\alpha' = \alpha^{(\infty)}$. To show this, it suffices to prove that their truncations match. We have

$$\theta_{n+i-h}\big(\alpha^{(\infty)}\big) = \pi \circ \theta_{n+i-h}\big(\varphi^{(\infty)}\big) = \pi \circ \theta_{n+i-h}\big(\varphi^{(i)}\big) = \theta_{n+i-h}\big(\alpha^{(i)}\big) = \theta_{n+i-h}(\alpha')\,.$$

Hence, $\alpha' = \alpha^{(\infty)}$. Lastly, we know $\varphi^{(\infty)} \in \mathcal{C}(k')$ because $\theta_{n-h}(\varphi^{(\infty)}) \simeq \theta_{n-h}(\varphi)$ and $n - h \geqslant \ell$. Since $\overline{\mathcal{C}}(k') \to \mathscr{L}(Y)(k')$ is injective, we then see that $\varphi' \simeq \varphi^{(\infty)}$. It follows that $\theta_{n-h}(\varphi') \simeq \theta_{n-h}(\varphi^{(\infty)}) \simeq \theta_{n-h}(\varphi)$.

It remains to construct the sequence of arcs $\{\varphi^{(i)}\}_i$. By induction, we may assume that we have constructed $\varphi^{(i)}$, and we show how to obtain $\varphi^{(i+1)}$. The set of lifts of $\theta_{n+i}(\alpha^{(i)})$ to $(2(n+i+1)-1)$-jets is a torsor for the deformation space $H := \mathrm{Hom}(\alpha^*\Omega^1_{Y/k}, \mathfrak{m}^{n+i+1}/\mathfrak{m}^{2(n+i+1)})$. Thus, after fixing a choice of deformation, namely $\theta_{2(n+i+1)-1}(\alpha^{(i)})$, we may identify the set of lifts with H. In particular, since $\theta_{n+i}(\alpha^{(i)}) = \theta_{n+i}(\alpha')$, we see that $\theta_{2(n+i+1)-1}(\alpha')$ defines an element $\Phi \in H$. Applying Proposition 5.2 and letting ϕ be as in the statement of the proposition, we obtain a lift $\widetilde{\phi} \in \mathrm{Ext}^0(L\varphi^* L_{\mathcal{X}/k}, \mathfrak{m}^{n+i+1-h}/\mathfrak{m}^{n+i+2})$. Note that ϕ is an element of $\mathrm{Hom}(\alpha^*\Omega^1_{Y/k}, \mathfrak{m}^{n+i+1}/\mathfrak{m}^{n+i+2})$, which is itself a deformation space parameterizing lifts of $\theta_{n+i}(\alpha^{(i)})$ to $(n+i+1)$-jets, and that ϕ corresponds to $\theta_{n+i+1}(\alpha')$. Similarly, the space $\mathrm{Ext}^0(L\varphi^* L_{\mathcal{X}/k}, \mathfrak{m}^{n+i+1-h}/\mathfrak{m}^{n+i+2})$ parameterizes lifts of $\theta_{n+i-h}(\varphi^{(i)})$ to $(n+i+1)$-jets of $\mathcal{X}$. Let β be the $(n+i+1)$-jet of $\mathcal{X}$ induced by $\widetilde{\phi}$; by construction, $\theta_{n+i-h}(\beta) = \theta_{n+i-h}(\varphi^{(i)})$ and $\pi\beta = \theta_{n+i+1}(\alpha')$ since $\widetilde{\phi}$ lifts ϕ. By the infinitesimal lifting criterion and smoothness of $\mathcal{X}$, we may lift β to an arc $\varphi^{(i+1)}$ of $\mathcal{X}$. Letting $\alpha^{(i+1)}$ be the induced arc of Y, we see that $\theta_{n+i+1}(\alpha^{(i+1)}) = \pi\beta = \theta_{n+i+1}(\alpha')$. $\square$

Proposition 5.2 will be proved after a series of lemmas. Throughout the rest of this section, we fix the notation $D = \mathrm{Spec}(k'[\![t]\!])$.

LEMMA 5.3. *The following hold:*

- $L^0\varphi^* L_{\mathcal{X}/k}$ *is free.*
- $L^i\varphi^* L_{\mathcal{X}/Y}$ *is torsion for any* $i \in \mathbb{Z}$.
- $L^{-1}\varphi^* L_{\mathcal{X}/Y} = (\alpha^*\Omega^1_{Y/k})_{\mathrm{tors}}$.

Moreover, the natural map $L\alpha^* L_{Y/k} \to L\varphi^* L_{\mathcal{X}/k}$ *induces an exact sequence*

$$0 \longrightarrow \left(\alpha^*\Omega^1_{Y/k}\right)_{\mathrm{tors}} \longrightarrow \alpha^*\Omega^1_{Y/k} \longrightarrow L^0\varphi^* L_{\mathcal{X}/k} \longrightarrow L^0\varphi^* L_{\mathcal{X}/Y} \longrightarrow 0\,. \tag{5.1}$$

Proof. Applying $L\varphi^*$ to the exact triangle $L\pi^* L_{Y/k} \to L_{\mathcal{X}/k} \to L_{\mathcal{X}/Y}$, we have an exact triangle

$$L\alpha^* L_{Y/k} \longrightarrow L\varphi^* L_{\mathcal{X}/k} \longrightarrow L\varphi^* L_{\mathcal{X}/Y}\,.$$

Taking cohomology sheaves, we have the exact sequence

$$0 \longrightarrow L^{-1}\varphi^* L_{\mathcal{X}/Y} \longrightarrow \alpha^*\Omega^1_{Y/k} \xrightarrow{\sigma} L^0\varphi^* L_{\mathcal{X}/k} \longrightarrow L^0\varphi^* L_{\mathcal{X}/Y} \longrightarrow 0\,.$$

To obtain (5.1), we must show that $L^{-1}\varphi^* L_{\mathcal{X}/Y} = (\alpha^*\Omega^1_{Y/k})_{\mathrm{tors}}$. This will follow upon showing that $L^0\varphi^* L_{\mathcal{X}/k}$ is torsion-free and $L^{-1}\varphi^* L_{\mathcal{X}/Y}$ is torsion. Indeed, then

$$\left(\alpha^*\Omega^1_{Y/k}\right)_{\mathrm{tors}} \subseteq \ker(\sigma) = L^{-1}\varphi^* L_{\mathcal{X}/Y} \subseteq \left(\alpha^*\Omega^1_{Y/k}\right)_{\mathrm{tors}}\,.$$

Next, consider the cartesian diagram

$$\begin{tikzcd} V \arrow[r, "\widetilde{\varphi}"] \arrow[d, "q"'] & \widetilde{X} \arrow[d, "p"] \\ D \arrow[r, "\varphi"] & \mathcal{X}\,, \end{tikzcd}$$

where p is a smooth cover. Then we have an exact triangle $p^*L_{\mathcal{X}/k} \to \Omega^1_{\widetilde{X}/k} \to \Omega^1_{\widetilde{X}/\mathcal{X}}$, and applying $L\widetilde{\varphi}^*$, we have an exact triangle

$$q^*L\varphi^*L_{\mathcal{X}/k} \longrightarrow \widetilde{\varphi}^*\Omega^1_{\widetilde{X}/k} \longrightarrow \widetilde{\varphi}^*\Omega^1_{\widetilde{X}/\mathcal{X}}.$$

As a result, $q^*L^0\varphi^*L_{\mathcal{X}/k} \to \widetilde{\varphi}^*\Omega^1_{\widetilde{X}}$ is injective, and since $\widetilde{\varphi}^*\Omega^1_{\widetilde{X}}$ is a vector bundle, $q^*L^0\varphi^*L_{\mathcal{X}/k}$ is torsion-free. Since q is smooth, this implies that $L^0\varphi^*L_{\mathcal{X}/k}$ is a torsion-free sheaf on D, hence free.

