Algebraic Geometry 12 (5) (2025) 601–660
doi:10.14231/AG-2025-017

On the Kodaira dimension of the moduli of deformation generalised Kummer varieties

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Dedicated to my parents

Abstract

We prove general-type results for orthogonal modular varieties associated with moduli spaces of compact hyperkähler manifolds of deformation generalised Kummer type (*'deformation generalised Kummer varieties'*). More precisely, we consider moduli spaces of deformation generalised Kummer fourfolds with split polarisation of degree $2d$. Our main result is that when d is prime or $2d$ is square-free, then the associated modular varieties are of general type when d exceeds certain bounds. As a corollary, we conclude that the corresponding moduli spaces are also of general type.

1. Introduction

1.1 Overview

Let $\mathrm{O}^+(L)$ be the kernel of the real spinor norm on $\mathrm{O}(L)$, where L is an even integral lattice of signature $(2, n)$. If $\mathcal{D}_L$ is the component of the Hermitian symmetric space

$$\Omega_L = \{[x] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (x, x) = 0,\ (x, \overline{x}) > 0\}$$

preserved by $\mathrm{O}^+(L)$ and $\Gamma \subset \mathrm{O}^+(L)$ is a subgroup of finite index, then the quotient $\mathcal{F}(\Gamma) = \mathcal{D}_L/\Gamma$ is known as an *orthogonal modular variety* (or an *orthogonal modular n-fold*). By the results of Baily and Borel, orthogonal modular varieties are irreducible normal quasi-projective varieties [BB66]. A classical problem in algebraic geometry is to determine their Kodaira dimension.

One expects most orthogonal modular varieties to be of general type. This is typically proved by constructing pluricanonical forms on a smooth projective model of $\mathcal{F}(\Gamma)$ using modular (that is, automorphic) forms for Γ and a technique known as the *'low-weight cusp form trick'*. The exact details of this approach depend on n. A general method is known for $n \geqslant 9$ which was initially developed for $n = 19$, see [Kon93, GHS07b], in connection with the moduli of K3 surfaces, finally culminating in the results of Ma [Ma18b] for $n = 17$ and $n > 20$. However, a naive application of these methods fails in small dimension due to problems related to singularities and the existence

Received 26 August 2020, accepted in final form 1 May 2024.
2020 Mathematics Subject Classification 11F03, 14J42, 14J15.
Keywords: orthogonal modular variety, general type, deformation generalised Kummer variety, modular forms, moduli spaces.

The author acknowledges partial support from the grant SFB/TR 45 '*Periods, Moduli Spaces and Arithmetic of Algebraic Varieties*' of the DFG (German Research Foundation).

of special modular forms. The purpose of this paper is to introduce techniques for dealing with these problems when $n = 4$, in the case of orthogonal modular varieties associated with moduli spaces of compact hyperkähler manifolds of deformation generalised Kummer type (*'deformation generalised Kummer varieties'*) with split polarisation. Our main result (Theorem 11.3) states that when the degree of polarisation exceeds a certain bound, the corresponding modular varieties are of general type, subject to arithmetic constraints on the degree. As a corollary, this implies that the associated moduli spaces are also of general type. To the best of our knowledge, these are the first general-type results for orthogonal modular fourfolds and moduli spaces of deformation generalised Kummer varieties.

1.2 Structure of the paper

In § 2, we introduce moduli spaces of deformation generalised Kummer varieties and the modular group Γ. In § 3, we introduce notation and definitions related to toroidal compactifications, modular forms and the low-weight cusp form trick. In § 4, we study singularities in orthogonal modular fourfolds and classify torsion in Γ by considering p-elementary lattices. In § 5, we study the branch divisor of $\mathcal{F}(\Gamma)$. In § 6, we obtain sufficient conditions for modular forms to define pluricanonical forms on $\mathcal{F}(\Gamma)$. In § 7, we prove the existence of low-weight cusp forms for Γ by studying a dimension formula for vector-valued modular forms and applying Borcherds–Gritsenko lifting. In § 8 and § 9, we calculate and bound Hirzebruch–Mumford volumes, which underpin the obstruction calculations of § 10. We conclude with general-type results in § 11.

2. Moduli of deformation generalised Kummer varieties

2.1 Elementary results on lattices and orthogonal groups

We start by introducing elementary results on lattices and orthogonal groups. Standard references for lattices are [CS99] and [Nik80]. Unless otherwise stated, all lattices are assumed to be even and integral.

Let L be a lattice. The *signature* of L is defined as the pair (m_+, m_-), where m_+ and m_- are the numbers of positive and negative terms in a diagonalisation of the Gram matrix of a basis of L, respectively. We will use $L_1 \oplus L_2$ to denote an orthogonal direct sum of lattices or finite quadratic forms L_1 and L_2, mL to denote the orthogonal direct sum of m copies of L and $L(n)$ to denote the lattice obtained by multiplying the quadratic form of L by n. We will use U to denote the hyperbolic plane and define a *standard basis* of U as a basis for U with Gram matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We will use $\langle x \rangle$ to denote the rank 1 lattice generated by a vector v of length $v^2 := (v, v) = x$, and we assume that all root lattices are negative definite.

Let $L^\vee := \operatorname{Hom}(L, \mathbb{Z})$ denote the *dual lattice* of L. For $u \in L$, the *divisor* $\operatorname{div}(u)$ of u in L is defined as the positive generator of the ideal (u, L), and we let $u^* := u/\operatorname{div}(u)$. The *discriminant group* $D(L)$ of L, defined by $D(L) := L^\vee/L$, inherits a $\mathbb{Q}/2\mathbb{Z}$-valued finite quadratic form q_L from L (the *discriminant form* of L), and the tuple $\operatorname{gen}(L) := (m_+, m_-, q_L)$ encodes the genus of L, which we also denote by $\operatorname{gen}(L)$; see [Nik80].

If $\mathrm{O}(L)$ is the orthogonal group of L, we use $\widetilde{\mathrm{O}}(L)$ to denote the *stable orthogonal group*

$$\widetilde{\mathrm{O}}(L) := \left\{ g \in \mathrm{O}(L) \mid gv = v + L \ \forall v \in L^\vee \right\},$$

and, for $G \subset \mathrm{O}(L)$, we let $\widetilde{G} := G \cap \widetilde{\mathrm{O}}(L)$. We will often use the property that if $S \subset L$ is a sublattice of L, then $\widetilde{\mathrm{O}}(S) \subset \widetilde{\mathrm{O}}(L)$; see [GHS13, Lemma 7.1]. For $g \in \mathrm{O}(L \otimes \mathbb{R})$ factored as a product of reflections $g = \sigma_{v_1} \cdots \sigma_{v_m}$, see [Cas78], where

$$\sigma_v \colon x \longmapsto x - \frac{2(x,v)v}{(v,v)} \in \mathrm{O}(L \otimes \mathbb{R}) \tag{2.1}$$

for $v \in L \otimes \mathbb{R}$, the *real spinor norm* [Kne02]

$$\mathrm{sn}_{\mathbb{R}} \colon \mathrm{O}(L \otimes \mathbb{R}) \longrightarrow \mathbb{R}^*/(\mathbb{R}^*)^2$$

is defined by

$$\mathrm{sn}_{\mathbb{R}}(g) = \left(\frac{-(v_1,v_1)}{2}\right) \cdots \left(\frac{-(v_m,v_m)}{2}\right) \in \mathbb{R}/(\mathbb{R}^*)^2 \,.$$

We will use a superscript $+$ to denote the intersection of a subgroup of $\mathrm{O}(L \otimes \mathbb{R})$ with the kernel of $\mathrm{sn}_{\mathbb{R}}$ (for example $\mathrm{O}^+(L)$, $\widetilde{\mathrm{O}}^+(L)$).

A sublattice $S \subset L$ is said to be *primitive* if L/S is torsion-free. Two primitive lattice embeddings $S_1 \hookrightarrow L_1$ and $S_2 \hookrightarrow L_2$ are said to be isomorphic if there exists a $g \in \mathrm{Iso}(L_1, L_2)$ restricting to an isomorphism between S_1 and S_2, see [Nik80], and if $L_1 = L_2$ and $\Gamma \subset \mathrm{O}(L_1)$, we say that S_1 and S_2 are *equivalent under* Γ if g can be taken in Γ. We will often determine the $\widetilde{\mathrm{SO}}^+(L)$ orbits of primitive vectors in L using the *Eichler criterion* (see [Eic52, § 10] and [GHS09, Proposition 3.3]), which states that if $u, v \in L$ are primitive and $L \cong 2U \oplus L_0$ for a lattice L_0, then u and v are equivalent under $\widetilde{\mathrm{SO}}^+(L)$ if and only if $u^2 = v^2$ and $u^* \equiv v^* \bmod L$.

2.2 Deformation generalised Kummer varieties

Our overview follows [GHS13]. If A is an abelian surface and $A^{[n+1]}$ is the Hilbert scheme parametrising $(n+1)$ points on A, there is a natural morphism

$$p \colon A^{[n+1]} \longrightarrow A$$

defined by addition on A. Deformations of the fibre $X := p^{-1}(0)$ are known as *deformation generalised Kummer varieties* and are examples of compact hyperkähler (irreducible holomorphic symplectic) manifolds [Bea83, Huy03] (that is, X is a simply connected compact Kähler manifold, and $H^0(X, \Omega^2_X)$ is generated by an everywhere non-degenerate holomorphic 2-form). The second integral cohomology of a compact hyperkähler manifold can be endowed with the *Beauville–Bogomolov lattice* [Bea83, Fuj87]. For deformation generalised Kummer varieties, the Beauville–Bogomolov lattice is given by [Rap08]

$$H^2(X, \mathbb{Z}) = 3U \oplus \langle -2(n+1) \rangle$$

and will be denoted by M when viewed as an abstract lattice.

To define moduli, we first fix some discrete invariants (further details can be found in [GHS13]). Let X be a deformation generalised Kummer variety of dimension $2n$. A choice of ample line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ defines a *polarisation* for X, and by taking the first Chern class, we obtain a vector $h \in M$, where $h := c_1(\mathcal{L}) \in H^2(X, \mathbb{Z}) \cong M$, and a lattice $L := h^\perp \subset M$ of signature $(2,4)$. We assume that all polarisations are primitive (that is, h is primitive in M). The *degree* $2d$ of $\mathcal{L}$ is defined as the length $2d := h^2$, the *polarisation type* $[h]$ of $\mathcal{L}$ is defined as the $\mathrm{O}(M)$-orbit of h, and the *numerical type* of $(X, \mathcal{L})$ is given by the tuple $N = (2n, L, [h])$. We will only consider polarisations of *split type*, which are those satisfying $\mathrm{div}(h) = 1$. In such a case, the lattice L is given by [Daw15, Corollary 4.0.3]

$$L = 2U \oplus \langle -2d \rangle \oplus \langle -2(n+1) \rangle \,, \tag{2.2}$$

which perhaps suggests the etymology of the term. By the Eichler criterion, the numerical type of a split polarisation is uniquely determined by the dimension $2n$ and the degree $2d$.

By the results of Viehweg [Vie95], Matsusaka's big theorem [Mat72] and a theorem of Kollár and Matsusaka [KM83], there exists a geometric invariant theory quotient $\mathcal{M}_N$ parametrising compact hyperkähler manifolds of fixed numerical type N. By the Hodge-theoretic Torelli theorem [Huy12, Mar11, Ver13], there is a holomorphic period map

$$\varphi\colon \mathcal{M}_N \longrightarrow \mathcal{D}_L/\operatorname{O}^+(L,h)\,,$$

where

$$\operatorname{O}^+(L,h) := \{g \in \operatorname{O}^+(L) \mid gh = h\}\,.$$

As $\mathcal{M}_N$ and $\mathcal{D}_L/\operatorname{O}^+(M,h)$ are quasi-projective, the map φ is a morphism of quasi-projective varieties [Bor72]. By replacing $\operatorname{O}^+(M,h)$ with

$$\operatorname{Mon}^2(M,h) := \operatorname{Mon}^2(M) \cap \operatorname{O}^+(M,h)\,,$$

where $\operatorname{Mon}^2(M)$ is the group of Markman's monodromy operators [Mar04, Mar08, Mar10], we obtain the stronger result that φ lifts to an open immersion $\widetilde{\varphi}\colon \mathcal{M}_N \to \mathcal{D}_L/\operatorname{Mon}^2(M,h)$; see [GHS13, Theorem 3.10].

2.3 The modular group Γ

From now on, as $\operatorname{O}^+(M,h)$ fixes $h \in M$, we regard $\operatorname{Mon}^2(M,h)$ as a subgroup of $\operatorname{O}(L)$ and use Γ to denote the image of $\operatorname{Mon}^2(M,h)$ in $\operatorname{O}^+(L)$.

Proposition 2.1. *Suppose that $h \in M$ is defined by a polarisation of split type, and let $2d := h^2 > 1$. Then,*

$$\Gamma = \{g \in \operatorname{O}^+(L) \mid gv^* = v^* + L,\ gw^* = \det(g)w^* + L\}\,, \tag{2.3}$$

where, if v and w are the respective generators of the $\langle -2d\rangle$ and $\langle -2(n+1)\rangle$ factors of L, then $v^*, w^* \in L^\vee$ are given by $v^* = (-1/2d)v$ and $w^* = (-1/2(n+1))w$. Furthermore, if $g \in \Gamma$, then $\chi_g(1) \equiv 0 \bmod 2d$ and $\chi_g(\det g) \equiv 0 \bmod 2(n+1)$, where χ_g is the characteristic polynomial of g.

Proof. Let $\{e_1, f_1, e_2, f_2, e_3, f_3, w\}$ be a basis for M, where $\{e_i, f_i\}$ is a standard basis for the ith copy of U in M and w generates the $\langle -2(n+1)\rangle$ factor of M. As $\operatorname{div}(h) = 1$, then by the Eichler criterion, we may assume that h is given by $e_3 + df_3 \in M$; we take $\{e_1, f_1, e_2, f_2, v, w\}$ as a basis for the sublattice $L \subset M$, where $v = e_3 - df_3$. If $\mathcal{W}$ is the group

$$\mathcal{W} = \{g \in \operatorname{O}^+(M) \mid g_{|D(M)} = \pm\operatorname{id}\}$$

and $\chi\colon \mathcal{W} \to \{\pm 1\}$ is the character defined by the action of $\mathcal{W}$ on $D(M)$, then by [Mon16],

$$\begin{aligned}\operatorname{Mon}^2(M) &= \operatorname{Ker}(\det\circ\chi) \\ &= \{g \in \operatorname{O}^+(M) \mid g(w^*) \equiv \det(g)w^* \bmod M\}\,.\end{aligned} \tag{2.4}$$

We now characterise the inclusion $\operatorname{O}^+(L) \subset \operatorname{O}^+(M,h)$. Following [Nik80, § 5], there is a series of finite abelian groups

$$M/(L \oplus \langle h\rangle) \subset (L^\vee/L) \oplus (\langle h\rangle^\vee/\langle h\rangle) = D(L) \oplus D(\langle h\rangle) \tag{2.5}$$

corresponding to the series of lattice inclusions

$$L \oplus \langle h\rangle \subset M \subset M^\vee \subset L^\vee \oplus \langle h\rangle^\vee\,.$$

If $H = M/(L \oplus \langle h \rangle)$, let P_L and P_h denote the projections $P_L\colon H \to D(L)$ and $P_h\colon H \to D(\langle h \rangle)$ in (2.5); we define $H_h := P_h(H)$, $H_L := P_L(H)$ and the map $\gamma^M_{h,L}$ by

$$\gamma^M_{L,h} := P_h \circ P_L^{-1}\colon H_L \longrightarrow H_h\,.$$

By [Nik80, Corollary 1.5.2], the element $g \in \mathrm{O}^+(L)$ extends to an element of $\mathrm{O}^+(M,h)$ if and only if $\gamma^M_{L,h} = \gamma^M_{L,h} \circ \overline{g}$, where $\overline{g}$ denotes the image of g under the natural homomorphism $\mathrm{O}(L) \to \mathrm{O}(D(L))$. As $e_3 = \frac{1}{2}(h+v)$ and $f_3 = (1/2d)(h-v)$, then H is generated by the classes of e_3 and f_3. We calculate $P_L(e_3) \equiv \frac{1}{2}v \equiv -dv^* \bmod L$ and $P_L(f_3) \equiv -(1/2d)v \equiv v^* \bmod L$, implying $H_{M,L} = \langle v^* \rangle$. Therefore,

$$\gamma^M_{L,h}(v^*) = P_h\big(P_L^{-1}(v^*)\big) = P_h(f_3) = P_h\left(\frac{1}{2d}(h-v)\right) \equiv \frac{h}{2d} \equiv h^* \bmod \langle h \rangle\,,$$

implying that $g \in \mathrm{O}^+(L)$ extends to $\mathrm{O}^+(M,h)$ if and only if

$$h^* = \gamma^M_{L,h}(\overline{g}(v^*))\,. \tag{2.6}$$

By (2.4), if $g \in \mathrm{O}^+(L)$ extends to $\mathrm{Mon}^2(M,h)$, then $g(w^*) = \det(g)w^* \bmod M$, and as $(v^*, w^*) = 0$, we have

$$(v^*, w^*) = (gv^*, gw^*) = (gv^*, \det(g)w^* + M) = (gv^*, w^*) = 0 \bmod \mathbb{Z}\,.$$

Therefore, $gv^* = \alpha v^*$ for some $\alpha \in \mathbb{Z}$ taken modulo $2d$, and so $g \in \mathrm{O}^+(L)$ extends to $\mathrm{Mon}^2(M,h)$ if and only if $h^* \equiv \alpha h^* \bmod \langle h \rangle$. Therefore, by (2.6), we have $\alpha \equiv 1 \bmod 2d$, and the first part of the claim follows.

For the second part of the claim, as in [Bra95], we note that the matrix of g is of the form

$$\begin{pmatrix}
* & * & * & * & 2d* & 2(n+1)* \\
* & * & * & * & 2d* & 2(n+1)* \\
* & * & * & * & 2d* & 2(n+1)* \\
* & * & * & * & 2d* & 2(n+1)* \\
* & * & * & * & 1+2d* & 2(n+1)* \\
* & * & * & * & 2d* & \det g + 2(n+1)*
\end{pmatrix},$$

where $*$ denotes an element of $\mathbb{Z}$. $\square$

3. Toroidal compactifications

In order to prove Theorem 3.2, we need an explicit description of a toroidal compactification $\overline{\mathcal{F}}(\Gamma)$ of $\mathcal{F}(\Gamma)$. We let L denote a lattice of signature $(2,n)$, where $n \geqslant 2$, and we suppose that $\Gamma \subset \mathrm{O}^+(L)$ is a subgroup of finite index.

3.1 Construction of toroidal compactifications

Toroidal compactifications are defined fully in the monograph [AMRT10]. Our overview follows [Kon93], in which more details can be found. Let $\overline{\mathcal{D}}_L$ denote the closure of $\mathcal{D}_L$ in the *compact dual* $\check{\mathcal{D}}_L := \{[w] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (w,w) = 0\}$ of $\mathcal{D}_L$. A *boundary component* (or a *cusp*) F of $\mathcal{D}_L$ is defined as a maximal connected complex analytic subset of $\overline{\mathcal{D}}_L \backslash \mathcal{D}_L$. A boundary component F is said to be *rational* if the stabiliser $N(F) := \{g \in G_{\mathbb{R}} \mid g(F) \subset F\}$ is defined over $\mathbb{Q}$, where $G_{\mathbb{R}}$ is a connected component of $\mathrm{O}(L \otimes \mathbb{R})$. We let $W(F)$ denote the unipotent radical of $N(F)$ and let $U(F)$ denote the centre of $W(F)$. We will use a subscript $\mathbb{C}$ to denote the complexification of a subgroup of $G_{\mathbb{R}}$ and a subscript $\mathbb{Z}$ to denote the intersection of a group with Γ (for example $N(F)_{\mathbb{C}}$ and $U(F)_{\mathbb{Z}}$). The stabiliser $N(F)$ is a maximal parabolic subgroup of $G_{\mathbb{R}}$, and conversely, every maximal parabolic subgroup of $G_{\mathbb{R}}$ is the stabiliser of a boundary component, subject to the condition that $G_{\mathbb{R}}$ is simple (which is true in our setting; cf. [AMRT10, Proposition III.3.9]). As $\Gamma \subset \mathrm{O}(2,n)$, the maximal parabolic subgroups of Γ are precisely the stabilisers of totally isotropic primitive sublattices $E \subset L$, with $\operatorname{rank} E = 1$ and $\operatorname{rank} E = 2$ corresponding to boundary points and boundary curves, respectively. (For more details, see [AMRT10, BJ06, Pya69, Sat80, GHS13].) For each rational boundary component F, the group $U(F)$ contains a self-dual homogeneous cone $C(F)$; see [AMRT10, § III.4]. To construct a toroidal compactification $\overline{\mathcal{F}}(\Gamma)$ of $\mathcal{F}(\Gamma)$, we start by taking an admissible polyhedral decomposition $\{\sigma_i\}$ of $C(F)$ for each rational boundary component F; see [AMRT10, § II.4]. The fan $\{\sigma_i\}$ defines a toric variety $T(F)_{\{\sigma_i\}} \supset T(F)$, where $T(F)$ is the torus $T(F) := U(F)_{\mathbb{C}}/U(F)_{\mathbb{Z}}$. If $\mathcal{D}_L(F) := U(F)_{\mathbb{C}}.\mathcal{D}_L (\subset \check{\mathcal{D}}_L)$, there exists a holomorphic isomorphism $\mathcal{D}_L(F) \cong U(F)_{\mathbb{C}} \times \mathbb{C}^m \times F$, where $m = 0$ if $\dim F = 0$ and $m = n-2$ if $\dim F = 1$. We define $(\mathcal{D}_L/U(F)_{\mathbb{Z}})_{\{\sigma_i\}}$ as the interior of the closure of $\mathcal{D}_L/U(F)_{\mathbb{Z}}$ in $(\mathcal{D}(F)/U(F)_{\mathbb{Z}}) \times_{T(F)} T(F)_{\{\sigma_i\}}$. By [AMRT10, Theorem III.5.2], there exists a compact analytic space $\overline{\mathcal{F}}(\Gamma)$ and a morphism

$$\pi_F \colon (\mathcal{D}_L/U(F)_{\mathbb{Z}})_{\{\sigma_i\}} \longrightarrow \overline{\mathcal{F}}(\Gamma)\,.$$

The space $\overline{\mathcal{F}}(\Gamma)$ has (at most) finite quotient singularities, and the modular variety $\mathcal{F}(\Gamma)$ is Zariski open in $\overline{\mathcal{F}}(\Gamma)$. Each point of $\overline{\mathcal{F}}(\Gamma)$ is contained in the image of π_F or the projection $\pi\colon \mathcal{D}_L \to \mathcal{F}(\Gamma)$.

3.2 Coordinates for $\overline{\mathcal{F}}(\Gamma)$

We introduce coordinates for $\overline{\mathcal{F}}(\Gamma)$ in the neighbourhood of a boundary curve, following [Kon93, GHS07b, Sca87]. Let $E \subset L$ be a primitive totally isotropic sublattice corresponding to the boundary curve F. As in [GHS07b, Lemma 2.23], there exists a $\mathbb{Z}$-basis $\{e'_i\}_{i=1}^{n+2}$ for L such that $\{e'_1, e'_2\}$ and $\{e'_i\}_{i=1}^{n}$ define $\mathbb{Z}$-bases for E and $E^{\perp}$, respectively. The Gram matrix of $\{e'_i\}_{i=1}^{n+2}$ is given by

$$Q' = (e'_i, e'_j) = \begin{pmatrix} 0 & 0 & A \\ 0 & B & C \\ {}^{\tau}A & {}^{\tau}C & D \end{pmatrix}, \tag{3.1}$$

where

$$A = \begin{pmatrix} \delta & 0 \\ 0 & \delta e \end{pmatrix} \tag{3.2}$$

and $\delta, e \in \mathbb{Z}$. As in [GHS07b, Lemma 2.24], by applying the change of basis

$$N = \begin{pmatrix} I & 0 & R' \\ 0 & I & R \\ 0 & 0 & I \end{pmatrix} \tag{3.3}$$

to $\{e'_i\}$, where $R = -B^{-1}C$ and R' satisfies $D - {}^{\tau}CB^{-1}C + {}^{\tau}R'A + {}^{\tau}AR' = 0$, we obtain a $\mathbb{Q}$-basis $\{e_i\}_{i=1}^{n+2}$ for L with Gram matrix

$$Q := (e_i, e_j) = \begin{pmatrix} 0 & 0 & A \\ 0 & B & 0 \\ A & 0 & 0 \end{pmatrix} \tag{3.4}$$

and A, B as before. With respect to $\{e_i\}$, the groups $N(F)$, $W(F)$, $U(F)$ are given by [GHS07b, Lemma 2.25]

$$N(F) = \left\{ \begin{pmatrix} U & V & W \\ 0 & X & Y \\ 0 & 0 & Z \end{pmatrix} \,\middle|\, \begin{matrix} {}^tUAZ = A, \ {}^tXBX = B, \ {}^tXBY + {}^tVAZ = 0, \\ {}^tYBY + {}^tZAW + {}^tWAZ = 0, \ \det U > 0 \end{matrix} \right\}, \tag{3.5}$$

$$W(F) = \left\{ \begin{pmatrix} I & V & W \\ 0 & I & Y \\ 0 & 0 & I \end{pmatrix} \,\middle|\, BY + {}^tVA = 0, \ {}^tYBY + AW + {}^tWA = 0 \right\},$$

$$U(F) = \left\{ \begin{pmatrix} I & 0 & \begin{pmatrix} 0 & ex \\ -x & 0 \end{pmatrix} \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\}. \tag{3.6}$$

As in [Kon93, GHS07b], we define coordinates for $\mathcal{D}_L(F)$. If $[t_1 : \cdots : t_{n+2}]$ are homogeneous coordinates for $\mathbb{P}(L \otimes \mathbb{C})$ on the basis $\{e_i\}$, then

$$\begin{aligned} \mathcal{D}_L(F) &\cong U(F)_{\mathbb{C}} \times \mathbb{C}^m \times F \\ &= \left\{ (z, \underline{w}, \tau) \in \mathbb{C} \times \mathbb{C}^m \times \mathbb{H}^+ \right\} \\ &= \left\{ [z : t_2 : \underline{w} : \tau : 1] \in \mathbb{P}(L \otimes \mathbb{C}) \mid 2\delta e t_2 = -2\delta z\tau - \underline{w} B^\tau \underline{w} \right\} \end{aligned} \tag{3.7}$$

with e and δ as in (3.2). By setting $u := \exp_e(z) := e^{2\pi i z/e}$, one obtains a coordinate for the torus $T(F) \cong \mathbb{C}^*$, and the partial compactification at F is obtained by allowing $u = 0$; see [GHS07b, § 2.3, p. 539]. Furthermore, for $g \in N(F)$ as in (3.5) with

$$Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the action of g on $\mathcal{D}_L(F)$ is given by

$$\begin{cases} z \longmapsto \dfrac{z}{\det Z} + (c\tau + d)^{-1} \left(\dfrac{c}{2\delta \det Z} {}^t\underline{w} B \underline{w} + \underline{V}_1 \underline{w} + W_{11}\tau + W_{12} \right), \\ \underline{w} \longmapsto (c\tau + d)^{-1} (X\underline{w} + Y \left(\begin{smallmatrix} \tau \\ 1 \end{smallmatrix}\right)), \\ \tau \longmapsto (a\tau + b)/(c\tau + d), \end{cases} \tag{3.8}$$

where $\underline{V}_i$ is the ith row of the matrix V in (3.5); see [GHS07b, Proposition 2.26] and [Kon93, § 2.12, p. 259].

3.3 The low-weight cusp form trick

We construct pluricanonical forms $\Omega(f)$ on a smooth projective model $\widetilde{\mathcal{F}}(\Gamma)$ for $\mathcal{F}(\Gamma)$ from modular forms f. The modular forms f must lie outside obstruction spaces EllObs and RefObs, which we define below.

Definition 3.1 ([GHS13, Definition 6.4]). Let $\Gamma \subset \mathrm{O}^+(L)$ be a subgroup of finite index with character $\chi\colon \Gamma \to \mathbb{C}^*$, and let $\mathcal{D}_L^\bullet$ be the affine cone of $\mathcal{D}_L$. A *modular form f of weight k and character* χ for Γ is a holomorphic function $f\colon \mathcal{D}_L^\bullet \to \mathbb{C}$ such that, for all $Z \in \mathcal{D}_L^\bullet$, both

(i) $f(gZ) = \chi(g) f(Z)$ for all $g \in \Gamma$,
(ii) $f(tZ) = t^{-k} f(Z)$ for all $t \in \mathbb{C}^*$

are satisfied. We denote the space of all such modular forms by $M_k(\Gamma, \chi)$ and the subspace of cusp forms by $S_k(\Gamma, \chi)$.

For $v \in L^\vee$ with $v^2 < 0$, the *rational quadratic divisor* $\mathcal{D}_L(v) \subset \mathcal{D}_L$ is defined by

$$\mathcal{D}_L(v) = \{[x] \in \mathcal{D}_L \mid (x, v) = 0\}\,.$$

We define the space of *reflective obstructions*

$$\mathrm{RefObs}_{(4-a)k}(\Gamma, \chi_2) \subset M_{(4-a)k}(\Gamma, \chi_2) \tag{3.9}$$

as the space of weight $(4-a)k$ modular forms vanishing to order less than k on $\mathcal{D}_L(v) \subset \mathcal{D}_L$ for all $\pm\sigma_v \in \Gamma$. If $\mathcal{E}$ is a set of primitive embeddings of signature $(2,3)$ lattices in L, the space of *elliptic obstructions*

$$\mathrm{EllObs}_{(4-a)k}(\Gamma, \mathcal{E}, \chi_2) \subset M_{(4-a)k}(\Gamma, \chi_2) \tag{3.10}$$

is defined as the space of weight $(4-a)k$ modular forms vanishing to order less than $2k$ on all rational quadratic divisors $\mathcal{D}_L(K^\perp)$ for $(K \hookrightarrow L) \in \mathcal{E}$. In later sections, we will mostly be concerned with the growth of $\mathrm{EllObs}_{(4-a)k}(\Gamma, \mathcal{E}, \chi_2)$, $\mathrm{RefObs}_{(4-a)k}(\Gamma)$ and $M_k(\Gamma, \chi)$ as $k \to \infty$: as this is independent of the character χ_2, we will typically ignore it, writing $\mathrm{EllObs}_{(4-a)k}(\Gamma, \mathcal{E})$, $\mathrm{RefObs}_{(4-a)k}(\Gamma)$ and $M_k(\Gamma)$ instead.

In Theorem 3.2, we will assume that $\mathcal{E}$ satisfies the conditions of Propositions 6.2 and 6.3. We will construct suitable $\mathcal{E}$ in Lemma 10.11.

