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Inversion of adjunction for quotient singularities II: Non-linear actions

Yusuke Nakamura and Kohsuke Shibata

Abstract

We prove the precise inversion of the adjunction formula for quotient singularities. As an application, we prove the semi-continuity of minimal log discrepancies for hyperquotient singularities. This paper is a continuation of our work [Algebr. Geom. **9** (2022), no. 2, 214–251]; we generalize the previous results to non-linear group actions.

1. Introduction

The minimal log discrepancy is an invariant of singularities defined in birational geometry. The importance of this invariant is that two conjectures on the invariant, the LSC (lower semi-continuity) conjecture and the ACC (ascending chain condition) conjecture, imply the termination of flips conjecture [Sho04]. This paper is a continuation of [NS22], and we focus on the LSC conjecture. We always work over an algebraically closed field k of characteristic zero.

The LSC conjecture was proposed by Ambro [Amb99]; it predicts that the minimal log discrepancies satisfy the lower semi-continuity.

Conjecture 1.1 (LSC conjecture, [Amb99]). Let $(X, \mathfrak{a})$ be a log pair with an $\mathbb{R}$-ideal $\mathfrak{a}$, and let $|X|$ be the set of all closed points of X for the Zariski topology. Then the function

$$|X| \longrightarrow \mathbb{R}_{\geqslant 0} \cup \{-\infty\}\,, \quad x \longmapsto \mathrm{mld}_x(X, \mathfrak{a})$$

is lower semi-continuous.

The LSC conjecture was proved when $\dim X \leqslant 3$ by Ambro [Amb99]. In [EMY03], Ein, Mustaţă and Yasuda prove the conjecture when X is smooth. In [EM04], Ein and Mustaţă generalize the argument to the case where X is a locally complete intersection variety. In [Nak16], the first author proves the conjecture when X has quotient singularities, more generally when X has a crepant resolution in the category of Deligne–Mumford stacks. In [NS22], the authors study the conjecture when X has hyperquotient singularities and prove the conjecture for the following X:

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- Suppose that a finite subgroup $G \subset \mathrm{GL}_N(k)$ acts linearly on $\mathbb{A}_k^N$ freely in codimension 1. Let $Y := \mathbb{A}_k^N/G$ be the quotient variety. Let $X \subset Y$ be a Kawamata log terminal (klt) subvariety of codimension c that is locally defined by c equations in Y.

The main purpose of this paper is to generalize the result in [NS22] to non-linear group actions. See Definition 8.1 for the definition of quotient singularities in this paper.

Theorem 1.2 (= Theorem 9.2). *Let Y be a variety with only quotient singularities. Let X be a klt subvariety of Y of codimension c that is locally defined by c equations in Y. Let $\mathfrak{a}$ be an $\mathbb{R}$-ideal sheaf on X. Then the function*

$$|X| \longrightarrow \mathbb{R}_{\geqslant 0} \cup \{-\infty\}, \quad x \longmapsto \mathrm{mld}_x(X, \mathfrak{a})$$

is lower semi-continuous.

In this paper, we also treat the PIA (precise inversion of adjunction) conjecture.

Conjecture 1.3 (PIA conjecture, [Kol92, Conjecture 17.3.1]). Let $(X, \mathfrak{a})$ be a log pair, and let D be a normal Cartier prime divisor. Let $x \in D$ be a closed point. Suppose that D is not contained in the cosupport of the $\mathbb{R}$-ideal sheaf $\mathfrak{a}$. Then we have

$$\mathrm{mld}_x\big(X, \mathfrak{a}\mathcal{O}_X(-D)\big) = \mathrm{mld}_x(D, \mathfrak{a}\mathcal{O}_D)\,.$$

In [EMY03], Ein, Mustaţă and Yasuda prove the PIA conjecture when X is smooth. In [EM04], Ein and Mustaţă generalize the argument to the case where X is a locally complete intersection variety. In [NS22], the authors prove the conjecture for the following X and D:

- Suppose that a finite subgroup $G \subset \mathrm{GL}_N(k)$ acts linearly on $\mathbb{A}_k^N$ freely in codimension 1. Let $Y := \mathbb{A}_k^N/G$ be the quotient variety, and let $x \in Y$ be the image of the origin. Let X be a subvariety of Y of codimension c that has only klt singularities and is locally defined by c equations in Y at x. Let D be a Cartier prime divisor on X through x with a klt singularity at $x \in D$.