It remains to prove that $L^m\varphi^*L_{\mathcal{X}/Y}$ is torsion for all $m \in \mathbb{Z}$. To do so, it is enough to show that $L^m\varphi^*L_{\mathcal{X}/Y}$ is supported on the closed fiber of D. In other words, we need only show $\eta^*L^m\varphi^*L_{\mathcal{X}/Y} = 0$, where $\eta\colon D^\circ \to D$ is the generic point. Since the generic point of φ factors through $\mathcal{U}$, we have a commutative diagram

$$\begin{tikzcd} D^\circ \arrow[r, "\varphi^\circ"] \arrow[d, "\eta"'] & \mathcal{U} \arrow[d, "j"] \\ D \arrow[r, "\varphi"] & \mathcal{X}. \end{tikzcd}$$

So, $\eta^*L^m\varphi^*L_{\mathcal{X}/Y} = L^m(\varphi^\circ)^*j^*L_{\mathcal{X}/Y}$. From the exact triangle $j^*L_{\mathcal{X}/Y} \to L_{\mathcal{U}/\mathcal{X}} \to L_{\mathcal{U}/Y}$ and the fact that $\mathcal{U} \to \mathcal{X}$ and $\mathcal{U} \to Y$ are open immersions, we have $L_{\mathcal{U}/\mathcal{X}} = L_{\mathcal{U}/Y} = 0$, and hence $j^*L_{\mathcal{X}/Y} = 0$. $\square$

Lemma 5.4. *Let M be a finitely generated $k'[\![t]\!]$-module, F be a finite-rank free $k'[\![t]\!]$-module, and*

$$0 \longrightarrow M_{\mathrm{tors}} \longrightarrow M \longrightarrow F \longrightarrow Q \longrightarrow 0$$

be an exact sequence. Let $h = \operatorname{len}(Q)$, $e' = \operatorname{len}(M_{\mathrm{tors}})$, $e \geqslant h$, and $n \geqslant \max(e, e')$. If $\Phi \in \operatorname{Hom}(M, \mathfrak{m}^{n+1}/\mathfrak{m}^{2(n+1)})$ and $\phi \in \operatorname{Hom}(M, \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2})$ is the induced map, then ϕ lifts to an element of $\operatorname{Hom}(F, \mathfrak{m}^{n+1-e}/\mathfrak{m}^{n+2})$.

Proof. To begin, we have $\Phi(M_{\mathrm{tors}}) \subseteq \mathfrak{m}^{2(n+1)-e'}/\mathfrak{m}^{2(n+1)}$, and since $n \geqslant e'$, it follows that ϕ kills M_{tors}. As a result, ϕ induces an element $\overline{\phi}$ of the group $\operatorname{Hom}(M/M_{\mathrm{tors}}, \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2})$. Replacing M with M/M_{tors}, we may assume that M is free, and so we have a short exact sequence $0 \to M \to F \to Q \to 0$.

Let $v_1, \dots, v_d$ be a basis for F. By the structure theorem for modules over a PID, we have $a_1, \dots, a_d \in k^*$ and nonnegative integers $h_1, \dots, h_d$ such that $a_1t^{h_1}v_1, \dots, a_dt^{h_d}v_d$ is a basis for M. Note that $h := \operatorname{len}(Q) = \sum_{i=1} h_i$. We have $\phi(a_it^{h_i}) = b_it^{n+1}$ for $b_i \in k$. Thus, we may extend ϕ to a map $\widetilde{\phi}\colon F \to \mathfrak{m}^{n+1-e}/\mathfrak{m}^{n+2}$ by letting $\widetilde{\phi}(v_i) = (b_i/a_i)t^{n+1-h_i}$. $\square$

We prove Proposition 5.2, completing the proof of Theorem 5.1.

Proof of Proposition 5.2. Let $\mathcal{M}$ be any coherent sheaf on D. We have a natural map $L\alpha^*L_{Y/k} \to L\varphi^*L_{\mathcal{X}/k}$ that induces a map

$$\operatorname{Ext}^0(L\varphi^*L_{\mathcal{X}/k}, \mathcal{M}) \to \operatorname{Ext}^0(L\alpha^*L_{Y/k}, \mathcal{M}) = \operatorname{Hom}\big(\alpha^*\Omega^1_{Y/k}, \mathcal{M}\big). \tag{5.2}$$

We first claim that $L\alpha^*L_{Y/k} \to L\varphi^*L_{\mathcal{X}/k}$ factors through $L^0\varphi^*L_{\mathcal{X}/k}$, and hence the map in (5.2) factors as

$$\operatorname{Ext}^0(L\varphi^*L_{\mathcal{X}/k}, \mathcal{M}) \xrightarrow{F} \operatorname{Ext}^0\big(L^0\varphi^*L_{\mathcal{X}/k}, \mathcal{M}\big) \xrightarrow{G} \operatorname{Hom}\big(\alpha^*\Omega^1_{Y/k}, \mathcal{M}\big). \tag{5.3}$$

Indeed, we have a morphism of exact triangles

$$\begin{tikzcd}
\tau_{\leqslant 0} L\alpha^* L_{Y/k} \arrow[r] \arrow[d] & L\alpha^* L_{Y/k} \arrow[r] \arrow[d] & \tau_{>0} L\alpha^* L_{Y/k} \arrow[d] \\
\tau_{\leqslant 0} L\varphi^* L_{\mathcal{X}/k} \arrow[r] & L\varphi^* L_{\mathcal{X}/k} \arrow[r] & \tau_{>0} L\varphi^* L_{\mathcal{X}/k} \,,
\end{tikzcd}$$

where τ denotes the canonical truncation. Since $L\alpha^* L_{Y/k}$ is concentrated in degrees at most 0, the map $\tau_{\leqslant 0} L\alpha^* L_{Y/k} \to L\alpha^* L_{Y/k}$ is the identity map; since $L\varphi^* L_{\mathcal{X}/k}$ is concentrated in degrees at least 0, we have $\tau_{\leqslant 0} L\varphi^* L_{\mathcal{X}/k} = L^0\varphi^* L_{\mathcal{X}/k}$.

