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A note on spherical bundles on K3 surfaces

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With an appendix by Genki Ouchi

Abstract

In this note we provide (partial) answers to two questions posed in Chapter 16 of D. Huybrechts' book *Lectures on $K3$ surfaces*, concerning the bounded derived category $\mathrm{D}^b(S)$ of a K3 surface S. The first question asks if for a spherical object E in $\mathrm{D}^b(S)$, there always exists a non-zero object F satisfying $\mathrm{RHom}(E,F)=0$. The second question asks if a spherical bundle E is always semistable with respect to some polarization on S and if there is a way to 'count' spherical bundles with a fixed Mukai vector. In this note we provide (partial) answers to these two questions. In the appendix Genki Ouchi shows that any spherical twist associated with an n-spherical object on a smooth projective n-dimensional variety is not conjugate to a standard autoequivalence.

1. Introduction

Let S be a smooth K3 surface and E be a spherical object in $\mathrm{D}^b(S)$; in other words,

$$\dim \mathrm{Hom}(E,E[i]) = \begin{cases} 1 & \text{when } i=0,2\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

The first question at the end of [Huy16, Chapter 16] asks whether $E^\perp \neq 0$; in other words, does there exist a non-trivial object F such that $\mathrm{Hom}(E,F[i])=0$ for all $i\in\mathbb{Z}$.

In the case where the K3 surface S is of Picard number 1, the question is answered in [Ouc20, Appendix A] by Bayer using tools and results from [BB17]. Here, we provide an affirmative answer to this question for all K3 surfaces and some more general cases.

Proposition 1.1 (Proposition 2.3). *Let X be a smooth proper n-dimensional variety over $\mathbb{C}$ and E be an n-spherical object in $\mathrm{D}^b(X)$. Then there exists a non-trivial object F orthogonal to E; in other words, $\mathrm{Hom}(E,F[k])=\mathrm{Hom}(F,E[k])=0$ for every $k\in\mathbb{Z}$.*

Proposition 1.1 has some immediate corollaries. For example, [Ouc20, Theorem 3.1] now holds for all spherical objects in $\mathrm{D}^b(X)$ without any extra assumptions. Another corollary is pointed

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out with this question in [Huy16, Chapter 16]: the composition of the spherical twist T_E^k is never a homological shift when $k \neq 0$. In fact, the autoequivalence T_E^k is not conjugate to a standard autoequivalence with respect to X in $\mathrm{Aut}\big(\mathrm{D}^b(X)\big)$. We refer to Theorem A.2 in the appendix for more details.

We may ask a consecutive question about $E^\perp$.

Question 1.2. Let S be a smooth projective surface and E be a 2-spherical object in $\mathrm{D}^b(S)$. Denote by $\langle E^\perp, E\rangle$ the full sub-triangulated category generated by E and objects in $E^\perp$. Do we always have an equivalence

$$\overline{\mathrm{D}^b(S)/\langle E^\perp, E\rangle} \cong \mathrm{D}_{\mathrm{sing}}\big(\mathrm{Spec}\big(\mathbb{C}[x]/\big(x^2\big)\big)\big)\,?$$

A few more details regarding this question will be discussed in Remark 2.6. Moving on to the next question, consider a smooth projective K3 surface S with Picard number greater than 1, and let E be a spherical bundle on S. The second question at the end of [Huy16, Chapter 16] asks if E is always Gieseker stable with respect to some polarization, and if there is a way to 'count' spherical bundles on S with a given Mukai vector. Definitions on stability and Mukai vectors will be recapped at the beginning of Section 3.

Unlike the first question, the naive answer to the above question is negative.

Example 1.3. (a) We can give an example of a K3 surface S of Picard number 2 and a spherical bundle F on S. The bundle F is not slope semistable (in particular, not Gieseker semistable) with respect to any polarization of S.

(b) We can give an example of a K3 surface S of Picard number 3 and a Mukai vector $v = (2, D, c)$ satisfying $\langle v, v\rangle = -2$. We can give examples of infinitely many spherical bundles $\{E_i\}$ with $v(E_i) = v$. Each bundle E_i is slope stable (in particular, Gieseker stable) with respect to some different polarization.

More details of these examples will be given in Examples 3.4 and 3.7. Here we point out that in Example 1.3(b), the nef cone of the surface S is circular. This is the main reason that the moduli space of a given numerical class may have infinitely many walls and chambers. Conversely, when the nef cone of the surface is rational polyhedral, this can never happen. More precisely, we have the following result in the more general case.

Proposition 1.4 (Proposition 4.4). *Let S be a smooth projective surface over $\mathbb{C}$ with a rational polyhedral nef cone. Let $v = (\mathrm{rk}, \ell, s) \in \tilde{\mathrm{H}}(S, \mathbb{Z})$ with $\mathrm{rk} > 0$ and (rk, ℓ) primitive. Then the moduli space $M^s(v)$ has only finitely many walls and chambers in the ample cone $\mathrm{Amp}(S)$ of S.*

Note that for any given polarization H and spherical Mukai vector v, there is at most one Gieseker p_H-stable (and slope μ_H-stable) vector bundle. Due to Proposition 1.4, the finite theory on walls and chambers, we can talk about 'counting' stable spherical bundles in the following sense for a large class of K3 surfaces.

Corollary 1.5. *Let S be a smooth projective K3 surface with $\mathrm{Nef}(S)$ rational polyhedral and v be a spherical Mukai vector with rank greater than 0. Then*

$$\mathsf{H}_S(v) := \#\{E \mid v(E) = v,\ E \text{ is } \mu_H\text{-stable with respect to some polarization } H\}$$

is finite.

We then show that the counting can be done purely numerically. In particular, the function $\mathsf{H}_S(-)$ is completely determined by the numerical information $(\mathrm{Nef}(S), \langle -,-\rangle_{\mathrm{intersection}})$ of the surface S. In Section 5, we make a case study on a generic elliptic K3 surface admitting a section. To convince the reader that such a counting theory may be not completely pointless, we highlight some interesting observations and results on $\mathsf{H}_S(-)$ when S is an elliptic K3 surface.

Let S be a generic elliptic K3 surface admitting a section. Assume $\mathrm{NS}(S) = \mathbb{Z}e \oplus \mathbb{Z}\sigma$, where e stands for the class of the fibre and σ for the class of the (-2)-curve. Then for some special spherical Mukai vectors, the value of $\mathsf{H}_S(v)$ can be explicitly computed and verified. In Example 5.4, we show that $\mathsf{H}_S((\mathrm{rk}, \sigma, 0)) = \mathrm{rk}$. In some other cases, the counting seems to have some pattern, but we cannot give a proof. For example, let

$$b_1 = b_2 = 1 \quad \text{and} \quad b_{n+1} := b_n + b_{n-1}$$

be the Fibonacci sequence; then

$$\mathsf{H}_S((b_{n+1}, b_n\sigma + *e, *)) = \left\lfloor \frac{n}{2} \right\rfloor^2 + 2\,,$$

where the $*$ are integers making the Mukai vector spherical.

In general, it seems unrealistic to give an explicit formula for $\mathsf{H}_S(v)$ for all v. The asymptotic behaviour of H_S with respect to the rank is already rather difficult. We have the following expectations on some of the asymptotic behaviour.

Question 1.6. Let S be a generic elliptic K3 surface admitting a section. Is it true that

$$\min_{\mathrm{rk}(v)=m}\{\mathsf{H}_S(v)\} = \Theta\big((\ln m)^2\big) \quad \text{and} \quad \mathrm{Average}_{\mathrm{rk}(v)=m}\{\mathsf{H}_S(v)\} = \Theta\big((\ln m)^2\big)\,?$$

Here we write $f(m) = \Theta(g(m))$ if there exist positive numbers C_1, C_2, and N such that $C_1 g(m) < f(m) < C_2 g(m)$ whenever $m > N$.

Note that $\mathsf{H}_S((\mathrm{rk}, \sigma, 0)) = \mathrm{rk}$; the spherical Mukai character $(\mathrm{rk}, \sigma, 0)$ is an outlier. In general, a spherical Mukai vector $v = (m, a\sigma + *e, *)$ has '$\mathsf{H}(v) \gg (\ln m)^2$' only when a/m is 'very close' to a rational number s/r with '$r \ll m$'. For example, the ratio $1/\mathrm{rk}$ is very close to $0/1$. Such outlier rational numbers consist of a tiny proportion of rational numbers, which tends to 0 as m tends to infinity.

As a piece of evidence for Question 1.6, we prove the following weaker form on the average estimate for H_S.

Proposition 1.7 (Proposition 5.7). *Let S be a generic elliptic K3 surface admitting a section. Then for any $\alpha > 0$, there are positive constants C_1 and C_2 such that for $m \gg 0$,*

$$C_1 \frac{\varphi(m)}{m} (\ln m)^2 \leqslant \mathrm{Average}_{\mathrm{rk}(v)=m}\{\mathsf{H}_S(v)\} \leqslant C_2 m^{\alpha}\,.$$

To get the asymptotic behaviour of $\mathrm{Average}_{\mathrm{rk}(v)=m}\{\mathsf{H}_S(v)\}$, we have reduced the main difficulty to the following analytic number theory question, which might hold independent interest.

Question 1.8. For $m \in \mathbb{Z}_{\geqslant 2}$, let $A_m := \big\{m^2 + r^2 - trm \in \mathbb{Z} \mid r, t \in \mathbb{Z}_{\geqslant 1}, \gcd(m, r) = 1\big\} \cap \big[1, m^2\big]$ and $\mathsf{G}'(m) := \sum_{a \in A_m} \tau(a)$, where $\tau(n) = \sum_{d|n} 1$ is the divisor function. What is the asymptotic behaviour of $\mathsf{G}'(m)$?

Notation. We adopt the standard notation from [Huy16] in general, and we always work over $\mathbb{C}$. We recap some notions at the beginning of each section for the convenience of the readers.

For a smooth proper variety X, we denote by $\mathrm{D}^b(X)$ the bounded derived category of coherent sheaves on X. Given two objects E and F in $\mathrm{D}^b(X)$, we write $\hom(E, F) := \dim \mathrm{Hom}(E, F)$. Let S be a smooth projective surface over $\mathbb{C}$. We adopt the following standard notation on the cone of S: Néron–Severi group $\mathrm{NS}(S) := \mathrm{Pic}(S)/\mathrm{Pic}^0(S)$ and $\mathrm{NS}(S)_{\mathbb{R}} := \mathrm{NS}(S) \otimes_{\mathbb{Z}} \mathbb{R}$, ample cone $\mathrm{Amp}(S)$, nef cone $\mathrm{Nef}(S) = \overline{\mathrm{Amp}}(S)$, effective cone $\mathrm{Eff}(S)$, and Mori cone $\overline{\mathrm{NE}}(S) = \overline{\mathrm{Eff}}(S)$.

2. Orthogonal component to a spherical object

Definition 2.1. Let X be a smooth proper n-dimensional variety over $\mathbb{C}$ ($n \geqslant 1$). A non-zero object E in $\mathrm{D}^b(X)$ is called *n-spherical* if $\mathsf{S}_X(E) = E[n]$ and

$$\mathrm{Hom}(E, E[i]) = \begin{cases} \mathbb{C} & \text{when } i = 0, n\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

An object G is defined to be in $E^{\perp}$ if $\mathrm{RHom}(E, G) = 0$.

Example 2.2 (An element in $\mathcal{O}_X^{\perp}$). Assume that X is a strict Calabi–Yau variety of dimension n; in other words, $\omega_X \cong \mathcal{O}_X$ and $H^i(\mathcal{O}_X) = 0$ when $1 \leqslant i \leqslant n-1$. Then the structure sheaf $\mathcal{O}_X$ is n-spherical.

Let p and q be two different closed points on X. Denote by $\mathcal{O}_p$ and $\mathcal{I}_q$ the skyscraper sheaf at p and the ideal sheaf of q, respectively. By Serre duality, $\mathrm{Hom}(\mathcal{O}_p, \mathcal{I}_q[n]) = (\mathrm{Hom}(\mathcal{I}_q, \mathcal{O}_p))^* = \mathbb{C}$. Let f be a non-zero morphism in $\mathrm{Hom}(\mathcal{O}_p, \mathcal{I}_q[n])$.

Claim. *The object* $F := \mathrm{Cone}\big(\mathcal{O}_p \xrightarrow{f} \mathcal{I}_q[n]\big)$ *is in* $\mathcal{O}_X^{\perp}$.

Proof. Apply $\mathrm{Hom}(\mathcal{O}_X, -)$ to the distinguished triangle

$$\mathcal{O}_p \xrightarrow{f} \mathcal{I}_q[n] \longrightarrow F \xrightarrow{+}\ . \tag{2.1}$$

We get the long exact sequence

$$\begin{aligned} \cdots \longrightarrow \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_p[i]) \longrightarrow \mathrm{Hom}(\mathcal{O}_X, \mathcal{I}_q[n+i]) \longrightarrow \mathrm{Hom}(\mathcal{O}_X, F[i]) \\ \longrightarrow \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_p[i+1]) \longrightarrow \cdots\,. \end{aligned}$$

Note that $\mathrm{RHom}(\mathcal{O}_X, \mathcal{I}_q[n]) = \mathbb{C}$ and $\mathrm{RHom}(\mathcal{O}_X, \mathcal{O}_p) = \mathbb{C}$. We get $\mathrm{Hom}(\mathcal{O}_X, F[i]) = 0$ when $i \neq 0, -1$. Moreover,

$$\mathrm{Hom}(\mathcal{O}_X, F) \cong \mathrm{Hom}(\mathcal{O}_X, F[-1]), \quad \text{and they are both 0 or } \mathbb{C}\,. \tag{2.2}$$

Applying $\mathrm{Hom}(-, \mathcal{O}_X)$ to (2.1), we get the long exact sequence

$$\begin{aligned} \cdots \longrightarrow \mathrm{Hom}(\mathcal{O}_p, \mathcal{O}_X[n-1]) = 0 \longrightarrow \mathrm{Hom}(F, \mathcal{O}_X[n]) \longrightarrow \mathrm{Hom}(\mathcal{I}_q[n], \mathcal{O}_X[n]) \\ \xrightarrow{-\circ f} \mathrm{Hom}(\mathcal{O}_p, \mathcal{O}_X[n]) \longrightarrow \cdots\,. \end{aligned} \tag{2.3}$$

Note that $\mathrm{Hom}(\mathcal{I}_q[n], \mathcal{O}_X[n]) = \mathbb{C}$. Let g be a non-zero element in $\mathrm{Hom}(\mathcal{I}_q[n], \mathcal{O}_X[n])$; then $\mathrm{Cone}(g) \cong \mathcal{O}_q[n]$. We apply $\mathrm{Hom}(\mathcal{O}_p, -)$ to the distinguished triangle

$$\mathcal{I}_q[n] \xrightarrow{g} \mathcal{O}_X[n] \longrightarrow \mathcal{O}_q[n] \xrightarrow{+}\ . \tag{2.4}$$

As $\mathrm{RHom}(\mathcal{O}_p, \mathcal{O}_q) = 0$, the map $g \circ -\colon \mathrm{Hom}(\mathcal{O}_p, \mathcal{I}_q[n]) \to \mathrm{Hom}(\mathcal{O}_p, \mathcal{O}_X[n])$ is an isomorphism. It follows that the composition $g \circ f$ is non-zero.