In this paper, this result from [NS22] is generalized to non-linear group actions.

Theorem 1.4 (= Corollary 9.1). *Suppose that a variety Y has a quotient singularity at a closed point $x \in Y$. Let X be a subvariety of Y of codimension c that is locally defined by c equations at x. Suppose that X is klt at x. Let $\mathfrak{a}$ be an $\mathbb{R}$-ideal sheaf on X. Let D be a prime divisor on X through x that is klt and Cartier at x. Suppose that D is not contained in the cosupport of the $\mathbb{R}$-ideal sheaf $\mathfrak{a}$. Then it follows that*

$$\mathrm{mld}_x\big(X, \mathfrak{a}\mathcal{O}_X(-D)\big) = \mathrm{mld}_x(D, \mathfrak{a}\mathcal{O}_D)\,.$$

Due to Theorem 1.4, Theorem 1.2 can be reduced to the known case where X has quotient singularities. Hence, this paper is mainly devoted to proving Theorem 1.4. If X has a quotient singularity at a closed point $x \in X$, then the completion $\widehat{\mathcal{O}}_{X,x}$ is isomorphic to $k[[x_1, \dots, x_N]]^G$ for some linear group action $G \subset \mathrm{GL}_N(k)$. Therefore, Theorem 1.4 can be proved by extending the proofs in [NS22] to the case of the formal power series ring. In what follows, we explain the main differences with the proofs in [NS22] and the difficulties that arise when dealing with formal power series rings.

The key ingredient of the proofs in [NS22] is the theory of arc spaces of quotient singularities established by Denef and Loeser in [DL02]. Suppose that a finite group $G \subset \mathrm{GL}_N(k)$ of order d acts on $\overline{A} = \operatorname{Spec} k[x_1, \dots, x_N]$. Let $\overline{X} \subset \overline{A}$ be a G-invariant closed subvariety, and let $X := \overline{X}/G$ be its quotient. Let $I \subset k[x_1, \dots, x_N]^G$ be the defining ideal of $X \subset \overline{A}/G$. Each $\gamma \in G$ can be diagonalized in some new basis $x_1^{(\gamma)}, \dots, x_N^{(\gamma)}$. Let $\mathrm{diag}(\xi^{e_1}, \dots, \xi^{e_N})$ be the diagonal matrix with $0 \leqslant e_i \leqslant d-1$, where ξ is a primitive dth root of unity in k. Then we define the ring homomorphism $\overline{\lambda}_\gamma^*$ by

$$\overline{\lambda}_\gamma^* \colon k[x_1, \dots, x_N]^G \longrightarrow k[t][x_1, \dots, x_N]\,, \quad x_i^{(\gamma)} \longmapsto t^{e_i/d} x_i^{(\gamma)}\,,$$

and define a $k[t]$-scheme $\overline{X}^{(\gamma)}$ by

$$\overline{X}^{(\gamma)} = \mathrm{Spec}\big(k[t][x_1, \dots, x_N]/\overline{I}^{(\gamma)}\big)\,,$$

where $\overline{I}^{(\gamma)}$ is the ideal generated by the elements of $\overline{\lambda}_\gamma^*(I)$. Then the theory of Denef and Loeser in [DL02] allows us to compare the spaces X_∞ and $\bigsqcup_{\gamma \in G} \overline{X}_\infty^{(\gamma)}$. In [NS22], using this theory, X_∞ is studied through each $\overline{X}_\infty^{(\gamma)}$.

In this paper, we deal with the case of formal power series rings, $\overline{A} = \mathrm{Spec}\, k[[x_1, \dots, x_N]]$. Let G, γ, $\overline{X}$ and X be as above. In this case, we take the e_i above to satisfy $1 \leqslant e_i \leqslant d$. Then we can define the two natural ring homomorphisms

$$\overline{\lambda}_\gamma^* \colon k[[x_1, \dots, x_N]]^G \longrightarrow k[x_1, \dots, x_N][[t]]\,, \quad x_i^{(\gamma)} \longmapsto t^{e_i/d} x_i^{(\gamma)}\,,$$
$$\overline{\lambda}_\gamma'^* \colon k[[x_1, \dots, x_N]]^G \longrightarrow k[t][[x_1, \dots, x_N]]\,, \quad x_i^{(\gamma)} \longmapsto t^{e_i/d} x_i^{(\gamma)}\,,$$

