



Two coniveau filtrations and algebraic equivalence over finite fields

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ABSTRACT

We extend the basic theory of the coniveau and strong coniveau filtrations to the ℓ -adic setting. By adapting the examples of Benoist–Ottem to the ℓ -adic context, we show that the two filtrations differ over any algebraically closed field of characteristic not 2. When the base field $\mathbb{F}$ is finite, we show that the equality of the two filtrations over the algebraic closure $\overline{\mathbb{F}}$ has some consequences for algebraic equivalence for codimension 2 cycles over $\mathbb{F}$. As an application, we prove that the third unramified cohomology group $H_{\text{nr}}^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ vanishes for a large class of rationally chain-connected threefolds X over $\mathbb{F}$, confirming a conjecture of Colliot-Thélène and Kahn.

1. Introduction

For a smooth complex projective variety X , the Betti cohomology $H_B^*(X, A)$ with coefficients in an abelian group A is endowed with the coniveau and strong coniveau filtrations $N^*H_B^*(X, A)$ and $\tilde{N}^*H_B^*(X, A)$, which satisfy

$$\tilde{N}^*H_B^*(X, A) \subset N^*H_B^*(X, A).$$

As shown by Deligne [Del74b, Corollaire 8.2.8] using the theory of mixed Hodge structures, the two filtrations coincide when $A = \mathbb{Q}$. In contrast, when $A = \mathbb{Z}$, Benoist–Ottem [BO21] recently proved that the two filtrations differ in general.

1.1 Two coniveau filtrations on ℓ -adic cohomology

Let X be a smooth projective variety over an algebraically closed field k . One may similarly define the coniveau and strong coniveau filtrations on $H^*(X, \mathbb{Z}_\ell)$ and $H^*(X, \mathbb{Q}_\ell)$. By replacing mixed Hodge structures by Galois-module structures and Hironaka’s resolution of singularities by De Jong’s alterations, it is not difficult to show that the two coniveau filtrations on $H^*(X, \mathbb{Q}_\ell)$ coincide; see Lemma 2.5.

Our first result shows that the two filtrations on $H^*(X, \mathbb{Z}_\ell)$ do not agree in general, generalizing the main results of Benoist–Ottem [BO21] to the ℓ -adic setting. Note that Theorem 1.1 only deals with $\ell = 2$, but similar examples may be constructed for any odd prime ℓ invertible in k ; see Remark 3.6.

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THEOREM 1.1. *Let k be an algebraically closed field of characteristic different from 2.*

- (1) *For all $c \geq 1$ and $l \geq 2c + 1$, there exists a smooth projective k -variety such that the inclusion*

$$\tilde{N}^c H^l(X, \mathbb{Z}_2) \subset N^c H^l(X, \mathbb{Z}_2)$$

is strict. Moreover, one can choose X with torsion canonical bundle, and if $c \geq 2$, one can choose Y rational.

- (2) *There exists a smooth projective fourfold X such that the inclusion*

$$\tilde{N}^1 H^3(X, \mathbb{Z}_2) \subset N^1 H^3(X, \mathbb{Z}_2)$$

is strict. Similarly, there exists a smooth projective fivefold X such that the inclusion

$$\tilde{N}^1 H^4(X, \mathbb{Z}_2) \subset N^1 H^4(X, \mathbb{Z}_2)$$

is strict.

- (3) *There exists a smooth projective k -variety X such that $H^4(X, \mathbb{Z}_2)$ is torsion-free and the inclusion*

$$\tilde{N}^1 H^4(X, \mathbb{Z}_2) \subset N^1 H^4(X, \mathbb{Z}_2)$$

is strict.

Theorem 1.1 will be proved in Section 3. As a key step in the proof of Theorem 1.1, we establish a relative version of Wu's theorem for Stiefel–Whitney classes; see Proposition 3.1.

Among the statements of Theorem 1.1, the discrepancy between $N^1 H^*$ and $\tilde{N}^1 H^*$ is the most interesting because $N^1 H^*(X, \mathbb{Z}_\ell) / \tilde{N}^1 H^*(X, \mathbb{Z}_\ell)$ is a stably birational invariant; see Lemma 2.6. This will serve as motivation for Theorem 1.2 and its applications below, which are independent of Theorem 1.1.

1.2 Algebraic equivalence over finite fields

The main purpose of this paper is to show that the *equality* of $N^1 H^3$ and $\tilde{N}^1 H^3$ has consequences for the algebraic equivalence for codimension 2 cycles over finite fields.

Let $\mathbb{F}$ be a finite field and $\bar{\mathbb{F}}$ be an algebraic closure of $\mathbb{F}$. Let ℓ be a prime number invertible in $\mathbb{F}$. Let X be a smooth projective geometrically connected $\mathbb{F}$ -variety and $\bar{X} = X \times_{\mathbb{F}} \bar{\mathbb{F}}$. We denote by $\mathrm{CH}^2(X)_{\mathrm{alg}}$ (respectively, $\mathrm{CH}^2(X)_{\mathrm{hom}}$) the subgroup of $\mathrm{CH}^2(X)$ consisting of all cycle classes whose pull-backs to $\mathrm{CH}^2(\bar{X})$ are algebraically (respectively, homologically) trivial. Recall that Jannsen [Jan90] constructed an ℓ -adic Abel–Jacobi map

$$\mathrm{cl}_{AJ}: \mathrm{CH}^2(X)_{\mathrm{hom}} \otimes \mathbb{Z}_\ell \longrightarrow H^1(\mathbb{F}, H^3(\bar{X}, \mathbb{Z}_\ell(2))).$$

Here is our main theorem.

THEOREM 1.2. *Let X be a smooth projective geometrically connected variety over a finite field $\mathbb{F}$ and ℓ be a prime number invertible in $\mathbb{F}$. Suppose*

$$N^1 H^3(\bar{X}, \mathbb{Z}_\ell(2)) = \tilde{N}^1 H^3(\bar{X}, \mathbb{Z}_\ell(2)). \quad (1.1)$$

Then the following statements hold:

- (1) *The ℓ -adic Abel–Jacobi map cl_{AJ} induces an isomorphism*

$$\mathrm{cl}_a: \mathrm{CH}^2(X)_{\mathrm{alg}} \otimes \mathbb{Z}_\ell \xrightarrow{\sim} H^1(\mathbb{F}, N^1 H^3(\bar{X}, \mathbb{Z}_\ell(2)))$$

and an isomorphism

$$\phi_\ell: \operatorname{Ker}(\operatorname{CH}^2(X) \rightarrow \operatorname{CH}^2(\overline{X}))\{\ell\} \xrightarrow{\sim} H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2))_{\operatorname{tors}}).$$

- (2) The base change map $\operatorname{CH}^2(X)_{\operatorname{alg}}\{\ell\} \rightarrow (\operatorname{CH}^2(\overline{X})_{\operatorname{alg}})^G\{\ell\}$ is surjective.
- (3) A cycle class in $\operatorname{CH}^2(X)\{\ell\}$ is algebraically trivial over $\mathbb{F}$ if and only if it is algebraically trivial over $\overline{\mathbb{F}}$.

Several remarks are in order.

- (i) The map cl_a in Theorem 1.2(1) may be thought of as a finite-field analogue of an algebraic Abel–Jacobi map over the complex numbers due to Lieberman [Lie72]. Note that the map cl_a is always injective (cf. Lemma 5.10), even without assuming (1.1), whereas over the complex numbers, the kernel of Lieberman’s map may be infinite-dimensional.
- (ii) The map ϕ_ℓ in Theorem 1.2(1) appears as the first map in an exact sequence established by Colliot-Thélène and Kahn [CK13, Théorème 6.8], which controls the Galois descent of codimension 2 cycles on X . In the course of the proof of Theorem 1.2, we refine this exact sequence; see Propositions 5.13 and 5.15.
- (iii) The assumption (1.1) is necessary for Theorem 1.2(1). Indeed, Theorem 1.1 and our previous work [SS24] provide examples of non-algebraic classes coming from $H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2))_{\operatorname{tors}})$.
- (iv) The authors do not know whether Theorem 1.2(2) and (3) remain true without assuming (1.1). More generally, the question of whether algebraic triviality over $\mathbb{F}$ is equivalent to algebraic triviality over $\overline{\mathbb{F}}$ is currently open.

1.3 A conjecture of Colliot-Thélène and Kahn

The surjectivity statements in Theorem 1.2(1) lead us to an application of the equality $N^1 H^3 = \tilde{N}^1 H^3$ to the vanishing of the third unramified cohomology group $H_{\operatorname{nr}}^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$. This abelian group is a stably birational invariant, defined in the framework of Bloch–Ogus theory [BO74].

Colliot-Thélène and Kahn posed the following conjecture.

CONJECTURE 1.3 ([CK13, Conjecture 5.7]). *For all smooth projective geometrically uniruled $\mathbb{F}$ -varieties X of dimension 3, we have $H_{\operatorname{nr}}^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$.*

Over the complex numbers, Voisin showed that $H_{\operatorname{nr}}^3(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$ for every uniruled threefold X . Over an algebraic closure of a finite field, the Tate conjecture for surfaces would imply the vanishing of the third unramified cohomology for a wider class of threefolds according to a result of Schoen [Sch98] (see [CK13, Propositions 3.2 and 4.2] for the precise statements). Conjecture 1.3 has been confirmed in the case of conic bundles over surfaces by Parimala–Suresh [PS16].

We confirm Conjecture 1.3 for a large class of rationally chain-connected threefolds.

THEOREM 1.4. *Let X be a smooth projective rationally connected threefold over a number field K . Suppose that $H_B^3(X(\mathbb{C}), \mathbb{Z})$ is torsion-free. For every prime $p \in \operatorname{Spec}(\mathcal{O}_K)$, we write k_p for the residue field of p .*

Then for all but finitely many primes $p \in \operatorname{Spec}(\mathcal{O}_K)$, there exists a finite field extension k'_p/k_p such that for all finite extensions E/k'_p , the E -variety X_E is smooth, projective, rationally chain connected and $H_{\operatorname{nr}}^3(X_E, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$ for all prime numbers ℓ invertible in k_p .

It should be noted that the implication

$$H_{\text{nr}}^3(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0 \implies H_{\text{nr}}^3(X_E, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0 \text{ for some finite extension } E/\mathbb{F}$$

is not expected to be true in general. Indeed, let X be a smooth projective geometrically connected $\mathbb{F}$ -variety as in the proof of [SS24, Theorem 1.4]. Then $H_{\text{nr}}^3(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ is divisible, but $H_{\text{nr}}^3(X_E, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ is not divisible for every finite extension $E/\mathbb{F}$. Moreover, $H_{\text{tr}}^3(\overline{X}, \mathbb{Z}_\ell) := H^3(\overline{X}, \mathbb{Z}_\ell)/N^1 H^3(\overline{X}, \mathbb{Z}_\ell) = 0$; hence by [Kah12, Théorème 5.2(c)] combined with [CK13, Propriétés 3.14(vii)], the (rational) Tate and Beilinson conjectures [CK13, Conjectures 3.11 and 3.12] together imply that $H_{\text{nr}}^3(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$ and $H_{\text{nr}}^3(X_E, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \neq 0$ for all finite extensions $E/\mathbb{F}$.

The assumptions of Theorem 1.4 are satisfied for Fano threefolds, as can be deduced from the classification. In the case of smooth Fano complete intersection threefolds, we show the following more precise statement, generalizing a result of Colliot-Thélène on smooth cubic threefolds [Col19, Theorem 5.1].

THEOREM 1.5. *Let $X \subset \mathbb{P}^N$ be a smooth Fano complete intersection threefold over a finite field $\mathbb{F}$. Suppose that the Fano scheme of lines on X is smooth of the expected dimension. Then $H_{\text{nr}}^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$.*

The Fano scheme of lines is smooth of the expected dimension if X is a general smooth Fano complete intersection threefold [DM98, Theorem 2.1] or if X is any smooth cubic threefold [AK77, Corollary 1.12].

We now discuss the main ingredients that go into the proofs of Theorems 1.4 and 1.5. Consider the ℓ -adic cycle maps

$$\text{CH}^2(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \longrightarrow H^4(X, \mathbb{Z}_\ell(2)), \quad (1.2)$$

$$\text{CH}^2(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \longrightarrow H^4(\overline{X}, \mathbb{Z}_\ell(2))^G. \quad (1.3)$$

By a theorem of Colliot-Thélène and Kahn [CK13, Theorem 2.2], the torsion subgroup of the cokernel of (1.2) is isomorphic to the quotient of $H_{\text{nr}}^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ by its maximal divisible subgroup. The maps (1.2) and (1.3) are related to each other by the following diagram, induced by the Hochschild–Serre spectral sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{CH}^2(X)_{\text{hom}} \otimes \mathbb{Z}_\ell & \longrightarrow & \text{CH}^2(X) \otimes \mathbb{Z}_\ell & & \\ & & \downarrow \text{cl}_{AJ} & & \downarrow (1.2) & \searrow (1.3) & \\ 0 & \longrightarrow & H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2))) & \xrightarrow{\iota} & H^4(X, \mathbb{Z}_\ell(2)) & \longrightarrow & H^4(\overline{X}, \mathbb{Z}_\ell(2))^G \longrightarrow 0. \end{array} \quad (1.4)$$

Thus the surjectivity of the map (1.2) is equivalent to the surjectivity of the two maps (1.3) and cl_{AJ} .

An important observation is that if $H^3(\overline{X}, \mathbb{Z}_\ell(2)) = \tilde{N}^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))$, then cl_{AJ} is surjective. Based on this, we deduce from results of Voisin on the coniveau filtrations of rationally connected complex threefolds [Voi06, Voi22] and a spreading-out argument that for X as in the statement of Theorem 1.4, the maps (1.3) and cl_{AJ} are surjective. Therefore, (1.2) is surjective, and a standard argument using the aforementioned [CK13, Theorem 2.2] and the fact that $\text{CH}_0(\overline{X}_{\overline{K}}) = \mathbb{Z}$, where K is the function field of $\overline{X}$, allows us to conclude $H_{\text{nr}}^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$.

We conclude by proposing the following question.

Question 1.6. For a smooth projective geometrically uniruled threefold X over a finite field $\mathbb{F}$ and for a prime number ℓ invertible in $\mathbb{F}$, do we have $H^3(\overline{X}, \mathbb{Z}_\ell(2)) = \tilde{N}^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))$?

Note that for all geometrically uniruled threefolds X over a finite field $\mathbb{F}$, the decomposition of the diagonal yields the identity $H^3(\overline{X}, \mathbb{Z}_\ell(2)) = N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))$.

This paper is organized as follows. In Section 2, we introduce the ℓ -adic coniveau and strong coniveau filtrations and discuss their basic properties. In Section 3, we prove a relative version of Wu's theorem for étale Stiefel–Whitney classes and Theorem 1.1. In Section 4, we construct the map cl_a . More generally, for every $i \geq 0$, we construct an ℓ -adic Walker Abel–Jacobi map with domain $\text{CH}^i(X)_{\text{alg}} \otimes \mathbb{Z}_\ell$; the map cl_a corresponds to the case $i = 2$. In Section 5, we prove basic properties of the ℓ -adic Walker Abel–Jacobi map for varieties over finite fields and then study cl_a in detail. In Section 6, we prove Theorems 1.2, 1.4, and 1.5.

1.4 Additional note on this paper

The results of this paper originally appeared on arXiv in June 2022 and have since been used by Tian [Tia25] to prove the surjectivity of the map (1.2) for fibrations into rational surfaces over curves defined over finite fields.

1.5 Notation

Let k be a field. We denote by $\overline{k}$ a separable closure of k and by $G := \text{Gal}(\overline{k}/k)$ the absolute Galois group of k . For a continuous G -module M , we denote by $H^i(k, M) := H^i(G, M)$ the continuous Galois cohomology of M . When k is a finite field, we denote it by $\mathbb{F}$.

If X is a k -scheme, we define $\overline{X} := X \times_k \overline{k}$. A k -variety is a separated k -scheme of finite type. For a smooth k -variety X of pure dimension and an integer $i \geq 0$, we denote by $\text{CH}^i(X)$ the Chow group of codimension i cycles on X modulo rational equivalence. If ℓ is a prime invertible in k , the notation $H^j(X, \mathbb{Z}_\ell(m))$ will mean $\varprojlim_n H^j(X, \mu_{\ell^n}^{\otimes m})$. (We will only use it when k is finite or separably closed.) If the smooth equidimensional k -variety is also projective and k is separably closed, we set

$$H_j(X, \mathbb{Z}_\ell(m)) := H^{2 \dim(X) - 2j}(X, \mathbb{Z}_\ell(\dim(X) - m)).$$

We write $H^j(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(m)) \simeq \varprojlim H^j(X, \mu_{\ell^n}^{\otimes m})$ for the étale cohomology of the sheaf $\mathbb{Q}_\ell/\mathbb{Z}_\ell(m)$. We let $H_{\text{nr}}^i(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(m))$ be the corresponding unramified cohomology group.

For an abelian group A , integer $n \geq 1$, and prime number ℓ , we set $A[n] := \{a \in A \mid na = 0\}$, denote by $A\{\ell\}$ the subgroup of ℓ -primary torsion elements of A , by A_{tors} the subgroup of torsion elements of A , and set $A_{\text{tf}} := A/A_{\text{tors}}$.