Theorem 3.2. *Assume that $\mathcal{E}$ satisfies the conditions of Propositions 6.2 and 6.3. Suppose that $0 \neq f_a \in S_a(\Gamma, \chi_1)$ for $0 < a < 4$ and $g \in M_{(4-a)k}\big(\Gamma, \big(\det^k\big)\chi_1^{-k}\big)$ lies outside the obstruction spaces $\mathrm{EllObs}_{(4-a)k}\big(\Gamma, \mathcal{E}, \big(\det^k\big)\chi_1^{-k}\big)$ and $\mathrm{RefObs}_{(4-a)k}\big(\Gamma, \big(\det^k\big)\chi_1^{-k}\big)$. Let*

$$f := f_a^k g \in M_{4k}\big(\Gamma, \det^k\big) \quad \text{and} \quad \Omega(f) := f\omega^{\otimes k}\,,$$

where ω is a holomorphic volume form for $\mathcal{D}_L$. Then there exists a toroidal compactification $\overline{\mathcal{F}}(\Gamma)$ of $\mathcal{F}(\Gamma)$ and a desingularisation $\widetilde{\mathcal{F}}(\Gamma)$ of $\overline{\mathcal{F}}(\Gamma)$ such that for all f as above, the Γ-invariant differential form $\Omega(f)$ defines a pluricanonical form on $\widetilde{\mathcal{F}}(\Gamma)$.

Proof. The argument follows [GHS07b, Theorem 1.1]. Take a toroidal compactification $\overline{\mathcal{F}}(\Gamma)$ of $\mathcal{F}(\Gamma)$ and a resolution of singularities $\widetilde{\mathcal{F}}(\Gamma) \to \overline{\mathcal{F}}(\Gamma)$. By [GHS07b, Corollaries 2.21 and 2.22] (and the correction of [Ma18b]), we may assume that all singularities of $\overline{\mathcal{F}}(\Gamma)$ are canonical above a 0-dimensional boundary component. If ω is a holomorphic volume form on $\mathcal{D}_L$, then as $f \in M_{4k}\big(\Gamma, \det^k\big)$, the differential form $\Omega(f) := f\omega^{\otimes k}$ is Γ-invariant on $\mathcal{D}_L$ and defines a section of the pluricanonical bundle $kK_{\mathcal{F}(\Gamma)}$ away from the cusps and branch locus. By [GHS07b, Corollary 2.13], the smooth part of the branch locus of $\pi\colon \mathcal{D}_L \to \mathcal{F}(\Gamma)$ coincides with the fixed locus of elements $\pm\sigma_v \in \Gamma$, and by the results of [Ma25], there are no fixed divisors in the boundary of $\overline{\mathcal{F}}(\Gamma)$. Therefore, as f vanishes to order k along the fixed loci of quasi-reflections, $\Omega(f)$ defines a pluricanonical form away from the singularities and cusps of $\mathcal{F}(\Gamma)$. By [AMRT10, Theorem IV.1.1], as f_a is a cusp form, then f has zeros of order k along the boundary divisor of $\overline{\mathcal{F}}(\Gamma)$. Therefore, $\Omega(f)$ defines a pluricanonical form on the smooth locus of $\overline{\mathcal{F}}(\Gamma)$. By Propositions 6.2 and 6.3, the differential form $\Omega(f)$ defines a pluricanonical form over all non-canonical singularities of $\overline{\mathcal{F}}(\Gamma)$, from which the result follows. □

4. Non-canonical singularities in $\overline{\mathcal{F}}(\Gamma)$

In this section, we classify non-canonical singularities in arbitrary orthogonal modular fourfolds, study elliptic elements of Γ in terms of their invariant lattices, calculate elliptic elements in Γ defining non-canonical singularities in $\mathcal{F}(\Gamma)$ and study non-canonical singularities in the boundary of $\overline{\mathcal{F}}(\Gamma)$.

4.1 Non-canonical singularities in orthogonal modular fourfolds

We classify non-canonical singularities in arbitrary orthogonal modular fourfolds $\mathcal{F}(\Delta)$, where $\Delta \subset \mathrm{O}^+(N)$ and N is a lattice of signature $(2,4)$. For $A, B \in M_n(\mathbb{C})$, we use $A \sim B$ to denote $A = {}^gB = gBg^{-1}$ for some $g \in \mathrm{GL}(n,\mathbb{C})$. For $m \in \mathbb{N}$, we use $(1/m)(a_1,\dots,a_n)$ to denote the diagonal matrix $\operatorname{diag}(\xi_m^{a_1},\dots,\xi_m^{a_n})$, where the $a_i \in \mathbb{Z}$ are taken modulo m and $\xi_m := e^{2\pi i/m}$. Where no confusion is likely to arise, we also use $(1/m)(a_1,\dots,a_n)$ to denote the quotient $\mathbb{C}^n/\langle g\rangle$.

As the singularities of $\overline{\mathcal{F}}(\Gamma)$ are finite quotient singularities, they can be studied using the Reid–Tai criterion [Rei87, Tai82], as in [Kon93, GHS07b]. For $g \sim (1/m)(a_1,\dots,a_n)$, the *Reid–Tai sum* $\Sigma(g)$ is defined as

$$\Sigma(g) = \begin{cases} \sum \left\{\dfrac{a_i}{m}\right\} & \text{if } g \neq I\,,\\ 1 & \text{if } g = I\,,\end{cases}$$

where $\{x\} := x - [x]$ denotes the fractional part of $x \in \mathbb{Q}$. The *Reid–Tai criterion* states that if $G \subset \mathrm{GL}(n,\mathbb{C})$ is a finite group containing no quasi-reflections (that is, no elements with precisely $n-1$ eigenvalues equal to 1), then $\mathbb{C}^n/G$ has canonical singularities [Rei87] if and only if $\Sigma(g) \geqslant 1$ for all $g \in G$; see [Rei80, Rei87, Tai82]. If G contains quasi-reflections, there is a modified version of the Reid–Tai criterion due to Katharina Ludwig [GHS13, Proposition 5.24]. If $g \sim (1/m)(a_1,\dots,a_n)$ and $m = st$, where t is the smallest positive integer such that g^t is the identity or a quasi-reflection and $\xi_m^{ka_i} = 1$ for all $1 \leqslant i \leqslant n-1$, where $\xi_m^{ka_n}$ is a primitive sth root of unity, then the *modified Reid–Tai sum* $\Sigma'(g)$ is defined by

$$\Sigma'(g) = \begin{cases} \left\{\dfrac{sa_n}{n}\right\} + \displaystyle\sum_{i=1}^{n-1}\left\{\dfrac{a_i}{m}\right\} & \text{if } g \neq I\,,\\ 1 & \text{if } g = I\,.\end{cases}$$

The *modified Reid–Tai criterion* states that $\mathbb{C}^n/G$ has canonical singularities if $\Sigma'(g) \geqslant 1$ for all $g \in G$.

Around the image of $[w] \in \mathcal{D}_N$, the modular variety $\mathcal{F}(\Delta)$ is locally isomorphic to $T_{[w]}\mathcal{D}_N/\operatorname{Iso}_\Delta([w])$, where $T_{[w]}\mathcal{D}_N$ is the tangent space of $\mathcal{D}_N$ at $[w]$ and $\operatorname{Iso}_\Delta([w])$ is the isotropy subgroup

$$\operatorname{Iso}_\Delta([w]) := \{g \in \Delta \mid g\cdot[w] = [w]\}\,.$$

As in [GHS07b], if $\mathbb{W} := \mathbb{C}\cdot w \subset N \otimes \mathbb{C}$, there exists an isomorphism

$$T_{[w]}\mathcal{D}_N \cong \operatorname{Hom}\big(\mathbb{W}, \mathbb{W}^\perp/\mathbb{W}\big)$$

and a character $\alpha\colon \operatorname{Iso}_\Delta([w]) \to \mathbb{C}^*$ defined by the action of $\operatorname{Iso}_\Delta([w])$ on $\mathbb{W}$. We refer to $[w] \in \mathcal{D}_N$ as a *canonical singularity of* $\mathcal{F}(\Delta)$ if $T_{[w]}\mathcal{D}_N/\operatorname{Iso}_\Delta([w])$ has only canonical singularities; otherwise, we refer to $[w]$ as a *non-canonical singularity of* $\mathcal{F}(\Delta)$.

Lemma 4.1. *Suppose that $g \in \Gamma$ fixes $[w] \in \mathcal{D}_N$. Then*

$$T_{[w]}\mathcal{D}_N/\langle g\rangle \tag{4.1}$$

is a non-canonical singularity if and only if the isomorphism class of (4.1) is given in Table 1. The corresponding values of $\alpha(g)$ and the characteristic polynomial $\chi_g(x)$ are also given in Table 1, where ϕ_r denotes the rth cyclotomic polynomial.

Proof. We study rational representations of the cyclic group using the Reid–Tai criterion, as in [Kon93, GHS07b]. Suppose that $g \in \operatorname{Iso}_\Delta([w])$ is of order m and g^k is of prime-power order

Table 1. Non-canonical cyclic quotient singularities in orthogonal modular 4-folds.

Singularity	χ_g	$\alpha(g)$	**Singularity**	χ_g	$\alpha(g)$
$\frac{1}{3}(0,0,1,1)$	ϕ_3^3	$\xi_3^{\pm 1}$	$\frac{1}{3}(0,0,1,1)$	ϕ_6^3	$\xi_6^{\pm 1}$
$\frac{1}{6}(0,0,2,2)$	$\phi_3\phi_6^2$	$\xi_6^{\pm 2}$	$\frac{1}{6}(0,0,2,2)$	$\phi_3\phi_6^2$	$\xi_6^{\pm 1}$
$\frac{1}{6}(0,1,1,2)$	$\phi_2^2\phi_3^2$	$\xi_6^{\pm 2}$	$\frac{1}{6}(1,1,1,1)$	$\phi_2^4\phi_3$	$\xi_6^{\pm 2}$
$\frac{1}{6}(0,1,1,2)$	$\phi_1^2\phi_6^2$	$\xi_6^{\pm 1}$	$\frac{1}{12}(0,2,2,4)$	$\phi_4\phi_6^2$	$\xi_{12}^{\pm 2}$
$\frac{1}{12}(2,2,2,2)$	$\phi_4^2\phi_6$	$\xi_{12}^{\pm 2}$	$\frac{1}{12}(0,2,2,4)$	$\phi_3\phi_4\phi_6$	$\xi_{12}^{\pm 4}$
$\frac{1}{12}(0,2,2,4)$	$\phi_3\phi_4\phi_6$	$\xi_{12}^{\pm 2}$	$\frac{1}{12}(2,2,2,2)$	$\phi_3\phi_4^2$	$\xi_{12}^{\pm 4}$
$\frac{1}{12}(0,2,2,4)$	$\phi_3^2\phi_4$	$\xi_{12}^{\pm 4}$	$\frac{1}{24}(4,4,4,4)$	$\phi_6\phi_8$	$\xi_{24}^{\pm 4}$
$\frac{1}{24}(4,4,4,4)$	$\phi_3\phi_8$	$\xi_{24}^{\pm 8}$	$\frac{1}{30}(5,5,5,5)$	$\phi_5\phi_6$	$\xi_{30}^{\pm 5}$
$\frac{1}{30}(5,5,5,5)$	$\phi_3\phi_{10}$	$\xi_{30}^{\pm 10}$			

p^r, and let $N \otimes \mathbb{Q} \cong \bigoplus_i \mathcal{V}_{p^{r_i}}$ be the decomposition of $N \otimes \mathbb{Q}$ as a g^k-module, where $\mathcal{V}_m$ is the unique faithful $\mathbb{Q}$-irreducible representation of the cyclic group C_m, see [Ser77], and $r_1 \leqslant \cdots \leqslant r_n = r$. As $\deg \mathcal{V}_{p^{r_i}} = \phi(p^{r_i}) = p^{r_i} - 1 \leqslant \dim N_{\mathbb{Q}} = 6$, then $p^r \in \{1,2,3,4,5,7,8,9\}$ and $m \leqslant 5 \cdot 7 \cdot 8 \cdot 9 = 2520$. (In fact, $m \in \{1,2,3,4,5,6,7,8,9,10,12,14,15,18,20,24,30\}$.) We then proceed to perform an exhaustive computer search using the modified Reid–Tai criterion over all faithful 6-dimensional $\mathbb{Q}$-representations of C_m for $m \leqslant 2520$. (Our search also returned the false positives $h = \frac{1}{6}(0,0,2,3)$ and $\frac{1}{6}(0,0,3,4)$: if such h were to exist, then h^2 and h^3 both define quasi-reflections of order different from 2, contradicting [GHS07b, Corollary 2.13].) The result follows. $\square$

4.2 Elliptic elements of Γ and p-elementary lattices

Suppose that L and Γ are as in (2.2) and (2.3), and let M denote the Beauville lattice. We now classify elliptic elements of Γ in terms of their invariant and perp-invariant lattices. For $g \in \mathrm{O}(N)$, where N is an arbitrary lattice, we define the *invariant* and *perp-invariant* lattices of g by

$$T := \{x \in N \mid gx = x\} \quad \text{and} \quad S := T^{\perp} \subset N\,,$$

respectively. We recall that a lattice N is said to be *p-elementary* if $D(N) \cong C_p^{\oplus a}$, where C_p is the cyclic group of prime order p. A partial classification of the genera of p-elementary lattices is given in [RS79, RS82] (see also [CS99]).

Assumption. From now on, we will assume (unless otherwise stated) that the parameters n and d defining the lattice L satisfy $n = 2$ and $d > 48$.

Proposition 4.2. *Suppose that $g \in \mathrm{O}(M)$ is of prime order p.*

(i) *If $p = 2$, then $D(S)$ or $D(T)$ is 2-elementary.*

(ii) *If $3 \leqslant p \leqslant 19$, then $D(S)$ is p-elementary.*

Proof. We follow [BNS13, Lemma 5.5]. (We also proved a version of this result in the unpublished thesis [Daw15].) If N is the g-module $N = M/(S \oplus T)$, then by [BNS13, Lemma 4.3 and

Remark 4.4], there exists an $a \in \mathbb{Z}$ such that $N \cong C_p^{\oplus a}$. By standard results on overlattices (see for example [BHP+04, Chapter I.1, Lemma 2.1]),

$$|M : S \oplus T|^2 = \det(S)\det(T)\det(M)^{-1}$$

and so

$$\det(S)\det(T) = 6p^{2a}\,, \tag{4.2}$$

implying $\det(S) = 2^\delta 3^\epsilon p^\alpha$ and $\det(T) = 2^{1-\delta}3^{1-\epsilon}p^\beta$, where $\alpha + \beta = 2a$ and $\epsilon, \delta \in \{0, 1\}$.

We now show that all elementary factors of $D(S)$ and $D(T)$ are isomorphic to C_2, C_3 or C_p. As in [Nik80, § 5], the overlattices

$$S \oplus T \subset M \subset M^\vee \subset S^\vee \oplus T^\vee$$

define an inclusion of finite abelian groups

$$N \subset D(S) \oplus D(T)$$

and natural projections $p_S\colon N \to D(S)$ and $p_T\colon N \to D(T)$. As both $S, T \subset L$ are primitive, then both p_S and p_T are injective [Nik80, § 1.5, p. 111], and so $C_p^{\oplus a} \subset D(S)$ and $C_p^{\oplus a} \subset D(T)$. Therefore, if $p = 2$, then $a \leqslant \alpha + \delta$ and $a \leqslant \beta + (1 - \delta)$. As $\alpha + \beta = 2a$, then $a - \delta \leqslant \alpha \leqslant a + (1 - \delta)$, and so $\alpha + \delta = a$ or $a + 1$. Then, as $N \subset D(S)$ and by using the classification of finite abelian groups, $D(S) = C_2^{\oplus a} \oplus C_3^\epsilon$ or $D(S) = C_4 \oplus C_2^{\oplus(a-1)} \oplus C_3^\epsilon$ if $\delta = 1$. The latter case cannot occur: as $D(S) \subset D(T) \oplus D(M)/G$, where $G \subset D(T) \oplus D(M)$, if $\delta = 1$, then we have $D(T) \cong C_2^{\oplus a} \oplus C_3^{\oplus(1-\epsilon)}$, implying that $D(T)$ and so $D(S)$ contain no 4-torsion. Similarly, if $p = 3$, then $D(S) = C_2^\delta \oplus C_3^{\oplus(a+\epsilon)}$. If $p > 3$, then as $a \leqslant \alpha$, $a \leqslant \beta$ and $2a = \alpha + \beta$, we have $\alpha = \beta = a$ and so $D(S) = C_2^{\oplus\delta} \oplus C_3^\epsilon \oplus C_p^{\oplus a}$.

We now prove the main result. Let $x_2, x_3 \in S^\vee$ represent generators for $C_2^{\oplus\delta}, C_3^{\oplus(a+\epsilon)} \subset D(S)$, respectively (if they exist), and define $\sigma := 1 + g + \cdots + g^{p-1}$. If $p = 2$, then by (4.2), the determinant $\det(T)$ or $\det(S)$ is coprime to 3, and so S or T is 2-elementary. If $p = 3$ and $\delta = 0$, then as $D(S)$ contains a single element of order 2, we have $g\overline{x_2} = \overline{x_2}$ and so $\sigma(\overline{x_2}) = \overline{x_2}$. On the other hand, from the $\mathbb{Q}$-representation theory of the cyclic group, σ is identically zero on S, implying $\overline{x_2} = 0$, which contradicts the assumption on x_2. Similarly, if $p > 3$, then $g\overline{x_2} = \overline{x_2}$ and $g\overline{x_3} = \pm\overline{x_3}$. Therefore, $\sigma(\overline{x_2}) = p\overline{x_2}$ and $\sigma(\overline{x_3}) = \overline{x_3}$ or $p\overline{x_3}$. However, as σ vanishes on S and $(2, p) = (3, p) = 1$, then both $\overline{x_2}$ and $\overline{x_3}$ must be zero. The result follows. □

Lemma 4.3. *Let $g \in \mathrm{O}(M, h)$, where $h^2 > 0$, and let g' be the restriction of g to the sublattice $L = h^\perp \subset M$. If $S \subset M$ is the perp-invariant lattice of g and $S' \subset L$ is the perp-invariant lattice of g', then $S = S'$.*

Proof. If T' is the invariant lattice of g', then

$$S' = \{x \in L \mid x \perp T'\} = \{x \in L \mid x \perp T\} = \{x \in M \mid x \perp h,\, x \perp T\}\,,$$

and as $h \in T$, then $S' = \{x \in M \mid x \perp T\} = S$. □

Lemma 4.4. *If $[w] \in \mathcal{D}_L$ is a non-canonical singularity of $\mathcal{F}(\Gamma)$, then $[w]$ is fixed by an element $g \in \Gamma$ of order 3 with characteristic polynomial $\chi_g = \phi_1^2\phi_3^2$ and perp-invariant lattice $2U$ or $U \oplus A_2(-1)$.*

Proof. By Lemma 4.1, the point $[w]$ is fixed by an element $h \in \Gamma$ such that $h^k =: g$ is of order 3 for some k. By eliminating cases using Proposition 2.1, we can assume that the characteristic polynomial χ_h equals $\phi_1^2\phi_6^2$. For $l \in \mathbb{Z}$, let $T_l, S_l \subset L$ denote the invariant and perp-invariant lattices of h^l, respectively. By Lemmas 4.2 and 4.3, we have $D(S_2) \cong C_3^a$ for some $a \in \mathbb{N}$. The genera of 3-elementary lattices are classified in [CS99, § 15.8.2, Theorem 13]. When $a \leqslant 4$, each genus of 3-elementary lattices contains at most one class, which follows from [CS99, § 15.9.7, Corollary 22] in the indefinite case and from [CS99, § 15.3.3, Table 15.1] and [Nip91] in the definite case. Therefore, as L is not 3-elementary, then any 3-elementary lattice admitting a primitive embedding in L must be one of $2U$, $U \oplus U(3)$, $2U(3)$, $U \oplus A_2(-1)$, $U(3) \oplus A_2(-1)$, U, $U(3)$, $A_2(\pm 1)$ or $2A_2(-1)$. By considering χ_h and the action of h on $L \otimes \mathbb{Q}$, we have $S_2 = S_3$ and $\operatorname{rank} S_2 = 4$. By Lemma 4.2, the lattice S_2 is 3-elementary, and either S_3 or T_3 is 2-elementary. If S_3 is 2-elementary, then $S_3 = 2U$; if S_3 is not 2-elementary, then S_3 is 3-elementary and so, in the notation of Proposition 4.2, we have $\delta = 0$, $a = 0$ and $\epsilon \in \{0, 1\}$, implying $S_3 = 2U$ or $U \oplus A_3(\pm 1)$. □

4.3 Elliptic elements of Γ

We now classify elliptic elements of Γ defining non-canonical singularities in $\mathcal{F}(\Gamma)$.

Lemma 4.5. *The group* $\mathrm{O}(1,3)$ *contains no element of order* 3 *with characteristic polynomial* ϕ_3^2.

Proof. We note that if $h \in \mathrm{O}(1,3)$ is of order 3, then $h \in \mathrm{SO}^+(1,3)$. Following [CSM95, § II.2, p. 55], we construct a double cover homomorphism $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^+(1,3)$ by identifying $\mathbb{R}^4$ with the 2×2 Hermitian matrices $\mathfrak{H}$ via

$$\mathbb{R}^4 \ni (t,x,y,z) \longmapsto \begin{pmatrix} t+x & y-iz \\ y+iz & t-x \end{pmatrix} \in \mathfrak{H}\,,$$

allowing det to define a signature $(1,3)$ quadratic form on $\mathfrak{H}^4$ and mapping each $g \in \mathrm{SL}(2,\mathbb{C})$ to $T_g \in \mathrm{SO}^+(1,3)$, where $T_g \colon A \mapsto gAg^{-1}$ for $A \in \mathfrak{H}$.

If $g \in \mathrm{SL}(2,\mathbb{C})$ is such that T_g is of order 3, then g is of finite order with conjugate eigenvalues α and $\overline{\alpha}$. Without loss of generality, we can assume $\alpha = e^{2\pi i/3}$ and $g = \mathrm{diag}(\alpha, \overline{\alpha})$, implying

$$T_g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

and $\chi_{T_g} = \phi_3 \phi_1^2$. □

Definition/Lemma 4.6. There is an isomorphism

$$\Delta \colon (\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z}))/\{\pm(I,I)\} \longrightarrow \mathrm{SO}^+(2U)$$

defined on a standard basis of $2U$ by

$$\Delta(A,I) \longmapsto \begin{pmatrix} a & 0 & 0 & b \\ 0 & d & -c & 0 \\ 0 & -b & a & 0 \\ c & 0 & 0 & d \end{pmatrix} \quad \text{and} \quad \Delta(I,A) \longmapsto \begin{pmatrix} d & 0 & c & 0 \\ 0 & a & 0 & -b \\ b & 0 & a & 0 \\ 0 & -c & 0 & d \end{pmatrix},$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\,.$$

Proof. The isomorphism is well known (see for example [Ste91]): we adopt the notation of [Bra95] and construct the homomorphism following the approach of [GHS09]. Identify $2U$ with the matrices $M_2(\mathbb{Z})$ via the map

$$U \oplus U \ni (w, x, y, z) \longmapsto \begin{pmatrix} w & -y \\ z & x \end{pmatrix} \in M_2(\mathbb{Z}),$$

and let $-\det$ define a quadratic form on $M_2(\mathbb{Z})$. Then, if $(A, B) \in \mathrm{SL}(2, \mathbb{Z}) \times \mathrm{SL}(2, \mathbb{Z})$, the element $\Delta(A, B) \in \mathrm{O}^+(2U)$ is defined by

$$\Delta(A, B)\colon \begin{pmatrix} w & -y \\ z & x \end{pmatrix} \longmapsto A \begin{pmatrix} w & -y \\ z & x \end{pmatrix} B^{-1}. \qquad \square$$

Lemma 4.7. *Let $g \in \mathrm{O}^+(2U)$ be of order 3 with characteristic polynomial $\chi_g = \phi_3^2$. Then, up to $\mathrm{O}^+(2U)$-conjugacy, g is represented by $\Delta(I, g_3)^{\pm 1}$ or $\Delta(g_3, I)^{\pm 1}$, where*

$$g_3 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Proof. We apply Lemma 4.6. If $g \in \mathrm{O}^+(2U)$ is of order 3, then $g \in \mathrm{SO}^+(2U)$. The conjugacy classes of 3-torsion in $\mathrm{SL}(2, \mathbb{Z})$ are represented by $g_3^{\pm 1}$, see [Ser73], where

$$g_3 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

The result follows by calculating the characteristic polynomials of $\Delta(g_3, I)^{\pm 1}$, $\Delta(I, g_3)^{\pm 1}$, $\Delta(g_3, g_3)^{\pm 1}$ and $\Delta\big(g_3^{-1}, g_3\big)^{\pm 1}$. $\square$

Definition 4.8. Every embedding $\iota\colon 2U \hookrightarrow L$ with $B := (2U)^\perp \subset L$ defines an embedding $\Delta_\iota\colon \mathrm{SL}(2, \mathbb{Z}) \times \mathrm{SL}(2, \mathbb{Z}) / \pm (I, I) \hookrightarrow \Gamma$ such that $\Delta_\iota(g, h)|_{2U} := \Delta(g, h)$ and $\Delta_\iota(g, h)|_B := I$ for all $g, h \in \mathrm{SL}(2, \mathbb{Z})$. The extension of $\Delta(g, h)$ to Δ_ι is well defined as $\widetilde{\mathrm{SO}}^+(2U) = \mathrm{SO}^+(2U)$.

Lemma 4.9. *Suppose that $[w] \in \mathcal{D}_L$ is fixed by $g \in \Gamma$. If $T_{[w]}\mathcal{D}_L/\langle g \rangle$ is a non-canonical singularity, then $g = \Delta_\iota(g_6, I)^{\pm 1}$ or $\Delta_\iota(I, g_6)^{\pm 1}$ for a primitive embedding $\iota\colon 2U \hookrightarrow L$, where*

$$g_6 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

Proof. If $T_{[w]}\mathcal{D}_L/\langle g \rangle$ is a non-canonical singularity, then by using Lemma 4.1 and eliminating cases using Proposition 2.1, we have $\chi_g = \phi_1^2\phi_6^2$. By Lemmas 4.4 and 4.5, the perp-invariant lattice of g is given by $2U$. Therefore, by Lemma 4.7, we have $g^2 = \Delta_\iota(g_3, I)^{\pm 1}$ or $\Delta_\iota(I, g_3)^{\pm 1}$ for a primitive embedding $\iota\colon 2U \hookrightarrow L$ (where primitivity follows by the definition of S). Hence, by Definition/Lemma 4.6, we have $g = \Delta_\iota(g_6, I)^{\pm 1}$ or $\Delta_\iota(I, g_6)^{\pm 1}$. $\square$

4.4 Non-canonical singularities in the boundary of $\overline{\mathcal{F}}(\Gamma)$

We now consider the non-canonical singularities of $\overline{\mathcal{F}}(\Gamma)$ above a boundary curve F. We let $G(F)$ denote the quotient $N(F)_\mathbb{Z}/U(F)_\mathbb{Z}$.

Lemma 4.10. *Suppose that $P \in \overline{\mathcal{F}}(\Gamma)$ lies above a boundary curve F. Let $G \subset G(F)$ be the isotropy subgroup of P, and let T_P be the tangent space of $(\mathcal{D}_L/U(F)_\mathbb{Z})_{\{\sigma_i\}}$ at P. If $g \in G$ is such that $T_P/\langle g \rangle$ is a non-canonical singularity, then g acts as $\frac{1}{6}(0,1,1,2)^{\pm 1}$ or $\frac{1}{6}(1,1,1,2)^{\pm 1}$ and is represented by*

$$g = \begin{pmatrix} U & V & W \\ 0 & I & Y \\ 0 & 0 & Z \end{pmatrix} \in N(F)_\mathbb{Z}, \tag{4.3}$$

where U and Z are of order 6.

Proof. By the modified Reid–Tai criterion, the isotropy subgroup G contains an element g such that $\Sigma'(g) < 1$. If g is represented by

$$g = \begin{pmatrix} U & V & W \\ 0 & X & Y \\ 0 & 0 & Z \end{pmatrix} \in N(F)$$

on the basis (3.4), then in the notation of (3.8) (following [Kon93, § 8.2, p. 292]), the transformation g acts on T_P by

$$\begin{pmatrix} \exp_e(t) & 0 & 0 \\ * & \xi^{-1}X & 0 \\ * & * & \xi^{-2} \end{pmatrix}, \tag{4.4}$$

where $\xi = (c\tau + d)$ is a root of unity [GHS07b, Proposition 2.28]. As one can verify by direct calculation, if $g \in G(F)$ is of finite order, then $g^m = I$ for $m > 0$ if and only if $U^m = X^m = Z^m = I$. Therefore, by (4.4), $\exp_e(t)$ is a 12th root of unity. By using this bound to perform an exhaustive computer search, we find that if $\Sigma'(g) < 1$, then $g^{\pm 1}$ acts as one of $\frac{1}{3}(0,0,1,1)$, $\frac{1}{6}(0,0,2,3)$, $\frac{1}{6}(0,1,1,2)$, $\frac{1}{6}(0,2,2,3)$ or $\frac{1}{6}(1,1,1,2)$. The case $\frac{1}{6}(0,0,2,3)$ is a false positive as the corresponding group is generated by quasi-reflections and the quotient is therefore smooth [Che55]. For each of the other cases, the characteristic polynomial χ_g is given by $\phi_3\phi_r^2$, $\phi_6\phi_r^2$, $\phi_2^2\phi_r^2$ or $\phi_1^2\phi_r^2$, where $r = 3$ or 6. By further eliminating cases using Proposition 2.1, we see that $g^{\pm 1}$ must act as either $\frac{1}{6}(0,1,1,2)$ or $\frac{1}{6}(1,1,1,2)$, where in both cases, $\xi = e^{5\pi i/3}$ and $X = I$. Then, by (3.5), $U \sim Z$ and so both must be of order 6. □

Our later calculations are simplified if we work with a basis for $L \otimes \mathbb{Q}$ adapted to $g \in G(F)$.

Lemma 4.11. *Assuming the conditions of Lemma 4.10, let $g \in N(F)_{\mathbb{Z}}$ be such that $T_P/\langle g \rangle$ is a non-canonical singularity. Then, without loss of generality, the basis (3.4) can be chosen so that*

$$g = \begin{pmatrix} U & 0 & W' \\ 0 & I & 0 \\ 0 & 0 & Z \end{pmatrix}$$

with U and Z as in (4.3).

Proof. We begin by proving some facts about U and V using the basis (3.4), on which g is given by (4.3). As g is of finite order in $G(F)_{\mathbb{Z}}$, then from the description of $U(F)_{\mathbb{Z}}$ in (3.6), the submatrix $\left(\begin{smallmatrix} U & V \\ 0 & I \end{smallmatrix}\right)$ is of finite order. By (3.3) and the definition of $N(F)$ in (3.5), we have $V \in M_2(\mathbb{Z})$, and as U is of order 6, then

$$U \sim \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^{\pm 1}$$

(following from the classification of torsion in $\mathrm{SL}(2,\mathbb{Z})$, see for example [DS05, Ser73]), implying $I - U \in \mathrm{SL}(2,\mathbb{Z})$.