Therefore, in (2.3), the map $- \circ f\colon \mathrm{Hom}(\mathcal{I}_q[n], \mathcal{O}_X[n]) \to \mathrm{Hom}(\mathcal{O}_p, \mathcal{O}_X[n])$ is non-zero. Note that $\hom(\mathcal{I}_q[n], \mathcal{O}_X[n]) = \hom(\mathcal{O}_p, \mathcal{O}_X[n]) = 1$, so the map $- \circ f$ is an isomorphism. It follows that we have $\mathrm{Hom}(F, \mathcal{O}_X[n]) = 0$. By Serre duality, we have $\mathrm{Hom}(\mathcal{O}_X, F) = 0$.

Together with (2.2), we get $\mathrm{RHom}(\mathcal{O}_X, F) = 0$; in other words, we have $F \in \mathcal{O}_X^{\perp}$. □

A similar construction to that in Example 2.2 works for all n-spherical objects in $\mathrm{D}^b(X)$. The statement reads as follows.

PROPOSITION 2.3. *Let X be an n-dimensional smooth proper variety ($n \geqslant 1$) over $\mathbb{C}$. Then for every n-spherical object E in $\mathrm{D}^b(X)$, its orthogonal category $E^\perp$ has non-trivial objects.*

Proof. By semicontinuity, we may pick two different closed points p and q on X (denote by $P = \mathcal{O}_p$ and $Q = \mathcal{O}_q$ the skyscraper sheaves) such that $\mathrm{RHom}(E, P) \cong \mathrm{RHom}(E, Q)$.

By Serre duality, we may consider the object

$$F := E \otimes \mathrm{RHom}(E, P) \cong E \otimes (\mathrm{RHom}(Q[-n], E))^* .$$

We complete the commutative square

$$\begin{tikzcd} Q[-n] \arrow[r] \arrow[d, "\mathsf{ev}_1"] & 0 \arrow[d] \\ F \arrow[r, "\mathsf{ev}_2"] & P \end{tikzcd}$$

to the following commutative diagrams of distinguished triangles (the arrows '$\xrightarrow{+}$' are all omitted):

$$\begin{tikzcd} Q[-n] \arrow[r, equal] \arrow[d] & Q[-n] \arrow[d, "\mathsf{ev}_1"] & \\ S_P \arrow[r] \arrow[d] & F \arrow[r, "\mathsf{ev}_2"] \arrow[d] & P \arrow[d, equal] \\ K \arrow[r] & S'_Q \arrow[r, "f"] & P . \end{tikzcd}$$

Our goal is to show that we have $K \in E^\perp$. To do this, we need to compute $\hom(E, S'_Q[i])$ for $i \in \mathbb{Z}$.

Apply $\mathrm{Hom}(-, E)$ to the distinguished triangle $Q[-n] \xrightarrow{\mathsf{ev}_1} F \to S'_Q \xrightarrow{+}$. We get surjective maps $- \circ \mathsf{ev}_1 \colon \mathrm{Hom}(F, E[i]) \to \mathrm{Hom}(Q[-n], E[i])$ for every $i \in \mathbb{Z}$ due to the nature of the evaluation map ev_1. So in the long exact sequence

$$\begin{aligned} \ldots \xrightarrow{-\circ \mathsf{ev}_1} \mathrm{Hom}(Q[-n], E[i-1]) \\ \xrightarrow{g_{i-1}} \mathrm{Hom}(S'_Q, E[i]) \longrightarrow \mathrm{Hom}(F, E[i]) \xrightarrow{-\circ \mathsf{ev}_1} \mathrm{Hom}(Q[-n], E[i]) \\ \xrightarrow{g_i} \mathrm{Hom}(S'_Q, E[i+1]) \longrightarrow \cdots , \end{aligned}$$

the map g_i is zero for every $i \in \mathbb{Z}$.

It follows that

$$\hom(F, E[i]) = \hom(Q[-n], E[i]) + \hom(S'_Q, E[i]) \tag{2.5}$$

for every $i \in \mathbb{Z}$.

As E is n-spherical, by the definition of $F = E \otimes \mathrm{RHom}(E, P)$, we have

$$\hom(F, E[i]) = \hom(E, P[-i]) + \hom(E, P[n-i]) \tag{2.6}$$

for every $i \in \mathbb{Z}$.

By Serre duality, $\hom(Q[-n], E[i]) = \hom(E, P[-i])$ for every $i \in \mathbb{Z}$. It follows by (2.5), (2.6), and Serre duality that

$$\hom(E, P[n-i]) = \hom(S'_Q, E[i]) = \hom(E, S'_Q[n-i]) \tag{2.7}$$

for every $i \in \mathbb{Z}$.

Finally, we apply $\mathrm{Hom}(E, -)$ to the commutative diagram

$$\begin{tikzcd} F \arrow[r, "\mathsf{ev}_2"] \arrow[d] & P \arrow[d, equal] \\ S'_Q \arrow[r, "f"] & P\,. \end{tikzcd}$$

By the definition of the evaluation map ev_2, the map $\mathsf{ev}_2 \circ - \colon \mathrm{Hom}(E, F[i]) \to \mathrm{Hom}(E, P[i])$ is surjective for every $i \in \mathbb{Z}$. It follows that the map $f \circ - \colon \mathrm{Hom}(E, S'_Q[i]) \to \mathrm{Hom}(E, P[i])$ is surjective for every $i \in \mathbb{Z}$ as well. By (2.7), this map is an isomorphism for every $i \in \mathbb{Z}$. We apply $\mathrm{Hom}(E, -)$ to the distinguished triangle

$$K \longrightarrow S'_Q \xrightarrow{f} P \xrightarrow{+}$$

at the bottom. It follows that $\mathrm{Hom}(E, K[i]) = 0$ for every $i \in \mathbb{Z}$. Therefore, we have $K \in E^{\perp}$.

Suppose that K is zero; then we have $S_P \cong Q[-n]$. As $p \neq q$, we have $F \cong P \oplus Q[-n]$, which is not the direct sum of spherical objects. We get a contradiction, so the object K is non-zero. □

Definition 2.4. Let E be a spherical object in $\mathrm{D}^b(X)$. The *spherical twist* T_E is defined to be the Fourier–Mukai transformation from $\mathrm{D}^b(X)$ to $\mathrm{D}^b(X)$ with kernel $\mathrm{Cone}(E^{\vee} \boxtimes E \xrightarrow{\mathsf{can}} \mathcal{O}_{\Delta})$.

The spherical twist T_E is an exact autoequivalence on $\mathrm{D}^b(X)$ (see [ST01, Theorem 1.2]). In practice, the spherical twist T_E on an object F can be computed by an equivalent definition to that in [ST01, Definition 2.5], namely $\mathsf{T}_E(F) = \mathrm{Cone}\big(E \otimes \mathrm{RHom}(E, F) \xrightarrow{\mathsf{ev}} F\big)$.

It is clear that $\mathsf{T}_E(E) = E[1-n]$ and T_E fixes objects in $E^{\perp}$. Proposition 2.3 immediately gives the following corollary. See also Theorem A.2 for a stronger statement.

Corollary 2.5. *Let X be an n-dimensional proper smooth variety over $\mathbb{C}$ and E be an n-spherical object. Then T_E^k is not a shift functor unless $k = 0$.*

Proof. When $n = 1$ and $g \neq 1$, only a skyscraper sheaf up to a shift $\mathcal{O}_p[m]$ can be 1-spherical. When $g = 1$, for every 1-spherical object E, there exists an exact autoequivalence Ψ of $\mathrm{D}^b(X)$ such that $\Psi(E) = \mathcal{O}_p$ for some skyscraper sheaf $\mathcal{O}_p$. As $\mathsf{T}^k_{\Psi(E)} = \Psi \circ \mathsf{T}^k_E \circ \Psi^{-1}$, this also reduces to the case where $E = \mathcal{O}_p$. Note that $\mathsf{T}_{\mathcal{O}_p} = \otimes \mathcal{O}_X(p)$ when X is a curve, so the statement holds.

When $n \neq 1$, by Proposition 2.3, the functor T^k_E fixes objects in $E^{\perp}$. Together with the fact that $\mathsf{T}_E(E) = E[1-n] \neq E$, this implies that the functor T^k_E is not a shift functor when $k \neq 0$. □

By the philosophy on the relation between even-dimensional nodal singularities and 2-spherical objects as in [CGL+23] and [KS23], one may reach Question 1.2 in the introduction. Namely, let S be a smooth projective surface with a 2-spherical object E in $\mathrm{D}^b(S)$. Do we always have the following isomorphism for the quotient category:

$$\overline{\mathrm{D}^b(S)/\langle E^{\perp}, E\rangle} \cong \mathrm{D}_{\mathrm{sing}}\big(\mathrm{Spec}\big(\mathbb{C}[x]/(x^2)\big)\big)\,?$$

Here we write $\mathrm{D}_{\mathrm{sing}}(X) = \mathrm{D}^b(X)/\mathrm{D}_{\mathrm{perf}}(X)$ for the singular category of a variety X, and $\overline{\mathcal{T}}$ denotes the idempotent completion of the triangulated category $\mathcal{T}$.

Remark 2.6. We finish this section with some remarks on Question 1.2.

(a) For a special case and the starting point of this question, assume that the surface S contains a smooth integral rational curve $C \cong \mathbf{P}^1$ with self-intersection -2. Then $E = \mathcal{O}_C(-1)$ is a 2-spherical object. Denote by $\pi\colon S \to T$ the morphism contracting C. Then π_* induces the equivalence $\mathrm{D}^b(S)/\langle \mathcal{O}_C(-1)\rangle \cong \mathrm{D}^b(T)$ and $\pi^*(\mathrm{D}_{\mathrm{perf}}(T)) = \langle E^\perp\rangle$. It follows that

$$\overline{\mathrm{D}^b(S)/\langle E^\perp, E\rangle} \cong \overline{\mathrm{D}^b(T)/\mathrm{D}_{\mathrm{perf}}(T)} = \overline{\mathrm{D}_{\mathrm{sing}}(T)} \cong \mathrm{D}_{\mathrm{sing}}\bigl(\mathrm{Spec}\bigl(\mathbb{C}[x]/\bigl(x^2\bigr)\bigr)\bigr)\,.$$

(b) The even-dimensional nodal singularity category $\mathrm{D}_{\mathrm{sing}}\bigl(\mathrm{Spec}\bigl(\mathbb{C}[x]/\bigl(x^2\bigr)\bigr)\bigr)$ is generated by the unique simple object A with $A = A[1]$. Each object in $\mathrm{D}_{\mathrm{sing}}\bigl(\mathrm{Spec}\bigl(\mathbb{C}[x]/\bigl(x^2\bigr)\bigr)\bigr)$ is of the form $A^{\oplus m}$ for some $m \in \mathbb{Z}_{\geqslant 0}$. By the argument in Proposition 2.3, one may check that $\overline{\mathcal{O}_p} = \overline{\mathcal{O}_q} = \overline{\mathcal{O}_q}[1]$ in $\overline{\mathrm{D}^b(S)/\langle E^\perp, E\rangle}$ for any points p and q on the support of E. One may further show that the category $\overline{\mathrm{D}^b(S)/\langle E^\perp, E\rangle}$ is classically generated by $\mathcal{O}_p$. This gives a piece of evidence for the question.

(c) The question may also have a more ambitious version in the general case. Replace S by an arbitrary smooth proper variety X and E by a Serre invariant full subcategory $\mathcal{A}$ in $\mathrm{D}^b(X)$. One may ask whether $\overline{\mathrm{D}^b(X)/\langle \mathcal{A}^\perp, \mathcal{A}\rangle} \cong \mathcal{A}^\dagger$, which is independent on the choice of X.

3. Spherical bundles and stability

From this section on, we will focus on smooth projective surfaces rather than varieties with arbitrary dimensions. We briefly recap some standard notions from [HL10] on the slope and Gieseker stability of coherent sheaves on a surface. As we will be mainly interested in torsion-free coherent sheaves, some notions are simplified in this paper.

3.1 Recap: Stability, Mukai vector

Definition 3.1. Let S be a smooth projective surface and H be an ample divisor. For a torsion-free sheaf E, the *reduced Hilbert polynomial* of E is defined to be $p_H(E, m) := \chi(E(mH))/\mathrm{rk}(E)$. The *slope* of E is defined to be $\mu_H(E) := H \cdot \mathrm{ch}_1(E)/\mathrm{rk}(E)$.

A torsion-free coherent sheaf E is called Gieseker stable with respect to H (or p_H-stable) if $p_H(F, m) < p_H(E, m)$ when $m \gg 0$, for all proper non-trivial subsheaves $F \subset E$. The sheaf E is called slope stable with respect to H (or μ_H-stable) if $\mu_H(F) < \mu_H(E)$ for all non-trivial subsheaves $F \subset E$ with $\mathrm{rk}(F) < \mathrm{rk}(E)$.

Analogously, we say that E is Gieseker semistable with respect to H (or p_H-semistable) if $p_H(F, m) \leqslant p_H(E, m)$ when $m \gg 0$, for all proper non-trivial subsheaves $F \subset E$. The sheaf E is called slope semistable with respect to H (or μ_H-semistable) if $\mu_H(F) \leqslant \mu_H(E)$ for all non-trivial subsheaves $F \subset E$ with $\mathrm{rk}(F) < \mathrm{rk}(E)$.