2. Coniveau and strong coniveau

2.1 Two coniveau filtrations

Let k be a separably closed field, ℓ be a prime number invertible in k , and X be a smooth projective connected k -variety of dimension d . The classical *coniveau* filtration is defined by

$$\begin{aligned} N^c H^j(X, \mathbb{Z}_\ell(m)) &:= \sum_{Z \subset X} \text{Im}(H_Z^j(X, \mathbb{Z}_\ell(m)) \longrightarrow H^j(X, \mathbb{Z}_\ell(m))) \\ &= \sum_{Z \subset X} \text{Ker}(H^j(X, \mathbb{Z}_\ell(m)) \longrightarrow H^j(X - Z, \mathbb{Z}_\ell(m))), \end{aligned} \tag{2.1}$$

where $Z \subset X$ runs over the closed subvarieties of codimension at least c . Similarly, the *strong coniveau* filtration is defined by

$$\tilde{N}^c H^j(X, \mathbb{Z}_\ell(m)) := \sum_{f: T \rightarrow X} \text{Im}(f_*: H^{j-2r}(T, \mathbb{Z}_\ell(m-r)) \longrightarrow H^j(X, \mathbb{Z}_\ell(m))), \quad (2.2)$$

where the sum is over all smooth projective connected k -varieties T of dimension $d-r$ with $r \geq c$ and morphisms $f: T \rightarrow X$; in fact, we may restrict to morphisms $f: T \rightarrow X$ where $\dim T = d-c$ by taking the product with a projective space of suitable dimension.

To establish the basic properties of the two coniveau filtrations in the ℓ -adic setting, the existence of prime-to- ℓ variants of De Jong's alterations, due to Gabber, will be crucial.

THEOREM 2.1. *Let Z be a projective irreducible k -variety. Then there exist a smooth projective connected k -variety $\tilde{Z}$ and a generically finite morphism $g: \tilde{Z} \rightarrow Z$ such that $\deg(g)$ is prime to ℓ .*

Proof. This is a special case of a theorem of Gabber [IT14, Theorem 2.1]. □

LEMMA 2.2. *We have*

$$\tilde{N}^c H^j(X, \mathbb{Z}_\ell(m)) = \sum_{(T, \Gamma)} \text{Im}(\Gamma_*: H^{j-2r}(T, \mathbb{Z}_\ell(m-r)) \longrightarrow H^j(X, \mathbb{Z}_\ell(m))), \quad (2.3)$$

where (T, Γ) runs over all smooth projective connected k -varieties T of dimension $e-r$ and $\Gamma \in \text{CH}^e(T \times X)$ for all $e \geq r \geq c$.

Proof. If $f: T \rightarrow X$ is a morphism of smooth projective varieties and $\Gamma \subset T \times X$ is the graph of f , then $f_* = \Gamma_*$. Therefore, the left side of (2.3) is contained in the right side of (2.3).

Conversely, let (T, Γ) be a pair as in (2.3). In particular, $\dim \Gamma = d-r$. Write $\Gamma = \sum_i n_i \Gamma_i$, where the Γ_i are the irreducible components of $\text{Supp}(\Gamma)$. For each i , let $g_i: \tilde{\Gamma}_i \rightarrow \Gamma_i$ be an ℓ' -alteration, which exists by Theorem 2.1; set $m_i = \deg(g_i)$, which is prime to ℓ . If we write $p: T \times X \rightarrow T$ and $q: T \times X \rightarrow X$ for the projections, then, for every $\alpha \in H^{j-2r}(T, \mathbb{Z}_\ell(m-c))$, we have

$$\Gamma_* \alpha = q_*(p^* \alpha \cdot \Gamma) = \sum_i n_i q_*(p^* \alpha \cdot \Gamma_i) = \sum_i \frac{n_i}{m_i} q_*(g_i)_*(g_i)^* p^* \alpha,$$

which shows that $\Gamma_* \alpha \in \tilde{N}^c H^j(X, \mathbb{Z}_\ell(m))$. The result follows. □

LEMMA 2.3. *The coniveau and strong coniveau filtrations are respected by correspondences. That is, for a smooth projective connected k -variety Y and $\Gamma \in \text{CH}^i(X \times Y)$, the action Γ_* of Γ on $H^j(X, \mathbb{Z}_\ell(m))$ induces homomorphisms*

$$\begin{aligned} \Gamma_*: N^c H^j(X, \mathbb{Z}_\ell(m)) &\longrightarrow N^{c+i-d} H^{j+2(i-d)}(Y, \mathbb{Z}_\ell(m+i-d)), \\ \Gamma_*: \tilde{N}^c H^j(X, \mathbb{Z}_\ell(m)) &\longrightarrow \tilde{N}^{c+i-d} H^{j+2(i-d)}(Y, \mathbb{Z}_\ell(m+i-d)). \end{aligned}$$

Proof. To show that Γ_* respects the coniveau filtration, we first choose a closed subvariety $Z \subset X$ of codimension at least c such that the homomorphism $H_Z^j(X, \mathbb{Z}_\ell(m)) \rightarrow N^c H^j(X, \mathbb{Z}_\ell(m))$ is surjective. We then apply the moving lemma [Sta24, Tag 0B0D] to Γ to ensure that it intersects properly with $Z \times Y$. Now the assertion follows from the fact that the image of $\Gamma_*: N^c H^j(X, \mathbb{Z}_\ell(m)) \rightarrow H^{j+2(i-d)}(Y, \mathbb{Z}_\ell(m+i-d))$ is contained in the image of the map $H_W^{j+2(i-d)}(Y, \mathbb{Z}_\ell(m+i-d)) \rightarrow H^{j+2(i-d)}(Y, \mathbb{Z}_\ell(m+i-d))$, where $W \subset Y$ is the image of $\text{Supp}(\Gamma) \cap (Z \times Y)$ under the projection $X \times Y \rightarrow Y$. The assertion for the strong coniveau filtration follows from Lemma 2.2 using the composition of correspondences. □

We give another description of the deepest part of the strong coniveau filtration in odd degree.

LEMMA 2.4. *For all $i \geq 1$, we have*

$$\tilde{N}^{i-1} H^{2i-1}(X, \mathbb{Z}_\ell(i)) = \sum_{(S, \Gamma)} \operatorname{Im}(\Gamma_*: H_1(S, \mathbb{Z}_\ell) \longrightarrow H^{2i-1}(X, \mathbb{Z}_\ell(i))), \quad (2.4)$$

where S runs over all smooth projective connected k -varieties and $\Gamma \in \operatorname{CH}^i(S \times X)$. Moreover, in (2.4) we may restrict to pairs (S, Γ) where $\dim S = 1$.

Proof. The proof is analogous to those of [Voi22, Lemma 1.2 and Proposition 1.3]. Lemma 2.2 yields

$$\tilde{N}^{i-1} H^{2i-1}(X, \mathbb{Z}_\ell(i)) = \sum_{(T, \Gamma)} \operatorname{Im}(\Gamma_*: H^1(T, \mathbb{Z}_\ell(1)) \longrightarrow H^{2i-1}(X, \mathbb{Z}_\ell(i))),$$

where (T, Γ) runs over all smooth projective connected k -varieties T of dimension $d - i + 1$ and $\Gamma \in \operatorname{CH}^d(T \times X)$. For any such pair (T, Γ) , the Poincaré line bundle $\mathcal{P} \in \operatorname{CH}^1(\operatorname{Pic}_{T/k}^0 \times T)$ gives an isomorphism

$$\mathcal{P}_*: H_1(\operatorname{Pic}_{T/k}^0, \mathbb{Z}_\ell) \xrightarrow{\sim} H^1(T, \mathbb{Z}_\ell(1)).$$

Now the composition $\Gamma \circ \mathcal{P} \in \operatorname{CH}^i(\operatorname{Pic}_{T/k}^0 \times X)$ gives a homomorphism

$$(\Gamma \circ \mathcal{P})_* = \Gamma_* \circ \mathcal{P}_*: H_1(\operatorname{Pic}_{T/k}^0, \mathbb{Z}_\ell) \longrightarrow H^{2i-1}(X, \mathbb{Z}_\ell(i))$$

whose image coincides with that of Γ_* . This shows that the left term of (2.4) is contained in the right term of (2.4).

It remains to show the opposite inclusion. One may first reduce to the case where $\dim S = 1$ by the Lefschetz hyperplane section theorem [Mil80, Theorem VI.7.1]. Then the assertion follows by taking ℓ' -alterations (Theorem 2.1) of the components of $\Gamma \in \operatorname{CH}^i(S \times X)$ for a smooth projective connected k -curve S , as in the proof of Lemma 2.2. The proof is complete. $\square$

2.2 Comparison of coniveau and strong coniveau

We maintain the notation of Section 2.1. Here we extend some of the results over the complex numbers shown in [BO21, §2.2] to an arbitrary separably closed field. The results below only concern the two coniveau filtrations on the integral cohomology $H^j(X, \mathbb{Z}_\ell)$ as groups. Since the twist is irrelevant here, we omit it.

LEMMA 2.5. *For all $j, c \geq 0$, the group $N^c H^j(X, \mathbb{Z}_\ell) / \tilde{N}^c H^j(X, \mathbb{Z}_\ell)$ is finite.*

Proof. It suffices to prove the equality

$$N^c H^j(X, \mathbb{Q}_\ell) = \tilde{N}^c H^j(X, \mathbb{Q}_\ell).$$

In order to prove this, it is in turn enough to prove that if Y is a smooth projective connected k -variety, $\iota: Z \hookrightarrow Y$ is an equidimensional closed subscheme, $\pi: \tilde{Z} \rightarrow Z$ is an alteration, and $q := \iota \circ \pi$, then the sequence

$$H^*(\tilde{Z}, \mathbb{Q}_\ell) \xrightarrow{q^*} H^*(Y, \mathbb{Q}_\ell) \longrightarrow H^*(Y \setminus Z, \mathbb{Q}_\ell)$$

is exact. This follows from a weight argument, an ℓ -adic analogue of the Hodge-theoretic proof of [Del74b, Corollaire 8.2.8]. We give the details below.

Using Poincaré duality, we reduce to showing that

$$\operatorname{Ker}(\iota^*: H^*(Y, \mathbb{Q}_\ell) \longrightarrow H^*(Z, \mathbb{Q}_\ell)) = \operatorname{Ker}(q^*: H^*(Y, \mathbb{Q}_\ell) \longrightarrow H^*(\tilde{Z}, \mathbb{Q}_\ell)).$$

By the rigidity property of étale cohomology, we may assume that the base field is an algebraic closure of a field of finite type over the prime field. A spreading-out argument combined with specialization maps for proper (not necessarily smooth) morphisms of schemes [Sta24, Tags 0GJ2 and 0DDF] then reduces the proof to the case where the base field is an algebraic closure of a finite field. In this case, we use Frobenius weights; see [Del80] for the definition and basic properties. By [Del74a, Theorem 1.6], the group $H^n(Y, \mathbb{Q}_\ell)$ is pure of weight n . Using the exactness of the functor Gr_n^W , it remains to show that the induced map

$$\mathrm{Gr}_n^W(\pi^*): \mathrm{Gr}_n^W(H^n(Z, \mathbb{Q}_\ell)) \longrightarrow \mathrm{Gr}_n^W(H^n(\tilde{Z}, \mathbb{Q}_\ell))$$

is injective. Let $D \subset Z$ be a Cartier divisor containing the singular locus of Z and $E := \pi^{-1}(D)$. Then π^* induces a map of long exact sequences of cohomology with compact support

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^{n-1}(D, \mathbb{Q}_\ell) & \longrightarrow & H_c^n(Z \setminus D, \mathbb{Q}_\ell) & \longrightarrow & H^n(Z, \mathbb{Q}_\ell) & \longrightarrow & H^n(D, \mathbb{Q}_\ell) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H^{n-1}(E, \mathbb{Q}_\ell) & \longrightarrow & H_c^n(\tilde{Z} \setminus E, \mathbb{Q}_\ell) & \longrightarrow & H^n(\tilde{Z}, \mathbb{Q}_\ell) & \longrightarrow & H^n(E, \mathbb{Q}_\ell) & \longrightarrow & \cdots \end{array}$$

In the above diagram, the second vertical map $\pi^*: H_c^n(Z \setminus D, \mathbb{Q}_\ell) \rightarrow H_c^n(\tilde{Z} \setminus E, \mathbb{Q}_\ell)$ is injective because the composition with $\pi_*: H_c^n(\tilde{Z} \setminus E, \mathbb{Q}_\ell) \rightarrow H_c^n(Z \setminus D, \mathbb{Q}_\ell)$ is the multiplication by the degree of $\pi: \tilde{Z} \setminus E \rightarrow Z \setminus D$ on each component. Using that $H^{n-1}(E, \mathbb{Q}_\ell)$ is mixed of weight at most $n-1$ by [Del80, Theorem 3.13], an easy diagram chase shows that

$$\begin{aligned} & \mathrm{Ker}(\mathrm{Gr}_n^W(\pi^*): \mathrm{Gr}_n^W(H^n(Z, \mathbb{Q}_\ell)) \longrightarrow \mathrm{Gr}_n^W(H^n(\tilde{Z}, \mathbb{Q}_\ell))) \\ & \subset \mathrm{Ker}(\mathrm{Gr}_n^W(\pi^*): \mathrm{Gr}_n^W(H^n(D, \mathbb{Q}_\ell)) \longrightarrow \mathrm{Gr}_n^W(H^n(E, \mathbb{Q}_\ell))). \end{aligned}$$

Using an alteration of E and induction on $\dim Z$, we see that the latter group is zero. The result now follows. $\square$

LEMMA 2.6. *The group $N^1 H^j(X, \mathbb{Z}_\ell) / \tilde{N}^1 H^j(X, \mathbb{Z}_\ell)$ is a stably birational invariant of smooth projective connected k -varieties X .*

Proof. We first show that $N^1 H^j(X, \mathbb{Z}_\ell) / \tilde{N}^1 H^j(X, \mathbb{Z}_\ell)$ is invariant under replacing X by $X \times \mathbb{P}^n$. Let $\pi: X \times \mathbb{P}^n \rightarrow X$ be the first projection. For the proof, the argument of [BO21, Lemma 2.3] works with a slight modification. The only necessary change to be made is where it is shown that, for a class $\alpha \in H^j(X, \mathbb{Z}_\ell)$, if $\pi^* \alpha \in H^j(X \times \mathbb{P}^n, \mathbb{Z}_\ell)$ has strong coniveau at least 1, then α has strong coniveau at least 1. (The issue is that Bertini's theorem for basepoint-free linear systems is not available in positive characteristic.) Here we may assume $\pi^* \alpha = f_* \beta$, where $f: T \rightarrow X \times \mathbb{P}^n$ is a morphism from a smooth projective connected k -variety T of dimension $d+n-1$ and $\beta \in H^{j-2}(T, \mathbb{Z}_\ell)$. Let $i: X \rightarrow X \times \mathbb{P}^n$ be the inclusion of a general fiber $X \times \{p\}$. Then $W = T \times_{X \times \mathbb{P}^n} X$ is a (possibly singular) subvariety of X of pure dimension $d-1$. As a cycle, write $W = \sum n_i W_i$, where the W_i are the irreducible components of W . For each i , let $g_i: \tilde{W}_i \rightarrow W_i$ be an ℓ' -alteration (Theorem 2.1), and set $m_i = \deg(g_i)$, which is prime to ℓ . It is straightforward to see that

$$\alpha = \pi_*(\pi^* \alpha \cdot i_* X) = \pi_*(f_* \beta \cdot i_* X) = \pi_* f_*(\beta \cdot W) = \sum_i \frac{n_i}{m_i} \pi_* f_*(g_i)_*(g_i)^* \beta,$$

which shows that $\alpha \in \tilde{N}^1 H^j(X, \mathbb{Z}_\ell)$.

We now show the birational invariance of $N^1 H^j(X, \mathbb{Z}_\ell) / \tilde{N}^1 H^j(X, \mathbb{Z}_\ell)$; this will complete the proof. We avoid the use of Hironaka's resolution of singularities or the weak factorization

theorem as in the proof of [BO21, Proposition 2.4], and instead adapt the strategy of [Ful98, Example 16.1.11] and apply ℓ' -alterations (Theorem 2.1). Let X, Y be two smooth projective connected k -varieties, $\varphi: X \dashrightarrow Y$ be a birational map, and Γ_φ be the closure of the graph of φ . The transpose Γ_φ^T agrees with the closure of the graph of φ^{-1} . Then the composition $\Gamma_\varphi^T \circ \Gamma_\varphi$ may be written as

$$\Gamma_\varphi^T \circ \Gamma_\varphi = \Delta_X + \Gamma \in \mathrm{CH}^d(X \times X), \quad (2.5)$$

where Γ is supported on $D \times D$ for some divisor $D \subset X$.

We claim that $\Gamma_*(N^1 H^j(X, \mathbb{Z}_\ell)) \subset \tilde{N}^1 H^j(X, \mathbb{Z}_\ell)$. Indeed, let $p, q: X \times X \rightarrow X$ be the projections. We write $\Gamma = \sum n_i \Gamma_i$, where the Γ_i are the irreducible components of $\mathrm{Supp}(\Gamma)$. For each i , let $Z_i = q(\Gamma_i)$ so that Z_i is a proper closed subset of X , and let $f_i: \widetilde{Z}_i \rightarrow Z_i$ be an ℓ' -alteration. Then choose an irreducible component Γ'_i of $\Gamma_i \times_{Z_i} \widetilde{Z}_i$ that dominates both $\widetilde{Z}_i$ and Γ_i and is such that $\deg(\Gamma'_i/\Gamma_i)$ is prime to ℓ ; such a component exists by the projection formula. Then take another ℓ' -alteration $\widetilde{\Gamma}_i \rightarrow \Gamma'_i$. Now let $g_i: \widetilde{\Gamma}_i \rightarrow \Gamma_i$ and $h_i: \widetilde{\Gamma}_i \rightarrow \widetilde{Z}_i$ be the resulting maps so that $q \circ g_i = f_i \circ h_i$. Set $m_i = \deg(g_i)$; it is an integer prime to ℓ . For every class $\alpha \in H^j(X, \mathbb{Z}_\ell)$, we have

$$\Gamma_* \alpha = \sum_i \frac{n_i}{m_i} q_*(g_i)_*(g_i)^* p^* \alpha = \sum_i \frac{n_i}{m_i} (f_i)_*(h_i)_*(g_i)^* p^* \alpha.$$

This shows that $\Gamma_* \alpha \in \tilde{N}^1 H^j(X, \mathbb{Z}_\ell)$. Thus $\Gamma_*(N^1 H^j(X, \mathbb{Z}_\ell)) \subset \tilde{N}^1 H^j(X, \mathbb{Z}_\ell)$, as claimed.