We now take the $\mathbb{Z}$-basis (3.1) for L, on which g is given by

$$g = \begin{pmatrix} U & V & W' \\ 0 & I & Y' \\ 0 & 0 & Z' \end{pmatrix}$$

with U and V as before (following from (3.3)). By applying the change of basis

$$N_1 := \begin{pmatrix} I & (I-U)^{-1}V & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix},$$

we obtain a $\mathbb{Z}$-basis $\{f_i\}$ for L such that

$$Q := ((f_i, f_j)) = \begin{pmatrix} 0 & 0 & A \\ 0 & B & C' \\ A & {}^{\tau}C' & D \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} U & 0 & W' \\ 0 & I & Y' \\ 0 & 0 & Z' \end{pmatrix}.$$

As ${}^{\tau}gQg = Q$, then $BY' = C'(I - Z')$, and as $Z' \in \mathrm{SL}(2,\mathbb{Z})$ is of order 6, then $I - Z' \in \mathrm{SL}(2,\mathbb{Z})$. Therefore, as $\det B \neq 0$, then $B^{-1}C' = Y'(I - Z')^{-1} \in M_2(\mathbb{Z})$, and we can apply the change of basis

$$N_2 := \begin{pmatrix} I & 0 & 0 \\ 0 & I & -B^{-1}C' \\ 0 & 0 & I \end{pmatrix}$$

to obtain a $\mathbb{Z}$-basis $\{f_i'\}$ for L such that

$$((f_i', f_j')) = \begin{pmatrix} 0 & 0 & A \\ 0 & B & 0 \\ A & 0 & D' \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} U & 0 & W' \\ 0 & I & Y'' \\ 0 & 0 & Z' \end{pmatrix}. \tag{4.5}$$

By (3.4), B is non-degenerate, and as g fixes $\langle f_3', f_4' \rangle$, then g acts on $\langle f_3', f_4' \rangle^{\perp} = \langle f_1', f_2', f_5', f_6' \rangle$, implying $Y'' = 0$. The result follows. □

5. The branch divisor of $\mathcal{F}(\Gamma)$

As explained in § 3, the branch divisor of $\mathcal{D}_L \to \mathcal{F}(\Gamma)$ is precisely the set of rational quadratic divisors

$$\{\mathcal{D}_L(z) \mid \pm\sigma_z \in \Gamma\}.$$

In this section, we classify vectors $z \in L$ defining $\pm\sigma_z \in \Gamma$ and the associated lattices $z^{\perp} \subset L$. We will use these results in § 8 to calculate Hirzebruch–Mumford volumes, which we use to bound the obstruction space $\mathrm{RefObs}_{(4-a)k}(\Gamma)$ in § 10. We will not necessarily assume $n = 2$, and we will adopt the convention that $\gcd(x, 0) = 1$ for $x \neq 0$.

Lemma 5.1. *Let $r, s, k, l \in \mathbb{N}$, where k and l are taken modulo $2r$ and modulo $2s$, respectively. If $u = m$ or $2m$, where $m := \mathrm{lcm}(2r/(2r,k), 2s/(2s,l))$, then the pair (k,l) satisfies conditions (5.1)–(5.3) given by*

$$ur \mid mk\,, \tag{5.1}$$

$$2rsu \mid m^2kl\,, \tag{5.2}$$

$$2s \mid m^2l^2u^{-1}s^{-1} - 2 \tag{5.3}$$

if and only if one of the following conditions is satisfied:

(i) *$k = 0$ and either*
  (a) *$u = m$ and $l^2 \equiv 1 \bmod s$ if l is odd;*
  (b) *$u = m$ and $l^2 \equiv 2 \bmod 2s$ if l is even;*
  (c) *$u = 2m$ and $l = 2l_1$ satisfies $2l_1^2 \equiv 2 \bmod 2s$.*

(ii) *$k = r$ and either*

   (a) *$u = m = 2s$ and $2l^2 \equiv 2 \bmod 2s$, where m_l is even and l is odd;*

   (b) *$u = m$ and $l^2 \equiv 1 \bmod s$ if m_l is odd and l is even.*

Additionally, in the above cases,

(i) *if $k = 0$ and*

   (a) *$u = m$, then $m = 2s$ when l is odd;*

   (b) *$u = m$, then $m = s$ when l is even;*

   (c) *$u = 2m$, then $m = s$.*

(ii) *If $k = r$ and*

   (a) *$u = m$ and l odd, then $m = 2s$;*

   (b) *$u = m$ and l even, then $m = 2s$.*

Proof. Let $m_k := 2r \gcd(2r, k)^{-1}$ and $m_l := 2s \gcd(2s, l)^{-1}$.

Case $u = m$. If $u = m$, then (5.1)–(5.3) are equivalent to

$$r \mid k\,, \tag{5.4}$$
$$2rs \mid mkl\,, \tag{5.5}$$
$$2s \mid ml^2 s^{-1} - 2\,, \tag{5.6}$$

and, by (5.4), we have $k = 0$ or r.

(i) If $k = 0$, then (5.5) is automatically satisfied. As $m_k = 1$, then $m = \operatorname{lcm}(m_k, m_l) = m_l$ and $ml = m_l l = 2sl \gcd(2s, l)^{-1}$. As $m = m_l$, then (5.6) is equivalent to $s^{-1} m l^2 \equiv 2l^2 \gcd(2s, l)^{-1} \equiv 2 \bmod 2s$, and so $l(l \gcd(2s, l)^{-1}) \equiv 1 \bmod s$. Therefore, $\gcd(s, l) = 1$ and (5.6) is equivalent to $l^2 \equiv 1 \bmod s$ if l is odd and to $l^2 \equiv 2 \bmod 2s$ if l is even.

(ii) If $k = r$, then (5.5) is equivalent to $2s \mid ml$. If m_l is odd, then $m = \operatorname{lcm}(2, m_l) = 2m_l$ and (5.5) is equivalent to $s \mid m_l l$. If m_l is even, then $m = m_l$ and (5.5) is equivalent to $2s \mid m_l l$. As $m_l l = \operatorname{lcm}(l, 2s)$, then (5.5) is satisfied in both cases. If m_l is odd, then (5.6) is equivalent to $s^{-1} m l^2 = 2 m_l l^2 s^{-1} = 4l^2 \gcd(2s, l)^{-1} \equiv 2 \bmod 2s$. Therefore, $2l^2 \gcd(2s, l)^{-1} \equiv 1 \bmod s$ and so $\gcd(s, l) = 1$. If m_l is even, then (5.6) is equivalent to $m_l l^2 s^{-1} = 2l^2 \gcd(2s, l)^{-1} \equiv 2 \bmod 2s$. Therefore, $l^2 \gcd(2s, l)^{-1} \equiv 1 \bmod s$ and $\gcd(s, l) = 1$. If m_l is odd, then as l cannot also be odd, (5.6) is equivalent to $l^2 \equiv 1 \bmod s$. When m_l is even, (5.6) is given by $l^2 \equiv 1 \bmod s$ if l is odd. The case of even l cannot occur: if it did, then s must be even and $l^2 \equiv 2 \bmod 2s$, implying the contradiction $0 \equiv 2 \bmod 4$.

In both cases, the statement about m is immediate.

Case $u = 2m$. If $u = 2m$, then (5.1)–(5.3) are equivalent to

$$2r \mid k\,, \tag{5.7}$$
$$4rs \mid mkl\,, \tag{5.8}$$
$$2s \mid (2s)^{-1} m l^2 - 2\,. \tag{5.9}$$

By (5.7), we have $k \equiv 0 \bmod 2r$ and therefore $m = m_l$. Condition (5.8) is automatically satisfied. Condition (5.9) is equivalent to $(2s)^{-1} m l^2 = l^2 \gcd(2s, l)^{-1} \equiv 2 \bmod 2s$. If $l = 2l_1$, then $\gcd(2s, l) = 2$ and (5.9) is equivalent to $2l_1^2 \equiv 2 \bmod 2s$. When l is odd, (5.9) admits no solution. The statement about m is immediate. $\square$

Lemma 5.2. *For primitive $z \in L$, let $e := \operatorname{div}(z)$ and define x and y by $z^* = xv^* + yw^* + L$, where x is taken modulo $2r$ and y is taken modulo $2s$.*

(i) *If $\sigma_z \in \mathrm{O}^+(L)$, then $\sigma_z \in \Gamma$ if and only if e, x, y, z^2 satisfy the conditions of Lemma 5.1 with $r = d$, $s = n+1$, $u = z^2$, $m = e$ and $z^2 = -e$ or $-2e$.*

(ii) *If $-\sigma_z \in \mathrm{O}^+(L)$, then $-\sigma_z \in \Gamma$ if and only if e, x, y, z^2 satisfy the conditions of Lemma 5.1 with $r = n+1$, $s = d$, $u = z^2$, $m = e$ and $z^2 = -e$ or $-2e$.*

Proof. By (2.1), we have $\pm\sigma_z \in \mathrm{O}^+(L)$ if and only if $z^2 < 0$ and $z^2 \mid 2e$. If $z = e(xv^* + yw^* + l)$ for $l \in L$, then

$$\begin{aligned}\sigma_z(v^*) &= v^* - \frac{2(v^*, exv^* + eyw^* + el)}{z^2} z\,,\\ \sigma_z(w^*) &= w^* - \frac{2(w^*, exv^* + eyw^* + el)}{z^2} z\,.\end{aligned}$$

As $z^2 \mid 2e$, then $z^2 \mid 2e(v^*, l)$, implying

$$\begin{aligned}\sigma_z(v^*) &= v^* + (exd^{-1}z^{-2} - 2e(v^*, l)z^{-2})z + L \qquad (5.10)\\ &= v^* + exd^{-1}z^{-2}(exv^* + eyw^* + el) + L \qquad (5.11)\\ &= (1 + e^2x^2d^{-1}z^{-2})v^* + e^2xyd^{-1}z^{-2}w^* + L\,. \qquad (5.12)\end{aligned}$$

Equation (5.11) follows from (5.10) by noting that $z^2 \mid 2e(v^*, l)$. Equation (5.12) follows from (5.11) as $e^2xd^{-1}z^{-2} = (ex2^{-1}d^{-1})(2ez^{-2})$, $2d \mid ex$ and $z^2 \mid 2e$. Similarly, $\sigma_z(w^*)$ is given by

$$\begin{aligned}\sigma_z(w^*) &= w^* + (ey(n+1)^{-1}z^{-2} - 2e(w^*, l)z^{-2})z + L \qquad (5.13)\\ &= w^* + (ey(n+1)^{-1}z^{-2})(exv^* + eyw^* + el) + L \qquad (5.14)\\ &= e^2xy(n+1)^{-1}z^{-2}v^* + (1 + e^2y^2(n+1)^{-1}z^{-2})w^* + L\,, \qquad (5.15)\end{aligned}$$

with (5.14) following from (5.13) because $z^2 \mid 2e(w^*, l)$ and (5.15) following from (5.14) because $(n+1)z^2 \mid e^2y$.

By Proposition 2.1, we have $\pm\sigma_z \in \Gamma$ if and only if $\pm\sigma_z(v^*) = v^* + L$ and $\pm\sigma_z(w^*) = -w^* + L$. Therefore, $\sigma_z \in \Gamma$ if and only if

$$\begin{cases} z^2 d \mid ex\,,\\ 2d(n+1)z^2 \mid e^2xy\,,\\ 2(n+1) \mid 2 + e^2y^2z^{-2}(n+1)^{-1} \end{cases} \qquad (5.16)$$

(following from (5.11), (5.15) and (5.15), respectively). Similarly, $-\sigma_z \in \Gamma$ if and only if

$$\begin{cases} z^2(n+1) \mid ey\,,\\ 2(n+1)dz^2 \mid e^2xy\,,\\ 2d \mid 2 + e^2x^2d^{-1}z^{-2}\,. \end{cases} \qquad (5.17)$$

Condition (5.16) is precisely (5.1)–(5.3) with $r = d$, $s = n+1$, $u = -z^2$, $m = e$, $x = k$ and $y = l$, and condition (5.17) is precisely (5.1)–(5.3) with $r = n+1$, $s = d$, $u = -z^2$, $m = e$, $x = l$ and $y = k$. The result follows. □

Proposition 5.3. *Let $L = 2U \oplus \langle -2r \rangle \oplus \langle -2s \rangle$, where both $r, s > 1$, and let v and w generate the $\langle -2r \rangle$ and $\langle -2s \rangle$ factors of L, respectively. Suppose that $z \in L$ is primitive with divisor $m := \operatorname{div}(z)$ and length $u := -z^2 > 0$, and assume $\pm\sigma_z \in \Gamma$. Let $z^* := z/m = kv^* + lw^* + L \in D(L)$, where k and l are taken modulo $2r$ and $2s$, respectively; let $K_{r,s}^{u,m}(k,l) := z^\perp \subset L$; and let*

$\mathrm{gen}(K^{u,m}_{r,s}(k,l)) := (2,3;G,-\delta)$, *where* $G := D(K^{u,m}_{r,s}(k,l))$ *and* δ *is a finite quadratic form on* G. *Then,*

(i) *if* $u = m = 2s$, $k = 0$ *and* l *is odd, then* $G \cong C_{2r}$ *and*
$$\delta(a) = \left(\frac{a^2}{2r}\right) \bmod 2\mathbb{Z} \quad \text{for } (a) \in G\,;$$

(ii) *if* $u = 2m = 2s$, $k = 0$ *and* l *is even, then* $G \cong C_2 \oplus C_2 \oplus C_{2r}$ *and* δ *is given by*
$$\delta(a,b,c) = \frac{(1-l_1^2)a^2}{2s} + \frac{(1-l_1^2)ab}{2s} + \frac{sb^2}{2} + \frac{c^2}{2r} \bmod 2\mathbb{Z} \quad \text{for } (a,b,c) \in G\,,$$
where $2l_1 := l$;

(iii) *if* $u = m = 2s$, $k = r$, l *odd and* $l^2 \equiv rs \bmod 2s$, *then* $G \cong C_{2r}$ *and*
$$\delta(a) = \frac{(1-sr)a^2}{2r} \bmod 2\mathbb{Z} \quad \text{for } (a) \in G\,;$$

(iv) *if* $u = m = 2s$, $k = r$, l *even and* r *odd, then* $G \cong C_2 \oplus C_r$ *and*
$$\delta(a,b) = \frac{(-l^2+1)a^2}{2s} + \frac{l(-l^2+1)ab}{2s} + \frac{(-sr+1)b^2}{2r} \bmod 2\mathbb{Z} \quad \text{for } (a,b) \in G\,.$$

Furthermore, no other cases can occur.

Proof. We calculate $q_{z^\perp}$, following the approach of [Nik80, Proposition1.15.1]. As explained in [Nik80, §§ 4–5], if $M \subset P$ is a primitive embedding of lattices and $N := M^\perp \subset P$, then the series of overlattices
$$M \oplus N \subset P \subset P^\vee \subset M^\vee \oplus N^\vee$$
defines a totally isotropic subgroup
$$H_P := \frac{P}{M \oplus N} \subset D(M) \oplus D(N) \tag{5.18}$$
and natural projections $p_M\colon H_P \to D(M)$ and $p_N\colon H_P \to D(N)$. The overlattice P is uniquely determined by the subgroup H_P, and the embeddings $M \subset P$ and $N \subset P$ are primitive if and only if P_M and P_N are injective [Nik80, § 1.5, p. 111].

Let $L \subset W$ be a primitive embedding, where W is unimodular (suitable W always exists by [Nik80, Corollary 1.12.3]), and define $K := \langle z\rangle^\perp \subset L$ and $T := L^\perp \subset W$. By (5.18), if $w_L \in L^\vee$ represents an element of $p_L(H_W)$, then there exists a $w \in W$ such that $w = w_L + w_T$ and $w_T \in T^\vee$ is unique modulo T. This correspondence defines an isomorphism $\gamma_{TL}\colon q_L \to -q_T$ defined on representatives in $L^\vee$ and $T^\vee$ by $\gamma_{TL}\colon w_L \mapsto w_T$; see [Nik80, § 5]. (In the terminology of [CS99], the map γ_{TL} defines *'glue vectors'*.)

By the above discussion, and by considering the overlattice
$$K \oplus \langle z\rangle \oplus T \subset W\,,$$
we see that if $V := K^\perp \subset W$, then V is an overlattice of $\langle z\rangle \oplus T$ and γ_{VK} defines an isomorphism $q_K \cong -q_V$, which we use to calculate q_K. As in (5.18), the lattice V is defined by the totally isotropic subgroup
$$H_V \subset D(\langle z\rangle) \oplus D(T)\,,$$
and we let P_z and P_T denote the associated projections. As T is primitive in W, then T is primitive in V. As $\langle z\rangle$ is primitive in L and L is primitive in W, then $\langle z\rangle$ is primitive in W, and so $\langle z\rangle$ is primitive in V, implying that P_z and P_T are injective. Let α denote $z/u \in D(\langle z\rangle)$, and

let $z^* = z/m = (k,l) \in D(L) \cong C_{2r} \oplus C_{2s}$. The group $P_z(H_V)$ is cyclic, and we take a generator $\xi\alpha$, where $\xi \in \mathbb{Z}$. By the injectivity of P_T, we have $\xi\alpha + v_T = v \in V$, where $v_T \in T^\vee$ is unique modulo T. As $q_L \cong -q_T$ and $\xi\alpha \in L^\vee$ (noting that $(l, \xi\alpha + v_T) = (l, \xi\alpha) \in \mathbb{Z}$ for all $l \in L^\vee$), then

$$\xi\alpha = \xi m u^{-1} z^* \equiv \xi m u^{-1}(k,l) \in D(L)\,. \tag{5.19}$$

As $p_z(H_v)$ is cyclic and $p_z(H_V) \cong H_V$, then $H_V = \langle(\xi\alpha, \xi m u^{-1}(k,l))\rangle \subset D(\langle z\rangle) \oplus D(T)$. By [Nik80, Proposition 1.4.1 and Corollary 1.6.2],

$$-q_K = q_V = \left(q_{\langle z\rangle} \oplus q_T | H_V^\perp\right)/H_V\,,$$

which can be calculated once a suitable ξ is determined: we claim one can take $\xi = u/m$.

We assume $\xi \neq 0$. If $z^* \equiv 0 \bmod L$, this is immediate from (5.19). If $z^* \not\equiv 0 \bmod L$ and $\xi = 0$, then $H_V = 0$, implying that $\langle z\rangle \oplus T$ has no overlattice in $K^\perp$ (that is, $D(\langle z\rangle) \oplus D(T)$ has no non-trivial isotropic subgroup by [Nik80, § 4]), which contradicts the assumption $z^* \not\equiv 0 \bmod L$. Each ξ defines an element $\xi u^{-1} z \in L^\vee$ (that is, $\xi^{-1}u | m$). Conversely, for all $0 \neq \xi \in \mathbb{N}$ such that $\xi' u^{-1} z \in L^\vee$ (by the definition of γ_{TL}), there exists a $v_T \in T^\vee$ such that $v = \xi'\alpha + v_T \in W$ and $v \in K^\perp$. If $\xi' \in \mathbb{N}$ satisfies $\xi' | \xi$, $\xi' \neq \xi$ and $\xi'^{-1} u | m$, then ξ' defines an overlattice $V \subset V' \subset K^\perp$ distinct from V, implying $V \neq K^\perp$. Therefore, we can take $\xi = u/m$.

For the rest of the proof, we identify $D(\langle z\rangle)$ and $D(T)$ with $\mathbb{Z}/u\mathbb{Z}$ and $\mathbb{Z}/(2r)\mathbb{Z} \oplus \mathbb{Z}/(2s)\mathbb{Z}$, respectively. We use $x = (x_1, x_2, x_3)$ to denote an element of $D(\langle z\rangle) \oplus D(T)$, and we define the map $\gamma\colon H_V \to D(T)$ by $\xi\alpha \mapsto \xi m u^{-1}(k,l)$, where H_V is identified with $\langle \xi\alpha\rangle \subset D(\langle z\rangle)$ for suitable $\xi \in \mathbb{Z}$. We now consider cases for u, m, k, l using Lemma 5.2.

Case $u = m = 2s$, $k = 0$, l odd. The discriminant form on $D(\langle z\rangle) \oplus D(T)$ is given by

$$\frac{-a^2}{2s} + \frac{b^2}{2r} + \frac{c^2}{2s} \bmod 2\mathbb{Z} \tag{5.20}$$

for $(a,b,c) \in D(\langle z\rangle) \oplus D(T)$. We take $\xi = 1$, and so H_V is generated by $y_3 := (1, 0, l)$. An element $x \in D(\langle z\rangle) \oplus D(T)$ belongs to $H_V^\perp \subset D(\langle z\rangle) \oplus D(T)$ if and only if

$$\frac{-x_1}{2s} + \frac{l x_3}{2s} \equiv 0 \bmod \mathbb{Z}$$

or, equivalently, if

$$x_1 \equiv l x_3 \bmod 2s\,. \tag{5.21}$$

The elements $y_1 = (l, 0, 1)$, $y_2 = (0, 1, 0)$ both satisfy (5.21), and so $H_V^\perp = \langle y_1, y_2\rangle$. To determine relations, if $a y_1 + b y_2 = c y_3$ for $a, b, c \in \mathbb{N}$, then $b \equiv 0 \bmod 2r$. By Lemma 5.1, we have $l^2 \equiv 1 \bmod s$, implying $2y_1 = 2l y_3$, and so $y_1 \bmod H_V$ is of order 1 or 2. We have $y_1 \in H_V$ if and only if $l^2 \equiv 1 \bmod 2s$, implying

$$H_V^\perp / H_V \cong \begin{cases} C_2 \oplus C_{2r} & \text{if } l^2 \equiv 1 + s \bmod 2s\,, \\ C_{2r} & \text{if } l^2 \equiv 1 \bmod 2s\,. \end{cases}$$

The case $l^2 \equiv 1 + s \bmod 2s$ cannot occur as, by considering $K^{u,m}_{r,s}(k,l) \otimes \mathbb{Z}_2$ and applying [Nik80, Theorem 1.9.1 and Corollary 1.9.3], we obtain that $K^{u,m}_{r,s}(k,l)$ is of even rank, which is impossible by assumption. Therefore, $H_V^\perp / H_V \cong C_{2r}$ and

$$q_V(a) = \frac{a^2}{2r} \bmod 2\mathbb{Z}$$

for $a \in C_{2r} \cong \mathbb{Z}/(2r)\mathbb{Z}$.

Case $u = m = s$, $k = 0$, l even. The discriminant form of $D(\langle z\rangle) \oplus D(T)$ is given by

$$\frac{-a^2}{s} + \frac{b^2}{2r} + \frac{c^2}{2s} \bmod 2\mathbb{Z}$$

for $(a, b, c) \in D(\langle z\rangle) \oplus D(T)$. We take $\xi = 1$ and $y_3 := (1, 0, l)$ as a generator of H_V. An element $x \in D(\langle z\rangle) \oplus D(T)$ belongs to $H_V^{\perp}$ if and only if

$$\frac{-x_1}{s} + \frac{lx_3}{2s} \equiv 0 \bmod \mathbb{Z}$$

or, equivalently as $l = 2l_1$, if

$$x_1 \equiv l_1 x_3 \bmod s\,.$$

Therefore, $H_V^{\perp} = \langle y_1, y_2\rangle$, where $y_1 := (l_1, 0, 1)$ and $y_2 = (0, 1, 0)$. To determine relations, suppose $ay_1 + by_2 = cy_3$, where $a, b, c \in \mathbb{N}$, a is taken modulo $2s$, b is taken modulo $2r$ and c is taken modulo s. Then $b = 0$ and $a \equiv cl \bmod 2s$, implying that a is even. By Lemma 5.1, we have $l^2 \equiv 2 \bmod 2s$, and if $a = 2$ and $c = l$, then $ay_1 = cy_3$. Therefore, y_1 and y_2 modulo H_V are of order 2 and $2r$, respectively, and satisfy no non-trivial relations, implying $H_V^{\perp}/H_V \cong C_2 \oplus C_{2r}$. However, this case cannot occur for rank reasons, as for $u = m = 2s$, $k = 0$, l odd.

Case $u = 2m$, $m = s$, $k = 0$, l even. The form on $D(\langle z\rangle) \oplus D(T)$ is as in (5.20), and $l = 2l_1$. We take $\xi = 2$ and $y_4 := (2, 0, l) \in D(\langle z\rangle) \oplus D(T)$ as a generator of H_V. An element $x \in D(\langle z\rangle) \oplus D(T)$ belongs to $H_V^{\perp}$ if and only if

$$\frac{-x_1}{s} + \frac{l_1 x_3}{s} \equiv 0 \bmod \mathbb{Z}$$

or, equivalently, if

$$x_1 \equiv l_1 x_3 \bmod s\,.$$

As $2l_1^2 \equiv 2 \bmod 2s$, then $H_V^{\perp} = \langle y_1, y_2, y_3\rangle$, where $y_1 = (l_1, 0, 1)$, $y_2 = (0, 1, 0)$ and $y_3 = (s, 0, 0)$. Suppose $ay_1 + by_2 + cy_3 = dy_4$, where a is taken modulo $2s$, b is taken modulo $2r$, c is taken modulo 2 and d is taken modulo s. Then $b = 0$ and $(a - dl)(l_1, 0, 1) = c(s, 0, 0)$, implying $a \equiv dl \bmod 2s$ and $c \equiv 0 \bmod 2$. As l is even, then $a =: 2a_1$ is even and $a \equiv 2dl_1 \bmod 2s$ is equivalent to $a_1 \equiv dl_1 \bmod s$, which always admits a solution as $2l_1^2 \equiv 2 \bmod 2s$ implies $\gcd(l_1, s) = 1$. Therefore, y_1, y_2 and y_3 modulo H_V are of orders 2, $2r$ and 2, respectively, and satisfy no non-trivial relations. Hence, $H_V^{\perp}/H_V \cong C_2 \oplus C_2 \oplus C_{2r}$ and

$$q_V(a, b, c) \equiv \frac{(1 - l_1^2)a^2}{2s} - \frac{(1 - l_1^2)l_1 ab}{2s} + \frac{sb^2}{2} + \frac{c^2}{2r} \bmod 2\mathbb{Z}$$

for $(a, b, c) \in C_2 \oplus C_2 \oplus C_{2r}$.

Case $u = m = 2s$, $k = r$, l odd. The form on $D(\langle z\rangle) \oplus D(T)$ is as in (5.20). We take $\xi = 1$ and $y_3 := (1, r, l)$ as a generator for H_V. We have $x \in H_V^{\perp}$ if and only if

$$\frac{-x_1}{2s} + \frac{rx_2}{2r} + \frac{lx_3}{2s} \equiv 0 \bmod \mathbb{Z}$$

or, equivalently, if

$$\begin{cases} x_1 \equiv lx_3 \bmod 2s & \text{if } x_2 \text{ even}\,, \\ x_1 \equiv lx_3 + sx_2 \bmod 2s & \text{if } x_2 \text{ odd}\,. \end{cases} \tag{5.22}$$

The elements $y_1 := (l, 0, 1)$ and $y_2 := (s, 1, 0)$ both satisfy (5.22), and one checks that all $x \in H_V^\perp$ satisfy $x = x_3 y_1 + x_2 y_2$, implying $H_V^\perp = \langle y_1, y_2 \rangle$. By Lemma 5.1, we have $2y_1 = (2l, 0, 2) = 2l(1, r, l) = 2ly_3$, and so y_1 modulo H_V is of order 1 or 2. If $y_1 = ay_3$ for $a \in \mathbb{N}$, then $ar \equiv 0 \bmod 2r$ and so a is even; on the other hand, $a \equiv l \bmod 2s$ implies that a is odd. Therefore, y_1 modulo H_V is of order 2. Now suppose $ay_2 = by_3$ for $a, b \in \mathbb{N}$ taken modulo $2r$ and $2s$, respectively. By Lemma 5.1, we have $2l^2 \equiv 2 \bmod 2s$, and as l is odd, then $\gcd(l, 2s) = 1$. Therefore, $b = 0$ and $a = 0$, implying that y_2 modulo H_V is of order $2r$.

We show that $y_1 + ay_2 = by_3$ if and only if $l^2 - 1 \equiv rs \bmod 2s$. Suppose $l - as - b \equiv 0 \bmod 2s$, $a - br \equiv 0 \bmod 2r$ and $1 - bl \equiv 0 \bmod 2s$. As $\gcd(l, 2s) = 1$, then $b \equiv l^{-1} \bmod 2s$. As l is odd, then b is odd, and so $a \equiv r \bmod 2r$. Therefore, $l - rs - l^{-1} \equiv 0 \bmod 2s$ and $l^2 - rs - 1 \equiv 0 \bmod 2s$. The converse is immediate, and so

$$H_V^\perp / H_V \cong \begin{cases} C_{2r} & \text{if } l^2 - 1 \equiv rs \bmod 2s\,, \\ C_2 \oplus C_{2r} & \text{if } l^2 - 1 \not\equiv rs \bmod 2s\,. \end{cases}$$

As for the case of $u = m = 2s$, $k = 0$, l odd, the case $l^2 - 1 \not\equiv rs \bmod 2s$ cannot occur for rank reasons. In the remaining case, q_V is given by

$$q_V(a) = \frac{(1 - sr)a^2}{2r} \bmod 2\mathbb{Z}$$

for $a \in C_{2r}$.

Case $u = m = 2s$, $k = r$, l even. The quadratic form on $D(\langle z \rangle) \oplus D(T)$ is as in (5.20). We take $\xi = 1$ and $y_3 := (1, r, l)$ as a generator of H_V. As for the case of $u = m = 2s$, $k = r$, l odd, $H_V^\perp = \langle y_1, y_2 \rangle$, where $y_1 := (l, 0, 1)$ and $y_2 := (s, 1, 0)$. By Lemma 5.1, we have $l^2 \equiv 1 \bmod s$ and so $2l^2 \equiv 2 \bmod 2s$. Therefore, $2y_1 = 2ly_3$ and so y_1 modulo H_V is of order 1 or 2. If $y_1 = by_3$ for b taken modulo $2s$, then $bl \equiv 1 \bmod 2s$, implying $\gcd(l, 2s) = 1$, which cannot hold as l is even, so y_1 modulo H_V is of order 2. Suppose $ay_2 = by_3$ for a and b taken modulo $2r$ and $2s$, respectively. Then $bl \equiv 0 \bmod 2s$, and as $\gcd(l, s) = 1$, we have $b \equiv 0 \bmod s$. If b is even, then $a = 0$; if b is odd, then as $l^2 \equiv 1 \bmod s$ and l is even, s is odd. As $as \equiv b \bmod 2s$, then $a \equiv b \bmod 2$, and so a is odd, and as $a \equiv br \bmod 2r$ and b is odd, then $a \equiv r \bmod 2r$, implying that r is odd. Hence, y_2 modulo H_V is of order $2r$ if r is even and of order r if r is odd. If $y_1 + ay_2 = by_3$ for $a, b \in \mathbb{N}$, then $1 - bl \equiv 0 \bmod 2s$, which cannot hold as l is even. Therefore, there are no non-trivial relations between y_1 and y_2 modulo H_V, implying

$$H_V^\perp / H_V \cong \begin{cases} C_2 \oplus C_{2r} & \text{if } r \text{ is even}\,, \\ C_2 \oplus C_r & \text{if } r \text{ is odd}\,. \end{cases}$$

As for $u = m = 2s$, $k = 0$, l odd, the case of even r cannot occur for rank reasons. Therefore, if r is odd, then q_V is given by

$$q_V(a, b) \equiv \frac{(-l^2 + 1)a^2}{2s} + \frac{l(l^2 - 1)ab}{2s} + \frac{(1 - sr)b^2}{2r} \bmod 2\mathbb{Z}$$

for $(a, b) \in C_2 \oplus C_r$. □

6. Extension criteria

We now prove the extension criteria used in Theorem 3.2, starting with an exercise on toric varieties. For a finite group $G \subset \mathrm{GL}(n, \mathbb{C})$, we let X_G denote the quotient $\mathbb{C}^n / G$.