For a given polarization H, the following implications are well known:

$$\text{slope stable} \implies \text{Gieseker stable} \implies \text{Gieseker semistable} \implies \text{slope semistable}\,. \tag{3.1}$$

When E is strictly Gieseker semistable, in other words, semistable but not stable, E admits a Jordan–Hölder filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$ such that all quotients E_{i+1}/E_i are stable with the same reduced Hilbert polynomial as that of E. The direct sum of all factors

$\bigoplus_{i=1}^{n}(E_i/E_{i-1})$ does not depend on the filtration. When E is strictly slope semistable, a Jordan–Hölder filtration exists such that all quotients E_{i+1}/E_i are slope stable with the same slope as that of E. The direct sum of all factors $\bigoplus_{i=1}^{n}(E_i/E_{i-1})$ is only unique up to a modification of skyscraper sheaves.

Definition 3.2. Let S be a smooth projective K3 surface. The *Mukai vector* of an object $E \in \mathrm{D}^b(S)$ is defined to be

$$v(E) := (\mathrm{rk}(E), \mathrm{ch}_1(E), \mathrm{ch}_2(E) + \mathrm{rk}(E)) \in \tilde{\mathrm{H}}(S, \mathbb{Z})\,.$$

The lattice product structure on $\tilde{\mathrm{H}}(S,\mathbb{Z}) = \mathrm{H}^0(S,\mathbb{Z}) \oplus \mathrm{NS}(S) \oplus \mathrm{H}^4(S,\mathbb{Z})$ is defined to be

$$\langle (r_1, \ell_1, c_1), (r_2, \ell_2, c_2)\rangle := \ell_1 \cdot \ell_2 - r_1 c_2 - r_2 c_1\,.$$

We call a Mukai vector $v = (\mathrm{rk}, \ell, c) \in \tilde{H}(S,\mathbb{Z})$ *spherical* if $\mathrm{rk} > 0$ and $\langle v, v\rangle = -2$.

Here we add the condition $\mathrm{rk} > 0$ in the definition to simplify some statements in the future. It is clear by the Riemann–Roch theorem that $\chi(E,F) = -\langle v(E), v(F)\rangle$. The Mukai vector $v(E)$ of a spherical bundle E is spherical.

3.2 Unstable spherical bundles

Before giving an example of a spherical bundle on a K3 surface that is never stable, it is worth mentioning that a spherical coherent sheaf can be not torsion-free but with positive rank.

Example 3.3. Let S be a K3 surface containing a smooth rational curve C. Then we may consider the object $\mathsf{T}_{\mathcal{O}_S}(\mathcal{O}_C(-2))$, which is given by the distinguished triangle

$$\longrightarrow \mathcal{O}_S[-1] \xrightarrow{\mathsf{ev}} \mathcal{O}_C(-2) \longrightarrow \mathsf{T}_{\mathcal{O}_S}(\mathcal{O}_C(-2)) \longrightarrow \mathcal{O}_S \longrightarrow\ . \tag{3.2}$$

As $\mathcal{O}_C(-2)$ is spherical and $\mathsf{T}_{\mathcal{O}_S}$ is an autoequivalence, the object $\mathsf{T}_{\mathcal{O}_S}(\mathcal{O}_C(-2))$ is spherical. As $\mathsf{T}_{\mathcal{O}_S}(\mathcal{O}_C(-2))$ is the extension of two coherent sheaves $\mathcal{O}_C(-2)$ and $\mathcal{O}_S$, it is a coherent sheaf. It is clear that $\mathsf{T}_{\mathcal{O}_S}(\mathcal{O}_C(-2))$ is not torsion-free.

We now make a further spherical twist on $\mathsf{T}_{\mathcal{O}_S}(\mathcal{O}_C(-2))$ to get an example of a spherical bundle that is never stable.

Example 3.4. Let S be a smooth projective K3 surface with Picard number 2 containing two smooth integral rational curves (the existence of such a K3 surface is ensured by [Kov94, Theorem 2]). In particular, the cone $\mathrm{Nef}(S)$ is strictly contained in $\overline{\mathrm{NE}}(S)$.

Denote one of the smooth rational curves by C and its divisor class by $[C]$. Let β be a non-zero divisor in $[C]^{\perp} \cap \partial\mathrm{Nef}(S)$. Then the two rays of $\partial\mathrm{Nef}(S)$ are spanned by β and $\beta - a_1[C]$ for some $a_1 > 0$.

The two rays of $\partial\overline{\mathrm{NE}}(S)$ are spanned by $[C]$ and $\beta - a_2[C]$ for some $a_2 > a_1 > 0$. Moreover, it follows from the duality between $\mathrm{Nef}(S)$ and $\overline{\mathrm{NE}}(S)$ that $\beta^2 = 2a_1a_2$. As $a_2 > a_1$, there exists a very ample divisor $[G] = s(\beta - a[C])$ for some $a \in \mathbb{Q}$ such that

$$a_2 - \sqrt{a_2^2 - a_1 a_2} < a < a_1\,.$$

Claim. *For every integer $m \gg 0$, the object $F_m := \mathsf{T}_{\mathcal{O}_S(-mG)}(\mathsf{T}_{\mathcal{O}_S}(\mathcal{O}_C(-2)))[-1]$ is a spherical bundle that is not slope semistable with respect to any polarization H.*

Proof. Apply $\mathsf{T}_{\mathcal{O}_{S(-mG)}}$ to the short exact sequence

$$0 \longrightarrow \mathcal{O}_C(-2) \longrightarrow \mathsf{T}_{\mathcal{O}_S}(\mathcal{O}_C(-2)) \longrightarrow \mathcal{O}_S \longrightarrow 0\,.$$

We have the following diagram of distinguished triangles (arrows '$\xrightarrow{+}$' are omitted for simplicity):

$$\begin{tikzcd}
A_m \arrow[r] \arrow[d] & F_m \arrow[r] \arrow[d] & B_m \arrow[d] \\
\mathcal{O}_S(-mG)^{\oplus mGC-1} \arrow[r] \arrow[d, "\mathsf{ev}_1"] & \mathcal{O}_S(-mG)^{\oplus mGC+m^2G^2/2+1} \arrow[r] \arrow[d, "\mathsf{ev}"] & \mathcal{O}_S(-mG)^{\oplus m^2G^2/2+2} \arrow[d, "\mathsf{ev}_2"] \\
\mathcal{O}_C(-2) \arrow[r] & \mathsf{T}_{\mathcal{O}_S}(\mathcal{O}_C(-2)) \arrow[r] & \mathcal{O}_S\,,
\end{tikzcd}$$

where $A_m = \mathsf{T}_{\mathcal{O}_S(-mG)}(\mathcal{O}_C(-2))[-1]$ and $B_m = \mathsf{T}_{\mathcal{O}_S(-mG)}(\mathcal{O}_S)[-1]$. Note that ev_1 and ev_2 are both surjective in $\mathrm{Coh}(S)$ when $m \gg 0$. It follows that the objects A_m and B_m are torsion-free spherical sheaves, hence spherical bundles. So F_m is a spherical bundle.

To finish the claim, we show that when $m \gg 0$, the slopes satisfy $\mu_H(A_m) > \mu_H(B_m)$ for every polarization H of S. It follows by a direct computation that

$$\begin{aligned}
\mu(A_m) - \mu(B_m) &:= \frac{\mathrm{ch}_1(A_m)}{\mathrm{rk}(A_m)} - \frac{\mathrm{ch}_1(B_m)}{\mathrm{rk}(B_m)} \\
&= \frac{-mG(mGC-1)-C}{mGC-1} - \frac{-mG\left(\frac{1}{2}m^2G^2+2\right)}{\frac{1}{2}m^2G^2+1} \\
&= \frac{2mG}{m^2G^2+2} - \frac{C}{mGC-1} \\
&= \frac{2ms}{m^2G^2+2}\left(\beta - aC - \frac{m^2G^2+2}{2ms(mGC-1)}C\right).
\end{aligned}$$

Note that

$$\begin{aligned}
\lim_{m\to+\infty} a + \frac{m^2G^2+2}{2ms(mGC-1)} &= a + \frac{G^2}{2sGC} = a + \frac{s^2\beta^2 - 2a^2s^2}{4s^2a} \\
&= \frac{a_1a_2 + a^2}{2a} < a_2\,.
\end{aligned}$$

Therefore, when $m \gg 0$, the divisor $\mu(A_m) - \mu(B_m)$ is effective. For every ample divisor H, it follows that $\mu_H(A_m) - \mu_H(B_m) = H(\mu(A_m) - \mu(B_m)) > 0$. By definition, the spherical bundle F_m is not μ_H-semistable. □

Remark 3.5. We do not know if there are such kind of never-(semi)stable spherical bundles on a K3 surface with other types of Mori cones, for example an elliptic K3 surface admitting a section.

For a given spherical Mukai vector, we also do not know if there can be infinitely many never-stable spherical bundles with that vector.

Question 3.6. A related question is: given a spherical object (or semi-rigid object, see [BB17, Definition 2.4]) $E \in \mathrm{D}^b(S)$, does there exist a Bridgeland stability condition σ such that E is σ-semistable?

Note that when the K3 surface S is of Picard number 1, the question above is true due to [BB17, Lemma 6.3] (or in the spherical case, more precisely stated in [BB17, Corollary 6.9]).

3.3 Infinitely many spherical bundles

In this section, we give examples of infinitely many stable spherical bundles with the same Mukai vector. In general, we expect that there exist such examples when the cone $\mathrm{Nef}(S)$ is circular.

Example 3.7. Let Λ be a rank 3 primitive sublattice in $U^{\oplus 3} \subset U^{\oplus 3} \oplus (-E_8)^{\oplus 2}$ with an orthogonal basis $\{\mathbf{h}, \mathbf{e}, \mathbf{f}\}$ satisfying $\mathbf{h}^2 = 12$, $\mathbf{e}^2 = -6$, and $\mathbf{f}^2 = -30$. By [Mor84, Corollary 1.9], there exists a projective K3 surface S such that $\mathrm{NS}(S) \cong \Lambda$.

Claim. *There exist infinitely many spherical bundles with Mukai vector* $(2, \mathbf{f}, -7)$ *on* S*. Each of them is slope stable with respect to some polarization on* S*.*

Proof. For every pair of positive integer solutions (a, b) to the Pell equation $2x^2 - y^2 = 1$, we may consider the divisors $D_a = a\mathbf{h} + b\mathbf{e} + \mathbf{f}$ and $E_a = -a\mathbf{h} - b\mathbf{e}$.

As $\mathrm{NS}(S)$ has no class with self-intersection -2, its effective cone is circular and is equal to the nef cone. A divisor $D = x\mathbf{h} + y\mathbf{e} + z\mathbf{f}$ is in $\overline{\mathrm{NE}}(S) = \mathrm{Nef}(S)$ if and only if $D^2 \geqslant 0$ and $x \geqslant 0$.

Note that

$$(D_a - E_a)^2 = 48a^2 - 24b^2 - 30 = -6\,;$$

neither $D_a - E_a$ nor $E_a - D_a$ is effective. It follows that

$$\mathrm{Hom}(\mathcal{O}_S(D_a), \mathcal{O}_S(E_a)) = \mathrm{Hom}(\mathcal{O}_S(E_a), \mathcal{O}_S(D_a)) = 0\,.$$

By the Riemann–Roch theorem,

$$\chi(\mathcal{O}_S(E_a), \mathcal{O}_S(D_a)) = \tfrac{1}{2}(D_a - E_a)^2 + 2 = -1\,.$$

So $\mathrm{Ext}^1(\mathcal{O}_S(E_a), \mathcal{O}_S(D_a)) = \mathbb{C}$, and there is a unique rank 2 bundle V_a, non-trivially extended by $\mathcal{O}_S(D_a)$ and $\mathcal{O}_S(E_a)$. Namely, it fits into the short exact sequence

$$0 \longrightarrow \mathcal{O}_S(D_a) \longrightarrow V_a \longrightarrow \mathcal{O}_S(E_a) \longrightarrow 0\,.$$

The bundle V_a is also given as $\mathsf{T}_{\mathcal{O}_S(E_a)}(\mathcal{O}_S(D_a))$ by definition. So it is spherical. Its Mukai vector is given as

$$v(V_a) = \big(1, D_a, \tfrac{1}{2}D_a^2 + 1\big) + \big(1, E_a, \tfrac{1}{2}E_a^2 + 1\big) = (2, \mathbf{f}, -7)\,.$$

For two different pairs of positive integer solutions (a_1, b_1) and (a_2, b_2) with $a_1 > a_2$, by the criterion of the effective divisor, $D_{a_2} - D_{a_1}$ and $E_{a_2} - D_{a_1}$ are not effective. It follows that $\mathrm{Hom}(\mathcal{O}_S(D_{a_1}), V_{a_2}) = 0$. Therefore, V_{a_1} and V_{a_2} are not isomorphic to each other when $a_1 \neq a_2$. As the Pell equation $2x^2 - y^2 = 1$ has infinitely many positive integer solutions, there are infinitely many spherical bundles V_a with the same Mukai vector $(2, \mathbf{f}, -7)$.

We then show the stability of the V_a. For every V_a, we may consider the divisor $H_a := 7a\mathbf{h} + 7b\mathbf{e} + 3\mathbf{f}$. By a direct computation, $H_a^2 = 588a^2 - 294b^2 - 270 = 24 > 0$ and $7a > 0$. So the divisor H_a is ample. By a direct computation,

$$H_a D_a = 42 - 90 = -48\,, \quad H_a E_a = -42\,, \quad \text{and} \quad \mu_{H_a}(V_a) = -45\,.$$

Suppose that V_a is not μ_{H_a}-stable; then it must be destabilized by a line bundle. So there exists a sub-line bundle $\mathcal{O}_S(C)$ with $H_a C = \mu_{H_a}(C) \geqslant \mu_{H_a}(V_a) = -45$.

