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The Euler characteristic of $\mathcal{A}_g$ via Hodge integrals

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Dedicated to Gerard van der Geer on the occasion of his 75th birthday

Abstract

We prove the Harder–Siegel formula for the Euler characteristic of $\mathcal{A}_g$ via the intersection theory of $\overline{\mathcal{M}}_g$ and a vanishing result for lambda classes on the boundary of the toroidal compactifications of $\mathcal{A}_g$, recently proven by Canning, Molcho, Oprea and Pandharipande.

1. Introduction

The moduli space of principally polarized abelian varieties of dimension g is a smooth Deligne–Mumford (DM for short) stack of dimension $\binom{g+1}{2}$, usually denoted by $\mathcal{A}_g$. Over the complex numbers, this is the same as saying that $\mathcal{A}_g$ is an orbifold, and it can be identified with the stack quotient of the Siegel upper half-space by the group of symplectic matrices with integer coefficients:

$$\mathcal{A}_g \cong [\mathcal{H}_g/\operatorname{Sp}(2g,\mathbb{Z})]\,. \tag{1.1}$$

The Gauss–Bonnet formula of Harder [Har71] shows that the Euler characteristic of $\mathcal{A}_g$ can be obtained from the volume of a fundamental domain for the action of $\operatorname{Sp}(2g,\mathbb{Z})$ on $\mathcal{H}_g$, which was known to Siegel [Sie36].

Theorem 1.1. *The Euler characteristic of $\mathcal{A}_g$ is*

$$\chi(\mathcal{A}_g) = \zeta(-1)\cdots\zeta(1-2g)\,. \tag{1.2}$$

We employ the *logarithmic Gauss–Bonnet formula* to reduce the calculation of the Euler characteristic of $\mathcal{A}_g$ to an intersection-theoretic problem on the toroidal compactifications of $\mathcal{A}_g$, which is solved via Hodge integrals and the Torelli map

On every toroidal compactification of $\mathcal{A}_g$, there is a universal semiabelian scheme $\pi\colon \mathcal{G}_g \to \overline{\mathcal{A}}_g$ with a zero section s. The *Hodge bundle* is a vector bundle of rank g defined by $\mathbb{E}_g = s^*\Omega_\pi$; its Chern classes are denoted by $\lambda_i = c_i(\mathbb{E}_g)$. Let $\overline{\mathcal{M}}_{g,n}$ denote the moduli space of stable curves of genus g with n markings, and let

$$\pi\colon \overline{\mathcal{C}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

be the universal curve, with sections $s_1,\ldots,s_n$ corresponding to the markings. It is a smooth DM stack of dimension $3g-3+n$, and it has a *Hodge bundle*, defined by $\mathbb{E}_g = \pi_*\omega_\pi$. We denote

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its Chern classes by λ_i as well. On some toroidal compactifications, there is a Torelli morphism

$$\mathrm{Tor}\colon \overline{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{A}}_g\,,$$

as shown in [Ale04, Nam76], and $\mathrm{Tor}^* \mathbb{E}_g = \mathbb{E}_g$. Esnault and Viehweg proved in [EV02] that the λ-classes satisfy *Mumford's relation*

$$(1+\lambda_1+\cdots+\lambda_g)\big(1-\lambda_1+\cdots+(-1)^g\lambda_g\big) = c\big(\mathbb{E}_g \oplus \mathbb{E}_g^\vee\big) = 1 \quad \text{in } \mathrm{CH}^*\big(\overline{\mathcal{A}}_g\big)\,. \tag{1.3}$$

On $\overline{\mathcal{M}}_{g,n}$ there are also line bundles $\mathbb{L}_i = s_i^*\omega_\pi$ representing the cotangent space at the ith marking; the first Chern class of $\mathbb{L}_i$ is ψ_i. The ψ-classes and λ-classes are part of the *tautological ring* of $\overline{\mathcal{M}}_{g,n}$ (see [FP13, Pan18] for the definition). Integrals of the form

$$\int_{\overline{\mathcal{M}}_{g,n}} \lambda_1^{a_1}\cdots\lambda_g^{a_g}\psi_1^{b_1}\cdots\psi_n^{b_n}$$

are *Hodge integrals*; they appear naturally in Gromov–Witten theory.