Next, we show that the map F in (5.3) is surjective. To see this, we apply the spectral sequence from [Sta21, Tag 07A9]

$$E_2^{pq} = \mathrm{Ext}^p(L^{-q}\varphi^* L_{\mathcal{X}/k}, \mathcal{M}) \Longrightarrow \mathrm{Ext}^{p+q}(L\varphi^* L_{\mathcal{X}/k}, \mathcal{M})\,,$$

which yields an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Ext}^1\big(L^1\varphi^* L_{\mathcal{X}/k}, \mathcal{M}\big) &\longrightarrow \mathrm{Ext}^0\big(L\varphi^* L_{\mathcal{X}/k}, \mathcal{M}\big) \\ &\xrightarrow{F} \mathrm{Hom}\big(L^0\varphi^* L_{\mathcal{X}/k}, \mathcal{M}\big) \longrightarrow \mathrm{Ext}^2\big(L^1\varphi^* L_{\mathcal{X}/k}, \mathcal{M}\big)\,. \end{aligned}$$

Since $k'[\![t]\!]$ is a PID, $\mathrm{Ext}^2(L^1\varphi^* L_{\mathcal{X}/k}, \mathcal{M})$ vanishes, and so F is surjective.

Lastly, note that $h \geqslant \mathrm{len}(L^0\varphi^* L_{\mathcal{X}/k})$ and $e' \geqslant \mathrm{len}((\alpha^*\Omega^1_{Y/k})_{\mathrm{tors}})$. So, by Lemmas 5.3 and 5.4, we see that ϕ extends to an element ϕ' of the group $\mathrm{Hom}(L^0\varphi^* L_{\mathcal{X}/k}, \mathfrak{m}^{n+1-e}/\mathfrak{m}^{n+2})$. Since F is surjective, ϕ' lifts to an element of $\mathrm{Ext}^0(L\varphi^* L_{\mathcal{X}/k}, \mathfrak{m}^{n+1-e}/\mathfrak{m}^{n+2})$. □

6. Motivic change of variables formula

In this section, we complete the proof of Theorem 1.2(b). We give the proof in Section 6.2 after proving a preliminary result in Section 6.1.

6.1 Stratifications by trivial gerbes

The goal of this subsection is to prove the following proposition. We expect this result to be well known, but we could not find it stated in the literature.

Proposition 6.1. *Let $\mathcal{Y}$ be a finite-type Artin stack over k such that the map $\mathcal{I}_{\mathcal{Y}} \to \mathcal{Y}$ is separated, let $r \in \mathbb{Z}_{\geqslant 0}$, and suppose that for every field extension k' of k and every $y \in \mathcal{Y}(k')$, the stabilizer of y is isomorphic to $\mathbb{G}^r_{a,k'}$. Then $\mathcal{Y}$ can be stratified into locally closed substacks $\mathcal{Y}_1, \dots, \mathcal{Y}_\ell$ such that each $\mathcal{Y}_i$ is isomorphic to $Y_i \times_k B\mathbb{G}^r_{a,k}$ for some finite-type k-scheme Y_i.*

We begin with a couple of preliminary lemmas.

Lemma 6.2. *Let S be a scheme, and let X and Y be S-schemes with X reduced and $Y \to S$ separated. If $f, g\colon X \to Y$ are two S-morphisms and for all fibers $s\colon \mathrm{Spec}(k(s)) \to S$, we have equality of the base change maps $f_s = g_s$, then $f = g$.*

Proof. Consider the Cartesian diagram

$$\begin{tikzcd}
Z \arrow[r] \arrow[d, "i"'] & Y \arrow[d, "\Delta_{Y/S}"] \\
X \arrow[r, "(f,g)"] & Y \times_S Y\,.
\end{tikzcd}$$

Note that $f = g$ if and only if i has a section. Since Y is separated over S, we see that i is a closed immersion. As X is reduced, i has a section if and only if i is surjective. Since surjectivity of i can be checked on fibers, it follows that $f = g$. $\square$

Lemma 6.3. *Let k' be a characteristic 0 field, and let S be a reduced k'-scheme that is locally of finite presentation. Suppose that $f\colon G \to S$ is a separated group algebraic space that is flat and locally of finite presentation. If every fiber $f_s\colon G_s \to \operatorname{Spec}(k'(s))$ is a commutative group scheme, then G is commutative.*

Proof. Let $m\colon G \times_S G \to G$ be the multiplication map and $\tau\colon G \times_S G \to G \times_S G$ be the map $\tau(g,h) = (h,g)$. To prove that G is commutative, we must show $m\circ\tau = m$. By [Sta21, Tag 0B8G], every fiber $f_s\colon G_s \to \operatorname{Spec}(k'(s))$ is a group scheme; hence by Cartier's theorem (see, for example, [Sta21, Tag 047N]), every fiber f_s is smooth. Since f flat and locally of finite presentation, [Sta21, Tag 01V8] shows that f is smooth; in particular, $G \times_S G \to S$ is smooth. Since S is reduced, $G \times_S G$ is as well. Lastly, the maps $m, m\circ\tau\colon G \times_S G \to G$ are equal on fibers, so $m \circ \tau = m$ by Lemma 6.2. $\square$

Proof of Proposition 6.1. First, replacing $\mathcal{Y}$ with an irreducible component, we may assume that $\mathcal{Y}$ is irreducible. We may also replace $\mathcal{Y}$ with $\mathcal{Y}_{\mathrm{red}}$, so we will assume that $\mathcal{Y}$ is reduced. By Noetherian induction, it suffices to show that there exists a nonempty open substack of $\mathcal{Y}$ of the form $Y \times_k B\mathbb{G}_{a,k}^r$ for some finite-type k-scheme Y. Since $\mathcal{I}_{\mathcal{Y}} \to \mathcal{Y}$ is quasi-compact, replacing $\mathcal{Y}$ with an open substack, we may assume that it is a gerbe by [Sta21, Tag 06RC]. Let $p\colon \mathcal{Y} \to W$ be the gerbe structure map, so W is an algebraic space. Because $\mathcal{Y}$ is reduced and gerbe structure maps are stable under base change, we may assume that W is reduced. Since $\mathcal{Y}$ is irreducible and of finite type, so is W. Since W is quasi-separated, it has a dense open scheme by [Sta21, Tag 06NH]. We therefore assume that W is a scheme.

Let $\operatorname{Spec}(K) \to W$ be the generic point. We claim that $\mathcal{Y}_K := \mathcal{Y} \times_W \operatorname{Spec}(K)$ is isomorphic to $B\mathbb{G}_{a,K}^r$. Assuming this claim for the moment, we have a morphism $\alpha\colon \mathcal{Y}_K \to B\mathbb{G}_{a,K}^r$. By [Ryd15, Proposition B.2], there exists an open subscheme $U \subset W$ for which α extends to a map $\mathcal{Y} \times_W U \to B\mathbb{G}_{a,U}^r$. Since $\mathcal{Y} \to W$ is of finite presentation and α is an isomorphism, by [Ryd15, Proposition B.3], after possibly shrinking U, we may assume that $\mathcal{Y} \times_Y U \to B\mathbb{G}_{a,U}^r$ is an isomorphism, thereby producing our desired open substack of $\mathcal{Y}$ isomorphic to $B\mathbb{G}_{a,U}^r = U \times_k B\mathbb{G}_{a,k}^r$.