By Lemma 2.3, the homomorphisms Γ_* , $(\Gamma_\varphi)_*$, and $(\Gamma_\varphi^T)_*$ respect the groups $N^1 H^j(X, \mathbb{Z}_\ell)$ and $\tilde{N}^1 H^j(X, \mathbb{Z}_\ell)$, and therefore induce endomorphisms of $N^1 H^j(X, \mathbb{Z}_\ell)/\tilde{N}^1 H^j(X, \mathbb{Z}_\ell)$. The claim shows that Γ_* is zero on $N^1 H^j(X, \mathbb{Z}_\ell)/\tilde{N}^1 H^j(X, \mathbb{Z}_\ell)$; hence by (2.5), we deduce that

$$(\Gamma_\varphi^T)_* \circ (\Gamma_\varphi)_* = (\Delta_X)_* = \mathrm{id} \quad \text{on } N^1 H^j(X, \mathbb{Z}_\ell)/\tilde{N}^1 H^j(X, \mathbb{Z}_\ell).$$

Repeating the argument with X replaced by Y , we obtain an isomorphism

$$(\Gamma_\varphi)_*: N^1 H^j(X, \mathbb{Z}_\ell)/\tilde{N}^1 H^j(X, \mathbb{Z}_\ell) \xrightarrow{\sim} N^1 H^j(Y, \mathbb{Z}_\ell)/\tilde{N}^1 H^j(Y, \mathbb{Z}_\ell).$$

This completes the proof. $\square$

COROLLARY 2.7. *If X is stably rational, then $N^1 H^j(X, \mathbb{Z}_\ell) = \tilde{N}^1 H^j(X, \mathbb{Z}_\ell)$.*

Corollary 2.7 may be strengthened using the cohomological decomposition of the diagonal.

LEMMA 2.8. *If X admits a decomposition of the diagonal in $H^{2d}(X \times X, \mathbb{Z}_\ell)$, then $N^1 H^j(X, \mathbb{Z}_\ell) = \tilde{N}^1 H^j(X, \mathbb{Z}_\ell)$.*

Proof. The assertion is clear when $j = 0$. Suppose $j \geq 1$. In this case, we are going to prove that $H^j(X, \mathbb{Z}_\ell) = N^1 H^j(X, \mathbb{Z}_\ell) = \tilde{N}^1 H^j(X, \mathbb{Z}_\ell)$.

Let $[\Delta_X] = [x \times X] + [\Gamma] \in H^{2d}(X, \mathbb{Z}_\ell)$ be the decomposition of the diagonal, where $x \in X$ is a closed point and the cycle Γ is supported on $X \times D$ for some divisor $D \subset X$. Applying ℓ' -alterations (Theorem 2.1) to Γ in the same way as in the proof of Lemma 2.6 yields that for every class $\alpha \in H^j(X, \mathbb{Z}_\ell)$, we have $\alpha = \Gamma_* \alpha \in \tilde{N}^1 H^j(X, \mathbb{Z}_\ell)$. The result now follows. $\square$

COROLLARY 2.9. *If X is retract rational, then $N^1 H^j(X, \mathbb{Z}_\ell) = \tilde{N}^1 H^j(X, \mathbb{Z}_\ell)$.*

Proof. Under the assumption, the diagonal element in $\mathrm{CH}_0(X_{k(X)})$ is contained in the image of the restriction map $\mathrm{CH}_0(X) \rightarrow \mathrm{CH}_0(X_{k(X)})$ by results of Merkurjev [Mer08, Theorem 2.11 and Proposition 2.15]. By taking the closure of the diagonal element in $X \times X$, we see that this property is equivalent to the integral Chow decomposition of the diagonal, which in turn implies the cohomological decomposition. $\square$

We conclude this section by pointing out a homological analogue of Lemma 2.6. We define $N_c H_j(X, \mathbb{Z}_\ell) := N^{d-c} H^{2d-j}(X, \mathbb{Z}_\ell)$.

LEMMA 2.10. *The quotient*

$$N_{i-1} H_i(X, \mathbb{Z}_\ell) / \tilde{N}_{i-1} H_i(X, \mathbb{Z}_\ell)$$

is a stably birational invariant of smooth projective k -varieties.

The invariant of Lemma 2.10 is zero if $\dim X < i$. To the authors' knowledge, it is zero in all known examples.

Proof. We will prove that $H_i(X, \mathbb{Z}_\ell) / \tilde{N}_{i-1} H_i(X, \mathbb{Z}_\ell)$ is a stably birational invariant. A similar argument will show that $H_i(X, \mathbb{Z}_\ell) / N_{i-1} H_i(X, \mathbb{Z}_\ell)$ is also a stably birational invariant, and the statement will follow.

We first establish the invariance of $H_i(X, \mathbb{Z}_\ell) / \tilde{N}_{i-1} H_i(X, \mathbb{Z}_\ell)$ under taking the product with $\mathbb{P}^1$. By the projective bundle formula and the Lefschetz hyperplane section theorem, one has

$$H_i(X \times \mathbb{P}^1, \mathbb{Z}_\ell) = H_i(X, \mathbb{Z}_\ell) + \tilde{N}_{i-1} H_i(X \times \mathbb{P}^1, \mathbb{Z}_\ell).$$

Hence the push-forward $H_i(X \times \mathbb{P}^1, \mathbb{Z}_\ell) \rightarrow H_i(X, \mathbb{Z}_\ell)$ induces an isomorphism

$$H_i(X \times \mathbb{P}^1, \mathbb{Z}_\ell) / \tilde{N}_{i-1} H_i(X \times \mathbb{P}^1, \mathbb{Z}_\ell) \xrightarrow{\sim} H_i(X, \mathbb{Z}_\ell) / \tilde{N}_{i-1} H_i(X, \mathbb{Z}_\ell).$$

We now prove the birational invariance of $H_i(X, \mathbb{Z}_\ell) / \tilde{N}_{i-1} H_i(X, \mathbb{Z}_\ell)$. Suppose that X and Y are birational and $\phi: X \dashrightarrow Y$ is a birational map. Let Γ_ϕ be the closure of the graph of the birational map ϕ . We aim to show the isomorphism

$$(\Gamma_\phi)_*: H_i(X, \mathbb{Z}_\ell) / \tilde{N}_{i-1} H_i(X, \mathbb{Z}_\ell) \xrightarrow{\sim} H_i(Y, \mathbb{Z}_\ell) / \tilde{N}_{i-1} H_i(Y, \mathbb{Z}_\ell).$$

The composition $\Gamma_\phi^T \circ \Gamma_\phi$ may be written as $\Gamma_\phi^T \circ \Gamma_\phi = \Delta_X + \Gamma \in \text{CH}^n(X \times X)$, where Γ is supported on $D \times D$ for some divisor $D \subset X$. It remains for us to show that Γ_* acts as zero on $H_i(X, \mathbb{Z}_\ell) / \tilde{N}_{i-1} H_i(X, \mathbb{Z}_\ell)$.

Arguing as in the proof of Lemma 2.10, for an ℓ' -prime alteration $f: \tilde{D} \rightarrow D$ (which exists by Theorem 2.1), one may find a correspondence $\tilde{\Gamma} \in \text{CH}^{n-1}(\tilde{D} \times X) \otimes \mathbb{Z}_\ell$ such that $\Gamma = f_* \tilde{\Gamma}$. Therefore, for every $\alpha \in H_i(X, \mathbb{Z}_\ell)$, one has $\Gamma_* \alpha = \tilde{\Gamma}_* f^* \alpha$. This, together with the Lefschetz hyperplane section theorem and the functoriality of the filtration $\tilde{N}_*$ (cf. Lemma 2.3) yields

$$\Gamma_* H_i(X, \mathbb{Z}_\ell) \subset \tilde{\Gamma}_* H_{i-2}(\tilde{D}, \mathbb{Z}_\ell) = \tilde{\Gamma}_* \tilde{N}_{i-2} H_{i-2}(\tilde{D}, \mathbb{Z}_\ell) \subset \tilde{N}_{i-1} H_i(X, \mathbb{Z}_\ell).$$

This concludes the proof. $\square$

2.3 Coniveau subgroups as Galois modules

Here we prepare for Sections 4, 5, 6, where we will work in an arithmetic setting. We let k be an arbitrary field (not necessarily separably closed), ℓ be a prime number invertible in k , and X be a smooth projective geometrically connected k -variety of dimension d . Recall that we denote by G the absolute Galois group of k . For all $j, m \geq 0$, the natural G -action on $\bar{X}$ makes $H^j(\bar{X}, \mathbb{Z}_\ell(m))$ into a continuous G -module.

LEMMA 2.11. *For all $j, m, c \geq 0$,*

$$N^c H^j(\bar{X}, \mathbb{Z}_\ell(m)) = \sum_{Z \subset X} \text{Im}(H_{\bar{Z}}^j(\bar{X}, \mathbb{Z}_\ell(m)) \rightarrow H^j(\bar{X}, \mathbb{Z}_\ell(m))), \quad (2.6)$$

where $Z \subset X$ runs over the closed subvarieties of codimension at least c , and

$$\tilde{N}^c H^j(\bar{X}, \mathbb{Z}_\ell(m)) = \sum_{f: T \rightarrow X} \text{Im}(f_*: H^{j-2c}(\bar{T}, \mathbb{Z}_\ell(m-c)) \longrightarrow H^j(\bar{X}, \mathbb{Z}_\ell(m))), \quad (2.7)$$

where the sum is over all smooth projective connected k -varieties T of dimension $d-c$ and morphisms $f: T \rightarrow X$. As a consequence, the subgroups $N^c H^j(\bar{X}, \mathbb{Z}_\ell(m))$ and $\tilde{N}^c H^j(\bar{X}, \mathbb{Z}_\ell(m))$ of $H^j(\bar{X}, \mathbb{Z}_\ell(m))$ are G -invariant. In particular, they are continuous G -modules.

Proof. For (2.6), let $Z' \subset \bar{X}$ be a closed subvariety. The Galois orbit $Z'' \subset \bar{X}$ of Z' under the G -action on $\bar{X}$ descends to a closed subvariety $Z \subset X$. Since $Z' \subset Z'' = \bar{Z}$, we have

$$\text{Im}(H_{Z'}^j(\bar{X}, \mathbb{Z}_\ell(m)) \longrightarrow H^j(\bar{X}, \mathbb{Z}_\ell(m))) \subset \text{Im}(H_{\bar{Z}}^j(\bar{X}, \mathbb{Z}_\ell(m)) \longrightarrow H^j(\bar{X}, \mathbb{Z}_\ell(m))).$$

This proves (2.6).

For (2.7), let $f': T' \rightarrow \bar{X}$ be a morphism of smooth projective connected $\bar{k}$ -varieties. Then f' descends to a morphism of smooth projective connected K -varieties $f'': T' \rightarrow X_K$ for some separable finite extension K/k , which may be regarded as a morphism of smooth projective connected k -varieties. Letting $\pi: X_K \rightarrow X$ be the natural projection, the composition

$$f: T \xrightarrow{f''} X_K \xrightarrow{\pi} X$$

is then a morphism of smooth projective connected k -varieties. Now $\bar{T} = T \times_k \bar{k}$ (respectively, $\bar{X}_K = X_K \times_k \bar{k}$) is the disjoint union of $[K:k]$ copies of T' (respectively, $\bar{X}$), and the base change $\bar{T} \rightarrow \bar{X}$ of f may be identified with the composition

$$\coprod_{n=1}^{[K:k]} T' \xrightarrow{\coprod i_n \circ f'} \coprod_{n=1}^{[K:k]} \bar{X} \xrightarrow{\coprod \text{id}} \bar{X}.$$

Hence we have

$$\begin{aligned} \text{Im}(f'_*: H^{j-2r}(T', \mathbb{Z}_\ell(m-c)) \longrightarrow H^j(\bar{X}, \mathbb{Z}_\ell(m))) \\ = \text{Im}(f_*: H^{j-2r}(\bar{T}, \mathbb{Z}_\ell(m-c)) \longrightarrow H^j(\bar{X}, \mathbb{Z}_\ell(m))). \end{aligned}$$

This proves (2.7). $\square$

We conclude this section by refining Lemma 2.4. This result shows an analogy between strong coniveau and algebraic equivalence, and it motivates the proof of Theorem 1.2; see also Lemma 4.3.

LEMMA 2.12. *For all $i \geq 1$, we have*

$$\tilde{N}^{i-1} H^{2i-1}(\bar{X}, \mathbb{Z}_\ell(i)) = \sum_{(C, \Gamma)} \text{Im}(\Gamma_*: H_1(\bar{C}, \mathbb{Z}_\ell) \longrightarrow H^{2i-1}(\bar{X}, \mathbb{Z}_\ell(i))),$$

where (C, Γ) runs over all pairs where C is a smooth projective geometrically connected k -curve and $\Gamma \in \text{CH}^i(C \times X)$.

Proof. We learned the idea of using symmetric powers of curves from [ACV19, §3.4]. Let S be a smooth projective connected k -curve and $\Gamma \in \text{CH}^i(S \times X)$. (We do not assume that S is geometrically connected.) Let $\bar{S}_1, \dots, \bar{S}_e$ be the connected components of $\bar{S}$, and let

$$\text{Sym}^\Delta(\bar{S}) = \bar{S}_1 \times \dots \times \bar{S}_e \subset \text{Sym}^e(\bar{S}).$$

Note that $\text{Sym}^e(\bar{S})$ is smooth and $\text{Sym}^\Delta(\bar{S})$ is a connected component of $\text{Sym}^e(\bar{S})$. Since $\text{Sym}^\Delta(\bar{S})$ is fixed by G , it descends to a geometrically connected component $\text{Sym}^\Delta(S)$ of $\text{Sym}^e(S)$.

Let $p: \text{Sym}^e(S) \times S \rightarrow \text{Sym}^e(S)$ and $q: \text{Sym}^e(S) \times S \rightarrow S$ be the projections. We view $\text{Sym}^e(S)$ as a closed subscheme of the Hilbert scheme of S and write $U \subset \text{Sym}^e(S) \times S$ for the universal family. Let

$$\text{Sym}^e(\Gamma) := (p|_U \times \text{id}_X)_*(q|_U \times \text{id}_X)^*\Gamma \in \text{CH}^i(\text{Sym}^e(S) \times X),$$

and let $\text{Sym}^\Delta(\Gamma) \in \text{CH}^i(\text{Sym}^\Delta(S) \times X)$ be the restriction of $\text{Sym}^e(\Gamma)$ over $\text{Sym}^\Delta(S)$. The choice of a closed point of degree e on S determines a morphism $f: S \rightarrow \text{Sym}^\Delta(S)$ by the universal property of symmetric powers, and we have

$$\text{Sym}^\Delta(\Gamma)_* \circ f_* = (f^* \text{Sym}^\Delta(\Gamma))_* = \Gamma_*$$

on $H_1(\overline{S}, \mathbb{Z}_\ell)$. It follows that

$$\begin{aligned} \text{Im}(\Gamma_*: H_1(\overline{S}, \mathbb{Z}_\ell) &\longrightarrow H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))) \\ &\subset \text{Im}(\text{Sym}^\Delta(\Gamma)_*: H_1(\text{Sym}^\Delta(\overline{S}), \mathbb{Z}_\ell) \longrightarrow H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))). \end{aligned}$$

Finally, let $C \subset \text{Sym}^\Delta(S)$ be a smooth complete intersection k -curve. (Such C exists, by Bertini's theorem when the field k is infinite, and by Poonen's theorem [Poo04] when k is finite.) In particular, C is geometrically connected. Moreover, letting $i: C \rightarrow \text{Sym}^\Delta(S)$ be the inclusion, the Lefschetz hyperplane section theorem [Mil80, Theorem VI.7.1] implies that the homomorphism

$$i_*: H_1(\overline{C}, \mathbb{Z}_\ell) \longrightarrow H_1(\text{Sym}^\Delta(\overline{S}), \mathbb{Z}_\ell)$$

is surjective. Now the pull-back $i^* \text{Sym}^\Delta(\Gamma) \in \text{CH}^i(C \times X)$ yields a homomorphism

$$(i^* \text{Sym}^\Delta(\Gamma))_* = \text{Sym}^\Delta(\Gamma)_* \circ i_*: H_1(\overline{C}, \mathbb{Z}_\ell) \longrightarrow H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))$$

whose image coincides with that of $\text{Sym}^\Delta(\Gamma)_*$ and contains that of Γ_* . Hence we may replace the pair (S, Γ) in Lemma 2.4 with the pair $(C, i^* \text{Sym}^\Delta \Gamma)$. This completes the proof. $\square$

3. A relative version of Wu's theorem and proof of Theorem 1.1

The purpose of this section is to prove Theorem 1.1, which extends [BO21, Theorem 4.3 and Proposition 4.6] to all algebraically closed fields of characteristic different from 2.