We will consider the locus $\mathcal{B}_g$ of semiabelian varieties that are trivial extensions of an abelian variety by a torus. The class λ_g is proportional to the class $[\mathcal{B}_g]$ when we restrict ourselves to an open subset of $\overline{\mathcal{A}}_g$ given by degenerations of torus rank at most 1; see [vdG99, EvdG05]. Then we compute the proportionality factor in two ways. The first is through the isomorphism $\mathcal{B}_g \cong \mathcal{A}_{g-1}\times \mathrm{B}\mathbb{Z}/2\mathbb{Z}$, and the second one is by pulling back via the Torelli map. We will see that the formula for the Euler characteristic (1.2) follows from the evaluation of the Hodge integrals

$$\int_{\overline{\mathcal{M}}_g} \lambda_g\lambda_{g-1}\lambda_{g-2} \quad \text{and} \quad \int_{\overline{\mathcal{M}}_{g,1}} \frac{\lambda_g\lambda_{g-1}c\big(\mathbb{E}_g^\vee\big)}{1-\psi}\,,$$

which have been computed by Faber and Pandharipande [FP00a, FP00b].

With a polarization on an abelian variety of dimension g, we can associate a list of numbers $\delta = (d_1,\dots,d_g)$, where $d_i \mid d_{i+1}$ and there is a moduli stack of abelian varieties of dimension g together with a polarization of type δ, denoted by $\mathcal{A}_{g,\delta}$. When $d_i = 1$ for all i, we recover $\mathcal{A}_g$. These moduli spaces are part of a tower of étale maps. By the degree calculations of [Iri24], the Euler characteristics of all the moduli spaces $\mathcal{A}_{g,\delta}$ are determined.

Theorem 1.2. *For a list of positive integers $\delta = (d_1,\dots,d_g)$ such that $d_i \mid d_{i+1}$ for all i,*

$$\chi(\mathcal{A}_{g,\delta}) = \left(d_g^{2g-2}d_{g-1}^{2g-6}\cdots d_1^{-2g+2} \prod_{1\leqslant i<j\leqslant g}\ \prod_{p\mid d_j/d_i} \frac{\big(1-p^{-2(j-i+1)}\big)}{\big(1-p^{-2(j-i)}\big)} \right)\chi(\mathcal{A}_g)\,,$$

where the second product is over primes p.

We will prove in Corollary 7.1 that the non-vanishing of $\chi(\mathcal{A}_g)$ implies the non-vanishing of $\lambda_{g-1}\cdots\lambda_1$ on $\mathrm{CH}^*(\mathcal{A}_g)$. This result was first established in [vdG99, Corollary 1.3] using the geometry of $\mathcal{A}_g$ over fields of positive characteristic.

Further directions

The main geometric input for this formula is that λ_g is proportional to $[\mathcal{B}_g]$ on $\mathcal{A}_g^{\leqslant 1}$, which follows from a residue calculation in [EvdG05] to express λ_g in terms of boundary strata of the toroidal compactifications of $\mathcal{A}_g$. Improvements of this result for the locus given by degenerations of abelian varieties of torus rank at most k for small k would lead to more connections to the intersection theory of $\overline{\mathcal{M}}_g$. Johannes Schmitt has checked that $[\mathcal{B}_2]$ is *not* proportional to λ_2 on $\overline{\mathcal{A}}_2$, so a deeper understanding is needed for torus rank at least 2.

Note that $\mathcal{B}_g$ is one of the two components of the closure of the *product locus* $\mathcal{A}_1 \times \mathcal{A}_{g-1}$ in $\mathcal{A}_g^{\leqslant 1}$. In [COP24], the authors compute $\mathrm{Tor}^*([\mathcal{A}_1 \times \mathcal{A}_{g-1}])$ and prove that it is a tautological[1] class on $\mathcal{M}_g^{\mathrm{ct}}$. Given our presentation of the fibered product $\mathcal{M}_g^{\leqslant 1} \times_{\mathcal{A}_g} \mathcal{B}_g$ in Lemma 5.1, which has the same form as the fibered product $\mathcal{M}_g^{\mathrm{ct}} \times_{\mathcal{A}_g} (\mathcal{A}_1 \times \mathcal{A}_{g-1})$ in [COP24], we think that it is reasonable to expect the following.

CONJECTURE 1.3. *The class* $\mathrm{Tor}^*\big([\overline{\mathcal{A}_1 \times \mathcal{A}_{g-1}}]\big)$ *lies in the tautological ring* $R^*\big(\mathcal{M}_g^{\leqslant 1}\big)$.