Lemma 6.1. *Take coordinates (y_1, y_2, y_3, y_4) for $\mathbb{C}^4$, and suppose that $g \in \mathrm{GL}(4, \mathbb{C})$ is given by $\frac{1}{6}(1,1,0,2)$, $\frac{1}{6}(0,1,1,2)$ or $\frac{1}{6}(1,1,1,2)$. Let*

$$\Omega = h(y_1, y_2, y_3, y_4)\frac{(dy_1 \wedge \cdots \wedge dy_4)^{\otimes k}}{(y_1 \dots y_4)^{\otimes k}}$$

be a g-invariant differential form for $\mathbb{C}^4$. If h vanishes to order at least $3k$ along $y_2 = 0$ and to order at least k along y_i for all $i \neq 2$, then Ω extends to a pluricanonical form on a desingularisation of $X_{\langle g\rangle}$.

Proof. Case $g = \frac{1}{6}(1,1,0,2)$. Let $g = \frac{1}{6}(1,1,0,2)$. As in [Tai82], the quotient $X_{\langle g\rangle}$ can be realised as a toric variety $X(\sigma)$ as follows. Let N be a lattice with basis $p_1 = (1,0,0,0)$, $p_2 = (0,1,0,0)$, $p_3 = (0,0,1,0)$, $p_4 = (0,0,0,1)$, and let $\overline{N}$ be an overlattice of N such that $\overline{N}/N$ is generated by $\frac{1}{6}(1,1,0,2) \in N \otimes \mathbb{Q}$. On the basis $\{p_i\}$, the lattice $\overline{N}$ is spanned by $q_1 = \frac{1}{6}(1,1,0,2)$, $q_2 = (0,1,0,0)$, $q_3 = (0,0,1,0)$, $q_4 = (0,0,0,1)$, and the cone σ, given by

$$\sigma = \left\{\sum_{i=1}^{4} \lambda_i p_i \mid \lambda_i \geqslant 0\right\} \subset \overline{N} \otimes \mathbb{R},$$

is generated by $(6,-1,0,-2)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$ on the basis $\{q_i\}$. The toric variety $X(\sigma)$ can be resolved by subdividing σ to obtain a fan Σ whose rays are given by $u_1 = (6,-1,0,-2)$, $u_2 = (0,1,0,0)$, $u_3 = (0,0,1,0)$, $u_4 = (0,0,0,1)$, $u_5 = (3,0,0,-1)$, $u_6 = (1,0,0,0)$ on the basis $\{q_i\}$ and whose maximal cones are given by $\{u_1, u_3, u_4, u_6\}$, $\{u_1, u_3, u_5, u_6\}$, $\{u_2, u_3, u_4, u_6\}$, $\{u_2, u_3, u_5, u_6\}$. (We used the package `NormalToricVarieties` for the computer algebra system `Macaulay2` to perform the calculation, but the result can be easily checked by hand using standard toric geometry [Ful93].)

As in [Tai82], if

$$\begin{aligned}\Omega &= h(y_1, \dots, y_4)\frac{(dy_1 \wedge \cdots \wedge dy_4)^{\otimes k}}{(y_1 \cdots y_4)^k} \\ &= h(y_1^*, \dots, y_4^*)\frac{(dy_1^* \wedge \cdots \wedge dy_4^*)^{\otimes k}}{(y_1^* \cdots y_4^*)^k},\end{aligned}$$

where $y_1^*, \dots, y_4^*$ are coordinates defined by the rays of a maximal cone of Σ, then Ω extends to a pluricanonical form on $X(\Sigma)$ if h vanishes to order $\mathrm{ord}_{y_j^*} h \geqslant k$ along each of the divisors $y_i^* = 0$. If y_i^* is given by the ray $v = \sum \lambda_j p_j$, where $\lambda_j \in \mathbb{Q}$, $\lambda_i \geqslant 0$, then

$$\mathrm{ord}_{y_i^*} h = \sum \lambda_j \,\mathrm{ord}_{y_j} h\,.$$

On the basis $\{p_i\}$, we have $u_1 = (1,0,0,0)$, $u_2 = (0,1,0,0)$, $u_3 = (0,0,1,0)$, $u_4 = (0,0,0,1)$, $u_5 = \frac{1}{2}(1,1,0,0)$ and $u_6 = \frac{1}{6}(1,1,0,2)$. From the statement of the lemma, $\mathrm{ord}_{y_2}(h) \geqslant 3k$ and $\mathrm{ord}_{y_i}(h) \geqslant k$ for $i \neq 2$, and we calculate $\mathrm{ord}_{y_1^*}(h) \geqslant k$, $\mathrm{ord}_{y_2^*}(h) \geqslant 3k$, $\mathrm{ord}_{y_3^*}(h) \geqslant k$, $\mathrm{ord}_{y_4^*}(h) \geqslant k$, $\mathrm{ord}_{y_5^*}(h) \geqslant 2k$ and $\mathrm{ord}_{y_6^*}(h) \geqslant k$.

Case $g = \frac{1}{6}(0,1,1,2)$. We proceed as for $g = \frac{1}{6}(1,1,0,2)$. For $\{p_i\}$ defined previously, we take generators $q_1 = \frac{1}{6}(0,1,1,2)$, $q_2 = (1,0,0,0)$, $q_3 = (0,0,1,0)$ and $q_4 = (0,0,0,1)$ for $\overline{N} \supset N$, and we set $X_{\langle g\rangle} = X(\sigma)$, where the cone σ is generated by the rays $(0,1,0,0)$, $(6,0,-1,-2)$, $(0,0,1,0)$ and $(0,0,0,1)$ on the basis $\{q_i\}$. We resolve $X(\sigma)$ by subdividing σ to obtain the fan Σ with rays $u_1 = (0,1,0,0)$, $u_2 = (6,0,-1,-2)$, $u_3 = (0,0,1,0)$, $u_4 = (0,0,0,1)$, $u_5 = (3,0,0,-1)$ and $u_6 = (1,0,0,0)$ on the basis $\{q_i\}$ and maximal cones $\{u_1, u_2, u_4, u_6\}$, $\{u_1, u_2, u_5, u_6\}$, $\{u_1, u_3, u_4, u_6\}$

and $\{u_1, u_3, u_5, u_6\}$. We have $u_1 = (1,0,0,0)$, $u_2 = (0,1,0,0)$, $u_3 = (0,0,1,0)$, $u_4 = (0,0,0,1)$, $u_5 = \frac{1}{2}(0,1,1,0)$ and $u_6 = \frac{1}{6}(0,1,1,2)$ on the basis $\{p_i\}$, from which we calculate $\mathrm{ord}_{z_i^*}(h) \geqslant k$ for all i and $\mathrm{ord}_{z_2^*}(h) \geqslant 3k$.

Case $g = \frac{1}{6}(1,1,1,2)$. Similarly, for $g = \frac{1}{6}(1,1,1,2)$, we realise $X_{\langle g\rangle}$ as the toric variety $X(\sigma)$, where $\overline{N}$ is generated by $q_1 = \frac{1}{6}(1,1,1,2)$, $q_2 = (0,1,0,0)$, $q_3 = (0,0,1,0)$, $q_4 = (0,0,0,1)$ on the basis $\{p_i\}$, where $\{p_i\}$ is as before and the cone σ is generated by the rays $(6,-1,-1,-2)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$ on $\{q_i\}$. We resolve $X(\sigma)$ by subdividing σ, obtaining a fan Σ with rays $u_1 = (6,-1,-1,-2)$, $u_2 = (0,1,0,0)$, $u_3 = (0,0,1,0)$, $u_4 = (0,0,0,1)$, $u_5 = (3,0,0,-1)$, $u_6 = (1,0,0,0)$ on $\{q_i\}$ and maximal cones $\{u_1,u_2,u_4,u_6\}$, $\{u_1,u_2,u_5,u_6\}$, $\{u_1,u_3,u_4,u_6\}$, $\{u_1,u_3,u_5,u_6\}$, $\{u_2,u_3,u_4,u_6\}$ and $\{u_2,u_3,u_5,u_6\}$. On the basis $\{p_i\}$, the rays u_i are given by $u_1 = (1,0,0,0)$, $u_2 = (0,1,0,0)$, $u_3 = (0,0,1,0)$, $u_4 = (0,0,0,1)$, $u_5 = \frac{1}{2}(1,1,1,0)$ and $u_6 = \frac{1}{6}(1,1,1,2)$, from which we conclude $\mathrm{ord}_{z_i^*}(h) \geqslant k$ for all i and $\mathrm{ord}_{z_2^*}(h) \geqslant 3k$. □

Proposition 6.2. *Suppose that $\mathcal{E}$ is taken so that for each primitive sublattice $2U \subset L$, there exists a embedding $2U \oplus \langle v\rangle \hookrightarrow L \in \mathcal{E}$ for $v \in (2U)^\perp$. Then the differential form $\Omega(f)$ of Theorem 3.2 extends to a pluricanonical form on $\widetilde{\mathcal{F}}(\Gamma)$ above the singular locus of $\mathcal{F}(\Gamma)$.*

Proof. By [Tai82, Proposition 3.2], a G-invariant pluricanonical form on $\mathbb{C}^n$ extends to a pluricanonical form on a desingularisation of X_G if and only if it extends to a pluricanonical form on a desingularisation of $X_{\langle g\rangle}$ for each $g \in G$. We use this criterion to show that $\Omega(F)$ extends to $\widetilde{\mathcal{F}}(\Gamma)$ by studying $\Omega(F)$ in a neighbourhood of a non-canonical singularity. By Lemma 4.9, non-canonical singularities of $\mathcal{F}(\Gamma)$ correspond to the fixed loci of $\Delta_\iota(g_6, I)$ or $\Delta_\iota(I, g_6) \in \Gamma$ for a primitive embedding $\iota\colon 2U \hookrightarrow L$. Here we consider the case of $g := \Delta_\iota(g_6, I)$, with $\Delta_\iota(I, g_6)$ following by an identical argument.

For an arbitrary primitive embedding $2U \subset L$, let $B := \iota(2U)^\perp \subset L$, and, for v as in the statement of the lemma, suppose that $u \in B$ is primitive and orthogonal to v. We begin by defining an isomorphism $\mathcal{H} \to \mathcal{D}_L$, following [GHS13, § 8.2, p. 508]. Let $C^+(U \oplus B)$ be a connected component of the cone

$$C(B \oplus U) = \{x \in (B \oplus U) \otimes \mathbb{R} \mid (x,x) > 0\}\,,$$

and let $\mathcal{H}$ be the tube domain

$$\mathcal{H} := (B \oplus U) \otimes \mathbb{R} + iC^+(B \oplus U)\,.$$

For $i = 1,2$, let $\{e_i, f_i\}$ be standard bases for the two copies of U in $\iota(2U)$, and let $(x_1,x_2,x_3,x_4) \mapsto x_1v + x_2u + x_3e_1 + x_4f_1 \in \mathcal{H} \subset (B \oplus U)\otimes\mathbb{C}$ define coordinates for $\mathcal{H}$. We define the isomorphism $\mathcal{H} \to \mathcal{D}_L$ by

$$\mathcal{H} \ni \underline{x} := (x_1,x_2,x_3,x_4) \longmapsto \left[x_1v + x_2u + x_3e_1 + x_4f_1 + e_2 + \frac{1}{2}x_6f_2\right] \in \mathcal{D}_L \subset \mathbb{P}(L \otimes \mathbb{C})\,,$$

where $x_6 := -\big(2x_3x_4 + x_1^2v^2 + 2x_1x_2(u,v) + x_2^2u^2\big)$. If

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\,,$$

then

$$\Delta_\iota(h, I)\colon \underline{x} \longmapsto \left(\frac{x_1}{a-bx_4}, \frac{x_2}{a-bx_4}, x_3 - \frac{b(x_1^2v^2 + x_2^2u^2)}{2(a-bx_4)}, \frac{dx_4 - c}{a-bx_4}\right). \tag{6.1}$$

As we fixed a component of the cone $C(U \oplus B)$, if $\Delta_\iota(g_6, I)\underline{x} = \underline{x}$, then $x_4 = e^{\pi i/3} =: \xi$, x_3 is free and $x_1 = x_2 = 0$.

Following the approach of [Car57] (and the explanation of [Huy16, § 15, p. 331]), we introduce local coordinates on $\mathcal{H}$ linearising the action of an order n element $g \in \Gamma$. For a fixed point $\underline{w} \in \mathcal{H}$ of g, let $\underline{x}' := \underline{x} - \underline{w}$ and $(g_1, \dots, g_4) := g(\underline{x}')$. If

$$\mathbf{d}_0(g) := \left.\left(\frac{\partial g_i}{\partial x'_j}\right)\right|_{\underline{x}'=0},$$

then

$$\underline{y}(\underline{x}') := \frac{1}{n}\sum_{i=1}^{n} \mathbf{d}_0(g)^{-i} g^i(\underline{x}')$$

defines local coordinates $\underline{y} = (y_1, y_2, y_3, y_4)$ around $\underline{w}$ on which g acts by $\mathbf{d}_0(g)$. Therefore, if $g = \Delta_\iota(g_6, I)$, then for a 6th root of unity ξ,

$$\mathbf{d}_0(g) = \operatorname{diag}\big(\xi, \xi, 1, \xi^2\big)\,, \tag{6.2}$$

and from (6.1) and (6.2),

$$\underline{y} = \big(x'_1 Q_1(x'_4), x'_2 Q_2(x'_4), Q_3(\underline{x}'), Q_4(\underline{x}')\big)\,, \tag{6.3}$$

where all Q_i are holomorphic and neither Q_1 nor Q_2 vanishes at zero.[1] As the modular form $f(\underline{x}')$ vanishes to order $2k$ along $x'_2 = 0$, then $f(\underline{y})$ vanishes to order $2k$ along $y_2 = 0$. By Lemma 6.1, the differential form $\Omega(f)$ extends to $\widetilde{\mathcal{F}}(\Gamma)$ above a fixed point of $\Delta_\iota(g_6, I)$, from which the result follows. □

Proposition 6.3. *Suppose that $\mathcal{E}$ is taken so that for each boundary curve F, there exists an embedding $u^\perp \hookrightarrow L \in \mathcal{E}$ for a vector u in the lattice B of (4.5). Then the differential form $\Omega(f)$ of Theorem 3.2 extends to a pluricanonical form on $\widetilde{\mathcal{F}}(\Gamma)$ above all boundary curves.*

Proof. We proceed as in Proposition 6.2. By Lemma 4.10, it suffices to consider points in the boundary fixed by $g \in G(F)$ acting as $g = \frac{1}{6}(1,1,1,2)$ or $\frac{1}{6}(0,1,1,2)$. In either case, we take the basis $\{f_i\}_{i=1}^6$ of $L \otimes \mathbb{Q}$ defined in Lemma 4.11, on which g is represented by

$$g = \begin{pmatrix} U & 0 & W' \\ 0 & I & 0 \\ 0 & 0 & Z \end{pmatrix} \in N(F)_{\mathbb{Z}} \quad \text{for } Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})\,.$$

If $(z, \underline{w}, \tau)$ are the coordinates for $\mathcal{D}_L(F)$ defined in (3.7) and $r = \exp_e(z)$ (where e is as in (3.2)), then $(r, \underline{w}, \tau)$ are coordinates for $(\mathcal{D}_L(F)/U(F)_{\mathbb{Z}})_{\{\sigma_i\}}$. By (3.8), for either of the above g fixing $P = (0, \underline{w}_0, \tau_0)$, we see that τ_0 is an elliptic point of $\mathbb{H}^+$, $(c\tau_0 + d)^{-1} = \xi$ and $\underline{w}_0 = 0$, where $\xi = e^{\pi i/3}$. As in Proposition 6.2, we define coordinates $\underline{y}$ on which g acts linearly. By (3.8), we have

$$\mathbf{d}_0(g) = \begin{pmatrix} * & 0 & * \\ 0 & \frac{-c}{c\tau + d} I & 0 \\ 0 & 0 & * \end{pmatrix},$$

[1]To obtain the terms

$$Q_1(x'_4) = Q_2(x'_4) = \frac{1}{3}\left(1 - \frac{1}{\xi^2(x'_4 + \xi)} + \frac{1}{\xi^4(x'_4 + \xi - 1)}\right)$$

in (6.3) with minimal calculation, one can simply determine $\underline{y}$ for $\Delta(g_3, I)$ and note that $\Delta(-I, I)$ acts by -1 on y_1, y_2 but fixes y_3, y_4, implying that $\underline{y}$ also gives linear coordinates for $\Delta(g_6, I)$.

and we obtain coordinates $\underline{y} = (y_i)$ where $y_2 = w_1$ and $y_3 = w_2$. In particular (by selecting an appropriate basis of B), we can assume that $y_2 = 0$ is a local equation for the closure of $\mathcal{D}_L(u)$ in a neighbourhood of P. By applying a further transformation of the form

$$\begin{pmatrix} * & 0 & 0 \\ 0 & I & 0 \\ * & 0 & * \end{pmatrix},$$

we obtain coordinates on which g acts as $\mathrm{diag}(*, \xi, \xi, \xi^2)$. As in Proposition 6.2, the result follows from Lemma 6.1. □

7. Low-weight cusp forms

We now prove the existence of low-weight cusp forms for Γ by studying dimension formulae for vector-valued modular forms, following a suggestion of the anonymous referee. In this section, we will assume that $L = 2U \oplus \langle -2(n+1) \rangle \oplus \langle -2d \rangle$, where $n, d > 0$. Our results should be compared with the work of Ma on $\mathrm{O}(D(L))$-invariant vector-valued modular forms [Ma18a].

7.1 Vector-valued modular forms

The *metaplectic group* $\mathrm{Mp}(2,\mathbb{Z})$ is a double cover of $\mathrm{SL}(2,\mathbb{Z})$ whose elements are given by pairs of the form $(M, \phi(\tau))$, where

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

and $\phi(\tau)$ is a holomorphic function on the upper-half plane $\mathbb{H}^+$ satisfying $\phi(\tau)^2 = c\tau + d$; see [Bor98]. The metaplectic group is generated by the elements

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right) \quad \text{and} \quad S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right),$$

and the centre of $\mathrm{Mp}(2,\mathbb{Z})$ is generated by

$$Z := S^2 = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \sqrt{-1} \right).$$

Definition 7.1 ([Bor98]). Suppose that $\rho\colon \mathrm{Mp}(2,\mathbb{Z}) \to \mathrm{GL}(n,\mathbb{C})$ is a representation of $\mathrm{Mp}(2,\mathbb{Z})$, and let $l \in \frac{1}{2}\mathbb{Z}$. A function $f\colon \mathbb{H}^+ \to \mathbb{C}^n$ that is holomorphic on $\mathbb{H}^+$ and at cusps is said to be a *vector-valued modular form* of type ρ and weight l if it satisfies

$$f(M\tau) = \rho(M,\phi)\phi(\tau)^{2l} f(\tau)$$

for all $(M,\phi) \in \mathrm{Mp}(2,\mathbb{Z})$. We denote the space of all such forms by $M_l(\rho)$ and the subspace of cusp forms by $S_l(\rho)$.

The dimensions of $M_k(\rho)$ and $S_k(\rho)$ can often be calculated using Theorem 7.2.

Theorem 7.2 ([Sko85], [ES95, § 4.2, Theorem (Dimension formula [23])]). *Let $\rho\colon \mathrm{Mp}(2,\mathbb{Z}) \to \mathrm{GL}(m,\mathbb{Z})$ be a representation with finite image such that $\rho(\xi_4^2 I, \xi_4) = \xi_4^{-2k}$ for all 4th roots of* *unity ξ_4. If $k \in \frac{1}{2}\mathbb{Z}$, then*

$$\begin{aligned}\dim S_{2-k}(\rho) - \dim M_k(\overline{\rho}) = \frac{1-k}{12}m - \frac{1}{4}\operatorname{Re}(e^{\pi i k/2}\operatorname{tr}\overline{\rho}(S)) \\ - \frac{2}{3\sqrt{3}}\operatorname{Re}(e^{\pi i(2k+1)/6}\operatorname{tr}(\overline{\rho}(ST))) - \frac{1}{2}\alpha(\overline{\rho}) + \sum_{j=1}^{m}\mathbb{B}_1(\lambda_j)\,, \end{aligned} \tag{7.1}$$

where the $\lambda_j \in \mathbb{C}$ are the eigenvalues of $\overline{\rho}(T)$, $\alpha(\overline{\rho})$ is the number of λ_j such that $e^{2\pi i\lambda_j} = 1$ and

$$\mathbb{B}_1(x) = \begin{cases} x' - 1/2 & \text{if } x \in x' + \mathbb{Z} \text{ for } 0 < x' < 1\,, \\ 0 & \text{if } x \in \mathbb{Z}\,. \end{cases}$$

Definition 7.3. The *Weil representation* ρ_M is defined for a lattice M of signature (b_+, b_-) by

$$\rho_A(T)(\mathbf{e}_x) := e((x,x)/2)\mathbf{e}_x\,, \tag{7.2}$$

$$\rho_A(S)(\mathbf{e}_x) := \frac{\xi_8^{b_- - b_+}}{|D(M)|^{1/2}} \sum_{y \in D(M)} e(-(x,y))\mathbf{e}_y\,, \tag{7.3}$$

where $\mathbf{e}_x \in \mathbb{C}[D(M)]$ is the basis element of the group algebra $\mathbb{C}[D(M)]$ corresponding to the element $x \in D(M)$ and $e(y) := e^{2\pi i y}$ for $y \in \mathbb{C}$.

If M is a lattice of signature $(2, m)$ with $m \geqslant 3$ and $2U \subset M$, then there exists an injective $\mathrm{O}(D(L))$-equivariant linear map (which, as explained in [Ma18b], is given by the Borcherds–Gritsenko lift [Gri95, Bor98])

$$S_l(\rho_M) \longrightarrow S_k(\widetilde{\mathrm{O}}^+(M))\,, \tag{7.4}$$

where $k = l + m/2 - 1$ and $l \in \frac{1}{2}\mathbb{Z}$ satisfies $l \equiv m/2 \bmod \mathbb{Z}$.

If $\Gamma^* := \langle \Gamma, \widetilde{\mathrm{O}}^+(L)\rangle$, then $\widetilde{\mathrm{O}}^+(L) \trianglelefteq \Gamma^*$ and, by Proposition 2.1, the quotient $G := \Gamma^*/\widetilde{\mathrm{O}}^+(L) \cong C_2$ is generated by the element $\sigma_{\underline{w}}$. If $H = \langle \rho(-I, i), \sigma_{\underline{w}}\rangle$, then the restriction of ρ_L to the subspace $S_2(\rho_L)^H \subset S_2(\rho_L)^G$ satisfies the conditions of Theorem 7.2, which we use in Theorem 7.14 and Corollary 7.15 to obtain conditions for $S_2(\rho_L)^H \neq 0$ and so, by (7.4), for $S_3(\Gamma) \neq 0$.

7.2 The representation ρ_L^{inv}

Let ρ_L^{inv} be the restriction of ρ_L to $\mathbb{C}[D(L)]^H$.

Definition/Lemma 7.4. If $\mathbb{C}[D(L)] \cong C_{2(n+1)} \oplus C_{2d}$, let

$$\mathcal{I} := \{(a,b) \in \mathbb{Z}^2 \mid 0 < a < n+1 \text{ and } 0 \leqslant b \leqslant d\}\,,$$

$$\nu_{(a,b)} := \begin{cases} \mathbf{e}_{(a,b)} - \mathbf{e}_{(-a,-b)} + \mathbf{e}_{(a,-b)} - \mathbf{e}_{(-a,b)} & \text{if } 0 < a < n+1 \text{ and } 0 < b < d\,, \\ \mathbf{e}_{(a,b)} - \mathbf{e}_{(-a,-b)} & \text{if } b = 0 \text{ or } d\,. \end{cases}$$

Then $\{\nu_{(a,b)} \mid (a,b) \in \mathcal{I}\}$ is a basis for $\mathbb{C}[D(L)]^H$.

Proof. This is immediate from the definition. □

Lemma 7.5. *We have $\rho_L^{\mathrm{inv}}(T)\nu_{(a,b)} = e((a,b)^2/2)\nu_{(a,b)}$.*

Proof. Immediate from (7.2) as $(a,b)^2 = (\pm a, \pm b)^2$. □

Lemma 7.6. *On the basis of Definition/Lemma 7.4, if $(a,b) \in \mathcal{I}$, then the $\nu_{(a,b)}$th coordinate of $\rho_L^{\mathrm{inv}}(S)\nu_{(a,b)}$ is given by*

$$\lambda \times \begin{cases} \displaystyle\sum_{\delta_1,\delta_2=\pm1} \delta_1 e\left(\frac{\delta_1 a^2}{2(n+1)} + \frac{\delta_2 b^2}{2d}\right) & \text{if } 0 < b < d\,, \\ e\left(\dfrac{a^2}{2(n+1)} + \dfrac{b^2}{2d}\right) - e\left(\dfrac{-a^2}{2(n+1)} + \dfrac{b^2}{2d}\right) & \text{if } b = 0 \text{ or } d\,, \end{cases}$$

and the $\nu_{(a,b)}$th coordinate of $\rho_L^{\mathrm{inv}}(ST)\nu_{(a,b)}$ is given by

$$\lambda \times \begin{cases} \displaystyle\sum_{\delta_1,\delta_2=\pm1} \delta_1 e\left(\frac{(2\delta_1-1)a^2}{4(n+1)} + \frac{(2\delta_2-1)b^2}{4d}\right) & \text{if } 0 < b < d\,, \\ e\left(\dfrac{a^2}{4(n+1)} + \dfrac{b^2}{4d}\right) - e\left(\dfrac{-3a^2}{4(n+1)} + \dfrac{b^2}{4d}\right) & \text{if } b = 0 \text{ or } d\,, \end{cases}$$

where $\lambda = i/2\sqrt{d(n+1)}$.

Proof. This follows from a routine calculation using (7.3) and Definition/Lemma 7.4. □

7.3 Bounds for traces

We now bound $\operatorname{tr}\rho_L^{\mathrm{inv}}(S)$ and $\operatorname{tr}\rho_L^{\mathrm{inv}}(ST)$. For the convenience of the reader, we begin by summarising some standard results on quadratic Gauss sums [BEW98]. For $a,b,c \in \mathbb{Z}$ with $b > 0$, we let $\mathbb{G}(a,b,c)$ denote the generalised quadratic Gauss sum

$$\mathbb{G}(a,b,c) := \sum_{m=0}^{c} e\left(\frac{am^2}{b}\right),$$

and we use $\mathbb{G}(a,b)$ to denote the classical quadratic Gauss sum $\mathbb{G}(a,b,b-1)$. We will often use the elementary property

$$\mathbb{G}(a,b) = (a,b)\mathbb{G}\left(\frac{a}{(a,b)}, \frac{b}{(a,b)}\right) \tag{7.5}$$

for $(a,b) \neq 1$ and $b > 0$.

Theorem 7.7 ([Gau08, BEW98]). *If $(a,b) = 1$, then*

$$\mathbb{G}(a,b) = \begin{cases} 0 & \text{if } b \equiv 2 \bmod 4\,, \\ \delta_b\sqrt{b}\left(\dfrac{a}{b}\right) & \text{if } b \text{ is odd}\,, \\ (1+i)\delta_a^{-1}\sqrt{b}\left(\dfrac{b}{a}\right) & \text{if } a \text{ is odd and } 4|b\,, \end{cases}$$

where $\delta_x = 1$ if $x \equiv 1 \bmod 4$ and $\delta_x = i$ if $x \equiv 3 \bmod 4$ and $(\frac{\cdot}{n})$ is the Jacobi symbol.