Let k be a field, X be a smooth k -variety, and E be a vector bundle over X . If $\iota: X \hookrightarrow Y$ is a codimension r closed embedding of smooth k -varieties, we denote by $s_{X/Y} \in H_X^{2r}(Y, \mu_\ell^{\otimes r})$ the cycle class of X , that is, the image of $1 \in H^0(X, \mathbb{Z}/\ell)$ under the Gysin homomorphism

$$\phi_{X/Y}: H^*(X, \mathbb{Z}/\ell) \longrightarrow H_X^{*+2r}(Y, \mu_\ell^{\otimes r}).$$

Now assume $\ell = 2$. We define the Steenrod squares

$$\text{Sq}^j: H_X^*(Y, \mathbb{Z}/2) \longrightarrow H_X^{*+j}(Y, \mathbb{Z}/2)$$

as in [Fen20, Definition 5.1]. These Steenrod squares satisfy all the basic properties of the topological Steenrod squares: compatibility with respect to pull-backs, Cartan formula, Adem relations, and for all $\alpha \in H_X^i(Y, \mathbb{Z}/2)$, the identities $\text{Sq}^0(\alpha) = \alpha$, $\text{Sq}^i(\alpha) = \alpha^2$, and $\text{Sq}^j(\alpha) = 0$ for all $j > i$; see for example [Fen20, Definition 3.4]. (Strictly speaking, [Fen20, Definition 3.4] is stated for topological spaces. The point is that combining this with Friedlander's result [Fen20, Proposition 3.5] shows that all of these properties immediately extend to X .)

The last property implies that the total Steenrod square

$$\mathrm{Sq}: H_X^*(Y, \mathbb{Z}/2) \longrightarrow H_X^*(Y, \mathbb{Z}/2), \quad \mathrm{Sq}(\alpha) = \sum_{j=0}^{\infty} \mathrm{Sq}^j(\alpha),$$

is well defined, and the Cartan property implies that it is a homomorphism of graded rings.

Following [Ura96] and [Fen20, Definition 5.2], for all $j \geq 0$, we define the j th Stiefel–Whitney class of E by

$$w_j(E) := \phi_{X/E}^{-1}(\mathrm{Sq}^j s_{X/E}).$$

The total Stiefel–Whitney class of E is

$$w(E) := \sum_{j=0}^{\infty} w_j(E).$$

If E is a vector bundle on X , we denote by $[E]$ its class in the Grothendieck group $K_0(X)$. Every class $\alpha \in K_0(X)$ can be written as $\alpha = [E] - [E']$, where E and E' are vector bundles on X ; we let $w(\alpha) := w(E)w(E')^{-1}$ be the total Stiefel–Whitney class of α .

The following is a relative version of Wu’s theorem. This complements [BO21, Proposition 3.2], where the setting of smooth manifolds is considered.

PROPOSITION 3.1. *Let k be an algebraically closed field of characteristic different from 2. Let $f: Y \rightarrow X$ be a morphism of smooth projective k -varieties and $N_f := [f^*T_X] - [T_Y] \in K_0(Y)$ be the virtual normal bundle of f . For all $\beta \in H^*(Y, \mathbb{Z}/2)$, we have*

$$\mathrm{Sq}(f_*\beta) = f_*(\mathrm{Sq}(\beta) \cup w(N_f))$$

in $H^*(X, \mathbb{Z}/2)$.

Proof. The following argument is similar to the proof of the Grothendieck–Riemann–Roch theorem. Let $g: Z \rightarrow Y$ be another morphism of smooth projective varieties. Then

$$N_{f \circ g} = (f \circ g)^*[T_X] - [T_Z] = g^*(f^*[T_X] - [T_Y]) + g^*[T_Y] - [T_X] = g^*N_f + N_g.$$

Since w is multiplicative, we have

$$w(N_{f \circ g}) = w(g^*N_f) \cup w(N_g) = g^*w(N_f) \cup w(N_g).$$

Therefore, if the conclusion of the proposition holds for f and g , then it holds for $f \circ g$. Indeed,

$$\begin{aligned} (f \circ g)_*(\mathrm{Sq}(\beta) \cup w(N_{f \circ g})) &= f_*g_*(\mathrm{Sq}(\beta) \cup g^*w(N_f) \cup w(N_g)) \\ &= f_*(g_*(\mathrm{Sq}(\beta) \cup w(N_g)) \cup w(N_f)) \\ &= f_*(\mathrm{Sq}(g_*\beta) \cup w(N_f)) \\ &= \mathrm{Sq}(f_*g_*(\beta)) = \mathrm{Sq}((f \circ g)_*\beta). \end{aligned}$$

In the second equality, we have used the projection formula in étale cohomology, and in the third and fourth equalities, we have used the fact that the conclusion holds for f and g .

Every morphism of smooth projective varieties factors as the composition of a closed embedding and a projection map $X \times \mathbb{P}_k^n \rightarrow X$. We may thus assume that f is either a closed embedding or the first projection $X \times \mathbb{P}_k^n \rightarrow X$.

First suppose that $f: X \times \mathbb{P}_k^n \rightarrow X$ is the first projection. Let $\pi: X \times \mathbb{P}_k^n \rightarrow \mathbb{P}_k^n$ be the second projection. Then

$$N_f = f^*[T_X] - [f^*T_X \oplus \pi^*T_{\mathbb{P}_k^n}] = f^*[T_X] - f^*[T_X] - \pi^*[T_{\mathbb{P}_k^n}] = -\pi^*[T_{\mathbb{P}_k^n}].$$

We have $H^*(\mathbb{P}_k^n, \mathbb{Z}/2) = (\mathbb{Z}/2)[h]/(h^{n+1})$, where $|h| = 2$ and

$$w(T_{\mathbb{P}_k^n}) = (1+h)^{n+1} \in H^*(T_{\mathbb{P}_k^n}, \mathbb{Z}/2).$$

Moreover, $H^*(X \times \mathbb{P}_k^n, \mathbb{Z}/2) \simeq H^*(X)[h]/(h^{n+1})$; hence there exist $b_i \in H^*(X)$ such that

$$\beta = \sum_{i=0}^n b_i h^i.$$

We have $f_*(\beta) = b_n$, hence $\text{Sq}(f_*\beta) = \text{Sq}(b_n)$. Moreover, we have $\text{Sq}(\beta) = \sum_i \text{Sq}(b_i) \text{Sq}(h)^i$ and $\text{Sq}(h) = h + h^2$; hence

$$f_*(\text{Sq}(\beta) \cup w(N_f)) = f_*\left(\sum_{i=0}^n \text{Sq}(b_i)(h + h^2)^i \cup (1+h)^{n+1}\right) = \text{Sq}(b_n) = \text{Sq}(f_*\beta).$$

Now suppose that $f: Y \rightarrow X$ is a closed embedding of codimension r . In this case, $N_f = [N_{Y/X}]$ is the class of the normal bundle of Y inside X . Letting $\mathcal{X} \rightarrow \mathbb{A}_k^1$ be the deformation of $Y \hookrightarrow X$ to the normal bundle $N_{Y/X}$, we have a commutative diagram

$$\begin{array}{ccccc} Y & \hookrightarrow & Y \times \mathbb{A}_k^1 & \hookleftarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ N_{Y/X} & \hookrightarrow & \mathcal{X} & \hookleftarrow & X. \end{array}$$

Passing to étale cohomology gives

$$\begin{array}{ccccc} H^*(Y, \mathbb{Z}/2) & \xleftarrow{\sim} & H^*(Y \times \mathbb{A}_k^1, \mathbb{Z}/2) & \xrightarrow{\sim} & H^*(Y, \mathbb{Z}/2) \\ \uparrow \wr & & \uparrow \wr & & \uparrow \wr \\ H_Y^*(N_{Y/X}, \mathbb{Z}/2) & \xleftarrow{\sim} & H_{Y \times \mathbb{A}_k^1}^{*+2r}(\mathcal{X}, \mathbb{Z}/2) & \xrightarrow{\sim} & H_Y^{*+2r}(X, \mathbb{Z}/2), \end{array}$$

where the vertical isomorphisms are Gysin maps. By [Fen20, Lemma 6.7], the isomorphisms of the first row send $N_f = [N_{Y/X}]$ to $[N_{Y/N_{Y/X}}]$.

Therefore, it is enough to prove the following: If $\phi: H^*(Y, \mathbb{Z}/2) \rightarrow H_Y^*(E, \mathbb{Z}/2)$ is the Gysin map associated with the zero-section of the vector bundle $\pi: E \rightarrow Y$, then

$$\phi(\text{Sq}(\beta) \cup w(E)) = \text{Sq}(\phi(\beta))$$

for all $\beta \in H^*(Y, \mathbb{Z}/2)$. To prove this, recall that ϕ is an $H^*(Y, \mathbb{Z}/2)$ -linear isomorphism given by pull-back and multiplication by a class U such that $w(E) = \phi^{-1}(\text{Sq}(U))$; that is, $\phi(\alpha) = \pi^*\alpha \cup U$ for all $\alpha \in H^*(Y, \mathbb{Z}/2)$. It follows that

$$\begin{aligned} \phi(\text{Sq}(\beta) \cup w(E)) &= \pi^*(\text{Sq}(\beta)) \cup \pi^*(w(E)) \cup U \\ &= \text{Sq}(\pi^*\beta) \cup \text{Sq}(U) = \text{Sq}(\pi^*\beta \cup U) = \text{Sq}(\phi(\beta)), \end{aligned}$$

as desired. $\square$

Using Proposition 3.1, we are able to extend the main results of [BO21, § 3] to fields of positive characteristic. For all $n \geq 1$, we define

$$S_n := \text{Sq}^{2^n-1} \text{Sq}^{2^n-5} \cdots \text{Sq}^7 \text{Sq}^3.$$

LEMMA 3.2. *For all $j \geq 1$ and $1 \leq i \leq 2^i - 1$, there exists a mod 2 cohomology operation S such that $\text{Sq}^{2^i-1} S_j = S \text{Sq}^1$.*

Proof. This follows from the Adem relations as in the proof of [BO21, Lemma 3.1]. $\square$

PROPOSITION 3.3. *Let $f: Y \rightarrow X$ be a morphism of smooth projective varieties over an algebraically closed field k of characteristic different from 2.*

(1) *For all $j \geq 1$ and all $\beta \in H^*(Y, \mathbb{Z}/2)$ such that $\text{Sq}^1(\beta) = 0$, we have*

$$S_j(f_*\beta) = f_*S_j(\beta).$$

(2) *Let $\beta \in H^m(Y, \mathbb{Z}/2)$ be such that $\text{Sq}^1(\beta) = 0$. Then for all $j \geq \max(m, 2)$, we have $S_j(f_*\beta) = 0$ in $H^*(X, \mathbb{Z}/2)$.*

Proof. (1) For every smooth k -variety B , every vector bundle $E \rightarrow B$, and every $i \geq 0$, the class $w^{2i+1}(E) \in H^{2i+1}(B, \mathbb{Z}/2)$ is trivial. Since w is multiplicative, $w^{2i+1}(N_f) = 0$ for all $i \geq 0$. The result now follows as in the proof of [BO21, Proposition 3.3].

(2) This follows from part (1) as in the proof of [BO21, Proposition 3.4]. $\square$

If X is a (simplicial) scheme of finite type over an algebraically closed field k of characteristic different from 2, we write π_2 for the homomorphism $H^i(X, \mathbb{Z}_2) \rightarrow H^i(X, \mathbb{Z}/2)$ given by reduction modulo 2.

PROPOSITION 3.4. *Let X be a smooth projective variety over an algebraically closed field k of characteristic different from 2, let $\alpha \in H^m(X, \mathbb{Z}_2)$, and let c, j be integers such that $m \leq 2c + j$, $j \geq 2$, and $S_j(\pi_2(\alpha)) \neq 0$. Then α has strong coniveau less than c .*

Proof. Let $d = \dim(X)$. We have $\text{Sq}^1(\pi_2(\alpha)) = 0$. Suppose that α has strong coniveau at least c , that is, that there exist a morphism $f: Y \rightarrow X$, where Y is a smooth projective variety of dimension $d - c$, and a class $\beta \in H^{m-2c}(Y, \mathbb{Z}_2)$ such that $f_*(\beta) = \alpha$. Then $\text{Sq}^1(f_*(\beta)) = 0$ and $f_*(\pi_2(\beta)) = \pi_2(\alpha)$. Since $j \geq \max(m - 2c, 2)$, by Proposition 3.3(2), we see that $S_j(\pi_2(\alpha)) = S_j(f_*\pi_2(\beta)) = 0$, which contradicts the assumption. Therefore, α has strong coniveau less than c , as desired. $\square$

If $\mathcal{G}$ is a linear algebraic group over a field k , we write $B\mathcal{G}$ for its classifying space: It is the simplicial scheme over k defined in [Fri82, Example 1.2]. If $\mathcal{G}$ is a split reductive group over $\mathbb{Z}$ and ℓ is a prime number invertible in k , then $H^*(B\mathcal{G}_k, \mathbb{Z}/\ell) \simeq H_B^*(B\mathcal{G}(\mathbb{C}); \mathbb{Z}/\ell)$ and $H^*(B\mathcal{G}_k, \mathbb{Z}_\ell) \simeq H_B^*(B\mathcal{G}(\mathbb{C}); \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ by [Fri82, Proposition 8.8].

Proof of Theorem 1.1(1) and (2). (1) Let $m \geq 1$ be an integer. By [Eke09, Theorem 1.3 and §1.1], there exist a smooth projective k -variety Z_m with torsion canonical bundle and a k -morphism $\varphi': Z_m \rightarrow B(\mathbb{Z}/2\mathbb{Z}) \times B\mathbb{G}_m$ such that, letting

$$\varphi: Z_m \rightarrow B(\mathbb{Z}/2\mathbb{Z}) \times B\mathbb{G}_m \longrightarrow B(\mathbb{Z}/2\mathbb{Z})$$

be the composition of φ' with the first projection, the induced map

$$\varphi^*: H^*(B(\mathbb{Z}/2\mathbb{Z}) \times B\mathbb{G}_m, \mathbb{Z}_2) \longrightarrow H^*(Z_m, \mathbb{Z}_2)$$

is injective in degrees less than or equal to m .

Define $s := l - 2c + 1$ and fix $m \geq 2^{s+1}$. By [BO21, Lemma 4.1], there exists a class ζ in $H^s(B(\mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z})$ such that $S_s \text{Sq}^1(\zeta) \neq 0$. We let $V := Z_m^s$, let $\varphi^s: Z_m^s \rightarrow B(\mathbb{Z}/2\mathbb{Z})^s$ be the product of s copies of φ , and set $\xi := (\varphi^s)^*(\zeta)$. By Künneth's theorem, $(\varphi^s)^*$ is injective in degrees less than or equal to m ; hence $S_l \text{Sq}^1(\xi) = (\varphi^s)^* S_l \text{Sq}^1(\zeta) \neq 0$.

Let T be a smooth projective k -variety of dimension $c - 1$. Define $X := V \times T$, and set $\alpha := \text{pr}_1^* \tilde{\beta}(\xi) \cup \text{pr}_2^*(\lambda)$, where $\lambda \in H^{2c-2}(T, \mathbb{Z}_2)$ is the class of a point $t \in T(k)$ and $\text{pr}_1: X \rightarrow V$ and $\text{pr}_2: X \rightarrow T$ are the projections.

Note that $\text{Sq}^i(\pi_2(\lambda)) = 0$ for all $i > 0$ for degree reasons, and recall that $\text{Sq}^1 = \pi_2 \circ \tilde{\beta}$. Therefore, by Cartan's formula,

$$S_l(\pi_2(\alpha)) = \text{pr}_1^* S_l \text{Sq}^1(\xi) \cup \text{pr}_2^*(\pi_2(\lambda)).$$

We deduce from Künneth's formula that $S_l(\pi_2(\alpha)) \neq 0$; hence by Proposition 3.4, the class α has strong coniveau less than c . On the other hand, since $\tilde{\beta}(\xi)$ is torsion, it has coniveau at least 1 by [Kah12, Lemme 3.12]. Since α is the push-forward of $\tilde{\beta}(\xi)$ by the codimension $c - 1$ closed immersion $V \times \{t\} \rightarrow V \times T$, it has coniveau at least c . To find X with torsion canonical bundle, it suffices to pick T with torsion canonical bundle. If $c \geq 2$, we may now find X rational following the procedure of [BO21, Theorem 4.3(ii)].

(2) We claim that there exist a smooth projective fourfold Z over k and a 2-torsion class $\sigma \in H^3(Z, \mathbb{Z}_2)$ such that the reduction modulo 2 of σ^2 is non-zero. Indeed, if $k = \mathbb{C}$, this is proved in [BO21, Proposition 5.3]. If k is an algebraically closed field of characteristic zero, the claim follows from the case $k = \mathbb{C}$, using the invariance of étale cohomology under extensions of algebraically closed fields and the fact that Z can be defined over $\overline{\mathbb{Q}}$. If k is the algebraic closure of a finite field of characteristic not 2, the claim was proved in the course of the proof of [SS24, Theorem 1.5]. Finally, if k is an algebraically closed field of characteristic not 2, the claim follows from the invariance of étale cohomology under extensions of algebraically closed fields.

The proof now follows from the previous claim, in the same way that [BO21, Theorem 5.4] follows from [BO21, Proposition 5.3]. More precisely, in the fourfold case, we let $X = Z$ and $\alpha = \sigma$. In the fivefold case, we choose an elliptic curve E and a class $\tau \in H^1(E, \mathbb{Z}_2)$ whose reduction modulo 2 is non-zero, and we let $X = Z \times E$ and $\alpha = p_1^* \sigma \cup p_2^* \tau$.