2. Logarithmic Euler characteristic

If $\mathcal{Y}$ is a smooth DM stack of dimension n with a smooth compactification $\overline{\mathcal{Y}}$ such that the complement D of $\mathcal{Y}$ is a normal crossing divisor, then the sheaf of meromorphic differentials having at most log poles along D is a vector bundle of rank n, denoted by $\Omega_{\overline{\mathcal{Y}}}(\log D)$. It is well known that

$$\chi(\mathcal{Y}) = (-1)^n \int_{\overline{\mathcal{Y}}} c_n(\Omega_{\overline{\mathcal{Y}}}(\log D))\,,$$

where χ is the Euler characteristic (see [CMZ22, Section 2] for a proof when $\overline{\mathcal{Y}}$ is a scheme, which generalizes step by step to smooth DM stacks). When $\mathcal{Y}$ is $\mathcal{A}_g$ and $\overline{\mathcal{Y}}$ is one of its smooth toroidal compactifications (constructed in [AMRT10] over the complex numbers and in [CF90] over the integers), $\Omega_{\overline{\mathcal{A}}_g}(\log D)$ is the *canonical extension* of $\Omega_{\mathcal{A}_g}$ (see [CF90, Examples VI.4.1] for details), so we have the following:

$$\Omega_{\overline{\mathcal{A}}_g}(\log D) = \mathrm{Sym}^2\,\mathbb{E}_g\,.$$

The top Chern class of $\mathrm{Sym}^2\,\mathbb{E}_g$ can be computed by the Giambelli formula [Ful84, Example 14.5.1] and equals

$$2^g \begin{vmatrix} \lambda_g & 0 & 0 & \dots & 0 \\ \lambda_{g-2} & \lambda_{g-1} & \lambda_g & \dots & 0 \\ \lambda_{g-4} & \lambda_{g-3} & \lambda_{g-2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_1 \end{vmatrix} = 2^g \Bigg(\lambda_g \lambda_{g-1} \cdots \lambda_1 + \sum_{k=0}^{g-1} \underbrace{\lambda_g \cdots \lambda_{g-k+1} \lambda_{g-k}^2}_{\substack{=0 \text{ by Mumford's} \\ \text{relation (1.3)}}} p_k(\lambda_1, \dots, \lambda_g) \Bigg),$$

so

$$\chi(\mathcal{A}_g) = (-1)^{\binom{g+1}{2}} 2^g \int_{\overline{\mathcal{A}}_g} \lambda_1 \cdots \lambda_g\,. \tag{2.1}$$

In particular, the integral on the right-hand side does not depend on the toroidal compactification, although more generally any integral of λ-classes is independent of the toroidal compactification since any two such compactifications are dominated by a third one, see [CF90, Definition IV.2.4], and the pullbacks of the Hodge bundles agree since they are the canonical extension of the Hodge bundle on the interior. Note that these integrals also make sense on non-smooth toroidal compactifications considering Chern classes as operational Chow classes.

An abelian variety of dimension 1 is an elliptic curve, so

$$\chi(\mathcal{A}_1) = -2 \int_{\overline{\mathcal{A}}_1} \lambda_1 = -2 \int_{\overline{\mathcal{M}}_{1,1}} \psi = \zeta(-1)\,. \tag{2.2}$$

[1] The tautological ring of $\mathcal{M}_g^{\mathrm{ct}}$, and, more generally, of any open subset $\mathcal{M} \subseteq \overline{\mathcal{M}}_{g,n}$, is defined by restriction.

3. λ_{g-k}-evaluations

Any toroidal compactification of $\mathcal{A}_g$ has a canonical map to the Satake compactification

$$\beta\colon \overline{\mathcal{A}}_g \longrightarrow \overline{\mathcal{A}}_g^{\mathrm{Sat}} = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \cdots \sqcup \mathcal{A}_0\,,$$

and we can obtain partial compactifications $\mathcal{A}_g^{\leqslant k} = \beta^{-1}(\mathcal{A}_g \sqcup \cdots \sqcup \mathcal{A}_{g-k})$ of semiabelian varieties of torus rank at most k. It is shown in [CMOP24] that $\lambda_{g-k}|_{\overline{\mathcal{A}}_g \smallsetminus \mathcal{A}_g^{\leqslant k}} = 0$.