As $\mathcal{O}_S(C) \subset V_a$, we have $\mathrm{Hom}(\mathcal{O}_S(C), V_a) \neq 0$. If $\mathrm{Hom}(\mathcal{O}_S(C), \mathcal{O}_S(D_a)) \neq 0$, then $D_a - C$ is effective. It follows that $H_a C \leqslant H_a D_a = -48$, which contradicts the conclusion that $H_a C \geqslant -45$. Hence, $\mathrm{Hom}(\mathcal{O}_S(C), \mathcal{O}_S(E_a)) \neq 0$. As the extension is non-trivial, $C \neq E_a$. So $E_a - C$ is non-zero effective. It follows that $H_a C = H_a E_a - H_a(E_a - C) \leqslant -42 - 6 < -45$, which again gives a contradiction. Therefore, V_a cannot be destabilized by any line bundle, and it is μ_{H_a}-stable. $\square$

4. Walls and chambers

4.1 A finite theory for walls and chambers

Let S be a smooth projective surface of Picard number at least 2. Let $v = (\mathrm{rk}, \ell, s) \in \widetilde{\mathrm{H}}(S, \mathbb{Z})$ with $\mathrm{rk} > 0$ and (rk, ℓ) primitive; in other words, $(\mathrm{rk}, \ell) \neq m(\mathrm{rk}', \ell')$ for any $m \in \mathbb{Z}_{\geqslant 2}$. Rescaling the polarization does not affect the stability of any sheaves, so for any ample divisor H and element $H' \in \mathbb{R}_{>0} \cdot H \subset \mathrm{Amp}(S)$, we may define the (semi)stability with respect to H' and $\mu_{H'}$-slope (semi)stability the same way as that with respect to H. Denote by $\mathrm{Amp}_{\mathbb{Q}}(S)$ the subset of all rational points in $\mathrm{Amp}(S)$. For every $H \in \mathrm{Amp}_{\mathbb{Q}}(S)$, denote by $M^s_H(v)$ (respectively, $M^{\mathrm{ss}}_H(v)$) the moduli space of μ_H-slope stable (respectively, semistable) torsion-free sheaves with class v.

In this section, we review the wall and chamber structure for $M^s_H(v)$ while the polarization H is changing in $\mathrm{Amp}(S)$. The upshot is a finite theory on the numerical walls when the nef cone is rational polyhedral. For simplicity, we focus on slope stability and give a remark on the theory of Gieseker stability at the end of this section.

For a coherent sheaf F on S, we set $\mathrm{ch}(F) := (\mathrm{rk}(F), \mathrm{ch}_1(F), \mathrm{ch}_2(F))$. Note that by a little abuse of notation, in this section, the vector $v \in \widetilde{\mathrm{H}}(S, \mathbb{Z})$ is assumed to be $\mathrm{ch}(F)$, which is different from the Mukai vector $v(F) := (\mathrm{rk}(F), \mathrm{ch}_1(F), \mathrm{ch}_2(F) + \mathrm{rk}(F))$ in the K3 surface case. The discriminant is denoted as $\Delta(v) := (\mathrm{ch}_1)^2 - 2\,\mathrm{rk}\,\mathrm{ch}_2$.

LEMMA 4.1 ([HL10, Lemma 4.C.5]). *Let F be a torsion-free sheaf that is μ_H-slope stable with respect to some $H \in \mathrm{Amp}_{\mathbb{Q}}(S)$. Let $H' \in \mathrm{Amp}_{\mathbb{Q}}(S)$.*

(1) *If F is $\mu_{H'}$-slope stable, then F is μ_L-slope stable for every $L = (1-t)H + tH'$, where $t \in [0, 1] \cap \mathbb{Q}$.*
(2) *If F is not $\mu_{H'}$-slope stable, then there exists an $L = (1-t)H + tH'$ for some $t \in (0, 1] \cap \mathbb{Q}$ such that F is strictly μ_L-slope semistable.*

For a pair of classes $v = (\mathrm{rk}, \ell, s)$ and $v' = (\mathrm{rk}', \ell', s') \in \tilde{\mathrm{H}}(S, \mathbb{Z})$, we set $\mathcal{W}(v, v') := (\mathrm{rk} \cdot \ell' - \mathrm{rk}' \cdot \ell)^{\perp}$ whenever this is a codimension 1 $\mathbb{R}$-linear subspace in $\mathrm{NS}(S)_{\mathbb{R}}$.

DEFINITION 4.2. Let $v = (\mathrm{rk}, \ell, s) \in \tilde{\mathrm{H}}(S, \mathbb{Z})$ with $\mathrm{rk} > 0$ and (rk, ℓ) primitive. We call a codimension 1 $\mathbb{R}$-linear subspace $\mathcal{W}$ in $\mathrm{NS}(S)_{\mathbb{R}}$ a *numerical wall* of v if $\mathcal{W} = \mathcal{W}(v, v_i)$ for some $v_i = (\mathrm{rk}_i, \ell_i, s_i) \in \widetilde{\mathrm{H}}(S, \mathbb{Z})$, $i = 1, 2$, such that

(a) $v = v_1 + v_2$, $\mathrm{rk}_i > 0$, and $\Delta(v_i) \geqslant 0$;
(b) $\mathcal{W} \cap \mathrm{Amp}(S) \neq \emptyset$.

LEMMA 4.3. *Let $v = (\mathrm{rk}, \ell, s) \in \tilde{\mathrm{H}}(S, \mathbb{Z})$ with $\mathrm{rk} > 0$ and (rk, ℓ) primitive. Let F be a strictly μ_H-slope semistable torsion-free sheave with class v. Then H is on a numerical wall of v.*

Proof. Let $F_1 \subset F$ be a proper saturated subsheaf of F with $\mu_H(F_1) = \mu_H(F)$. Let $v_1 = \mathrm{ch}(F_1)$ and $v_2 = \mathrm{ch}(F_2)$. Then the only non-trivial part $\Delta(v_i) \geqslant 0$ follows from the Bogomolov inequality; see [HL10, Theorem 3.4.1]. □

PROPOSITION 4.4. *Let S be a smooth projective surface over $\mathbb{C}$ with $\mathrm{Nef}(S)$ rational polyhedral. Then for every class $v = (\mathrm{rk}, \ell, s) \in \tilde{\mathrm{H}}(S, \mathbb{Z})$ with $\mathrm{rk} > 0$ and (rk, ℓ) primitive, there are only finitely many numerical walls of v. In particular, there are finitely many walls and chambers for the moduli space $M^s_H(v)$ in $\mathrm{Amp}(S)$.*

We will reduce the argument for the proposition to some elementary lattice theory.

Let $N = \mathbb{R}^n$ be an n-dimensional real vector space and $\Lambda = \mathbb{Z}^n$ the integer points in it. Let $|\bullet|$ be a standard norm on N. For a vector $\mathbf{d} \in N$ and $r \in \mathbb{R}_{>0}$, denote by $B(\mathbf{d}, r) := \{\mathbf{c} \in N \mid |\mathbf{d} - \mathbf{c}| < r\}$ the neighbourhood at $\mathbf{d}$. Denote by $\tilde{B}(\mathbf{d}, r) := \{\mathbb{R}_{>0} \cdot \mathbf{c} \mid \mathbf{c} \in B(\mathbf{d}, r)\}$ the cone neighbourhood.

Let $Q \in \mathrm{Mat}(n \times n, \mathbb{Z})$ be a non-degenerate symmetric bilinear form on N with $\mathbb{Z}$-coefficients and signature $(1, n-1)$. Let $C_0 = \{x \in N \mid Q(x) > 0\} = C_0^+ \coprod C_0^-$, where $C_0^\pm$ are the two connected components of C. It follows by Hölder's inequality that $Q(\mathbf{d}, \mathbf{d}') > 0$ for all $\mathbf{d}, \mathbf{d}' \in C_0^+$. For a vector $\mathbf{d} \in N$, we set $\mathbf{d}^\perp := \{\mathbf{c} \in N \mid Q(\mathbf{d}, \mathbf{c}) = 0\}$.

Lemma 4.5. *Let A be a non-empty open convex cone in C_0^+ such that*

$$\partial A \cap \partial C_0^+ = \cup_{i=1}^k \mathbb{R}_{>0} \cdot \mathbf{d}_i \tag{4.1}$$

for finitely many $\mathbf{d}_i$ in Λ such that $\overline{A}$ is locally polyhedral at every $\mathbf{d}_i$. Then for any real value $t < 0$,

$$\#\{\mathbf{d} \in \Lambda \mid Q(\mathbf{d}, \mathbf{d}) \geqslant t,\, \exists \mathbf{h} \in A,\, Q(\mathbf{d}, \mathbf{h}) = 0\} < +\infty\,.$$

Proof. Denote by $A^* := \{\mathbf{d} \in N \mid Q(\mathbf{d}, \mathbf{h}) \geqslant 0 \text{ for all } \mathbf{h} \in A\}$ the closed dual cone of A. It is clear that $A^* \supset C_0^+$ and

$$\partial A^* \cap \partial C_0^+ = \partial A \cap \partial C_0^+ = \bigcup_{i=1}^k \mathbb{R}_{>0} \cdot \mathbf{d}_i\,, \tag{4.2}$$

$$\{\mathbf{d} \in \Lambda \mid \exists \mathbf{h} \in A,\, Q(\mathbf{d}, \mathbf{h}) = 0\} = \{0\} \cup (\Lambda \setminus \mathrm{Int}(A^* \cup (-A^*))\,. \tag{4.3}$$

For each $\mathbf{d}_i$ spanning a ray of $\partial A \cap \partial C_0^+$, as A is locally polyhedral and C_0^+ is circular, the boundary of the dual cone A^* is given by $\mathbf{d}_i^\perp$ locally at $\mathbf{d}_i$. In other words, there exists an $r_i > 0$ such that

$$\partial A^* \cap B(\mathbf{d}_i, r_i) = \mathbf{d}_i^\perp \cap B(\mathbf{d}_i, r_i)\,, \tag{4.4}$$

$$Q(\mathbf{d}, \mathbf{d}_i) < 0 \quad \text{and} \quad \mathbf{d} - \tfrac{1}{2}\mathbf{d}_i \notin A^* \cup (-A^*) \quad \text{for all } \mathbf{d} \in B(\mathbf{d}_i, r_i) \setminus A^*\,. \tag{4.5}$$

Since $\mathbf{d}_i \in \Lambda$ and $Q(-,-)$ is with coefficients in $\mathbb{Z}$, the sublattice $\mathbf{d}_i^\perp \cap \Lambda$ is of rank $n-1$, and there exists a $c_i \in \mathbb{Q}_{>0}$ such that for every $\mathbf{d} \in \Lambda$, the distance from $\mathbf{d}$ to $\mathbf{d}_i^\perp$ is nc_i for some $n \in \mathbb{Z}$.

Let $\epsilon_i := \min\{r_i, -c_i/t\}$; then for every $\mathbf{c} \in \big(\tilde{B}(\mathbf{d}_i, \epsilon_i) \cap \Lambda\big) \setminus A^*$, by (4.4), the distance from $\mathbf{c}$ to $\mathbb{R} \cdot \mathbf{d}_i$ is at least c_i. It follows that $\mathbf{c} \in B(m\mathbf{d}_i, m\epsilon_i)$ for some $m\epsilon_i \geqslant c_i$.

By (4.5), the vector $\mathbf{c} - \frac{1}{2}m\mathbf{d}_i$ is not in $A^* \cup (-A^*)$, so $Q\big(\mathbf{c} - \frac{1}{2}m\mathbf{d}_i, \mathbf{c} - \frac{1}{2}m\mathbf{d}_i\big) < 0$. As $Q(-,-)$ is with $\mathbb{Z}$-coefficients and both $\mathbf{d}_i$ and $\mathbf{c}$ are elements of Λ, by (4.5), we have $Q(\mathbf{c}, \mathbf{d}_i) \leqslant -1$. It follows that

$$Q(\mathbf{c}, \mathbf{c}) = Q\left(\mathbf{c} - \frac{m}{2}\mathbf{d}_i, \mathbf{c} - \frac{m}{2}\mathbf{d}_i\right) + mQ(\mathbf{c}, \mathbf{d}_i) - \frac{m^2}{4}Q(\mathbf{d}_i, \mathbf{d}_i) < -m \leqslant -\frac{c_i}{\epsilon_i} \leqslant t\,. \tag{4.6}$$

So

$$Q(\mathbf{c}, \mathbf{c}) < t \quad \text{for all } \mathbf{c} \in \big(\pm\tilde{B}(\mathbf{d}_i, \epsilon_i) \cap \Lambda\big) \setminus (\pm A^*) = \pm\big(\big(\tilde{B}(\mathbf{d}_i, \epsilon_i) \cap \Lambda\big) \setminus A^*\big)\,. \tag{4.7}$$

Let $M := \partial B(\mathbf{0}, 1) \setminus \big(\pm\big(\mathrm{Int}A^* \cup \big(\bigcup_{i=1}^k \tilde{B}_i(\mathbf{d}_i, \epsilon_i)\big)\big)\big)$. By (4.2), we have

$$\mathrm{Int}A^* \cup \left(\bigcup_{i=1}^k \tilde{B}_i(\mathbf{d}_i, \epsilon_i)\right) \cup \{\mathbf{0}\} \supset \overline{C_0^+}\,,$$

so $M \cap \overline{C}_0 = \emptyset$. Since M is compact, the function $Q(\mathbf{d}) := Q(\mathbf{d}, \mathbf{d})$ has minimum value on M. As $\overline{C}_0 = \{\mathbf{d} \in N \mid Q(\mathbf{d}, \mathbf{d}) \geqslant 0\}$, we have $\min_{\mathbf{d} \in M}\{Q(\mathbf{d}, \mathbf{d})\} = -\delta < 0$. It follows that

$$Q(\mathbf{d}, \mathbf{d}) < t \quad \text{for all } \mathbf{d} \notin \mathrm{Int} A^* \cup \left(\bigcup_{i=1}^{k} \tilde{B}_i(\mathbf{d}_i, \epsilon_i) \right) \text{ with } |\mathbf{d}| > \sqrt{-t/\delta}\,. \tag{4.8}$$

In summary, by (4.3), (4.7), (4.8),

$$\{\mathbf{d} \in \Lambda \mid Q(\mathbf{d}, \mathbf{d}) \geqslant t,\, \exists \mathbf{h} \in A,\, Q(\mathbf{d}, \mathbf{h}) = 0\} \subset \Lambda \cap B\left(\mathbf{0}, \sqrt{-t/\delta}\right),$$

which is finite. □

Now we can prove the main result on finite walls.

Proof of Proposition 4.4. Assume $v = v_1 + v_2$ for some $v_i = (r_i, \ell_i, s_i)$ satisfying the conditions in Definition 4.2. Set $\delta_i = r \cdot \ell_i - r_i \cdot \ell$; then by Definition 4.2(b), there exist an $H \in \mathrm{Amp}(S)$ such that $\delta_i \cdot H = 0$. By the Hodge index theorem, $\delta_i^2 \leqslant 0$.