In either case, α is 2-torsion, hence has coniveau at least 1 by [Kah12, Lemme 3.12]. In the fourfold case, we have $\text{Sq}^3(\pi_2(\alpha)) = \pi_2(\sigma)^2 \neq 0$. In the fivefold case, we have $\text{Sq}^1(\pi_2(\sigma)) = \text{Sq}^1(\pi_2(\tau)) = \text{Sq}^3(\pi_2(\tau)) = 0$; Cartan's formula implies $\text{Sq}^3(\pi_2(\alpha)) = p_1^* \pi_2(\sigma)^2 \cup p_2^* \pi_2(\tau) \neq 0$. Therefore, in both cases, Proposition 3.4 shows that α has strong coniveau 0. $\square$

LEMMA 3.5. *Let k be a field of positive characteristic, let $i \geq 0$ be an integer, and let ℓ be a prime number invertible in k . Suppose that $\mathcal{G}$ be a reductive group scheme over $\mathbb{Z}$ and that $\alpha \in H^{2i}(B_{\overline{\mathbb{Q}}} \mathcal{G}, \mathbb{Z}_\ell(i))$ belongs to the image of the cycle map*

$$\text{CH}^i(B_{\overline{\mathbb{Q}}} \mathcal{G}) \otimes \mathbb{Z}_\ell \longrightarrow H^{2i}(B_{\overline{\mathbb{Q}}} \mathcal{G}, \mathbb{Z}_\ell(i)).$$

Then the specialization of α in $H^{2i}(B_{\overline{k}} \mathcal{G}, \mathbb{Z}_\ell(i))$ belongs to the image of

$$\text{CH}^i(B_k \mathcal{G}) \otimes \mathbb{Z}_\ell \longrightarrow H^{2i}(B_{\overline{k}} \mathcal{G}, \mathbb{Z}_\ell(i)).$$

Proof. The proof is entirely analogous to that of [SS24, Lemma 4.4(a)]. By the invariance of étale cohomology under extensions of algebraically closed fields, we may suppose $k = \mathbb{F}_p$. There exist a $\mathcal{G}_{\mathbb{Z}_p}$ -representation V and an open subscheme $U \subset V$ such that $V_{\mathbb{F}_p} - U_{\mathbb{F}_p}$ and $V_{\mathbb{Q}_p} - U_{\mathbb{Q}_p}$ have codimension at least $i + 1$ in $V_{\mathbb{F}_p}$ and $V_{\mathbb{Q}_p}$, respectively, and a $\mathcal{G}_{\mathbb{Z}_p}$ -torsor $U \rightarrow B$, where B is a smooth $\mathbb{Z}_p$ -scheme.

Let $R := W(\overline{\mathbb{F}_p})$. Fix an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ and an inclusion of $\mathbb{Z}_p$ -algebras $R \subset \overline{\mathbb{Q}_p}$.

We obtain a commutative diagram

$$\begin{array}{ccc}
 Z^i(B_{\mathbb{Q}_p}) \otimes \mathbb{Z}_\ell & \longrightarrow & H^{2i}(B_{\overline{\mathbb{Q}_p}}, \mathbb{Z}_\ell(i)) \\
 \uparrow & & \uparrow \\
 Z_{\text{flat}}^i(B/\mathbb{Z}_p) \otimes \mathbb{Z}_\ell & \longrightarrow & H^{2i}(B_R, \mathbb{Z}_\ell(i)) \\
 \downarrow & & \downarrow \\
 Z^i(B_{\mathbb{F}_p}) \otimes \mathbb{Z}_\ell & \longrightarrow & H^{2i}(B_{\overline{\mathbb{F}_p}}, \mathbb{Z}_\ell(i)).
 \end{array} \tag{3.1}$$

Here $Z_{\text{flat}}^i(B/\mathbb{Z}_p)$ is the free abelian group generated by the classes of integral subschemes of B that are flat (that is, dominant) over $\mathbb{Z}_p$. The horizontal map in the middle is defined as the inverse limit in n of the cycle maps

$$Z_{\text{flat}}^i(B/\mathbb{Z}_p) \longrightarrow H^{2i}(B, \mu_{\ell^n}^{\otimes i}) \longrightarrow H^{2i}(B_R, \mu_{\ell^n}^{\otimes i})$$

of [Del77, Chapitre 4, § 2.3] (where we take $S = \text{Spec}(\mathbb{Z}_p)$ in the definition). The top and bottom horizontal homomorphisms are the usual cycle maps, the top vertical maps are pull-backs along the flat morphisms $B_{\mathbb{Q}_p} \rightarrow B$ and $B_{\overline{\mathbb{Q}_p}} \rightarrow B_R$, and the bottom vertical maps are pull-backs along the closed embeddings $B_{\mathbb{F}_p} \hookrightarrow B$ and $B_{\overline{\mathbb{F}_p}} \hookrightarrow B_R$. The top-left vertical map is surjective: Indeed, if $Z \subset B_{\mathbb{Q}_p}$ is an integral subscheme, its closure inside B is irreducible, hence flat over R .

Since $V_{\mathbb{F}_p} - U_{\mathbb{F}_p}$ and $V_{\mathbb{Q}_p} - U_{\mathbb{Q}_p}$ have codimension at least $i+1$ in $V_{\mathbb{F}_p}$ and $V_{\mathbb{Q}_p}$, respectively, by definition we have $\text{CH}^i(B_{\mathbb{Q}_p}) = \text{CH}^i(B_{\mathbb{Q}_p}G)$ and $\text{CH}^i(B_{\mathbb{F}_p}) = \text{CH}^i(B_{\mathbb{F}_p}G)$. Moreover, the natural morphism $B \rightarrow B_{\mathbb{Z}_p}G$ induces a commutative diagram

$$\begin{array}{ccc}
 H^{2i}(B_{\overline{\mathbb{Q}_p}}, \mathbb{Z}_\ell(i)) & \xrightarrow{\sim} & H^{2i}(B_{\overline{\mathbb{Q}_p}}G, \mathbb{Z}_\ell(i)) \\
 \uparrow & & \uparrow \\
 H^{2i}(B_R, \mathbb{Z}_\ell(i)) & \xrightarrow{\sim} & H^{2i}(B_RG, \mathbb{Z}_\ell(i)) \\
 \downarrow & & \downarrow \\
 H^{2i}(B_{\overline{\mathbb{F}_p}}, \mathbb{Z}_\ell(i)) & \xrightarrow{\sim} & H^{2i}(B_{\overline{\mathbb{F}_p}}G, \mathbb{Z}_\ell(i)).
 \end{array}$$

By [FP81, Corollary 2], the two vertical maps on the right are isomorphisms. The conclusion follows from (3.1). $\square$

Proof of Theorem 1.1(3). Let $\mathcal{G}$ be a semisimple k -group of type G_2 . By [Fri82, Proposition 8.8] and [AK16, § 2.4, Theorem 2.19], the $\mathbb{Z}_2$ -module $H^4(B\mathcal{G}, \mathbb{Z}_2)$ is torsion-free and contains a class whose reduction modulo 2 is not killed by Sq^3 . By the Künneth formula, the same holds for $B(\mathcal{G} \times \mathbb{G}_m)$. By [Eke09, Theorem 1.3], there exist a smooth projective k -variety X and a morphism $f: X \rightarrow B(\mathcal{G} \times \mathbb{G}_m)$ such that the pull-back map

$$f^*: H^*(B(\mathcal{G} \times \mathbb{G}_m), \mathbb{Z}_2) \longrightarrow H^*(X, \mathbb{Z}_2)$$

is an isomorphism in degrees less than or equal to 8. In particular, $H^4(X, \mathbb{Z}_2)$ is torsion-free. Moreover, the five lemma implies that

$$f^*: H^*(B(\mathcal{G} \times \mathbb{G}_m), \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^*(X, \mathbb{Z}/2)$$

is an isomorphism in degrees less than or equal to 7. Therefore, there exists a class in $H^4(X, \mathbb{Z}_2)$ whose reduction modulo 2 is not killed by Sq^3 . The class α has strong coniveau 0 by Proposi-

tion 3.4. A multiple of α is algebraic: If $k = \mathbb{C}$, this is due to Edidin and Graham [EG97]; if $\text{char}(k) = 0$, this follows from the invariance of étale cohomology under extensions of algebraically closed field extensions; and if $\text{char}(k) > 2$, this follows from Lemma 3.5. As a consequence, α restricts to a torsion class on a dense open subset $U \subset X$, hence has coniveau at least 1 by [Kah12, Lemme 3.12] applied to U . $\square$

Remark 3.6. Arguing as in [BO21, Remark 4.5], we obtain counterexamples as in Theorem 1.1, but where 2-adic cohomology is replaced by ℓ -adic cohomology for any odd prime ℓ invertible in k .

4. Walker Abel–Jacobi map in ℓ -adic cohomology: Construction

In this section, we let k be a field, ℓ be a prime number invertible in k , and X be a smooth projective geometrically connected k -variety of dimension d .

In Section 4.1, we will recall two notions of algebraic equivalence for cycles on X and summarize their basic properties. In Section 4.2, we will construct Walker Abel–Jacobi maps in ℓ -adic cohomology, defined on cycles on X that are algebraically trivial.

4.1 Two notions of algebraic equivalence

We say that a cycle class $\alpha \in \text{CH}^i(X)$ is *algebraically trivial over k* , or *algebraically trivial in the sense of Fulton* [Ful98, Definition 10.3], if there exist a smooth integral (not necessarily separated) scheme S of finite type over k , a cycle class $\Gamma \in \text{CH}^i(S \times X)$, and k -rational points $s_1, s_2 \in S$ such that $\alpha = \Gamma_{s_1} - \Gamma_{s_2}$ in $\text{CH}^i(X)$, where $\Gamma_{s_1}, \Gamma_{s_2}$ are the refined Gysin fibers of the cycle class Γ (see [Ful98, § 6.2]). We also say that a cycle class $\alpha \in \text{CH}^i(X)$ is *algebraically trivial over $\bar{k}$* , or simply *algebraically trivial*, if α becomes algebraically trivial in the sense of Fulton over some separable finite extension K/k , or equivalently over the separable closure $\bar{k}$. Finally, we say that a cycle class $\alpha \in \text{CH}^i(X)$ is *homologically trivial* if α is in the kernel of the cycle class map $\text{CH}^i(X) \rightarrow \prod_{\ell \neq \text{char } k} H^{2i}(\bar{X}, \mathbb{Z}_\ell(i))$.

We define $\text{CH}^i(X)_{\text{F-alg}}, \text{CH}^i(X)_{\text{alg}}, \text{CH}^i(X)_{\text{hom}} \subset \text{CH}^i(X)$ as the subgroups of cycle classes that are algebraically trivial in the sense of Fulton, algebraically trivial, and homologically trivial, respectively. By definition, we have

$$\text{CH}^i(X)_{\text{F-alg}} \subset \text{CH}^i(X)_{\text{alg}} \subset \text{CH}^i(X)_{\text{hom}} \subset \text{CH}^i(X)$$

and $\text{CH}^i(\bar{X})_{\text{F-alg}} = \text{CH}^i(\bar{X})_{\text{alg}}$.

LEMMA 4.1. *The groups $\text{CH}^i(X)_{\text{F-alg}}$ and $\text{CH}^i(X)_{\text{alg}}, \text{CH}^i(X)_{\text{hom}}$ are stable under proper push-forward, flat pull-back, refined Gysin homomorphisms, and Chern class operations. In particular, they are stable under the action of correspondences.*

Proof. For $\text{CH}^i(X)_{\text{F-alg}}$, see [Ful98, Proposition 10.3]. The properties for $\text{CH}^i(X)_{\text{alg}}$ and $\text{CH}^i(X)_{\text{hom}}$ follow from the corresponding properties [Ful98, Proposition 10.3 and Chapter 19] for $\text{CH}^i(\bar{X})_{\text{alg}}$ and $\text{CH}^i(\bar{X})_{\text{hom}}$, respectively, and from the fact that the operations in question are compatible with the base change map $\text{CH}^i(X) \rightarrow \text{CH}^i(\bar{X})$, which may be deduced from the compatibility of the operations with flat pull-backs [Ful98, Lemma 1.7.1, Proposition 1.7, and Theorems 6.2(b) and 3.2(d)]. $\square$

LEMMA 4.2. *The groups $\text{CH}^1(X)_{\text{alg}}$ and $\text{CH}^1(X)_{\text{hom}}$ coincide in characteristic zero, and they*

coincide up to inverting p in characteristic $p > 0$. In arbitrary characteristic, we have

$$\mathrm{CH}^d(X)_{\mathrm{F}\text{-alg}} = \mathrm{CH}^d(X)_{\mathrm{alg}} = \mathrm{CH}^d(X)_{\mathrm{hom}} = \mathrm{Ker}(\deg: \mathrm{CH}_0(X) \longrightarrow \mathbb{Z}).$$

Proof. In codimension 1, for every prime number ℓ invertible in k , the Kummer exact sequence induces a short exact sequence

$$0 \longrightarrow NS(\overline{X}) \otimes \mathbb{Z}_\ell \xrightarrow{\mathrm{cl}} H^2(\overline{X}, \mathbb{Z}_\ell(1)) \longrightarrow \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, \mathrm{Br}(\overline{X})) \longrightarrow 0,$$

where $NS(\overline{X}) = \mathrm{CH}^1(\overline{X})/\mathrm{CH}^1(\overline{X})_{\mathrm{alg}}$ is the Néron–Severi group and $\mathrm{Br}(\overline{X})$ is the Brauer group. Hence, if $\mathrm{char} k = 0$, then $\mathrm{CH}^1(\overline{X})_{\mathrm{alg}} = \mathrm{CH}^1(\overline{X})_{\mathrm{hom}}$, which yields $\mathrm{CH}^1(X)_{\mathrm{alg}} = \mathrm{CH}^1(X)_{\mathrm{hom}}$. If $\mathrm{char} k = p > 0$, we obtain the same result after inverting p .

In codimension d , we have $\mathrm{CH}^d(X)_{\mathrm{hom}} = \mathrm{Ker}(\deg: \mathrm{CH}_0(X) \rightarrow \mathbb{Z})$ in arbitrary characteristic. The assertion then follows from a result of Achter–Casalaina–Martin–Vial [ACV19, Proposition 3.11]. $\square$

LEMMA 4.3. *Suppose that k is perfect. Then*

$$\mathrm{CH}^i(X)_{\mathrm{F}\text{-alg}} = \sum_{(C, \Gamma)} \mathrm{Im}(\Gamma_*: \mathrm{CH}_0(C)_{\mathrm{hom}} \longrightarrow \mathrm{CH}^i(X)), \quad (4.1)$$

where (C, Γ) runs over all pairs where C is a smooth projective geometrically connected k -curve and $\Gamma \in \mathrm{CH}^i(C \times X)$.

Proof. This is a restatement of results of Achter–Casalaina–Martin–Vial [ACV19]. They showed in [ACV19, Theorem 1] that one may use smooth projective connected k -curves as parameter spaces in the definition of algebraic equivalence in the sense of Fulton; such k -curves are automatically geometrically connected because they have a k -rational point. Hence the left term of (4.1) is contained in the right term of (4.1). The opposite inclusion follows from [ACV19, Lemma 3.9 and Proposition 3.10]. $\square$

4.2 Construction of an ℓ -adic Walker Abel–Jacobi map

Recall that Jannsen [Jan90, §9] constructed an ℓ -adic Abel–Jacobi map

$$\mathrm{cl}_{AJ}: \mathrm{CH}^i(X)_{\mathrm{hom}} \otimes \mathbb{Z}_\ell \longrightarrow H^1(k, H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))).$$

The map cl_{AJ} is related to the cycle class map via the Hochschild–Serre spectral sequence [Jan90, Lemma 9.4]:

$$E_2^{r,s} := H^r(k, H^s(\overline{X}, \mathbb{Z}_\ell(m))) \implies H^{r+s}(X, \mathbb{Z}_\ell(m)).$$

For instance, if $k = \mathbb{F}$ is a finite field, then we have a Cartesian square

$$\begin{array}{ccc} \mathrm{CH}^i(X)_{\mathrm{hom}} \otimes \mathbb{Z}_\ell & \hookrightarrow & \mathrm{CH}^i(X) \otimes \mathbb{Z}_\ell \\ \downarrow \mathrm{cl}_{AJ} & & \downarrow \mathrm{cl} \\ H^1(\mathbb{F}, H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))) & \hookrightarrow & H^{2i}(X, \mathbb{Z}_\ell(i)), \end{array}$$

where the bottom horizontal map is an edge homomorphism in the Hochschild–Serre spectral sequence.

We wish to construct an ℓ -adic Walker Abel–Jacobi map

$$\mathrm{cl}_W: \mathrm{CH}^i(X)_{\mathrm{alg}} \otimes \mathbb{Z}_\ell \longrightarrow H^1(k, N^{i-1} H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))) \quad (4.2)$$

compatible with the ℓ -adic Abel–Jacobi map. Our construction is analogous to the construction of the Walker Abel–Jacobi map for the singular cohomology of smooth proper complex varieties given in [Suz21, § 2].

Here is the construction. For every codimension i closed subset $Z \subset X$, the long exact sequence for cohomology groups with supports, together with purity (cf. [AGV73, Corollaire XVI.3.9], [BO74, (2.1)]), yields a short exact sequence of G -modules

$$0 \longrightarrow H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i)) \longrightarrow H^{2i-1}(\overline{X} - \overline{Z}, \mathbb{Z}_\ell(i)) \longrightarrow Z_{\overline{Z}}^i(\overline{X})_{\text{hom}} \otimes \mathbb{Z}_\ell \longrightarrow 0,$$

which is functorial in Z , where $Z_{\overline{Z}}^i(\overline{X})_{\text{hom}}$ denotes the free abelian group of codimension i cycles on $\overline{X}$ that are supported on $\overline{Z}$ and homologically trivial. This restricts to another short exact sequence of G -modules,

$$0 \longrightarrow N^{i-1}H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i)) \longrightarrow N^{i-1}H^{2i-1}(\overline{X} - \overline{Z}, \mathbb{Z}_\ell(i)) \longrightarrow Z_{\overline{Z}}^i(\overline{X})_{\text{alg}} \otimes \mathbb{Z}_\ell \longrightarrow 0,$$

which is functorial in Z , where $Z_{\overline{Z}}^i(\overline{X})_{\text{alg}}$ denotes the free abelian group of codimension i cycles on $\overline{X}$ that are supported on $\overline{Z}$ and algebraically trivial. Indeed, the argument in [Suz21, § 2] goes through using the coniveau spectral sequence for ℓ -adic cohomology, where in order to extract information of algebraic equivalence, we need [Kah12, Proposition 4.5] as an ℓ -adic analogue of [BO74, Theorem 7.3]. Passing to Galois cohomology then yields a connecting homomorphism

$$(Z_{\overline{Z}}^i(\overline{X})_{\text{alg}})^G \otimes \mathbb{Z}_\ell \longrightarrow H^1(k, N^{i-1}H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))),$$

which is natural in Z . Pre-composing with the map $Z_Z^i(X)_{\text{alg}} \rightarrow (Z_{\overline{Z}}^i(\overline{X})_{\text{alg}})^G$ induced by pull-back, we obtain a homomorphism

$$Z_Z^i(X)_{\text{alg}} \otimes \mathbb{Z}_\ell \longrightarrow H^1(k, N^{i-1}H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))),$$

which is functorial in Z . Since $Z(X)_{\text{alg}}$ is the direct limit of the $Z_Z(X)_{\text{alg}}$, these maps induce a unique homomorphism

$$Z^i(X)_{\text{alg}} \otimes \mathbb{Z}_\ell \longrightarrow H^1(k, N^{i-1}H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i)))$$

compatible with all the previous maps. Moreover, this homomorphism factors through cycles modulo rational equivalence by an argument analogous to [Suz21, Lemma 2.2 and Corollary 2.3]. We define (4.2) as the induced map.

