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The dual Lagrangian fibration of known hyper-Kähler manifolds

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Abstract

Given a Lagrangian fibration $\pi\colon X \to \mathbb{P}^n$ of a compact hyper-Kähler manifold of $\mathrm{K3}^{[n]}$-, Kum_n-, OG10-, or OG6-type, we construct a natural compactification of its dual torus fibration. Specifically, this compactification is given by a quotient of X by certain automorphisms acting trivially on the second cohomology and respecting the Lagrangian fibration. It is a compact hyper-Kähler orbifold with identical period mapping behavior to X.

1. Introduction

Let Y be a compact Calabi–Yau manifold with a fixed Kähler class and $\pi\colon Y \to B$ its Lagrangian fibration structure. A general fiber of π is a torus by the classical Arnold–Liouville theorem. Any torus has its dual, so one may wonder if we can systematically dualize general fibers of π to obtain a new fibration $\check{\pi}$. The mirror symmetry conjecture in [SYZ96] predicts that this should be possible for certain situations. More specifically, one expects that there exists a "dual Lagrangian fibration" $\check{\pi}\colon \check{Y} \to B$ satisfying: (1) $\check{Y}$ is a compact Calabi–Yau orbifold and $\check{\pi}$ is its Lagrangian fibration, and (2) the smooth fibers of $\check{\pi}$ are dual tori to the smooth fibers of π. When the Calabi–Yau manifold of interest is a K3 surface, there is a holomorphic variant of this question. The Kähler class and Lagrangian fibration are replaced by a holomorphic symplectic form and holomorphic elliptic fibration $\pi\colon X \to B$. Unfortunately, elliptic curves are self-dual, so the original $\pi\colon X \to B$ satisfies both conditions (1) and (2) and the conjecture becomes rather uninteresting.

A compact hyper-Kähler manifold is a higher-dimensional generalization of a K3 surface. It is a simply connected compact Kähler manifold with a unique global holomorphic symplectic form up to scaling. Let $\pi\colon X \to B$ be a holomorphic Lagrangian fibration of a compact hyper-Kähler manifold X. By the same reasons as for K3 surfaces, [GTZ13, § 2] claimed that π should be considered as self-dual *if* the following two conditions hold: (a) all the torus fibers of π are principally polarized abelian varieties, and (b) π admits at least one section. The role of assumption (a) is to say that π is fiberwise self-dual, and the role of assumption (b) is to single out a uniform dualization of complex tori as a family. If the assumptions are dropped, there is a priori no reason why one should believe the existence of a good notion of a dual Lagrangian

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fibration $\check{\pi}\colon \check{X} \to B$. The goal of this paper is to give, without assumptions (a) and (b), one distinguished candidate for a dual Lagrangian fibration $\check{\pi}$ that has all the expected properties. Unfortunately, we were able to realize our strategy only for the currently known deformation types of hyper-Kähler manifolds (Theorem 1.1), but we believe that similar results should hold in the most general setup. Once assumption (a) fails, the construction yields a compact hyper-Kähler orbifold $\check{X}$ that is not homeomorphic to X. The technical assumption (b) will be completely overcome.

Let again X be a compact hyper-Kähler manifold of dimension $2n$. In this paper, a *Lagrangian fibration* of X will mean a holomorphic surjective morphism $\pi\colon X \to B$ with connected fibers to a complex manifold B with $0 < \dim B < 2n$. By [Hwa08, GL14], the base B is necessarily isomorphic to $\mathbb{P}^n$. It is well known that any smooth fiber of π is a complex Lagrangian subtorus of X, so by restricting the Lagrangian fibration π to the locus of smooth fibers $B_0 \subset B$, we get a torus fibration $\pi_0\colon X_0 \to B_0$, a smooth proper family of complex tori.

The dual Lagrangian fibration $\check{\pi}$ will be obtained by a suitable compactification of the "dual torus fibration" $\check{\pi}_0\colon \check{X}_0 \to B_0$ that fiberwise dualizes the original torus fibration π_0. Sawon [Saw04, § 4.2] and Nagai (Ph.D. thesis, Tokyo, 2005) proposed to define the dual torus fibration as the neutral relative Picard scheme of π_0. While this definition behaves well when π_0 admits a section, it behaves in a slightly unsatisfactory way when π_0 has no sections. We thus start by proposing a new definition of $\check{\pi}_0$. Recall the fact that all torus fibers of π_0 are naturally polarized (see, for example, Voisin's argument in [Cam06, Proposition 2.1]). That is, each torus fiber F of π_0 admits a natural isogeny $F \to \check{F}$ to its dual torus $\check{F}$. Let us denote the kernel of this isogeny by (ker) and obtain an isomorphism $\check{F} \cong F/(\text{ker})$. The idea is to make this discussion global over the entire base B_0. In Theorem 3.1, we will attach a naturally polarized abelian scheme $P_0 \to B_0$ to π_0 so that X_0 becomes a P_0-torsor (this combines the results of Arinkin–Fedorov and van Geemen–Voisin). Let K_0 be the kernel of this natural polarization $P_0 \to \check{P}_0$. It is a group scheme over B_0 acting on both P_0 and X_0. Take the K_0-quotient of both spaces; on the one hand, we recover the dual abelian scheme $\check{P}_0 \cong P_0/K_0$, and on the other hand, we obtain a new space

$$\check{\pi}_0\colon \check{X}_0 \longrightarrow B_0 \quad \text{for } \check{X}_0 = X_0/K_0\,.$$

By construction, $\check{X}_0$ is a $\check{P}_0$-torsor, a smooth proper family of complex tori that are fiberwise dual to the original fibration π_0. This $\check{\pi}_0$ is our definition of the *dual torus fibration*. If π_0 has at least one section, then X_0 is a trivial P_0-torsor, which would imply that $\check{X}_0$ is a trivial $\check{P}_0$-torsor. Since $\check{P}_0 = \mathrm{Pic}^0_{X_0/B_0}$ (see Example 3.12), this recovers Sawon and Nagai's definition.

It is important to notice that the group scheme K_0 is only a finite étale group scheme over B_0. One can think of this as the total space of a finite local system on B_0; there is a monodromy issue hiding in the background, and *a priori* K_0 may not be a constant group scheme. We are now ready to state the main result of this paper.

Theorem 1.1. *Let $\pi\colon X \to B$ be a Lagrangian fibration of a compact hyper-Kähler manifold. Assume that X is of $K3^{[n]}$-, Kum_n-, OG10-, or OG6-type.*

(i) *The kernel group scheme $K_0 \to B_0$ extends to a constant group scheme $K \to B$ that acts on the entire Lagrangian fibration $\pi\colon X \to B$. Moreover, K is a subgroup of the group*

$$\mathrm{Aut}^\circ(X/B) = \left\{ f \in \mathrm{Aut}(X) : \pi \circ f = \pi,\ f^* \text{ acts as the identity on } H^2(X,\mathbb{Z}) \right\}.$$

(ii) *The quotient $\check{\pi}\colon \check{X} \to B$ for $\check{X} = X/K$ compactifies the dual torus fibration $\check{\pi}_0$.*

(iii) *The space $\check{X}$ is a compact hyper-Kähler orbifold, and $\check{\pi}$ is its Lagrangian fibration. Moreover, $\check{X}$ has the same period mapping/deformation behavior as X.*

Note. We will frequently view a finite constant group scheme $K \to B$ as a finite group, and vice versa. We will denote them by the same letter K if no confusions arise.

If X is of $\mathrm{K3}^{[n]}$- or OG10-type, then the group K (or the constant group scheme $K \to B$) is in fact trivial and these hyper-Kähler manifolds are self-dual. On the other hand, if X is of Kum_n- or OG6-type, then K is nontrivial and $\check{X}$ is not even homeomorphic to X. We will provide explicit computations for the group K in Theorem 5.1 and Remark 6.2. Also note that $\check{X}$ is a global quotient of X by automorphisms acting trivially on $H^2(X, \mathbb{Z})$. As a result, the second rational cohomologies of $\check{X}$ and X are isometric as Beauville–Bogomolov–Fujiki quadratic spaces. The higher cohomology of $\check{X}$ may be strictly smaller than that of X by [Ogu20], but they are still tightly connected via their Looijenga–Lunts–Verbitsky (LLV) structures (see [LL97, Ver95, GKLR22]). Finally, the singularities of $\check{X}$ are quotient singularities of high codimensions (at least 4), so they do not admit any symplectic resolutions. We briefly recall for the reader's convenience the notion of a singular hyper-Kähler variety and its Lagrangian fibration in Appendix A.

Remark 1.2. There were several previous results on the constructions of dual Lagrangian fibrations of compact hyper-Kähler manifolds. In particular, [Saw20, Theorem 24] announced the construction of a dual Lagrangian fibration of certain Kum_n-type hyper-Kähler manifolds (without a proof). Although Sawon's method is different from ours, it is isomorphic to our construction when π admits a section and the polarization type is $(1, \dots, 1, n+1)$. This can be shown by using the computations in Section 5.4. Sawon [Saw04] and Nagai (Ph.D. thesis) discussed a possible hyper-Kähler structure on a partial compactification of the relative Picard scheme of π_0. These are different from our direction because our dual torus fibration $\check{\pi}_0 \colon \check{X}_0 \to B_0$ is not isomorphic to the relative Picard scheme when π_0 does not have any section. Markushevich–Tikhomirov, and Menet [MT07, Men14] introduced an explicit geometric construction of certain 4-dimensional Lagrangian fibered hyper-Kähler orbifolds and realized their dual Lagrangian fibrations using the same construction. It would be interesting to find a connection between their results and our perspective. Finally, [Ver99] discussed certain self-dualities of hyper-Kähler manifolds at the level of cohomology.

There are two key ingredients for our proof of Theorem 1.1: the group $\mathrm{Aut}^\circ(X/B)$ and the notion of a polarization type. The definition of the group $\mathrm{Aut}^\circ(X/B)$ is inspired by the similar group $\mathrm{Aut}^\circ(X)$, which has already played an important role in the theory of hyper-Kähler manifolds. The two main properties of $\mathrm{Aut}^\circ(X)$ are its finiteness [Huy99] and deformation invariance [HT13]. The group $\mathrm{Aut}^\circ(X)$ is also computed for all known deformation types of hyper-Kähler manifolds (see [Bea83a, BNS11, MW17]). We provide similar results for the group $\mathrm{Aut}^\circ(X/B)$: It is finite abelian (Proposition 3.22) and deformation invariant (Theorem 2.2). We also compute $\mathrm{Aut}^\circ(X/B)$ for all known deformation types in Theorem 5.1. The polarization type of the fibers of π_0 are studied in [Wie16, Wie18], though the idea in that work has already been used before. We relate the polarization type to the study of our group scheme K_0.

1.1 Structure of the paper

In Section 2, we prove that the group $\mathrm{Aut}^\circ(X/B)$ is deformation invariant on the Lagrangian fibration π. This is inspired by Hassett–Tschinkel's proof of the deformation invariance of $\mathrm{Aut}^\circ(X)$ in [HT13, Theorem 2.1]. In Section 3, we start by attaching an abelian scheme P_0 to any Lagrangian fibration of a hyper-Kähler manifold: P_0 is the identity component of the relative automorphism scheme of π. There exists a unique primitive polarization λ on P_0, and we can define its kernel group scheme K_0. We then try to relate K_0 and $\mathrm{Aut}^\circ(X/B)$ in general. This section also discusses the notion of the polarization type of a Lagrangian fibration. In essence, the polarization type is the study of a single fiber of the group scheme K_0.

The goal of Section 5 is twofold. First, we compute the group $\mathrm{Aut}^\circ(X/B)$ for all currently known deformation types of hyper-Kähler manifolds. Second, we prove an inclusion $K_0 \subset \mathrm{Aut}^\circ(X/B)$ for special constructions of Kum_n-type hyper-Kähler manifolds. The material here will be mostly concrete computations. Section 4 introduces a slightly more systematic method to assist these computations. In Section 6, we prove the main result of this article: There exists a natural compactification of the dual torus fibration for all currently known deformation types of hyper-Kähler manifolds.

We provide two appendices. Appendix A contains various definitions of singular hyper-Kähler varieties appearing in the literature. In Appendix B, we discuss certain special quotients of compact hyper-Kähler manifolds. The quotient $\check{X} = X/K$ will be a special instance of this more general setup.

1.2 Notation and conventions

In this paper, every hyper-Kähler manifold X will be assumed to be compact but not necessarily projective unless stated explicitly. When X further admits a Lagrangian fibration $\pi\colon X \to B$, it is helpful to keep in mind that X is projective if and only if π admits at least one rational multisection. Indeed, if X is projective, then a general scheme-theoretic fact says that any smooth morphism between algebraic varieties admits an étale local section. The converse is [Saw09, Lemma 2].

Assume that X has dimension $2n$. Any Lagrangian fibration $\pi\colon X \to B$ in this paper will always have base $B = \mathbb{P}^n$ since we are assuming that B is smooth and $0 < \dim B < 2n$ (see [Hwa08, GL14]). The *Beauville–Bogomolov–Fujiki form* and the *Fujiki constant* of X are a unique primitive symmetric bilinear form $q\colon H^2(X,\mathbb{Z}) \otimes H^2(X,\mathbb{Z}) \to \mathbb{Z}$ and a positive rational number c_X satisfying the *Fujiki relation*

$$\int_X x^{2n} = c_X \cdot \frac{(2n)!}{2^n \cdot n!} \cdot q(x)^n \quad \text{for } x \in H^2(X,\mathbb{Z})\,. \tag{1.1}$$

The Fujiki constant is computed for all currently known deformation types of hyper-Kähler manifolds: (1) $c_X = 1$ for $\mathrm{K3}^{[n]}$- or OG10-type, and (2) $c_X = n+1$ for Kum_n- or OG6-type (see [Bea83b, Rap07, Rap08]). In practice, we will mostly need a stronger version of the Fujiki relation, which follows from the polarization process:

$$\int_X x_1 \cdots x_{2n} = c_X \sum_\sigma q(x_{\sigma(1)}, x_{\sigma(2)}) \cdots q(x_{\sigma(2n-1)}, x_{\sigma(2n)}) \quad \text{for } x_i \in H^2(X,\mathbb{Z})\,.$$

Here $\sigma \in \mathfrak{S}_{2n}$ runs through all the $2n$-permutations but up to $2^n \cdot n!$ ambiguities inducing the same expression in the summation. The *divisibility* of $x \in H^2(X,\mathbb{Z})$ is defined to be the positive integer

$$\mathrm{div}(x) = \gcd\bigl\{q(x,y) : y \in H^2(X,\mathbb{Z})\bigr\}\,. \tag{1.2}$$

The study of the full cohomology $H^*(X,\mathbb{Q})$ will need the notion of the *LLV algebra* $\mathfrak{g}$, introduced by Looijenga–Lunts [LL97] and Verbitsky [Ver95]. For its concrete computations, we will follow the representation-theoretic notation used in [GKLR22, §§ 2–3].

Throughout, group schemes will be used both in algebraic and analytic context. An *abelian scheme* $P \to S$ is an analytically proper connected commutative group scheme over S with complex torus fibers. Every abelian scheme P admits its *dual abelian scheme* $\check{P}$. A *polarization* of an abelian scheme P is a finite étale homomorphism $\lambda\colon P \to \check{P}$ over S such that for each fiber F, the restriction $\lambda_{|F}\colon F \to \check{F}$ is of the form $x \mapsto \left[t_x^* L \otimes L^{-1}\right]$ for an ample line bundle L on F. Given a group scheme $G \to S$, an *analytic G-torsor* is a morphism $Y \to S$ equipped with a G-action such that there exists an analytic covering $\tilde{S} = \bigsqcup_\alpha U_\alpha \to S$ where the base changes $\tilde{Y} = Y \times_S \tilde{S}$ and $\tilde{G} = G \times_S \tilde{S}$ are $\tilde{G}$-equivariantly isomorphic over $\tilde{S}$. In the algebraic setting, one can use a different topology, for example étale topology, to define an étale torsor. Our references for the theory of abelian schemes are [MFK94, BLR90, FC90]. For the notion of torsors, see [Mil80] or [BLR90].

2. Deformation invariance of the H^2-trivial automorphisms

Let X be a compact hyper-Kähler manifold. Consider the group of *H^2-trivial automorphisms*

$$\operatorname{Aut}^\circ(X) = \ker\bigl(\operatorname{Aut}(X) \longrightarrow \mathrm{O}\bigl(H^2(X,\mathbb{Z}), q\bigr),\ f \longmapsto f_*\bigr)\,.$$

Here $\operatorname{Aut}(X)$ is the group of biholomorphic automorphisms of X. Huybrechts [Huy99, Proposition 9.1] together with Hassett–Tschinkel [HT13, Theorem 2.1] proved that $\operatorname{Aut}^\circ(X)$ is a finite group that is invariant under deformations of X.

Let us now further assume that X admits a Lagrangian fibration $\pi\colon X \to B$ and denote by $\operatorname{Aut}(X/B)$ the group of automorphisms of X acting fiberwise on π. We can restrict our attention to H^2-trivial automorphisms that acts fiberwise on the Lagrangian fibration

$$\operatorname{Aut}^\circ(X/B) = \operatorname{Aut}(X/B) \cap \operatorname{Aut}^\circ(X)\,. \tag{2.1}$$

Since $\operatorname{Aut}^\circ(X)$ is finite, so is $\operatorname{Aut}^\circ(X/B)$. In fact, we can further prove that $\operatorname{Aut}^\circ(X/B)$ is abelian: This will be shown later in Proposition 3.22. In Section 3, we will reinterpret $\operatorname{Aut}^\circ(X/B)$ as global sections of the "translation automorphism scheme" $P_0 \to B_0$. In Section 5, we will compute $\operatorname{Aut}^\circ(X/B)$ for all Lagrangian fibrations $\pi\colon X \to B$ from a hyper-Kähler manifold X of known deformation type.

But before doing so, let us establish a more basic fact in this section; we prove that $\operatorname{Aut}^\circ(X/B)$ is invariant under deformations of π. To make this more precise, we first need to define the notion of a family of Lagrangian fibered hyper-Kähler manifolds.

Definition 2.1. A *family of Lagrangian fibered compact hyper-Kähler manifolds* is a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[d, "p"'] \arrow[dr, "\pi"] & \\ S & \mathcal{B} \arrow[l, "q"] \end{tikzcd}$$

with the following conditions:

(i) The map $p\colon \mathcal{X} \to S$ is a smooth proper family of compact hyper-Kähler manifolds of dimension $2n$ over a connected complex space S.
(ii) The map $q\colon \mathcal{B} \to S$ is the projectivization of a rank $n+1$ holomorphic vector bundle on S.
(iii) For all $t \in S$, the fiber $\pi\colon X_t \to B_t$ is a Lagrangian fibration.

Note that the second condition ensures that $\mathcal{B}$ is projective over S and admits a relative ample line bundle $\mathcal{O}_{\mathcal{B}/S}(1)$. The pullback $\mathcal{H} = \pi^*\mathcal{O}_{\mathcal{B}/S}(1)$ can be considered as a family of line bundles on $\mathcal{X}$. Therefore, Definition 2.1 induces a family of pairs (X, H), where $H = \pi^*\mathcal{O}_B(1)$.

If we assumed that $q\colon \mathcal{B} \to S$ is only a $\mathbb{P}^n$-bundle, then we would not have had a family of line bundles $\mathcal{H}$.

As usual, two Lagrangian fibrations $\pi\colon X \to B$ and $\pi'\colon X' \to B'$ are *deformation equivalent* if there exists a family of Lagrangian fibered compact hyper-Kähler manifolds $\mathcal{X}/\mathcal{B}/S$ over a connected union of 1-dimensional open disks S, realizing them as two fibers at $t, t' \in S$. Matsushita [Mat16, Corollary 1.3] proved that such a deformation problem admits a local universal deformation.

We can now state the main theorem of this section.

Theorem 2.2. *The group* $\mathrm{Aut}^\circ(X/B)$ *is invariant under deformations of* $\pi\colon X \to B$.

The rest of this section will be devoted to the proof of Theorem 2.2. The sketch of the proof is as follows. First, we descend the $\mathrm{Aut}^\circ(X)$-action on X to B so that the Lagrangian fibration $\pi\colon X \to B$ becomes an equivariant morphism. This means that we have a group homomorphism $\mathrm{Aut}^\circ(X) \to \mathrm{Aut}(B)$ whose kernel is precisely $\mathrm{Aut}^\circ(X/B)$. Descending such an action is a nontrivial problem (this is quite similar to the result of [Bri11, Proposition 2.1]), so we need to overcome this issue using the notion of a G-linearizability of line bundles. Next, we need to sheafify the discussions as we are interested in their deformation behavior. The result will follow from formal properties of the kernel of the sheaf homomorphism.

2.1 *G*-linearizability of a line bundle

Before we get into the proof of Theorem 2.2, let us recall the notion of G-linearizability of a line bundle on a complex manifold. For simplicity we only consider finite group actions. Our references are [Bri18, § 3], [Dol03, § 7], and [MFK94, § 1.3], but we need to take some additional care since these references only consider the algebraic setting.