This result guarantees that the natural integration maps

$$\epsilon_{\mathcal{A}_g^{\leqslant k}}\colon\ \mathrm{CH}^{\binom{g+1}{2}+k-g}\big(\mathcal{A}_g^{\leqslant k}\big) \longrightarrow \mathbb{Q}\,, \quad \alpha \longmapsto \int_{\overline{\mathcal{A}}_g} \overline{\alpha}\lambda_{g-k}\,,$$

where $\overline{\alpha}$ is any extension of α to $\overline{\mathcal{A}}_g$, are well defined.

On the moduli space of curves, we consider the open subsets $\mathcal{M}_g^{\leqslant k}$ of curves whose dual graph has Betti number less than or equal to k. Since the Torelli map can be extended to some toroidal compactification of $\mathcal{A}_g$, see [Nam76, Ale04], and satisfies

$$\mathrm{Tor}\big(\overline{\mathcal{M}}_g \setminus \mathcal{M}_g^{\leqslant k}\big) \subseteq \overline{\mathcal{A}}_g \setminus \mathcal{A}_g^{\leqslant k}\,,$$

we see that λ_{g-k} vanishes outside of $\mathcal{M}_g^{\leqslant k}$. We can construct integration maps

$$\epsilon_{\mathcal{M}_g^{\leqslant k}}\colon\ \mathrm{CH}^{2g-3+k}\big(\mathcal{M}_g^{\leqslant k}\big) \longrightarrow \mathbb{Q}$$

analogously.

When the Torelli morphism extends to $\overline{\mathcal{A}}_g$ and $\big[\mathcal{J}_g^{\leqslant k}\big]$ is the pushforward of 1 under this Torelli morphism to $\mathcal{A}_g^{\leqslant k}$, the integration maps are related by the identity

$$\epsilon_{\mathcal{M}_g^{\leqslant k}}(\mathrm{Tor}^*(\alpha)) = \epsilon_{\mathcal{A}_g^{\leqslant k}}\big(\alpha \cdot \big[\mathcal{J}_g^{\leqslant k}\big]\big)\,. \tag{3.1}$$

4. The locus $\mathcal{B}_g$

There is a natural map $j\colon \mathcal{A}_{g-1} \to \partial\mathcal{A}_g^{\leqslant 1}$ sending A to the semiabelian variety $A \times \mathbb{G}_m$; we denote its image by $\mathcal{B}_g$. It is isomorphic to $\mathcal{A}_{g-1} \times \mathrm{B}\mathbb{Z}/2\mathbb{Z}$ because of the extra automorphism of $\mathbb{G}_m$. The map j extends to toroidal compactifications of $\mathcal{A}_{g-1}$ and satisfies $j^*\mathbb{E}_g = \mathbb{E}_{g-1} \oplus \mathcal{O}$. Therefore, the normal bundle to j is

$$N_j = \mathrm{Sym}^2\big(\mathbb{E}_{g-1}^\vee \oplus \mathcal{O}_{\mathcal{A}_{g-1}}\big) - \mathrm{Sym}^2\big(\mathbb{E}_{g-1}^\vee\big) = \mathbb{E}_{g-1}^\vee \oplus \mathcal{O}_{\mathcal{A}_{g-1}}\,.$$

We have the following result, first proven in cohomology[2] [vdG99, Proposition 1.10] and then in the Chow groups [EvdG05, Theorem 1.1].

Theorem 4.1. *In the Chow ring of $\mathcal{A}_g^{\leqslant 1}$, the following holds:*

$$\lambda_g = \frac{|B_{2g}|}{2g}[\mathcal{B}_g]\,,$$

where B_{2g} is the gth even Bernoulli number.

We will give a new proof of the proportionality factor under the assumption that the two cycles are proportional. Let $\tau(g) \in \mathbb{Q}$ be such that $\lambda_g = \tau(g)[\mathcal{B}_g]$.

[2] In fact, the result in cohomology is enough for our calculation since the λ_{g-k}-evaluation maps factor through the cycle class map.

5. Pullback of $\mathcal{B}_g$ to $\mathcal{M}_g^{\leqslant 1}$

Consider the Cartesian diagram

$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow & \mathcal{M}_g^{\leqslant 1} \\ \downarrow & & \downarrow \scriptstyle{\mathrm{Tor}} \\ \mathcal{B}_g & \xrightarrow{j} & \mathcal{A}_g^{\leqslant 1}\,, \end{array} \tag{5.1}$$

where j is a regular embedding.