It remains to prove our claim; that is, we have reduced to showing that if $W = \operatorname{Spec}(K)$, then $\mathcal{Y}$ is the trivial $\mathbb{G}_{a,K}^r$-gerbe. Note that because k and thus K have characteristic 0, the gerbe structure map $\mathcal{Y} \to W$ is smooth, so in this case where $W = \operatorname{Spec}(K)$, the stack $\mathcal{Y}$ is reduced. Let $\rho\colon \widetilde{Y} \to \mathcal{Y}$ be a smooth cover. We first show that all S-valued points of $\mathcal{Y}$ have commutative automorphism groups. This may be checked smooth locally on S, and hence it is enough to check the commutativity of the automorphism group G of the $\widetilde{Y}$-valued point corresponding to ρ. We see that $G = I_{\mathcal{Y}} \times_{\mathcal{Y}} \widetilde{Y}$. Since the stabilizer groups of all field-valued points are commutative, $G \to \widetilde{Y}$ is a commutative group scheme by Lemma 6.3.

Therefore, [Sta21, Tag 0CJY] tells us that $\mathcal{Y}$ is a $\mathcal{G}$-gerbe for some sheaf of abelian groups $\mathcal{G}$ on $\operatorname{Spec}(K)$. Let K'/K be an étale extension that trivializes the gerbe $\mathcal{Y}$. Since the K'-point of $\mathcal{Y}$ yielding the trivialization has stabilizer group scheme given by $\mathbb{G}_{a,K'}^r$, we see that $\mathcal{G}$ is a twisted form of $\mathbb{G}_{a,K}^r$. Such twisted forms are classified by $H^1_{\mathrm{et}}(K, \operatorname{Aut}(\mathbb{G}_a^r)) = H^1_{\mathrm{et}}(K, \mathrm{GL}_r)$, which vanishes by Hilbert's Theorem 90. Thus, $\mathcal{G} = \mathbb{G}_a^r$. Lastly, $\mathbb{G}_a^r$-gerbes on $\operatorname{Spec}(K)$ are classified by $H^2(\operatorname{Spec}(K), \mathcal{O}^{\oplus r}) = 0$. It follows that $\mathcal{Y}$ is the trivial $\mathbb{G}_a^r$-gerbe. $\square$

6.2 Proof of Theorem 1.2(b)

For the remainder of this paper, let $\pi\colon \mathcal{X} \to Y$, $\mathcal{U}$, $\mathcal{C}$, D, and σ be as in the statement of Theorem 1.2, let $\mathscr{J}_Y$ be the $(\dim Y)$th Fitting ideal of $\Omega^1_{Y/k}$, let $\mathscr{J}$ denote the ideal sheaf on $\mathcal{I}_{\mathcal{X}}$ defined by the closed substack $\mathcal{X} \xrightarrow{\sigma} \mathcal{I}_{\mathcal{X}}$, and let $\mathscr{I}$ be a quasi-coherent ideal sheaf on $\mathcal{X}$ such that $\mathscr{I}\mathcal{O}_{\mathcal{I}_{\mathcal{X}}} \cdot \mathscr{J} = 0$ and the closed substack of $\mathcal{X}$ defined by $\mathscr{I}$ has underlying topological space equal to $|\mathcal{X}| \setminus |\mathcal{U}|$. Note that such an ideal $\mathscr{I}$ is guaranteed to exist by Theorem 4.1(a).

We begin with the following key case of Theorem 1.2(b), where the functions of interest are all constant on $\mathcal{C}$.

PROPOSITION 6.4. *Suppose that* $\mathrm{ord}_{\mathscr{J}_Y}$ *is constant on* D *and that* $\mathrm{ht}^{(0)}_{L_{\mathcal{X}/Y}} + \mathrm{ht}^{(1)}_{L_{\mathcal{X}/Y}}$, $\mathrm{ht}^{(1)}_{L_{\mathcal{X}/Y}}$, $\mathrm{ht}^{(0)}_{L\sigma^* L_{\mathcal{I}_{\mathcal{X}}/\mathcal{X}}}$, *and* $\mathrm{ord}_{\mathscr{I}}$ *are constant on* $\mathcal{C}$. *Then*

$$\mu_Y(D) = \int_{\mathcal{C}} \mathbb{L}^{\mathrm{ht}^{(0)}_{L\sigma^* L_{\mathcal{I}_{\mathcal{X}}/\mathcal{X}}} - \mathrm{ht}^{(0)}_{L_{\mathcal{X}/Y}}} \mathrm{d}\mu_{\mathcal{X}} \,.$$

The proof of Proposition 6.4 is rather technical, so we first pause and attempt to shed light on the "big picture." In the case of schemes, the analogous proposition is accomplished by using deformation theory and (the scheme case of) Theorem 5.1 to show that the reduced fibers of $\theta_n(\mathcal{C}) \to \theta_n(D)$ are affine spaces of appropriate dimension. Then applying (the scheme version of) [SU22, Lemma 3.35] gives the relationships in $\widehat{\mathscr{M}_k}$ that lead to the change of variables formula. This direct argument does not seem to work in the case of stacks: Deformation theory and Theorem 5.1 can be used to show that each reduced fiber of $\theta_n(\mathcal{C}) \to \theta_n(D)$ is a quotient, by a vector group, of the product of an affine space and a vector group. These fibers all have the same class in $\widehat{\mathscr{M}_k}$. However, without knowing more about the action of that first vector group, it is not clear that one can apply [SU22, Lemma 3.35]. This issue leads to the technical nature of the following proof, which involves stratifying $\theta_n(\mathcal{C})$ into certain $\mathcal{W}_i$ that each admit certain vector group torsors $\mathcal{Z}_i$. The imprecise intuition is that the torsors $\mathcal{Z}_i \to \mathcal{W}_i$ globalize the vector group actions referred to above, providing just enough uniformity to apply [SU22, Lemma 3.35]. However, this only serves as motivating intuition. The actual idea is that the specific construction of these strata allows us, using deformation theory and Theorem 5.1, to compute the fibers $\mathcal{Z}_i \to \theta_n(D)$ with sufficient precision to apply [SU22, Lemma 3.35]. With care, as elaborated below, this eventually leads to the desired change of variables formula.

We note to the reader that in a few locations below, we will identify certain spaces of jets with appropriate deformation spaces. These identifications are as stacks over a field k', and they are obtained by using deformation theory to compute the category of A-valued points of these spaces of jets, where A varies over all k'-algebras. See the proof of Theorem 4.1(b), where this is done in a slightly simpler context. We give more specific justifications in the appropriate locations below.