Let G be an arbitrary finite group and $\mathcal{X}$ be a complex manifold equipped with a holomorphic G-action. A *G-linearized line bundle* on $\mathcal{X}$ is a holomorphic line bundle $\mathcal{L}$ together with a collection of isomorphisms $\Phi_g\colon \mathcal{L} \to g^*\mathcal{L}$ for $g \in G$ such that $\Phi_{gg'} = \Phi_{g'} \circ g'^*\Phi_g$ for $g, g' \in G$. A *G-invariant line bundle* on $\mathcal{X}$ is a holomorphic line bundle $\mathcal{L}$ such that $\mathcal{L} \cong g^*\mathcal{L}$ for all $g \in G$ (without any condition). We denote by $\mathrm{Pic}^G(\mathcal{X})$ and $\mathrm{Pic}(\mathcal{X})^G$ the groups of G-linearized line bundles and G-invariant line bundles on $\mathcal{X}$ up to isomorphisms. The second group is precisely the G-invariant subgroup of $\mathrm{Pic}(\mathcal{X})$.

There is a forgetful homomorphism $\mathrm{Pic}^G(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X})^G$, which is neither injective nor surjective in general. To understand the obstruction to its surjectivity, one considers an exact sequence of abelian groups ([Dol03, Lemma 7.1 and Remark 7.2] or [Bri18, Proposition 3.4.5])

$$\mathrm{Pic}^G(\mathcal{X}) \longrightarrow \mathrm{Pic}(\mathcal{X})^G \longrightarrow H^2(G, \Gamma)\,, \quad \Gamma = H^0(\mathcal{X}, \mathcal{O}^*_{\mathcal{X}})\,.$$

Both Dolgachev's and Brion's discussions are for algebraic varieties, but their proofs consist in checking the cocycle/coboundary conditions of group cohomology, so they apply to our analytic setting as well. With this exact sequence in hand, we have the following.

Lemma 2.3. *Every G-invariant line bundle $\mathcal{H}$ on $\mathcal{X}$ is G-linerizable up to a suitable tensor power.*

Proof. It is a general fact in the theory of group cohomology (for finite groups) that all the higher-degree cohomologies $H^{\geqslant 1}(G, \Gamma)$ are $|G|$-torsion for any G-module Γ (see, for example, [Ser79, Corollary VIII.1]). Hence by the exact sequence above, the $|G|$th tensor $\mathcal{H}^{\otimes |G|}$ vanishes in $H^2(G, \Gamma)$ and hence comes from $\mathrm{Pic}^G(\mathcal{X})$. ☐

For us, the importance of the G-linearizability of a line bundle comes from the induced G-action on the higher direct images of a linearized line bundle. If $\mathcal{L}$ is a G-linearized line bundle on $\mathcal{X}$ and $p\colon \mathcal{X} \to S$ is a G-invariant holomorphic map, then we have a contravariant G-action on all the higher direct image sheaves

$$g^*\colon R^k p_*\mathcal{L} \to R^k p_*\mathcal{L}\,, \quad (g \circ g')^* = g'^* \circ g^*\,.$$

Now assume further that $\mathcal{L}$ is globally generated over S and $p_*\mathcal{L}$ is a vector bundle on S. Then we have a G-action on $\mathbb{P}_S(p_*\mathcal{L})$ making the holomorphic map $\mathcal{X} \to \mathbb{P}_S(p_*\mathcal{L})$ G-equivariant over S. See [MFK94, Proposition 1.7].

2.2 The automorphism sheaves and deformation invariance of the H^2-trivial automorphisms

Suppose that we have a smooth proper family of hyper-Kähler manifolds $p\colon \mathcal{X} \to S$. Let $U \subset S$ be an analytic open subset, and denote by $p\colon \mathcal{X}_U = p^{-1}(U) \to U$ the restricted family over U. We define the sheaf of H^2-trivial automorphism groups $\underline{\mathrm{Aut}}^\circ_{\mathcal{X}/S}$ on S by

$$\underline{\mathrm{Aut}}^\circ_{\mathcal{X}/S}(U) = \bigl\{U\text{-automorphisms } f\colon \mathcal{X}_U \to \mathcal{X}_U \text{ such that } f^*\colon R^2p_*\mathbb{Z} \to R^2p_*\mathbb{Z} \text{ is the identity}\bigr\}\,.$$

By the work of Huybrechts and Hassett–Tschinkel, this sheaf is a local system of finite groups. We can consider it as a family of groups $\mathrm{Aut}^\circ(X_t)$ for $t \in S$. Similarly, given a family of Lagrangian fibered hyper-Kähler manifolds, we can define a family of groups $\mathrm{Aut}^\circ(X_t/B_t)$.

Definition 2.4. Given a family of Lagrangian fibered hyper-Kähler manifolds $p\colon \mathcal{X} \xrightarrow{\pi} \mathcal{B} \xrightarrow{q} S$, we define a sheaf of groups $\underline{\mathrm{Aut}}^\circ_{\mathcal{X}/\mathcal{B}/S}$ on S by

$$\begin{aligned}\underline{\mathrm{Aut}}^\circ_{\mathcal{X}/\mathcal{B}/S}(U) = \bigl\{&\mathcal{B}_U\text{-automorphisms } f\colon \mathcal{X}_U \longrightarrow \mathcal{X}_U \\ &\text{such that } f^*\colon R^2p_*\mathbb{Z} \longrightarrow R^2p_*\mathbb{Z} \text{ is the identity}\bigr\}.\end{aligned}$$

Equivalently, we may define $\underline{\mathrm{Aut}}^\circ_{\mathcal{X}/\mathcal{B}/S} = q_*\underline{\mathrm{Aut}}_{\mathcal{X}/\mathcal{B}} \cap \underline{\mathrm{Aut}}^\circ_{\mathcal{X}/S}$.

As mentioned, the sheaf of H^2-trivial automorphisms $\underline{\mathrm{Aut}}^\circ_{\mathcal{X}/S}$ is a local system. The sheaf $\underline{\mathrm{Aut}}^\circ_{\mathcal{X}/\mathcal{B}/S}$ is a subsheaf of $\underline{\mathrm{Aut}}^\circ_{\mathcal{X}/S}$, and our goal is to prove that it is locally constant as well. The question is local on the base S, so we may assume that S is a small open ball. Then $\underline{\mathrm{Aut}}^\circ_{\mathcal{X}/S}$ becomes a constant sheaf, so we may consider it as an abstract finite group

$$G = \mathrm{Aut}^\circ(X)$$

acting on $\mathcal{X} \to S$ fiberwise.

Consider the automorphism sheaf $\underline{\mathrm{Aut}}_{\mathcal{B}/S}$ of the $\mathbb{P}^n$-bundle $\mathcal{B} \to S$. It is the sheaf of analytic local sections of the $\mathrm{PGL}(n+1,\mathbb{C})$-group scheme $\mathrm{Aut}_{\mathcal{B}/S} \to S$.

Proposition 2.5. *Assume that S is an open ball. Then the G-action on $\mathcal{X}$ descends to $\mathcal{B}$ and makes $\pi\colon \mathcal{X} \to \mathcal{B}$ a G-equivariant morphism. In other words, there exists a homomorphism of sheaves*

$$G = \underline{\mathrm{Aut}}^\circ_{\mathcal{X}/S} \longrightarrow \underline{\mathrm{Aut}}_{\mathcal{B}/S} \tag{2.2}$$

with kernel $\underline{\mathrm{Aut}}^\circ_{\mathcal{X}/\mathcal{B}/S}$.

We use the G-linearizability of line bundles to prove this proposition. The following lemma proves that every line bundle on $\mathcal{X}$ is G-invariant.

Lemma 2.6. *The group G acts trivially on* $\mathrm{Pic}(\mathcal{X})$.

Proof. We first claim that G acts trivially on $H^2(\mathcal{X},\mathbb{Z})$. Apply the Leray spectral sequence

$$E_2^{p,q} = H^p\big(S, R^q p_*\mathbb{Z}\big) \implies H^{p+q}(\mathcal{X},\mathbb{Z})\,.$$

Noticing that $R^0p_*\mathbb{Z} = \mathbb{Z}$, $R^1p_*\mathbb{Z} = 0$, and S is an open ball, we obtain an isomorphism $H^2(\mathcal{X},\mathbb{Z}) \cong H^0\big(S, R^2p_*\mathbb{Z}\big)$. This isomorphism respects the G-action as the Leray spectral sequence is functorial. Now G acts on $H^2(X_t,\mathbb{Z})$ trivially for any fiber X_t, so G acts on $R^2p_*\mathbb{Z}$ trivially and the claim follows.

Let us prove that the first Chern class map $\operatorname{Pic}(\mathcal{X}) \to H^2(\mathcal{X},\mathbb{Z})$ is injective to conclude. From the usual exponential sequence $0 \to \mathbb{Z} \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}^* \to 0$, it suffices to prove $H^1(\mathcal{X},\mathcal{O}_{\mathcal{X}}) = 0$. Again, use the Leray spectral sequence

$$E_2^{p,q} = H^p\big(S, R^q p_*\mathcal{O}_{\mathcal{X}}\big) \implies H^{p+q}(\mathcal{X},\mathcal{O}_{\mathcal{X}})\,.$$

This time, we have $R^0p_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_S$ and $R^1p_*\mathcal{O}_{\mathcal{X}} = 0$. This implies $H^1(\mathcal{X},\mathcal{O}_{\mathcal{X}}) = 0$. $\square$

Consider the line bundle $\mathcal{H} = \pi^*\mathcal{O}_{\mathcal{B}/S}(1)$ on $\mathcal{X}$. Since $\operatorname{Pic}(\mathcal{X})$ is G-invariant, we can apply Lemma 2.3 to $\mathcal{H}$ and conclude that $\mathcal{H}^{\otimes m}$ is G-linearizable for some positive integer m. As a result, we have a G-equivariant morphism $\pi_m\colon \mathcal{X} \to \mathcal{B}_m$, where $\mathcal{B}_m = \mathbb{P}_S\big(p_*\mathcal{H}^{\otimes m}\big)$ is the dual of the complete linear system associated with $\mathcal{H}^{\otimes m}$. Consider the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[dr, "\pi"] \arrow[drr, bend left, "\pi_m"] \arrow[dd, "p"'] & & \\ & \mathcal{B} \arrow[r, hook] \arrow[dl, "q"] & \mathcal{B}_m \arrow[dll, bend left, "q_m"] \\ S & & \end{tikzcd}\tag{2.3}$$

Proof of Proposition 2.5. Let us first prove that the mth relative Veronese embedding $\mathcal{B} \hookrightarrow \mathcal{B}_m$ makes the diagram (2.3) commute. Let $t \in S$ be an arbitrary point. The morphism $\pi\colon X_t \to B_t$ over t has connected fibers, so it is the Iitaka fibration associated with the line bundle H_t (see, for example, [Laz04, § 2.1.B]). This in particular implies that the morphism $\pi_m\colon X_t \to (B_m)_t$ defined by $H_t^{\otimes m}$ factors through the Iitaka fibration π, where the morphism $B_t \hookrightarrow (B_m)_t$ is precisely the mth Veronese embedding. In other words, the mth relative Veronese embedding makes the diagram (2.3) commute.

Note that π_m is equivariant because $\mathcal{H}^{\otimes m}$ was G-linearized. This implies that π is equivariant since the image of π_m is $\mathcal{B}$. In other words, we have a homomorphism of sheaves $G \to \underline{\operatorname{Aut}}_{\mathcal{B}/S}$. Its kernel consists of elements acting fiberwise on $\pi\colon \mathcal{X} \to \mathcal{B}$, which is $\underline{\operatorname{Aut}}^\circ_{\mathcal{X}/\mathcal{B}/S}$. $\square$

We now present the proof of the main theorem.

Proof of Theorem 2.2. Let $\mathcal{X} \to \mathcal{B} \to S$ be a family of Lagrangian fibered hyper-Kähler manifolds over an open ball S. The sheaves $\underline{\operatorname{Aut}}^\circ_{\mathcal{X}/S}$ and $\underline{\operatorname{Aut}}_{\mathcal{B}/S}$ are represented by constant group schemes

$$\operatorname{Aut}^\circ_{\mathcal{X}/S} \cong G \times S\,, \quad \operatorname{Aut}_{\mathcal{B}/S} \cong \operatorname{PGL}(n+1) \times S\,.$$

The homomorphism (2.2) is thus representable by a homomorphism $\alpha\colon \operatorname{Aut}^\circ_{\mathcal{X}/S} \to \operatorname{Aut}_{\mathcal{B}/S}$ of group schemes, so $\ker\alpha = \underline{\operatorname{Aut}}^\circ_{\mathcal{X}/\mathcal{B}/S}$ is representable by a subgroup scheme of $G \times S$.

To prove that $\ker\alpha$ is a constant subgroup scheme of $G \times S$, fix a connected component S' of $G \times S$. Consider the restriction of α followed by the projection

$$\beta\colon S' \longrightarrow \operatorname{PGL}(n+1) \times S \longrightarrow \operatorname{PGL}(n+1)\,.$$

Then we claim that either $\beta(S') = \{\mathrm{id}\}$ or $\beta(S') \not\ni \mathrm{id}$. Notice that the image $\beta(S')$ consists of $|G|$-torsion matrices in $\mathrm{PGL}(n+1)$. Since the set of $|G|$-torsion matrices is a disjoint union of $\mathrm{PGL}(n+1)$-adjoint orbits (classified by eigenvalues), the connected set $\beta(S')$ has to lie in a single orbit. The adjoint orbit containing the identity matrix is a singleton set $\{\mathrm{id}\}$. Hence the claim follows and shows that $\ker\alpha$ is constant. □

3. Abelian schemes associated with Lagrangian fibrations

The aim of this section is to associate a polarized abelian scheme with every Lagrangian fibered compact hyper-Kähler manifold, and to discuss its consequences. The following is the first main theorem of this section.

Theorem 3.1. *Let $\pi\colon X \to B$ be a Lagrangian fibration of a compact hyper-Kähler manifold and $B_0 \subset B$ the open subset where π is smooth. Set $X_0 = \pi^{-1}(B_0)$ so that it becomes a smooth proper family of complex tori over B_0.*

(i) *There exists a unique projective abelian scheme $\nu\colon P_0 \to B_0$ making $\pi\colon X_0 \to B_0$ an analytic torsor under ν.*

(ii) *Moreover, the abelian scheme is simple and has a unique choice of a primitive polarization*

$$\lambda\colon P_0 \longrightarrow \check{P}_0\,. \tag{3.1}$$

Here $\check{P}_0 \to B_0$ is the dual abelian scheme of $P_0 \to B_0$.

Recall that an abelian scheme $P_0 \to B_0$ is called simple if it does not contain any nontrivial proper abelian subscheme $Q_0 \subsetneq P_0$. Theorem 3.1 is a combination (with a slight generalization) of [AF16, Theorem 2] and [vGV16]. See also [AR21].

Definition 3.2. The abelian scheme $\nu\colon P_0 \to B_0$ in Theorem 3.1 is called the *abelian scheme associated with* π.

One application of Theorem 3.1 is a systematic study of the polarization type of the smooth (torus) fibers of π. The study of their polarization type goes back to at least [Saw03], which in turn references an earlier idea of Mukai (see Proposition 5.3 in op. cit.). However, to our knowledge, Wieneck's series of papers [Wie16, Wie18] were the first work to consider the polarization type as an invariant attached to a Lagrangian fibration and study them in great details for $\mathrm{K3}^{[n]}$- and Kum_n-type hyper-Kähler manifolds. Using Theorem 3.1, we can give a slightly more refined definition of the polarization type.

Definition 3.3. (i) The *polarization scheme* of π is the kernel $K_0 = \ker\lambda$ of the polarization (3.1).

(ii) The *polarization type* of π is an n-tuple of positive integers $(d_1, \dots, d_n)$ with $d_1 \mid \cdots \mid d_n$ such that the fibers of the polarization scheme are isomorphic to $(\mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_n)^{\oplus 2}$.

The polarization scheme K_0 is a finite, étale, and commutative group scheme over B_0. Hence its fibers are all isomorphic, and the polarization type is well defined. The polarization type will be crucially used in our method. We therefore devote the short Section 3.3 to collect its properties.

The second main theme of this section is a relation between the group $\mathrm{Aut}^\circ(X/B)$ and the polarization scheme K_0. We will see in Proposition 3.21 that $\mathrm{Aut}^\circ(X/B)$ can be interpreted as a constant subgroup scheme of P_0:

$$\mathrm{Aut}^\circ(X/B) \hookrightarrow P_0 \,. \tag{3.2}$$

We expect that the image of this homomorphism will *contain* the polarization scheme K_0. This is a nontrivial claim; this would imply that $K_0 \to B_0$ is extendable to a constant group scheme $K \to B$ acting on the entire $X \to B$. We were not able to prove this claim in general, and a large part of this paper will be devoted to showing this for known deformation types of hyper-Kähler manifolds. The following propositions will be our technical tools for doing this. It will be convenient to introduce a temporary notation

$$K_0[a] = \ker\bigl(a\lambda \colon P_0 \longrightarrow \check{P}_0\bigr) \,,$$

a finite étale commutative group scheme over B_0.

Proposition 3.4. *Let $\pi \colon X \to B$ and $\pi' \colon X' \to B'$ be two deformation-equivalent Lagrangian fibrations of compact hyper-Kähler manifolds. Let a be a positive integer. Then the inclusion (3.2) factors through*

$$\mathrm{Aut}^\circ(X/B) \hookrightarrow K_0[a] \tag{3.3}$$

if and only if the same holds for π'.

Proposition 3.5. *Let $\pi \colon X \to B$ be a Lagrangian fibration of a compact hyper-Kähler manifold and $(d_1, \dots, d_n)$ its polarization type. Assume that we have an equality $c_X = d_1 \cdots d_n$. Then (3.3) holds for $a = \mathrm{div}(h)$, where $h \in H^2(X, \mathbb{Z})$ is the class of $\pi^*\mathcal{O}_B(1)$ and its divisibility $\mathrm{div}(h)$ is as defined in (1.2).*

Note that the inclusion (3.3) has a different direction from our desired $K_0 \hookrightarrow \mathrm{Aut}^\circ(X/B)$. Our strategy will be to first show (3.3) for a certain value of a, and then deduce the relation between two subgroup schemes $K_0, \mathrm{Aut}^\circ(X/B) \subset K_0[a]$. The first proposition says that the inclusion (3.3) is deformation invariant on π. The second proposition provides at least one such an integer a, though this may not be the minimal possible value. The unfortunate assumption $c_X = d_1 \cdots d_n$ will be satisfied for all known deformation types of hyper-Kähler manifolds, so it will not be a huge problem. See Theorem 3.16.

3.1 Abelian scheme associated with a Lagrangian fibration

In this subsection, we present the proof of Theorem 3.1. Note again that we are assuming neither that X is projective nor that π has a rational section.

Recall that every smooth closed fiber F of π is a complex torus (holomorphic Arnold–Liouville theorem). In fact, F is necessarily an abelian variety as observed by Voisin [Cam06, Proposition 2.1]. It would be helpful for us to first review this fact. The key idea is the following cohomological lemma, which has been discovered several times independently in [Voi92, Ogu09, Mat16] and recently generalized into higher-degree cohomologies by Shen–Yin and Voisin [SY22]. We follow the version stated in [Mat16, Lemma 2.2].

Lemma 3.6. *Let F be any smooth fiber of π and $h \in H^2(X, \mathbb{Z})$ the cohomology class of $\pi^*\mathcal{O}_B(1)$. Then the restriction map*

$$-_{|F} \colon H^2(X, \mathbb{Z}) \longrightarrow H^2(F, \mathbb{Z})$$

has $\ker(-_{|F}) = h^\perp$. Consequently, it has $\mathrm{im}(-_{|F}) \cong \mathbb{Z}$.

Corollary 3.7 (Voisin). *The integral generator of $\mathrm{im}(-_{|F})$ in Lemma 3.6 is an ample class of F. As a result, F is an abelian variety.*

Proof. Say y is an integral generator of Lemma 3.6. Choose any Kähler class $\omega \in H^2(X, \mathbb{R})$, and consider its restriction $\omega_{|F}$, a Kähler class on F. It has to be a nonzero real multiple of y. This means that, up to a sign, y has to be a Kähler class on F. Hence y is an integral Kähler class, so it is ample. □

We caution the reader to be aware that the ample generator y of $\operatorname{im}(-_{|F})$ may be nonprimitive (see Proposition 3.17). One reasonable choice of a polarization on an abelian variety fiber F is a unique *primitive* ample class in $H^2(F, \mathbb{Z})$ parallel to y. Theorem 3.1 is essentially a more global way to formulate this over the whole base B_0.

We divide the proof of Theorem 3.1 into three parts: (1) an explicit construction of the polarized abelian scheme P_0, (2) a proof that such a construction makes X_0 a torsor under P_0, and finally (3) a proof of its uniqueness. The uniqueness should be a more general fact about arbitrary torsors, at least in the algebraic case (see Moret-Bailly's answer in [Mor18]). The construction of P_0 works for any proper family of complex tori. The uniqueness of the polarization is the only part that needs the fact that X_0 is obtained from a Lagrangian fibered hyper-Kähler manifold X.

The proof of the construction part closely follows [vGV16], but for completeness we reproduce their argument here.