Lemma 5.1. *For a partition $\mu = (g_1, \ldots, g_l)$ of $g-1$ with $g_1 \leqslant \cdots \leqslant g_l$, let $\mathcal{M}_{1,l}^{\mathrm{cycle}}$ be the substack of $\overline{\mathcal{M}}_{1,l}$ given by curves without rational tails and whose normalization is a union of rational curves, and consider the gluing map*

$$\xi_\mu \colon \mathcal{M}_{g_1,1}^{\mathrm{ct}} \times \cdots \times \mathcal{M}_{g_l,1}^{\mathrm{ct}} \times \mathcal{M}_{1,l}^{\mathrm{cycle}} \longrightarrow \mathcal{M}_g^{\leqslant 1}$$

that attaches the marked point of the ith moduli space of compact type to the ith marked point of $\mathcal{M}_{1,l}^{\mathrm{cycle}}$. The images of the morphisms ξ_μ when μ runs through all the partitions of $g-1$ are disjoint and cover $\mathcal{Z}$.

Proof. This follows from the results of [CV11], but we give a direct proof here.

We first recall the structure of semiabelian varieties of torus rank 1. For every line bundle $\mathcal{L}$ on an abelian variety A, the *Theta group* is an algebraic group $G(\mathcal{L})$ sitting in an exact sequence

$$0 \longrightarrow \mathbb{G}_m \longrightarrow G(\mathcal{L}) \longrightarrow K(\mathcal{L}) \longrightarrow 0\,.$$

If $\mathcal{L} \in A^\vee$, then $K(\mathcal{L}) = A$, and so $G(\mathcal{L})$ is a semiabelian variety. This defines a morphism

$$A^\vee \longrightarrow \mathrm{Ext}^1_{ab}(A, \mathbb{G}_m)$$

that is in fact an isomorphism [Oor66, Proposition 17.6].

Consider a prestable curve D of genus h whose dual graph is a cycle of length k. Denote by $D_1, \ldots, D_k$ the irreducible components of $\widetilde{D}$, and consider points $p_i, q_i \in D_i$ such that q_i and p_{i+1} are identified in C. Let D^{part} be the partial normalization of D at the edge $p_1 = q_k$. Then, since a line bundle on D is equivalent to a line bundle $\mathcal{L}$ on D^{part} and an isomorphism $\mathcal{L}_{p_1} \cong \mathcal{L}_{q_k}$, we have a short exact sequence

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{Jac}(D) \longrightarrow \mathrm{Jac}(D^{\mathrm{part}}) \longrightarrow 0\,.$$

By the above discussion, this corresponds to a unique line bundle $\mathcal{M} \in \mathrm{Jac}(D^{\mathrm{part}})^\vee$, which we can identify by pullback along the Abel–Jacobi map

$$\begin{aligned} aj \colon D^{\mathrm{part}} &\longrightarrow \mathrm{Jac}(D^{\mathrm{part}})\,, \\ x \in D_i &\longmapsto (\mathcal{O}_{D_1}(p_1 - q_1), \ldots, \mathcal{O}_{D_i}(x - q_i), \ldots, \mathcal{O}_{D_k}(p_k - q_k))\,. \end{aligned}$$

We assume $k = 1$ for simplicity. In this case, D^{part} is a smooth curve, and $aj^*(\mathcal{M})$ is the line bundle over the D^{part} bundle whose fiber over a point x is the space of isomorphisms

$$\mathcal{O}(x-p)_p \cong \mathcal{O}(x-p)_q\,,$$

which is $\mathcal{O}(p-q)$. When $k \geqslant 1$, the line bundle $aj^*(\mathcal{M})$ is

$$(\mathcal{O}_{D_1}(p_1 - q_1), \ldots, \mathcal{O}_{D_k}(p_k - q_k)) \in \mathrm{Jac}(D^{\mathrm{part}})\,.$$

Therefore, the semiabelian variety $\mathrm{Jac}(D)$ lies in $\mathcal{B}_h$ if and only if $\mathcal{O}_{D_i}(p_i - q_i) = \mathcal{O}_{D_i}$, that is, if and only if all the D_i are rational.