Proof of Proposition 6.4. Let $e' \in \mathbb{Z}_{\geqslant 0}$ be such that $\mathrm{ord}_{\mathscr{J}_Y}$ is equal to e' on D, and let $h_{01}, h_1, h_0, r \in \mathbb{Z}_{\geqslant 0}$ be such that $\mathrm{ht}^{(0)}_{L_{\mathcal{X}/Y}} + \mathrm{ht}^{(1)}_{L_{\mathcal{X}/Y}}$, $\mathrm{ht}^{(1)}_{L_{\mathcal{X}/Y}}$, $\mathrm{ht}^{(0)}_{L\sigma^* L_{\mathcal{I}_{\mathcal{X}}/\mathcal{X}}}$, and $\mathrm{ord}_{\mathscr{I}}$ are equal to h_{01}, h_1, h_0, and r, respectively, on $\mathcal{C}$. Because $\mathcal{C}$ is a cylinder, there exists some $\ell \in \mathbb{Z}_{\geqslant 0}$ such that $\theta_\ell^{-1}(\theta_\ell(\mathcal{C})) = \mathcal{C}$. Set $h = h_{01} - h_1$, and note that $\mathrm{ht}^{(0)}_{L_{\mathcal{X}/Y}}$ is equal to h on $\mathcal{C}$, so in particular $h \in \mathbb{Z}_{\geqslant 0}$. Set $e = h_{01} + h_1$. Because the (isomorphic) image of $\mathcal{U}$ in Y is smooth, we have that $D \cap \mathscr{L}(Y \setminus Y_{\mathrm{sm}}) = \emptyset$, where Y_{sm} is the smooth locus of Y. By [SU22, Lemma 7.5], there exists some $n_D \in \mathbb{Z}_{\geqslant 0}$ that satisfies the following: For any field extension k' of k, any $n \geqslant n_D$, and any $\psi_n \in (\theta_n(D))(k')$, there exists some $\psi \in D(k')$ such that $\theta_n(\psi) = \psi_n$.

Fix any $n \geqslant \max(\ell+e, r+e, 2\max(r,h_0)-1+e, e', n_D, 2e-1)$. Because $\dim \mathcal{X} = \dim Y$, it is sufficient to show that

$$\mathrm{e}(\theta_n(D)) = \mathbb{L}^{h_0-(h_{01}-h_1)}\,\mathrm{e}(\theta_n(\mathcal{C}))\,.$$

Set $\mathcal{C}_{n-e} = \theta_{n-e}(\mathcal{C}) \subset |\mathscr{L}_n(\mathcal{X})|$. We prove the above equality working over a suitable stratification.

Step 1: *Giving stratifications of* $\theta_n(\mathcal{C})$, $\theta_{n-e}(\mathcal{C})$, *and* $\theta_n(D)$. For every field extension k' of k, every k' valued point of $\mathcal{C}_{n-e}$ lifts to a k'-valued point of $\mathcal{C}$ because $\mathcal{X}$ is smooth and $n-e \geqslant \ell$. Thus because $n-e \geqslant \max(r, 2\max(r,h_0)-1)$, Theorem 4.1(b) implies that every k'-valued point of $\mathcal{C}_{n-e}$ has stabilizer isomorphic to $\mathbb{G}_{a,k}^{h_0}$. Because $\mathcal{X}$ has separated diagonal, $\mathscr{L}_{n-e}(\mathcal{X})$ has separated diagonal [Ryd11, Proposition 3.8(xx)]. Thus Proposition 6.1 implies that $\mathcal{C}_{n-e}$ can be stratified into (finitely many) locally closed substacks $\{\mathcal{V}_i\}_i$ of $\mathscr{L}_{n-e}(\mathcal{X})$ such that each $\mathcal{V}_i$ is isomorphic to $V_i \times_k B\mathbb{G}_{a,k}^{h_0}$ for some finite-type k-scheme V_i. Let each $V_i \to \mathcal{V}_i$ be the quotient map by the trivial $\mathbb{G}_{a,k}^{h_0}$-action on V_i, and let each $\mathcal{V}_i \hookrightarrow \mathscr{L}_{n-e}(\mathcal{X})$ be the inclusion. Also for each i, let $\mathcal{Z}_i \to \mathcal{W}_i \hookrightarrow \mathscr{L}_n(\mathcal{X})$ be the base change of $V_i \to \mathcal{V}_i \hookrightarrow \mathscr{L}_{n-e}(\mathcal{X})$ along $\theta^n_{n-e}\colon \mathscr{L}_n(\mathcal{X}) \to \mathscr{L}_{n-e}(\mathcal{X})$. Because $\mathcal{Z}_i \to \mathcal{W}_i$ is a $\mathbb{G}_{a,k}^{h_0}$-torsor and $\mathbb{G}_{a,k}^{h_0}$ is a special group, [Eke25, Propositions 1.1(iii) and 1.4(i)] show that

$$\mathrm{e}(\mathcal{Z}_i) = \mathbb{L}^{h_0}\,\mathrm{e}(\mathcal{W}_i)\,.$$

Because $n-e \geqslant \ell$ and $\mathcal{X}$ is smooth, $\theta_n(\mathcal{C}) = (\theta^n_{n-e})^{-1}(\mathcal{C}_{n-e})$, so the $\mathcal{W}_i$ form a partition of $\theta_n(\mathcal{C})$. Thus

$$\sum_i \mathrm{e}(\mathcal{Z}_i) = \mathbb{L}^{h_0}\,\mathrm{e}(\theta_n(\mathcal{C}))\,. \tag{6.1}$$

Now set $Q_i = \mathscr{L}_n(\pi)(|\mathcal{W}_i|) \subset |\mathscr{L}_n(Y)|$. By Chevalley's theorem, the Q_i are constructible subsets of $|\mathscr{L}_n(Y)|$. We will now show that the Q_i partition $\theta_n(D)$. The Q_i cover $\theta_n(D)$ because $\mathcal{C}$ surjects onto D and because the $|\mathcal{W}_i|$ cover $\theta_n(\mathcal{C})$. We just need to show that the Q_i are mutually disjoint. To do this, it is sufficient to show that if k' is a field extension of k and $\varphi_n, \varphi'_n \in \theta_n(\mathcal{C})(k')$ have the same image in $\mathscr{L}_n(Y)$, then $\theta^n_{n-e}(\varphi_n) \cong \theta^n_{n-e}(\varphi'_n)$. Because $\mathcal{X}$ is smooth and $n \geqslant \ell$, there exist $\varphi, \varphi' \in \mathcal{C}(k')$ that truncate to φ_n, φ'_n, respectively. We have

$$\theta_n(\mathscr{L}(\pi)(\varphi)) = \mathscr{L}_n(\pi)(\varphi_n) = \mathscr{L}_n(\pi)(\varphi'_n) = \theta_n(\mathscr{L}(\pi)(\varphi'))\,,$$

$n \geqslant \max(\ell+e, e') \geqslant \max(\ell+h, e')$, and $n \geqslant e \geqslant h$. Therefore, Theorem 5.1 implies that $\theta_{n-e}(\varphi) \cong \theta_{n-e}(\varphi')$, as desired. We have now shown that the Q_i partition $\theta_n(D)$. Therefore,

$$\sum_i \mathrm{e}(Q_i) = \mathrm{e}(\theta_n(D))\,. \tag{6.2}$$

In light of equations (6.1) and (6.2), we must relate $\mathrm{e}(Q_i)$ and $\mathrm{e}(\mathcal{Z}_i)$. Because the composition

$$\mathcal{Z}_i \longrightarrow \mathcal{W}_i \hookrightarrow \mathscr{L}_n(\mathcal{X}) \xrightarrow{\mathscr{L}_n(\pi)} \mathscr{L}_n(Y)$$

maps $|\mathcal{Z}_i|$ onto Q_i, we will accomplish this by controlling the fiber (with reduced structure) of $\mathcal{Z}_i \to \mathscr{L}_n(Y)$ over every point of Q_i. To this end, fix some i, some field extension k' of k, and some $\psi_n \in Q_i(k')$. Let $\mathcal{F}$ be the fiber of $\mathcal{Z}_i \to \mathscr{L}_n(Y)$ over ψ_n. We will relate $\mathcal{F}$ to the better-behaved fibers of $\mathcal{Z}_i \to V_i$.