Proof of Theorem 3.1, construction. Apply the global invariant cycle theorem (for proper maps between compact Kähler manifolds [Del71]) and Lemma 3.6 to obtain

$$H^0\big(B_0, R^2\pi_*\mathbb{Q}\big) = \operatorname{im}\big(H^2(X, \mathbb{Q}) \longrightarrow H^2(F, \mathbb{Q})\big) \cong \mathbb{Q}\,. \tag{3.4}$$

It follows that there exists a unique homomorphism $\mathbb{Q} \to R^2\pi_*\mathbb{Q}$ of local systems on B_0 up to scaling. Hence, there exists a unique primitive homomorphism $\mathbb{Z} \to R^2\pi_*\mathbb{Z}$ of $\mathbb{Z}$-local systems. This is a homomorphism of variations of $\mathbb{Z}$-Hodge structures ($\mathbb{Z}$-VHS for short) since fiberwise, Corollary 3.7 proves that the image of $H^2(X, \mathbb{Q}) \to H^2(F, \mathbb{Q})$ is a Hodge cycle. Taking the dual and using $R^2\pi_*\mathbb{Z} = \wedge^2 R^1\pi_*\mathbb{Z}$ (as $\pi_0 \colon X_0 \to B_0$ is a family of complex tori), we obtain a primitive polarization

$$\big(R^1\pi_*\mathbb{Z}\big)^\vee \otimes \big(R^1\pi_*\mathbb{Z}\big)^\vee \longrightarrow \mathbb{Z}\,. \tag{3.5}$$

We have constructed a weight -1 $\mathbb{Z}$-VHS $\big(R^1\pi_*\mathbb{Z}\big)^\vee$ equipped with a polarization (3.5). Now use a formal equivalence of categories between the category of polarized weight -1 $\mathbb{Z}$-VHS and that of polarized abelian schemes (see, for example, [Del72, § 5.2] and [Del71, § 4.4]). This constructs our desired abelian scheme $\nu \colon P_0 \to B_0$ with a unique primitive polarization $\lambda \colon P_0 \to \check{P}_0$ over B_0. To prove that P_0 is simple, we may prove that the corresponding VHS $R^1\pi_*\mathbb{Q}$ is simple. This is tacitly proved in [vGV16] and later explicitly stated in [Voi18, Lemma 4.5]. The idea is that if $R^1\pi_*\mathbb{Q}$ splits as a direct sum $\mathcal{V}_1 \oplus \mathcal{V}_2$ of two VHS, then each of them has their own polarizations, forcing $h^0\big(B_0, R^2\pi_*\mathbb{Q}\big) \geqslant h^0\big(B_0, \wedge^2\mathcal{V}_1\big) + h^0\big(B_0, \wedge^2\mathcal{V}_2\big) \geqslant 2$. We omit the details here. □

Proof of Theorem 3.1, torsor. The following is a standard descent argument for torsors. Consider a covering $\bigsqcup_{i \in I} B_i \to B_0$ in analytic topology so that each $X_i = \pi^{-1}(B_i) \to B_i$ admits a holomorphic section $s_i \colon B_i \to X_i$. Consider $\pi \colon X_i \to B_i$ as an abelian scheme with a zero section s_i. By the equivalence of abelian schemes and $\big(R^1\pi_*\mathbb{Z}\big)^\vee$, we have an isomorphism $\phi_i \colon X_i \to P_i$ of abelian schemes.

The group law of the abelian schemes under the isomorphism ϕ_i can be made into a P_i-action on X_i:

$$\rho_i \colon P_i \times_{B_i} X_i \longrightarrow X_i\,, \quad (p_i, x_i) \longmapsto \phi_i^{-1}(\phi_i(x_i) + p_i)\,.$$

To patch the ρ_i together to descend them to $\rho\colon P \times_B X \to X$, we need to check that ρ_i and ρ_j coincide over the intersection $B_{ij} = B_i \cap B_j$, that is,

$$\phi_i^{-1}(\phi_i(x_{ij}) + p_{ij}) = \phi_j^{-1}(\phi_j(x_{ij}) + p_{ij}) \quad \text{for all } (p_{ij}, x_{ij}) \in P_{ij} \times_{B_{ij}} X_{ij}\,. \tag{3.6}$$

Over B_{ij}, notice that the automorphism $\phi_j \circ \phi_i^{-1}\colon P_{ij} \to P_{ij}$ is a translation automorphism by $\phi_j \circ \phi_i^{-1}(0)$, the discrepancy between the two zero sections. This means that

$$\begin{aligned} \phi_j(x_{ij}) + p_{ij} &= \phi_j \circ \phi_i^{-1}(\phi_i(x_{ij})) + p_{ij} = \big(\phi_i(x_{ij}) + \phi_j \circ \phi_i^{-1}(0)\big) + p_{ij} \\ &= (\phi_i(x_{ij}) + p_{ij}) + \phi_j \circ \phi_i^{-1}(0) = \phi_j \circ \phi_i^{-1}(\phi_i(x_{ij}) + p_{ij})\,, \end{aligned}$$

proving (3.6). The group action axioms are easily verified, and X_0 is clearly a P_0-torsor. □

Proof of Theorem 3.1, uniqueness. Let $\nu\colon P_0 \to B_0$ be a (not necessarily projective) abelian scheme so that π becomes a torsor under ν. We claim that $R^1\nu_*\mathbb{Z} \cong R^1\pi_*\mathbb{Z}$ as VHS over B_0. Consider the group scheme action map

$$\begin{tikzcd} P_0 \times_{B_0} X_0 \arrow[r, "\rho"] \arrow[dr, "\mu"'] & X_0 \arrow[d, "\pi"] \\ & B_0\,. \end{tikzcd}$$

From the diagram, we have a pullback morphism between the VHS $\rho^*\colon R^1\pi_*\mathbb{Z} \to R^1\mu_*\mathbb{Z}$. The latter is isomorphic to the direct sum $R^1\nu_*\mathbb{Z} \oplus R^1\pi_*\mathbb{Z}$ by the Künneth formula (see, for example, [Ive86, § VII.2.7]) and decomposition theorem for smooth proper morphisms. Hence composing with the first projection, we obtain a morphism $R^1\pi_*\mathbb{Z} \to R^1\nu_*\mathbb{Z}$.

Over an open subset $U \subset B_0$ where $\pi\colon X_U \to U$ admits a holomorphic section, X_U is identified with P_U as a trivial torsor. Hence ρ becomes the addition operation of the abelian scheme $X_U \times_U X_U \to X_U$, and the pullback morphism is fiberwise $\rho^*\colon H^1(F,\mathbb{Z}) \to H^1(F,\mathbb{Z}) \oplus H^1(F,\mathbb{Z})$, $x \mapsto (x,x)$. Hence the morphism $R^1\pi_*\mathbb{Z} \to R^1\nu_*\mathbb{Z}$ is an isomorphism over U, and the claim follows. □

Remark 3.8. (i) A posteriori, one has an interpretation of Corollary 3.7 in terms of Theorem 3.1. The abelian scheme ν is projective, and ν and π are fiberwise isomorphic. Hence the smooth fibers of π are projective, even when the hyper-Kähler manifold X is not.

(ii) Theorem 3.1 is also related to [Ogu09, Theorem 1.1] in the following sense. Let L be the function field of B, and consider the generic fiber of the abelian scheme $\nu\colon P_0 \to B_0$, an abelian variety $P_L \to \operatorname{Spec} L$. Since P_0 has a unique polarization up to scaling, so does P_L. Now ampleness is an open condition in $\mathrm{NS}(P_L)_{\mathbb{R}}$, so this implies $\rho(P_L) = 1$.

(iii) If we further assume that X is projective, then the smooth morphism $\pi\colon X_0 \to B_0$ admits étale local sections, that is, π is an étale torsor under ν.

The unique abelian scheme P_0 above should be considered as the identity component of the automorphism scheme of π (see [AF16, § 8.3]). Define a sheaf of relative automorphisms acting by translations on each fiber by

$$\underline{\mathrm{Aut}}^{\mathrm{tr}}_{X_0/B_0}(U) = \{U\text{-automorphisms } f\colon X_U \longrightarrow X_U \text{ acting by translation on each fiber}\}\,. \tag{3.7}$$

Proposition 3.9. *The abelian scheme $\nu\colon P_0 \to B_0$ represents $\underline{\mathrm{Aut}}^{\mathrm{tr}}_{X_0/B_0}$.*

Proof. Denote by $\underline{P_0}$ the sheaf of analytic local sections of ν. Since P_0 acts on X_0 by fiberwise translation, we have a sheaf homomorphism $\underline{P_0} \to \underline{\mathrm{Aut}}^{\mathrm{tr}}_{X_0/B_0}$. The homomorphism is injective because the P_0-action is effective. Over a small analytic open subset $U \subset B_0$, the morphism $X_U \to U$ admits a section, so it is identified with an abelian scheme $P_U \to U$, and thus $\underline{P_0}(U) \to \underline{\mathrm{Aut}}^{\mathrm{tr}}_{X_0/B_0}(U)$ is an isomorphism. □

3.2 Examples

Before getting into the next discussion, let us devote a short subsection to collecting a few examples.

Example 3.10. Let X be a smooth projective moduli space of stable torsion coherent sheaves on a K3 surface with a fixed Mukai vector, so that it is a hyper-Kähler manifold of $\mathrm{K3}^{[n]}$-type equipped with a Lagrangian fibration $\pi\colon X \to B$. In this case, it is known that the torus fibration $\pi\colon X_0 \to B_0$ is isomorphic to a relative Jacobian $\mathrm{Pic}^d_{\mathcal{C}_0/B_0}$ associated with a certain universal family $\mathcal{C}_0/B_0$ of smooth curves on the K3 surface. Now the relative Jacobian $\mathrm{Pic}^d_{\mathcal{C}_0/B_0}$ is a torsor under the numerically trivial relative Jacobian $\mathrm{Pic}^0_{\mathcal{C}_0/B_0}$; see [BLR90, Theorem 9.3.1]. By the uniqueness assertion of Theorem 3.1, this is the associated abelian scheme P_0.

Example 3.11. When $\pi\colon X \to B = \mathbb{P}^1$ is an elliptic K3 surface, Theorem 3.1 is a weaker version of the relative Jacobian fibration construction of π (see, for example, [Huy16, § 11.4]). In this case, there exists a group scheme $P \to B$ extending the one in Theorem 3.1 over the entire base (Néron model) so that the smooth locus of π becomes a torsor under P. Arinkin–Fedorov generalized this result to higher-dimensional projective hyper-Kähler manifolds when π has integral fibers.

Example 3.12. For higher-dimensional compact hyper-Kähler manifolds, we can still consider the relative Picard scheme $\mathrm{Pic}^0_{X_0/B_0} \to B_0$. However, in general, this is the *dual* of the abelian scheme P_0. This means that P_0 is the "double relative Picard scheme" of the original X_0 (this is done in [Saw04, § 4.2]). However, for us it will be more important to consider P_0 as the identity component of the relative automorphism scheme of X_0/B_0 as in Proposition 3.9.

Example 3.13. When π admits at least one rational section, the abelian scheme P_0 is in fact isomorphic to X_0. This is because the rational section must be defined over B_0 (see Remark 3.20) so that X_0 becomes a trivial P_0-torsor. In some sense, Theorem 3.1 is thus a generalization of certain properties of X_0 to the case where π does not have any rational section. For example, one can study the Mordell–Weil group of ν, generalizing the study of the Mordell–Weil group of π.

Example 3.14. Assume again that π admits a rational section and hence $X_0 \cong P_0$. We will see later in Proposition 3.21 (or (3.2)) that there exists an inclusion $\mathrm{Aut}^\circ(X/B) \subset P_0$. Since $\mathrm{Aut}^\circ(X/B)$ is finite, this implies the existence of certain torsion rational sections of the Lagrangian fibration

$$\mathrm{Aut}^\circ(X/B) \subset \mathrm{MW}(X/B)\,.$$

For example, let us consider the case when X is of Kum_n-type. We will prove in Theorem 5.1 that the order of $\mathrm{Aut}^\circ(X/B)$ is at least $(n+1)^2$. Thus any Lagrangian fibration of a Kum_n-type hyper-Kähler manifold *must* have at least $(n+1)^2$ torsion rational sections (once it admits a single rational section), and the dual hyper-Kähler orbifold $\check{X}$ is precisely the quotient of X by these special torsion rational sections. To our knowledge, this phenomenon was not known, and it became one of our original motivations. See also [Sac23, §§ 3–5] for some related ideas on the Mordell–Weil group and birational automorphisms defined by torsion rational sections.

3.3 Polarization type and divisibility of $\pi^*\mathcal{O}_B(1)$

The purpose of this section is to study two numerical invariants associated with π and study their relations: (1) the polarization type of π in Definition 3.3, and (2) the divisibility of the line bundle $\pi^*\mathcal{O}_B(1)$. Throughout, we will write $h \in H^2(X,\mathbb{Z})$ for the first Chern class of $\pi^*\mathcal{O}_B(1)$ and $\operatorname{div}(h)$ for the divisibility

$$\operatorname{div}(h) = \gcd\big\{q(h,y) : y \in H^2(X,\mathbb{Z})\big\}\,. \qquad \text{(1.2 restated)}$$

The polarization type of π is an n-tuple of positive integers $(d_1,\dots,d_n)$ with $d_1 \mid \cdots \mid d_n$ such that each fiber of the polarization scheme K_0 is isomorphic to $(\mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_n)^{\oplus 2}$. Since we are assuming that the polarization $\lambda\colon P_0 \to \check{P}_0$ is primitive, we always have $d_1 = 1$. The polarization type was already computed for all currently known deformation types of hyper-Kähler manifolds. The known computations are based on its original definition in [Wie16, § 4, p. 318]. Therefore, to use the previous results, we first need to show our definition is equivalent to the original one.

Lemma 3.15. *Definition 3.3 of the polarization type is equivalent to Wieneck's definition.*

Proof. Our definition of polarization is from the primitive homomorphism $\mathbb{Z} \to R^2\pi_*\mathbb{Z}$ of local systems. Fixing a smooth fiber F of π, the VHS $R^2\pi_*\mathbb{Z}$ is identified with $H^2(F,\mathbb{Z})$ as a $\pi_1(B_0)$-module. In this setting, the primitive morphism $\mathbb{Z} \to R^2\pi_*\mathbb{Z}$ of local systems corresponds to the generator of $H^2(F,\mathbb{Z})^{\pi_1(B_0)}$. By the global invariant cycle theorem, this is the primitive polarization type of F coming from the image of $H^2(X,\mathbb{Q}) \to H^2(F,\mathbb{Q})$, which coincides with the definition in [Wie16]. $\square$

We can now use the previous results on computations of the polarization type of π. By [Wie16, Theorem 1.1], the polarization type is invariant under deformations of π. We will recover this result later in Corollary 3.24. The polarization type is computed for all Lagrangian fibrations of hyper-Kähler manifolds of known deformation types. The computations for $\mathrm{K3}^{[n]}$- and Kum_n-types are the main results of [Wie16, Wie18]. The results for OG10- and OG6-types are in [MO22, Theorem 2.2] and [MR21, Theorem 7.2].

Theorem 3.16 ([Wie16, Wie18, MO22, MR21]). *Let $\pi\colon X \to B$ be a Lagrangian fibered compact hyper-Kähler manifold. Then the polarization type of π is*

$$\begin{cases} (1,\dots,1) & \textit{if } X \textit{ is of } K3^{[n]}\textit{-type}\,,\\ (1,1,1,1,1) & \textit{if } X \textit{ is of OG10-type}\,m,\\ (1,\dots,1,d_1,d_2) & \textit{if } X \textit{ is of } \mathrm{Kum}_n\textit{-type}\,, \textit{ and}\\ (1,2,2) & \textit{if } X \textit{ is of OG6-type}\,. \end{cases}$$

When X is of Kum_n-type, we set $d_1 = \operatorname{div}(h)$ in $H^2(X,\mathbb{Z})$ and $d_2 = (n+1)/d_1$.

Observe from Theorem 3.16 that we have an equality $c_X = d_1 \cdots d_n$ for all known deformation types of hyper-Kähler manifolds. In this sense, the polarization type should be considered as a refinement of the Fujiki constant c_X. This is also related to the nonprimitiveness of the image of the restriction homomorphism $H^2(X,\mathbb{Z}) \to H^2(F,\mathbb{Z})$.

Proposition 3.17. *Assume that we have an equality $c_X = d_1 \cdots d_n$. Then the image of the restriction homomorphism $H^2(X,\mathbb{Z}) \to H^2(F,\mathbb{Z})$ in Lemma 3.6 is generated by $a\theta$, where $a = \operatorname{div}(h)$ and θ is a primitive ample class representing the natural polarization of F.*

Proof. Choose a cohomology class $x \in H^2(X,\mathbb{Z})$ with $q(h,x) = a$. By Lemma 3.6, the class $x_{|F} \in H^2(F,\mathbb{Z})$ must be an integer multiple of the primitive polarization class θ. Set $x_{|F} = b\theta$ for $b \in \mathbb{Z}_{\neq 0}$. Now the claim directly follows from the Fujiki relation

$$d_1 \cdots d_n = \frac{1}{n!}\int_F \theta^n = \frac{1}{n!}\int_X h^n \Big(\frac{1}{b}x\Big)^n = c_X \cdot q\Big(h, \frac{1}{b}x\Big)^n = c_X \left(\frac{a}{b}\right)^n . \qquad \square$$

Though not used in this paper, the divisibility of h is also related to the existence of a rational section of π. We end this subsection with the following observation.

Proposition 3.18. *Assume $c_X = d_1 \cdots d_n$ and that π admits at least one rational section. Then* $\operatorname{div}(h) = 1$ *or* 2.

Proof. If π admits a rational section, then $X_0 \cong P_0$ is a projective abelian scheme (Example 3.13). By the general theory of abelian schemes, twice a polarization is always associated with a line bundle (see, for example, [MFK94, Proposition 6.10] or [FC90, Definition I.1.6]). This means that $2\theta \in H^2(F,\mathbb{Z})$ is contained in the image of $\operatorname{Pic}(X) \subset H^2(X,\mathbb{Z}) \to H^2(F,\mathbb{Z})$. By Proposition 3.17, this implies $\operatorname{div}(h) = 1$ or 2. $\square$

For every positive integer d with $d^2 \mid n-1$, there exists a Lagrangian fibration of a $\mathrm{K3}^{[n]}$-type hyper-Kähler manifold with $\operatorname{div}(h) = d$; see [Mar14, Theorem 1.5]. Similarly, for every d with $d^2 \mid n+1$, there exists a Lagrangian fibration of a Kum_n-type hyper-Kähler manifold with $\operatorname{div}(h) = d$; see [Wie18, Theorem 1.2]. Therefore, there are examples of Lagrangian fibrations π with $\operatorname{div}(h) > 2$; such π (and any of their deformations) would never admit any rational section, and the notion of torsor is necessary.

3.4 The polarization scheme and H^2-trivial automorphisms

We present the proofs of Propositions 3.4 and 3.5 in this subsection.

Lemma 3.19. *Every rational section of $\nu\colon P_0 \to B_0$ is a regular section.*

Proof. Assume that $s\colon B_0 \dashrightarrow P_0$ is a rational section undefined at $b \in B_0$. Let $S \subset P_0$ be the closure of the image of s so that we obtain a proper birational morphism $\nu_{|S}\colon S \to B_0$. Since B_0 is smooth and s is undefined at b, the fiber $S_b = (\nu_{|S})^{-1}(b)$ is a uniruled variety (see, for example, [Kol96, Theorem VI.1.2]). This means that an abelian variety $\nu^{-1}(b)$ contains a uniruled variety S_b. We have a contradiction. See [BLR90, Corollary 8.4.6] for an alternative proof. $\square$

Remark 3.20. The same argument applies to π and proves the following: Any rational section of π is necessarily defined over B_0.

Proposition 3.21. *Every H^2-trivial automorphism in* $\operatorname{Aut}^\circ(X/B)$ *defines a global section of $P_0 \to B_0$. That is, we have a closed immersion of group schemes*

$$\operatorname{Aut}^\circ(X/B) \hookrightarrow P_0 .$$

Proof. We need to prove that $\operatorname{Aut}^\circ(X/B)$ acts on $\pi\colon X_0 \to B_0$ by fiberwise translations (Proposition 3.9). Consider the quotient $\bar{X} = X/\operatorname{Aut}^\circ(X/B)$ with a commutative diagram

$$\begin{tikzcd} X \arrow[dd, "\pi"'] \arrow[dr, "p"] & \\ & \bar{X} \arrow[dl, "\bar{\pi}"] \\ B & \end{tikzcd} .$$

We first claim that p is étale on general fibers over B. Let $S \subset X$ be the ramified locus of p. It has codimension at least 2 because p is quasi-étale by Proposition B.1. Let $b \in B$ be a general point, so that the fibers $F = X_b$ and $\bar{F} = \bar{X}_b$ are both smooth. Observe that the ramification locus of $p\colon F \to \bar{F}$ is precisely $S \cap F$, which is of codimension at least 2 since b is general. The purity of the branch locus theorem forces $p\colon F \to \bar{F}$ to be étale.