Now consider a general stable curve C of genus g whose dual graph Γ_C has Betti number 1. There is a subgraph of Γ_C that is minimal among all the subgraphs that have Betti number 1; let $D \subset C$ be the curve that corresponds to such a graph. Then $\overline{C \smallsetminus D}$ is a disjoint union of l curves C_i of compact type, and by stability $g(C_i) > 0$. Moreover,

$$\mathrm{Jac}(C) = \mathrm{Jac}(D) \times \prod \mathrm{Jac}(C_i)\,,$$

so $\mathrm{Jac}(C)$ lies in $\mathcal{B}_g$ if and only if $\mathrm{Jac}(D)$ lies in $\mathcal{B}_g$, which happens if and only if $\widetilde{D}$ is a union of rational curves (and in particular $g(D) = 1$), so C lies in the image of ξ_μ for the partition $\mu = (g(C_1), \ldots, g(C_l))$. $\square$

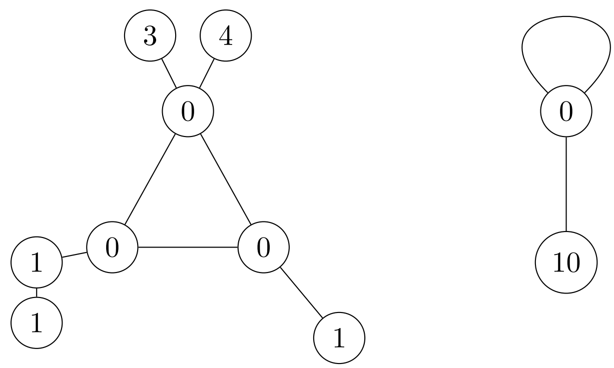

Figure 1. Two examples of the dual graph of a curve of genus 11 in $\mathcal{Z}$. They correspond to the partitions $(1, 2, 3, 4)$ and (10).

Remark 5.2. Note that

$$\dim\bigl(\mathcal{M}^{\mathrm{ct}}_{g_1,1} \times \cdots \times \mathcal{M}^{\mathrm{ct}}_{g_l,1} \times \mathcal{M}^{\mathrm{cycle}}_{1,l}\bigr) = \sum_{i=1}^{l} (3(g_i - 1) + 1) + (l - 1) = \dim\bigl(\mathcal{M}^{\leqslant 1}_g\bigr) - (l + 1)$$

and that $\mathcal{M}^{\mathrm{cycle}}_{1,l}$ is not proper except in the case $l = 1$, where it is isomorphic to $\mathrm{B}\mathbb{Z}/2\mathbb{Z}$. In particular, $\int_{\mathcal{M}^{\mathrm{cycle}}_{1,1}} 1 = \frac{1}{2}$.

When $\mu = (g - 1)$, the image of ξ_μ is defined to be the *principal locus*.

Lemma 5.3. *The principal locus is closed in $\mathcal{Z}$.*

Proof. If $\mu = (g_1, \ldots, g_l)$, then the dual graph of a curve in the image of ξ_μ is a stable cycle with l markings and a collection of rooted trees together with an assignment of a tree to each marking on the cycle. There are five types of edges:

- edges belonging to the cycle, when it has length at least 2,
- edges belonging to the tree,
- edges between a vertex of the cycle and the root of the associated tree, when the genus of the root is not 0,
- edges between a vertex of the cycle and the root of the tree, when the genus of the root is 0,
- the edge of the cycle, when it has length 1.

If we contract an edge of the first two types, the curve stays in ξ_μ. If we contract an edge of the third type, the new curve lies in the image of $\xi_{\mu'}$, where μ' is a refinement of μ, and the last two contractions take the curve outside of $\mathcal{Z}$. The partition $(g-1)$ is not the refinement of any other partition, so $\mathcal{Z}\setminus\operatorname{im}(\xi_{(g-1)})$ is closed under generalization, and so the principal locus is closed. □

In particular, $\xi_{(g-1)}$ is proper. Let $\mathcal{Z}'$ be the closure of the complement of the principal locus in $\mathcal{Z}$.

Lemma 5.4. *The class $\lambda_{g-2} \in \mathrm{CH}^{g-2}(\mathcal{M}_g^{\leqslant 1})$ vanishes when restricted to $\mathcal{Z}'$.*

Proof. From the discussion in Lemma 5.3, the points in $\xi_{(g-1)}(\mathcal{M}_{g-1,1}\times\mathcal{M}_{1,1}^{\mathrm{cycle}})$ cannot be in $\mathcal{Z}'$, so the normalization of any curve in $\mathcal{Z}'$ has to be a union of rational curves and curves of genus $g_1,\dots,g_l$ with $\sum g_i = g-1$ and $l\geqslant 2$, but note that

$$c_{g-2}\left(\bigoplus \mathbb{E}_{g_i}\right) = \sum_{i=1}^{l} c_{g_i-1}(\mathbb{E}_{g_i})\cdot \prod_{\substack{j=1,\dots,l\\ j\neq i}} c_{g_j}(\mathbb{E}_{g_j})\,,$$

which vanishes because the domain of all the morphisms ξ_μ is a space of curves of compact type and $\lambda_h|_{\mathcal{M}_h^{\mathrm{ct}}}=0$. Note that the products on the right-hand side of the equation are over a non-empty set because $l\geqslant 2$. □