We summarize the basic properties of the ℓ -adic Walker Abel–Jacobi map below.

LEMMA 4.4. (1) *The ℓ -adic Walker Abel–Jacobi map is compatible with the ℓ -adic Abel–Jacobi map; that is, we have a commutative diagram*

$$\begin{array}{ccc} \text{CH}^i(X)_{\text{alg}} \otimes \mathbb{Z}_\ell & \xrightarrow{\text{cl}_W} & H^1(k, N^{i-1}H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))) \\ \downarrow & & \downarrow \\ \text{CH}^i(X)_{\text{hom}} \otimes \mathbb{Z}_\ell & \xrightarrow{\text{cl}_{AJ}} & H^1(k, H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))). \end{array} \quad (4.3)$$

For $i \in \{1, \dim X\}$, the two maps coincide.

(2) *The ℓ -adic Walker Abel–Jacobi map is compatible with correspondences. That is, for all smooth projective geometrically connected varieties X, Y and every codimension $(i - j + \dim Y)$*

cycle Γ on $Y \times X$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^j(Y)_{\mathrm{alg}} \otimes \mathbb{Z}_\ell & \xrightarrow{\mathrm{cl}_W} & H^1(k, N^{j-1} H^{2j-1}(\overline{Y}, \mathbb{Z}_\ell(j))) \\ \downarrow \Gamma_* & & \downarrow \Gamma_* \\ \mathrm{CH}^i(X)_{\mathrm{alg}} \otimes \mathbb{Z}_\ell & \xrightarrow{\mathrm{cl}_W} & H^1(k, N^{i-1} H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))) \end{array}$$

(3) The ℓ -adic Walker–Abel–Jacobi map commutes with field extensions and norm maps. More precisely, for every separable finite field extension K/k , we have commutative diagrams

$$\begin{array}{ccc} \mathrm{CH}^i(X)_{\mathrm{alg}} \otimes \mathbb{Z}_\ell & \xrightarrow{\mathrm{cl}_W} & H^1(k, N^{i-1} H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))) \\ \downarrow \rho_{K/k} & & \downarrow \rho_{K/k} \\ \mathrm{CH}^i(X_K)_{\mathrm{alg}} \otimes \mathbb{Z}_\ell & \xrightarrow{\mathrm{cl}_W} & H^1(K, N^{i-1} H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{CH}^i(X_K)_{\mathrm{alg}} \otimes \mathbb{Z}_\ell & \xrightarrow{\mathrm{cl}_W} & H^1(K, N^{i-1} H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))) \\ \downarrow \nu_{K/k} & & \downarrow \nu_{K/k} \\ \mathrm{CH}^i(X)_{\mathrm{alg}} \otimes \mathbb{Z}_\ell & \xrightarrow{\mathrm{cl}_W} & H^1(k, N^{i-1} H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))) \end{array},$$

where $\rho_{K/k}$ is the pull-back map and $\nu_{K/k}$ is the norm map.

Proof. (1) Note that for every codimension i closed subset $Z \subset X$, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & N^{i-1} H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i)) & \longrightarrow & N^{i-1} H^{2i-1}(\overline{X} - \overline{Z}, \mathbb{Z}_\ell(i)) & \longrightarrow & Z_{\overline{Z}}^i(\overline{X})_{\mathrm{alg}} \otimes \mathbb{Z}_\ell \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i)) & \longrightarrow & H^{2i-1}(\overline{X} - \overline{Z}, \mathbb{Z}_\ell(i)) & \longrightarrow & Z_{\overline{Z}}^i(\overline{X})_{\mathrm{hom}} \otimes \mathbb{Z}_\ell \longrightarrow 0. \end{array}$$

By passing to Galois cohomology, the commutativity of cl_W and cl_{AJ} follows. As for the latter assertion, by Lemma 4.2, we have

$$\begin{aligned} \mathrm{CH}^1(X)_{\mathrm{alg}}[1/p] &= \mathrm{CH}^1(X)_{\mathrm{hom}}[1/p], \\ \mathrm{CH}^d(X)_{\mathrm{F-alg}} &= \mathrm{CH}^d(X)_{\mathrm{alg}} = \mathrm{CH}^d(X)_{\mathrm{hom}}, \end{aligned}$$

where $p = \mathrm{char} k$. Moreover, we have

$$N^0 H^1(\overline{X}, \mathbb{Z}_\ell(1)) = H^1(\overline{X}, \mathbb{Z}_\ell(1)), \quad N^{d-1} H^{2d-1}(\overline{X}, \mathbb{Z}_\ell(d)) = H^{2d-1}(\overline{X}, \mathbb{Z}_\ell(d)),$$

where the latter is a consequence of the Lefschetz hyperplane section theorem [Mil80, Theorem VI.7.1]. Thus for $i \in \{1, \dim X\}$, the two vertical maps in (4.3) are the identity maps, which proves that cl_W and cl_{AJ} coincide.

(2) The argument is entirely analogous to [Suz21, Lemma 2.4]. For every codimension i closed subset $W \subset Y$ and every codimension $(i - j + \dim Y)$ cycle Γ on $Y \times X$ such that $W \times X$ and Γ intersect properly, the argument in the proof of [Suz21, Lemma 2.2] yields a commutative

diagram with exact rows

$$\begin{array}{ccccccc}
 0 & \longrightarrow & N^{j-1}H^{2j-1}(\overline{W}, \mathbb{Z}_\ell(j)) & \longrightarrow & N^{j-1}H^{2j-1}(\overline{Y} - \overline{W}, \mathbb{Z}_\ell(j)) & \longrightarrow & Z_{\overline{W}}^j(\overline{Y})_{\text{alg}} \otimes \mathbb{Z}_\ell \longrightarrow 0 \\
 & & \downarrow \Gamma_* & & \downarrow (\pi_X)_* j^* (\cup \Gamma)' (\pi_Y)^* & & \downarrow \Gamma_* \\
 0 & \longrightarrow & N^i H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i)) & \longrightarrow & N^i H^{2i-1}(\overline{X} - \overline{Z}, \mathbb{Z}_\ell(i)) & \longrightarrow & Z_{\overline{Z}}^i(\overline{X})_{\text{alg}} \otimes \mathbb{Z}_\ell \longrightarrow 0,
 \end{array}$$

where Z is the image of $(W \times X) \cap \text{Supp}(\Gamma)$ under the projection $Y \times X \rightarrow X$ and the middle vertical map is defined in the same way as in the proof of [Suz21, Lemma 2.2]. Note that it is enough to consider the proper intersection case as above by the moving lemma [Sta24, Tag 0B0D]. By passing to Galois cohomology, the assertion follows.

(3) This follows from the functoriality of Galois cohomology with respect to restriction and corestriction and from the construction of the ℓ -adic Walker Abel–Jacobi map. $\square$

LEMMA 4.5. *Suppose that k is perfect. Then the image of $\text{CH}^i(X)_{\text{F-alg}} \otimes \mathbb{Z}_\ell$ under cl_W is contained in*

$$\text{Im}(H^1(k, \tilde{N}^{i-1} H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))) \rightarrow H^1(k, N^{i-1} H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i)))) .$$

Proof. Lemmas 2.12 and 4.3 yield

$$\tilde{N}^{i-1} H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i)) = \sum_{(C, \Gamma)} \text{Im}(\Gamma_* : H_1(\overline{C}, \mathbb{Z}_\ell) \rightarrow H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i)))$$

and

$$\text{CH}^i(X)_{\text{F-alg}} = \sum_{(C, \Gamma)} \text{Im}(\Gamma_* : \text{CH}_0(C)_{\text{hom}} \rightarrow \text{CH}^i(X)) ,$$

where (C, Γ) runs over all pairs where C is a smooth projective geometrically connected k -curve and $\Gamma \in \text{CH}^i(C \times X)$. By Lemma 4.4(1) and (2), we obtain a commutative diagram

$$\begin{array}{ccc}
 \bigoplus_{(C, \Gamma)} \text{CH}_0(C)_{\text{hom}} \otimes \mathbb{Z}_\ell & \xrightarrow{\text{cl}_{AJ}} & \bigoplus_{(C, \Gamma)} H^1(k, H_1(\overline{C}, \mathbb{Z}_\ell)) \\
 \downarrow (\Gamma_*) & & \downarrow \\
 \text{CH}^i(X)_{\text{F-alg}} \otimes \mathbb{Z}_\ell & \xrightarrow{\text{cl}_W} & H^1(k, \tilde{N}^{i-1} H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))) \\
 & & \downarrow \\
 & & H^1(k, N^{i-1} H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))) .
 \end{array}$$

$\curvearrowright (\Gamma_*)$

This finishes the proof. $\square$

Finally, we define the ℓ -adic Walker intermediate Jacobian by

$$J_{W, \ell}^{2i-1}(\overline{X}) := \varinjlim_{K/k} H^1(K, N^{i-1} H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))) ,$$

where the K/k runs over all finite separable extensions of k . Taking the direct limit of (4.2), we obtain a map

$$\text{CH}^i(\overline{X})_{\text{alg}} \otimes \mathbb{Z}_\ell \longrightarrow J_{W, \ell}^{2i-1}(\overline{X}) .$$

We call this map the ℓ -adic Walker Abel–Jacobi map over $\overline{k}$.

5. Walker Abel–Jacobi map in ℓ -adic cohomology: Over a finite field

In this section, we let $\mathbb{F}$ be a finite field, ℓ be a prime number invertible in $\mathbb{F}$, and X be a smooth projective geometrically connected $\mathbb{F}$ -variety of dimension d .

5.1 Some lemmas over a finite field

LEMMA 5.1. *The groups $\mathrm{CH}^i(X)_{\mathbb{F}\text{-alg}}$ and $\mathrm{CH}^i(X)_{\mathrm{alg}}$ are torsion.*

Proof. By Lemma 4.3, the group $\mathrm{CH}^i(X)_{\mathbb{F}\text{-alg}}$ is generated by cycles of the form $\Gamma_*(\alpha)$, where Γ is a codimension i correspondence on $C \times X$ for some smooth projective geometrically connected $\mathbb{F}$ -curve C and α belongs to $\mathrm{CH}_0(C)_{\mathrm{hom}}$. Recall that $\mathrm{CH}_0(C)_{\mathrm{hom}}$ is isomorphic to the group of $\mathbb{F}$ -rational points of the Jacobian variety of C and hence is finite. It follows that $\mathrm{CH}^i(X)_{\mathbb{F}\text{-alg}}$ is torsion.

The group $\mathrm{CH}^i(X)_{\mathrm{alg}}$ is torsion because $\mathrm{CH}^i(\overline{X})_{\mathrm{alg}} = \varinjlim_{K/\mathbb{F}} \mathrm{CH}^i(X_K)_{\mathbb{F}\text{-alg}}$, where $K/\mathbb{F}$ runs over all finite extensions, is torsion and the kernel of the base change map $\mathrm{CH}^i(X) \rightarrow \mathrm{CH}^i(\overline{X})$ is torsion by a restriction-corestriction argument. $\square$

LEMMA 5.2. *The groups $\mathrm{CH}^1(X)_{\mathbb{F}\text{-alg}}$ and $\mathrm{CH}^1(X)_{\mathrm{alg}}$ coincide.*

Proof. The Poincaré line bundle exists for the Picard scheme $\mathrm{Pic}_{X/\mathbb{F}}^0$ (equivalently, $\mathrm{Pic}_{X/\mathbb{F}}^0$ represents the Picard functor $\mathcal{P}ic_{X/\mathbb{F}}^0$ of algebraically trivial line bundles). Indeed, the Poincaré line bundle exists if X has a zero-cycle of degree 1 (see [ACV23, §7.1]), and this is the case by the Lang–Weil estimate. This shows that $\mathrm{CH}^1(X)_{\mathrm{hom}} \subset \mathrm{CH}^1(X)_{\mathbb{F}\text{-alg}}$; hence $\mathrm{CH}^1(X)_{\mathbb{F}\text{-alg}} = \mathrm{CH}^1(X)_{\mathrm{alg}} = \mathrm{CH}^1(X)_{\mathrm{hom}}$. $\square$

LEMMA 5.3. *Let M be a continuous G -module of finite type over $\mathbb{Z}_\ell$. Then $H^2(\mathbb{F}, M) = 0$.*

Remark 5.4. Lemma 5.3 shows that $H^1(\mathbb{F}, -)$ is right exact when it is restricted to continuous G -modules of finite type over $\mathbb{Z}_\ell$. This is analogous to the right exactness of the intermediate Jacobian functor on the category of mixed Hodge structures.

Proof of Lemma 5.3. We have an isomorphism $G \simeq \widehat{\mathbb{Z}}$, and we let $\sigma \in G$ be a topological generator of G . Let M be a continuous G -module of finite type over $\mathbb{Z}_\ell$. We may write $M = \varprojlim_n M_n$, where M_n is a finite G -module for all $n \geq 0$. The groups $H^2(G, M_n)$ are trivial since $\mathrm{cd}(G) = 1$, and the groups $H^1(G, M_n) = M_n/(\sigma - 1)M_n$ are finite. By [NSW08, (2.7.6)], we have $H^2(G, M) = \varprojlim_n H^2(G, M_n) = 0$. $\square$

LEMMA 5.5. *Let $j \geq 0$ and m be integers such that $j \neq 2m$, and let M be a subquotient of $H^j(\overline{X}, \mathbb{Z}_\ell(m))$. Then the groups $H^i(\mathbb{F}, M)$ are finite for $i = 0, 1$.*

Proof. Let $\sigma \in G \simeq \widehat{\mathbb{Z}}$ be the Frobenius map. Recall that a continuous G -module N of finite type over $\mathbb{Z}_\ell$ is pure of weight n if the eigenvalues of σ^{-1} on $N \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ are algebraic numbers with all archimedean absolute values equal to $q^{n/2}$, where q is the order of $\mathbb{F}$. By Deligne’s proof of the Weil conjecture [Del74a, Theorem 1.6], the G -module $H^j(\overline{X}, \mathbb{Z}_\ell(m))$ is of pure weight $j - 2m$. Thus M is also of pure weight $j - 2m$.

Since $j - 2m \neq 0$, we deduce that 1 is not an eigenvalue of σ on $M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Therefore, $(M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)^G = 0$; that is, $H^0(\mathbb{F}, M)$ is finite.

To prove that $H^1(\mathbb{F}, M)$ is finite, we follow the argument in [CSS83, §2, proof of Theorem 2, p. 781]. Write $M = \varprojlim_n M_n$, where the M_n are finite G -modules. Then we have

$$H^1(\mathbb{F}, M) = \varprojlim_n H^1(\mathbb{F}, M_n) = \varprojlim_n \mathrm{Coker}(M_n \xrightarrow{\sigma-1} M_n) = \mathrm{Coker}(M \xrightarrow{\sigma-1} M),$$

where the last identity holds because the finite G -modules M_n are finite and so satisfy the Mittag-Leffler condition. Since 1 is not an eigenvalue of σ on $M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, the map $\sigma - 1: M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \rightarrow M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is an isomorphism, and the conclusion follows. $\square$

5.2 In arbitrary codimension

Here is a key proposition of the paper.

PROPOSITION 5.6. *The ℓ -adic Walker Abel–Jacobi map*

$$\mathrm{cl}_W: \mathrm{CH}^i(X)_{\mathrm{alg}} \otimes \mathbb{Z}_\ell \longrightarrow H^1(\mathbb{F}, N^{i-1} H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i)))$$

restricts to a surjection

$$\mathrm{CH}^i(X)_{\mathrm{F}\text{-alg}} \otimes \mathbb{Z}_\ell \longrightarrow \mathrm{Im}(H^1(\mathbb{F}, \tilde{N}^{i-1} H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))) \longrightarrow H^1(\mathbb{F}, N^{i-1} H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i)))) .$$

Remark 5.7. As shown by [SS24, Theorems 1.4 and 1.5], the ℓ -adic Walker Abel–Jacobi map is not necessarily surjective. The combination of Theorem 1.1 and [SS24, Proposition 3.8] provides further examples of non-surjectivity. This is in contrast to the fact that the complex Walker Abel–Jacobi map is always surjective.

For the proof, we need the following lemma.