Theorem 7.8 (Reciprocity for generalised quadratic Gauss sums [BEW98, Theorem 1.2.2]). *If $a,b \in \mathbb{Z}\backslash\{0\}$, where $b > 0$ and ab is even, then*

$$\sum_{m=0}^{b-1} e^{\pi i a m^2/b} = e^{\operatorname{sgn}(a)\pi i/4}\left|\frac{b}{a}\right|^{1/2} \sum_{m=0}^{|a|-1} e^{-\pi i b m^2/a}.$$

Lemma 7.9. *If a and d are positive integers, then*

$$\mathbb{G}(a,4d,d-1) = \frac{1}{2}\left(1 - e\left(\frac{ad}{4}\right) + \xi_8\mathbb{G}(-d,a)\sqrt{\frac{2d}{a}}\right).$$

Proof. By Theorem 7.8,

$$\sum_{m=0}^{2d-1} e^{\pi i(am^2/2d)} = \xi_8 \sqrt{2d/a} \sum_{m=0}^{a-1} e^{-2\pi i dm^2/a}.$$

As

$$\sum_{m=0}^{2d-1} e^{\pi i(am^2/2d)} = \sum_{m=0}^{d-1} e^{\pi i(am^2/2d)} + \sum_{m=1}^{d} e^{\pi i(a(2d-m)^2/2d)} = -1 + e^{a\pi i d/2} + 2\sum_{m=0}^{d-1} e^{\pi i(am^2/2d)},$$

then

$$\sum_{m=0}^{d-1} e\left(\frac{am^2}{4d}\right) = \frac{1}{2}\left(1 - e\left(\frac{ad}{4}\right) + \xi_8\sqrt{2d/a}\sum_{m=0}^{a-1} e\left(\frac{-dm^2}{a}\right)\right)$$

and the result follows. □

LEMMA 7.10. *If $d, n > 0$, then $\operatorname{Re}\operatorname{tr}\rho_L^{\mathrm{inv}}(S) = 0$.*

Proof. By Lemma 7.6,

$$\lambda^{-1}\operatorname{tr}\rho_L^{\mathrm{inv}}(S) = \left(\sum_{a=1}^{n} e\left(\frac{a^2}{2(n+1)}\right) - e\left(\frac{-a^2}{2(n+1)}\right)\right)\left\{2 + 2e(d/2) + \sum_{b=1}^{d-1} e\left(\frac{b^2}{2d}\right) + e\left(\frac{-b^2}{2d}\right)\right\},$$

which is the product of a purely imaginary term and a purely real term. □

LEMMA 7.11. *We have*

$$\begin{aligned}\left|\operatorname{tr}\rho_L^{\mathrm{inv}}(ST)\right| \leqslant \frac{1}{2}&\left(2/\sqrt{d(n+1)} + 1/\sqrt{2d} + \sqrt{3/2d}\right)\\ &\times\left(4/\sqrt{d(n+1)} + 1/\sqrt{2(n+1)} + \sqrt{3/2(n+1)}\right).\end{aligned}$$

Proof. By Lemma 7.6, we have

$$\begin{aligned}\operatorname{tr}\rho_L^{\mathrm{inv}}(ST) &= \frac{i}{2\sqrt{d(n+1)}}\left\{\left(\sum_{\delta_1,\delta_2=\pm1}\sum_{a=1}^{n}\sum_{b=1}^{d-1}\delta_1 e\left(\frac{(2\delta_1-1)a^2}{4(n+1)} + \frac{(2\delta_2-1)b^2}{4d}\right)\right)\right.\\ &\qquad\left. + (1+e(d/4))\sum_{a=1}^{n} e\left(\frac{a^2}{4(n+1)}\right) - e\left(\frac{-3a^2}{4(n+1)}\right)\right\}\\ &= \frac{i}{2\sqrt{d(n+1)}}\left\{\sum_{a=1}^{n}\left(e\left(\frac{a^2}{4(n+1)}\right) - e\left(\frac{-3a^2}{4(n+1)}\right)\right)\right\}\\ &\quad\times\left\{(1+e(d/4)) + \sum_{b=1}^{d-1}\left(e\left(\frac{b^2}{4d}\right) + e\left(\frac{-3b^2}{4d}\right)\right)\right\}\\ &= \frac{i}{2\sqrt{d(n+1)}}\left\{\sum_{a=1}^{n}\left(e\left(\frac{a^2}{4(n+1)}\right) - e\left(\frac{-3a^2}{4(n+1)}\right)\right)\right\}\\ &\quad\times\left\{\mathbb{G}(1,4d,d-1) + \mathbb{G}(-3,4d,d-1) + e(d/4) - 1\right\}.\end{aligned}$$

By Theorem 7.7 and Lemma 7.9,

$$|\mathbb{G}(1,4d,d-1) - 1| \leqslant 1 + \sqrt{d/2} \quad\text{and}\quad |\mathbb{G}(3,4d,d-1) - 1| \leqslant 1 + \sqrt{3d/2}.$$

Therefore,

$$\begin{aligned}|\operatorname{tr}\rho_L^{\mathrm{inv}}(ST)| &\leqslant \frac{1}{2\sqrt{d(n+1)}}|\mathbb{G}(1,4(n+1),n)-\mathbb{G}(-3,4(n+1),n)|\\ &\qquad\times|\mathbb{G}(1,4d,d-1)+\mathbb{G}(-3,4d,d-1)+e(d/4)-1|\\ &\leqslant \frac{1}{2\sqrt{d(n+1)}}(2+\sqrt{(n+1)/2}+\sqrt{3(n+1)/2})(4+\sqrt{d/2}+\sqrt{3d/2}),\end{aligned}$$

from which the result follows. □

Lemma 7.12. *In (7.1), we have* $|\alpha(\rho_L^{\mathrm{inv}})| \leqslant \big(1+2^{\nu(d)}\big)(n+2)$.

Proof. By Lemma 7.5, it suffices to count solutions to

$$\frac{a^2}{4(n+1)}+\frac{b^2}{4d}\equiv 0 \bmod \mathbb{Z} \tag{7.6}$$

for $(a,b)\in\mathcal{I}$. If (7.6) is satisfied for fixed a, then $b^2\equiv -a^2d(n+1)^{-1} \bmod d$, which has at most $1+2^{\nu(d)}$ solutions. The bound then follows by noting that there are at most $n+2$ choices for a. □

7.4 Bounds for $\sum_{j=1}^m \mathbb{B}_1(\lambda_j)$

We now obtain bounds for the sum $\sum_{j=1}^m \mathbb{B}_1(\lambda_j)$ in (7.1), in terms of

$$\mathbb{B}(\mu,d):=\sum_{x=0}^{d}\mathbb{B}_1\left(\mu-\frac{x^2}{4d}\right),$$

where $\mu\in\mathbb{Q}$ and $d\in\mathbb{Z}_{>0}$.

Lemma 7.13. *If $\mu=\mu_1/\mu_2$, where $(\mu_1,\mu_2)=1$ and $\mu_2>0$, let $N:=\operatorname{lcm}(\mu_2,4d)$ and $k=N/4d$. Then,*

$$|\mathbb{B}(\mu,d)|\leqslant|\mathbb{B}_1(\mu-d/4)|+\frac{1}{8\pi}\big(9+4k+8\log(N)+2^{4+\nu(4d)}\sqrt{d}(2k(1+\log(4d))+1)\big).$$

Proof. We closely follow [Bru02, Lemma 5]. As the function $\mathbb{B}_1$ is 1-periodic with Fourier expansion

$$\mathbb{B}_1(x)=-\frac{1}{2\pi i}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\frac{e(nx)}{n},$$

then

$$\begin{aligned}\mathbb{B}(\mu,d)&=-\frac{1}{2\pi i}\sum_{x=0}^{d}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\frac{1}{n}e\left(n\mu-\frac{nx^2}{4d}\right)\\ &=\mathbb{B}_1\left(\mu-\frac{d}{4}\right)-\frac{1}{2\pi i}\sum_{x=0}^{d-1}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\frac{1}{n}e\left(n\mu-\frac{nx^2}{4d}\right),\end{aligned}$$

and under the change of index $n\mapsto -n$,

$$=\mathbb{B}_1\left(\mu-\frac{d}{4}\right)+\frac{1}{2\pi i}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\frac{1}{n}e(-n\mu)\mathbb{G}(n,4d,d-1).$$

As $\mathbb{B}_1(\mu, d) \in \mathbb{R}$, then

$$\mathbb{B}(\mu, d) = \mathbb{B}_1\left(\mu - \frac{d}{4}\right) + \frac{1}{2\pi} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{1}{n} \operatorname{Im}(e(-n\mu)\mathbb{G}(n, 4d, d-1)),$$

and as $\mathbb{G}(-n, 4d, d-1) = \overline{\mathbb{G}(n, 4d, d-1)}$, then

$$\mathbb{B}(\mu, d) = \mathbb{B}_1\left(\mu - \frac{d}{4}\right) + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Im}(e(-n\mu)\mathbb{G}(n, 4d, d-1)).$$

By definition, we have the equalities $e((n+N)\mu)\mathbb{G}((n+N), 4d, d-1) = e(n\mu)\mathbb{G}(n, 4d, d-1)$ and $\operatorname{Im}(e(N\mu)\mathbb{G}(N, 4d, d-1)) = 0$. Therefore,

$$\begin{aligned}
\mathbb{B}(\mu, 2d) - \mathbb{B}_1\left(\mu - \frac{d}{4}\right) &= \frac{1}{\pi} \sum_{m=0}^{\infty} \sum_{j=1}^{N-1} \frac{1}{Nm+j} \operatorname{Im}(e(-j\mu)\mathbb{G}(j, 4d, d-1)) \\
&= \frac{1}{2\pi} \sum_{m=0}^{\infty} \sum_{j=1}^{N-1} \left(\frac{1}{Nm+j} \operatorname{Im}(e(-j\mu)\mathbb{G}(j, 4d, d-1)) \right. \\
&\qquad\qquad \left. + \frac{1}{N(m+1)-j} \operatorname{Im}(e(j\mu)\mathbb{G}(-j, 4d, d-1)) \right) \\
&= \frac{1}{2\pi} \sum_{m=0}^{\infty} \sum_{j=1}^{N-1} \left(\frac{1}{Nm+j} - \frac{1}{N(m+1)-j} \right) \operatorname{Im}(e(-j\mu)\mathbb{G}(j, 4d, d-1)).
\end{aligned}$$

We obtain the expression

$$\mathbb{B}(\mu, d) = \lambda_1 + \lambda_2 + \mathbb{B}_1\left(\mu - \frac{d}{4}\right),$$

where

$$\lambda_1 := \frac{1}{2\pi} \sum_{j=1}^{N-1} \left(\frac{1}{j} - \frac{1}{N-j} \right) \operatorname{Im}(e(-j\mu)\mathbb{G}(j, 4d, d-1)),$$

$$\lambda_2 := \frac{1}{2\pi} \sum_{m=1}^{\infty} \sum_{j=1}^{N-1} \left(\frac{1}{Nm+j} - \frac{1}{N(m+1)-j} \right) \operatorname{Im}(e(-j\mu)\mathbb{G}(j, 4d, d-1)).$$

We now bound λ_1 and λ_2. As $\mathbb{G}(j, 4d, d-1) = \mathbb{G}(N+j, 4d, d-1)$, then

$$|\lambda_1| \leqslant \frac{1}{2\pi} \sum_{j=1}^{N-1} \frac{1}{j} |\mathbb{G}(j, 4d, d-1)| + \frac{1}{2\pi} \sum_{j=1}^{N-1} \frac{1}{N-j} |\mathbb{G}(j, 4d, d-1)| \leqslant \frac{1}{\pi} \sum_{j=1}^{N-1} \frac{1}{j} |\mathbb{G}(j, 4d, d-1)|.$$

By Lemma 7.9,

$$\begin{aligned}
|\mathbb{G}(j, 4d, d-1)| &\leqslant 1 + \frac{1}{2} |\mathbb{G}(-d, j)| \left| \frac{2d}{j} \right|^{1/2} \\
&\leqslant 1 + \frac{1}{2} (d, j) \left| \mathbb{G}\left(\frac{-d}{(d,j)}, \frac{j}{(d,j)} \right) \right| \left| \frac{2d}{j} \right|^{1/2},
\end{aligned}$$

and by Theorem 7.7,

$$|\mathbb{G}(j, 4d, d-1)| \leqslant 1 + \sqrt{d(d, j)} \tag{7.7}$$

(where the last two lines follow from (7.5)). Therefore,

$$|\lambda_1| \leqslant \frac{1}{\pi}\sum_{j=1}^{N-1}\frac{1}{j} + \frac{\sqrt{d}}{\pi}\sum_{j=1}^{N-1}\frac{(j,4d)^{1/2}}{j}\,,$$

and by noting that $\sum_{n=1}^{N}\frac{1}{n} \leqslant 1+\log(N)$, we obtain

$$|\lambda_1| \leqslant \frac{1+\log(N)}{\pi} + \frac{k-1}{2\pi} + \frac{k\sqrt{d}}{\pi}\sum_{j=1}^{4d-1}\frac{(4d,j)^{1/2}}{j}\,.$$

By elementary considerations, each element of $\{1,\dots,4d-1\}$ can be expressed uniquely as xy, where $x|4d$, $x \neq 4d$ and $1 \leqslant y \leqslant 4d/x$ with $(y,4d/x)=1$. Therefore,

$$\begin{aligned}|\lambda_1| &\leqslant \frac{1+\log(N)}{\pi} + \frac{k-1}{2\pi} + \frac{k\sqrt{d}}{\pi}\sum_{\substack{x|4d\\ x\neq 4d}}\sum_{\substack{y=1\\ (y,4d/x)=1}}^{4d/x}\frac{\sqrt{x}}{xy}\\ &\leqslant \frac{1+\log(N)}{\pi} + \frac{k-1}{2\pi} + \frac{k\sqrt{d}}{\pi}\sum_{\substack{x|4d\\ x\neq 4d}}(1+\log(4d/x))x^{-1/2}\\ &\leqslant \frac{1+\log(N)}{\pi} + \frac{k-1}{2\pi} + \frac{k\sqrt{d}}{\pi}(1+\log(4d))\sigma_{-1/2}(4d)\,,\end{aligned}$$

where $\sigma_r(d)$ is the divisor function

$$\sigma_r(n) = \sum_{d|n} d^r\,.$$

By considering values at prime powers, we have $\sigma_{-1/2}(4d) \leqslant 2^{\nu(4d)+2}$. Therefore,

$$|\lambda_1| \leqslant \frac{1+\log(N)}{\pi} + \frac{k-1}{2\pi} + \frac{4k\sqrt{d}}{\pi}(1+\log(4d))2^{\nu(4d)}\,.$$

We now bound λ_2. By definition,

$$|\lambda_2| \leqslant \frac{1}{2\pi}\sum_{m=1}^{\infty}\sum_{j=1}^{N-1}\left(\frac{N-2j}{(Nm+j)(N(m+1)-j)}\right)|\mathbb{G}(j,4d,d-1)|\,.$$

As $|N-2j|<N$ and $N^2m(m+1)+Nj-j^2 \geqslant N^2m(m+1)$, then

$$|\lambda_2| \leqslant \frac{1}{2\pi N}\sum_{m=1}^{\infty}\sum_{j=1}^{N-1}\frac{|\mathbb{G}(j,4d,d-1)|}{m(m+1)}\,,$$

and as $1 = \sum_{m=1}^{\infty}\frac{1}{m(m+1)}$, then

$$|\lambda_2| \leqslant \frac{1}{2\pi N}\sum_{j=1}^{N-1}|\mathbb{G}(j,4d,d-1)|\,.$$

By (7.7),

$$|\lambda_2| \leqslant \frac{N-1}{2\pi N} + \frac{(k-1)d}{2\pi N} + \frac{d}{2\pi N}\sum_{j=1}^{4d-1}(d,j)^{1/2}$$
$$\leqslant \frac{1}{2\pi N} + \frac{\sqrt{d}}{2\pi N}\left((k-1)\sqrt{d} + k\sum_{j=1}^{4d-1}(d,j)^{1/2}\right).$$

As for λ_1,

$$|\lambda_2| \leqslant \frac{1}{2\pi N} + \frac{\sqrt{d}}{2\pi N}\left((k-1)\sqrt{d} + k\sum_{\substack{x|4d\\ x\neq 4d}}\sum_{\substack{y=1\\ (y,4d/x)=1}}^{4d/x}\sqrt{x}\right)$$
$$\leqslant \frac{1}{2\pi N} + \frac{\sqrt{d}}{2\pi N}\left((k-1)\sqrt{d} + 4dk + \sum_{\substack{x|4d\\ x\neq 4d}}\frac{1}{\sqrt{x}}\right)$$
$$\leqslant \frac{1}{2\pi N} + \frac{\sqrt{d}}{2\pi N}\left((k-1)\sqrt{d} + 4dk\sigma_{-1/2}(4d)\right)$$
$$\leqslant \frac{1}{8\pi}\left(5 + 4d^{1/2}\sigma_{-1/2}(4d)\right).$$

Therefore,

$$|\mathbb{B}(\mu,d)| \leqslant |\mathbb{B}_1(\mu - d/4)| + \frac{1}{8\pi}(9 + 4k + 8\log(N) + 2^{4+\nu(4d)}\sqrt{d}(2k(1+\log(4d))+1))\,. \qquad \square$$

7.5 Bounds for low-weight cusp forms

We now show that low-weight cusp forms exist for all $n > 0$ when $d \gg 0$, obtaining an explicit bound for d when $n = 2$.

Theorem 7.14. *If n is fixed and $d \gg 0$, then $S_3(\Gamma) \neq 0$.*

Proof. By Theorem 7.2 and Definition/Lemma 7.4, with $\rho = \rho_L^{\mathrm{inv}}$, $k = 0$ and $m = n(d+1)$, we have

$$\dim S_2(\rho_L^{\mathrm{inv}}) = \frac{n(d+1)}{12} - \frac{1}{4}\operatorname{Re}(\operatorname{tr}\overline{\rho}_L^{\mathrm{inv}}(S)) - \frac{2}{3\sqrt{3}}\operatorname{Re}(e^{\pi i/6}\operatorname{tr}(\overline{\rho}_L^{\mathrm{inv}}(ST)))$$
$$- \frac{1}{2}a(\overline{\rho}_L^{\mathrm{inv}}) + \sum_j \mathbb{B}_1(\lambda_j) + \dim M_0(\overline{\rho}_L^{\mathrm{inv}})\,.$$

By Lemma 7.10, we have $\operatorname{Re}(\operatorname{tr}\overline{\rho}_L^{\mathrm{inv}}(S)) = 0$; by Lemma 7.11, we have $\operatorname{tr}(\overline{\rho}_L^{\mathrm{inv}}(ST))) = O\big(\sqrt{d}\big)$; by Lemmas 7.12 and 10.8, we have $a(\overline{\rho}_L^{\mathrm{inv}}) = O(d^{\epsilon})$; and by Lemmas 7.13 and 10.8, we have $\sum_j \mathbb{B}_1(\lambda_j) = O\big(d^{1/2+\epsilon}\log(d)\big)$. Therefore, for $\epsilon < 1/2$, we have

$$\dim S_2(\rho_L^{\mathrm{inv}}) = \frac{n(d+1)}{12} + \dim M_0(\overline{\rho}_L^{\mathrm{inv}}) + O(d^{1/2+\epsilon}\log(d))\,,$$

and the result follows from (7.4). $\square$

Corollary 7.15. *Let $n = 2$. Then, for all $d > 2209016321562407$ and prime $d > 11943004$, we have $S_3(\Gamma) \neq 0$.*

Proof. By Lemmas 7.13 and 10.8,

$$|\mathbb{B}(\mu, d)| \leqslant \frac{1}{8\pi}\big((32)2^{\nu(d)}d^{1/2}(12\log(2) + 6\log(d) + 7) + 8\log(12) + 8\log(d) + 21\big) + \frac{1}{2}\,.$$

If $d > 5$, then by Lemma 7.11, we have $|\operatorname{Re}(e^{\pi i/12}\operatorname{tr}(\overline{\rho}(ST))| \leqslant \big(3\sqrt{3}\big)/2$. In (7.1),

$$\sum_j \mathbb{B}_1(\lambda_j) \leqslant |\mathbb{B}(1/12, d)| + |\mathbb{B}(1/3, d)|\,,$$

and by Theorem 7.2 and Lemmas 7.10 and 7.12, we have

$$\begin{aligned}\dim(S_2(\rho_L^{\mathrm{inv}})) \geqslant\ & -\frac{1}{4\pi}\big(1200d^{2/3}(12\log(2) + 6\log(d) + 7) + 8\log(12) + 8\log(d) + 21\big) \\ & + \frac{d}{6} - 75d^{1/6} - \frac{23}{6}\,. \end{aligned} \tag{7.8}$$

We then locate the roots of (7.8) using a computer (setting $2^{\nu(4d)} = 2$ in the case of prime d). ☐

8. Hirzebruch–Mumford volumes

Let L be an indefinite lattice of signature $(2, n)$, and let $\Lambda \subset \mathrm{O}(L\otimes\mathbb{Q})$ be an arithmetic subgroup. The Hirzebruch–Mumford proportionality principle [GHS08] states that

$$\dim M_k(\Lambda, \chi) = \mathrm{Vol}_{\mathrm{HM}}(\Lambda)k^m + O\big(k^{m-1}\big)\,,$$

where $\mathrm{Vol}_{\mathrm{HM}}(\Lambda)$ is a constant known as the *Hirzebruch–Mumford volume* of Λ. In this section, we calculate the Hirzebruch–Mumford volumes of $\mathrm{O}(L)$ and $\mathrm{O}\big(K_{r,s}^{u,m}(k,l)\big)$ in terms of local densities, following the approach of [GHS07a]. Throughout, we will use $\nu(x)$ to denote the number of prime divisors of $x \in \mathbb{N}$ counted without multiplicity.

Theorem 8.1 ([GHS07a, Theorem 2.1]). *If L is an indefinite lattice of signature $(2, m)$, then*

$$\mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}(L)) = \frac{2}{|\operatorname{Sg}(L)|}|\det L|^{(m+3)/2}\prod_{k=1}^{m+2}\pi^{-k/2}\Gamma(k/2)\prod_p \alpha_p(L)^{-1}\,,$$

where $\alpha_p(L)$ is the local density of the lattice L at the prime p and $|\operatorname{Sg}(L)|$ denotes the number of spinor genera in $\operatorname{gen}(L)$.

As noted in [GHS07a] (adopting the convention of [Ma18b]), if $\Lambda \subset \mathrm{O}(L)$ is of finite index and L contains a copy of the hyperbolic plane U, then

$$\mathrm{Vol}_{\mathrm{HM}}(\Lambda) = |\langle \mathrm{O}(L), -I\rangle : \langle \Lambda, -I\rangle|\,\mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}(L))\,. \tag{8.1}$$

8.1 Local densities

We recall the calculation of the local densities $\alpha_p(L)$ for an indefinite lattice L of signature $(2, m)$, following [GHS07a]. These results can also be found in [Kit93].

Suppose that

$$L \otimes \mathbb{Z}_p = \bigoplus_{j\in\mathbb{Z}} L_j \tag{8.2}$$

is the Jordan decomposition of the $\mathbb{Z}_p$-lattice $L \otimes \mathbb{Z}_p$, where for $n_j := \operatorname{rank} L_j$, we define $N_j := p^{-j}L_j \in \mathrm{GL}(\mathbb{Z}_p, n_j)$ if $n_j > 0$ and $N_j := 0$ otherwise.

Case $p > 2$. If p is an odd prime, then $\alpha_p(L)$ is given by

$$\alpha_p(L) = 2^{s-1} p^w P_p(L) E_p(L)\,,$$

where s is the number of non-zero factors in (8.2) and $P_p(L) := \prod_j P_p([n_j/2])$, where

$$P_p(n) := \begin{cases} \prod_{i=1}^{n} \left(1 - p^{-2i}\right) & \text{if } n \in \mathbb{N}\backslash\{0\}\,, \\ 1 & \text{if } n = 0 \end{cases}$$

and $[n_j/2]$ denotes the integer part of $n_j/2$. The term w is given by

$$w = \sum_j j n_j \left(\frac{(n_j+1)}{2} + \sum_{k>j} n_k \right),$$

and $E_p(L)$ is defined by

$$E_p(L) = \prod_{L_j \neq 0} \left(1 + \chi(N_j/pN_j) p^{-n_j/2}\right)^{-1},$$

where, for a finite quadratic space W over $\mathbb{F}_p$,

$$\chi(W) = \begin{cases} 0 & \text{if } \dim(W) \text{ is odd}\,, \\ 1 & \text{if } W \text{ is a hyperbolic space}\,, \\ -1 & \text{otherwise}\,. \end{cases}$$

Case $p = 2$. The formula needs to be modified when $p = 2$. Over $\mathbb{Z}_2$, the lattices N_j decompose as the orthogonal direct sum

$$N_j = N_j^{\text{even}} \oplus N_j^{\text{odd}}\,,$$

where N_j^{even} is even, N_j^{odd} is odd and $\operatorname{rank} N_j^{\text{odd}} \leqslant 2$. We let $E_j := \frac{1}{2}\left(1 + \chi\left(N_j^{\text{even}}\right) 2^{-\operatorname{rank} N_j^{\text{even}}/2}\right)$ if both N_{j-1} and N_{j+1} are even and $N_j^{\text{odd}} \not\cong \langle \epsilon_1 \rangle \oplus \langle \epsilon_2 \rangle$, where $\epsilon_1 \cong \epsilon_2 \bmod 4$; otherwise, we let $E_j := 1/2$. We define

$$q_j := \begin{cases} 0 & \text{if } N_j \text{ is even}\,, \\ n_j & \text{if } N_j \text{ is odd and } N_{j+1} \text{ is even}\,, \\ n_j + 1 & \text{if } N_j \text{ and } N_{j+1} \text{ are both odd}\,, \end{cases}$$

and set $q := \sum_j q_j$. Then, if

$$P_2(L) := \prod_j P_2\left(\operatorname{rank} N_j^{\text{even}}/2\right) \quad \text{and} \quad E_2(L) := \prod_j E_j^{-1}\,,$$

the local density $\alpha_2(L)$ is given by

$$\alpha_2(L) = 2^{m+1+w-q} P_2(L) E_2(L)\,.$$

The calculation of E_j can often be simplified by noting that, see [Kit93],

$$E_j = \begin{cases} 1 & \text{if } N_{j-1} = N_j = N_{j+1} = \{0\}\,, \\ 1 & \text{if } N_{j-1} = N_j = \{0\} \text{ and } N_{j+1} \text{ is even}\,, \\ 1 & \text{if } N_{j-1} \text{ is even and } N_j = N_{j+1} = \{0\}\,, \\ \frac{1}{2} & \text{if } N_{j-1} = N_j = \{0\} \text{ and } N_{j+1} \text{ is odd}\,, \\ \frac{1}{2} & \text{if } N_{j-1} \text{ is odd and } N_j = N_{j+1} = \{0\}\,. \end{cases}$$

8.2 The Hirzebruch–Mumford volume of $\mathrm{O}(L)$

Proposition 8.2. *If $L = 2U \oplus \langle -6\rangle \oplus \langle -2d\rangle$, then*

$$\mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}(L)) = \frac{(12d)^{5/2}\pi^3}{137781} L\left(3, \left(\frac{-12d}{*}\right)\right)^{-1} \times 2^{-\nu(12d)}|12d|_2^{-1}|12d|_3^{-1}\alpha_2(L)^{-1}\alpha_3(L)^{-1}\prod_{p|12d}\left(1-p^{-6}\right),$$

where expressions for $\alpha_2(L)$ and $\alpha_3(L)$ are given in (8.3) and (8.4).

Proof. For prime p, let $L_p := L \otimes \mathbb{Z}_p$ and define a by $|12d|_p = p^{-a}$.

Case $p=2$. If $p=2$, then $D(L_2) = C_2 \oplus C_{2^{a-1}}$, $q_2(x,y) = -3x^2/2 - dy^2/2^{2a-3} \bmod 2\mathbb{Z}_2$ for $(x,y)\in D(L_2)$ and

$$L_2 = 2U \oplus \langle\theta_1\rangle(2) \oplus \langle\theta_{a-1}\rangle\left(2^{a-1}\right),$$

where $\theta_1, \theta_{a-1} \in \mathbb{Z}_p^*$.

(i) If $a=2$, then $s=2$, $q=2$, $w=3$ and $P_2(L) = P_2(\operatorname{rank} N_0^{\mathrm{even}}/2)P_2(\operatorname{rank} N_1^{\mathrm{even}}/2)$. As $\operatorname{norm} N_1 = \mathbb{Z}_2$, then $P_2(\operatorname{rank} N_1^{\mathrm{even}}/2) = P_2(0)$ and $P_2(\operatorname{rank} N_0^{\mathrm{even}}/2) = P_2(2)$. Moreover, $E_i = 1$ when $i<0$ or $i>2$, $E_0 = 1/2$, $E_1 = 1$ if $d \not\equiv 3 \bmod 4$, $E_1 = 1/2$ if $d\equiv 3 \bmod 4$ and $E_2 = 1/2$. Therefore,

$$E_2(L) = \begin{cases} 4 & \text{if } d \not\equiv 3 \bmod 4\,, \\ 8 & \text{if } d \equiv 3 \bmod 4\,. \end{cases}$$

(ii) If $a>2$, then $w = a+1$, $q=3$ if $a=3$, $q=2$ if $a \geqslant 4$ and $P_2(L) = P_2(2)P_2(0)^2$.

(iii) If $a=3$, then $E_i = 1$ if $i<-1$ or $i>3$, $E_{-1}=1$, $E_0 = 1/2$, $E_1 = 1/2$, $E_2 = 1/2$, $E_3 = 1/2$ and $E_2(L) = 16$.

(iv) If $a=4$, then $E_i = 1$ if $i<-1$ or $i>4$, $E_{-1}=1$, $E_0=1/2$, $E_1 = 1$, $E_2 = 1/2$, $E_3 = 1$, $E_4 = 1/2$ and $E_2(L) = 8$.

(v) If $a=5$, then $E_i=1$ if $i<-1$ or $i>5$, $E_{-1}=1$, $E_0 = 1/2$, $E_1=1$, $E_2 = 1/2$, $E_3 = 1/2$, $E_4 = 1$, $E_5 = 1/2$ and $E_2(L) = 16$.

(vi) If $a \geqslant 6$, then $E_2(L) = E_{-1}^{-1}E_0^{-1}E_1^{-1}E_2^{-1}E_{a-2}^{-1}E_{a-1}^{-1}E_a^{-1}$ and $E_{-1}=1$, $E_0 = 1/2$, $E_1 = 1$, $E_2 = 1/2$, $E_{a-2} = 1/2$, $E_{a-1} = 1$, $E_a = 1/2$ and $E_2(L) = 16$.

Therefore,

$$\alpha_2(L) = \begin{cases} 256P_2(2) & \text{if } a=2 \text{ and } d\not\equiv 3 \bmod 4\,, \\ 512P_2(2) & \text{if } a=2 \text{ and } d \equiv 3 \bmod 4\,, \\ 2^{7+a}P_2(2) & \text{if } a = 3 \text{ or } 4\,, \\ 2^{8+a}P_2(2) & \text{if } a \geqslant 5\,. \end{cases} \tag{8.3}$$

Case $p=3$. Let $p=3$. If $a=1$, then $D(L_3) = C_3$ and $q_3(x) = -2x^2/3 \bmod \mathbb{Z}_3$ for $x \in D(L_3)$; if $a=2$, then $D(L_3) = C_3 \oplus C_3$ and $q_3(x,y) = -2x^2/3 - 2y^2d/3^2 \bmod 2\mathbb{Z}_3$ for $(x,y)\in D(L_3)$; if $a>2$, then $D(L_3) = C_3 \oplus C_{3^{a-1}}$ and $q_3(x,y) = -2x^2/3 - 2y^2d/3^{2a-2} \bmod \mathbb{Z}_3$ for

$(x, y) \in D(L_3)$. By [Nik80, Theorem 1.9.1 and Corollary 1.9.3],

$$L_3 = \begin{cases} \langle 1 \rangle^{\oplus 4} \oplus \langle \theta_0 \rangle \oplus \langle \theta_1 \rangle(3) & \text{if } a = 1\,, \\ \langle 1 \rangle^{\oplus 3} \oplus \langle \theta_0 \rangle \oplus \langle \theta_1 \rangle(3) \oplus \langle \theta_1' \rangle(3) & \text{if } a = 2\,, \\ \langle 1 \rangle^{\oplus 3} \oplus \langle \theta_0 \rangle \oplus \langle \theta_1 \rangle(3) \oplus \langle \theta_{a-1} \rangle(3^{a-1}) & \text{if } a > 2 \end{cases}$$

for θ_1, θ_1', $\theta_a \in \mathbb{Z}_p^*$.

(i) If $a = 1$, then $s = 2$, $w = 1$, $P_3(L) = P_3(2)$ and $E_3(L) = 1$ as $\chi(N_0) = \chi(N_1) = 0$.

(ii) If $a = 2$, then $s = 2$, $w = 3$ and $P_3(L) = P_3(2)P_3(1)$. As N_0 is obtained from $2U$, then N_0 is hyperbolic and $\chi(N_0) = 1$. As

$$\chi(N_1) = \left(-\frac{4 \cdot 3^{-1} d}{3} \right),$$

then $E_3(L) = (1 + 3^{-2})^{-1}(1 + \frac{1}{3}(-4 \cdot 3^{-1} d)3^{-1})^{-1}$.