Now we have a finite étale quotient $p\colon F \to \bar{F} = F/\operatorname{Aut}^\circ(X/B)$ between smooth projective varieties. Its Galois group $\operatorname{Aut}^\circ(X/B)$ acts on F by fixed-point-free automorphisms. Since F and $\bar{F}$ are both abelian varieties (see [Sch20, Theorem 3]), this means that $\operatorname{Aut}^\circ(X/B)$ acts on F by translations. The conclusion is that on a general fiber of π, the group $\operatorname{Aut}^\circ(X/B)$ acts by translation. This means that $\operatorname{Aut}^\circ(X/B)$ defines a rational section of $\nu\colon P_0 \to B_0$ by Proposition 3.9. Such a rational section must be a regular section by Lemma 3.19. That is, $\operatorname{Aut}^\circ(X/B)$ acts by translations over the entire B_0. □

An immediate byproduct is that $\operatorname{Aut}^\circ(X/B)$ is abelian.

Proposition 3.22. *The group* $\operatorname{Aut}^\circ(X/B)$ *is finite abelian.*

We next understand the behavior of the polarization λ under deformations of π. Recall that the polarization scheme K_0 was defined to be the kernel of the polarization $\ker\lambda$. To deal with the more technical Propositions 3.4 and 3.5, we have defined $K_0[a] = \ker(a\lambda)$ for each positive integer a:

$$0 \longrightarrow K_0[a] \longrightarrow P_0 \xrightarrow{a\lambda} \check{P}_0 \longrightarrow 0\,.$$

Since the abelian scheme P_0 was associated with the VHS $\big(R^1\pi_*\mathbb{Z}\big)^\vee$, there is a VHS version of this sequence:

$$0 \longrightarrow \big(R^1\pi_*\mathbb{Z}\big)^\vee \xrightarrow{a\lambda_*} R^1\pi_*\mathbb{Z} \longrightarrow \underline{K_0[a]} \longrightarrow 0\,. \tag{3.8}$$

The group scheme $K_0[a]$ and the local system $\underline{K_0[a]}$ are related as follows: $\underline{K_0[a]}$ is a sheaf of analytic sections of the group scheme $K_0[a] \to B_0$, and $K_0[a]$ is the total space of the local system $\underline{K_0[a]}$.

Lemma 3.23. *Let $p\colon \mathcal{X} \xrightarrow{\pi} \mathcal{B} \xrightarrow{q} \Delta$ be a family of Lagrangian fibered compact hyper-Kähler manifolds over a complex open disc Δ. Let $\mathcal{B}_0 \subset \mathcal{B}$ be the locus where π is smooth. Then for each positive integer a, there exists a finite étale group scheme $\mathcal{K}_0[a]$ over $\mathcal{B}_0$ parametrizing the group schemes $(K_0[a])_t$ over $(B_0)_t$ for all $t \in \Delta$.*

Proof. We prove the statement for $a = 1$ for simplicity; the same proof works for any a. For each $t \in \Delta$, the fibration $\pi_t\colon X_t \to B_t$ has its own abelian scheme $(P_t)_0 \to (B_t)_0$ coming from the VHS $R^1(\pi_t)_*\mathbb{Z}$. Hence the VHS $R^1\pi_*\mathbb{Z}$ of the entire family $\pi\colon \mathcal{X}_0 \to \mathcal{B}_0$ will construct the family of abelian schemes $\mathcal{P}_0 \to \mathcal{B}_0$. The kernel group scheme $\mathcal{K}_0 \subset \mathcal{P}_0$ being a finite subgroup scheme, the question is (analytically) local on Δ, so we may shrink Δ if necessary and assume that $\mathcal{X}$ is diffeomorphic to $X \times \Delta$ by Ehresmann's theorem. In particular, we have $H^2(\mathcal{X},\mathbb{Z}) \cong H^2(X_t,\mathbb{Z})$ for all $t \in \Delta$.

Applying the invariant cycle theorem to $\pi\colon \mathcal{X} \to \mathcal{B}$, we have

$$H^0\big(\mathcal{B}_0, R^2\pi_*\mathbb{Q}\big) = \operatorname{im}\big(H^2(\mathcal{X},\mathbb{Q}) \longrightarrow H^2(F,\mathbb{Q})\big)\,,$$

where F is any fiber of $\pi\colon \mathcal{X}_0 \to \mathcal{B}_0$. Take $t \in \Delta$ such that $F \subset X_t$. Since we have an identification $H^2(\mathcal{X},\mathbb{Q}) = H^2(X_t,\mathbb{Q})$, the above homomorphism coincides with the description (3.4), and we have an isomorphism $H^0\big(\mathcal{B}_0, R^2\pi_*\mathbb{Q}\big) \cong \mathbb{Q}$ by Lemma 3.6. Now consider the primitive element in

$H^0\big(\mathcal{B}_0, R^2\pi_*\mathbb{Z}\big) \cong \mathbb{Z}$, and run the same argument as in the construction proof of Theorem 3.1; we construct a primitive polarization $\lambda\colon \mathcal{P}_0 \to \check{\mathcal{P}}_0$, which over any $t \in \Delta$ coincides with the primitive polarization $\lambda_t\colon (P_0)_t \to (\check{P}_0)_t$. Taking the kernel, this constructs $\mathcal{K}_0 \subset \mathcal{P}_0$ over $\mathcal{B}_0$. □

Lemma 3.23 in particular recovers [Wie16, Theorem 1.1].

Corollary 3.24. *The polarization type of π is invariant under deformations of π.*

The following final observation is elementary but nontrivial. We match its notation to our original discussion.

Lemma 3.25. *Let $\mathcal{P}_0 \to \mathcal{B}_0$ be an abelian scheme over a complex manifold $\mathcal{B}_0$ and $a\lambda\colon \mathcal{P}_0 \to \check{\mathcal{P}}_0$ a polarization with $\mathcal{K}_0[a] = \ker(a\lambda)$. Assume that there exists a torsion section $f\colon \mathcal{B}_0 \to \mathcal{P}_0$. If $f(\mathcal{B}_0) \cap \mathcal{K}_0[a] \neq \emptyset$, then $f(\mathcal{B}_0) \subset \mathcal{K}_0[a]$.*

Proof. The statement is topological and local on the base $\mathcal{B}_0$, so we may assume that $\mathcal{B}_0$ is a complex open ball S and $\mathcal{P}_0 \to \mathcal{B}_0$ is homeomorphic to a topological constant group scheme $(\mathbb{R}/\mathbb{Z})^{2n} \times S \to S$. In this setting, the kernel $\mathcal{K}_0[a]$ is a constant subgroup scheme and the *torsion* section f is a constant section. Hence $f(S) \cap \mathcal{K}_0[a] \neq \emptyset$ if and only if $f(S) \subset \mathcal{K}_0[a]$. □

Proof of Proposition 3.4. Consider a one-parameter family of Lagrangian fibered hyper-Kähler manifolds $\mathcal{X} \to \mathcal{B} \to \Delta$ over a complex disc Δ. By Lemma 3.23, there exists notions of a family of abelian schemes $\mathcal{P}_0 \to \mathcal{B}_0$ and a family of finite étale group schemes $\mathcal{K}_0[a] \subset \mathcal{P}_0$. Proposition 3.21 proves that we have a closed immersion $\mathrm{Aut}^\circ(X/B) \hookrightarrow P_0$ for a single fiber. In fact, the argument applies to the entire family and produces $\mathrm{Aut}^\circ(X/B)$-global sections of $\mathcal{P}_0 \to \mathcal{B}_0$, or equivalently an embedding

$$\mathrm{Aut}^\circ(X/B) \longhookrightarrow \mathcal{P}_0\,.$$

Since $\mathrm{Aut}^\circ(X/B)$ is finite, the global sections are torsion. Suppose that we had $\mathrm{Aut}^\circ(X/B) \hookrightarrow K_0[a]$ for the original Lagrangian fibration over $0 \in \Delta$. Then this forces $\mathrm{Aut}^\circ(X/B) \hookrightarrow \mathcal{K}_0[a]$ over the entire Δ by Lemma 3.25. The claim follows. □

Proof of Proposition 3.5. Recall from Proposition 3.17 that the restriction map $H^2(X,\mathbb{Z}) \to H^2(F,\mathbb{Z})$ has a rank 1 image generated by the class $a\theta$, where $a = \mathrm{div}(h)$ and θ is the primitive ample class corresponding to our polarization $\lambda\colon F \to \check{F}$. The preimage of $a\theta \in H^2(F,\mathbb{Z})$ under this restriction homomorphism is precisely $S = \big\{x \in H^2(X,\mathbb{Z}) : q(x,h) = a\big\}$. By Proposition 3.4, the claim is invariant under deformations of π. We may thus deform π and assume $\mathrm{Pic}(X) \cap S \neq \emptyset$. In other words, we may assume that the composition $\mathrm{Pic}(X) \subset H^2(X,\mathbb{Z}) \to H^2(F,\mathbb{Z})$ is generated by $a\theta$.

The assertion $\mathrm{Aut}^\circ(X/B) \hookrightarrow K_0[a] = \ker(a\lambda)$ is equivalent to $a\lambda(\mathrm{Aut}^\circ(X/B)) = 0$. The latter equality may be verified fiberwise, so we may concentrate on a single fiber $F = \nu^{-1}(b) = \pi^{-1}(b)$. Let L be any line bundle on X such that its image under $\mathrm{Pic}(X) \to H^2(F,\mathbb{Z})$ is $a\theta$. This means that the polarization $a\lambda$ can be described as

$$a\lambda\colon F \longrightarrow \check{F}\,, \quad t_x \longmapsto \big[t_x^*(L_{|F}) \otimes L_{|F}^{-1}\big]\,.$$

If we assume that $t_x = f_{|F}$ is from a global H^2-trivial automorphism $f \in \mathrm{Aut}^\circ(X/B)$, then we have a sequence of identities

$$t_x^*(L_{|F}) = (f_{|F})^*(L_{|F}) = (f^*L)_{|F} \cong L_{|F}\,,$$

where the last isomorphism follows from the fact that f acts on $\mathrm{Pic}(X) \subset H^2(X,\mathbb{Z})$ trivially. This proves that $a\lambda$ sends $\mathrm{Aut}^\circ(X/B)$ to 0, and the claim follows. □

4. The minimal split covering and H^2-trivial automorphisms

This section discusses an explicit construction of certain H^2-trivial automorphisms. This will be conveniently used in the next section when we describe the $\mathrm{Aut}^\circ(X)$-action for certain examples of Kum_n-type hyper-Kähler manifolds. Recall that the group $\mathrm{Aut}^\circ(X)$ is computed for all known deformation types of hyper-Kähler manifolds; see [Bea83a, Proposition 10] for $\mathrm{K3}^{[n]}$-types, [BNS11, Corollary 5] for Kum_n-types, and the main results of [MW17] for OG10- and OG6-types. The strategy is to compute the group for a specific model in such deformation types and then use the deformation equivalence in [HT13, Theorem 2.1]. Unfortunately, this argument does not tell us precisely how $\mathrm{Aut}^\circ(X)$ acts on X. The goal of this section is to prove Proposition 4.4 to partially resolve this problem.

Throughout the section, we stick to the following setting. Let M be a *projective* holomorphic symplectic manifold, not necessarily irreducible. By the Beauville–Bogomolov decomposition theorem, M admits a finite étale covering $X \times T \to M$, called a *split covering*, where X is a finite product of projective hyper-Kähler manifolds and T is an abelian variety. In fact, Beauville in [Bea83a, Proposition 3] also considered the smallest possible split covering. A *minimal split covering* of M is the smallest possible split covering of M, in the sense that every split covering factors through it. The minimal split covering of M always exists and is unique up to a (nonunique) isomorphism. Moreover, it is a Galois covering. We refer to [Bea83a, § 3] for more details about minimal split coverings.

On the other hand, Kawamata [Kaw85, Theorem 8.3] proved that if M is a K-trivial smooth projective variety, then its Albanese morphism $\mathrm{Alb}\colon M \to \mathrm{Alb}(M)$ has to be smooth projective and isotrivial. More precisely, there exists an isogeny $\phi\colon T \to \mathrm{Alb}(M)$ of abelian varieties such that the base change of Alb becomes a trivial fiber bundle over T. We obtain a cartesian diagram

$$\begin{tikzcd} X \times T \arrow[r, "\Phi"] \arrow[d, "\mathrm{pr}_2"] & M \arrow[d, "\mathrm{Alb}"] \\ T \arrow[r, "\phi"] & \mathrm{Alb}(M)\,, \end{tikzcd} \tag{4.1}$$

where X is a fiber of the Albanese morphism. In particular, one sees that $\Phi\colon X \times T \to M$ is a split covering of M. Combining the two results, we get the following.

Proposition 4.1. *Let M be a projective holomorphic symplectic manifold and $\mathrm{Alb}\colon M \to \mathrm{Alb}(M)$ its Albanese morphism, an isotrivial family of a product of hyper-Kähler manifolds $X = \mathrm{Alb}^{-1}(0)$. Then there exists a unique isogeny $\phi\colon T \to \mathrm{Alb}(M)$ such that the morphism Φ in the fiber diagram (4.1) is the minimal split covering.*

Proof. Use Kawamata's result to construct an isogeny $\phi'\colon T' \to \mathrm{Alb}(M)$ trivializing the Albanese map as in (4.1). Since ϕ' is a finite Galois covering, Φ' is also a finite Galois covering with $\mathrm{Gal}(\Phi') = \mathrm{Gal}(\phi')$. Denote these common groups by G. The G-action on T' is by translations since ϕ' is a morphism between abelian varieties, that is, we can consider $G \subset T'$. The first lemma in [Bea83a, § 3] claims $\mathrm{Aut}(X \times T') = \mathrm{Aut}(X) \times \mathrm{Aut}(T')$. In conclusion, $g \in G$ acts on $t \in T'$ and $(x, t) \in X \times T'$ by

$$g \cdot t = t + g\,, \quad g \cdot (x, t) = (f_g(x), t + g) \quad \text{for } f_g \in \mathrm{Aut}(X)\,. \tag{4.2}$$

Consider the homomorphism $G \to \mathrm{Aut}(X)$ by $g \mapsto f_g$. Its kernel defines a factorization of ϕ':

$$\begin{tikzcd} X \times T' \arrow[r] \arrow[d, "\mathrm{pr}_2"] & X \times T \arrow[r, "\Phi"] \arrow[d, "\mathrm{pr}_2"] & M \arrow[d, "\mathrm{Alb}"] \\ T' \arrow[r] & T \arrow[r, "\phi"] & \mathrm{Alb}(M)\,. \end{tikzcd}$$

By construction, $\mathrm{Gal}(\phi)$ acts on X faithfully, or equivalently Φ is the minimal split covering. The uniqueness of ϕ follows from the uniqueness of the minimal split covering. ☐

Proposition 4.1 in particular proves that the minimal split covering can always be realized by an isogeny $\phi\colon T \to \mathrm{Alb}(M)$ and the base change (4.1).

DEFINITION 4.2. We call $\phi\colon T \to \mathrm{Alb}(M)$ in Proposition 4.1 the *minimal isogeny* trivializing the Albanese morphism $\mathrm{Alb}\colon M \to \mathrm{Alb}(M)$. It is unique up to a (nonunique) isomorphism.

In fact, the proof of Proposition 4.1 says more about an arbitrary isogeny ϕ'.

COROLLARY 4.3. *We use the notation of Proposition 4.1. Let $\phi'\colon T' \to \mathrm{Alb}(M)$ be an isogeny trivializing the Albanese morphism.*

(i) *The isogeny ϕ' factors though the minimal isogeny ϕ.*
(ii) *There exists a canonical $\mathrm{Gal}(\phi')$-action on X.*
(iii) *The isogeny ϕ' is minimal if and only if the $\mathrm{Gal}(\phi')$-action on X is effective.*

Now we can state the main result of this section. The ideas here were already contained in [Bea83a, Bea83b].

PROPOSITION 4.4. *We use the notation of Proposition 4.1 and Corollary 4.3. The group $\mathrm{Gal}(\phi')$ acts on X by H^2-trivial automorphisms. That is, we have a canonical homomorphism*

$$\mathrm{Gal}(\phi') \longrightarrow \mathrm{Aut}^\circ(X)\,,$$

which is injective if and only if ϕ' is minimal.

Proof. By Corollary 4.3, we may assume that $\phi' = \phi$ is minimal and $G = \mathrm{Gal}(\phi) \subset \mathrm{Aut}(X)$. The content of the proposition is that it is further a subgroup of $\mathrm{Aut}^\circ(X)$, or the G-action on $H^2(X, \mathbb{Q})$ is trivial.

Consider the diagram (4.1). Our first step is to define T-actions on all the four spaces to make the diagram T-equivariant. Define a T-action on T by translation, and on $X \times T$ only on the second factor, again by translation. The T-action on $\mathrm{Alb}(M)$ is by translation via the morphism ϕ: If $a \in T$ and $z \in \mathrm{Alb}(M)$, then we define $a \cdot z = z + \phi(a)$. To equip M with a T-action, we claim that the T-action on $X \times T$ descends to M via Φ. The descent works if the G-action on $X \times T$ commutes with the T-action (recall that $G = \mathrm{Gal}(\phi) = \mathrm{Gal}(\Phi)$). Recall from (4.2) that $g \in G$ acts diagonally on $X \times T$ by $g \cdot (x,t) = (f_g(x), t+g)$, where f_g is an automorphism of X. For any $g \in G$ and $a \in T$, we have a sequence of equalities

$$a \cdot (g \cdot (x,t)) = (f_g(x), t+g+a) = g \cdot (a \cdot (x,t))\,,$$

proving that the T- and G-actions commute.

Notice that the Galois group $G = \mathrm{Gal}(\phi)$ was a subgroup of T, but our G- and T-actions (restricted to $G \subset T$) on $X \times T$ are different: We had $g \cdot (x,t) = (f_g(x), t+g)$ for $g \in G$ and $a \cdot (x,t) = (x, t+a)$ for $a \in T$. Due to this fact, we call the restriction of the T-action to $G \subset T$

a Γ-action. Since $a \in T$ acts on $\mathrm{Alb}(M)$ by translation by $\phi(a)$, its stabilizer at $0 \in \mathrm{Alb}(M)$ is precisely $\Gamma \subset T$. Since Alb is T-equivariant, this defines a Γ-action on $X = \mathrm{Alb}^{-1}(0)$. Combining our definitions, we can check that $a \in \Gamma$ acts on X by

$$a \cdot x = (f_a)^{-1}(x) \quad (= f_{-a}(x))\,. \tag{4.3}$$

That is, the Γ- and G-actions are inverse to each other.

Notice that any T-action on M is isotopic to the identity map because T is path connected. In particular, T acts trivially on the cohomology $H^*(M, \mathbb{Q})$. The embedding $X \subset M$ is Γ-equivariant, so we have a Γ-equivariant restriction homomorphism

$$H^2(M, \mathbb{Q}) \longrightarrow H^2(X, \mathbb{Q})\,.$$

By (4.3), the Γ- and G-actions on X are inverse. Hence it suffices to prove that the Γ-action on $H^2(X, \mathbb{Q})$ is trivial. Since Γ acted trivially $H^2(M, \mathbb{Q})$, it is enough to prove that this restriction homomorphism is surjective.

The question now becomes topological. Deform the complex structure of the hyper-Kähler manifold X very generally so that $H^2(X, \mathbb{Q})$ becomes a simple $\mathbb{Q}$-Hodge structure (we will have to lose the projectiveness of X). The complex structure of M can be correspondingly chosen in a way that the finite covering map $\Phi \colon X \times T \to M$ becomes holomorphic. Therefore, the Hodge structure morphism $H^2(M, \mathbb{Q}) \to H^2(X, \mathbb{Q})$ is either 0 or surjective. We only need to rule out the former possibility.

To prove that it is nonzero, consider any global holomorphic symplectic form σ on M. Pulling it back to $X \times T$ gives a holomorphic *symplectic* form $\tilde{\sigma} \in H^{2,0}(X \times T) = H^{2,0}(X) \oplus H^{2,0}(T)$. Since $\tilde{\sigma}$ is symplectic, it has nonzero components in both $H^{2,0}(X)$ and $H^{2,0}(T)$. In particular, $\sigma_{|X} = \tilde{\sigma}_{|X}$ is nonzero, and the claim follows. $\square$

Remark 4.5. An alternative way to state the results in this section is as follows. Any isogeny $\phi' \colon T' \to \mathrm{Alb}(M)$ trivizalizing the Albanese morphism defines a group homomorphism $\mathrm{Gal}(\phi') \to \mathrm{Aut}^\circ(X)$. The image of this homomorphism is independent on the choice of ϕ', which we denote by

$$\mathrm{Aut}'(X) \subset \mathrm{Aut}^\circ(X)\,.$$

It is a finite abelian group, isomorphic to $\mathrm{Gal}(\phi)$ for a minimal isogeny ϕ, and is deformation invariant on X. For example, we will later see that when X is of Kum_n-type, then

$$\mathrm{Aut}'(X) \cong (\mathbb{Z}/n+1)^{\oplus 4}\,, \quad \mathrm{Aut}^\circ(X) \cong \mathbb{Z}/2 \ltimes (\mathbb{Z}/n+1)^{\oplus 4}\,.$$

Our main result can be more directly stated with this definition. See Remark 6.2.

5. The H^2-trivial automorphisms and polarization scheme for generalized Kummer varieties

The goal of this section is an explicit computation of the group $\mathrm{Aut}^\circ(X/B)$ and the polarization scheme K_0 for certain Lagrangian fibrations of Kum_n-type hyper-Kähler manifolds. Since the group $\mathrm{Aut}^\circ(X/B)$ will be of interest for all known deformation types of hyper-Kähler manifolds, we state the result in a more general form. The following is the first main theorem of this section.