It follows by Definition 4.2(a) that

$$\begin{aligned}
\Delta(v_i) &= \ell_i^2 - 2r_i s_i \geqslant 0 \\
&\Longrightarrow \delta_i^2 + 2rr_i\ell \cdot \ell_i - r_i^2\ell^2 - 2r_i r^2 s_i \geqslant 0 \iff \frac{r}{r_i}\delta_i^2 + 2r^2\ell \cdot \ell_i - r_i r\ell^2 - 2r^3 s_i \geqslant 0 \\
&\Longrightarrow \delta_i^2 + 2r^2\ell \cdot \ell_i - r_i r\ell^2 - 2r^3 s_i \geqslant \frac{r_i - r}{r_i}\delta_i^2 \geqslant 0 \\
&\Longrightarrow \delta_1^2 + \delta_2^2 + r^2\ell^2 - 2r^3 s = \sum_{i=1,2} \left(\delta_i^2 + 2r^2\ell \cdot \ell_i - r_i r\ell^2 - 2r^3 s_i\right) \geqslant 0 \\
&\Longrightarrow \delta_1^2 \geqslant -r^2\ell^2 + 2r^3 s - \delta_2^2 \geqslant -r^2\ell^2 + 2r^3 s\,.
\end{aligned}$$

Let $NS(S)_{\mathbb{R}}$ be N and $NS(S)$ be Λ equipped with the intersection as the bilinear form. Let $\mathrm{Nef}(S)$ be A; as it is rational polyhedral, assumption (4.1) is satisfied. If $-r^2\ell^2 + 2r^3 s \geqslant 0$, then it can be replaced by -1. So we may assume $-r^2\ell^2 + 2r^3 s < 0$. By Lemma 4.5,

$$\#\left\{\delta_1 \in \mathrm{NS}(S) \mid \delta_1^2 \geqslant -r^2\ell^2 + r^3 s,\, \exists H \in \mathrm{Amp}(S),\, \delta_1 \cdot H = 0\right\} < +\infty\,.$$

As a numerical wall of v is given by $\delta_1^\perp$, there are only finitely many of them. □

4.2 Walls and chambers of a spherical Mukai vector

From now on we will focus on the case of K3 surfaces and spherical bundles.

The existence of semistable spherical bundles is due to Kuleshov [Kul90].

THEOREM 4.6 ([Huy16, Theorem 10.2.7]). *Let S be a smooth projective K3 surface over $\mathbb{C}$ with an ample line bundle H. For any spherical Mukai vector v, there exists a μ_H-semistable vector bundle E with $v(E) = v$. If E is μ_H-stable, then it is the unique μ_H-semistable bundle with Mukai vector v.*

PROPOSITION AND DEFINITION 4.7. *Let S be a smooth projective K3 surface with rational polyhedral nef cone and v be a spherical Mukai vector. Then*

$$\mathsf{H}_S(v) := \#\{E \in \mathrm{Coh}(S) \mid v(E) = v,\, E \text{ is } \mu_H\text{-stable for some ample divisor } H\}$$

is finite.

Proof. As v is spherical, it is clear that (r, ℓ) is primitive. By Proposition 4.4, there are finitely many numerical walls of v. As $\mathrm{Amp}(S)$ is convex, it is divided by the numerical walls into finitely many numerical chambers.

For all H and H' in the same numerical chamber, by Lemmas 4.1 and 4.3, we have $M_H^{\mathrm{ss}}(v) = M_H^s(v) = M_{H'}^s(v)$. By Theorem 4.6, in each numerical chamber there is a unique slope stable bundle with Mukai vector v.

For each polarization H on the numerical wall of v, if $M_H^s(v) = \{E\}$ is non-empty, then by Lemmas 4.1 and 4.3, the bundle E is slope stable in all neighbour numerical chambers. So slope stable bundles on the numerical wall never contribute to the count. Therefore, $\mathsf{H}(v)$ is bounded by the number of numerical chambers, which is finite. □

Proposition 4.8. *Let S be a smooth projective K3 surface and v be a spherical Mukai vector. Let $\mathcal{W}$ be a numerical wall of v. Then the following are equivalent:*

(1) *For all $H \in \mathcal{W}$, we have $M_H^s(v) = \emptyset$.*
(2) *We have $v = v_1 + \cdots + v_s$ with $s \geqslant 2$ for some spherical Mukai vectors v_i such that $\mathcal{W}(v, v_i) = \mathcal{W}$ and $M_H^s(v_i) \neq \emptyset$ for some $H \in \mathcal{W}$ and all i.*
(3) *We have $\mathcal{W} = \mathcal{W}(v, w)$ for some spherical Mukai vector w satisfying $\mathrm{rk}(w) < \mathrm{rk}(v)$ and $\langle w, v\rangle < 0$.*
(3′) *We have $\mathcal{W} = \mathcal{W}(v, w)$ for some spherical Mukai vector w satisfying $\mathrm{rk}(w) \leqslant \mathrm{rk}(v)/2$, $\langle w, v\rangle < 0$, and $M_H^s(w) \neq \emptyset$ for some $H \in \mathcal{W}$.*
(3″) *We have $\mathcal{W} = \mathcal{W}(v, w)$ for some spherical Mukai vector w satisfying $\mathrm{rk}(w) \leqslant \frac{1}{2}\mathrm{rk}(v)$ and $\langle w, v\rangle < 0$.*
(4) *For all $H_\pm \in \mathrm{Amp}(S)$ in two different components of $\mathrm{Amp}(S) \setminus \mathcal{W}$, there is no spherical bundle E with $v(E) = v$ both μ_{H^+}- and μ_{H^-}-slope stable.*

Definition 4.9. We call a $\mathcal{W}$ with the equivalent properties of Proposition 4.8 an *actual wall* of v and each connected component of $\mathrm{Amp}(S) \setminus (\bigcup_{\text{actual walls}} \mathcal{W})$ a *chamber* of v. We call a spherical Mukai vector w such as that in item (3) a *destabilizing factor* of v.

Proof. (1)⇒(2) Let H be an ample divisor on $\mathcal{W}$ but not on any other numerical walls of v. Let H_+ be in the neighbour numerical chamber of H. By Theorem 4.6, there exists a μ_{H_+}-stable and μ_H-slope semistable spherical bundle E with Mukai vector v. By item (1), the bundle E is strictly μ_H-semistable. By the same argument as that in [BM14, Lemma 6.2], the Jordan–Hölder filtration factors E_i of E are all spherical. Their Mukai vectors give the v_i satisfying the condition in item (2).

We may write $v = a_1 w_1 + \cdots + a_t w_t$ by combining similar terms of Mukai vectors of Jordan–Hölder factors of E. The following lemma will be useful for the rest of the argument.

Lemma 4.10. *For every $1 \leqslant i \leqslant t$, we have*

$$a_i + \langle v, w_i\rangle \geqslant 0\,. \tag{4.9}$$

Proof of Lemma 4.10. Denote by E_i the unique object in $M_{H_0}^s(w_i)$. As both E_i and E are μ_{H_+}-slope stable, the spaces $\mathrm{Hom}(E, E_i)$ and $\mathrm{Hom}(E_i, E)$ cannot both be non-zero. If both of them are zero, then $\langle v, w_i\rangle \geqslant 0$ and (4.9) holds automatically.

Without loss of generality, we may assume that we have $\mathrm{Hom}(E_i, E) \neq 0$ and $\mathrm{Hom}(E, E_i) = \mathrm{Hom}(E_i, E[2]) = 0$. It follows that

$$\hom(E_i, E) + \langle v_i, v\rangle = \hom(E_i, E) - \chi(E_i, E) = \hom(E_i, E[1]) \geqslant 0\,. \tag{4.10}$$

We may exhaust the E_i as subfactors in the Jordan–Hölder filtration of E. As $\operatorname{Hom}(E_i, E_i[1]) = 0$, this subfactor is $E_i^{\oplus a}$ for some $a \in \mathbb{Z}_{\geqslant 0}$. It follows that E fits into the short exact sequence $0 \to E_i^{\oplus a} \to E \to E' \to 0$ of bundles for some $\operatorname{Hom}(E_i, E') = 0$. In particular, $a_i \geqslant a = \hom(E_i, E)$. Together with (4.10), this implies that inequality (4.9) holds. □

(1)⇒(3′) Note that $\sum_{i=1}^{s} \langle v_i, v\rangle = \langle v, v\rangle = -2$, so there exists a v_i such that $\langle v_i, v\rangle < 0$. Without loss of generality, we may assume $i = 1$. If $\operatorname{rk}(v_1) \leqslant \frac{1}{2}\operatorname{rk}(v)$, then v_1 satisfies all conditions on w in item (3′).

Assume $\operatorname{rk}(v_1) > \frac{1}{2}\operatorname{rk}(v)$; then $v_1 \neq v_i$ for any $i \geqslant 2$.

By Lemma 4.10, we have $\langle v_1, v\rangle = -1$. It then follows that $\sum_{i=2}^{s} \langle v_i, v\rangle = -1$. There exists an $i \geqslant 2$ such that $\langle v_i, v\rangle < 0$. Note that $\operatorname{rk}(v_1) > \frac{1}{2}\operatorname{rk}(v)$, so $\operatorname{rk}(v_i) < \frac{1}{2}\operatorname{rk}(v)$ and v_i satisfies all conditions on w in item (3′).

(3′)⇒(3″)⇒(3) This is obvious.

(2)⇒(3) Note that $\sum_{i=1}^{s} \langle v_i, v\rangle = \langle v, v\rangle = -2$, so there exists a v_i such that $\langle v_i, v\rangle < 0$.

(3)⇒(4) Suppose that E with $v(E) = v$ is both μ_{H_+}- and μ_{H_-}-stable. Then by Lemma 4.1, the bundle E is μ_H-stable for some H on $\mathcal{W}$. Let w be a spherical Mukai vector as that in item (3). By Theorem 4.6, there exists a μ_H-semistable spherical bundle F with $v(F) = w$.

Since $H \in \mathcal{W} = \mathcal{W}(v, w)$, the slope satisfies $\mu_H(E) = \mu_H(F)$. Since $\chi(F, E) = -\langle w, v\rangle > 0$, we have $\operatorname{Hom}(E, F) \neq 0$ or $\operatorname{Hom}(F, E) \neq 0$. As E is a μ_H-stable vector bundle, this can only happen if E is a subbundle or quotient bundle of F. However, $\operatorname{rk}(E) = \operatorname{rk}(v) > \operatorname{rk}(w) = \operatorname{rk}(F)$ by assumption, so we get a contradiction. So assertion (4) holds.

(4)⇒(1) Suppose that E is μ_H-slope stable for some $H \in \mathcal{W}$. Then by Lemmas 4.1 and 4.3, the bundle E is slope stable in all neighbour numerical chambers of H. This contradicts the assumption of item (4). So assertion (1) holds. □

Remark 4.11. Note that v might have different decompositions to that in Proposition 4.8(2) even with respect to a very general $H \in \mathcal{W}$. Lemma 4.10 only holds for the sum of Mukai vectors of Jordan–Hölder factors of a μ_{H_+}-slope stable (or more generally, a μ_H-semistable) E. In general, Lemma 4.10 does not hold for an arbitrary decomposition of v as that in Proposition 4.8(2).

Corollary 4.12. *Let S be a smooth projective K3 surface with a rational polyhedral nef cone and $v = (\operatorname{rk}, \ell, s)$ be a spherical Mukai vector. Then $\mathsf{H}(v)$ is the number of actual chambers.*

Let S' be another K3 surface. Assume that there exists an injective group homomorphism $f\colon \operatorname{NS}(S') \to \operatorname{NS}(S)$ preserving the intersection numbers and satisfying $f(\operatorname{Nef}(S')) \subseteq \operatorname{Nef}(S)$. Then we have $\mathsf{H}_{S'}((\operatorname{rk}, \ell, s)) \leqslant \mathsf{H}_S((\operatorname{rk}, f(\ell), s))$. In particular, the number $\mathsf{H}_S(v)$ only depends on $(\operatorname{Nef}(S), (-,-)_{\text{intersection}})$.

Proof. The first statement follows by Proposition 4.8(4). For the second statement, by Proposition 4.8(3), all actual walls of v in $\operatorname{NS}(S)$ are mapped into actual walls of $f(v) := (\operatorname{rk}, f(\ell), s)$ in $\operatorname{NS}(S')$. By Proposition 4.8(4), two different actual chambers of v in $\operatorname{NS}(S)$ are still separated by some actual wall in $\operatorname{NS}(S')$. The conclusion follows. □

We finish the section with a remark on the difference between Gieseker and slope stabilities.

Remark 4.13. Let S be a smooth projective surface and $v = (\operatorname{rk}, \ell, s) \in \tilde{\mathrm{H}}(S, \mathbb{Z})$ with $\operatorname{rk} > 0$ and $(\operatorname{rk}, \ell)$ primitive. For $H \in \operatorname{Amp}_{\mathbb{Q}}(S)$, we denote by $M^s_{H,\text{slope}}(v)$ and $M^s_{H,\text{Gieseker}}(v)$ the spaces of all μ_H-slope stable and p_H-stable coherent sheaves, respectively.

(a) If H is in an open numerical chamber, then we always have $M^s_{H,\mathrm{slope}}(v) = M^s_{H,\mathrm{Gieseker}}(v)$. If F is μ_H-slope stable, then F is $\mu_{H'}$-slope stable for H' in an open neighbourhood of H. This is not the case for p_H-stability. See Example 4.14 below for a spherical bundle that is p_H-stable only for mH but not with respect to any other polarization.

(b) If H and H' are both in $\cap_{i=1}^s \mathcal{W}_i$ for some numerical chambers $\mathcal{W}_i$ of v and the line segment $\{(1-t)H + tH' \mid t \in [0,1]\}$ does not intersect any other numerical chamber, then $M^s_{H,\mathrm{Gieseker}}(v) = M^s_{H',\mathrm{Gieseker}}(v)$. In particular, if S is a K3 surface with rational polyhedral nef cone and v is a spherical Mukai vector, then $\mathsf{H}_{\mathrm{Gieseker}}(v)$ is also finite. It is clear that $\mathsf{H}_{\mathrm{Gieseker}}(v) \geqslant \mathsf{H}(v)$. The value $\mathsf{H}_{\mathrm{Gieseker}}(v)$ will be strictly greater than $\mathsf{H}(v)$ for some v if isolated p_H-stable sheaves as that in Example 4.14 appear on the numerical wall.