(iii) If $a > 2$, then $s = 3$, $w = a + 1$ and $P_3(L) = P_3(2)$. As N_0 is hyperbolic, then $\chi(N_0) = 1$ and $E_3(L) = (1 + 3^{-2})^{-1}$.

Therefore,

$$\alpha_3(L) = \begin{cases} 6P_3(2) & \text{if } a = 1\,, \\ 54P_3(2)P_3(1)(1 + 3^{-2})^{-1} \left(1 + \left(\frac{-4 \cdot 3^{-1} d}{3}\right) 3^{-1}\right)^{-1} & \text{if } a = 2\,, \\ 2^2 3^{a+1} P_3(2)(1 + 3^{-2})^{-1} & \text{if } a > 2\,. \end{cases} \tag{8.4}$$

Case $p > 3$. If $p > 3$, then by [Nik80, Proposition 1.7.1], we have $D(L_p) = C_{p^a}$ with discriminant form $q_p(x) = -2dp^{-2a}x^2 \bmod \mathbb{Z}_p$ for $x \in D(L_p)$. By [Nik80, Theorem 1.9.1 and Corollary 1.9.3],

$$L_p = \begin{cases} \langle 1 \rangle^{\oplus 5} \oplus \langle \theta_0 \rangle & \text{if } a = 0\,, \\ \langle 1 \rangle^{\oplus 4} \oplus \langle \theta_0 \rangle \oplus \langle \theta_a \rangle(p^a) & \text{if } a > 0\,, \end{cases}$$

where θ_0, $\theta_a \in \mathbb{Z}_p^*$. If $a = 0$, then $s = 1$, $P_p(L) = P_p(3)$, $w = 0$ and $E_p(L) = (1 + \chi(N_0)p^{-3})^{-1}$. By the classification of non-degenerate quadratic forms over $\mathbb{F}_p$ (see for example [Die71, § II.10, p. 63]), the quotient space L_p/pL_p is hyperbolic if and only if

$$\left(\frac{\theta_0}{p}\right) = \left(\frac{-12d}{p}\right) = 1\,,$$

where $\left(\frac{*}{p}\right)$ denotes the Legendre symbol, implying

$$\chi(N_0) = \left(\frac{-12d}{p}\right).$$

If $a > 0$, then $s = 2$, $P_p(L) = P_p(2)$, $w = a$ and $\chi(N_0) = \chi(N_a) = 0$. Therefore,

$$E_p(L) = \begin{cases} \left(1 + \left(\frac{-12d}{p}\right) p^{-3}\right)^{-1} & \text{if } a = 0\,, \\ 1 & \text{if } a > 0 \end{cases}$$

and

$$\alpha_p(L) = \begin{cases} P_p(3) \left(1 + \left(\frac{-12d}{p}\right) p^{-3}\right)^{-1} & \text{if } a = 0\,, \\ 2p^a P_p(2) & \text{if } a > 0\,. \end{cases}$$

We now calculate $\prod \alpha_p^{-1}$. For prime p, let

$$\beta_p(L) := P_p(3)\left(1+\left(\frac{-12d}{p}\right)p^{-3}\right)^{-1} \quad \text{and} \quad \gamma_p(L) := 2p^a P_p(2)\,.$$

Then,

$$\prod_p \alpha_p^{-1} = \frac{\gamma_2(L)\gamma_3(L)}{\alpha_2(L)\alpha_3(L)}\left(\prod_p \beta_p(L)^{-1}\right)\left(\prod_{p\mid 12d}\frac{\beta_p(L)}{\gamma_p(L)}\right)$$

and

$$\begin{aligned}\prod_p \beta_p(L)^{-1} &= \prod_p (1-p^{-2})^{-1}\prod_p (1-p^{-4})^{-1}\prod_p (1-p^{-6})^{-1}\prod_p\left(1+\left(\frac{-12d}{p}\right)p^{-3}\right)\\ &= \zeta(2)\zeta(4)\zeta(6)L\left(3,\left(\frac{-12d}{*}\right)\right)^{-1}.\end{aligned}$$

Using the familiar values $\zeta(2)=\pi^2/6$, $\zeta(4)=\pi^4/90$ and $\zeta(6)=\pi^6/945$, we have

$$\prod_p \beta_p(L)^{-1} = \frac{\pi^{12}}{510300}L\left(3,\left(\frac{-12d}{*}\right)\right)^{-1},$$

where $L\big(s,\big(\frac{D}{*}\big)\big)$ is the Dirichlet series defined by $p \mapsto \big(\frac{D}{p}\big)$. Similarly,

$$\prod_{p\mid 12d}\frac{\beta_p(L)}{\gamma_p(L)} = \prod_{p\mid 12d}\frac{P_p(3)}{2p^aP_p(2)} = 2^{-\nu(12d)}(12d)^{-1}\prod_{p\mid 12d}\left(1-p^{-6}\right).$$

As

$$\begin{aligned}\frac{\gamma_3(L)}{\alpha_3(L)}\frac{\gamma_2(L)}{\alpha_2(L)} &= 2^2|12d|_3^{-1}|12d|_2^{-1}P_3(2)P_2(2)\alpha_3(L)^{-1}\alpha_2(L)^{-1}\\ &= \frac{200}{81}|12d|_3^{-1}|12d|_2^{-1}\alpha_3(L)^{-1}\alpha_2(L)^{-1},\end{aligned}$$

then

$$\begin{aligned}\prod_p \alpha_p(L)^{-1} = &\left(\frac{\pi^{12}}{510300}\right)\left(\frac{200}{81}\right)L\left(3,\left(\frac{-12d}{*}\right)\right)^{-1}2^{-\nu(12d)}(12d)^{-1}|12d|_2^{-1}|12d|_3^{-1}\\ &\times \alpha_2(L)^{-1}\alpha_3(L)^{-1}\prod_{p\mid 12d}\left(1-p^{-6}\right).\end{aligned}$$

As noted in [GHS07a], by the Legendre duplication formula (8.5) (see for example [Dav80, § 8, p. 73]),

$$\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\pi^{1/2}\Gamma(2z)\,, \tag{8.5}$$

we have

$$\prod_{k=1}^{6}\pi^{-k/2}\Gamma(k/2) = \frac{3}{4\pi^9}\,. \tag{8.6}$$

Hence, by Theorem 8.1,

$$\begin{aligned}
\mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}(L)) &= \frac{3(12d)^{7/2}}{2\pi^9}\prod_p \alpha_p(L)^{-1} \\
&= (12d)^{7/2}\left(\frac{3}{2\pi^9}\right)\left(\frac{\pi^{12}}{510300}\right)\left(\frac{200}{81}\right)L\left(3,\left(\frac{-12d}{*}\right)\right)^{-1}2^{-\nu(12d)}(12d)^{-1} \\
&\quad\times\left(\prod_{p\mid 12d}(1-p^{-6})\right)|12d|_3^{-1}|12d|_2^{-1}\alpha_3(L)^{-1}\alpha_2(L)^{-1} \\
&= \frac{(12d)^{5/2}\pi^3}{(137781)2^{\nu(12d)}|12d|_3|12d|_2}L\left(3,\left(\frac{-12d}{*}\right)\right)^{-1}\alpha_3(L)^{-1}\alpha_2(L)^{-1} \\
&\quad\times\prod_{p\mid 12d}\left(1-p^{-6}\right).
\end{aligned}$$

□

8.3 Hirzebruch–Mumford volumes of $\mathrm{O}(K_{r,s}^{u,m}(k,l))$

We now calculate the Hirzebruch–Mumford volumes of $\mathrm{O}\big(K_{r,s}^{u,m}(k,l)\big)$ for the lattices $K_{r,s}^{u,m}(k,l)$ in Proposition 5.3.

Proposition 8.3. *Let K denote the lattice $K_{r,s}^{u,m}(k,l)$ of Proposition 5.3, and let N_0 denote the unimodular factor of $K\otimes\mathbb{Z}_p$ in (8.2). Then,*

$$\mathrm{Vol}_{\mathrm{HM}}\big(\mathrm{O}\big(K_{r,s}^{2s,2s}(0,l)\big)\big) = \frac{r^2}{160}2^{-\nu(2r)}\big(\alpha_2(K)^{-1}|2r|_2^{-1}\big)\prod_{p\mid 2r}\big(1+p^{-2}\big) \qquad \text{if } l \text{ is odd},$$

$$\mathrm{Vol}_{\mathrm{HM}}\big(\mathrm{O}\big(K_{r,l}^{2s,s}(0,l)\big)\big) \leqslant \frac{r^2}{6}2^{-\nu(8r)}\prod_{p\mid 8r}\big(1+\chi(N_0)p^{-2}\big) \qquad \text{if } l \text{ is even},$$

$$\mathrm{Vol}_{\mathrm{HM}}\big(\mathrm{O}\big(K_{r,s}^{2s,2s}(r,l)\big)\big) = \frac{r^2}{160}2^{-\nu(2r)}\big(\alpha_2(K)^{-1}|2r|_2^{-1}\big)\prod_{p\mid 2r}\big(1+\chi(N_0)p^{-2}\big) \qquad \text{if } l \text{ is odd},$$

$$\mathrm{Vol}_{\mathrm{HM}}\big(\mathrm{O}\big(K_{r,l}^{2s,2s}(r,l)\big)\big) = \frac{r^2}{3600}2^{-\nu(2r)}\prod_{p\mid 2r}\big(1+\chi(N_0)p^{-2}\big) \qquad \text{if } l \text{ is even}.$$

Proof. We use N_i to denote unimodular $\mathbb{Z}_p$-lattices, and for prime p, we define a by $|\det K|_p = p^{-a}$. We note that in each case $\mathrm{Sg}(K)$ contains exactly one class, which follows from [Nik80, Theorem 1.13.2].

Case $K_{r,s}^{2s,2s}(0,l)$, l odd. Let $K = K_{r,s}^{2s,2s}(0,l)$, where l is odd. If $p=2$, then by [Nik80, Theorem 1.9.1 and Corollary 1.9.3],

$$K_2 = 2U \oplus \langle\theta_a\rangle(2^a),$$

implying $m=3$, $w=a$ and $q=1$.

(i) If $a=1$, then $E_j=1$ for $j<-1$ or $j>3$, $E_{-1}=1$, $E_0=1/2$, $E_1=1$, $E_2=1/2$ and $E_3=1$.

(ii) If $a=2$, then $E_j=1$ for $j<-1$ or $j>3$, $E_{-1}=1$, $E_0=\frac{1}{2}\big(1+2^{-2}\big)$, $E_1=1/2$, $E_2=1$ and $E_3=1/2$.

(iii) If $a = 3$, then $E_j = 1$ for $j < -1$ or $j > 5$, $E_{-1} = 1$, $E_0 = \frac{1}{2}(1 + 2^{-2})$, $E_1 = 1$, $E_2 = 1/2$, $E_3 = 1$, $E_4 = 1/2$ and $E_5 = 1$.

(iv) If $a > 3$, then $E_j = 1$ for $j < -1$ or $j > a + 1$, $E_{-1} = 1$, $E_0 = \frac{1}{2}(1 + 2^{-2})$, $E_1 = 1$, $E_j = 1$ for $1 < j < a - 1$, $E_{a-1} = 1/2$, $E_a = 1$ and $E_{a+1} = 1/2$.

Therefore,

$$E_2(K) = \begin{cases} 4 & \text{if } a = 1\,, \\ 8\big(1 + 2^{-2}\big)^{-1} & \text{if } a > 1 \end{cases}$$

and

$$\alpha_2(K) = \begin{cases} 64P_2(2) & \text{if } a = 1\,, \\ 2^{6+a}P_2(2)\big(1 + 2^{-2}\big)^{-1} & \text{if } a > 1\,. \end{cases} \tag{8.7}$$

If $p \neq 2$, then by [Nik80, Theorem 1.9.1 and Corollary 1.9.3],

$$K_p = \begin{cases} \langle 1\rangle^{\oplus 4} \oplus \langle \theta_0\rangle & \text{if } a = 0\,, \\ \langle 1\rangle^{\oplus 3} \oplus \langle \theta_0\rangle \oplus \langle \theta_a\rangle(p^a) & \text{if } a > 0\,. \end{cases} \tag{8.8}$$

If $a = 0$, then $s = 1$, $w = 0$, $P_p(K) = P_p(2)$ and $E_p(K) = 1$. If $a > 0$, then $s = 2$, $w = a$, $P_p(K) = P_p(2)$ and $E_p(K) = (1 + p^{-2})^{-1}$ (noting that $K = 2U \oplus \langle -2r\rangle$ by [Nik80, Corollary 1.13.4]). Therefore,

$$\alpha_p(K) = \begin{cases} P_p(2) & \text{if } a = 0\,, \\ 2p^a P_p(2)\big(1 + p^{-2}\big)^{-1} & \text{if } a > 0\,. \end{cases}$$

As

$$\begin{aligned} \prod_p \alpha_p(K)^{-1} &= \left(\frac{2\alpha_2(K)^{-1}|2r|_2^{-1}}{1 + 2^{-2}}\right)\left(\prod_{p|2r} 2^{-1}p^{-a}\big(1 + p^{-2}\big)\right)\left(\prod_p P_p(2)^{-1}\right)P_2(2) \\ &= \frac{2\zeta(2)\zeta(4)}{1 + 2^{-2}}2^{-\nu(2r)}(2r)^{-1}\big(\alpha_2(K)^{-1}|2r|_2^{-1}\big)P_2(2)\prod_{p|2r}\big(1 + p^{-2}\big) \\ &= \frac{\pi^6}{480}2^{-\nu(2r)}(2r)^{-1}\big(\alpha_2(K)^{-1}|2r|_2^{-1}\big)\prod_{p|2r}\big(1 + p^{-2}\big) \end{aligned}$$

and, as in (8.6),

$$\prod_{k=1}^{5} \pi^{-k/2}\Gamma(k/2) = \frac{3}{8\pi^6}\,, \tag{8.9}$$

then

$$\mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}(K)) = \frac{r^2}{160}2^{-\nu(2r)}\big(\alpha_2(K)^{-1}|2r|_2^{-1}\big)\prod_{p|2r}\big(1 + p^{-2}\big)\,.$$

Case $K^{2s,s}_{r,s}(0, l)$, l even. Let $K = K^{2s,s}_{r,s}(0, l)$, where l is even. If $p = 2$, then by [Nik80, Corollary 1.9.3],

$$K_2 = \begin{cases} U \oplus N_1(2) \text{ or } V \oplus N_1(2) & \text{if } a = 3\,, \\ U \oplus N_1(2) \oplus N_{a-2}\big(2^{a-2}\big) \text{ or } V \oplus N_1(2) \oplus N_{a-2}\big(2^{a-2}\big) & \text{if } a > 3\,, \end{cases}$$

where $V = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. If $a = 3$, then $q \leqslant 4$, $w = 6$, $P_2(K) \geqslant P_2(1)^2$ and $E_2(K) \geqslant 16/9$. If $a > 3$, then $q \leqslant 5$, $w = a + 3$, $P_2(K) \geqslant P_2(1)^2$ and $E_2(K) \geqslant 16/9$, implying $\alpha_2(K) \geqslant 2^a$.

If $p \neq 2$, then K_p is as in (8.8). If $a = 0$, then $s = 1$, $w = 0$, $P_p(K) = P_p(2)$ and $E_p(K) = 1$; if $a > 0$, then $s = 2$, $w = a$, $P_p(K) = P_p(2)$ and $E_p(K) = (1 + \chi(N_0)p^{-2})^{-1}$. Hence,

$$\alpha_p(K) = \begin{cases} P_p(2) & \text{if } a = 0\,, \\ 2p^a P_p(2)\bigl(1 + \chi(N_0)p^{-2}\bigr)^{-1} & \text{if } a > 0 \end{cases}$$

and

$$\begin{aligned} \prod_p \alpha_p(K)^{-1} &\leqslant \left(\prod_p P_p(2)^{-1}\right)\left(\prod_{p \mid 8r} 2^{-1}|8r|_p\bigl(1 + \chi(N_0)p^{-2}\bigr)\right) \\ &\quad \times \bigl(2|8r|_2^{-1} P_2(2)\bigl(1 + \chi(N_0)2^{-2}\bigr)^{-1}\bigr)|8r|_2 \\ &\leqslant (15/8)2^{-\nu(8r)}(8r)^{-1}\left(\prod_p P_p(2)^{-1}\right)\left(\prod_{p \mid 8r}\bigl(1 + \chi(N_0)p^{-2}\bigr)\right). \end{aligned}$$

Therefore, by (8.9) and the standard values for $\zeta(2)$ and $\zeta(4)$,

$$\mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}(K)) \leqslant \frac{r^2}{6} 2^{-\nu(8r)} \prod_{p \mid 8r}\bigl(1 + \chi(N_0)p^{-2}\bigr)\,.$$

Case $K^{2s,2s}_{r,s}(r,l)$, l odd. If $p = 2$, then by [Nik80, Corollary 1.9.3],

$$K_2 = 2U \oplus \langle \theta_a \rangle (2^a)\,,$$

implying that K_2 and $\alpha_2(K)$ are as for $K^{2r,2s}_{r,s}(0,l)$ with l odd. If $p \neq 2$, then K_p and $\alpha_p(K)$ are as for $K^{2s,2s}_{r,s}(0,l)$ with l even. Therefore,

$$\begin{aligned} \prod_p \alpha_p(K)^{-1} &= \left(\prod_p P_p(2)^{-1}\right)\left(\prod_{p \mid 2r} 2^{-1}|2r|_p\bigl(1 + \chi(N_0)p^{-2}\bigr)\right) \\ &\quad \times 2|2r|_2^{-1}\alpha_2(K)^{-1}\bigl(1 + 2^{-2}\bigr)^{-1} P_2(2)\,. \end{aligned}$$

Hence, by (8.9),

$$\mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}(K)) = \frac{r^2}{160} 2^{-\nu(2r)}\left(\prod_{p \mid 2r} 1 + \chi(N_0)p^{-2}\right)\alpha_2(K)^{-1}|2r|_2^{-1}\,.$$

Case $K^{2s,2s}_{r,s}(r,l)$, l even. Let $K = K^{2s,2s}_{r,s}(r,l)$, where l is even. If $p = 2$, then by [Nik80, Theorem 1.9.1, Corollary 1.9.3 and Propositions 1.8.1 and 1.82],

$$K_2 = 2U \oplus \langle \theta_1 \rangle (2)$$

(where we have used the fact that r is odd). Then, as in the case of $K^{2s,2s}_{r,s}(0,l)$ with l odd, $|2r|_2 = 2^{-1}$, $E_2(K) = 4$ and $\alpha_2(K) = 64P_2(2)$.

If $p \neq 2$, then by [Nik80, Theorem 1.9.1 and Corollary 1.9.3], we see that K_p is as in (8.8). If $a = 0$, then $s = 1$, $w = 0$, $P_p(K) = P_p(2)$ and $E_p(K) = 1$. If $a > 0$, then $s = 2$, $w = a$,

$P_p(K) = P_p(2)$ and $E_p(K) = (1+\chi(N_0)p^{-2})^{-1}$. Therefore,

$$\alpha_p(K) = \begin{cases} P_p(2) & \text{if } a = 0\,, \\ 2p^a P_p(2)\bigl(1+\chi(N_0)p^{-2}\bigr)^{-1} & \text{if } a > 0\,, \end{cases}$$

and so

$$\begin{aligned}\prod_p \alpha_p(K)^{-1} &= \frac{1}{32|2r|_2\bigl(1+\chi(N_0)2^{-2}\bigr)}\left(\prod_p P_p(2)^{-1}\right)\left(\prod_{p|2r} 2^{-1}|2r|_p\bigl(1+\chi(N_0)p^{-2}\bigr)\right)\\ &= \frac{\zeta(2)\zeta(4)}{16\bigl(1+2^{-2}\bigr)2^{\nu(2r)}(2r)}\prod_{p|2r}\bigl(1+\chi(N_0)p^{-2}\bigr)\end{aligned}$$

(where the denominator $\bigl(1+\chi(N_0)2^{-2}\bigr)$ is obtained from the decomposition of K_2). Hence,

$$\begin{aligned}\mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}(K)) &= \left(\frac{6(2r)^3}{8\pi^6}\right)\left(\frac{\zeta(2)\zeta(4)}{16(1+2^{-2})2^{\nu(2r)}(2r)}\right)\prod_{p|2r}\bigl(1+\chi(N_0)p^{-2}\bigr)\\ &= \frac{r^2}{3600}2^{-\nu(2r)}\prod_{p|2r}\bigl(1+\chi(N_0)p^{-2}\bigr)\,. \qquad \square\end{aligned}$$

9. Bounds for Hirzebruch–Mumford volumes

We now produce bounds for the Hirzebruch–Mumford volumes of § 8. We will use these bounds in § 10 in connection with obstruction calculations. In this section, L is the lattice

$$L = 2U \oplus \langle -2d\rangle \oplus \langle -6\rangle$$

(that is, (2.2) with $n = 2$) and $\Gamma \subset \mathrm{O}^+(L)$ is the modular group defined in § 2.1.

9.1 Bounds for $\mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}(L))$

Lemma 9.1. *If $D \in \mathbb{N}$, $s > 1$ and χ is an arithmetic function taking values in $\{0, \pm 1\}$, then*

$$\zeta(2s)\zeta(s)^{-1} \leqslant L(s,\chi) \leqslant \zeta(s) \quad \text{and} \quad \zeta(2s)\zeta(s)^{-1} \leqslant \prod_{p|D}(1-\chi(p)p^{-s})^{-1} \leqslant \zeta(s)\,.$$

Proof. If λ is the Liouville function, then

$$L(s,\lambda) = \prod_p (1+p^{-s})^{-1} \leqslant \prod_{p|D}(1+p^{-s})^{-1} \leqslant \prod_{p|D}(1-\chi(p)p^{-s})^{-1} \leqslant \prod_{p|D}(1-p^{-s})^{-1} \leqslant \zeta(s)\,.$$

As $L(s,\lambda) = \zeta(2s)\zeta(s)^{-1}$, see [HW08, § 17.8, p. 335], then,

$$\zeta(2s)\zeta(s)^{-1} \leqslant \prod_{p|D}(1-\chi(p)p^{-s})^{-1} \leqslant \zeta(s)\,,$$

from which the result follows. $\square$

Lemma 9.2. *We have*

$$\mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}(L)) \geqslant \frac{(12d)^{5/2}}{248832 \cdot \zeta(3)\pi^3 2^{\nu(12d)}}\,.$$

Proof. By Proposition 8.2,

$$\mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}(L)) = \frac{(12d)^{5/2}\pi^3}{137781} L\left(3, \left(\frac{-12d}{*}\right)\right)^{-1} \times 2^{-\nu(12d)}|12d|_3^{-1}|12d|_2^{-1}\alpha_3(L)^{-1}\alpha_2(L)^{-1}\prod_{p|12d}\left(1-p^{-6}\right),$$

where

$$\alpha_2(L) = \begin{cases} 45|12d|_2^{-1} & \text{if } a=2 \text{ and } d \not\equiv 3 \bmod 4\,, \\ 90|12d|_2^{-1} & \text{if } a=2 \text{ and } d \equiv 3 \bmod 4\,, \\ 90|12d|_2^{-1} & \text{if } a=3 \text{ or } a=4\,, \\ 180|12d|_2^{-1} & \text{if } a \geqslant 5 \end{cases}$$

and

$$\alpha_3(L) = \begin{cases} 2|12d|_3^{-1}P_3(2) & \text{if } a=1\,, \\ 6|12d|_3^{-1}P_3(2)P_3(1)\left(1+3^{-2}\right)^{-1}\left(1+\left(\frac{-4\cdot 3^{-1}d}{p}\right)3^{-1}\right)^{-1} & \text{if } a=2\,, \\ 12|12d|_3^{-1}P_3(2)\left(1+3^{-2}\right)^{-1} & \text{if } a>2\,. \end{cases}$$

As $P_2(2) = 45/64$, $P_3(1) = 8/9$ and $P_3(2) = 640/729$, then

$$\alpha_2(L)^{-1} \geqslant \frac{|12d|_2}{256P_2(2)} = \frac{|12d|_2}{180} \quad \text{and} \quad \alpha_3(L)^{-1} \geqslant \frac{27|12d|_3}{256}\,,$$

implying

$$|12d|_3^{-1}|12d|_2^{-1}\alpha_3(L)^{-1}\alpha_2(L)^{-1} \geqslant \frac{3}{5120}\,.$$

By Lemma 9.1,

$$L\left(3, \left(\frac{-12d}{*}\right)\right)^{-1} \geqslant \zeta(3)^{-1} \quad \text{and} \quad \prod_{p|12d}\left(1-p^{-6}\right) \geqslant \zeta(6)^{-1}\,,$$

from which the result follows. □

9.2 Bounds for $\mathrm{Vol}_{\mathrm{HM}}(\Gamma)$

Lemma 9.3. *We have*

$$\mathrm{Vol}_{\mathrm{HM}}(\Gamma) \geqslant \frac{\gamma(12d)^{5/2}}{124416\cdot\zeta(3)\pi^3}\,, \quad \textit{where } \gamma = \begin{cases} 1/4 & \textit{if } (6,d)=1\,, \\ 1 & \textit{if } (6,d)=6\,, \\ 1/2 & \textit{otherwise}\,. \end{cases}$$

Proof. By Proposition 2.1, we have $\Gamma = \widetilde{\mathrm{SO}}^+(L) \rtimes \langle\sigma_v\rangle$, implying

$$\left|\mathrm{O}(L) : \widetilde{\mathrm{SO}}^+(L)\right| = 2|\mathrm{O}(L) : \Gamma| = 2\left|\mathrm{O}(L) : \widetilde{\mathrm{O}}^+(L)\right|.$$

By [GHS07a, Lemma 3.2], we have $\left|\mathrm{O}^+(L) : \widetilde{\mathrm{O}}^+(L)\right| = |\mathrm{O}(q_L)|$, and as $-I \notin \Gamma$, then

$$|\langle\mathrm{O}(L), -I\rangle : \langle\Gamma, -I\rangle| = \frac{1}{2}|\mathrm{O}(L) : \Gamma| = \left|\mathrm{O}^+(L) : \widetilde{\mathrm{O}}^+(L)\right| = |\mathrm{O}(q_L)|\,.$$

We now produce a lower bound for $|\mathrm{O}(q_L)|$. If $(a,b) \in D(L) \cong C_6 \oplus C_{2d}$, then

$$q_L(a,b) \equiv -\frac{a^2}{6} - \frac{b^2}{2d} \bmod 2\mathbb{Z}\,.$$

As in Lemma 10.4, there are $2^{\nu(d)}$ solutions to

$$x^2 \equiv 1 \bmod 4d \tag{9.1}$$

and two solutions to

$$y^2 \equiv 1 \bmod 12\,, \tag{9.2}$$

where $x \in \mathbb{Z}/(2d)\mathbb{Z}$ and $y \in \mathbb{Z}/6\mathbb{Z}$. As each solution to (9.1) and (9.2) is a generator for $\mathbb{Z}/(2d)\mathbb{Z}$ or $\mathbb{Z}/6\mathbb{Z}$, then $(0,1) \mapsto (0,x)$ and $(1,0) \mapsto (y,0)$ define elements of $\mathrm{O}(q_L)$, implying the inequality $|\mathrm{O}(q_L)| \geqslant 2^{\nu(d)+1}$. By Lemma 9.2,

$$\mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}(L)) \geqslant \left(\frac{(12d)^{5/2}}{124416 \cdot \zeta(3)\pi^3} \right) 2^{-\nu(12d)}\,,$$

and the result follows from (8.1), with γ as in the statement of the lemma. □

9.3 Bounds for $\mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}(K))$

As in Proposition 5.3, we will often use K to denote the lattice $K^{u,m}_{r,s}(k,l)$.

Lemma 9.4. *For $K^{u,m}_{r,s}(k,l)$ as in Proposition 5.3,*

$$\mathrm{Vol}_{\mathrm{HM}}\big(\mathrm{O}\big(K^{2s,2s}_{r,s}(0,l)\big)\big) \leqslant \frac{r^2}{240\pi^2} 2^{-\nu(2r)} \quad \text{if } l \text{ is odd}\,,$$

$$\mathrm{Vol}_{\mathrm{HM}}\big(\mathrm{O}\big(K^{2s,s}_{r,s}(0,l)\big)\big) \leqslant \frac{5r^2}{2\pi^2} 2^{-\nu(2r)} \quad \text{if } l \text{ is even}\,,$$

$$\mathrm{Vol}_{\mathrm{HM}}\big(\mathrm{O}\big(K^{2s,2s}_{r,s}(r,l)\big)\big) \leqslant \frac{r^2}{240\pi^2} 2^{-\nu(2r)} \quad \text{if } l \text{ is odd}\,,$$

$$\mathrm{Vol}_{\mathrm{HM}}\big(\mathrm{O}\big(K^{2s,2s}_{r,s}(r,l)\big)\big) \leqslant \frac{r^2}{240\pi^2} 2^{-\nu(2r)} \quad \text{if } l \text{ is even}\,.$$

Proof. (i) If l is odd and $K = K^{2s,2s}_{r,s}(0,l)$ or $K^{2s,2s}_{r,s}(r,l)$, then by Proposition 8.3,

$$\mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}(K)) = \frac{r^2}{160} 2^{-\nu(2r)} \big(\alpha_2(K)^{-1}|2r|_2^{-1}\big) \prod_{p|2r} \big(1 + \chi_p p^{-2}\big)\,,$$

where $\chi_p = \pm 1$. By (8.7), $\alpha_2(K)^{-1} \leqslant (2/45)|2r|_2$, and by Lemma 9.1,

$$\prod_{p|2r} \big(1 + \chi_p p^{-2}\big) \leqslant \zeta(2)\zeta(4)^{-1} = 15\pi^{-2}\,,$$

implying

$$\mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}(K)) \leqslant \frac{r^2}{240\pi^2} 2^{-\nu(2r)}\,.$$

(ii) If l is odd and $K = K^{2s,s}_{r,s}(0,l)$, then by Proposition 8.3 and Lemma 9.1,

$$\mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}(K)) \leqslant \frac{r^2}{6} 2^{-\nu(8r)} \prod_{p|8r} \big(1 + \chi(N_0) p^{-2}\big) \leqslant \frac{5r^2}{2\pi^2} 2^{-\nu(2r)}\,.$$

(iii) Similarly, if l is even and $K = K^{2s,2s}_{r,s}(r,l)$, then by Proposition 8.3 and Lemma 9.1,

$$\mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}(K)) = \frac{(2r)^2}{14400} 2^{-\nu(2r)} \prod_{p|2r} \big(1 + \chi(N_0) p^{-2}\big) \leqslant \frac{r^2}{240\pi^2} 2^{-\nu(2r)}\,.$$ □

9.4 Hirzebruch–Mumford volumes and elliptic obstructions

We now produce bounds for Hirzebruch–Mumford volumes of groups related to elliptic obstructions.