Theorem 5.1. *Let $\pi \colon X \to B$ be a Lagrangian fibration of a compact hyper-Kähler manifold.*

(i) *We have* $\mathrm{Aut}^\circ(X/B) \cong \begin{cases} \{\mathrm{id}\} & \text{if } X \text{ is of } \mathrm{K3}^{[n]}\text{- or } OG10\text{-type}\,, \\ (\mathbb{Z}/2)^{\oplus 4} & \text{if } X \text{ is of } OG6\text{-type}\,. \end{cases}$

(ii) *Assume that X is of Kum_n-type and $(1,\dots,1,d_1,d_2)$ is the polarization type of π in Theorem 3.16. Then*

$$\mathrm{Aut}^\circ(X/B) \cong \begin{cases} (\mathbb{Z}/2)^{\oplus 5} & \text{if } n=3 \text{ and the polarization type is } (1,2,2)\,, \\ (\mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2)^{\oplus 2} & \text{otherwise}\,. \end{cases}$$

Notice that the bigger group $\mathrm{Aut}^\circ(X)$ is already trivial for $\mathrm{K3}^{[n]}$- and OG10-types (see [Bea83a, Proposition 10] and [MW17, Theorem 3.1]), so the theorem is clear in these cases. For Kum_n- and OG6-types, recall from Theorem 2.2 that $\mathrm{Aut}^\circ(X/B)$ is invariant under deformations of π. By [Wie18, § 6.28], every Lagrangian fibration of a Kum_n-type hyper-Kähler manifold is deformation equivalent to the *moduli construction*, which will be recalled in Section 5.1. By [MR21, Theorem 7.2], all Lagrangian fibrations of an OG6-type hyper-Kähler manifold are deformation equivalent to one another. Therefore, Theorem 5.1 follows from the following two more concrete results.

Let S be an abelian surface, $l \in \mathrm{NS}(S)$ an ample cohomology class, and $s \in H^4(S,\mathbb{Z})$ a cohomology class making the Mukai vector $v=(0,l,s) \in H^*(S,\mathbb{Z})$ is primitive. In this situation, we will see in Definition 5.5 (called the *moduli construction* or the *Debarre system*) that there is an explicit construction of a Lagrangian fibration of a Kum_n-type hyper-Kähler manifold.

PROPOSITION 5.2. *Let $\pi\colon X \to B$ be a Lagrangian fibration of a Kum_n-type hyper-Kähler manifold, obtained by the moduli construction from a triple (S,l,s) in Definition 5.5. Let (d_1,d_2) be the polarization type of the ample class l. Then*

$$\mathrm{Aut}^\circ(X/B) \cong \begin{cases} (\mathbb{Z}/2)^{\oplus 5} & \text{if } n=3 \text{ and } d_1=d_2=2\,, \\ (\mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2)^{\oplus 2} & \text{otherwise}\,. \end{cases}$$

PROPOSITION 5.3 (Mongardi–Wandel). *Let $\pi\colon X \to B$ be a Lagrangian fibration of an OG6-type hyper-Kähler manifold, obtained by the moduli of sheaves construction. Then*

$$\mathrm{Aut}^\circ(X/B) \cong (\mathbb{Z}/2)^{\oplus 4}\,.$$

We note that the latter computation for OG6-type was already done by Mongardi–Wandel in [MW17, § 5], as an intermediate step for their computation of the larger group $\mathrm{Aut}^\circ(X) \cong (\mathbb{Z}/2)^{\oplus 8}$. Thus proving Proposition 5.2 will be enough to conclude Theorem 5.1. As mentioned before, the proof will be done by an explicit computation. In fact, the computation can be extended further and calculates the polarization scheme K_0 as well. This is the second main result of this section.

PROPOSITION 5.4. *Let $\pi\colon X \to B$ be a Lagrangian fibration of a Kum_n-type hyper-Kähler manifold, obtained by the moduli of sheaves construction from a triple (S,l,s) in Definition 5.5. Let (d_1,d_2) be the polarization type of the ample class l.*

(i) *If $n=3$ and $d_1=d_2=2$, then*

$$K_0 \longhookrightarrow \mathrm{Aut}^\circ(X/B) \longhookrightarrow K_0[2]\,.$$

(ii) *Otherwise, we have*

$$K_0 = \mathrm{Aut}^\circ(X/B)\,.$$

Contrary to the previous sections, all the discussions in this section will be algebraic. In particular, algebraic Chern classes and Chow groups will be used. Given a coherent sheaf E on a smooth projective variety S, we denote by

$$c_i(E) \in H^{2i}(S,\mathbb{Z})\,, \quad \tilde{c}_i(E) \in \mathrm{CH}^i(S)$$

the ith cohomological and algebraic Chern classes of E, respectively.

5.1 Moduli of coherent sheaves on an abelian variety

In this subsection, we recall the construction of Kum_n-type hyper-Kähler manifolds obtained from certain moduli spaces of sheaves on abelian varieties. We will mostly follow [Yos01].

Let S be an abelian surface and $l \in \mathrm{NS}(S)$ an ample cohomology class with $\int_S l^2 = 2n+2$. Fix a nonzero class $s \in H^4(S,\mathbb{Z})$ so that we have a *primitive* Mukai vector

$$v = (0, l, s) \in H^*_{\mathrm{even}}(S,\mathbb{Z})\,. \tag{5.1}$$

Then the moduli space M of stable coherent sheaves on S with Mukai vector v, with respect to a v-generic polarization, is a smooth projective holomorphic symplectic variety of dimension $\langle v,v\rangle + 2 = 2n+4$. (Since the polarization is v-generic, every semistable sheaf with Mukai vector v is stable; that is, M is projective. See [Yos01, Theorem 0.1] and [KLS06, § 2.4].) Denote by Pic^l_S a connected component of the Picard scheme of S with cohomological first Chern class l. Yoshioka proved that the Albanese variety of M is isomorphic to $S \times \mathrm{Pic}^l_S$, so that we can define the Albanese morphism $\mathrm{Alb}\colon M \to S \times \mathrm{Pic}^l_S$.

To be more precise, we first need to choose a specific reference line bundle L_0 and coherent sheaf E_0 on S. We choose the line bundle L_0 to be a *symmetric* ample line bundle with $c_1(L) = l$ (there are precisely 16 of them). Fix a smooth curve $i\colon C_0 \hookrightarrow S$ in the linear system $|L_0|$, and define a reference coherent sheaf by $E_0 = i_* D$ for a line bundle D on C_0 with degree $s+n+1$. The Riemann–Roch computation gives $\mathrm{ch}(E_0) = v = (0,l,s)$ and $\tilde{c}_1(E_0) = \tilde{c}_1(L_0)$. Say $\Sigma\colon\ \mathrm{CH}^2(S) \to S$ is the summation map. We can choose an appropriate line bundle D on C_0 such that $\Sigma(\tilde{c}_2(E_0)) = \Sigma\big(\frac{1}{2}\big(L_0^2\big) + i_*\tilde{c}_1(D)\big) = 0$. After choosing this specific reference point, the Albanese morphism can be explicitly described by

$$\mathrm{Alb}\colon M \longrightarrow S \times \mathrm{Pic}^l_S\,, \quad [E] \longmapsto (\mathfrak{c}(E), \tilde{c}_1(E))\,, \tag{5.2}$$

where we define $\mathfrak{c}(E) = \Sigma(\tilde{c}_2(E))$. It sends the reference point $[E_0]$ to the point $(0,[L_0])$. The Albanese morphism is isotrivial with Kum_n-type projective hyper-Kähler manifold fibers. We will work with the central fiber

$$X = \mathrm{Alb}^{-1}(0,[L_0])\,.$$

Due to our choice of the Mukai vector v in (5.1), the above construction further comes with a Lagrangian fibration. Consider a connected component $\tilde{B}$ of the Chow variety of effective divisors on S with cohomological first Chern class l. Le Potier [LeP93] constructed a morphism

$$\mathrm{Supp}\colon M \longrightarrow \tilde{B}\,, \quad [E] \longmapsto [\mathrm{Fitt}_0 E]\,,$$

where $\mathrm{Fitt}_0 E$ is the Fitting support of a coherent sheaf E. Finally, consider the Poincaré line bundle $\mathcal{P}$ on $S \times \mathrm{Pic}^l_S$, the universal family of line bundles with the cohomological Chern class l. Denote by $r\colon S \times \mathrm{Pic}^l_S \to \mathrm{Pic}^l_S$ the second projection; the pushforward $r_*\mathcal{P}$ is a vector bundle of rank $n+1$. Its projectivization is a Zariski locally trivial $\mathbb{P}^n$-bundle

$$\mathrm{LB}\colon \tilde{B} \longrightarrow \mathrm{Pic}^l_S\,, \quad [C] \longmapsto [\mathcal{O}_S(C)]\,.$$

Gathering all the morphisms together, we obtain a commutative diagram

$$\begin{tikzcd} M \arrow[dr, "{(\mathfrak{c}, \mathrm{Supp})}"] \arrow[dd, "\mathrm{Alb}"'] & \\ & S \times \tilde{B}\,. \arrow[dl, "\mathrm{id} \times \mathrm{LB}"] \\ S \times \mathrm{Pic}^l_S & \end{tikzcd}$$

This is an isotrivial family of Lagrangian fibered hyper-Kähler manifolds in the sense of Definition 2.1. Its central fiber is a Lagrangian fibration $\pi\colon X \to B$, where $B = \mathrm{LB}^{-1}([L_0]) = |L_0| \cong \mathbb{P}^n$.

Definition 5.5. Let (S, l) be a degree $n+1$ polarized abelian surface with polarization type (d_1, d_2) and $s \in H^4(S, \mathbb{Z})$ be any nonzero class with $\gcd(d_1, s) = 1$ (so that the Mukai vector $v = (0, l, s)$ primitive). Then the above construction $\pi\colon X \to B$ is called the *moduli construction of* Kum_n*-type* (or of a *Debarre system*) associated with the triple (S, l, s). It is a Lagrangian fibration of a projective hyper-Kähler manifold of Kum_n-type to a projective space.

5.2 The H^2-trivial automorphisms of Kum_n-type moduli constructions

Recall that [BNS11, Corollary 5] and [HT13, Theorem 2.1] computed that every Kum_n-type hyper-Kähler manifold has

$$\mathrm{Aut}^\circ(X) \cong \mathbb{Z}/2 \ltimes (\mathbb{Z}/n+1)^{\oplus 4}\,.$$

The goal of this subsection is to explicitly describe such automorphisms for the moduli construction in Definition 5.5. Note that Lagrangian fibrations will temporarily play no role in this subsection.

Recall that we have fixed the origin $[L_0] \in \mathrm{Pic}^l_S$, a symmetric ample line bundle on S. By the general theory of abelian varieties, there exists a dual ample line bundle $\check{L}_0$ on the dual abelian variety $\check{S}$ (see [BL04, § 14.4]). The ample line bundles L_0 and $\check{L}_0$ induce polarization isogenies

$$\varphi\colon S \longrightarrow \check{S}\,, \quad \check{\varphi}\colon \check{S} \longrightarrow S\,, \tag{5.3}$$

making their compositions the multiplication endomorphisms

$$[n+1]\colon S \xrightarrow{\varphi} \check{S} \xrightarrow{\check{\varphi}} S\,, \quad [n+1]\colon \check{S} \xrightarrow{\check{\varphi}} S \xrightarrow{\varphi} \check{S}\,. \tag{5.4}$$

Since L_0 has polarization type (d_1, d_2), the dual line bundle $\check{L}_0$ has polarization type (d_1, d_2) as well. In particular, we have an isomorphism

$$\ker \check{\varphi} \cong (\mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2)^{\oplus 2}\,. \tag{5.5}$$

A closed point x on S defines a translation automorphism by x. Our notation for the translation automorphism is

$$t_x\colon S \longrightarrow S\,, \quad y \longmapsto y + x\,.$$

A closed point ξ on $\check{S}$ represents a numerically trivial line bundle on S. Considering ξ as both a closed point on $\check{S}$ and a line bundle on S can possibly lead to confusion. Thus, we will write

$$P_\xi \coloneqq \text{the numerically trivial line bundle on } S \text{ corresponding to } \xi \in \check{S}\,.$$

With this notation in mind, we can explicitly realize the $\mathrm{Aut}^\circ(X)$-action for the moduli constructions.

Proposition 5.6. *Let X be a Kum_n-type moduli construction associated with a triple (S, l, s) in Definition 5.5.*

(i) *We have an isomorphism*

$$\mathrm{Aut}^\circ(X) = \{\pm 1\} \ltimes \big\{(x,\xi) \in S[n+1] \times \check{S}[n+1] : \varphi(x) = 0,\, \check{\varphi}(\xi) = sx\big\}\,.$$

(ii) *With the above identification, the* $\mathrm{Aut}^\circ(X)$*-action on* X *is defined by*

$$(1,x,\xi)\cdot[E] = [t_x^* E \otimes P_\xi]\,, \quad (-1,x,\xi)\cdot[E] = [t_x^*\big([-1]^*E\big) \otimes P_\xi]\,,$$

where $[-1]\colon S \to S$ *is the multiplication by* -1 *automorphism on* S*.*

The rest of this subsection is devoted to the proof of Proposition 5.6. To start, we note that Yoshioka has already computed an explicit trivialization of the Albanese morphism $\mathrm{Alb}\colon M \to S \times \mathrm{Pic}^l_S$. Yoshioka's trivialization is obtained by the base change $[n+1]\colon S\times \mathrm{Pic}^l_S \to S \times \mathrm{Pic}^l_S$, which is a degree $(n+1)^8$ isogeny. As we will see in a moment, this is not a minimal isogeny in the sense of Definition 4.2. Using the methods in Section 4, we first prove that the morphism

$$\phi\colon S\times \mathrm{Pic}^l_S \longrightarrow S\times \mathrm{Pic}^l_S\,, \quad (y,[L]) \longmapsto \big(sy - \check{\varphi}\big(L\otimes L_0^{-1}\big), [L_0 \otimes P_{\varphi(y)}]\big) \tag{5.6}$$

is the minimal isogeny trivializing the Albanese morphism.

Proposition 5.7. *The base change (5.6) is the minimal isogeny trivializing the Albanese morphism* $\mathrm{Alb}\colon M \to S\times \mathrm{Pic}^l_S$ *in the sense of Definition 4.2.*

Proof. Start from Yoshioka's diagram [Yos01, §4.1, (4.10)] trivializing the Albanese morphism, which is a cartesian diagram

$$\begin{array}{ccc} X \times \big(S\times \mathrm{Pic}^l_S\big) & \xrightarrow{\Phi'} & M \\ \Big\downarrow{\scriptstyle \mathrm{pr}_2} & & \Big\downarrow{\scriptstyle \mathrm{Alb}} \\ S\times \mathrm{Pic}^l_S & \xrightarrow{[n+1]} & S\times \mathrm{Pic}^l_S\,. \end{array} \tag{5.7}$$

Here $\Phi'\colon X \times \big(S \times \mathrm{Pic}^l_S\big) \to M$ is

$$\Phi'([E],y,[L]) = \Big[t^*_{\check{\varphi}(L\otimes L_0^{-1})} E \otimes \big(L\otimes L_0^{-1}\big)^{\otimes s} \otimes P_{-\varphi(y)}\Big]\,.$$

Note that our convention differs by a sign to Yoshioka's original paper because Yoshioka's dual line bundle $\check{L}_0$ differs to ours by a sign.

The Galois group of the base change $[n+1]$ is the group of $(n+1)$-torsion points, namely $S[n+1]\times \check{S}[n+1] \cong (\mathbb{Z}/n+1)^{\oplus 8}$. By Proposition 4.4, it acts on $X\times \big(S\times \mathrm{Pic}^l_S\big)$ by translation on the second factor:

$$(x,\xi)\cdot([E],y,[L]) = ([E], y+x, [L\otimes P_\xi])\,.$$

One computes the descent of this action to M via Φ':

$$(x,\xi)\cdot[E] = \big[t^*_{\check{\varphi}(\xi)} E \otimes P_{s\xi - \varphi(x)}\big]\,.$$

This is the $(S[n+1]\times \check{S}[n+1])$-action on $X = \mathrm{Alb}^{-1}(0,[L_0])$ in Proposition 4.4. One sees that this action is not an effective action, and the kernel of the action is precisely

$$\big\{(x,\xi) \in S[n+1]\times \check{S}[n+1] : \check{\varphi}(\xi) = 0,\, s\xi - \varphi(x) = 0\big\}\,.$$

To kill the kernel and obtain an effective action, take a Galois quotient corresponding to the kernel (via the Galois correspondence). This is an isogeny $\psi\colon S\times \mathrm{Pic}^l_S \to S\times \mathrm{Pic}^l_S$ defined by

$$\psi(y,[L]) = \big(\check{\varphi}\big(L\otimes L_0^{-1}\big), \big(L\otimes L_0^{-1}\big)^{\otimes s} \otimes P_{-\varphi(y)}\big)\,.$$

One can check that the morphism ϕ in (5.6) is precisely the isogeny making $\phi\circ\psi=[n+1]$ (here one needs to use (5.4)), but we omit the computation). The result is a factorization of (5.7) into the minimal isogeny

$$\begin{tikzcd}
X\times\left(S\times\mathrm{Pic}^l_S\right) \arrow[r, "\Psi"] \arrow[d, "\mathrm{pr}_2"] & X\times\left(S\times\mathrm{Pic}^l_S\right) \arrow[r, "\Phi"] \arrow[d, "\mathrm{pr}_2"] & M \arrow[d, "\mathrm{Alb}"] \\
S\times\mathrm{Pic}^l_S \arrow[r, "\psi"] & S\times\mathrm{Pic}^l_S \arrow[r, "\phi"] & S\times\mathrm{Pic}^l_S\,.
\end{tikzcd}$$

Here our new morphism Φ, Beauville's minimal split covering of M, turns out to have a neater form than the original Φ':

$$\Phi([E],y,[L])=\left[t_y^*E\otimes\left(L\otimes L_0^{-1}\right)\right]. \tag{5.8}$$

The claim follows. $\square$

Again thanks to Proposition 4.4, we have a canonical, effective, and H^2-trivial $\mathrm{Gal}(\phi)$-action on X. The Galois group $\mathrm{Gal}(\phi)$ is captured by the kernel of ϕ, so we have

$$\mathrm{Gal}(\phi)=\left\{(x,\xi)\in S[n+1]\times\check{S}[n+1]:\varphi(x)=0,\ \check{\varphi}(\xi)=sx\right\}. \tag{5.9}$$

This explains the isomorphism in Proposition 5.6. The $\mathrm{Gal}(\phi)$-action on the fiber X is obtained via the description of Φ in (5.8). This explains how we obtained the group action in Proposition 5.6.

Lemma 5.8. *More explicitly, we have* $\mathrm{Gal}(\phi)\cong(\mathbb{Z}/n+1)^{\oplus4}$.

Proof. Let us compute the group (5.9) explicitly. The expression involves the abelian surfaces S and its dual $\check{S}$, their $(n+1)$-torsion points, and their polarization isogenies φ and $\check{\varphi}$. Therefore, the expression is independent of the complex structure of S, and the question is topological. We may fix polarization bases $H_1(S,\mathbb{Z})=\mathbb{Z}\{e_1,\dots,e_4\}$ and $H_1(\check{S},\mathbb{Z})=\mathbb{Z}\{e_1^*,\dots,e_4^*\}$ so that we can identify $S=(\mathbb{R}/\mathbb{Z})\{e_1,\dots,e_4\}$ and $\check{S}=(\mathbb{R}/\mathbb{Z})\{e_1^*,\dots,e_4^*\}$. The polarization isogenies with respect to them are

$$\varphi=\begin{pmatrix}0&0&d_1&0\\0&0&0&d_2\\-d_1&0&0&0\\0&-d_2&0&0\end{pmatrix},\quad\check{\varphi}=\begin{pmatrix}0&0&-d_2&0\\0&0&0&-d_1\\d_2&0&0&0\\0&d_1&0&0\end{pmatrix}. \tag{5.10}$$

Writing the coordinates as $(a_1,\dots,a_4)$ for $S=(\mathbb{R}/\mathbb{Z})^4$ and $(b_1,\dots,b_4)$ for $\check{S}=(\mathbb{R}/\mathbb{Z})^4$, we can explicitly compute

$$\mathrm{Gal}(\phi)=\left\{(a_i,b_i)_{i=1}^4\in\left(\tfrac{1}{n+1}\mathbb{Z}/\mathbb{Z}\right)^{\oplus8}:d_1a_1=0,\ sa_1+d_2b_3=0,\ \dots\right\}\cong A^{\oplus4},$$

where the abelian group A is defined by

$$A=\left\{(a,b)\in(\mathbb{Z}/n+1)^{\oplus2}:d_1a=0,\ sa+d_2b=0\right\}.$$

Notice that $\gcd(d_1,s)=1$ by the very assumption we had in Definition 5.5. Now $A\cong\mathbb{Z}/n+1$ by the following simple computational lemma, and the desired isomorphism is proved. $\square$

Lemma 5.9. *Let p, q, s be nonzero integers. Set $m=pq$ and assume either* $\gcd(p,s)=1$ *or* $\gcd(q,s)=1$. *Then the abelian group*

$$A=\left\{(a,b)\in(\mathbb{Z}/m)^{\oplus2}:pa=0,\ sa+qb=0\right\}$$

is isomorphic to $\mathbb{Z}/m$.