Example 4.14. Let S be a generic elliptic K3 surface with a section and adopt the notation of Section 5. We may consider the spherical bundle $B := \mathsf{T}_{\mathcal{O}_S(E-F)}(\mathsf{T}_{\mathcal{O}_S(F-E)}(\mathcal{O}_S(2E-2F)))$, which is with Mukai vector $v(B) = (113, 82e - 82\sigma, -119)$. Then B is $p_{3e+\sigma}$-stable, but B is not slope stable with respect to any polarization. The spherical bundle B is not Gieseker stable with respect to any polarization other than $3e+\sigma$ up to a scalar.

5. Case study: Counting stable spherical bundles on an elliptic K3 surface

Let S be a generic elliptic K3 surface $\pi\colon S \to \mathbb{P}^1$ with a section $C \subset S$. Denote by $E = \pi^{-1}(p)$ a generic fibre, and set $\sigma := [C]$ and $e := [E]$. Assume $\mathrm{NS}(S) = \mathbb{Z}e \oplus \mathbb{Z}\sigma$. The intersection numbers are given as

$$e^2 = 0\,, \quad e\cdot\sigma = 1\,, \quad \sigma^2 = -2\,. \tag{5.1}$$

The cones are $\mathrm{Nef}(S) = \mathbb{R}_{\geqslant 0}\cdot e + \mathbb{R}_{\geqslant 0}\cdot(2e+\sigma)$ and $\overline{\mathrm{NE}}(S) = \mathbb{R}_{\geqslant 0}\cdot e + \mathbb{R}_{\geqslant 0}\cdot\sigma$.

5.1 Accurate counting

In this section, we compute the values of $\mathsf{H}_S(v)$ for such an elliptic K3 surface S. We first give a simple parametrization for all spherical Mukai vectors of S.

Lemma 5.1. *Let S be an elliptic K3 surface as above; then there is a one-to-one correspondence*

$$v\colon \mathbb{Q}\times\mathbb{Z} \longrightarrow \{\textit{spherical Mukai vectors in } \mathbb{Z}\oplus\mathrm{NS}(S)\oplus\mathbb{Z}\}\,,$$
$$\left(\frac{n}{m}, k\right) \longmapsto \left(m, n\sigma + \left(km + n - n^{\varphi(m)-1}\right)e, \frac{-n^2 + n\left(km + n - n^{\varphi(m)-1}\right) + 1}{m}\right).$$

Here n/m is in the unique irreducible form in the sense that $m \geqslant 1$ and $\gcd(n,m) = 1$. We write $\varphi(-)$ for Euler's totient function. In particular, $n^{\varphi(m)} \equiv 1 \pmod{m}$. When $m = 1$ and $n = 0$, we set the term $n^{\varphi(m)-1}$ to be 0.

Proof. By definition, the rank satisfies $m > 0$. By a direct computation, $\langle v(n/m,k), v(n/m,k)\rangle = -2$. So v is well defined. As every rational number has a unique irreducible form, the map v is clearly injective.

For every spherical Mukai vector $v = (m, n\sigma + be, s)$, by definition, the rank satisfies $m > 0$. The self-intersection is

$$-2 = \langle v, v\rangle = -2n^2 + 2bn - 2ms\,. \tag{5.2}$$

It follows that $m|-n^2+bn+1$. So $\gcd(m,n)=1$ and

$$bn \equiv n^2-1 (\operatorname{mod} m) \implies b \equiv n - n^{\varphi(m)-1} (\operatorname{mod} m)\,.$$

So there exists a $k \in \mathbb{Z}$ such that $b = km + n - n^{\varphi(m)-1}$. By (5.2), the last factor s equals $(1/m)\bigl(-n^2+bn+1\bigr)$. Therefore, the map v is surjective. ☐

Note that if a spherical bundle E is μ_H-stable, then the dual bundle E^* and $E\otimes\mathcal{L}$ are μ_H-stable for every line bundle $\mathcal{L}$. It follows that $\mathsf{H}_S(v(a+n,m)) = \mathsf{H}_S(v(a,0)) = \mathsf{H}_S(v(-a,0))$ for all $a\in\mathbb{Q}$, and $n,m\in\mathbb{Z}$.

NOTATION 5.2. By abuse of notation, we write the counting function as $\mathsf{H}(a) := \mathsf{H}_S(v(a,0))$ for every $a\in\mathbb{Q}$. It is clear that $\mathsf{H}(a+n) = \mathsf{H}(a) = \mathsf{H}(-a)$ for every $n\in\mathbb{Z}$.

We can describe the destabilizing factor as follows.

LEMMA 5.3. *A spherical Mukai vector $v(s/r,k)$ is a destabilizing factor of $v(n/m,0)$ if and only if $r<m$ and*

$$\left(\frac{n}{m}-\frac{s}{r}\right)^2 - \frac{1}{r^2} < \left(\frac{n}{m}-\frac{s}{r}\right)\left(\frac{n-n^{\varphi(m)-1}}{m} - \frac{s-s^{\varphi(r)-1}}{r} - k\right) < 0\,. \tag{5.3}$$

Proof. By Proposition 4.8(3), the spherical Mukai vector $v(s/r,k)$ is a destabilizing factor of $v(n/m,0)$ if and only if $r<m$ and

(a) $\mathcal{W}(v(s/r,k), v(n/m,0)) \cap \operatorname{Amp}(S) \neq \emptyset$,

(b) $\langle v(s/r,k), v(n/m,0)\rangle \leqslant -1$ (or equivalently <0).

Define the divisors $D_1 = n\sigma + \bigl(n-n^{\varphi(m)-1}\bigr)e$ and $D_2 = s\sigma + \bigl(kr+s-s^{\varphi(r)-1}\bigr)e$. Then condition (a) holds if and only if the divisor $rD_1 - mD_2$ is not in $\overline{\mathrm{NE}}(S)$ or $-\overline{\mathrm{NE}}(S)$, which holds if and only if

$$(rn-ms)\bigl(r\bigl(n-n^{\varphi(m)-1}\bigr) - m\bigl(kr+s-s^{\varphi(r)-1}\bigr)\bigr) < 0\,.$$

Condition (b) is equivalent to

$$\begin{aligned}
D_1\cdot D_2 - r\frac{D_1^2+2}{2m} - m\frac{D_2^2+2}{2r} &< -\frac{r}{m}\\
\iff 2rmD_1\cdot D_2 - r^2D_1^2 - 2r^2 - m^2D_2^2 - 2m^2 &< -2r^2\\
\iff -(rD_1-mD_2)^2 &< 2m^2.\\
\iff 2(rn-ms)^2 - 2(rn-ms)\bigl(r\bigl(n-n^{\varphi(m)-1}\bigr) - m\bigl(kr+s-s^{\varphi(r)-1}\bigr)\bigr) &< 2m^2\,.
\end{aligned}$$

Combining this with condition (a) and dividing all terms by r^2m^2, we see that this is equivalent to inequality (5.3). ☐

Example 5.4. Note that (5.3) in particular implies $|n/m - s/r| < 1/r$ and

$$\left|\frac{n-n^{\varphi(m)-1}}{m} - \frac{s-s^{\varphi(r)-1}}{r} - k\right| < \frac{m}{r}\,.$$

For each given n/m, these two conditions reduce the number of possible destabilizing factors to around $2m\ln m$. With the help of the computer, we may list all destabilizing factors $v(s/r,k)$ for $v(n/m,0)$ when m is small. Note that the wall $\mathcal{W}(v(s/r,k), v(n/m,0))$ is given by

$$\bigl((rn-ms)\cdot\sigma + \bigl(r\bigl(n-n^{\varphi(m)-1}\bigr) - m\bigl(kr+s-s^{\varphi(r)-1}\bigr)\bigr)\cdot e\bigr)^{\perp} \cap \operatorname{Amp}(S)\,, \tag{5.4}$$

which is determined by the ratio of two coefficients. We may then compute the number of all actual walls. We list a few explicit computations here:

$\mathsf{H}(1)=1,\quad \mathsf{H}(\tfrac{1}{2})=2,\quad \mathsf{H}(\tfrac{1}{3})=3,\quad \mathsf{H}(\tfrac{1}{4})=4,\quad \mathsf{H}(\tfrac{1}{5})=5,\quad \mathsf{H}(\tfrac{2}{5})=6,\quad \mathsf{H}(\tfrac{1}{6})=6,$
$\mathsf{H}(\tfrac{1}{7})=7,\quad \mathsf{H}(\tfrac{2}{7})=7,\quad \mathsf{H}(\tfrac{3}{7})=7,\quad \mathsf{H}(\tfrac{1}{8})=8,\quad \mathsf{H}(\tfrac{3}{8})=6,\quad \mathsf{H}(\tfrac{1}{9})=9,\quad \mathsf{H}(\tfrac{2}{9})=7,$
$\mathsf{H}(\tfrac{4}{9})=8,\quad \mathsf{H}(\tfrac{1}{10})=10,\quad \mathsf{H}(\tfrac{3}{10})=9,$

$\ldots\ldots$

$\mathsf{H}(\tfrac{1}{21})=21,\quad \mathsf{H}(\tfrac{2}{21})=13,\quad \mathsf{H}(\tfrac{4}{21})=12,\quad \mathsf{H}(\tfrac{5}{21})=13,\quad \mathsf{H}(\tfrac{8}{21})=11,\quad \mathsf{H}(\tfrac{10}{21})=14,$

$\ldots\ldots$

$\mathsf{H}(\tfrac{1}{32})=32,\quad \mathsf{H}(\tfrac{3}{32})=16,\quad \mathsf{H}(\tfrac{5}{32})=16,\quad \mathsf{H}(\tfrac{7}{32})=14,\quad \mathsf{H}(\tfrac{9}{32})=15,\quad \mathsf{H}(\tfrac{11}{32})=16,$
$\mathsf{H}(\tfrac{13}{32})=14,\quad \mathsf{H}(\tfrac{15}{32})=13,$

$\ldots\ldots$

$\mathsf{H}(\tfrac{1}{93})=93,\quad \mathsf{H}(\tfrac{2}{93})=49,\quad \mathsf{H}(\tfrac{4}{93})=30,\quad \mathsf{H}(\tfrac{5}{93})=28,\quad \mathsf{H}(\tfrac{7}{93})=25,\quad \mathsf{H}(\tfrac{8}{93})=25,$
$\mathsf{H}(\tfrac{10}{93})=26,\quad \mathsf{H}(\tfrac{11}{93})=24,\quad \mathsf{H}(\tfrac{13}{93})=23,\quad \mathsf{H}(\tfrac{14}{93})=24,\quad \mathsf{H}(\tfrac{16}{93})=21,\quad \mathsf{H}(\tfrac{17}{93})=23,$
$\mathsf{H}(\tfrac{19}{93})=22,\quad \mathsf{H}(\tfrac{20}{93})=22,\quad \mathsf{H}(\tfrac{22}{93})=22,\quad \mathsf{H}(\tfrac{23}{93})=31,\quad \mathsf{H}(\tfrac{25}{93})=23,\quad \mathsf{H}(\tfrac{26}{93})=24,$
$\mathsf{H}(\tfrac{28}{93})=20,\quad \mathsf{H}(\tfrac{29}{93})=25,\quad \mathsf{H}(\tfrac{32}{93})=20,\quad \mathsf{H}(\tfrac{34}{93})=24,\quad \mathsf{H}(\tfrac{35}{93})=21,\quad \mathsf{H}(\tfrac{37}{93})=26,$
$\mathsf{H}(\tfrac{38}{93})=22,\quad \mathsf{H}(\tfrac{40}{93})=23,\quad \mathsf{H}(\tfrac{41}{93})=23,\quad \mathsf{H}(\tfrac{43}{93})=26,\quad \mathsf{H}(\tfrac{44}{93})=25,\quad \mathsf{H}(\tfrac{46}{93})=50,$

$\ldots\ldots$

$\mathsf{H}(\tfrac{92}{811})=48,\quad \mathsf{H}(\tfrac{93}{811})=47,\quad \mathsf{H}(\tfrac{94}{811})=46,\quad \mathsf{H}(\tfrac{95}{811})=43,\quad \mathsf{H}(\tfrac{96}{811})=45,\quad \mathsf{H}(\tfrac{97}{811})=50,$
$\mathsf{H}(\tfrac{98}{811})=48,\quad \mathsf{H}(\tfrac{99}{811})=46,\quad \mathsf{H}(\tfrac{100}{811})=53,\quad \ldots,\quad \mathsf{H}(\tfrac{266}{811})=50,\quad \mathsf{H}(\tfrac{267}{811})=51,$
$\mathsf{H}(\tfrac{268}{811})=58,\quad \mathsf{H}(\tfrac{269}{811})=83,\quad \mathsf{H}(\tfrac{270}{811})=276,\quad \mathsf{H}(\tfrac{271}{811})=146,\quad \mathsf{H}(\tfrac{272}{811})=72,$
$\mathsf{H}(\tfrac{273}{811})=60,\quad \mathsf{H}(\tfrac{274}{811})=60,\quad \mathsf{H}(\tfrac{275}{811})=47,\quad \mathsf{H}(\tfrac{276}{811})=46,\quad \mathsf{H}(\tfrac{277}{811})=45.$

We give the computation of $\mathsf{H}(n/m)$ for some explicit value of n/m as follows.

Example 5.5. Let S be a generic elliptic K3 surface admitting a section and adopt Notation 5.2. Then (a) $\mathsf{H}(1/m)=m$, (b) $\mathsf{H}(n/(2n+1))=n+4$ when $n\geqslant 2$.

Proof. (a) By Proposition 4.8(3′′), we only need to consider all $v(s/r,k)$ with $r\leqslant\frac{1}{2}m$ satisfying (5.3). When $r\geqslant 2$, as $|1/m-1/r|<1/r^2$ and $\gcd(s,r)=1$, the destabilizing factor is of the form of $v(1/r,k)$.

Note that $n=s=1$ in (5.3), so by the second inequality, $k\leqslant -1$. By the first inequality, $(1/m-1/r)^2-1/r^2<(1/m-1/r)(-k)\leqslant -1/2r\leqslant -1/r^2$. So there is no solution to (5.3) when $r\geqslant 2$.

We only need to consider all destabilizing factors of the form $v(s,k)$. It is clear that $s=0$ or $s=1$. The inequalities in (5.3) become $(1/m-s)^2-1<(1/m-s)(-k)<0$. When $s=1$, there is no solution k to (5.3). When $s=0$, the solutions to (5.3) are $k=1,\ldots,m-1$. Note that the slopes of the lines through $(1/m,0)$ and each $(0,k)$ are all different. So the $m-1$ destabilizing factors of $v(1/m,0)$ correspond to $m-1$ different actual walls. So $\mathsf{H}(1/m)=m$.