The map $\xi_{(g-1)}$ is a regular embedding with normal bundle $\mathbb{L}^\vee\boxplus\mathcal{O}_{\mathcal{M}_{1,1}^{\mathrm{cycle}}}$, where $\mathbb{L}$ is the cotangent line, and therefore the excess class for the principal component in the diagram (5.1) is

$$c_{\mathrm{top}}\left(N_j - N_{\xi_{(g-1)}}\right) = c_{g-2}\left(\mathbb{E}_{g-1}^\vee - \mathbb{L}^\vee\right).$$

By the residual intersection formula [Ful84, Corollary 9.2.3],

$$j^!([\mathcal{B}_g]) = \xi_{(g-1),*}\left(\left[\frac{c(\mathbb{E}_{g-1}^\vee)}{1-\psi}\right]_{g-2}\right) + \mathbf{R}\,,$$

where $\mathbf{R}$ is the residual class supported on $\mathcal{Z}'$. By Lemma 5.4, the class $\lambda_{g-2}j^!([\mathcal{B}_g])$ has a natural extension to $\overline{\mathcal{M}}_g$, namely,

$$\lambda_{g-2}\left[\overline{\xi}_{(g-1),*}\left(\frac{c(\mathbb{E}_{g-1}^\vee)}{1-\psi}\right)\right]_g \in \mathrm{CH}^{2g-2}(\overline{\mathcal{M}}_g)\,,$$

where $\overline{\xi}_{(g-1)}\colon \overline{\mathcal{M}}_{g-1,1}\times\mathcal{M}_{1,1}^{\mathrm{cycle}}\to\overline{\mathcal{M}}_g$ is the gluing map.

Proof of Theorem 4.1. Using equation (3.1), we obtain

$$\epsilon_{\mathcal{A}_g^{\leqslant 1}}\left(\lambda_{g-2}[\mathcal{B}_g]\cdot[\mathcal{J}_g^{\leqslant 1}]\right) = \left(\int_{\mathcal{M}_{1,1}^{\mathrm{cycle}}} 1\right)\left(\int_{\overline{\mathcal{M}}_{g-1,1}} \frac{\lambda_{g-1}\lambda_{g-2}c(\mathbb{E}_{g-1}^\vee)}{1-\psi}\right) = \frac{1}{2}\frac{1}{(2g-2)!}\frac{|B_{2g-2}|}{2g-2}\,,$$

by [FP00b, Theorem 3]. In [FP00a, Theorem 4], the authors also showed that

$$\epsilon_{\mathcal{A}_g^{\leqslant 1}}\left(\lambda_{g-2}\lambda_g\cdot[\mathcal{J}_g^{\leqslant 1}]\right) = \int_{\overline{\mathcal{M}}_g} \lambda_g\lambda_{g-1}\lambda_{g-2} = \frac{1}{2(2g-2)!}\frac{|B_{2g-2}|}{2g-2}\frac{|B_{2g}|}{2g}\,.$$

Dividing the last two equations, we obtain the value of $\tau(g)$. □

6. The Euler characteristic of $\mathcal{A}_{g,\delta}$

First, we see that the evaluation of $\tau(g)$ for all g is equivalent to the knowledge of the Euler characteristic of $\mathcal{A}_g$.

Proof of Theorem 1.1. Since $j^*\mathbb{E}_g = \mathbb{E}_{g-1} \oplus \mathcal{O}$, where $j\colon \mathcal{A}_{g-1} \to \mathcal{A}_g^{\leqslant 1}$, we see that

$$\int_{\overline{\mathcal{A}}_g} \lambda_g \cdots \lambda_1 = \epsilon_{\mathcal{A}_g^{\leqslant 1}}([\tau(g)[\mathcal{B}_g]\lambda_{g-1} \cdots \lambda_1]) = \frac{\tau(g)}{2} \int_{\overline{\mathcal{A}}_{g-1}} \lambda_{g-1} \cdots \lambda_1 \,,$$

and so, by the logarithmic Gauss–Bonnet formula (2.1), we see that $\chi(\mathcal{A}_g) = (-1)^g \tau(g) \chi(\mathcal{A}_{g-1})$, and by Theorem 4.1, we have $(-1)^g \tau(g) = (-1)^g |B_{2g}|/2g = \zeta(1-2g)$. □