LEMMA 5.8. *Let $i \in \{1, d\}$. Then the ℓ -adic Walker Abel–Jacobi map and ℓ -adic Abel–Jacobi map for cycles of codimension i coincide, and they are isomorphisms.*

Proof. By Lemma 4.4(1), the maps cl_W and cl_{AJ} coincide. We now prove that cl_{AJ} is an isomorphism. If $i = 1$, this is [BST97, Lemma 1.12]. The case $i = d$ follows from (1.4) and the fact that the cycle class map (1.2) is an isomorphism for zero-cycles; see [CSS83, Theorem 5]. $\square$

Proof of Proposition 5.6. By Lemma 4.5, it remains to show the surjectivity of the induced map. By Lemma 2.11,

$$\tilde{N}^{i-1} H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i)) = \sum_{f: T \rightarrow X} \mathrm{Im}(f_*: H^1(\overline{T}, \mathbb{Z}_\ell(1)) \longrightarrow H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))) ,$$

where the sum is over all smooth projective connected $\mathbb{F}$ -varieties T of dimension $d - i + 1$ and $f: T \rightarrow X$ is a morphism. Using Lemma 4.4(1) and (2) and Lemmas 5.2, 5.3, and 5.8, we obtain a commutative diagram

$$\begin{array}{ccc} \bigoplus_{f: T \rightarrow X} \mathrm{CH}^1(T)_{\mathrm{hom}} \otimes \mathbb{Z}_\ell & \xrightarrow[\sim]{\mathrm{cl}_{AJ}} & \bigoplus_{f: T \rightarrow X} H^1(\mathbb{F}, H^1(\overline{T}, \mathbb{Z}_\ell(1))) \\ \downarrow (f_*) & & \downarrow \\ & & H^1(\mathbb{F}, \tilde{N}^{i-1} H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))) \\ & & \downarrow \\ \mathrm{CH}^i(X)_{\mathrm{F}\text{-alg}} \otimes \mathbb{Z}_\ell & \xrightarrow{\mathrm{cl}_W} & H^1(\mathbb{F}, N^{i-1} H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))) . \end{array}$$

$\curvearrowright (f_*)$

This finishes the proof. $\square$

COROLLARY 5.9. *For all $i \geq 0$, the ℓ -adic Walker Abel–Jacobi map over $\overline{\mathbb{F}}$*

$$\mathrm{cl}_W: \mathrm{CH}^i(\overline{X})_{\mathrm{alg}} \otimes \mathbb{Z}_\ell \longrightarrow J_{W, \ell}^{2i-1}(\overline{X})$$

is surjective.

Proof. Note that $\mathrm{CH}^i(\overline{X})_{\mathrm{alg}} = \varinjlim_{K/\mathbb{F}} \mathrm{CH}^i(X_K)_{\mathrm{F}\text{-alg}}$, where $K/\mathbb{F}$ runs over all finite extensions. By Proposition 5.6, it suffices to show that the natural map

$$\varinjlim_{K/\mathbb{F}} H^1(K, \tilde{N}^{i-1} H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))) \longrightarrow J_{W,\ell}^{2i-1}(\overline{X})$$

is surjective. For this, it is enough to show that the group

$$\varinjlim_{K/\mathbb{F}} H^1(K, N^{i-1} H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i)) / \tilde{N}^{i-1} H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i)))$$

is zero. By Lemma 2.5, the group $N^{i-1} H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i)) / \tilde{N}^{i-1} H^{2i-1}(\overline{X}, \mathbb{Z}_\ell(i))$ is finite; hence the result follows from [Ser97, § 2.2, Corollary 1]. $\square$

5.3 In codimension 2

We now specialize to the case when $i = 2$. We call the ℓ -adic Walker Abel–Jacobi map in codimension 2 the ℓ -adic algebraic Abel–Jacobi map and write cl_a (respectively, $J_{a,\ell}^3(\overline{X})$) for cl_W (respectively, $J_{W,\ell}^3(\overline{X})$). This is in analogy with the situation over the complex numbers, where for codimension 2 cycles, the Walker Abel–Jacobi map agrees with the algebraic Abel–Jacobi map constructed by Lieberman [Lie72]; see [Suz21, Introduction].

Recall that, by the Merkurjev–Suslin theorem [MS83] (see [Kah12, Lemma 3.12]),

$$\text{the group } H^3(\overline{X}, \mathbb{Z}_\ell(2)) / N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2)) \text{ is torsion-free.} \quad (5.1)$$

It follows that

$$[H^3(\overline{X}, \mathbb{Z}_\ell(2)) / N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))]^G = 0. \quad (5.2)$$

Indeed, this group is torsion-free, and it is finite by Lemma 5.5.

We have the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{CH}^2(X)_{\mathrm{alg}} \otimes \mathbb{Z}_\ell & \hookrightarrow & \mathrm{CH}^2(X)_{\mathrm{hom}} \otimes \mathbb{Z}_\ell & \hookrightarrow & \mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \\ \downarrow \mathrm{cl}_a & & \downarrow \mathrm{cl}_{A,J} & & \downarrow (1.2) \\ H^1(\mathbb{F}, N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))) & \hookrightarrow & H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2))) & \hookrightarrow & H^4(X, \mathbb{Z}_\ell(2)), \end{array} \quad (5.3)$$

where the injectivity of the bottom-left horizontal map follows from (5.2).

LEMMA 5.10. *The ℓ -adic algebraic Abel–Jacobi map is injective:*

$$\mathrm{cl}_a: \mathrm{CH}^2(X)_{\mathrm{alg}} \otimes \mathbb{Z}_\ell \hookrightarrow H^1(\mathbb{F}, N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))).$$

Proof. By a theorem of Colliot-Thélène–Sansuc–Soulé [CSS83, Theorem 4], the map (1.2) is injective on torsion. Since $\mathrm{CH}^2(X)_{\mathrm{alg}}$ is torsion by Lemma 5.1, the result follows from the diagram (5.3). $\square$

Proposition 5.6 now implies the following.

PROPOSITION 5.11. *The ℓ -adic algebraic Abel–Jacobi map restricts to an isomorphism:*

$$\mathrm{CH}^2(X)_{\mathrm{F}\text{-alg}} \otimes \mathbb{Z}_\ell \xrightarrow{\sim} \mathrm{Im}(H^1(\mathbb{F}, \tilde{N}^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))) \longrightarrow H^1(\mathbb{F}, N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2)))) .$$

Over the algebraic closure $\overline{\mathbb{F}}$, we have the following.

COROLLARY 5.12. *The ℓ -adic algebraic Abel–Jacobi map over $\overline{\mathbb{F}}$ is an isomorphism:*

$$\mathrm{CH}^2(\overline{X})_{\mathrm{alg}} \otimes \mathbb{Z}_\ell \xrightarrow{\sim} J_{a,\ell}^3(\overline{X}).$$

Furthermore, it induces an isomorphism

$$(\mathrm{CH}^2(\overline{X})_{\mathrm{alg}})^G \otimes \mathbb{Z}_\ell \xrightarrow{\sim} H^1(\mathbb{F}, N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))_{\mathrm{tf}}).$$

Proof. The first statement is immediate from Proposition 5.11 and an argument as in the proof of Corollary 5.9. The second statement is then a consequence of the identity

$$J_{a,\ell}^3(\overline{X})^G = H^1(\mathbb{F}, N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))_{\mathrm{tf}}),$$

which follows from [Sch99, Lemma 1.1.5 (ii)] and Lemma 5.5. $\square$

Recall the following exact sequence, due to Colliot-Thélène and Kahn [CK13, Theorem 6.8]:

$$\begin{aligned} 0 &\longrightarrow \mathrm{Ker}(\mathrm{CH}^2(X) \longrightarrow \mathrm{CH}^2(\overline{X}))\{\ell\} \xrightarrow{\varphi_\ell} H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2))_{\mathrm{tors}}) \\ &\longrightarrow \mathrm{Ker}(H_{\mathrm{nr}}^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \longrightarrow H_{\mathrm{nr}}^3(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \\ &\longrightarrow \mathrm{Coker}(\mathrm{CH}^2(X) \longrightarrow \mathrm{CH}^2(\overline{X})^G)\{\ell\} \longrightarrow 0. \end{aligned} \quad (5.4)$$

We now refine the sequence (5.4).

PROPOSITION 5.13. *There is an exact sequence*

$$\begin{aligned} 0 &\longrightarrow \mathrm{Ker}(\mathrm{CH}^2(X) \longrightarrow \mathrm{CH}^2(\overline{X}))\{\ell\} \xrightarrow{\psi_\ell} H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2))_{\mathrm{tors}}) \\ &\longrightarrow \mathrm{Coker}(\mathrm{CH}^2(X)_{\mathrm{alg}} \otimes \mathbb{Z}_\ell \xrightarrow{\mathrm{cl}_a} H^1(\mathbb{F}, N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2)))) \\ &\longrightarrow \mathrm{Coker}(\mathrm{CH}^2(X)_{\mathrm{alg}} \longrightarrow (\mathrm{CH}^2(\overline{X})_{\mathrm{alg}})^G)\{\ell\} \longrightarrow 0. \end{aligned} \quad (5.5)$$

The map ψ_ℓ in (5.5) will be constructed in the proof.

Proof. By (5.1), we have $H^3(\overline{X}, \mathbb{Z}_\ell(2))_{\mathrm{tors}} \subset N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))$ and hence a short exact sequence of G -modules

$$0 \longrightarrow H^3(\overline{X}, \mathbb{Z}_\ell(2))_{\mathrm{tors}} \longrightarrow N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2)) \longrightarrow N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))_{\mathrm{tf}} \longrightarrow 0.$$

Note that $(N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))_{\mathrm{tf}})^G = 0$ by Lemma 5.5. Hence we have a short exact sequence

$$0 \longrightarrow H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2))_{\mathrm{tors}}) \longrightarrow H^1(\mathbb{F}, N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))) \longrightarrow H^1(\mathbb{F}, N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))_{\mathrm{tf}}) \longrightarrow 0.$$

On the other hand, Corollary 5.12 shows that the composition

$$\mathrm{CH}^2(X)_{\mathrm{alg}} \otimes \mathbb{Z}_\ell \xrightarrow{\mathrm{cl}_a} H^1(\mathbb{F}, N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))) \longrightarrow H^1(\mathbb{F}, N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))_{\mathrm{tf}})$$

can be identified with the pull-back map $\mathrm{CH}^2(X)_{\mathrm{alg}} \otimes \mathbb{Z}_\ell \rightarrow (\mathrm{CH}^2(\overline{X})_{\mathrm{alg}})^G \otimes \mathbb{Z}_\ell$. Thus we obtain a commutative diagram of short exact sequences

$$\begin{array}{ccccccc} 0 \rightarrow \mathrm{Ker}(\mathrm{CH}^2(X) \rightarrow \mathrm{CH}^2(\overline{X}))\{\ell\} & \longrightarrow & \mathrm{CH}^2(X)_{\mathrm{alg}} \otimes \mathbb{Z}_\ell & \longrightarrow & \mathrm{Im}(\mathrm{CH}^2(X)_{\mathrm{alg}} \otimes \mathbb{Z}_\ell \rightarrow \mathrm{CH}^2(\overline{X})_{\mathrm{alg}} \otimes \mathbb{Z}_\ell) & \longrightarrow & 0 \\ & \downarrow \psi_\ell & & \downarrow \mathrm{cl}_a & & \downarrow & \\ 0 & \longrightarrow & H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2))_{\mathrm{tors}}) & \longrightarrow & H^1(\mathbb{F}, N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))) & \rightarrow & H^1(\mathbb{F}, N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))_{\mathrm{tf}}) \simeq (\mathrm{CH}^2(\overline{X})_{\mathrm{alg}})^G \otimes \mathbb{Z}_\ell \rightarrow 0, \end{array}$$

where the vertical arrows are all injective. One may now construct (5.5) by a simple diagram chase. $\square$

The following lemma collects results due to Colliot-Thélène and Kahn [CK13].

LEMMA 5.14. *The Hochschild–Serre spectral sequence for étale motivic cohomology*

$$E_2^{r,s} = H^r(\mathbb{F}, \mathbb{H}^s(\overline{X}, \mathbb{Z}(2))) \implies \mathbb{H}^{r+s}(X, \mathbb{Z}(2))$$

yields a short exact sequence

$$0 \longrightarrow H^1(\mathbb{F}, \mathbb{H}^3(\overline{X}, \mathbb{Z}(2))) \longrightarrow \mathbb{H}^4(X, \mathbb{Z}(2)) \longrightarrow \mathbb{H}^4(\overline{X}, \mathbb{Z}(2))^G \longrightarrow 0. \quad (5.6)$$

Moreover, the natural map $\mathbb{H}^4(X, \mathbb{Z}(2)) \otimes \mathbb{Z}_\ell \rightarrow H^4(X, \mathbb{Z}_\ell(2))$ restricts to an isomorphism

$$H^1(\mathbb{F}, \mathbb{H}^3(\overline{X}, \mathbb{Z}(2))) \otimes \mathbb{Z}_\ell \xrightarrow{\sim} H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2))_{\text{tors}}).$$

Finally, we have a natural injection

$$\varinjlim_{K/\mathbb{F}} \mathbb{H}^4(X_K, \mathbb{Z}(2)) \otimes \mathbb{Z}_\ell \hookrightarrow \mathbb{H}^4(\overline{X}, \mathbb{Z}(2)) \otimes \mathbb{Z}_\ell,$$

where $K/\mathbb{F}$ ranges over all finite field extensions.

Proof. The first assertion is shown at the end of the proof of [CK13, Theorem 6.3], and the second assertion follows from [CK13, Proposition 6.7] by tensoring with $\mathbb{Z}_\ell$. As for the third assertion, note that since the G -module $H^3(\overline{X}, \mathbb{Z}_\ell(2))_{\text{tors}}$ is finite, by [Ser97, § 2.2, Corollary 1], we have $\varinjlim_{K/\mathbb{F}} H^1(K, H^3(\overline{X}, \mathbb{Z}_\ell(2))_{\text{tors}}) = 0$. Hence taking the direct limit of (5.6) concludes the proof. $\square$

The following proposition, which will not be needed in the rest of the paper, clarifies the relationship between the exact sequences (5.5) and (5.4).

PROPOSITION 5.15. *There is a monomorphism of exact sequences from (5.5) to (5.4), which identifies ψ_ℓ with φ_ℓ .*

Remark 5.16. It follows from Proposition 5.15 that if the group $H_{\text{nr}}^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ vanishes, then $\text{cl}_a: \text{CH}^2(X)_{\text{alg}} \otimes \mathbb{Z}_\ell \rightarrow H^1(\mathbb{F}, N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2)))$ is an isomorphism.

Proof of Proposition 5.15. We claim that, for every torsion group T and every homomorphism $f: T \rightarrow H^4(X, \mathbb{Z}_\ell(2))$, there exists a unique lift $f': T \rightarrow \mathbb{H}^4(X, \mathbb{Z}(2)) \otimes \mathbb{Z}_\ell$ of f along the natural map

$$\theta: \mathbb{H}^4(X, \mathbb{Z}(2)) \otimes \mathbb{Z}_\ell \longrightarrow H^4(X, \mathbb{Z}_\ell(2)).$$

To prove the claim, we first note that $f(T)$ is contained in the image of θ because $\text{Coker}(\theta)$ is torsion-free by [Kah12, Corollary 3.5]. Now let $K(2)$ be the mapping cone of the morphism

$$\mathbb{Z}(2) \overset{L}{\otimes} \mathbb{Z}_\ell \longrightarrow \mathbb{Z}_\ell(2)^c = R \varprojlim_r \mu_{\ell^r}^{\otimes 2}$$

of complexes of abelian sheaves on $X_{\text{ét}}$ given in [Kah02, § 1.4]. Then $\mathbb{H}^4(X, K(2))$ is uniquely divisible by [Kah12, Proposition 3.4]. On the other hand, the group $H^3(X, \mathbb{Z}_\ell(2))$ is finite by Lemma 5.5 and the Hochschild–Serre spectral sequence. Thus, in the exact sequence

$$H^3(X, \mathbb{Z}_\ell(2)) \longrightarrow \mathbb{H}^4(X, K(2)) \longrightarrow \mathbb{H}^4(X, \mathbb{Z}(2)) \otimes \mathbb{Z}_\ell \xrightarrow{\theta} H^4(X, \mathbb{Z}_\ell(2)),$$

the first map is zero. Therefore, $\text{Ker}(\theta)$ is uniquely divisible. Combining this with the fact that $f(T) \subset \text{Im } \theta$, it follows that the surjection $\theta^{-1}(f(T)) \rightarrow f(T)$ admits a unique splitting $\sigma: f(T) \rightarrow \theta^{-1}(f(T))$ and $f' := \sigma \circ f$ is the unique lift of f along θ . This proves the claim.

By Lemma 5.5, the subgroup $H^1(\mathbb{F}, N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))) \subset H^4(X, \mathbb{Z}_\ell(2))$ is finite: We let $H^1(\mathbb{F}, N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2)))' \subset \mathbb{H}^4(X, \mathbb{Z}(2)) \otimes \mathbb{Z}_\ell$ be its unique lift along θ . By Lemma 5.1, the group $\text{CH}^2(X)_{\text{alg}}$ is torsion, and hence cl_a uniquely lifts to a homomorphism

$$\text{cl}'_a: \text{CH}^2(X)_{\text{alg}} \otimes \mathbb{Z}_\ell \longrightarrow H^1(\mathbb{F}, N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2)))'.$$

The lift cl'_a coincides with the restriction of the cycle map $\text{CH}^2(X) \otimes \mathbb{Z}_\ell \rightarrow \mathbb{H}^4(X, \mathbb{Z}(2)) \otimes \mathbb{Z}_\ell$ to $\text{CH}^2(X)_{\text{alg}} \otimes \mathbb{Z}_\ell$ by the uniqueness property.