Step 2: *Viewing* $\mathcal{F}_{\mathrm{red}}$ *as a closed substack of a fiber of* $\mathcal{Z}_i \to V_i$. Since $n \geqslant n_D$, there exists some $\psi \in D(k')$ that truncates to ψ_n. By the assumption on $\overline{\mathcal{C}}(k') \to D(k')$, there is some $\varphi \in \mathcal{C}(k')$ whose image is ψ. Set $\varphi_n = \theta_n(\varphi) \in \theta_n(\mathcal{C})(k')$ and $\varphi_{n-e} = \theta_{n-e}(\varphi)$. We have already shown that the $|\mathcal{W}_j|$ have disjoint images in $\mathscr{L}_n(Y)$, so the fact that φ_n has image $\psi_n \in Q_i(k')$ implies that $\varphi_n \in \mathcal{W}_i(k')$. Because $\mathcal{Z}_i \to \mathcal{W}_i$, as a base change of $V_i \to \mathcal{V}_i$, is a $\mathbb{G}_{a,k}^{h_0}$-torsor and $\mathbb{G}_{a,k}^{h_0}$ is a special group, there exists some $z \in \mathcal{Z}_i(k')$ whose image in $\mathcal{W}_i$ is 2-isomorphic to φ_n. Set $v \in V_i(k')$ to be the image of z along $\mathcal{W}_i \to V_i$. Now consider the following 2-commutative diagram:

$$\begin{tikzcd}
 & \mathcal{Z}_i \arrow[r] \arrow[d] & \mathcal{W}_i \arrow[r] \arrow[d] & \mathscr{L}_n(\mathcal{X}) \arrow[r, "\mathscr{L}_n(\pi)"] \arrow[d, "\theta_n"] & \mathscr{L}_n(Y) \arrow[d, "\theta_n"] \\
\operatorname{Spec}(k') \arrow[ur, "z"] \arrow[r, "v"] \arrow[urrr, bend left, "\varphi_n"] \arrow[rrr, bend right, "\varphi_{n-e}"'] & V_i \arrow[r] & \mathcal{V}_i \arrow[r] & \mathscr{L}_{n-e}(\mathcal{X}) \arrow[r, "\mathscr{L}_{n-e}(\pi)"] & \mathscr{L}_{n-e}(Y) .
\end{tikzcd}$$

Let $\mathcal{H}$ be the fiber of $\mathcal{Z}_i \to V_i$ over v. Because $\mathscr{L}_n(Y)$ and V_i are schemes, $\mathcal{F}$ and $\mathcal{H}$ are closed substacks of $\mathcal{Z}_i \otimes_k k'$. We will show that $|\mathcal{F}| \subset |\mathcal{H}|$.

Suppose that for some field extension k'', we have $z' \in \mathcal{F}(k'')$, so the image of z' in $\mathscr{L}_n(Y)$ is $\psi_n \otimes_{k'} k''$. We already saw (when proving that the Q_j were disjoint) that any two k''-points of $\theta_n(\mathcal{C})$ that have the same image in $\mathscr{L}_n(Y)$ must have 2-isomorphic $(n-e)$-truncations (this was where we used Theorem 5.1 above). Thus z' and $z \otimes_{k'} k''$ have 2-isomorphic images in $\mathscr{L}_{n-e}(\mathcal{X})(k'')$. Because the quotient map $V_i \to \mathcal{V}_i$ is a section of $\mathcal{V}_i = V_i \times_k B\mathbb{G}_{a,k}^{h_0} \to V_i$, it induces an injection $V_i(k'') \to \overline{\mathcal{V}_i}(k'')$. Also, $\mathcal{V}_i \hookrightarrow \mathscr{L}_{n-e}(\mathcal{X})$ induces an injection $\overline{\mathcal{V}_i}(k'') \to \overline{\mathscr{L}_{n-e}(\mathcal{X})}(k'')$. Therefore, z' and $z \otimes_{k'} k''$ have the same image in V_i, namely $v \otimes_{k'} k''$. Thus $z' \in \mathcal{H}(k'')$, and we have shown that $|\mathcal{F}| \subset |\mathcal{H}|$, so $\mathcal{F}_{\mathrm{red}}$ is a closed substack of $\mathcal{H}$.

Step 3: *Computing* $\mathcal{F}_{\mathrm{red}}$. We determine $\mathcal{F}_{\mathrm{red}}$ by computing $\mathcal{H}$ and understanding $|\mathcal{F}|$ as a subset of $|\mathcal{H}|$. Because $n \geqslant 2e-1$, we have that $\operatorname{Spec}(k'[t]/(t^{n-e+1})) \hookrightarrow \operatorname{Spec}(k'[t]/(t^{n+1}))$ is defined by a square zero ideal. Thus $\mathcal{H}$, which is isomorphic to the fiber of $\mathscr{L}_n(\mathcal{X}) \to \mathscr{L}_{n-e}(\mathcal{X})$ over $\theta_{n-e}(\varphi_n)$, can be identified with the deformation space (see, for example, the proof of [SU22, Lemma 3.34])

$$\operatorname{Ext}^0(L\varphi_{n-e}^* L_{\mathcal{X}/k}, I) \times_{k'} B\operatorname{Ext}^{-1}(L\varphi_{n-e}^* L_{\mathcal{X}/k}, I),$$

where $I = (t^{n-e+1})/(t^{n+1})$ and by slight abuse of notation, we are using $L\varphi_{n-e}^*$ to mean the derived pullback along the associated jet $\operatorname{Spec}(k'[t]/(t^{n-e+1})) \to \mathcal{X}$. Similarly, the fiber H of $\mathscr{L}_n(Y) \to \mathscr{L}_{n-e}(Y)$ over $\theta_{n-e}^n(\psi_n)$ can be identified with the deformation space

$$\operatorname{Ext}^0(L\varphi_{n-e}^* L\pi^* L_{Y/k}, I)$$

so that the map $\mathcal{H} \to H$ is identified with the projection

$$\operatorname{Ext}^0(L\varphi_{n-e}^* L_{\mathcal{X}/k}, I) \times_{k'} B\operatorname{Ext}^{-1}(L\varphi_{n-e}^* L_{\mathcal{X}/k}, I) \longrightarrow \operatorname{Ext}^0(L\varphi_{n-e}^* L_{\mathcal{X}/k}, I)$$

followed by the linear map

$$f\colon \operatorname{Ext}^0(L\varphi_{n-e}^* L_{\mathcal{X}/k}, I) \longrightarrow \operatorname{Ext}^0(L\varphi_{n-e}^* L\pi^* L_{Y/k}, I)$$

induced by $L\pi^* L_{Y/k} \to L_{\mathcal{X}/k}$, and $\mathcal{F}_{\mathrm{red}}$ is identified with the closed substack of $\mathcal{H}$

$$\ker f \times_{k'} B\operatorname{Ext}^{-1}(L\varphi_{n-e}^* L_{\mathcal{X}/k}, I).$$