Proof. The group A is realized by the kernel of a homomorphism $f\colon (\mathbb{Z}/m)^{\oplus 2} \to (\mathbb{Z}/m)^{\oplus 2}$, $f = \left(\begin{smallmatrix} p & 0 \\ s & q \end{smallmatrix}\right)$. Adjusting the bases of both the domain and codomain (that is, performing elementary row and column operations), the matrix can be transformed into $\left(\begin{smallmatrix} 1 & 0 \\ 0 & pq \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Here one needs the assumption $\gcd(p, s) = 1$ or $\gcd(q, s) = 1$ to apply the Euclidean algorithm. The claim follows. □

We have described $\mathrm{Gal}(\phi) \cong (\mathbb{Z}/n{+}1)^{\oplus 4}$-action on X acting trivially on H^2. Since $\mathrm{Aut}^\circ(X) \cong \mathbb{Z}/2 \ltimes (\mathbb{Z}/n+1)^{\oplus 4}$, we still need to describe an additional $\mathbb{Z}/2$-part. Fortunately, this is not hard to guess. Construct an involution ι on $X \times \big(S \times \mathrm{Pic}^l_S\big)$ by

$$\iota([E], y, [L]) = ([[-1]^* E], -y, [[-1]^* L])\,.$$

Because we are not relying on the general theory anymore, we need to check that ι acts on M. We omit the typical Chern class computation.

The involution does not commute with the $\big(S \times \check{S}\big)$-action on $X \times \big(S \times \mathrm{Pic}^l_S\big)$, and this is the reason why $\mathbb{Z}/2$ should act on $(\mathbb{Z}/n+1)^{\oplus 4}$ nontrivially and leads to the semi-direct product. The action descends to M as

$$\iota([E]) = [[-1]^* E]\,.$$

To check that ι acts on the fiber $X = \mathrm{Alb}^{-1}(0, [L_0])$, we need to check that $\mathfrak{c}([-1]^* E) = 0$ and $\tilde{c}_1([-1]^* E) = \tilde{c}_1(L_0)$ for all $[E] \in X$. The former follows by definition, and the latter follows from the fact that L_0 is a *symmetric* line bundle. It remains to prove that ι acts on the second cohomology of X as the identity. We have already proved in Proposition 4.4 that $H^2(M, \mathbb{Q}) \to H^2(X, \mathbb{Q})$ is surjective. Hence we only need to prove that ι acts on $H^2(M, \mathbb{Q})$ as the identity. This follows because ι is induced from the automorphism $[-1]$ on S and $[-1]^*$ acts on $H^2(S, \mathbb{Q})$ trivially, and finally the Hodge structure $H^2(M, \mathbb{Q})$ is obtained by a tensor construction of $H^2(S, \mathbb{Q})$ by [Bül20]. This exhausts the entire $\mathrm{Aut}^\circ(X)$-action description on X and hence completes the proof of Proposition 5.6.

5.3 Automorphisms respecting the Lagrangian fibration

With Proposition 5.6 at hand, the proof of Proposition 5.2 becomes fairly straightforward. Any H^2-trivial automorphism is of the form

$$f = (\pm 1, x, \xi) \quad \text{for } x \in \ker \varphi \text{ and } \xi \in \check{S}[n+1] \text{ with } \check{\varphi}(\xi) = sx\,.$$

Let us split the proof it into two lemmas to make it more organized. Assume that $\pi\colon X \to B$ is the moduli construction of Kum_n-type.

Lemma 5.10. *Suppose that $f = (1, x, \xi) \in \mathrm{Aut}^\circ(X)$ in the description of Proposition 5.6 respects the Lagrangian fibration π. Then $x = 0$ and $\xi \in \ker \check{\varphi}$.*

Proof. Recall that f acts on X by $f \cdot [E] = [t_x^* E \otimes P_\xi]$ and that $\pi\colon X \to B$ is the (Fitting) support map $\mathrm{Supp}\colon [E] \mapsto [\mathrm{Fitt}_0 E]$. The support of $t_x^* E \otimes P_\xi$ is $\mathrm{Supp}\, E - x$, so f respects π if and only if $\mathrm{Supp}\, E = \mathrm{Supp}\, E - x$. This means $x = 0$ and thus $\xi \in \ker \check{\varphi}$. □

Lemma 5.11. *Suppose that $f = (-1, x, \xi) \in \mathrm{Aut}^\circ(X)$ in Proposition 5.6 respects π. Then this forces $n = 3$, $d_1 = d_2 = 2$, $x = 0$, and $\xi \in \ker \check{\varphi}$.*

Proof. Recall that f acts on X by $f \cdot [E] = [t_x^*\big([-1]^* E\big) \otimes P_\xi]$. A similar argument shows that $\mathrm{Supp}\, E = [-1]^* \mathrm{Supp}\, E - x$ for all $[E] \in X$. In other words, we have $D = [-1]^* D - x$ for all $D \in |L_0|$. Fix any $y \in S$ with $2y = x$ (there are 16 of them). Then this condition is equivalent to

$$D + y = [-1]^*(D + y) \quad \text{for all } D \in |L_0|\,,$$

or equivalently to every divisor in $|t_{-y}^*L_0|$ being symmetric. In particular, $t_{-y}^*L_0$ is a symmetric line bundle. We have chosen L_0 to be a symmetric line bundle, so this implies that y is a 2-torsion point, or $x = 0$. Hence $\xi \in \ker \check{\varphi}$.

Now we can say that every $D \in |L_0|$ is symmetric. In the following lemma, we will prove that there are only three possible polarization types of L_0. We have assumed from the very beginning that $\int_S l^2 = 2n + 2 = 2d_1d_2$ and $n \geqslant 2$. The first two cases are thus excluded. The only possible case is the polarization type $(2, 2)$, that is, when $n = 3$ and $d_1 = d_2 = 2$. $\square$

Lemma 5.12. *Let S be an abelian surface and L_0 a symmetric ample line bundle on it. Then every divisor in the complete linear system $|L_0|$ is symmetric if and only if L_0 has a polarization type $(1, 1)$, $(1, 2)$, or $(2, 2)$.*

Proof. Assume that L_0 has one of the three given polarization types. When L_0 is a principal polarization, $|L_0|$ consists of a single symmetric divisor. When L_0 is twice a principal polarization, this is [BL04, Theorem 4.8.1]. When L_0 has a polarization type $(1, 2)$, the statement can be found in [Bar87, Proposition 1.6].

Conversely, let us assume that every divisor in $|L_0|$ is symmetric. Denote by $H^0(S, L_0)_\pm$ the ± 1-eigenspaces of the involution $[-1]^*$ on $H^0(S, L_0)$. Every divisor in $|L_0|$ is symmetric if and only if either $H^0(S, L_0)_+ = 0$ or $H^0(S, L_0)_- = 0$ by [BL04, Lemma 4.7.1]. The dimensions of $H^0(S, L_0)_\pm$ are computed in [BL90, § 0] (or [BL04, Example 4.12.11]): If we let the polarization type of L_0 be (d_1, d_2), then depending on the *characteristic* of L_0, we have

$$h^0(L_0)_+ = \tfrac{1}{2}h^0(L_0) \quad \text{or} \quad \tfrac{1}{2}h^0(L_0) \pm 2^{1-s}\,,$$

where $0 \leqslant s \leqslant 2$ is an integer where $d_1, \dots, d_s$ are odd and d_{s+1} is even. There are three possibilities making $h^0(L_0)_+ = 0$ or $h^0(L_0)_- = 0$:

(i) $h^0(L_0) = 1$ and $s = 2$,

(ii) $h^0(L_0) = 2$ and $s = 1$, or

(iii) $h^0(L_0) = 4$ and $s = 0$.

Using $h^0(L_0) = d_1d_2$, it is easy to check that these are the three desired cases in the statement. $\square$

Proof of Proposition 5.2. If we are not in the exceptional case, then the above discussion concludes that $\mathrm{Aut}^\circ(X/B) = \ker \check{\varphi} \cong (\mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2)^{\oplus 2}$ by (5.5). In the exceptional case $n = 3$ and $d_1 = d_2 = 2$, we conclude that $\mathrm{Aut}^\circ(X/B) = \{\pm 1\} \ltimes \ker \check{\varphi} \cong \{\pm 1\} \ltimes (\mathbb{Z}/2)^{\oplus 4}$. The $\{\pm 1\}$-action on $\ker \check{\varphi}$ is trivial, either by direct computation or by using Proposition 3.22 to show that $\mathrm{Aut}^\circ(X/B)$ is abelian. $\square$

5.4 The polarization scheme of generalized Kummer varieties

This subsection will be devoted to the proof of Proposition 5.4. Assume that $\pi\colon X \to B$ is a Kum_n-type moduli construction. The computations in this subsection are highly influenced by [Wie18, § 6]. Recall from Section 5.1 that we had a Fitting support morphism $\mathrm{Supp}\colon M \to \tilde{B}$ over Pic_S^l. Fix a point $[L_0] \in \mathrm{Pic}_S^l$, and consider the fibers of M and $\tilde{B}$ over it. We obtain a morphism

$$\mathrm{Supp}\colon Y \longrightarrow B\,,$$

where $B = |L_0|$ is a complete linear system and $Y \subset M$ consists of torsion coherent sheaves E on S with $\mathrm{ch}(E) = v = (0, l, s)$ and $\tilde{c}_1(E) = \tilde{c}_1(L_0)$. The Kum_n-type hyper-Kähler manifold X is obtained by a fiber of the isotrivial fiber bundle $\mathfrak{c}\colon Y \to S$.

Consider the universal family $\mathcal{C} \to B$ of curves on S parametrizing effective divisors in $B = |L_0|$. Since L_0 is ample, by Bertini's theorem, there exists a Zariski-dense open subset $B_0 \subset B$ parametrizing smooth curves. The restriction of the universal family $\mathcal{C}_0 \to B_0$ is a smooth projective family of curves. The following lemma is standard, and we only sketch its proof.

Lemma 5.13. *The morphism* $\mathrm{Supp}\colon Y_0 = \mathrm{Supp}^{-1}(B_0) \to B_0$ *is isomorphic to the relative Picard scheme of the universal family of curves* $\mathrm{Pic}^d_{\mathcal{C}_0/B_0} \to B_0$ *for* $d = s + n + 1$.

Proof. Define a morphism $\mathrm{Pic}^d_{\mathcal{C}_0/B_0} \to Y_0$ over B_0 by $[L] \mapsto [i_*L]$, where L is a line bundle of degree $d = s + n + 1$ (Riemann–Roch computation) on a curve $i\colon C \hookrightarrow S$. Over a closed point $[C] \in B_0$ on the base, Pic^d_C is an abelian variety of dimension $n + 2$, so $\mathrm{Pic}^d_C \to Y_{[C]}$ must be an isomorphism. Any morphism of complex smooth varieties bijective on closed points is an isomorphism by Zariski's main theorem. $\square$

Lemma 5.13 in particular says that $Y_0 \to B_0$ is a torsor under the numerically trivial relative Picard scheme

$$J_0 = \mathrm{Pic}^0_{\mathcal{C}_0/B_0} \longrightarrow B_0\,.$$

Since $\mathcal{C}_0/B_0$ is a smooth projective family of curves, its relative Picard scheme J_0 is a canonically principally polarized abelian scheme. As standard, we will call it a relative Jacobian of the family and identify $\check{J}_0 = J_0$. Notice that we now have four different spaces over B_0: P_0, X_0, J_0, and Y_0. The space X_0 is a P_0-torsor as usual, and we also have Y_0 as a J_0-torsor. Since P_0 and J_0 are translation automorphism schemes of $X_0 \subset Y_0$, we have an inclusion $P_0 \subset J_0$. Our first goal is to describe the quotient of this inclusion.

Proposition 5.14. *There exists a short exact sequence of abelian schemes over* B_0

$$0 \longrightarrow P_0 \longrightarrow J_0 \longrightarrow S \times B_0 \longrightarrow 0\,. \tag{5.11}$$

Proof. The universal family $\mathcal{C}_0 \to B_0$ is a subvariety of the product $i\colon \mathcal{C}_0 \hookrightarrow S \times B_0$. This induces a pullback morphism $i^*\colon \check{S} \times B_0 \to J_0$ between their relative Picard schemes over B_0. The morphism $J_0 \to S \times B_0$ can be constructed by the dual of i^*. Fiberwise, it is the morphism $J_C \to S$ induced by the universal property of the Albanese morphism applied to $i\colon C \hookrightarrow S$.

We prove that the kernel of the morphism $J_0 \to S \times B_0$ is P_0. The claim can be verified fiberwise. Fix a closed point $[C] \in B_0$ corresponding to a smooth curve $i\colon C \hookrightarrow S$. Over it, a closed point of Y_0 (respectively, J_0) is represented by a degree d line bundle L on C (respectively, degree 0 line bundle M). The J_0-action on Y_0 is given by $[i_*M] \cdot [i_*L] = [i_*(L \otimes M)]$. Recall that X is a fiber of the morphism $\mathfrak{c}\colon Y \to S$. Hence the abelian scheme P_0 consists of translation automorphisms of J_0 invariant under the morphism $\mathfrak{c}$. Recall the definition of $\mathfrak{c}$ in (5.2). A Riemann–Roch computation gives us

$$\mathfrak{c}([i_*(L \otimes M)]) = \mathfrak{c}\big([i_*L]\big) - \Sigma\big(i_*\tilde{c}_1(M)\big)\,,$$

where $\tilde{c}_1(M) \in \mathrm{CH}^1(S)$, $i_*\colon\ \mathrm{CH}^1(S) \to \mathrm{CH}^2(S)$, and $\Sigma\colon\ \mathrm{CH}^2(S) \to S$ is a summation map. This proves that the $[i_*M]$-action on the fiber of Y_0 is $\mathfrak{c}$-invariant if and only if $\Sigma(i_*\tilde{c}_1(M)) = 0$. The claim follows by the following lemma, which is already proved in [Wie18, (6.8)]. $\square$

Lemma 5.15. *The morphism* $J_C \to S$ *sends a closed point* $[M] \in J_C$ *to* $\Sigma(i_*\tilde{c}_1(M)) \in S$.

The dual of (5.11) is automatically (see, for example, [BL04, Proposition 2.4.2]) a short exact sequence of abelian schemes

$$0 \longrightarrow \check{S} \times B_0 \longrightarrow J_0 \longrightarrow \check{P}_0 \longrightarrow 0\,. \tag{5.12}$$

In particular, P_0 and $\check{S} \times B_0$ are both abelian subschemes of a bigger abelian scheme J_0. The following proposition describes the polarization scheme K_0 more explicitly for the moduli constructions.

Proposition 5.16. *We have the following two additional descriptions of the polarization scheme K_0 as a B_0-group scheme:*

$$K_0 = P_0 \cap \big(\check{S} \times B_0\big) = \ker\big(\check{\varphi} \times \mathrm{id}\colon \check{S} \times B_0 \longrightarrow S \times B_0\big)\,.$$

Proof. Fiberwise at a closed point $[C] \in B_0$, the sequences (5.11) and (5.12) are short exact sequences of abelian varieties

$$0 \longrightarrow F \longrightarrow J_C \longrightarrow S \longrightarrow 0\,, \quad 0 \longrightarrow \check{S} \longrightarrow J_C \longrightarrow \check{F} \longrightarrow 0\,.$$

Here $F = \nu^{-1}([C])$ is a fiber of P_0, and J_C is the Jacobian of the curve C. The two abelian subvarieties F and $\check{S}$ of the principally polarized abelian variety J_C are the so-called complementary abelian subvarieties (see [BL04, § 12.1] or [Wie18, § 6.4]). In this case, we have an equality [BL04, Corollary 12.1.4]

$$\ker\big(F \longrightarrow J_C \longrightarrow \check{F}\big) = F \cap \check{S} = \ker\big(\check{S} \longrightarrow J_C \longrightarrow S\big)\,.$$

We will soon prove in Lemma 5.17 that the composition $\check{S} \to J_C \to S$ is precisely the polarization isogeny $\check{\varphi}$ in (5.3), regardless of the choice of a closed point $[C] \in B_0$. Given this, we obtain a sequence of identities of group schemes

$$\ker\big(P_0 \longrightarrow J_0 \longrightarrow \check{P}_0\big) = P_0 \cap \big(\check{S} \times B_0\big) = \ker\big(\check{\varphi} \times \mathrm{id}\colon \check{S} \times B_0 \longrightarrow S \times B_0\big)\,.$$

By the last description and (5.5), this group scheme is a constant group scheme with fibers $(\mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2)^{\oplus 2}$. On the other hand, the first description is the polarization scheme K_0; combine the uniqueness of the polarization in Theorem 3.1 and the computation of polarization types in Theorem 3.16. The claim follows. $\square$

Lemma 5.17. *The composition $\check{S} \to J_C \to S$ is the polarization isogeny $\check{\varphi}$ in (5.3).*

Proof. Denote by $i\colon C \hookrightarrow S$ the closed immersion. At the level of first homologies, the composition $\check{S} \to J_C \to S$ becomes a Hodge structure homomorphism

$$H_1\big(\check{S}, \mathbb{Z}\big) = H^1(S, \mathbb{Z}) \xrightarrow{i^*} H^1(C, \mathbb{Z}) \xrightarrow{i_*} H^3(S, \mathbb{Z}) = H_1(S, \mathbb{Z})\,.$$

Hence the composition is $i_* \circ i^*$, which is the multiplication map by $c_1(\mathcal{O}_S(C)) \in H^2(S, \mathbb{Z})$. Because we have chosen $[C]$ in a complete linear system $|L_0|$, it is multiplication by $c_1(L_0) = l$.

Therefore, the question reduces to the following claim: The dual polarization $\check{\varphi}\colon \check{S} \to S$ is given by $l \cup -\colon H^1(S, \mathbb{Z}) \to H^3(S, \mathbb{Z})$. Again choose polarization bases $H_1(S, \mathbb{Z}) = \mathbb{Z}\{e_1, \ldots, e_4\}$ and $H_1\big(\check{S}, \mathbb{Z}\big) = H^1(S, \mathbb{Z}) = \mathbb{Z}\{e_1^*, \ldots, e_4^*\}$ as in Lemma 5.8. The polarization isogenies φ and $\check{\varphi}$ have the matrix forms (5.10). The ample class l is the skew-symmetric bilinear map

$$\varphi\colon H_1(S, \mathbb{Z}) \otimes H_1(S, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

considered as an element of $H^2(S, \mathbb{Z})$. Hence it is $l = d_1 e_1^* \wedge e_3^* + d_2 e_2^* \wedge e_4^*$.

We can now explicitly compute the map $l \cup -\colon H^1(S, \mathbb{Z}) \to H^3(S, \mathbb{Z})$:

$$\begin{aligned} e_1^* &\longmapsto d_2 e_1^* \wedge e_2^* \wedge e_4^*\,, & e_2^* &\longmapsto -d_1 e_1^* \wedge e_2^* \wedge e_3^*\,,\\ e_3^* &\longmapsto -d_2 e_2^* \wedge e_3^* \wedge e_4^*\,, & e_4^* &\longmapsto d_1 e_1^* \wedge e_3^* \wedge e_4^*\,. \end{aligned}$$

The Poincaré duality $H_1(S, \mathbb{Z}) = H^3(S, \mathbb{Z})$ yields the basis of $H^3(S, \mathbb{Z})$:

$$\{e_2^* \wedge e_3^* \wedge e_4^*,\ -e_1^* \wedge e_3^* \wedge e_4^*,\ e_1^* \wedge e_2^* \wedge e_4^*,\ -e_1^* \wedge e_2^* \wedge e_3^*\}\,.$$

With respect to it, the matrix form of the multiplication coincides with precisely the matrix form of $\check{\varphi}$ above. (Compare this lemma with [Wie18, Lemma 6.14].) $\square$

Proof of Proposition 5.4. Recall from Section 5.3 the complete description of $\mathrm{Aut}^\circ(X/B)$. Let us assume $n \neq 3$ or $(d_1, d_2) \neq (2, 2)$ so that every automorphism $f \in \mathrm{Aut}^\circ(X/B)$ is of the form $(1, 0, \xi)$ for $\xi \in \ker \check{\varphi}$. It acts on Y by $f \cdot [E] = [E \otimes P_\xi]$, where P_ξ is the numerically trivial line bundle on S represented by $\xi \in \ker \check{\varphi} \subset \check{S}$. On Y_0, closed points are of the form $[E] = [i_* L]$, where L is a line bundle on a smooth curve $i \colon C \hookrightarrow S$. Hence f acts on it by

$$f \cdot [i_* L] = [i_* L \otimes P_\xi] = [i_*(L \otimes i^* P_\xi)] .$$

This means that the global section of $J_0 \to B_0$ defined by f represents a line bundle $[i^* P_\xi]$ over $[C] \in B_0$. The inclusion $\check{S} \times B_0 \subset J_0$ was by definition the pullback morphism of line bundles. Hence f in fact defines a global section $\xi = [P_\xi] \in \check{S}$ of the constant group scheme $\check{S} \times B_0$. This coincides with the description of the polarization scheme K_0 in Proposition 5.16, proving the desired equality $K_0 = \mathrm{Aut}^\circ(X/B)$.