The construction of cl'_a is functorial in $\mathbb{F}$. Moreover, by the uniqueness property, for all finite field extensions $\mathbb{F} \subset K \subset L$, the pull-back map

$$\mathbb{H}^4(X_K, \mathbb{Z}(2)) \otimes \mathbb{Z}_\ell \longrightarrow \mathbb{H}^4(X_L, \mathbb{Z}(2)) \otimes \mathbb{Z}_\ell$$

sends $H^1(K, N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2)))'$ to $H^1(L, N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2)))'$. This shows that the cohomology groups $H^1(K, N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2)))'$ form a directed system, where $K/\mathbb{F}$ ranges over all finite field extensions. Let

$$J_{a,\ell}^3(\overline{X})' := \varinjlim_{K/\mathbb{F}} H^1(K, N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2)))' \subset \varinjlim_{K/\mathbb{F}} \mathbb{H}^4(X_K, \mathbb{Z}(2)) \otimes \mathbb{Z}_\ell,$$

where $K/\mathbb{F}$ ranges over all finite field extensions. Using Lemma 5.14, we obtain a lift of the ℓ -adic algebraic Abel–Jacobi map over $\overline{\mathbb{F}}$

$$\text{CH}^2(\overline{X})_{\text{alg}} \otimes \mathbb{Z}_\ell \xrightarrow{\sim} J_{a,\ell}^3(\overline{X})' \hookrightarrow \mathbb{H}^4(\overline{X}, \mathbb{Z}(2)) \otimes \mathbb{Z}_\ell,$$

which coincides with the restriction of the cycle map

$$\text{CH}^2(\overline{X}) \otimes \mathbb{Z}_\ell \longrightarrow \mathbb{H}^4(\overline{X}, \mathbb{Z}(2)) \otimes \mathbb{Z}_\ell.$$

Now the commutative diagram with exact rows in the proof of [CK13, Theorem 6.3]

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{CH}^2(X) \otimes \mathbb{Z}_\ell & \longrightarrow & \mathbb{H}^4(X, \mathbb{Z}(2)) \otimes \mathbb{Z}_\ell & \longrightarrow & H_{\text{nr}}^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{CH}^2(\overline{X})^G \otimes \mathbb{Z}_\ell & \longrightarrow & \mathbb{H}^4(\overline{X}, \mathbb{Z}(2))^G \otimes \mathbb{Z}_\ell & \longrightarrow & H_{\text{nr}}^3(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))^G \end{array} \quad (5.7)$$

restricts to the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{CH}^2(X)_{\text{alg}} \otimes \mathbb{Z}_\ell & \xrightarrow{\text{cl}'_a} & H^1(\mathbb{F}, N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2)))' & \longrightarrow & \text{Coker}(\text{cl}'_a) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (\text{CH}^2(\overline{X})_{\text{alg}})^G \otimes \mathbb{Z}_\ell & \xrightarrow{\sim} & (J_{a,\ell}^3(\overline{X})')^G & \longrightarrow & 0. \end{array} \quad (5.8)$$

Applying the snake lemma to (5.7) and (5.8) and using Lemma 5.14, we deduce the conclusion from the constructions of (5.4) in [CK13, Theorem 6.8] and (5.5) in Proposition 5.13. $\square$

6. Proofs of Theorems 1.2, 1.4, and 1.5

Proof of Theorem 1.2. If the equality $N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2)) = \tilde{N}^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))$ holds, Lemma 5.10 and Proposition 5.11 show that $\text{CH}^2(X)_{\text{F-alg}} \otimes \mathbb{Z}_\ell = \text{CH}^2(X)_{\text{alg}} \otimes \mathbb{Z}_\ell$ and that the ℓ -adic algebraic Abel–Jacobi map

$$\text{cl}_a: \text{CH}^2(X)_{\text{alg}} \otimes \mathbb{Z}_\ell \longrightarrow H^1(\mathbb{F}, N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2)))$$

is an isomorphism. Proposition 5.13 now implies that the homomorphism

$$\varphi_\ell: \text{Ker}(\text{CH}^2(X) \longrightarrow \text{CH}^2(\overline{X}))\{\ell\} \longrightarrow H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2))_{\text{tors}})$$

is an isomorphism and the base change map $\text{CH}^2(X)_{\text{alg}}\{\ell\} \rightarrow (\text{CH}^2(\overline{X})_{\text{alg}})^G\{\ell\}$ is surjective. This finishes the proof. $\square$

Remark 6.1. Let X be a smooth projective geometrically connected $\mathbb{F}$ -variety as in [SS24, Theorems 1.4 and 1.5]. It follows from Theorem 1.2 that the inclusion $\tilde{N}^1 H^3(\overline{X}, \mathbb{Z}_\ell(2)) \subset N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))$ is strict.

Suppose that X is as in the proof of [SS24, Theorem 1.4]. By construction, X is a projective approximation of $B_{\mathbb{F}}(\mathrm{PGL}_\ell \times \mathbb{G}_m)$; see [SS24, § 4.2, proof of Theorem 1.4]. The cohomology and Chow ring of $B_{\mathbb{F}} \mathrm{PGL}_\ell$ have been computed by Vistoli [Vis07]. In particular, $H^3(\overline{X}, \mathbb{Z}_\ell(2)) = \mathbb{Z}/\ell$ and $H^4(\overline{X}, \mathbb{Z}_\ell(2)) = \mathbb{Z}_\ell$. By [Kah12, Lemme 3.12], the group $H_{\mathrm{nr}}^3(\overline{X}, \mathbb{Z}_\ell(2))$ is torsion-free, and hence

$$H^3(\overline{X}, \mathbb{Z}_\ell(2)) = N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2)) = \mathbb{Z}/\ell.$$

Since the inclusion $\tilde{N}^1 H^3(\overline{X}, \mathbb{Z}_\ell(2)) \subset N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))$ is strict, we deduce that

$$\tilde{N}^1 H^3(\overline{X}, \mathbb{Z}_\ell(2)) = 0.$$

Meanwhile, since $H^4(\overline{X}, \mathbb{Z}_\ell(2)) = \mathbb{Z}_\ell$ is torsion-free, [CSS83, Theorem 4] shows that $\mathrm{CH}^2(X)\{\ell\}$ injects into $H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2))) = H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2))_{\mathrm{tors}}) = \mathbb{Z}/\ell$ via (1.2). Since every non-zero element of the latter group is not algebraic, we conclude that

$$\mathrm{CH}^2(X)\{\ell\} = \mathrm{CH}^2(X)_{\mathrm{alg}}\{\ell\} = \mathrm{CH}^2(X)_{\mathrm{F}\text{-alg}}\{\ell\} = 0.$$

PROPOSITION 6.2. *Let X be a smooth projective geometrically connected variety over a finite field $\mathbb{F}$ and ℓ be a prime number invertible in $\mathbb{F}$. Consider the following statements:*

- (1) $H_{\mathrm{nr}}^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$.
- (2) The map (1.2) has torsion-free cokernel.
- (3) The map (1.3) has torsion-free cokernel.

Then item (1) implies item (2), and item (2) implies item (3). Moreover, items (1) and (2) are equivalent if $\mathrm{CH}_0(\overline{X})$ is supported in dimension at most 2, and items (2) and (3) are equivalent if $H^3(\overline{X}, \mathbb{Z}_\ell(2)) = \tilde{N}^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))$.

Remark 6.3. As recalled in the introduction, the relation between statements (1) and (2) of Proposition 6.2 is due to Colliot-Thélène and Kahn [CK13]. The relation between statements (2) and (3) is new and follows from our work in Section 5.

Statement (1) (respectively, (2)) in Proposition 6.2 is conjecturally equivalent to the surjectivity of (1.2) (respectively, (1.3)). More precisely, the Tate conjecture for codimension 2 cycles on X predicts that (1.3) has finite cokernel. Using Lemma 5.5, this is equivalent to the prediction that (1.2) has finite cokernel.

Proof of Proposition 6.2. By [CK13, Theorem 2.2], statement (1) implies statement (2). Since the natural map $H^4(X, \mathbb{Z}_\ell(2)) \rightarrow H^4(\overline{X}, \mathbb{Z}_\ell(2))^G$ is surjective, statement (2) implies statement (3).

If $\mathrm{CH}_0(\overline{X})$ is supported in dimension ≤ 2 , then $H_{\mathrm{nr}}^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ is finite by [CK13, Proposition 3.2]; statements (1) and (2) are then equivalent by [CK13, Theorem 2.2]. If $H^3(\overline{X}, \mathbb{Z}_\ell(2)) = \tilde{N}^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))$, then the equivalence of statements (2) and (3) follows from (1.4), (5.3), and Proposition 5.11. $\square$

Example 6.4. Let X be a smooth projective geometrically retract rational $\mathbb{F}$ -variety. Then, as we recalled in the proof of Corollary 2.9, the variety $\overline{X}$ admits an integral Chow decomposition of the diagonal. As a consequence, we have $\mathrm{CH}_0(\overline{X}_K) = \mathbb{Z}$, where K is the function field of $\overline{X}$ (in particular, $\mathrm{CH}_0(\overline{X})$ is supported in dimension at most 2) and $H^3(\overline{X}, \mathbb{Z}_\ell(2)) = \tilde{N}^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))$. Hence statements (1)–(3) in Proposition 6.2 are equivalent for X .

Example 6.5. Pirutka [Pir11] constructed a geometrically rational $\mathbb{F}$ -variety X of dimension 5 for which statements (1)–(3) in Proposition 6.2 fail for $\ell = 2$. Indeed, Pirutka showed that the pull-back map $\mathrm{CH}^2(X) \rightarrow \mathrm{CH}^2(\overline{X})^G$ is not surjective. By the exact sequence of Colliot-Thélène and Kahn [CK13, Théorème 6.8], this implies that $H_{\mathrm{nr}}^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \neq 0$; that is, statement (1) of Proposition 6.2 is false for X . Since $\overline{X}$ is a split quadric, $\mathrm{CH}_0(\overline{X}_K) = \mathbb{Z}$, where K is the function field of $\overline{X}$, and $H^3(\overline{X}, \mathbb{Z}_\ell(2)) = 0$. Therefore, statements (2) and (3) are also false for X .

Example 6.6. Using Proposition 6.2, we now recover some of the results of [CS23] and [Sca22].

Let X be the product of a smooth projective geometrically connected surface S such that $b_2(\overline{S}) = \rho(\overline{S})$ and a smooth projective geometrically connected curve C over $\mathbb{F}$. We claim that

$$H^3(\overline{X}, \mathbb{Z}_\ell(2)) = \tilde{N}^1 H^3(\overline{X}, \mathbb{Z}_\ell(2)). \quad (6.1)$$

Indeed, the Künneth formula yields

$$H^3(\overline{X}, \mathbb{Z}_\ell(2)) = [H^1(\overline{S}, \mathbb{Z}_\ell(1)) \otimes H^2(\overline{C}, \mathbb{Z}_\ell(1))] \oplus [H^2(\overline{S}, \mathbb{Z}_\ell(1)) \otimes H^1(\overline{C}, \mathbb{Z}_\ell(1))] \oplus H^3(\overline{S}, \mathbb{Z}_\ell(2)). \quad (6.2)$$

Since $\dim(C) = 1$, we have

$$H^2(\overline{C}, \mathbb{Z}_\ell(1)) = \tilde{N}^1 H^2(\overline{C}, \mathbb{Z}_\ell(1)). \quad (6.3)$$

The cokernel of the cycle map $\mathrm{Pic}(\overline{S}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \rightarrow H^2(\overline{S}, \mathbb{Z}_\ell(1))$ is torsion-free by the Kummer sequence; it is torsion because $b_2(\overline{S}) = \rho(\overline{S})$. Thus the cycle map $\mathrm{Pic}(\overline{S}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \rightarrow H^2(\overline{S}, \mathbb{Z}_\ell(1))$ is surjective, and hence

$$H^2(\overline{S}, \mathbb{Z}_\ell(1)) = N^1 H^2(\overline{S}, \mathbb{Z}_\ell(1)).$$

Since $\overline{S}$ is projective, every element of $\mathrm{Pic}(\overline{S})$ can be written as the difference of two very ample divisor classes. By Bertini's theorem [Har77, Theorem II.8.18], a very ample divisor class on $\overline{S}$ may be represented by a smooth very ample divisor. Thus $N^1 H^2(\overline{S}, \mathbb{Z}_\ell(1)) = \tilde{N}^1 H^2(\overline{S}, \mathbb{Z}_\ell(1))$ and hence

$$H^2(\overline{S}, \mathbb{Z}_\ell(1)) = \tilde{N}^1 H^2(\overline{S}, \mathbb{Z}_\ell(1)). \quad (6.4)$$

Let $Z \subset \overline{S}$ be a smooth very ample divisor. By the Lefschetz hyperplane section theorem, the push-forward $H_1(Z, \mathbb{Z}_\ell) \rightarrow H_1(\overline{S}, \mathbb{Z}_\ell)$ is surjective, and hence

$$H^3(\overline{S}, \mathbb{Z}_\ell(2)) = \tilde{N}^1 H^3(\overline{S}, \mathbb{Z}_\ell(2)). \quad (6.5)$$

Now equality (6.1) follows from the combination of (6.2)–(6.5).

By Proposition 6.2, we conclude that statements (2) and (3) of Proposition 6.2 are equivalent for X . This recovers [CS23, Théorème 1.3], where the equivalence of statements (2) and (3) was proved when S is geometrically CH_0 -trivial (see [CS23, Introduction] for the definition), and also [Sca22, Theorem 1.3], where the equivalence was proved under the weaker assumption $b_2(\overline{S}) = \rho(\overline{S})$.

Suppose further that S is an Enriques surface. In this case, the assumptions of Proposition 6.2 are all satisfied, hence statements (1), (2), and (3) are equivalent for X . When no point of order 2 of the Jacobian of C is defined over $\mathbb{F}$, this was proved in [CS23, Théorème 1.4] conditionally on the Tate conjecture for 1-cycles on surfaces over finite fields, and unconditionally in [Sca22, Corollary 1.5]. (Note that [Sca22, Corollary 1.5] is stated when C has genus 1 for simplicity.)

Proof of Theorem 1.4. Let X be a smooth projective rationally connected threefold over a number field K such that $H_B^3(X(\mathbb{C}), \mathbb{Z})$ is torsion-free. Results of Voisin, [Voi06, Theorem 2] and [Voi22, Theorem 0.2], imply that $H_B^3(X(\mathbb{C}), \mathbb{Z}) = \tilde{N}^1 H_B^3(X(\mathbb{C}), \mathbb{Z})$ and that $H_B^4(X(\mathbb{C}), \mathbb{Z})$ is

spanned by algebraic classes. For every prime p of O_K , let k_p be the residue field of p . A spreading-out argument now shows that for all but finitely many prime numbers p , if X_p is the mod p reduction of X , then

- (i) X_p is a smooth projective rationally chain-connected k_p -variety (note that rational chain connectivity specializes by [Kol96, Corollary IV.3.5.2]),
- (ii) $H^3(\overline{X}_p, \mathbb{Z}_\ell(2)) = \tilde{N}^1 H^3(\overline{X}_p, \mathbb{Z}_\ell(2))$ for all prime numbers ℓ invertible in k_p , and
- (iii) there exists a finite field extension k'_p/k_p such that for all finite extensions E/k'_p and for all prime number ℓ invertible in k_p , the cycle map $\mathrm{CH}^2(X_E) \otimes \mathbb{Z}_\ell \rightarrow H^4(\overline{X}_p, \mathbb{Z}_\ell(2))$ is surjective.

By assertion (iii), the E -variety X_E satisfies statement (3) of Proposition 6.2. By assertion (ii), the E -variety X_E also satisfies statement (2). Finally, assertion (i) implies that $\mathrm{CH}_0((\overline{X}_p)_K) = \mathbb{Z}$, where K is the function field of $\overline{X}_p$, hence X_E also satisfies statement (1) of Proposition 6.2. $\square$

Proof of Theorem 1.5. Let $X \subset \mathbb{P}^N$ be a smooth Fano complete intersection threefold over a finite field $\mathbb{F}$. Since X is Fano, X is rationally chain connected [Kol96, Theorem V.2.13], hence $\mathrm{CH}_0(\overline{X}_K) = \mathbb{Z}$, where K is the function field of $\overline{X}$.

Suppose that the Fano scheme $F(X)$ of lines on X is smooth of the expected dimension. Let $S = F(X)$ be the Fano scheme of lines and $\Gamma \subset S \times X$ be the universal line. Since X is a complete intersection, it admits a lift to a smooth complete intersection over a discrete valuation ring with fraction field of characteristic zero. Then S lifts to the Fano scheme of lines of the lift of X , and Γ lifts to the universal line (this follows immediately from the definition of the Hilbert scheme functor). A result of Voisin [Voi22, Theorem 1.13] and the smooth proper base change theorem in étale cohomology now imply that $\Gamma_*: H_1(\overline{S}, \mathbb{Z}_\ell) \rightarrow H^3(\overline{X}, \mathbb{Z}_\ell(2))$ is surjective. This shows that $H^3(\overline{X}, \mathbb{Z}_\ell(2)) = \tilde{N}^1 H^3(\overline{X}, \mathbb{Z}_\ell(2))$ by Lemma 2.4. Thus statements (1), (2), and (3) in Proposition 6.2 are all equivalent for X .

By the Lefschetz hyperplane theorem, $H^4(\overline{X}, \mathbb{Z}_\ell(2)) \simeq \mathbb{Z}_\ell$ is generated by the class of a line. The Fano scheme $F(X)$ is geometrically connected by [DM98, Theorem 2.1] and admits a zero-cycle of degree 1 by the Lang–Weil estimates; hence (1.3) is surjective. It follows that statements (1), (2), and (3) in Proposition 6.2 are all satisfied by X . In particular, $H_{\mathrm{nr}}^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$. $\square$

Remark 6.7. In the case of cubic threefolds, Theorem 1.5 recovers a theorem of Colliot-Thélène [Col19, Theorem 5.1], without appealing to theorems of Parimala–Suresh [PS16] and Kato–Saito [KS83].

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