(b) Note that $n^{\varphi(2n+1)-1}\equiv -2 \pmod{2n+1}$. Replacing $v(n/(2n+1),0)$ by $v(n/(2n+1),k_n')$, we may assume $v(n/(2n+1),k_n')=(2n+1,n\cdot\delta+(n+2)\cdot e,*)$. The term $(n-n^{\varphi(2n+1)-1})/(2n+1)$ in (5.3) is accordingly replaced by $(n+2)/(2n+1)$.

If $r=1$, then $s=0$ or $s=1$. The solutions to (5.3) are $(s,k)=(0,1)$, $(0,2)$, or $(1,0)$.

If $r=2$, then $s=1$. The solutions to (5.3) are $k=0,-1,\ldots,-n+1$.

If $r = 2d$ with $2 \leqslant d \leqslant n/2$, then as $|n/(2n+1) - s/2d| < 1/2d$, the numerator s can only be $d-1$. It follows that $2|d$, so $(d-1)^{\varphi(2d)-1} \equiv -d-1 \pmod{2d}$. It is clear that (5.3) has no solution for $k \in \mathbb{Z}$.

If $r = 2d+1$ with $1 \leqslant d \leqslant n/2$, then s can only be d. There is a unique $k_d \in \mathbb{Z}$ satisfying (5.3). More precisely, $v(d/(2d+1), k_d) = (2d+1, d\cdot\delta + (d+2)\cdot e, *)$. Note that the wall $\mathcal{W}\big((v(n/(2n+1), k_n'), v\big(d/(2d+1), k_d\big)\big)$ is always spanned by $\delta + 5e$, which is the same as that given by $\mathcal{W}\big(v\big(n/(2n+1), k_n'\big), v(0,2)\big)$.

To sum up, the actual walls correspond to the destabilizing factors $v(0,1)$, $v(0,2)$, $v(1,0)$, $v(1/2,k)$, where $k = 0, -1, \ldots, -n+1$. It is clear that these walls are all different from one another when $n \geqslant 2$. So there are $n+4$ chambers in total for $v\big(n/(2n+1), k_n'\big)$. □

Remark 5.6. There are some other patterns of $\mathsf{H}(n/m)$. We write down some typical ones in this remark. Here we adopt the notation of continuous fraction

$$[a_1, a_2, \ldots, a_s] := \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots + \frac{1}{a_s}}}}.$$

(a) $\mathsf{H}\big(\frac{n}{m}\big) \leqslant m$ when $m \geqslant 6$.

(b) $\mathsf{H}\big(\frac{2}{2n+1}\big) = \begin{cases} n+4 & \text{when } n \equiv 0, 1 \pmod 3, \\ n+3 & \text{when } n \equiv 2 \pmod 3. \end{cases}$

(c) $\mathsf{H}([a,b]) = \begin{cases} b+2a & \text{when } b \geqslant a \geqslant 2, \\ a+2b \text{ or } a+2b-1 & \text{when } a \geqslant b \geqslant 2. \end{cases}$

(d) $\mathsf{H}([a_1, \ldots, a_s, N]) = N + 2\sum a_i$ when $a_1 \geqslant 2$ and N is sufficiently large with respect to all a_i.

(e) Let $b_1 = b_2 = 1$ and $b_{n+1} := b_n + b_{n-1}$; then $\mathsf{H}(b_n/b_{n+1}) = \mathsf{H}([1, 1, \ldots, 1]) = \lfloor n/2 \rfloor^2 + 2$, where the middle term has n ones.

Note that all these patterns are observed via the help of a computer. We cannot give mathematical proofs for any of these conjectural statements.

5.2 Asymptotic behaviour

We also notice the following conjectural general behaviour of $\mathsf{H}(n/m)$ when m tends to infinity:

$$\begin{aligned} \mathsf{H}_{\min}(m) &:= \min_{1 \leqslant n \leqslant m,\, \gcd(n,m)=1} \left\{ \mathsf{H}\left(\frac{n}{m}\right) \right\} = \Theta\big((\ln m)^2\big), \\ \mathsf{H}_{\text{ave}}(m) &:= \text{Average}_{1 \leqslant n \leqslant m,\, \gcd(n,m)=1} \left\{ \mathsf{H}\left(\frac{n}{m}\right) \right\} = \Theta\big((\ln m)^2\big). \end{aligned} \tag{5.5}$$

In the rest of the paper, we prove the following weaker version of the average estimate (5.5) on the counting $\mathsf{H}(v)$.

Proposition 5.7. *Let S be a generic elliptic K3 surface admitting a section. Adopt Notation 5.2, then for any $\alpha > 0$, there exist constants C_1 and C_2 such that for $m \gg 0$,*

$$\frac{C_1}{m}(\varphi(m)\ln m)^2 \leqslant \sum_{1 \leqslant n < m,\, \gcd(n,m)=1} \mathsf{H}\left(\frac{n}{m}\right) \leqslant C_2 m^{1+\alpha}. \tag{5.6}$$

We give some notation and lemmas on elementary number theory before the proof. For a pair of rational numbers n/m and s/r with $1 \leqslant r < m$, we write

$$\mathsf{F}\left(\frac{n}{m}, \frac{s}{r}\right) := \#\{k \in \mathbb{Z} \mid k \text{ satisfies the inequalities in } (5.3)\}\,. \tag{5.7}$$

LEMMA 5.8. *The function* F *and* H *satisfy the following properties:*

(1) $\mathsf{F}(n/m, s/r) = \mathsf{F}(1 - n/m, 1 - s/r)$.

(2) *When* $|rn - ms| \geqslant m$, *we have* $\mathsf{F}(n/m, s/r) = 0$.
When $|rn - ms| < m$, *we have*

$$\mathsf{F}\left(\frac{n}{m}, \frac{s}{r}\right) \geqslant \left\lfloor \frac{m^2 - (rn - ms)^2}{mr|rn - ms|} \right\rfloor .$$

(3) *For every* $r < m$ *and* $s/r \in \mathbb{Q}$, *we have* $\mathsf{H}(n/m) \geqslant \mathsf{F}(n/m, s/r) + 1$.

(4) *Assume* $0 < n < m$; *then*

$$\mathsf{H}\left(\frac{n}{m}\right) \leqslant 1 + \sum_{1 \leqslant r \leqslant \lfloor \frac{m}{2} \rfloor} \sum_{\substack{0 \leqslant s \leqslant r \\ \gcd(s,r)=1}} \mathsf{F}\left(\frac{n}{m}, \frac{s}{r}\right) . \tag{5.8}$$

(5) *Let* $a = \gcd(m, r)$; *then* $\mathsf{F}(n/m, s/r) \leqslant \mathsf{F}(an/m, as/r)$.

(6) *When* $\gcd(m, r) = 1$,

$$\mathsf{F}\left(\frac{n}{m}, \frac{s}{r}\right) = \#\left\{t \in \mathbb{Z}_{\geqslant 1} \mid (rn - ms)|m^2 + r2 - tmr > (rn - ms)^2\right\} . \tag{5.9}$$

Proof. Statement (1) follows directly by (5.3).

(2) When $|rn - ms| \geqslant m$, we have $(n/m - s/r)^2 - 1/r^2 \geqslant 0$. So there is no k satisfying (5.3). When $|rn - ms| < m$, the inequality follows by dividing (5.3) by $|n/m - s/r|$.

(3) By Lemma 5.3, there are $\mathsf{F}(n/m, s/r)$ different k_i such that the spherical Mukai vector $v(s/r, k_i)$ destabilizes $v(n/m, 0)$. By (5.4), their walls are different from one another. By Proposition 4.8, the inequality holds.

(4) By Proposition 4.8(3″), each actual wall of $v(n/m, 0)$ corresponds to at least one destabilizing factor $v(s/r, k)$ with $r \leqslant m/2$. When $s/r \notin [0, 1]$, we have $|s/r - n/m| \geqslant 1/r$. By property (1), we have $\mathsf{F}(n/m, s/r) = 0$. Inequality (5.8) follows.

(5) Assume $m = am'$ and $r = ar'$; then $m' | n^{\varphi(m)-1} - n^{\varphi(m')-1}$ and $r' | s^{\varphi(r)-1} - s^{\varphi(r')-1}$. Set

$$c = \frac{n^{\varphi(m)-1} - n^{\varphi(m')-1}}{m'} - \frac{s^{\varphi(r)-1} - s^{\varphi(r')-1}}{r'} \in \mathbb{Z}\,.$$

For every $k \in \mathbb{Z}$ satisfying (5.3) with respect to $(n/m, s/r)$, the integer $c + ka$ satisfies (5.3) with respect to $(n/m', s/r')$.

(6) An integer k satisfies (5.3)

$$\iff (rn - ms)^2 - m^2 < (rn - ms)\big(r\big(n - n^{\varphi(m)-1}\big) - m\big(s - s^{\varphi(r)-1}\big) - kmr\big) < 0$$
$$\iff m^2 > (rn - ms)\big(rn^{\varphi(m)-1} - ms^{\varphi(r)-1} + kmr\big) > (rn - ms)^2.$$

When $\gcd(m, r) = 1 = \gcd(n, m) = \gcd(s, r)$, it is clear that $\gcd(rn - ms, mr) = 1$.

Note that

$$(rn - ms)\big(rn^{\varphi(m)-1} - ms^{\varphi(r)-1}\big) \equiv r^2 n^{\varphi(m)} + m^2 s^{\varphi(r)} \equiv r^2 + m^2 (\operatorname{mod} mr)\,.$$

By the Chinese remainder (SunZi) theorem,

$$\begin{aligned}&\left\{a \in \mathbb{Z} \mid a \equiv m^2 + r^2 (\operatorname{mod} mr),\ (rn - ms) | a\right\}\\ &\quad = \left\{(rn - ms)\left(rn^{\varphi(m)-1} - ms^{\varphi(r)-1}\right) + k(rn - ms)mr \mid k \in \mathbb{Z}\right\}.\end{aligned}$$

So

$$\begin{aligned}\mathsf{F}\left(\frac{n}{m}, \frac{s}{r}\right) &= \#\left\{k \in \mathbb{Z} \mid (rn - ms)\left(rn^{\varphi(m)-1} - ms^{\varphi(r)-1}\right) + k(rn - ms)mr \in \left((rn - ms)^2, m^2\right)\right\}\\ &= \#\left\{a \in \mathbb{Z} \mid a \equiv m^2 + r^2 (\operatorname{mod} mr),\ (rn - ms) | a,\ a \in \left((rn - ms)^2, m^2\right)\right\}\\ &= \#\left\{t \in \mathbb{Z} \mid (rn - ms) | a = m^2 + r^2 - tmr \in \left((rn - ms)^2, m^2\right)\right\}\\ &= \#\left\{t \in \mathbb{Z}_{\geqslant 1} \mid (rn - ms) | m^2 + r^2 - tmr > (rn - ms)^2\right\}.\end{aligned}$$

The claim follows. □

For a pair of positive integers (m, r) satisfying $m > r$, we set

$$\begin{aligned}S_{m,r} &:= \left\{(n, s) \in \mathbb{Z}^2 \mid 1 \leqslant n \leqslant m - 1,\ 0 \leqslant s \leqslant r,\ \gcd(n, m) = \gcd(s, r) = 1\right\},\\ A_{m,r} &:= \left\{m^2 + r^2 - trm \in \mathbb{Z} \cap \left[1, m^2\right] \mid t \in \mathbb{Z}\right\},\\ \mathsf{G}(m, r) &:= 2 \sum_{a \in A_{m,r}} \left\lfloor \frac{\tau(a)}{2} \right\rfloor \quad \text{and} \quad \mathsf{G}(m) := \sum_{1 \leqslant r < m,\, \gcd(r,m)=1} \mathsf{G}(m, r),\end{aligned}$$

where $\tau(n) = \sum_{d|n} 1$ is the divisor function.

LEMMA 5.9. *Let $m \geqslant 2$ be a positive integer.*

(1) *For all positive integers $r, d \in [1, m - 1]$, we have*

$$\#\{(n, s) \in S_{m,r} \mid rn - ms = d\} \begin{cases} = 1 & \text{if } \gcd(m, r) = 1 \text{ and } \gcd(mr, d) = 1,\\ \leqslant \dfrac{\varphi(m)}{\varphi(m/a)} & \text{if } \gcd(m, r) = a,\ a|d, \text{ and } \gcd\left(mr/a^2, d/a\right) = 1,\\ = 0 & \text{otherwise}.\end{cases} \tag{5.10}$$

(2) *For every $r < m$ with $\gcd(m, r) = 1$,*

$$\sum_{(n,s) \in S_{m,r}} \mathsf{F}\left(\frac{n}{m}, \frac{s}{r}\right) = \mathsf{G}(m, r). \tag{5.11}$$

(3)

$$\sum_{\substack{1 \leqslant n \leqslant m-1\\ \gcd(n,m)=1}} \mathsf{H}\left(\frac{n}{m}\right) \leqslant \varphi(m) + \sum_{d>1, d|m} \frac{\varphi(m)}{\varphi(d)} \mathsf{G}(d). \tag{5.12}$$

Proof. (1) Assume $\gcd(m, r) = a$. As $\gcd(n, m) = \gcd(s, r) = 1$, it is clear that $a | rn - ms$ and $\gcd(mr/a^2, (rn - ms)/a) = 1$. So the ‘otherwise’ case in the equation holds.

When $r = 1$ and $(n, s) \in S_{m,r}$, it is clear that $n - ms = d$ has a unique solution $(n, s) = (d, 0)$. We may assume $r \geqslant 2$ for the rest of the argument.

Assume $\gcd(m, r) = 1$. By the Chinese remainder (SunZi) theorem, for every d satisfying $\gcd(mr, d) = 1$, there is a unique pair $(n, s) \in S_{m,r}$ such that $rn - ms \equiv d \pmod{mr}$. Note that $rn - ms \in [r + m - rm, rm - r]$. If $rn - ms \equiv d \pmod{mr}$ for some $d \in [1, m - 1]$, then $rn - ms = d$. The first equality in the equation holds.