The proof for the exceptional case $n = 3$ and $d_1 = d_2 = 2$ is nearly identical. The only difference is that the automorphisms $f \in \mathrm{Aut}^\circ(X/B)$ of the form $(1, 0, \xi)$ make up an index 2 subgroup $\ker \check{\varphi} \subset \mathrm{Aut}^\circ(X/B)$. So the same argument proves that $K_0 = \ker \check{\varphi} \subset \mathrm{Aut}^\circ(X/B)$ is an index 2 subgroup. The second inclusion $\mathrm{Aut}^\circ(X/B) \subset K_0[2]$ follows from Proposition 3.5 since we have $\mathrm{div}(h) = d_1 = 2$. $\square$

6. The dual Lagrangian fibration of a compact hyper-Kähler manifold

Combining the previous results, we can prove that the polarization scheme extends to a constant subgroup scheme of $\mathrm{Aut}^\circ(X/B)$ over B for known hyper-Kähler manifolds.

Theorem 6.1. *Let $\pi \colon X \to B$ be a Lagrangian fibration of a compact hyper-Kähler manifold of $K3^{[n]}$-, Kum_n-, OG10-, or OG6-type. Then the polarization scheme $K_0 \to B_0$ uniquely extends to a constant group scheme $K \to B$ that is a subgroup scheme of the constant group scheme $\mathrm{Aut}^\circ(X/B)$.*

Proof. When X is of $\mathrm{K3}^{[n]}$- or OG10-type, both the polarization scheme K and the global sections defined by $\mathrm{Aut}^\circ(X/B)$ are the zero section of the abelian scheme P_0. Hence the claim is trivial. When X is of OG6-type, lattice theory forces $\mathrm{div}(h) = 1$ as shown in [MR21, Lemma 7.1]. Proposition 3.5 applies, and we get an inclusion $\mathrm{Aut}^\circ(X/B) \hookrightarrow K_0$. Combining Theorems 3.16 and 5.1, we see that the inclusion is forced to be an equality fiberwise. Hence we get the global equality $K_0 = \mathrm{Aut}^\circ(X/B)$. In particular, K_0 extends over B to a constant group scheme $\mathrm{Aut}^\circ(X/B)$.

Assume that X is of Kum_n-type and the polarization type of π is not $(1, 2, 2)$. In this case, Proposition 5.4, together with Proposition 3.4, implies an equality of group schemes $K_0 = \mathrm{Aut}^\circ(X/B)$. The remaining case is when X is of Kum_3-type and the polarization type of π is $(1, 2, 2)$. In this case, we have $\mathrm{div}(h) = 2$ by Theorem 3.16, so Proposition 3.4 guarantees $\mathrm{Aut}^\circ(X/B) \subset K_0[2]$, where $K_0[2] = \ker(2\lambda)$ is slightly bigger than K_0. Both $\mathrm{Aut}^\circ(X/B)$ and $K_0 = 2 \cdot K_0[2]$ are contained in $K_0[2]$ and are invariant under deformations, so the inclusion $K_0 \subset \mathrm{Aut}^\circ(X/B)$ in Proposition 5.4 is preserved under deformation. The claim follows. $\square$

Remark 6.2. We may state Theorem 6.1 in the following simpler way: We have an equality of group schemes

$$K_0 = \mathrm{Aut}'(X/B) \quad \big(:= \mathrm{Aut}^\circ(X/B) \cap \mathrm{Aut}'(X)\big) ,$$

where $\mathrm{Aut}'(X) \subset \mathrm{Aut}^\circ(X)$ is a group defined in Remark 4.5. For most of the known examples of Lagrangian fibered hyper-Kähler manifolds, we have $\mathrm{Aut}'(X/B) = \mathrm{Aut}^\circ(X/B)$. There is a single known example where the inclusion $\mathrm{Aut}'(X/B) \subset \mathrm{Aut}^\circ(X/B)$ is strict, when X is of Kum_3-type and π has polarization type $(1,2,2)$. In this case, $\mathrm{Aut}'(X/B) \cong (\mathbb{Z}/2)^{\oplus 4}$ and $\mathrm{Aut}^\circ(X/B) \cong (\mathbb{Z}/2)^{\oplus 5}$.

A direct consequence of this theorem is a promised compactification of the dual torus fibration $\check{\pi}\colon \check{X}_0 \to B_0$.

Theorem 6.3. *Let $\pi\colon X \to B$ be a Lagrangian fibration of a compact hyper-Kähler manifold of $K3^{[n]}$-, Kum_n-, OG10-, or OG6-type. Then*

$$\check{\pi}\colon \check{X} \longrightarrow B \quad \text{for } \check{X} = X/K$$

defines a compactification of the dual torus fibration $\check{\pi}\colon \check{X}_0 \to B_0$.

Proof. As explained in the introduction, we have defined the dual torus fibration by $\check{X}_0 = X_0/K_0$. For known deformation types, Theorem 6.1 proves that K_0 extends to a constant group scheme K over B acting on X. Therefore, the group scheme quotient $X_0/K_0 \to B_0$ can be compactified into $X/K \to B$. Since $K \to B$ is a constant group scheme, the quotient X/K may be considered as either a group scheme quotient over B or a finite group quotient over $\mathbb{C}$. □

When X is of $\mathrm{K3}^{[n]}$- or OG10-type, $\check{X}$ is identical to X, and there is nothing more to say. Let us further study the space $\check{X}$ when X is of Kum_n- or OG6-type. Being a quotient by H^2-trivial automorphisms, $\check{X}$ inherits many interesting properties from X. We provide Appendix B to collect their properties in a more general setup; the following proposition is a direct consequence of this more general discussion. For the definitions of a primitive symplectic orbifold and irreducible symplectic variety used in the following proposition, see Appendix A.

Proposition 6.4. *Keep the notation from Theorem 6.3, and assume that X is of either* Kum_n*- or OG6-type.*

(i) *The variety $\check{X}$ is a compact primitive symplectic orbifold and also an irreducible symplectic variety.*
(ii) *The variety $\check{X}$ does not admit a symplectic resolution.*
(iii) *The variety $\check{X}$ is simply connected. It has Fujiki constant $c_{\check{X}} = 1/c_X$.*
(iv) *The groups $H^2(\check{X}, \mathbb{Q})$ and $H^2(X, \mathbb{Q})$ are Hodge isomorphic and Beauville–Bogomolov–Fujiki isometric.*
(v) *The Looijenga–Lunts–Verbitsky (LLV) algebras and Mumford–Tate algebras of X and $\check{X}$ are isomorphic.*
(vi) *The pullback $H^*(\check{X}, \mathbb{Q}) \to H^*(X, \mathbb{Q})$ is an injective map of LLV structures.*
(vii) *There exists a canonical isomorphism $\mathrm{Def}^{\mathrm{lt}}(\bar{X}) = \mathrm{Def}(X)$. If $\mathcal{X} \to \mathrm{Def}(X)$ is the universal deformation of X, then $\mathcal{X}/K \to \mathrm{Def}(X)$ is the (locally trivial) universal deformation of $\check{X}$.*

Proof. Everything is a direct consequence of Propositions B.1 and B.2. Only the first three items need further explanations. For the first and second items, it is enough to show that $\operatorname{codim} X^f \geqslant 4$ for all $f \in K \setminus \{\mathrm{id}\}$. The fixed loci of H^2-trivial automorphisms deform when X deforms, by Proposition B.3. Hence we may prove this for any model in the deformation class on X. For OG6, the fixed loci are computed in [MW17, § 6]; they are either K3 surfaces or points. For Kum_n, the fixed loci are computed in [Ogu20, Lemma 3.5], and similarly one can deduce that their codimension is always at least 4. For the third item, simply notice that the group K has order c_X^2 in all cases. □

Proposition 6.4 shows that $\check{X}$ has quotient singularities when X is of Kum_n- or OG6-type. Therefore, $\check{X}$ cannot be homeomorphic to X. We call the corresponding $\check{X}$ the *dual Kummer variety* and *dual OG6 variety*, respectively.

Finally, Proposition 6.4 shows, in particular, that the local deformation behavior and period domains of X and $\check{X}$ are identical. Therefore, one can still apply the method in [GTZ13, § 2] at the level of period domains and obtain similar conclusions to it for all known deformation types of hyper-Kähler manifolds. One subtlety here is that the quotient construction works for any deformation X' of X, even if X' does not admit any Lagrangian fibration; the quotient X'/K is still well defined because we have considered K as an abstract subgroup of the group $\mathrm{Aut}^\circ(X)$. The local universal deformation space of the Lagrangian fibration $\pi\colon X \to B$ is a hyperplane $\mathrm{Def}(X,H) \subset \mathrm{Def}(X)$ (see [Mat16]). Once we choose a deformation X' by respecting the Lagrangian fibration $[\pi'\colon X' \to B'] \in \mathrm{Def}(X,H)$, we can say that $\check{\pi}'\colon X'/K \to B'$ is the dual Lagrangian fibration of $\pi'\colon X' \to B'$.

Appendix A. Various notions of singular hyper-Kähler varieties

Many of the important properties of compact hyper-Kähler manifolds have been generalized to singular settings. There are several definitions of singular hyper-Kähler varieties in the current literature. To make our discussion less ambiguous, we collect some definitions and compare them. Our main references are [BL22, Sch20, Men20].

If X is a normal complex space, then its sheaf of reflexive k-forms is defined to be the reflexive closure of the sheaf of k-forms $\Omega_X^{[k]} = \big(\Omega_X^k\big)^{\vee\vee}$, or equivalently $\Omega_X^{[k]} = j_*\Omega^k_{X_{\mathrm{reg}}}$, where $j\colon X_{\mathrm{reg}} \hookrightarrow X$ is the smooth locus of X. A quasi-étale morphism is a morphism étale outside of a closed subvariety of codimension at least 2.

Definition A.1 ([BL22, Definition 3.1 and Theorem 3.4], [Sch20, Definition 1], [Men20, Definition 3.1]). Let X be a compact normal Kähler space (see, for example, [BL22, § 2.3]) and $\sigma \in H^0\big(X, \Omega_X^{[2]}\big)$ a reflexive 2-form.

(i) (X,σ) is called a *symplectic variety* if X has rational Gorenstein singularities and σ is nondegenerate on X_{reg}.

(ii) X is called a *primitive symplectic variety* if

$$H^0\big(X,\Omega_X^{[1]}\big) = 0\,,\quad H^0\big(X,\Omega_X^{[2]}\big) = \mathbb{C}\sigma\,,$$

and (X,σ) is a symplectic variety.

(iii) X is called an *irreducible symplectic variety* if it is a primitive symplectic variety with the following condition: For any finite quasi-étale cover $f\colon X' \to X$, we have

$$H^0\big(X',\Omega_{X'}^{[2k+1]}\big) = 0\,,\quad H^0\big(X',\Omega_{X'}^{[2k]}\big) = \mathbb{C}\cdot f^*\sigma^{[k]} \quad \text{for } k \geqslant 0\,.$$

(iv) X is called a *Namikawa symplectic variety* if it is a $\mathbb{Q}$-factorial and terminal primitive symplectic variety.

(v) X is called a *primitive symplectic orbifold* if it is Namikawa symplectic with only finite quotient singularities.

We have a series of implications

$$\begin{array}{ccccc} & \text{primitive symplecitc orbifold} & \Longrightarrow & \text{Namikawa symplectic} & \\ & & & \Downarrow & \\ \text{irreducible symplectic} & \Longrightarrow & & \text{primitive symplectic} & \Longrightarrow \text{symplectic}\,. \end{array}$$

Eventually, the dual hyper-Kähler variety $\check{X}$ in Theorem 1.1 will be both a primitive symplectic orbifold and an irreducible symplectic variety (Proposition 6.4). Hence all of the discussions here apply.

Many of the interesting properties of compact hyper-Kähler manifolds generalize to their singular analogues, especially to primitive symplectic varieties. We highlight some of their properties that will be useful to our discussion. Let X be a primitive symplectic variety of dimension $2n$.

- The normalization of the singular locus X_{sing} is again symplectic [Kal06, Theorem 2.5]. In particular, X_{sing} is always even-dimensional.
- There exists a notion of the Beauville–Bogomolov–Fujiki form and Fujiki constant of X, so that the Fujiki relation (1.1) holds; see [Sch20, Theorem 2], [BL22, Lemma 5.7].
- The variety X is Namikawa symplectic if and only if it is $\mathbb{Q}$-factorial and $\operatorname{codim} X_{\mathrm{sing}} \geqslant 4$ [Nam01, Corollary 1] and [BL22, Theorem 3.4].
- Every morphism $\pi\colon X \to B$ with connected fibers to a normal base B (with $0 < \dim B < 2n$) is a Lagrangian fibration [Sch20, Theorem 3]. That is, all the irreducible components of the fibers of π are Lagrangian subvarieties of X.
- The Hodge structure $H^2(X,\mathbb{Z})$ is pure; see [Sch20, Theorem 8] and [BL22, Corollary 3.5]. If X is a primitive symplectic orbifold, then the full cohomology $H^*(X,\mathbb{Q})$ is a pure Hodge structure [PS08, Theorem 2.43].
- There exists a universal locally trivial deformation $\mathcal{X} \to \mathrm{Def}^{\mathrm{lt}}(X)$ over a smooth complex germ $\mathrm{Def}^{\mathrm{lt}}(X)$ of dimension $h^{1,1}(X)$; see [BL22, Theorem 4.7]. If X is Namikawa symplectic, then any deformation is automatically locally trivial [Nam06, Main Theorem].

We will use these facts in Section 6 and Appendix B, without mentioning them explicitly.

Appendix B. Quotient of a hyper-Kähler manifold by H^2-trivial automorphisms

Let X be a compact hyper-Kähler manifold and $\mathrm{Aut}^\circ(X)$ the finite group of H^2-trivial automorphisms. Throughout the appendix, we always let $G \subset \mathrm{Aut}^\circ(X)$ to be *any* subgroup and write

$$p\colon X \longrightarrow \bar{X} = X/G\,. \tag{B.1}$$

The goal of this appendix is to gather basic geometric and cohomological properties of the quotient $\bar{X}$. Note that Lagrangian fibrations play no role in this appendix. The results are summarized in Propositions B.1, B.2, and B.3.

Proposition B.1. *Consider the quotient (B.1) of a compact hyper-Kähler manifold X.*

(i) *The morphism p is a finite quasi-étale symplectic quotient.*

(ii) *The quotient $\bar{X}$ is a $\mathbb{Q}$-factorial irreducible symplectic variety whose singularity locus is $p\big(\bigcup_{f\in G\setminus\{\mathrm{id}\}} X^f\big)$. If $\operatorname{codim} X^f > 2$ for all $f \in G\setminus\{\mathrm{id}\}$, then $\bar{X}$ is also a primitive symplectic orbifold.*

(iii) *The quotient $\bar{X}$ is simply connected.*

(iv) *There exists a canonical isomorphism $\mathrm{Def}^{\mathrm{lt}}(\bar{X}) = \mathrm{Def}(X)$. Moreover, if $\mathcal{X} \to \mathrm{Def}(X)$ is the universal deformation of X, then the quotient $\mathcal{X}/G \to \mathrm{Def}(X)$ is the universal locally trivial deformation of $\bar{X}$.*

The quotient $\bar{X} = X/G$ being an irreducible symplectic variety, its behavior is intimately related to its (second) cohomology. By [BL22, Lemma 5.7], there exists a notion of the Beauville–Bogomolov–Fujiki form $q_{\bar{X}}$ that is unique *up to scaling*. Typically, a primitive symmetric bilinear form is chosen. However, to compute the Fujiki form $c_{\bar{X}}$ explicitly, we will choose instead

$$q_{\bar{X}}(x, y) = q_X(p^*x, p^*y) \quad \text{for } x, y \in H^2(\bar{X}, \mathbb{Z}) .$$

We do not know if such $q_{\bar{X}}$ is a primitive bilinear form.

Proposition B.2. *We use the same notation as above.*

(i) *The Fujiki constant of $\bar{X}$ is $c_{\bar{X}} = c_X/|G|$.*

(ii) *The pullback*

$$p^* \colon H^2(\bar{X}, \mathbb{Z})/(\text{torsion}) \longrightarrow H^2(X, \mathbb{Z})$$

is an injective Hodge structure homomorphism and a Beauville–Bogomolov–Fujiki isometry. It is an isomorphism over $\mathbb{Q}$.

(iii) *Let $\mathfrak{g}$ be the LLV algebra of X. Then with respect to the injective homomorphism*

$$p^* \colon H^*(\bar{X}, \mathbb{Q}) \longrightarrow H^*(X, \mathbb{Q}) ,$$

the cohomology of $\bar{X}$ is closed under the $\mathfrak{g}$-action.

(iv) *If $H^k(\bar{X}, \mathbb{Q}) \neq 0$, then its special Mumford–Tate algebra is isomorphic to that of $H^2(X, \mathbb{Q})$. As a consequence, any $\mathfrak{g}$-module decomposition of $H^*(\bar{X}, \mathbb{Q})$ is a pure Hodge structure decomposition.*

Proposition B.3. *Let X be a compact hyper-Kähler manifold and $G \subset \mathrm{Aut}^\circ(X)$. If X' is deformation equivalent to X, then $(X')^G$ is deformation equivalent to X^G.*

Proof. Let $p \colon \mathcal{X} \to \mathrm{Def}(X)$ be a universal deformation of X. Since G acts fiberwise on p, the morphism $\mathcal{X}^G \to \mathrm{Def}(X)$ gives a family of fixed loci $(X_t)^G$. Because G is a finite group acting on a complex manifold $\mathcal{X}$, its fixed locus $\mathcal{X}^G$ is a complex manifold proper over $\mathrm{Def}(X)$. Similarly, each $(X_t)^G$ is a (symplectic) manifold. Hence $\mathcal{X}^G \to \mathrm{Def}(X)$ is a smooth proper family, and the claim follows. □

We note that, at the moment this paper is written, the definition of the LLV algebra for primitive symplectic varieties is missing. The third and fourth items of Proposition B.2 suggest that $\mathfrak{g}$ is a good candidate for the LLV algebra of $\bar{X} = X/G$.

Note again that the subgroup $G \subset \mathrm{Aut}^\circ(X)$ was taken arbitrary. Hence we have a family of irreducible symplectic varieties corresponding to each subgroup of $\mathrm{Aut}^\circ(X)$. That is, we get a Galois correspondence between the subgroups $G \subset \mathrm{Aut}^\circ(X)$ and the symplectic quotients $\bar{X} = X/G$ with the same rational Beauville–Bogomolov–Fujiki forms. In particular, their deformation behaviors are all identical.

The rest of this appendix is devoted to the proof of Propositions B.1 and B.2. Most of the proofs will be straightforward, so we will be brief.

Proof of Proposition B.1(i), (iii), (iv). Let us present the proof of the proposition without the second item. The second item will be proved separately.

The group G acts trivially on $H^2(X, \mathbb{Z})$, so it acts symplectically on X. Hence p is a symplectic quotient. The ramified locus of p is contained in the union of the fixed loci $\bigcup_{f \in G \setminus \{\mathrm{id}\}} X^f$, which is of codimension at least 2 as every $f \in G$ is a symplectic automorphism. This means that p is quasi-étale and σ descends to $\bar{X}$; the first item follows.

The third item is a direct consequence of the second item because any irreducible symplectic variety is simply connected by [GGK19, Corollary 13.3]. The last item again follows directly from [Fuj83, Theorem 3.5 and Lemma 3.10]. Since G acts holomorphically on $\mathcal{X}$ and trivially on $H^2(X, \mathbb{Z})$, the morphism $\mathcal{X} \to \mathrm{Def}(X)$ equipped with a G-action is the universal deformation of the pair (X, G). Once we have a universal deformation of the pair (X, G), the quotient $\mathcal{X}/G \to \mathrm{Def}(X)$ is the locally trivial universal family of X/G. $\square$

LEMMA B.4. *Let (X, σ) be a compact symplectic variety and $f\colon X' \to X$ a finite quasi-étale morphism. Then $(X', f^*\sigma)$ is a compact symplectic variety.*

Proof. By [KM98, Proposition 5.20] or [GKP16, Remark 3.4], the variety X' is Gorenstein and canonical. Therefore, it has rational singularities by [KM98, Corollary 5.24]. Now $f^*\sigma \in H^0\big(X', \Omega^{[2]}_{X'}\big)$ is a symplectic form in codimension 1 as f is étale in codimension 1. The claim follows. $\square$

Proof of Proposition B.1(ii). We prove the second item here. As a finite quotient of a smooth variety X, the space $\bar{X}$ is certainly $\mathbb{Q}$-factorial and has quotient singularities. Fix a point $x \in X$, and let $\bar{x} = p(x)$. According to the Chevalley–Shephard–Todd theorem, the quotient $\bar{X}$ is smooth at $\bar{x}$ if and only if the stabilizer group G_x acting on the tangent space T_xX is generated by pseudoreflections (that is, linear automorphisms on T_xX with codimension 1 fixed loci). If $x \in X$ has a nontrivial stabilizer G_x, any nontrivial automorphism $f \in G_x$ is symplectic so has fixed locus of codimension at least 2 . This means that G_x cannot be generated by pseudoreflections. Therefore, $\bar{X}$ is singular at $\bar{x}$. If we further assume $\operatorname{codim} X^f \geqslant 4$ for all nontrivial $f \in G$, then $\operatorname{codim} \bar{X}_{\mathrm{sing}} \geqslant 4$ and $\bar{X}$ becomes Namikawa symplectic.