Therefore,

$$\mathcal{F}_{\mathrm{red}} \cong \mathbb{A}_{k'}^{\dim_{k'} \ker f} \times_{k'} B\mathbb{G}_{a,k'}^{\dim_{k'} \mathrm{Ext}^{-1}(L\varphi_{n-e}^* L_{\mathcal{X}/k}, I)}.$$

By the exact triangle $L\pi^* L_{Y/k} \to L_{\mathcal{X}/k} \to L_{\mathcal{X}/Y}$ and the fact that Y is a scheme, we have that

$$\ker f \cong \mathrm{Ext}^0(L\varphi_{n-e}^* L_{\mathcal{X}/Y}, I) \quad \text{and} \quad \mathrm{Ext}^{-1}(L\varphi_{n-e}^* L_{\mathcal{X}/k}, I) \cong \mathrm{Ext}^{-1}(L\varphi_{n-e}^* L_{\mathcal{X}/Y}, I).$$

Thus Lemma 3.6 implies that

$$\dim_{k'} \ker f = \mathrm{ht}^{(0)}_{e-1, L_{\mathcal{X}/Y}}(\theta_{e-1}(\varphi)) \quad \text{and} \quad \dim_{k'} \mathrm{Ext}^{-1}(L\varphi_{n-e}^* L_{\mathcal{X}/k}, I) = \mathrm{ht}^{(1)}_{e-1, L_{\mathcal{X}/Y}}(\theta_{e-1}(\varphi)),$$

so

$$\mathcal{F}_{\mathrm{red}} \cong \mathbb{A}_{k'}^{\mathrm{ht}^{(0)}_{e-1, L_{\mathcal{X}/Y}}(\theta_{e-1}(\varphi))} \times_{k'} B\mathbb{G}_{a,k'}^{\mathrm{ht}^{(1)}_{e-1, L_{\mathcal{X}/Y}}(\theta_{e-1}(\varphi))}.$$

Because $e \geqslant h_{01} + h_1$ and $\mathrm{ht}^{(2)}_{L_{\mathcal{X}/Y}}(\varphi) = 0$, Theorem 3.2(b) implies that $\mathrm{ht}^{(0)}_{e-1, L_{\mathcal{X}/Y}}(\theta_{e-1}(\varphi)) = h_{01}$ and $\mathrm{ht}^{(1)}_{e-1, L_{\mathcal{X}/Y}}(\theta_{e-1}(\varphi)) = h_1$, so $\mathcal{F}_{\mathrm{red}} \cong \mathbb{A}_{k'}^{h_{01}} \times_{k'} B\mathbb{G}_{a,k'}^{h_1}$.

Step 4: *Completing the proof.* Now that we have computed $\mathcal{F}_{\mathrm{red}}$ and shown that it does not depend on ψ_n, we can finally relate $\mathrm{e}(\mathcal{Z}_i)$ and $\mathrm{e}(Q_i)$. Specifically, [SU22, Lemma 3.35] implies that

$$\mathrm{e}(\mathcal{Z}_i) = \mathrm{e}\big(\mathbb{A}_k^{h_{01}} \times_k B\mathbb{G}_{a,k}^{h_1}\big)\, \mathrm{e}(Q_i) = \mathbb{L}^{h_{01} - h_1}\, \mathrm{e}(Q_i),$$

where the second equality is because $\mathbb{G}_{a,k}^{h_1}$ is a special group. This, along with equations (6.1) and (6.2), gives $\mathrm{e}(\theta_n(D)) = \mathbb{L}^{h_0 - (h_{01} - h_1)}\, \mathrm{e}(\theta_n(\mathcal{C}))$. completing the proof of this proposition. ☐

We can now finish the proof of Theorem 1.2(b) by reducing to Proposition 6.4. The remainder of the argument is essentially standard, but we still briefly outline it for completeness.

Proof of Theorem 1.2(b). Because the (isomorphic) image of $\mathcal{U}$ in Y is smooth, the intersection $D \cap \mathscr{L}(Y \setminus Y_{\mathrm{sm}})$ is empty, where Y_{sm} is the smooth locus of Y, so $\mathrm{ord}_{\mathscr{J}_Y} : D \to \mathbb{Z}$ is constructible. Thus D can be partitioned into finitely many cylinders where $\mathrm{ord}_{\mathscr{J}_Y}$ is constant, and because the preimage of a cylinder along $\mathscr{L}(\pi)$ is a cylinder, it is straightforward to reduce to the case where $\mathrm{ord}_{\mathscr{J}_Y}$ is constant on D. Thus we assume that $\mathrm{ord}_{\mathscr{J}_Y}$ is constant on D.

Recall that $\mathrm{ht}^{(0)}_{L_{\mathcal{X}/Y}} + \mathrm{ht}^{(1)}_{L_{\mathcal{X}/Y}}$, $\mathrm{ht}^{(1)}_{L_{\mathcal{X}/Y}}$, and $\mathrm{ht}^{(0)}_{L\sigma^* L_{\mathcal{I}_{\mathcal{X}}/\mathcal{X}}}$ are constructible functions on $\mathcal{C}$ (see the proof of Theorem 1.2(a)). Also, $\mathrm{ord}_{\mathscr{J}} : \mathcal{C} \to \mathbb{Z}$ is constructible by Remark 2.6. Thus there exists a finite partition of $\mathcal{C}$ into cylinders $\{\mathcal{C}_i\}_i$ such that $\mathrm{ht}^{(0)}_{L_{\mathcal{X}/Y}} + \mathrm{ht}^{(1)}_{L_{\mathcal{X}/Y}}$, $\mathrm{ht}^{(1)}_{L_{\mathcal{X}/Y}}$, $\mathrm{ht}^{(0)}_{L\sigma^* L_{\mathcal{I}_{\mathcal{X}}/\mathcal{X}}}$, and $\mathrm{ord}_{\mathscr{J}}$ are all constant on each $\mathcal{C}_i$.