Proof of Theorem 1.2. We want to compute the Euler characteristic of $\mathcal{A}_{g,\delta}$ from that of $\mathcal{A}_g$. In order to compare the two moduli spaces, we introduce a level structure. Let θ be a polarization on an abelian variety. Then

$$\ker(\theta) \cong (\mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_g\mathbb{Z})^2$$

as symplectic groups, where $(d_1, \ldots, d_g)$ is the type of θ. The moduli space of triplets (A, θ, F), where $(A, \theta) \in \mathcal{A}_{g,\delta}$ and F is a symplectic basis of $\ker(\theta)$, is denoted by $\mathcal{A}_{g,\delta}^{\mathsf{lev}}$ and has étale morphisms

$$\begin{array}{ccccc} & & \mathcal{A}_{g,\delta}^{\mathsf{lev}} & & \\ & \swarrow^{\pi_\delta} & & \searrow^{\varphi_\delta} & \\ \mathcal{A}_{g,\delta} & & & & \mathcal{A}_g \,, \end{array}$$

where π_δ forgets the symplectic basis and φ_δ sends an abelian variety X to its quotient by a Lagrangian subgroup of $\ker(\theta)$; see [Iri24, Section 2] for details.

It follows that $\chi(\mathcal{A}_{g,\delta}) = (\deg(\varphi_\delta)/\deg(\pi_\delta))\chi(\mathcal{A}_g)$. The equality

$$\frac{\deg(\varphi_\delta)}{\deg(\pi_\delta)} = d_g^{2g-2} d_{g-1}^{2g-6} \cdots d_1^{-2g+2} \prod_{1 \leqslant i < j \leqslant g} \prod_{p \mid d_j/d_i} \frac{\left(1 - p^{-2(j-i+1)}\right)}{\left(1 - p^{-2(j-i)}\right)}$$

was computed in [Iri24, Propositions 26 and 27]. This proves Theorem 1.2. □

7. The Hirzebruch–Mumford proportionality theorem

Consider the Lagrangian Grassmanian LG_g, which parametrizes Lagrangian subspaces of dimension g inside a symplectic vector space of dimension $2g$. It is a smooth projective variety of dimension $\binom{g+1}{2}$. The universal Lagrangian subspace $\mathbb{S}_g \to \mathrm{LG}_g$ defines a vector bundle of rank g, with Chern classes $x_i = c_i(\mathbb{S}_g)$, and $\mathrm{Sym}^2(\mathbb{S}_g)$ is the cotangent bundle to LG_g. Mumford shows in [Mum77] that there is a constant $K(g)$ such that for any $a_i \in \mathbb{N}$,

$$\int_{\overline{\mathcal{A}}_g} \lambda_1^{a_1} \cdots \lambda_g^{a_i} = K(g) \int_{\mathrm{LG}_g} x_1^{a_1} \cdots x_n^{a_n} \,.$$

This constant has been determined in [vdG99, Theorem 1.13] by the Gauss–Bonnet formula:

$$\chi(\mathcal{A}_g) = K(g)\chi(\mathrm{LG}_g) \,.$$

An approach to prove the formula for $\chi(\mathcal{A}_g)$ could be to determine $K(g)$ in an alternative way.

Van der Geer also shows that the assignment $\lambda_i \mapsto x_i$ defines an isomorphism between the subring of $\mathrm{CH}^*(\mathcal{A}_g)$ generated by the classes $\lambda_1, \ldots, \lambda_{g-1}$ (also known as the *tautological ring*) and the cohomology ring of LG_{g-1}. One of the steps of the proof given there relies on a characteristic p argument and the ampleness of λ_1. We give a different proof, which was also known by Dragos Oprea.

COROLLARY 7.1 ([vdG99, Corollary 1.4]). *We have* $\lambda_1 \cdots \lambda_{g-1} \neq 0$ *in* $\mathrm{CH}_g(\mathcal{A}_g)$.

Proof. If the product is 0, then

$$0 = \epsilon_{\mathcal{A}_g}(\lambda_1 \cdots \lambda_{g-1}) = \int_{\overline{\mathcal{A}}_g} \lambda_1 \cdots \lambda_g = (-1)^{\binom{g+1}{2}} 2^{-g} \chi(\mathcal{A}_g)\,,$$

and we have a contradiction. □

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