Lemma 9.5. *Let $K^{\mathrm{II}} = U \oplus U(3) \oplus \langle -2r\rangle$ and $K^{\mathrm{III}} = U \oplus P \oplus \langle -2r\rangle$, where P is an indefinite binary quadratic form satisfying $D(P) = C_9$. If $K = K^{\mathrm{II}}$ or K^{III}, then*

$$\mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}(K)) \leqslant \frac{3r^2}{5\pi^2} 2^{-\nu(18r)}\,.$$

Proof. We use θ_0, θ_a, θ'_a, θ''_a, θ'''_a to denote elements of $\mathbb{Z}_p^*$ and $D(L)_p$ to denote the p-component of the abelian group $D(L)$. If $K = K^{\mathrm{II}}$ or K^{III}, $p = 2$ and $x \in D(K)_2 \cong \mathbb{Z}/(2^a)\mathbb{Z}$, then

$$q_K(x)_2 = \left(\frac{\theta_2 x^2}{2^a}\right) \bmod 2\mathbb{Z}_2.$$

Therefore, by [Nik80, Proposition 1.8.1],

$$K_2 \cong U \oplus V \oplus \langle\theta_a\rangle(2^a) \quad \text{or} \quad K_2 \cong U \oplus U \oplus \langle\theta_a\rangle(2^a)\,,$$

where $V = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2\end{smallmatrix}\right)$. Similarly, if $p = 3$, then

$$K_3^{\mathrm{II}} = \begin{cases} \langle 1\rangle^{\oplus 2} \oplus \langle\theta_0\rangle \oplus (\langle\theta'_3\rangle \oplus \langle\theta''_3\rangle)(3) & \text{if } a = 2\,,\\ \langle 1\rangle \oplus \langle\theta_0\rangle \oplus (\langle\theta'_3\rangle \oplus \langle\theta''_3\rangle \oplus \langle\theta'''_3\rangle)(3) & \text{if } a = 3\,,\\ \langle 1\rangle \oplus \langle\theta_0\rangle \oplus (\langle\theta'_3\rangle \oplus \langle\theta''_3\rangle)(3) \oplus \langle\theta_{3^{a-2}}\rangle(3^{a-2}) & \text{if } a > 3\end{cases}$$

and

$$K_3^{\mathrm{III}} = \begin{cases} \langle 1\rangle^{\oplus 3} \oplus \langle\theta_0\rangle \oplus \langle\theta_a\rangle(9) & \text{if } a = 2\,,\\ \langle 1\rangle^{\oplus 2} \oplus \langle\theta_0\rangle \oplus \langle\theta_3\rangle(3) \oplus \langle\theta_9\rangle(9) & \text{if } a = 3\,,\\ \langle 1\rangle^{\oplus 2} \oplus \langle\theta_0\rangle \oplus (\langle\theta'_9\rangle \oplus \langle\theta''_9\rangle)(9) & \text{if } a = 4\,,\\ \langle 1\rangle^{\oplus 2} \oplus \langle\theta_0\rangle \oplus (\langle\theta_9\rangle)(9) \oplus (\langle\theta_{3^{a-2}}\rangle)(3^{a-2}) & \text{if } a > 4\,.\end{cases}$$

If $K = K^{\mathrm{II}}$ or K^{III} and $p > 3$, then (by direct calculation or [Nik80, Proposition 1.8.1]),

$$q_K(x)_p = \left(\frac{\theta_a x^2}{p^a}\right) \bmod 2\mathbb{Z}_p\,,$$

where $x \in D(K)_p \cong \mathbb{Z}/(p^a)\mathbb{Z}$, implying

$$K_p = \begin{cases} \langle 1\rangle^{\oplus 4} \oplus \langle\theta_0\rangle & \text{if } a = 0\,,\\ \langle 1\rangle^{\oplus 3} \oplus \langle\theta_0\rangle \oplus \langle\theta_a\rangle(p^a) & \text{if } a > 0\,.\end{cases}$$

We now calculate the terms $\alpha_p(K)$. Suppose $K = K^{\mathrm{II}}$ or K^{III}, and let $p = 2$. Then $q = 1$, $w = a$ and $P_2(K) = P_2(2)$.

(i) If $a = 1$, then $E_{-1} = 1$, $E_0 = 1/2$, $E_1 = 1$, $E_2 = 1/2$ and $E_2(K) \geqslant 4$.

(ii) If $a = 2$, then $E_{-1} = 1$, $E_0 \leqslant 5/8$, $E_1 = 1/2$, $E_2 = 1$, $E_3 = 1/2$ and $E_2(K) \geqslant 32/5$.

(iii) If $a = 3$, then $E_{-1} = 1$, $E_0 \leqslant 5/8$, $E_1 = 1$, $E_2 = 1/2$, $E_3 = 1$, $E_4 = 1/2$ and $E_2(K) \geqslant 32/5$.

(iv) If $a \geqslant 4$, then $E_{-1} = 1$, $E_0 \leqslant 5/8$, $E_1 = 1$, $E_{a-1} = 1/2$, $E_a = 1$, $E_{a+1} = 1/2$ and $E_2(K) \geqslant 32/5$.

Therefore, $E_2(K) \geqslant 4$ and

$$\alpha_2(K) \geqslant 2^{5+a} P_2(2)\,.$$

Suppose $K = K^{\mathrm{II}}$.

(i) If $a = 2$, then $s = 2$, $w = 3$, $P_3(K) = P_3(1)^2$ and $E_3(K) = 3/4$.

(ii) If $a = 3$, then $s = 2$, $w = 6$, $P_3(K) = P_3(1)^2$ and $E_3(K) = 3/4$.

(iii) If $a > 3$, then $s = 3$, $w = a + 3$, $P_3(K) = P_3(1)^2$ and $E_3(K) = 9/16$.

Therefore, $s \geqslant 2$, $w \geqslant a + 1$, $P_3(K) \geqslant P_3(1)^2$ and $E_3(K) \geqslant 9/16$.

Suppose $K = K^{\mathrm{III}}$.

(i) If $a = 2$, then $s = 2$, $w = 2$, $P_3(K) = P_3(2)$ and $E_3(K) \geqslant 9/10$.

(ii) If $a = 3$, then $s = 3$, $w = 4$, $P_3(K) = P_3(1)$ and $E_3(K) = 1$.

(iii) If $a = 4$, then $s = 2$, $w = 6$, $P_3(K) = P_3(1)^2$ and $E_3(K) \geqslant 3/4$.

(iv) If $a > 4$, then $s = 3$, $w = a + 2$, $P_3(K) = P_3(1)$ and $E_3(K) = 1$.

Therefore, $s \geqslant 2$, $w \geqslant a$, $P_3(K) \geqslant P_3(1)^2$ and $E_3(K) \geqslant 3/4$. Hence, for $K = K^{\mathrm{II}}$ or K^{III},

$$\alpha_3(K) \geqslant 2^{-3} 3^{a+2} P_3(1)^2 .$$

Similarly, if $p > 3$, then

$$\alpha_p(K) \geqslant \begin{cases} P_p(2) & \text{if } a = 0\,, \\ 2p^a P_p(2)\bigl(1 + p^{-2}\bigr)^{-1} & \text{if } a > 0 \end{cases}$$

(as in the case of $K^{2s,2s}_{r,s}(0, l)$ with l odd in Theorem 8.3).

Therefore,

$$\begin{aligned}
\prod_p \alpha_p(K)^{-1} &= \Bigl(\prod_{\substack{p \\ p \neq 2,3}} \alpha_p(K)^{-1}\Bigr) \alpha_2(K)^{-1} \alpha_3(K)^{-1} \\
&\leqslant \Bigl(\prod_p P_p(2)^{-1}\Bigr)\Bigl(\prod_{p \mid 18r} 2^{-1} |18r|_p \bigl(1 + p^{-2}\bigr)\Bigr)\Bigl(\frac{16 P_3(2)}{9 P_3(1)^2 (1 + 3^{-2})}\Bigr)\Bigl(\frac{1}{2^4(1 + 2^{-2})}\Bigr) \\
&\leqslant \frac{4}{45}\Bigl(\prod_p P_p(2)^{-1}\Bigr)\Bigl(\prod_{p \mid 18r} 2^{-1} |18r|_p \bigl(1 + p^{-2}\bigr)\Bigr) .
\end{aligned}$$

From which we obtain,

$$\mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}(K)) \leqslant \Bigl(\frac{108 r^2}{5\pi^6}\Bigr)\zeta(2)\zeta(4) 2^{-\nu(18r)} \prod_{p \mid 18r} \bigl(1 + p^{-2}\bigr) ,$$

and by Lemma 9.1,

$$\mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}(K)) \leqslant \Bigl(\frac{3r^2}{5\pi^2}\Bigr) 2^{-\nu(18r)} . \qquad \square$$

10. Obstruction calculations

In this section, we study the obstruction spaces $\mathrm{RefObs}_{(4-a)k}(\Gamma)$ and $\mathrm{EllObs}_{(4-a)k}(\Gamma, \mathcal{E})$ defined in (3.9) and (3.10). We begin by characterising the spaces in Propositions 10.1 and 10.10; we exhibit a collection of vectors $\mathcal{E}$ satisfying the conditions of Propositions 6.2 and 6.3 in Lemma 10.11; we study the growth of $\mathrm{RefObs}_{(4-a)k}(\Gamma)$ and $\mathrm{EllObs}_{(4-a)k}(\Gamma, \mathcal{E})$ in Propositions 10.7, 10.13, 10.14

and 10.15, and we examine their codimensions in Proposition 11.2, following [GHS08]. Throughout, we assume $L = 2U \oplus \langle -6\rangle \oplus \langle -2d\rangle$ and use $x\|y$ to denote $x|y$ and $(x, y/x) = 1$.

10.1 Reflective obstructions

Proposition 10.1 ([GHS08, Proposition 4.1]). *If $0 < a < 4$ is integral and $k > 0$ is even, then*

$$\mathrm{RefObs}_{k(4-a)}(\Gamma) \subset \bigoplus_{z\in\mathcal{R}/\Gamma} \bigoplus_{i=0}^{k/2-1} M_{k(4-a)+2i}\big(\Gamma \cap \mathrm{O}^+(z^\perp)\big)\,,$$

where $\mathcal{R}/\Gamma$ is a set of representatives for the Γ-orbits of

$$\mathcal{R} := \{z \in L \mid z \text{ is primitive and } \pm\sigma_z \in \Gamma\}\,.$$

Definition 10.2. By Hirzebruch–Mumford proportionality, we define $\alpha(a)$ by

$$\dim \bigoplus_{z\in\mathcal{R}/\Gamma} \bigoplus_{i=0}^{k/2-1} M_{k(4-a)+2i}\big(\Gamma \cap \mathrm{O}^+(z^\perp)\big) = \alpha(a)k^4 + O\big(k^3\big)\,. \tag{10.1}$$

We bound $\alpha(a)$ in Proposition 10.7, starting with bounds for $\mathcal{R}/\Gamma$ in Lemma 10.3.

Lemma 10.3. *If $L = 2U \oplus \langle -2r\rangle \oplus \langle -2s\rangle$, let $J(K)$ denote the number of $[z] \in \mathcal{R}/\Gamma$ such that $z^\perp \cong K$.*

(i) *If $l^2 \equiv 1 \bmod s$, l is odd and $K = K^{2s,2s}_{r,s}(0,l)$, then $J(K) \leqslant 2^{\nu(s)+1}$ if s is odd, $J(K) \leqslant 2^{\nu(s)}$ if $2\|s$, $J(K) \leqslant 2^{\nu(s)+1}$ if $4\|s$ and $J(K) \leqslant 2^{\nu(s)+2}$ if $8|s$.*

(ii) *If $2l_1^2 \equiv 2 \bmod 2s$, $l = 2l_1$ is even and $K = K^{2s,s}_{r,s}(0,l)$, then $J(K) \leqslant 2^{\nu(s)}$ if s is odd, $J(K) \leqslant 2^{\nu(s)-1}$ if $2\|s$, $J(K) \leqslant 2^{\nu(s)}$ if $4\|s$ and $J(K) \leqslant 2^{\nu(s)+1}$ if $8|s$.*

(iii) *If $l^2 \equiv 1 \bmod s$, l is even and $K = K^{2s,2s}_{r,s}(r,l)$, then $J(K) \leqslant 2^{\nu(s)+1}$.*

(iv) *If $l^2 \equiv 2 \bmod 2s$, l is odd and $K = K^{2s,2s}_{r,s}(r,l)$, then $J(K) \leqslant 2^{\nu(s)+1}$ if s is odd, $J(K) \leqslant 2^{\nu(s)}$ if $2\|s$, $J(K) \leqslant 2^{\nu(s)+1}$ if $4\|s$ and $J(K) \leqslant 2^{\nu(s)+2}$ if $8|s$.*

Proof. By Proposition 5.3, if $z \in \mathcal{R}/\Gamma$ with $u := -(z,z)$, $m := \operatorname{div}(z)$ and $z^* = z/\operatorname{div}(z) = (k,l) \in D(L)$, then $z^\perp \cong K^{u,m}_{r,s}(k,l)$. As $\widetilde{\mathrm{SO}}^+(L) \subset \Gamma$, then by the Eichler criterion, it suffices to bound the number of $l \in \mathbb{Z}/(2s)\mathbb{Z}$ satisfying the conditions of Lemma 5.2. We count solutions as in [GHS07a, Lemma 3.3].

(i) Suppose $l^2 \equiv 1 \bmod s$, where l is odd and $z^\perp \cong K^{2s,2s}_{r,s}(0,l)$. We count solutions to $(l-1)(l+1) \equiv 0 \bmod p_i^{a_i}$, where $\{p_i^{a_i}\}$ are the distinct prime-power divisors of s. As $m = \operatorname{div}(z) = 2s$ is the order of $(0,l) \in D(L)$, then $(l,2s) = 1$ and so, if p_i is odd, then p_i cannot divide both $l-1$ and $l+1$ as otherwise $p_i|l$ and $(l,s) \neq 1$. Therefore, $l - 1 \equiv 0 \bmod p_i^{a_i}$ or $l + 1 \equiv 0 \bmod p_i^{a_i}$. If $2^a\|s$, then $l^2 - 1 \equiv 0 \bmod 2^a$ has exactly one solution if $a = 0$ or 1 and two solutions if $a = 2$. If $a \geqslant 3$, then only one of $4|l-1$ or $4|l+1$ can occur (otherwise $2|(l,s)$) and so $l \equiv \pm 1 \bmod 2^a$ or $l \equiv \pm 1 + 2^{a-1} \bmod 2^a$. Therefore, by the Chinese remainder theorem, $l^2 - 1 \equiv 0 \bmod s$ has $2^{\nu(s)}$ solutions if s is odd, $2^{\nu(s)-1}$ solutions if $2\|s$, $2^{\nu(s)}$ solutions if $4\|s$ and $2^{\nu(s)+1}$ solutions if $8|s$. If $(l,s) = 1$ and s is even, then $(l,2s) = (l+s,2s) = 1$. If $(l,s) = 1$ and s is odd, then precisely one of $(l,2s) = 1$ and $(l+s,2s) = 2$ is satisfied. We therefore obtain a bound for the solutions to $l^2 \equiv 1 \bmod s$ with $l \in \mathbb{Z}/(2s)\mathbb{Z}$ and $(l,2s) = 1$.

(ii) Suppose $l = 2l_1$, where $2l_1^2 \equiv 2 \bmod 2s$ and $z^\perp \cong K^{2s,s}_{r,s}(0,l)$. As l is uniquely determined by $l_1 \bmod s$, we count solutions to $l_1^2 \equiv 1 \bmod s$ as in item (i). There are $2^{\nu(s)}$ solutions if s is odd, $2^{\nu(s)-1}$ solutions if $2\|s$, $2^{\nu(s)}$ solutions if $4\|s$ and $2^{\nu(s)+1}$ solutions if $8|s$.

(iii) Suppose $l^2 \equiv 1 \bmod s$ with l even and $z^\perp \cong K^{-2s,2s}_{r,s}(r,l)$. As l is even, then s is odd. As in item (i), there are $2^{\nu(s)}$ solutions to $l^2 \equiv 1 \bmod s$ and $2^{\nu(s)+1}$ solutions if l is taken in $\mathbb{Z}/(2s)\mathbb{Z}$.

(iv) See the proof of item (i). ☐

To use the Hirzebruch–Mumford bounds of § 9, we will also need bounds for $|\mathrm{O}(q_K)|$, where $K \cong z^\perp$ for $z \in \mathcal{R}$.

LEMMA 10.4. *If K is a lattice and $D(K) \cong C_{2r}$, then $|\mathrm{O}(q_K)| \leqslant 2^{\nu(r)}$.*

Proof. Suppose that $x \in \mathbb{Z}/(2r)\mathbb{Z}$ is a generator for $D(K)$. By [Nik80, Proposition 1.8.1], the discriminant form satisfies

$$q_K(x) = \frac{x^2\theta}{2r} \bmod 2\mathbb{Z}\,,$$

where $\theta \in \mathbb{Z}$ and $(\theta, 2r) = 1$. As $(\theta, 2r) = 1$, the condition $q_K(y) = q_K(x)$ is equivalent to

$$x^2 \equiv y^2 \bmod 4r\,. \tag{10.2}$$

Suppose $4r = p_1^{a_1} \cdots p_n^{a_n}$, where the p_i are distinct primes, $p_1 = 2$ and $a_i \neq 0$. By the Chinese remainder theorem, (10.2) is satisfied if and only if

$$(x-y)(x+y) \equiv 0 \bmod p_i^{a_i} \quad \forall p_i^{a_i} \| 4r\,.$$

If $p > 2$ and $p \mid (x-y)$ and $p \mid (x+y)$, then $p|2x$, contradicting $(x, 2r) = 1$. Therefore, precisely one of $x-y \equiv 0 \bmod p_i^{a_i}$ or $x+y \equiv 0 \bmod p_i^{a_i}$ is true for each i. Let $p = 2$. If $a_1 = 2$, then we have $x \equiv 1 \mod 2$; if $a_1 = 3$, then (as x is odd) $2\|x-y$ and $4\|x+y$, or $4\|x-y$ and $2\|x+y$, implying $x \equiv \pm y \bmod 4$; if $a > 3$, then 4 cannot divide both $x-y$ and $x+y$; otherwise, $4|2x$, implying that x is even. Therefore, $2^{a_1-1}\|x+y$ or $2^{a_1-1}\|x-y$, and so $x \equiv \pm y \bmod 2^{a_1-1}$. As $D(K)$ is cyclic, then any element of $\mathrm{O}(q_k)$ is defined by its image on a generator of $D(K)$. Therefore, by counting solutions to (10.2) for $x \in \mathbb{Z}/(2r)\mathbb{Z}$, we conclude that $|\mathrm{O}(q_K)| \leqslant 2^{\nu(r)}$. ☐

LEMMA 10.5. *If K is a lattice such that $D(K) \cong C_2^{\oplus 3} \oplus C_r$, where r is odd, then $|\mathrm{O}(q_K)| \leqslant 2^{\nu(r)+3}$.*

Proof. Let q_K^2 and q_K^r be the restrictions of the discriminant form q_K to $C_2^{\oplus 3} \subset D(K)$ and $C_r \subset D(K)$, respectively. As r is odd, then as explained in [Nik80, § 1], we have $q_K = q_K^2 \oplus q_K^r$ and therefore $\mathrm{O}(q_K) \cong \mathrm{O}(q_K^2) \times \mathrm{O}(q_K^r)$. As $\mathrm{O}(q_K^2) \subset \mathrm{Aut}(C_2^{\oplus 3})$ and $\left|\mathrm{Aut}(C_2^{\oplus 3})\right| = 8$, then $|\mathrm{O}(q_K)| \leqslant 8|\mathrm{O}(q_K^r)|$. By [Nik80, Proposition 1.8.1], if $x \in \mathbb{Z}/r\mathbb{Z}$, then

$$q_K^r(x) = \frac{x^2\theta}{r} \bmod 2\mathbb{Z}\,, \tag{10.3}$$

where $(\theta, r) = 1$. To bound $|\mathrm{O}(q_K^r)|$, we count solutions to $q_K^r(x) = q_K^r(y)$ satisfying $(x, r) = (y, r) = 1$ for fixed y. As $(\theta, r) = 1$, then (10.3) is equivalent to $(x-y)(x+y) \equiv 0 \bmod 2r$ or to

$$(x-y)(x+y) \equiv 0 \bmod p_i^{a_i} \tag{10.4}$$

for all primes p_i satisfying $p_i^{a_i}\|2r$. If the prime p satisfies $p|2r$, $p|x-y$ and $p|x+y$, then $p|2x$, which cannot hold unless $p = 2$, implying that (10.4) has exactly two solutions for each $p_i \neq 2$. If $p_i = 2$ and $p_i^{a_i}\|2r$, then $a_i = 1$ and (10.4) has exactly one solution. Therefore, (10.3) has no more than $2^{\nu(r)}$ solutions, from which the result follows. ☐

We will need the following lemma in § 10.2.

LEMMA 10.6. *If $K = U \oplus U(3) \oplus \langle -2r \rangle$, then $|\mathrm{O}(q_K)| \leqslant (624)2^{\nu(2r)}$. If $K = U \oplus P \oplus \langle -2r \rangle$, where P is an indefinite binary quadratic form satisfying $D(P) = C_9$, then $|\mathrm{O}(q_K)| \leqslant (72)2^{\nu(2r)}$.*

Proof. If $K = U \oplus U(3) \oplus \langle -2r \rangle$, then $D(K) \cong C_3 \oplus C_3 \oplus C_{2r}$. The discriminant form q_K is given by

$$q_K(a,b,c) = \frac{2ab}{3} - \frac{c^2}{2r} \bmod 2\mathbb{Z} \tag{10.5}$$

for $(a,b,c) \in D(K)$, and we define $x, y, z \in D(K)$ by $x = (1,0,0)$, $y = (0,1,0)$ and $z = (0,0,1)$. If $g \in \mathrm{O}(q_K)$, then gx and gy belong to the unique subgroups $C_3^{\oplus 2} \subset D(K)$ if $(2r,3) = 1$ and $C_3^{\oplus 3} \subset D(K)$ if $(2r,3) \neq 1$. In each case, there are no more than $\left|\mathrm{Aut}\left(C_3^{\oplus 2}\right)\right| = 48$ or $\left|\mathrm{Aut}\left(C_3^{\oplus 3}\right)\right| = 624$ possible images for (x,y). By considering (10.5) as in Lemma 10.4, we see that there are at most $2^{\nu(2r)}$ images for $gz \in \langle gx, gy \rangle^{\perp} \cong C_{2r}$, implying $|\mathrm{O}(q_K)| \leqslant (624)2^{\nu(2r)}$. Similarly, if $K = U \oplus P \oplus \langle -2r \rangle$, then $D(K) \cong C_9 \oplus C_{2r}$ and

$$q_K(a,b) = \frac{\theta a^2}{9} - \frac{b^2}{2r} \bmod 2\mathbb{Z} \tag{10.6}$$

for $(a,b) \in D(K)$ and $\theta \in \mathbb{Z}$. We define $x', y' \in D(K)$ by $x' = (1,0)$ and $y' = (0,1)$. There are at most 6, 18 or 72 elements of order 9 in $D(K)$ if $(2r,9)$ equals 1, 3 or 9, respectively. By (10.6), there are at most $2^{\nu(2r)}$ images for y' in $(gx')^{\perp}$, and the result follows. □

Proposition 10.7. *Suppose $2^{\nu(x)} \leqslant C_\epsilon x^\epsilon$ for some $\epsilon, C_\epsilon > 0$. If d is odd, then*

$$\alpha(a) \leqslant \left(\alpha_1 d^2 + \alpha_2 d^\epsilon\right) C(a)\,,$$

where $\alpha_1 = 108199 \cdot \left(675\pi^2\right)^{-1}$, $\alpha_2 = 216263 \cdot \left(300\pi^2\right)^{-1} C_\epsilon$ and

$$C(a) = \frac{1}{8}\left(4(4-a)^3 + 6(4-a)^2 + 4(4-a) + 1\right). \tag{10.7}$$

Proof. Let

$$I(K) := \mathrm{Max}\left\{|\langle \mathrm{O}(z^{\perp}), -I\rangle : \langle \mathrm{O}\left(z^{\perp}\right) \cap \Gamma, -I\rangle| \mid z \in \mathcal{R}\right\}$$

and $J(K) := \left|\left\{z \in \mathcal{R}/\Gamma \mid z^{\perp} \cong K\right\}\right|$, where $K = z^{\perp} \subset L$ for $z \in \mathcal{R}$. As $C(a)$ in (10.7) is the k^4-coefficient of the polynomial $\sum_{j=0}^{k/2-1}((4-a)k + 2j)^3$, then $E(K) := I(K)J(K)\,\mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}(K))$ is an upper bound for the contribution from $[z] \in \mathcal{R}/\Gamma$ with $z^{\perp} \cong K$ to $\alpha(a)/C(a)$ in (10.1). We produced bounds for $J(K)$ in Lemma 10.3. To bound $I(K)$ we will show that

$$|\langle \mathrm{O}(K), -I\rangle : \langle \mathrm{O}(K) \cap \Gamma, -I\rangle| \leqslant \mathrm{O}(q_K)$$

and apply Lemmas 10.4 and 10.5.

By Proposition 2.1, we have $\widetilde{\mathrm{SO}}^+(K) \subset \mathrm{O}(K) \cap \Gamma$, and so

$$|\langle \mathrm{O}(K), -I\rangle : \langle \mathrm{O}(K) \cap \Gamma, -I\rangle| \leqslant |\mathrm{O}(K) : \mathrm{O}(K) \cap \Gamma| \leqslant \left|\mathrm{O}(K) : \widetilde{\mathrm{SO}}^+(K)\right|.$$

As $|\mathrm{O}(L) : \mathrm{O}^+(L)| \leqslant 2$, $\left|\widetilde{\mathrm{O}}^+(L) : \widetilde{\mathrm{SO}}^+(L)\right| \leqslant 2$ and $\left|\mathrm{O}^+(L) : \widetilde{\mathrm{O}}^+(L)\right| \leqslant |\mathrm{O}(q_L)|$, then

$$|\mathrm{O}(K) : \widetilde{\mathrm{SO}}^+(K)| \leqslant 4|\mathrm{O}(q_K)|\,. \tag{10.8}$$

Therefore, by Lemma 9.4,

$$E\left(K_{d,3}^{6,3}(0,l)\right) \leqslant \frac{160d^2}{\pi^2}\,, \qquad E\left(K_{d,3}^{6,6}(0,l)\right) \leqslant \frac{64d^2}{675\pi^2}\,,$$
$$E\left(K_{3,d}^{2d,d}(0,l)\right) \leqslant \frac{720 \cdot (2^{\nu(d)})}{\pi^2}\,, \qquad E\left(K_{3,d}^{2d,2d}(0,l)\right) \leqslant \frac{32 \cdot (2^{\nu(d)})}{75\pi^2}\,.$$

If l is odd,

$$E\big(K_{d,3}^{6,6}(0,l)\big) \leqslant \frac{2d^2}{15\pi^2}\,, \qquad\qquad E\big(K_{3,d}^{2d,2d}(3,l)\big) \leqslant \frac{3\cdot 2^{\nu(d)}}{10\pi^2}\,,$$

and if l is even,

$$E\big(K_{d,3}^{6,6}(0,l)\big) \leqslant \frac{d^2}{15\pi^2}\,, \qquad\qquad E\big(K_{3,d}^{2d,2d}(3,l)\big) \leqslant \frac{3\cdot 2^{\nu(d)}}{20\pi^2}\,.$$

Hence,

$$\alpha(a) \leqslant \big(\alpha_1 d^2 + \alpha_2 d^{\epsilon}\big)C(a)\,,$$

where $\alpha_1 = 108199\cdot\big(675\pi^2\big)^{-1}$ and $\alpha_2 = 216263\cdot\big(300\pi^2\big)^{-1}C_{\epsilon}$. $\square$

To produce the general-type results of § 11, we will need an explicit bound for $2^{\nu(x)}$.

Lemma 10.8. *If $\epsilon > 0$, then $2^{\nu(x)} = O(x^{\epsilon})$. In particular, $2^{\nu(x)} \leqslant (75/2)x^{1/6}$ for all $x \in \mathbb{N}$.*

Proof. Let $m \in \mathbb{N}$ be such that $1/m < \epsilon$. We proceed by induction on the number of primes n in the prime-power decomposition of $x \in \mathbb{N}$. If p_i denotes the ith prime, let $x = p_{i_1}^{a_1}\cdots p_{i_n}^{a_n}$, where the p_{i_j} are distinct and $a_i \in \mathbb{Z}_{>0}$. The condition $2^n \leqslant cx^{1/m}$ is equivalent to

$$n \leqslant \frac{\log(c)}{\log(2)} + \frac{1}{m\log(2)}\sum_{j=1}^{n} a_j \log(p_{i_j})\,, \tag{10.9}$$

and if N is minimal such that $\log(p_{N+1}) \geqslant m\log(2)$ and c is chosen such that (10.9) is satisfied for all $n \leqslant N$, then (10.9) is satisfied for all $n > N$. If

$$S(n) := \sum_{i=1}^{n}\log(p_i)\,,$$

then any c satisfying

$$\left(n - \frac{\log(C)}{\log(2)}\right)m\log(2) \leqslant S(n)$$

for all $n \leqslant N$ suffices. The result follows. $\square$

We note that Lemma 10.8 improves on a well-known bound of Robin [Rob83] when x is small.

Corollary 10.9. *We have*

$$\alpha(a) \leqslant \left(\frac{108199}{675\pi^2}d^2 + \frac{216263}{8\pi^2}d^{1/6}\right)C(a)\,.$$

10.2 Elliptic obstructions

We now produce bounds for $\mathrm{EllObs}_{k(4-a)}(\Gamma,\mathcal{E})$ and exhibit a set $\mathcal{E}$ satisfying the conditions of Propositions 6.2 and 6.3. We will use $\mathcal{E}/\Gamma$ to denote a set of representatives for the orbits of $\mathcal{E}$ for the action of Γ, and where no confusion is likely to arise, we will use $K \in \mathcal{E}/\Gamma$ to denote an embedding $(K \hookrightarrow L) \in \mathcal{E}/\Gamma$.