Now assume $\gcd(m,r) = a > 1$, and let $m = am'$ and $r = ar'$. For a given number $d = ad' \in [a, m-a]$, let (n_0, s_0) be the smallest solution to $rn - ms = ad'$ for all $(n,s) \in S_{m,r}$. Then other solutions to $rn - ms = ad'$ are all in the form of $(n_0 + tr', s_0 + tm')$, where $t = 0, 1, \ldots, a-1$. Note that there are exactly $\varphi(am')/\varphi(m')$ values of t satisfying $\gcd(s_0 + tm', a) = 1$, so there are at most $\varphi(am')/\varphi(m')$ pairs of $(n_0 + tr', s_0 + tm')$ in $S_{m,r}$. The second inequality (5.10) holds.

(2) By Lemma 5.8(1)–(2), the left-hand side of (5.11) equals

$$2 \cdot \left(\sum_{(n,s)\in S_{m,r}\,, 1\leqslant rn-ms\leqslant m-1} \mathsf{F}\left(\frac{n}{m}, \frac{s}{r}\right) \right).$$

By item (1) and Lemma 5.8(6), this is equal to

$$2 \cdot \left(\sum_{1\leqslant d\leqslant m-1,\, \gcd(d,mr)=1} \#\big\{ k \in \mathbb{Z}_{\geqslant 1} \mid d | m^2 + r^2 - kmr > d^2 \big\} \right).$$

As $\gcd(m,r) = 1$, it is clear that $\gcd\big(mr, m^2 + r^2 - kmr\big) = 1$. In particular, any divisor of $m^2 + r^2 - kmr$ is coprime to mr. As $m > r$ and $k \geqslant 1$, we have $m^2 + r^2 - kmr < m^2$. The equation holds.

(3) By Lemma 5.8(4), the left-hand side of (5.12) is less than or equal to

$$\varphi(m) + \sum_{1\leqslant r\leqslant \left\lfloor \frac{m}{2} \right\rfloor} \sum_{(n,s)\in S_{m,r}} \mathsf{F}\left(\frac{n}{m}, \frac{s}{r}\right) \leqslant \varphi(m) + \sum_{1\leqslant a<m,\, a|m} \sum_{\substack{1\leqslant r<m,\\ \gcd(r,m)=a}} \sum_{(n,s)\in S_{m,r}} \mathsf{F}\left(\frac{n}{m}, \frac{s}{r}\right). \tag{5.13}$$

Assume $\gcd(m,r) = a$, and let $m = am'$ and $r = ar'$. Then by items (1) and (2) and Lemma 5.8(5), we have

$$\sum_{(n,s)\in S_{m,r}} \mathsf{F}\left(\frac{n}{m}, \frac{s}{r}\right) \leqslant \frac{2\varphi(m)}{\varphi(m')} \sum_{\substack{(n',s')\in S_{m',r'}\\ 1\leqslant r'n'-m's'\leqslant m'-1}} \mathsf{F}\left(\frac{n'}{m'}, \frac{s'}{r'}\right) = \frac{\varphi(m)}{\varphi(m')} \mathsf{G}(m', r').$$

Substituting this into (5.13), we get

$$\varphi(m) + \sum_{1\leqslant a<m,\, a|m} \sum_{\substack{1\leqslant r<m,\\ \gcd(r,m)=a}} \frac{\varphi(m)}{\varphi(m')} \mathsf{G}(m', r') = \varphi(m) + \sum_{1\leqslant a<m,\, a|m} \frac{\varphi(m)}{\varphi(m')} \mathsf{G}(m').$$

The inequality follows. $\square$

Proof of Proposition 5.7. We first prove the upper bound. Note that for any $\epsilon > 0$, we have $\tau(n) = o(n^\epsilon)$, it follows that $\mathsf{G}(m) = m\log m \cdot o(m^\epsilon) = o\big(m^{1+2\epsilon}\big)$. When $d|m$, we have the inequality $\varphi(m)/\varphi(d) \leqslant m/d$. The upper bound follows from (5.12).

We then prove the lower bound. Set

$$B_m := \bigcup_{\substack{1\leqslant r<m^{1/4},\, 0\leqslant s\leqslant r-1\\ \gcd(s,r)=1}} \left(\frac{s}{r}, \frac{s}{r} + \frac{1}{\sqrt{m}} \right).$$

Note that when $r < m^{1/4}$, the intervals $\big(s/r, s/r + 1/\sqrt{m}\big)$ do not intersect one another. So for every $n/m \in B_m$, there is a unique s/r, denoted as $q(n/m)$, associated with n/m such that

$n/m \in \left(s/r, s/r + 1/\sqrt{m}\right)$. We can estimate the lower bound of (5.6) as follows:

$$
\begin{aligned}
\sum_{\substack{1\leqslant n\leqslant m-1\\ \gcd(n,m)=1}} \mathsf{H}\left(\frac{n}{m}\right) &\geqslant \sum_{\frac{n}{m}\in B_m} \mathsf{H}\left(\frac{n}{m}\right) \geqslant \sum_{\frac{n}{m}\in B_m} 1+\mathsf{F}\left(\frac{n}{m}, q\left(\frac{n}{m}\right)\right) && \text{by Lemma 5.8(3)}\\
&\geqslant \sum_{\frac{n}{m}\in B_m,\, q(\frac{n}{m})=\frac{s}{r}} \frac{m^2-(rn-ms)^2}{mr|rn-ms|} && \text{by Lemma 5.8(2)}\\
&= \sum_{1\leqslant r<m^{1/4}} \sum_{\substack{0\leqslant s\leqslant r-1\\ \gcd(s,r)=1}} \sum_{\substack{0<rn-sm<r\sqrt{m}\\ \gcd(n,m)=1}} \frac{m^2-(rn-ms)^2}{mr(rn-ms)}\\
&\geqslant \sum_{\substack{1\leqslant r<m^{1/4}\\ \gcd(m,r)=1}} \sum_{\substack{1\leqslant d<r\sqrt{m}\\ \gcd(d,mr)=1}} \frac{m^2-d^2}{mrd} && \text{by Lemma 5.9(1)}\\
&\geqslant m\Bigg(\sum_{\substack{1\leqslant r<m^{1/4}\\ \gcd(m,r)=1}} \sum_{\substack{1\leqslant d<r\sqrt{m}\\ \gcd(d,mr)=1}} \frac{1}{rd}\Bigg) - \sum_{1\leqslant r<m^{1/4}} \sum_{1\leqslant d<r\sqrt{m}} \frac{d}{mr}\\
&\geqslant m \sum_{\substack{1\leqslant r<m^{1/4}\\ \gcd(m,r)=1}} \frac{\varphi(rm)}{rm}\frac{1}{r}\ln\left(r\sqrt{m}\right) - \Theta\left(\sqrt{m}\right)\\
&\geqslant \frac{1}{2}\varphi(m)\ln m \cdot \sum_{\substack{1\leqslant r<m^{1/4}\\ \gcd(m,r)=1}} \frac{\varphi(r)}{r^2} - \Theta\left(\sqrt{m}\right)\\
&\geqslant \frac{\varphi(m)^2}{2m}\ln m \cdot \sum_{1\leqslant r<m^{1/4}} \frac{\varphi(r)}{r^2} - \Theta\left(\sqrt{m}\right) = \Theta\left(\frac{\varphi(m)^2}{m}(\ln m)^2\right).
\end{aligned}
$$

The lower bound follows. $\square$

Remark 5.10. As we have mentioned in Remark 3.5, it is not clear to us whether there exists a spherical bundle that is never slope semistable with respect to any polarization. We are also curious whether other kinds of 'counting' are better than the naive one. For example, one may put different weights on spherical bundles. It is not clear to us whether one can build up any meaningful power series from these counting values.

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Appendix. Spherical twists and standard autoequivalences

Genki Ouchi

In this appendix, we prove that the spherical twists of spherical objects are not standard for smooth projective varieties of dimensions greater than 1. Let X be a smooth projective n-dimensional variety over $\mathbb{C}$.

Definition A.1. We define the *standard autoequivalence group* $A(X)$ as the subgroup

$$A(X) := \mathbb{Z}[1] \times (\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X))$$

of the autoequivalence group $\mathrm{Aut}\big(\mathrm{D}^b(X)\big)$. An autoequivalence Φ of $\mathrm{D}^b(X)$ is called *standard with respect to* X if $\Phi \in A(X)$ holds.

The main theorem in this appendix is as follows.

Theorem A.2. *Assume $n \geqslant 2$ and $k \neq 0$. Let E be an n-spherical object in $\mathrm{D}^b(X)$ and k be a non-zero integer. Let Y be a smooth projective variety with an exact equivalence $\Psi\colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$. There is no standard autoequivalence $\Phi \in A(Y)$ such that $\mathsf{T}_E^k = \Psi^{-1} \circ \Phi \circ \Psi$ holds in $\mathrm{Aut}\big(\mathrm{D}^b(X)\big)$. In particular, the autoequivalence T_E^k is not conjugate to a standard autoequivalence with respect to X in $\mathrm{Aut}\big(\mathrm{D}^b(X)\big)$.*

Note that $\Psi \circ \mathsf{T}_E \circ \Psi^{-1} = \mathsf{T}_{\Psi(E)}$. Theorem A.2 can be phrased entirely as T_E^k not being standard.

In the proof of Theorem A.2, we use the notion of categorical entropy of autoequivalences of $\mathrm{D}^b(X)$ introduced in [DHKK14].

Let Φ be an autoequivalence of $\mathrm{D}^b(X)$. By [DHKK14, Definition 2.4], we have the categorical entropy $h_t(\Phi)\colon \mathbb{R} \to [-\infty, \infty)$. By [DHKK14, Definition 2.4, proof of Lemma 2.5], we have $h_0(\Phi) \in \mathbb{R}_{\geqslant 0}$. An object $G \in \mathrm{D}^b(X)$ is called a *split generator* of $\mathrm{D}^b(X)$ if $\mathrm{D}^b(X) = \langle G \rangle_{\mathrm{thick}}$ holds, where $\langle G \rangle_{\mathrm{thick}}$ is the smallest thick subcategory of $\mathrm{D}^b(X)$ containing G. The categorical entropy $h_t(\Phi)$ is computed by the following formula.

Theorem A.3 ([DHKK14, Theorem 2.6]). (1) *For a split generator G of $\mathrm{D}^b(X)$, we have*

$$h_t(\Phi) = \lim_{N \to \infty} \frac{1}{N} \log\Big(\sum_{m \in \mathbb{Z}} \hom\big(G, \Phi^N(G)[m]\big) e^{-mt} \Big)$$

for any $t \in \mathbb{R}$.

(2) *Let ℓ be an integer. Then we have $h_t(\Phi[\ell]) = h_t(\Phi) + \ell t$ for any $t \in \mathbb{R}$.*

(3) *For a positive integer k, we have $h_t(\Phi^k) = k h_t(\Phi)$.*

In Theorem A.3, statements (2) and (3) are easily deduced from statement (1). See also [Kik17, Lemma 2.7(v)] and [DHKK14, Section 2.2]. By [Kik17, Lemma 2.9], if autoequivalences Φ_1, Φ_2 of $\mathrm{D}^b(X)$ are conjugate in $\mathrm{Aut}\mathrm{D}^b(X)$, we have $h_t(\Phi_1) = h_t(\Phi_2)$ for any $t \in \mathbb{R}$. To prove Theorem A.2, we compare $h_t(\mathsf{T}_E)$ with $h_t(\Phi)$ for a standard autoequivalence $\Phi \in A(Y)$, where Y is a Fourier–Mukai partner of X. The following is the explicit formula for $h_t(\mathsf{T}_E)$.

Theorem A.4. *The following equality holds:*

$$h_t(\mathsf{T}_E) = \begin{cases} 0\,, & t \geqslant 0\,, \\ (1-n)t\,, & t < 0\,. \end{cases}$$

Proof. This is a direct consequence of [Ouc20, Theorem 1.4] and Proposition 1.1. □

Next, we compute the categorical entropy of standard autoequivalences. Let Y be a smooth projective variety with an exact equivalence $\Psi\colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$. Take $\Phi \in A(Y)$. There exist an automorphism f of Y, a line bundle $\mathcal{L}$ on Y and an integer ℓ such that $\Phi = f^* \circ (\otimes \mathcal{L})[\ell]$. Note that $h_t(\Psi^{-1} \circ \Phi \circ \Psi) = h_t(\Phi)$ holds for any $t \in \mathbb{R}$. We have the following formula.

PROPOSITION A.5. *The equality $h_t(\Phi) = h_0(f^* \circ (\otimes\mathcal{L})) + \ell t$ holds for any $t \in \mathbb{R}$.*

Proof. Fix $t \in \mathbb{R}$. By Theorem A.3(2), we have $h_t(\Phi) = h_t(f^* \circ (\otimes\mathcal{L})) + \ell t$.

Consider the object $G := \bigoplus_{i=0}^{n} \mathcal{O}_Y(i)$, where $\mathcal{O}_Y(1)$ is a very ample line bundle on Y. Then G is a split generator of $\mathrm{D}^b(Y)$. For any positive integer N, the objects G and $(f^* \circ (\otimes\mathcal{L}))^N(G)$ are coherent sheaves on Y. By [DHKK14, Lemma 2.10], the categorical entropy $h_t(f^* \circ (\otimes\mathcal{L}))$ is constant with respect to t. Hence, we obtain $h_t(f^* \circ (\otimes\mathcal{L})) = h_0(f^* \circ (\otimes\mathcal{L}))$. □

Proof of Theorem A.2. It is enough to prove the case $k > 0$. Assume that there is a standard autoequivalence $\Phi \in A(Y)$ such that $\mathsf{T}_E^k = \Psi^{-1} \circ \Phi \circ \Psi$ holds in $\mathrm{Aut}(\mathrm{D}^b(X))$. By Theorem A.3(3), we have $kh_t(\mathsf{T}_E) = h_t(\mathsf{T}_E^k) = h_t(\Phi)$ for any $t \in \mathbb{R}$. By Theorem A.4 and Proposition A.5, this gives a contradiction since $n \geqslant 2$. □

Remark A.6. Let C be a smooth projective curve. For a closed point $x \in C$, the skyscraper sheaf $\mathcal{O}_x$ is a 1-spherical object in $\mathrm{D}^b(C)$. As in [Huy06, Example 8.1(i)], we have $\mathsf{T}_{\mathcal{O}_x} \simeq \otimes\mathcal{O}_C(x)$. In particular, $\mathsf{T}_{\mathcal{O}_x} \in A(C)$ holds.

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