To prove that $\bar{X}$ is irreducible symplectic, we follow the argument of Matsushita [Mat15, Lemma 2.2]. Let $f\colon \bar{Y} \to \bar{X}$ be an arbitrary finite quasi-étale morphism. Consider the diagram

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "q"] & X \arrow[d, "p"] \\ \bar{Y} \arrow[r, "f"] & \bar{X}, \end{tikzcd}$$

where Y is the normalization of the fiber product $X \times_{\bar{X}} \bar{Y}$. We claim that g and q are finite quasi-étale. The finiteness is clear, so we concentrate on their quasi-étaleness. Notice that the quasi-étale property is stable under base change, so we need to prove that the normalization in this case is quasi-étale. But notice that X is smooth and f is quasi-étale, so that $X \times_{\bar{X}} \bar{Y}$ is smooth in codimension 1. Hence its normalization is in fact an isomorphism in codimension 1. This proves that g and q are quasi-étale.

Now X is smooth, Y is normal, and $g\colon Y \to X$ is finite quasi-étale. By the Zariski–Nagata purity theorem of branch loci (see, for example, [Sta24, Tag 0BMB]), this forces g to be étale. The hyper-Kähler manifold X is simply connected, so this means that Y must be a disjoint union of several isomorphic copies of X. Let us fix a connected component Y_0 of Y. It is a hyper-Kähler manifold isomorphic to X.

Consider the morphism q restricted to the connected component $q\colon Y_0 \to \bar{Y}$. It is a finite quasi-étale morphism, so in particular it is surjective. Note that the target $\bar{Y}$ is canonical (Lemma B.4), so [GKKP11, Theorem 4.3] guarantees the existence of a reflexive pullback $q^*\colon H^0\big(\bar{Y}, \Omega^{[k]}_{\bar{Y}}\big) \to H^0\big(Y_0, \Omega^{[k]}_{Y_0}\big)$. Since q is quasi-étale, this morphism is injective. But recall that $Y_0 \cong X$ is a hyper-Kähler manifold, so this forces $\bar{Y}$ to satisfy the dimension condition of the definition of irreducible symplectic varieties. This proves that $\bar{X}$ is an irreducible symplectic variety. □

Proof of Proposition B.2. The following sequence of identities proves that $q_{\bar{X}}$ is the Beauville–Bogomolov–Fujiki form with Fujiki constant $c_{\bar{X}} = c_X/|G|$:

$$\int_{\bar{X}} x^{2n} = \frac{1}{|G|}\int_X (p^*x)^{2n} = \frac{c_X}{|G|}\cdot\frac{(2n)!}{2^n\cdot n!}\cdot q_X(p^*x)^n = \frac{c_X}{|G|}\cdot\frac{(2n)!}{2^n\cdot n!}\cdot q_{\bar{X}}(x)^n\,.$$

Since $\bar{X}$ is a compact Kähler orbifold, its rational singular cohomology admits a pure Hodge structure [PS08, Theorem 2.43], and $p^*\colon H^*\big(\bar{X}, \mathbb{Q}\big) \to H^*(X, \mathbb{Q})$ is an injective Hodge structure homomorphism with image $H^*(X, \mathbb{Q})^G$. In particular, p^* is an isomorphism in degree 2.

To prove that $H^*\big(\bar{X}, \mathbb{Q}\big) = H^*(X, \mathbb{Q})^G$ is closed under the $\mathfrak{g}$-action, it is enough to prove that the G-action and $\mathfrak{g}$-action on $H^*(X, \mathbb{Q})$ commute. Recall that the LLV structure is invariant under homeomorphisms. In other words, if $f\colon X_1 \to X_2$ is a homeomorphism between two compact hyper-Kähler manifolds, then we have

$$f^*(L_x(\xi)) = L_{f^*x}(f^*\xi)\,,\quad f^*(\Lambda_x(\xi)) = \Lambda_{f^*x}(f^*\xi)\quad \text{for } \xi \in H^*(X_2, \mathbb{Q})$$

for any $x \in H^2(X_2, \mathbb{Q})$ with $q_{X_2}(x) \neq 0$. Here L_x and Λ_x are Lefschetz and inverse Lefschetz operators associated with x. If we set $X_1 = X_2 = X$ and let $f \in G$ be an H^2-trivial automorphism, then this means that f^* commutes with the operators L_x and Λ_x. That is, G commutes with $\mathfrak{g}$.

To obtain the results about the Mumford–Tate algebras, one imitates the method used in [GKLR22, § 2] and deduces $f \in \mathfrak{g}$ for f a Weil operator on the cohomology $H^*\big(\bar{X}, \mathbb{Q}\big)$ (which is the restriction of Weil operator on $H^*(X, \mathbb{Q})$). This proves that all the special Mumford–Tate algebras of nonzero $H^k\big(\bar{X}, \mathbb{Q}\big)$ are the same and even the same result for that of $H^2(X, \mathbb{Q})$. □

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References

AR21 A. Abasheva and V. Rogov, *Shafarevich-Tate groups of holomorphic Lagrangian fibrations*, Math. Z. **311** (2025), no. 1, Paper No. 4; doi:10.1007/s00209-025-03776-w.

AF16 D. Arinkin and R. Fedorov, *Partial Fourier-Mukai transform for integrable systems with applications to Hitchin fibration*, Duke Math. J. **165** (2016), no. 15, 2991–3042; doi:10.1215/00127094-3645223.

BL22 B. Bakker and C. Lehn, *The global moduli theory of symplectic varieties*, J. reine angew. Math. **790** (2022), 223–265; doi:10.1515/crelle-2022-0033.

Bar87 W. Barth, *Abelian surfaces with* $(1,2)$*-polarization*, Algebraic geometry (Sendai, 1985), Adv. Stud. Pure Math., vol. 10 (North-Holland, Amsterdam, 1987), 41–84; doi:10.2969/aspm/01010041.

Bea83a A. Beauville, *Some remarks on Kähler manifolds with* $c_1 = 0$, Classification of algebraic and analytic manifolds (Katata, 1982), Progr. Math., vol. 39 (Birkhäuser, Boston, MA, 1983), 1–26.

Bea83b ———, *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Differential Geom. **18** (1983), no. 4, 755–782; doi:10.4310/jdg/1214438181.

BL90 C. Birkenhake and H. Lange, *Symmetric theta-structures*, Manuscripta Math. **70** (1990), no. 1, 67–91; doi:10.1007/BF02568362.

BL04 ———, *Complex abelian varieties*, 2nd ed., Grundlehren Math. Wiss., vol. 302 (Springer-Verlag, Berlin, 2004); doi:10.1007/978-3-662-06307-1.

BNS11 S. Boissière, M. Nieper-Wißkirchen, and A. Sarti, *Higher dimensional Enriques varieties and automorphisms of generalized Kummer varieties*, J. Math. Pures Appl. (9) **95** (2011), no. 5, 553–563; doi:10.1016/j.matpur.2010.12.003.

BLR90 S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models*, Ergeb. Math. Grenzgeb. (3), vol. 21 (Springer, Berlin, 1990); doi:10.1007/978-3-642-51438-8.

Bri11 M. Brion, *On automorphism groups of fiber bundles*, Publ. Mat. Urug. **12** (2011), 39–66.

Bri18 ———, *Linearization of algebraic group actions*, in *Handbook of group actions. Vol. IV*, Adv. Lect. Math. (ALM), vol. 41 (Int. Press, Somerville, MA, 2018), 291–340.

Bül20 T.-H. Bülles, *Motives of moduli spaces on K3 surfaces and of special cubic fourfolds*, Manuscripta Math. **161** (2020), no. 1-2, 109–124; doi:10.1007/s00229-018-1086-0.

Cam06 F. Campana, *Isotrivialité de certaines familles kählériennes de variétés non projectives*, Math. Z. **252** (2006), no. 1, 147–156; doi:10.1007/s00209-005-0851-4.

Del71 P. Deligne, *Théorie de Hodge. II*, Inst. Hautes Études Sci. Publ. Math. **40** (1971), 5–57; doi:10.1007/BF02684692.

Del72 ———, *La conjecture de Weil pour les surfaces* $K3$, Invent. Math. **15** (1972), 206–226; doi:10.1007/BF01404126.

Dol03 I. Dolgachev, *Lectures on invariant theory*, London Math. Soc. Lecture Note Ser., vol. 296 (Cambridge Univ. Press, Cambridge, 2003); doi:10.1017/CBO9780511615436.

FC90 G. Faltings and C. L. Chai, *Degeneration of abelian varieties*, Ergeb. Math. Grenzgeb. (3), vol. 22 (Springer-Verlag, Berlin, 1990); doi:10.1007/978-3-662-02632-8.

Fuj83 A. Fujiki, *On primitively symplectic compact Kähler* V*-manifolds of dimension four*, Classification of algebraic and analytic manifolds (Katata, 1982), Progr. Math., vol. 39 (Birkhäuser Boston, Boston, MA, 1983), 71–250.

vGV16 B. van Geemen and C. Voisin, *On a conjecture of Matsushita*, Int. Math. Res. Not. IMRN **10** (2016), 3111–3123; doi:10.1093/imrn/rnv230.

GGK19 D. Greb, H. Guenancia, and S. Kebekus, *Klt varieties with trivial canonical class: holonomy, differential forms, and fundamental groups*, Geom. Topol. **23** (2019), no. 4, 2051–2124; doi:10.2140/gt.2019.23.2051.

GKKP11 D. Greb, S. Kebekus, S. Kovács, and T. Peternell, *Differential forms on log canonical spaces*, Publ. Math. Inst. Hautes Études Sci. **114** (2011), 87–169; doi:10.1007/s10240-011-0036-0.

GKP16 D. Greb, S. Kebekus, and T. Peternell, *Singular spaces with trivial canonical class*, Minimal models and extremal rays (Kyoto, 2011), Adv. Stud. Pure Math., vol. 70 (Math. Soc. Japan, Tokyo, 2016), 67–113; doi:10.2969/aspm/07010067.

GL14 D. Greb and C. Lehn, *Base manifolds for Lagrangian fibrations on hyperkähler manifolds*, Int. Math. Res. Not. IMRN **19** (2014), 5483–5487; doi:10.1093/imrn/rnt133.

GKLR22 M. Green, Y.-J. Kim, R. Laza, and C. Robles, *The LLV decomposition of hyper-Kähler cohomology (the known cases and the general conjectural behavior)*, Math. Ann. **382** (2022), no. 3-4, 1517–1590; doi:10.1007/s00208-021-02238-y.

GTZ13 M. Gross, V. Tosatti, and Y. Zhang, *Collapsing of abelian fibered Calabi–Yau manifolds*, Duke Math. J. **162** (2013), no. 3, 517–551; doi:10.1215/00127094-2019703.

HT13 B. Hassett and Y. Tschinkel, *Hodge theory and Lagrangian planes on generalized Kummer fourfolds*, Mosc. Math. J. **13** (2013), no. 1, 33–56; doi:10.17323/1609-4514-2013-13-1-33-56.

Huy99 D. Huybrechts, *Compact hyper-Kähler manifolds: basic results*, Invent. Math. **135** (1999), no. 1, 63–113; doi:10.1007/s002220050280.

Huy16 ______, *Lectures on $K3$ surfaces*, Cambridge Stud. Adv. Math., vol. 158 (Cambridge Univ. Press, 2016); doi:10.1017/CBO9781316594193.

Hwa08 J.-M. Hwang, *Base manifolds for fibrations of projective irreducible symplectic manifolds*, Invent. Math. **174** (2008), no. 3, 625–644; doi:10.1007/s00222-008-0143-9.

Ive86 B. Iversen, *Cohomology of sheaves*, Universitext (Springer-Verlag, Berlin, 1986); doi:10.1007/978-3-642-82783-9.

Kal06 D. Kaledin, *Symplectic singularities from the Poisson point of view*, J. reine angew. Math. **600** (2006), 135–156; doi:10.1515/CRELLE.2006.089.

KLS06 D. Kaledin, M. Lehn, and C. Sorger, *Singular symplectic moduli spaces*, Invent. Math. **164** (2006), no. 3, 591–614; doi:10.1007/s00222-005-0484-6.

Kaw85 Y. Kawamata, *Minimal models and the Kodaira dimension of algebraic fiber spaces*, J. reine angew. Math. **363** (1985), 1–46; doi:10.1515/crll.1985.363.1.

Kol96 J. Kollár, *Rational curves on algebraic varieties*, Ergeb. Math. Grenzgeb. (3), vol. 32 (Springer-Verlag, Berlin, 1996); doi:10.1007/978-3-662-03276-3.

KM98 J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Math., vol. 134 (Cambridge Univ. Press, Cambridge, 1998); doi:10.1017/CBO9780511662560.

Laz04 R. Lazarsfeld, *Positivity in algebraic geometry. I*, Ergeb. Math. Grenzgeb. (3), vol. 48 (Springer-Verlag, Berlin, 2004); doi:10.1007/978-3-642-18808-4.

LeP93 J. Le Potier, *Faisceaux semi-stables de dimension* 1 *sur le plan projectif*, Rev. Roumaine Math. Pures Appl. **38** (1993), no. 7-8, 635–678.

LL97 E. Looijenga and V. A. Lunts, *A Lie algebra attached to a projective variety*, Invent. Math. **129** (1997), no. 2, 361–412; doi:10.1007/s002220050166.

Mar14 E. Markman, *Lagrangian fibrations of holomorphic-symplectic varieties of $K3^{[n]}$-type*, in *Algebraic and complex geometry*, Springer Proc. Math. Stat., vol. 71 (Springer, Cham, 2014), 241–283; doi:10.1007/978-3-319-05404-9_10.

MT07 D. Markushevich and A. S. Tikhomirov, *New symplectic V-manifolds of dimension four via the relative compactified Prymian*, Internat. J. Math. **18** (2007), no. 10, 1187–1224; doi:10.1142/S0129167X07004503.

Mat15 D. Matsushita, *On base manifolds of Lagrangian fibrations*, Sci. China Math. **58** (2015), no. 3, 531–542; doi:10.1007/s11425-014-4927-7.

Mat16 ______, *On deformations of Lagrangian fibrations*, $K3$ surfaces and their moduli, Progr. Math., vol. 315 (Birkhäuser/Springer, Cham, 2016), 237–243; doi:10.1007/978-3-319-29959-4_9.

Men14 G. Menet, *Duality for relative Prymians associated to K3 double covers of del Pezzo surfaces of degree 2*, Math. Z. **277** (2014), no. 3-4, 893–907; doi:10.1007/s00209-014-1283-9.

Men20 ______, *Global Torelli theorem for irreducible symplectic orbifolds*, J. Math. Pures Appl. (9) **137** (2020), 213–237; doi:10.1016/j.matpur.2020.03.010.

Mil80 J. S. Milne, *Étale cohomology*, Princeton Math. Ser., vol. 33 (Princeton Univ. Press, Princeton, NJ, 1980).

MO22 G. Mongardi and C. Onorati, *Birational geometry of irreducible holomorphic symplectic tenfolds of O'Grady type*, Math. Z. **300** (2022), no. 4, 3497–3526; doi:10.1007/s00209-021-02966-6.

MR21 G. Mongardi and A. Rapagnetta, *Monodromy and birational geometry of O'Grady's sixfolds*, J. Math. Pures Appl. (9) **146** (2021), 31–68; doi:10.1016/j.matpur.2020.12.006.

MW17 G. Mongardi and M. Wandel, *Automorphisms of O'Grady's manifolds acting trivially on cohomology*, Algebr. Geom. **4** (2017), no. 1, 104–119; doi:10.14231/AG-2017-005.

Mor18 L. Moret-Bailly, response to J. Rosen, *To what extent does a torsor determine a group*, version: 2018-01-06, https://mathoverflow.net/q/290094.

MFK94 D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 3rd ed., Ergeb. Math. Grenzgeb. (2), vol. 34 (Springer-Verlag, Berlin, 1994); doi:10.1007/978-3-319-65907-7.

Nam01 Y. Namikawa, *A note on symplectic singularities*, 2001, arXiv:math/0101028.

Nam06 ———, *On deformations of $\mathbb{Q}$-factorial symplectic varieties*, J. reine angew. Math. **599** (2006), 97–110; doi:10.1515/CRELLE.2006.079.

Ogu09 K. Oguiso, *Picard number of the generic fiber of an abelian fibered hyperkähler manifold*, Math. Ann. **344** (2009), no. 4, 929–937; doi:10.1007/s00208-009-0335-7.

Ogu20 ———, *No cohomologically trivial nontrivial automorphism of generalized Kummer manifolds*, Nagoya Math. J. **239** (2020), 110–122; doi:10.1017/nmj.2018.29.

PS08 C. A. M. Peters and J. H. M. Steenbrink, *Mixed Hodge structures*, Ergeb. Math. Grenzgeb. (3), vol. 52 (Springer-Verlag, Berlin, 2008); doi:10.1007/978-3-540-77017-6.

Rap07 A. Rapagnetta, *Topological invariants of O'Grady's six dimensional irreducible symplectic variety*, Math. Z. **256** (2007), no. 1, 1–34; doi:10.1007/s00209-006-0022-2.

Rap08 ———, *On the Beauville form of the known irreducible symplectic varieties*, Math. Ann. **340** (2008), no. 1, 77–95; doi:10.1007/s00208-007-0139-6.

Sac23 G. Saccà, *Birational geometry of the intermediate Jacobian fibration of a cubic fourfold* (with an appendix by C. Voisin), Geom. Topol. **27** (2023), no. 4, 1479–1538; doi:10.2140/gt.2023.27.1479.

Saw03 J. Sawon, *Abelian fibred holomorphic symplectic manifolds*, Turkish J. Math. **27** (2003), no. 1, 197–230.

Saw04 ———, *Derived equivalence of holomorphic symplectic manifolds*, in *Algebraic structures and moduli spaces*, CRM Proc. Lecture Notes, vol. 38 (Amer. Math. Soc., Providence, RI, 2004), 193–211; doi:10.1090/crmp/038/09.

Saw09 ———, *Deformations of holomorphic Lagrangian fibrations*, Proc. Amer. Math. Soc. **137** (2009), no. 1, 279–285; doi:10.1090/S0002-9939-08-09473-2.

Saw20 ———, *Lagrangian fibrations by Prym varieties*, Mat. Contemp. **47** (2020), 182–227; doi:10.21711/231766362020/rmc479.

Sch20 M. Schwald, *Fujiki relations and fibrations of irreducible symplectic varieties*, Épijournal Géom. Algébrique **4** (2020), article no. 7; doi:10.46298/epiga.2020.volume4.4557.

Ser79 J.-P. Serre, *Local fields*, Grad. Texts in Math., vol. 67 (Springer-Verlag, New York-Berlin, 1979); doi:10.1007/978-1-4757-5673-9.

SY22 J. Shen and Q. Yin, *Topology of Lagrangian fibrations and Hodge theory of hyper-Kähler manifolds* (with Appendix B by C. Voisin), Duke Math. J. **171** (2022), no. 1, 209–241; doi:10.1215/00127094-2021-0010.

Sta24 The Stacks Project Authors, *The Stacks Project*, 2024, https://stacks.math.columbia.edu.

SYZ96 A. Strominger, S.-T. Yau, and E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Phys. B **479** (1996), no. 1-2, 243–259; doi:10.1016/0550-3213(96)00434-8.

Ver95 M. Verbitsky, *Cohomology of compact hyperkähler manifolds*, Ph.D. thesis, 1995, Harvard University, arXiv:alg-geom/9501001.

Ver99 ———, *Mirror symmetry for hyper-Kähler manifolds*, Mirror symmetry, III (Montreal, PQ, 1995), AMS/IP Stud. Adv. Math., vol. 10 (Amer. Math. Soc., Providence, RI, 1999), 115–156; doi:10.1090/amsip/010/04.

Voi92 C. Voisin, *Sur la stabilité des sous-variétés lagrangiennes des variétés symplectiques holomorphes*, Complex projective geometry (Trieste, 1989/Bergen, 1989), London Math. Soc. Lecture Note Ser. (Cambridge Univ. Press, 1992), 294–303; doi:10.1017/CBO9780511662652.022.

Voi18 ———, *Torsion points of sections of Lagrangian torus fibrations and the Chow ring of hyper-Kähler manifolds*, in *Geometry of moduli*, Abel Symp., vol. 14 (Springer, Cham, 2018), 295–326; doi:10.1007/978-3-319-94881-2_10.

Wie16 B. Wieneck, *On polarization types of Lagrangian fibrations*, Manuscripta Math. **151** (2016), no. 3-4, 305–327; doi:10.1007/s00229-016-0845-z.

Wie18 ———, *Monodromy invariants and polarization types of generalized Kummer fibrations*, Math. Z. **290** (2018), no. 1-2, 347–378; doi:10.1007/s00209-017-2020-y.

Yos01 K. Yoshioka, *Moduli spaces of stable sheaves on abelian surfaces*, Math. Ann. **321** (2001), no. 4, 817–884; doi:10.1007/s002080100255.

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