Lemma 10.10. *The obstruction space $\mathrm{EllObs}_{k(4-a)}(\Gamma,\mathcal{E})$ is a subspace of*

$$\bigoplus_{K\in\mathcal{E}/\Gamma}\bigoplus_{i=0}^{2k-1} M_{(4-a)k+i}(\Gamma\cap \mathrm{O}(K))\,.$$

Proof. See [GHS08, Proposition 4.1]. □

Lemma 10.11. *Suppose that $2d$ is square-free. Then there exists a set $\mathcal{E}$ satisfying the conditions of Propositions 6.2 and 6.3. The set $\mathcal{E}$ is given by*

$$\mathcal{E} = \mathcal{E}^{\mathrm{I}} \sqcup \mathcal{E}^{\mathrm{II}} \sqcup \mathcal{E}^{\mathrm{III}},$$

where $\mathcal{E}^{\mathrm{I}}$, $\mathcal{E}^{\mathrm{II}}$, $\mathcal{E}^{\mathrm{III}}$ are sets of primitive embeddings $\iota_K\colon K \hookrightarrow L$ satisfying the following conditions:

(i) *If $\iota_K \in \mathcal{E}^{\mathrm{I}}$, then $K = 2U \oplus \langle -2r\rangle$ and $0 < r \leqslant 2\sqrt{d}$.*
(ii) *If $\iota_K \in \mathcal{E}^{\mathrm{II}}$, then $K = U \oplus U(3) \oplus \langle -2r\rangle$ and $0 < r \leqslant (2\sqrt{d})/3$.*
(iii) *If $\iota_K \in \mathcal{E}^{\mathrm{III}}$, then $K = U \oplus P \oplus \langle -2r\rangle$, $0 < r \leqslant (2\sqrt{d})/3$ and P is an indefinite binary quadratic form with $D(P) = C_9$.*

Proof. We first show that $\mathcal{E}$ satisfies the conditions of Proposition 6.3. We take the $\mathbb{Z}$-basis of L given in (4.5) and take $g \in N(F)_{\mathbb{Z}}$ as in Lemma 4.11. By (3.2) and (4.5), we have $(\det A)^2 \det B = \det L = 12d$ and, as $2d$ is square-free, the pair $(\delta, \delta e)$ defining the matrix A is given by $(1,1)$, $(1,2)$, $(1,3)$ or $(1,6)$. By (3.5) (as noted in [GHS07b, Proposition 2.27]), we have $Z \in \Gamma_0(e)$, where $\Gamma_0(e) \subset \mathrm{SL}(2,\mathbb{Z})$ is a congruence subgroup. By considering $\mathrm{SL}(2,\mathbb{Z})/\Gamma(2)$ and noting that $\Gamma(2)$ contains no torsion other than $\pm I$ (or by the results of [Seb01]), we see that the group $\Gamma_0(2)$ contains no 6-torsion, allowing us to assume $(\delta, \delta e) = (1,1)$ or $(1,3)$. The lattice B in Lemma 4.11 is negative definite of determinant $(12d)(\delta^4 e^2)^{-1}$ and so admits a Gauss decomposition [CS99, § 15.3.2] of the form

$$B = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix},$$

where $\alpha < -2\beta \leqslant -\alpha \leqslant -\gamma$ and $\beta \leqslant 0$ if $\alpha = \gamma$. Therefore, there exist $t, v \in B$ such that $t \neq 0$, v is primitive, $(t,v) = 0$ and $|(v,v)| \leqslant 2\sqrt{(4d)(\delta^4 e^2)^{-1}}$. If $(\delta, \delta e) = (1,1)$, then the Gram matrix of $B^{\perp} \subset L$ is given by

$$B^{\perp} = \begin{pmatrix} 0 & I \\ I & D \end{pmatrix}$$

and, by the classification of unimodular lattices, $B^{\perp} \cong 2U$ and $K = 2U \oplus \langle -2r\rangle$, where $0 < r \leqslant 2\sqrt{d}$. Similarly, if $(\delta, \delta e) = (1,3)$, then B is of signature $(2,2)$ with $D(B) = C_3 \oplus C_3$ or C_9. If $D(B) = C_3 \oplus C_3$, then by the classification of 3-elementary lattices [CS99, § 15.8.2], we have $B^{\perp} \cong U \oplus U(3)$. If $D(B) = C_9$ then, by [Nik80, Corollary 1.13.5], we have $B^{\perp} = U \oplus P$, where P is an indefinite binary quadratic form of determinant 9. Therefore, $K = U \oplus U(3) \oplus \langle -2r\rangle$ or $K = U \oplus P \oplus \langle -2r\rangle$ with $0 < r \leqslant (2/3)\sqrt{d}$ and $\mathcal{E}$ satisfies the conditions of Proposition 6.3. The case of Proposition 6.2 follows by a similar argument. □

Definition 10.12. As for α, we define β by

$$\dim \bigoplus_{K \in \mathcal{E}/\Gamma} \bigoplus_{i=0}^{2k-1} M_{k(4-a)+i}(\Gamma \cap \mathrm{O}(K)) = \beta(a)k^4 + O\big(k^3\big).$$

We use $\beta^{\mathrm{I}}(a)$, $\beta^{\mathrm{II}}(a)$ and $\beta^{\mathrm{III}}(a)$ to denote the total contribution to $\beta(a)$ from embeddings of $K \in \mathcal{E}^{\mathrm{I}}$, $\mathcal{E}^{\mathrm{II}}$ and $\mathcal{E}^{\mathrm{III}}$, respectively. We will use $\beta_K(a)$ (or, if no confusion is likely to arise, β_K) to denote the contribution to $\beta(a)$ from a fixed embedding $K \in \mathcal{E}/\Gamma$.

Proposition 10.13. *For $\epsilon > 0$, suppose that A_ϵ satisfies $2^{\nu(i)} \leqslant A_\epsilon i^\epsilon$ for all $i \in \mathbb{N}\backslash\{0\}$. If $2d$ is* *square-free, then*

$$\beta^{\mathrm{I}}(a) := \sum_{K \in \mathcal{E}^{\mathrm{I}}/\Gamma} \beta_K(a) \leqslant \left(\frac{6144 \cdot 8^{\epsilon} A_{\epsilon}^3 D(a)}{5(3+\epsilon)\pi^2} \right) d^{(3+3\epsilon)/2} \left(\log\left(4\sqrt{d}\right) + 1 \right),$$

where

$$D(a) := 2(4-a)^3 + 6(4-a)^2 + 8(4-a) + 4 . \tag{10.10}$$

Proof. We estimate $\beta^{\mathrm{I}}(a)$ starting by bounding the number of embeddings $K \hookrightarrow L$ up to Γ-equivalence. Let $K \in \mathcal{E}^{\mathrm{I}}$, written as $K \cong 2U \oplus \langle -2r \rangle$. By [Nik80, Proposition 1.15.1], the primitive embeddings $K \hookrightarrow L$ are determined by tuples $(H_K, H_L, \gamma; T, \gamma_T)$, where $H_K \subset D(K)$ and $H_L \subset D(L)$ are subgroups, $\gamma\colon q_K|_{H_K} \xrightarrow{\sim} q_L|_{H_L}$ is an isomorphism of groups preserving the restriction of q_K and q_L to H_K and H_L, respectively, T is a lattice such that $\mathrm{gen}(T) = (0, 1, -\delta)$, where

$$\delta \cong (q_K \oplus (-q_L)|\Gamma_{\gamma}^{\perp})/\Gamma_{\gamma} \tag{10.11}$$

and Γ_γ is the pushout of γ in $D(K) \oplus D(L)$, and $\gamma_T\colon q_T \to (-\delta)$ is an isomorphism of finite quadratic forms. As explained in [Nik80, § 1.15, pp. 129–130], the embeddings defined by the tuples $(H_K, H_L, \gamma; T, \gamma_T)$ and $(H'_K, H'_L, \gamma'; T', \gamma'_T)$ are equivalent under $\widetilde{\mathrm{O}}(L)$ if and only if $\gamma = \gamma'$ and $\gamma_T = \pm\gamma'_T$. Therefore, as $\left|\widetilde{\mathrm{O}}(L) : \widetilde{\mathrm{SO}}^+(L)\right| = 4$ and $\widetilde{\mathrm{SO}}^+(L) \subset \Gamma$, there are at most four Γ-equivalent embeddings $K \subset L$ for each tuple $(H_K, H_L, \gamma; T, \gamma_T)$, implying

$$\beta^{\mathrm{I}}(a) \leqslant 4 \sum_{\substack{K \cong 2U \oplus \langle -2r \rangle \\ 0 < r \leqslant 2\sqrt{d}}} \sum_{(H_K, H_L, \gamma; T, \gamma_T)} \beta_K(a) . \tag{10.12}$$

(We have adopted the convention that when K, H_K, H_L, γ, T and γ_T occur as indices in a summation, they depend on the summand to the left.)

We now count the groups H_K and H_L. As $D(K)$ is cyclic, the subgroup H_K is uniquely determined by its order s, and as $H_K \cong H_L \subset D(L)$, then $s|6d$. As $2d$ is square-free, then

$$D(L) \cong \begin{cases} C_2^{\oplus 2} \oplus C_3^{\oplus 2} \oplus C_{d/3} & \text{if } 3|d , \\ C_2^{\oplus 2} \oplus C_3 \oplus C_d & \text{otherwise} , \end{cases}$$

implying that there are at most 1, 3, 4 or 12 embeddings of $H_L \cong C_s \subset D(L)$ if $(s, 6) = 1$, 2, 3 or 6, respectively. Therefore, by (10.12),

$$\beta^{\mathrm{I}}(a) \leqslant 48 \sum_{\substack{s|12d \\ s \leqslant 4\sqrt{d} \\ H_K \cong C_s}} \sum_{\substack{K \cong 2U \oplus \langle -2r \rangle \\ 0 < r \leqslant 2\sqrt{d}}} \sum_{\gamma, T, \gamma_T} \beta_K , \tag{10.13}$$

where the constraint $s \leqslant 4\sqrt{d}$ is obtained by noting that $C_s \cong H_K \subset D(K) \cong C_{2r}$ and $r \leqslant 2\sqrt{d}$ (which follows from Lemma 10.11).

We now count the isomorphisms γ. As $\gamma\colon H_K \xrightarrow{\sim} H_L$ preserves the restrictions of q_K and q_L, there are at most $|\mathrm{O}(H_L)|$ choices for γ for a given (H_K, H_L). To bound $|\mathrm{O}(H_L)|$, suppose $H_L \cong \langle (x, y) \rangle \subset D(L) \cong C_6 \oplus C_{2d}$, and let

$$\frac{q_1}{q_2} := \frac{-dx^2 - 3y^2}{6d} ,$$

where $q_1, q_2 \in \mathbb{Z}$ are coprime. We show that

$$\frac{q_1}{q_2} = \frac{\theta_s}{3s} \tag{10.14}$$

for some $\theta_s \in \mathbb{Z}$. As $(s(x,y))^2 \equiv 0 \bmod 2\mathbb{Z}$, then $q_2|s^2$. Suppose $p|q_2$ for prime p. If $p \neq 3$, then $p|q_2$, $q_2|s$ and $s|6d$ and, as $2d$ is square-free, $|q_2|_p = p^{-1}$. Noting that $q_2|s^2$, we have $|s|_p \leqslant p^{-1}$, implying $|3s|_p = |s|_p \leqslant |q_2|_p$. If $p = 3$, then as $2d$ is square-free, $|6d|_3 \geqslant 3^{-2}$ and $|q_2|_3 \geqslant 3^{-2}$ as $q_2|6d$. From $q_2|s^2$, we obtain $|s|_3 \leqslant 3^{-1}$ and $|3s|_3 \leqslant 3^{-2} \leqslant |q_2|_3$, hence $q_2|3s$, and (10.14) follows. As H_L is cyclic, then $|\mathrm{O}(H_L)|$ is equal to the number of solutions of $(z^2-1)\theta_s \equiv 0 \bmod 6s$, where $z \in (\mathbb{Z}/s\mathbb{Z})^*$. We can obtain an upper bound by counting solutions to

$$(z^2 - 1)\theta_s \equiv 0 \bmod s$$

or, noting that

$$s = \mathrm{lcm}\left(\frac{6}{(6,x)}, \frac{2d}{(2d,y)}\right) \tag{10.15}$$

is square-free, by counting solutions to

$$(z^2 - 1)\theta_s \equiv 0 \bmod p \tag{10.16}$$

for each prime $p|s$. We first show that $(p,\theta_s) = 1$, where $p \neq 2,3$ is prime. As $s|6d$, then

$$dx^2 + 3y^2 \equiv 3y^2 \bmod p,$$

where $p \neq 2,3$ and $p|s$. We must have $(p,y) = 1$; otherwise, as $2d$ is square-free, by (10.15), we obtain $(p,s) = 1$, which cannot hold by the choice of p. As $\theta_s|dx^2 + 3y^2$, then $(p,\theta_s) = 1$ and (10.16) has two solutions if $p \neq 2,3$, at most two solutions if $p = 2$ and at most three solutions if $p = 3$, implying that $|\mathrm{O}(H_L)| \leqslant 3 \cdot 2^{\nu(s)-1}$. Therefore, by (10.13),

$$\beta^{\mathrm{I}}(a) \leqslant 72 \sum_{\substack{s|12d \\ s \leqslant 4\sqrt{d} \\ H_K \cong C_s}} 2^{\nu(s)} \sum_{\substack{K \cong 2U \oplus \langle -2r \rangle \\ 0 < r \leqslant 2\sqrt{d}}} \sum_{T,\gamma_T} \beta_K\,. \tag{10.17}$$

As $H_K \subset D(K) \cong C_{2r}$, then $s|2r$, $0 < r \leqslant 2\sqrt{d}$ and, by (10.17),

$$\beta^{\mathrm{I}}(a) \leqslant 72 \sum_{\substack{s|12d \\ s \leqslant 4\sqrt{d}}} 2^{\nu(s)} \left(\sum_{\substack{i=1 \\ K \cong 2U \oplus \langle -si \rangle \\ 2|si}}^{(4\sqrt{d})/s} \sum_{T,\gamma_T} \beta_K \right). \tag{10.18}$$

We now count T and γ_T. The lattice T is of rank 1 and so uniquely determined by the data $(D(K), D(L), \gamma)$ defining δ. By (10.11), for fixed K, we have $\det T|(12d)(2r)$, and so, as in Lemma 10.4, there are at most $2^{\nu(6dr)}$ choices for γ_T for a given K. Therefore, by (10.18),

$$\beta^{\mathrm{I}}(a) \leqslant 72 \sum_{\substack{s|12d \\ s \leqslant 4\sqrt{d}}} 2^{\nu(s)} \sum_{\substack{i=1 \\ K \cong 2U \oplus \langle -si \rangle \\ 2|si}}^{(4\sqrt{d})/s} 2^{\nu(6dsi)} \beta_K\,. \tag{10.19}$$

We now bound β_K. As

$$\mathrm{Vol}_{\mathrm{HM}}(\Gamma \cap \mathrm{O}(K)) = |\langle \mathrm{O}(K), -I\rangle : \langle \Gamma \cap \mathrm{O}(K), -I\rangle| \, \mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}(K))$$

and $\widetilde{\mathrm{SO}}^+(K) \subset \Gamma \cap \mathrm{O}(K)$, then by (10.8) and Lemma 10.4,

$$|\langle \mathrm{O}(K), -I\rangle : \langle \Gamma \cap \mathrm{O}(K), -I\rangle| \leqslant 4|\mathrm{O}(q_K)| \leqslant 2^{\nu(r)+2}\,.$$

As $\mathrm{gen}(K) = \mathrm{gen}(K_{r,s}^{2s,2s}(0,l))$, we can bound $\mathrm{Vol}_{\mathrm{HM}}(\mathrm{O}(K))$ using Lemma 9.4, obtaining

$$\mathrm{Vol}_{\mathrm{HM}}(\Gamma \cap \mathrm{O}(K)) \leqslant \frac{r^2}{60\pi^2}\,. \tag{10.20}$$

From the identities

$$\sum_{k=1}^{n} k = n(n+1)/2\,, \quad \sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6 \quad \text{and} \quad \sum_{k=1}^{n} k^3 = n^2(n+1)^2/4\,,$$

we have $\beta_K \leqslant D(a)\big(16r^2/675\pi^2\big)$, where

$$D(a) := 2(4-a)^3 + 6(4-a)^2 + 8(4-a) + 4$$

is the k^4-coefficient of the polynomial $\sum_{j=0}^{2k-1}((4-a)k+j)^3$. Then, from (10.19), noting that $s|6d$ implies $\nu(6dsi) = \nu(6di)$ and bounding β_K as in (10.20), we obtain

$$\beta^{\mathrm{I}}(a) \leqslant \frac{12}{5\pi^2} D(a) 2^{\nu(3d)} \sum_{\substack{s|12d \\ s\leqslant 4\sqrt{d}}} 2^{\nu(s)} \sum_{i=1}^{(4\sqrt{d})/s} s^2 i^2 2^{\nu(i)}\,.$$

As $2^{\nu(i)} \leqslant A_\epsilon i^\epsilon$, we have

$$\sum_{i=1}^{(4\sqrt{d})/s} 2^{\nu(i)} i^2 \leqslant \sum_{i=1}^{(4\sqrt{d})/s} A_\epsilon i^{2+\epsilon} \leqslant \frac{A_\epsilon}{3+\epsilon}\frac{(8\sqrt{d})^{3+\epsilon}}{s^{3+\epsilon}}\,, \tag{10.21}$$

(with (10.21) following by comparing with an integral and noting that $s \leqslant 4\sqrt{d}$). By (10.21),

$$\begin{aligned}\sum_{\substack{s|12d \\ s\leqslant 4\sqrt{d}}} 2^{\nu(s)} \left(\sum_{i=1}^{(4\sqrt{d})/s} s^2 i^2 2^{\nu(i)}\right) &\leqslant \left(\frac{8^{3+\epsilon}A_\epsilon}{3+\epsilon}\right) d^{(3+\epsilon)/2} \sum_{\substack{s\leqslant 4\sqrt{d} \\ s|12d}} \frac{2^{\nu(s)}}{s^{1+\epsilon}} \\ &\leqslant \left(\frac{8^{3+\epsilon}A_\epsilon^2}{3+\epsilon}\right) d^{(3+\epsilon)/2} \sum_{s=1}^{4\sqrt{d}} s^{-1}\,,\end{aligned}$$

and as $\sum_{i=1}^{N} i^{-1} \leqslant 1 + \log(N)$, then

$$\leqslant \left(\frac{8^{3+\epsilon}A_\epsilon^2}{3+\epsilon}\right) d^{(3+\epsilon)/2}\big(\log\big(4\sqrt{d}\big)+1\big)\,.$$

We therefore obtain

$$\beta^{\mathrm{I}}(a) \leqslant \left(\frac{6144\cdot 8^\epsilon A_\epsilon^3 D(a)}{5(3+\epsilon)\pi^2}\right) d^{(3+3\epsilon)/2}\big(\log\big(4\sqrt{d}\big)+1\big)\,. \qquad \square$$

Proposition 10.14. *For $\epsilon > 0$, suppose that A_ϵ satisfies $2^{\nu(i)} \leqslant A_\epsilon i^\epsilon$ for all $i \in \mathbb{N}\backslash\{0\}$. If $2d$ is square-free, then*

$$\beta^{\mathrm{II}}(a) \leqslant \frac{3726508032\cdot A_\epsilon^3 8^\epsilon}{5(3+\epsilon)} D(a) d^{3(1+\epsilon)/2}\left(\log\left(\frac{4\sqrt{d}}{3}\right)+1\right),$$

where $D(a)$ is as in (10.10).

Proof. Let $K \cong U \oplus U(3) \oplus \langle -2r \rangle \in \mathcal{E}^{\mathrm{II}}$. As in Proposition 10.13,

$$\beta^{\mathrm{II}}(a) \leqslant 4 \sum_{\substack{K \cong U \oplus U(3) \oplus \langle -2r \rangle \\ 0<r\leqslant (2\sqrt{d})/3}} \sum_{(H_K, H_L, \gamma; T, \gamma_T)} \beta_K\,, \tag{10.22}$$

and we bound the number of tuples $(H_K, H_L, \gamma; T, \gamma_T)$ encoding Γ-equivalent embeddings of $K \in \mathcal{E}^{\mathrm{II}}$. We begin by characterising the groups H_K. Suppose

$$D(K) = C_3^{\oplus 2} \oplus C_{3^\mu} \oplus C_{2r'} \quad \text{and} \quad D(L) = C_2^{\oplus 2} \oplus C_3 \oplus C_{3^\delta} \oplus C_{d'}\,, \tag{10.23}$$

where $3^\mu \| 2r$, $3^\delta \| 2d$ and $r' := 3^{-\mu} r$, $d' := 3^{-\delta} d$. As explained in [Pet11], if G_1 and G_2 are finite groups of coprime order, then by a theorem of Suzuki [Suz51] (see also [Sch94, § 4]), subgroups of $G_1 \times G_2$ are of the form $G_1' \times G_2'$, where $G_1' \subset G_1$ and $G_2' \subset G_2$. Therefore,

$$H_K = H_K^3 \oplus C_s\,, \tag{10.24}$$

where $H_K^3 \subset C_3^{\oplus 2} \oplus C_{3^\mu}$ and $C_s \subset C_{2r'}$ for some s. As $H_L \cong H_K$, then by (10.23), we have $H_K^3 \subset C_3 \oplus C_{3^\delta}$ and, as $2d$ is square-free, δ is 0 or 1 and H_K^3 is isomorphic to $\{e\}$, C_3 or $C_3^{\oplus 2}$.

We now bound the number of embeddings $H_K \subset D(K)$, where H_K is as in (10.24) and C_s is fixed. There is a single embedding $H_K \subset D(K)$ if $H_K^3 = \{e\}$, there are at most 13 embeddings if $H_K^3 = C_3$, and there are at most 13 embeddings if $H_K^3 = C_3^{\oplus 2}$. Therefore (by using an identical argument for $H_L \subset D(L)$ and noting that there are 3 elements of order 2 in $C_2^{\oplus 2}$), for fixed s, there are at most 3 pairs (H_K, H_L) if $H_K^3 = \{e\}$, 156 pairs if $H_K^3 = C_3$ and 39 pairs if $H_K^3 = C_3^{\oplus 2}$.

We now count the maps γ. For fixed (H_K, H_L), there are at most $|\mathrm{O}(H_L)|$ choices for $\gamma\colon H_K \xrightarrow{\sim} H_L$. As in Proposition 10.13, we have $|\mathrm{O}(H_L)| \leqslant (3)2^{\nu(s)-1}$ if $H_K^3 = \{e\}$, we have $|\mathrm{O}(H_L)| \leqslant (3)2^{\nu(s)}$ if $H_K^3 = C_3$, and we have $|\mathrm{O}(H_L)| \leqslant (9)2^{\nu(s)+3}$ if $H_K^3 = C_3^{\oplus 2}$. Hence, by (10.22),

$$\beta^{\mathrm{II}}(a) \leqslant 13122 \sum_{\substack{K = U \oplus U(3) \oplus \langle -2r \rangle \\ 0<r\leqslant (2\sqrt{d})/3}} \sum_{H_K = H_K^3 \oplus C_s} 2^{\nu(s)} \sum_{(T, \gamma_T)} \beta_K\,.$$

As in Proposition 10.13, the lattice T is uniquely determined by (H_K, H_L, γ), implying that there are at most $2^{\nu(6dr)}$ choices for γ_T for a given T. If $H_K \cong H_K^3 \oplus C_s$, then $s|2r$, and as $H_K \cong H_L$, then $s|2d$. As $r \leqslant (2\sqrt{d})/3$, then

$$\beta^{\mathrm{II}}(a) \leqslant 13122 \sum_{\substack{s|2d \\ 0<s\leqslant (4\sqrt{d})/3}} 2^{\nu(s)} \sum_{\substack{i=1 \\ K = U \oplus U(3) \oplus \langle -si \rangle \\ 2|si}}^{4\sqrt{d}/3s} 2^{\nu(6dsi)} \beta_K\,.$$

We now bound β_K. By (10.8) and Lemma 10.6,

$$|\langle \mathrm{O}(K), -I \rangle : \langle \Gamma \cap \mathrm{O}(K), -I \rangle| \leqslant 4|\mathrm{O}(q_K)| \leqslant (2496)2^{\nu(2r)}\,,$$

and by Lemma 9.5, we have $\beta_K(a) \leqslant \frac{1}{5} \cdot 7488 D(a) r^2$. Therefore,

$$\beta^{\mathrm{II}}(a) \leqslant \frac{98257536}{5} 2^{\nu(3d)} D(a) \left(\sum_{\substack{s|2d \\ 0<s\leqslant (4\sqrt{d})/3}} 2^{\nu(s)} s^2 \sum_{i=1}^{(4\sqrt{d})/3s} 2^{\nu(i)} i^2 \right),$$

where $D(a)$ is as in (10.10). As in (10.21),

$$\sum_{i=1}^{(4\sqrt{d})/3s} 2^{\nu(i)} i^2 \leqslant \frac{A_\epsilon (8\sqrt{d})^{3+\epsilon}}{(3+\epsilon)(3s)^{3+\epsilon}}$$

and so

$$\beta^{\mathrm{II}}(a) \leqslant \frac{3726508032 \cdot A_\epsilon^3 24^\epsilon}{5(3+\epsilon)} D(a) d^{3(1+\epsilon)/2} \left(\log\left(\frac{4\sqrt{d}}{3} \right) + 1 \right). \qquad \square$$

Proposition 10.15. *For $\epsilon > 0$, suppose that A_ϵ satisfies $2^{\nu(i)} \leqslant A_\epsilon i^\epsilon$ for all $i \in \mathbb{N}\backslash\{0\}$. If $2d$ is square-free, then*

$$\beta^{\mathrm{III}}(a) \leqslant \frac{47775744 \cdot A_\epsilon^3 8^\epsilon D(a)}{5(3+\epsilon)} d^{3(1+\epsilon)/2} \left(\log\left(\frac{4\sqrt{d}}{3} \right) + 1 \right),$$

where $D(a)$ is as in (10.10).

Proof. Our argument proceeds as in Proposition 10.14. Let $K \cong U \oplus P \oplus \langle -2r \rangle \in \mathcal{E}^{\mathrm{III}}$, where P is an indefinite binary quadratic form of determinant 9; then

$$\beta^{\mathrm{III}}(a) \leqslant 4 \sum_{\substack{K \cong U \oplus P \oplus \langle -2r \rangle \\ 0 < r \leqslant (2\sqrt{d})/3}} \sum_{(H_K, H_L, \gamma; T, \gamma_T)} \beta_K ,$$

and we begin by bounding the number of tuples $(H_K, H_L, \gamma; T, \gamma_T)$ for each K. We have

$$H_K^3 \oplus C_s \cong H_K \subset D(K),$$

where $H_K^3 = \{e\}$, C_3 or $C_3^{\oplus 2}$ and $s|2r$. For fixed C_s, there are at most 3 pairs (H_K, H_L) if $H_K^3 = \{e\}$ or $C_3^{\oplus 2}$ and at most 48 pairs if $H_K^3 = C_3$. The number of $\gamma \colon H_L \xrightarrow{\sim} H_K$ preserving the restriction of q_L and q_K is bounded as in Proposition 10.13. Hence, as in Proposition 10.14,

$$\begin{aligned}\beta^{\mathrm{III}}(a) &\leqslant \frac{1259712}{5} D(a) \left(\sum_{\substack{s|2d \\ 0 < s \leqslant (4\sqrt{d})/3}} 2^{\nu(s)} s^2 \sum_{i=1}^{(4\sqrt{d})/3s} 2^{\nu(6dsi)} i^2 \right) \\ &\leqslant \frac{47775744 \cdot A_\epsilon^3 8^\epsilon D(a)}{5(3+\epsilon)} d^{3(1+\epsilon)/2} \left(\log\left(\frac{4\sqrt{d}}{3} \right) + 1 \right),\end{aligned}$$

where $D(a)$ is as in (10.10). $\square$

11. General-type results

Proposition 11.1. *Suppose that $2d$ is square-free, and take $\mathcal{E}$ as in Lemma 10.11. If $a = 2$ and $d > 2.04134 \times 10^{28}$, or $a = 3$ and $d > 2.5112 \times 10^{29}$, then*

$$\dim M_{k(4-a)}(\Gamma) - \dim \mathrm{RefObs}_{k(4-a)}(\Gamma) - \dim \mathrm{EllObs}_{k(4-a)}(\Gamma, \mathcal{E}) > 0 \qquad (11.1)$$

for $k \gg 0$.

Proof. By Hirzebruch–Mumford proportionality, (11.1) is satisfied for $k \gg 0$ if

$$\gamma(4-a)^4 - \alpha(a) - \beta^{\mathrm{I}}(a) - \beta^{\mathrm{II}}(a) - \beta^{\mathrm{III}}(a) > 0. \qquad (11.2)$$

As $\log(d) \leqslant 6d^{1/12}$ for $d > 10^{10}$, by using the bounds of Corollary 10.9 and Propositions 10.13, 10.14 and 10.15, we obtain a polynomial in $d^{1/12}$ bounding (11.2) from below. By Descartes' rule of signs [Des86], this polynomial has at most one root, which we locate using a computer. □

Because of the range of values taken by $\nu(d)/\log(d)$, the bounds obtained in Proposition 11.1 are quite large. However, as a consequence of the Erdős–Kac theorem [EK40], the value of $\nu(d)/\log(d)$ is typically much smaller than the worst case assumed in Proposition 11.1, allowing smaller bounds to be obtained by constraining $\nu(d)$.

Proposition 11.2. *Suppose that d is prime, and take $\mathcal{E} = \mathcal{E}^{\mathrm{I}}$ as in Lemma 10.11. If $a = 2$ and $d > 409679370162$, or $a = 3$ and $d > 3468755940829$, then*

$$\dim M_{k(4-a)}(\Gamma) - \dim \mathrm{RefObs}_{k(4-a)}(\Gamma) - \dim \mathrm{EllObs}_{k(4-a)}(\Gamma, \mathcal{E}) > 0$$

for $k \gg 0$.

Proof. We proceed as in Proposition 11.1, noting that we can assume $\mathcal{E} = \mathcal{E}^{\mathrm{I}}$ and using the improved bound

$$\beta^{\mathrm{I}}(a) \leqslant \frac{16128 \cdot (8^{3+\epsilon}) A_\epsilon D(a) d^{(3+\epsilon)/2}}{85\pi^2(3+\epsilon)},$$

which is obtained by setting $\nu(d) = 1$ in (10.19). □

Theorem 11.3. *The orthogonal modular varieties $\mathcal{F}_L(\Gamma)$ are of general type for all odd square-free $d > 2.5112 \times 10^{29}$ and for all prime $d > 3468755940829$.*

Proof. This is immediate from Propositions 11.1 and 11.2, Corollary 7.15 and Theorem 3.2. □

Remark. It was questioned in [CS88] whether the local density α_2, see [Wat76] (stated in [Kit93, Theorem 5.6.3]), *might* differ from its actual value by a factor of 2^n for a rank n lattice. We do not know if this has been resolved. The cautious reader may therefore wish to multiply the bounds of Propositions 11.1 and 11.2 and Theorem 11.3 by a factor of $4^{1/4}$.

Corollary 11.4. *Under the conditions of Theorem 11.3, the moduli $\mathcal{M}$ of deformation generalised Kummer varieties of dimension 4 with polarisation of degree $2d$ and split type is of general type if the orthogonal modular variety $\mathcal{F}(\Gamma)$ is.*

Proof. By [GHS13, Theorem 3.10], there is an open immersion $\mathcal{M} \to \mathcal{F}(\Gamma)$. □

Acknowledgements

I thank Professor Gregory Sankaran for piquing my interest in general-type results back when I was a PhD student and for all manner of interesting conversations over the years. I also thank the anonymous referee for suggesting the approach of § 7, which greatly improved upon my earlier attempts. Part of the paper was completed when I worked at the University of Bonn. I am especially grateful to Professor Huybrechts for providing excellent working conditions and a very stimulating environment.

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