For each i, let ℓ_i be such that $\mathcal{C}_i$ is the preimage along θ_{ℓ_i} of a constructible subset of $\mathscr{L}_{\ell_i}(\mathcal{X})$, and let h_i be such that $\mathrm{ht}^{(0)}_{L_{\mathcal{X}/Y}} = \big(\mathrm{ht}^{(0)}_{L_{\mathcal{X}/Y}} + \mathrm{ht}^{(1)}_{L_{\mathcal{X}/Y}}\big) - \mathrm{ht}^{(1)}_{L_{\mathcal{X}/Y}}$ is equal to h_i on $\mathcal{C}_i$. Set $\ell = \max_i \ell_i$ and $h = \max_i h_i$, and let e' be such that $\mathrm{ord}_{\mathscr{J}_Y} \circ \mathscr{L}(\pi)$ is equal to e' on $\mathcal{C}$. Set $n = \max(\ell + h, e')$. We will now show that the $\mathscr{L}_n(\pi)(\theta_n(\mathcal{C}_i))$ are pairwise disjoint. To do so, it is sufficient to show that if k' is a field extension of k, $\varphi \in \mathcal{C}_i(k')$, $\varphi' \in \mathcal{C}_j(k')$, and $\theta_n(\mathscr{L}(\pi)(\varphi)) = \theta_n(\mathscr{L}(\pi)(\varphi'))$, then $i = j$. By our choice of ℓ, h, e', n, Theorem 5.1 applied to $\mathcal{C}$ implies that $\theta_{n-h}(\varphi) \cong \theta_{n-h}(\varphi')$. Because $n - h \geqslant \ell \geqslant \ell_i$, this implies that $\varphi' \in \mathcal{C}_i(k')$. Thus $\mathcal{C}_i \cap \mathcal{C}_j \neq \emptyset$, so $i = j$.

We have thus shown that the $\mathscr{L}_n(\pi)(\theta_n(\mathcal{C}_i))$ are pairwise disjoint. Therefore, D is partitioned into the cylinders $D_i = D \cap \theta_n^{-1}(\mathscr{L}_n(\pi)(\theta_n(\mathcal{C}_i)))$, where we note that each $\mathscr{L}_n(\pi)(\theta_n(\mathcal{C}_i))$ is constructible by Chevalley's theorem. Then $\mathcal{C}_i \to D_i$ satisfies the hypotheses of Proposition 6.4, and the theorem follows by summing over i. $\square$

Acknowledgements

It is a pleasure to thank Dan Abramovich and Dan Edidin for helpful conversations. We would especially like to thank Ariyan Javanpeykar and Siddharth Mathur for helpful discussions concerning Proposition 6.1.

References

ATW24 D. Abramovich, M. Temkin, and J. Włodarczyk, *Functorial embedded resolution via weighted blowings up*, Algebra Number Theory **18** (2024), no. 8, 1557–1587; doi:10.2140/ant.2024.18.1557.

Alp13 J. Alper, *Good moduli spaces for Artin stacks*, Ann. Inst. Fourier (Grenoble) **63** (2013), no. 6, 2349–2402; doi:10.5802/aif.2833.

BFK19 M. Ballard, D. Favero, and L. Katzarkov, *Variation of geometric invariant theory quotients and derived categories*, J. reine angew. Math. **746** (2019), 235–303; doi:10.1515/crelle-2015-0096.

Bal08 C. T. Balwe, *Geometric motivic integration on Artin N-stacks*, Ph.D. thesis, University of Pittsburgh, 2008, available at https://www.proquest.com/docview/304496985.

Bat98 V. V. Batyrev, *Stringy Hodge numbers of varieties with Gorenstein canonical singularities*, Integrable systems and algebraic geometry (Kobe/Kyoto, 1997) (World Sci. Publ., River Edge, NJ, 1998), 1–32.

CNS18 A. Chambert-Loir, J. Nicaise, and J. Sebag, *Motivic integration*, Progr. Math., vol. 325 (Birkhäuser/Springer, New York, 2018); doi:10.1007/978-1-4939-7887-8.

DL99 J. Denef and F. Loeser, *Germs of arcs on singular algebraic varieties and motivic integration*, Invent. Math. **135** (1999), no. 1, 201–232; doi:10.1007/s002220050284.

DL02 ———, *Motivic integration, quotient singularities and the McKay correspondence*, Compos. Math. **131** (2002), no. 3, 267–290; doi:10.1023/A:1015565912485.

DS14 W. Donovan and E. Segal, *Window shifts, flop equivalences and Grassmannian twists*, Compos. Math. **150** (2014), no. 6, 942–978; doi:10.1112/S0010437X13007641.

Eke25 T. Ekedahl, *The Grothendieck group of algebraic stacks*, in *Perspectives on Four Decades of Algebraic Geometry, Volume 1. In Memory of Alberto Collino*, Progr. Math., vol. 351 (Birkhäuser, Cham, 2025), 233–261; doi:10.1007/978-3-031-66230-0_9.

Hal15 D. Halpern-Leistner, *The derived category of a GIT quotient*, J. Amer. Math. Soc. **28** (2015), no. 3, 871–912; doi:10.1090/S0894-0347-2014-00815-8.

HS20 D. Halpern-Leistner and S. V Sam, *Combinatorial constructions of derived equivalences*, J. Amer. Math. Soc. **33** (2020), no. 3, 735–773; doi:10.1090/jams/940.

Huy06 D. Huybrechts, *Fourier–Mukai transforms in algebraic geometry*, Oxford Math. Monogr. (The Clarendon Press, Oxford Univ. Press, Oxford, 2006); doi:10.1093/acprof:oso/9780199296866.001.0001.

Loo02 E. Looijenga, *Motivic measures*, Astérisque **276** (2002), 267–297.

MM19 M. McQuillan and G. Marzo, *Very fast, very functorial, and very easy resolution of singularities*, 2019, arXiv:1906.06745.

Ryd11 D. Rydh, *Representability of Hilbert schemes and Hilbert stacks of points*, Comm. Algebra **39** (2011), no. 7, 2632–2646; doi:10.1080/00927872.2010.488678.

Ryd15 ———, *Noetherian approximation of algebraic spaces and stacks*, J. Algebra **422** (2015), 105–147; doi:10.1016/j.jalgebra.2014.09.012.

SU22 M. Satriano and J. Usatine, *Stringy invariants and toric Artin stacks*, Forum Math. Sigma **10** (2022), Paper No. e9; doi:10.1017/fms.2022.3.

SU23a ———, *Beyond twisted maps: applications to motivic integration on stacks*, 2023, arXiv:2309.11434.

SU23b ———, *Motivic integration for singular Artin stacks*, 2023, arXiv:2309.11442.

SU24 ———, *Crepant resolutions of log-terminal singularities via Artin stacks*, Duke Math. J. **173** (2024), no. 18, 3605–3646; doi:10.1215/00127094-2024-0021.

Sta21 The Stacks Project Authors, *The Stacks Project*, 2021, https://stacks.math.columbia.edu.

Yas04 T. Yasuda, *Twisted jets, motivic measures and orbifold cohomology*, Compos. Math. **140** (2004), no. 2, 396–422; doi:10.1112/S0010437X03000368.

Yas06 ———, *Motivic integration over Deligne–Mumford stacks*, Adv. Math. **207** (2006), no. 2, 707–761; doi:10.1016/j.aim.2006.01.004.

Yas24 ———, *Motivic integration over wild Deligne–Mumford stacks*, Algebr. Geom. **11** (2024), no. 2, 178–255; doi:10.14231/AG-2024-007.

Matthew Satriano msatriano@uwaterloo.ca
Department of Pure Mathematics, University of Waterloo, 200 University Ave W, Waterloo, ON N2V 2P1 Canada

Jeremy Usatine jusatine@fsu.edu
Department of Mathematics, Florida State University, 208 Love Building, 1017 Academic Way, Tallahassee, FL 32306-4510 USA