and we define $k[t]$-schemes

$$\overline{X}^{(\gamma)} = \mathrm{Spec}\big(k[x_1, \dots, x_N][[t]]/\overline{I}^{(\gamma)}\big)\,, \quad \overline{X}'^{(\gamma)} = \mathrm{Spec}\big(k[t][[x_1, \dots, x_N]]/\overline{I}'^{(\gamma)}\big)\,,$$

where $\overline{I}'^{(\gamma)}$ is the ideal generated by $\overline{\lambda}_\gamma'^*(I)$. In this paper, we use both arc spaces $\overline{X}_\infty^{(\gamma)}$ and $\overline{X}_\infty'^{(\gamma)}$ to study X_∞. We also explain below how to use $\overline{X}_\infty^{(\gamma)}$ and $\overline{X}_\infty'^{(\gamma)}$ differently.

First, the theory of Denef and Loeser in [DL02] can be generalized to formal power series rings, and X_∞ can be compared with $\overline{X}_\infty^{(\gamma)}$. Indeed, we will see in Proposition 6.3 that $\overline{\lambda}_\gamma^*$ gives a map $\bigsqcup_{\gamma \in G} \overline{X}_\infty^{(\gamma)} \to X_\infty$ that is surjective outside a thin set. We note that it is not enough to consider spaces $\overline{X}_\infty'^{(\gamma)}$ in this respect (see Remark 6.4 for the details).

On the other hand, the spaces $\overline{X}_\infty'^{(\gamma)}$ will be used with the following motivation. In [NS22], a k-arc $\beta \in \overline{X}_\infty^{(\gamma)}$ is called the "trivial arc" when it corresponds to the $k[t]$-ring homomorphism $\beta^* \colon k[t][x_1, \dots, x_N] \to k[[t]]$ satisfying $\beta^*(x_i) = 0$ for each i. Another key point of the argument in [NS22] is showing that the trivial arc always has a lift on a resolution W of $\overline{X}^{(\gamma)}$. The existence of such a lift is proved by combining a result by Graber, Harris and Starr [GHS03] and the rational chain connectedness of the fibers of the resolution proved by Hacon and McKernan [HM07]. In our formal power series ring setting, this argument does not work directly on $\overline{X}^{(\gamma)}$ because each closed fiber $\overline{X}^{(\gamma)} \to \mathrm{Spec}\, k[t]$ over $t = a \in k^\times$ is empty. However, the same argument works on $\overline{X}'^{(\gamma)}$ and proves that the trivial arc has a lift on a desingularization W' of $\overline{X}'^{(\gamma)}$ (see Claim 8.4). It should be noted that it is not clear whether the results of [GHS03] and [HM07] can be applied to the formal power series ring setting. However, this difficulty can be avoided by using the functorial desingularization due to Temkin [Tem12].

A large part of this paper (Sections 2, 4 and 5) is devoted to proving and listing the basic facts for dealing with the arc spaces $\overline{X}_\infty^{(\gamma)}$ and $\overline{X}_\infty'^{(\gamma)}$. Firstly, $\overline{X}_\infty^{(\gamma)}$ can be seen as the arc space (Greenberg scheme) of a formal $k[[t]]$-scheme, and the theory developed by Sebag in [Seb04] can be applied (cf. [CNS18, Yas24]). On the other hand, as far as we know, the arc spaces of the form $\overline{X}'^{(\gamma)}$ have not been fully discussed so far. In Section 5.1, we discuss the basic facts on the arc spaces of $k[t][[x_1, \dots, x_N]]$-schemes of finite type. Furthermore, in Section 2, we discuss the theory of sheaves of special differentials introduced by de Fernex, Ein and Mustaţă in [dFEM11] and the theory of derivations that are needed in Section 5. The theory of the arc spaces of $k[t][[x_1, \dots, x_N]]$-schemes has the following technical difficulties (see Remark 5.16 for the details). For a k-variety X, it is almost trivial that Z_∞ is a thin set of X_∞ for the closed subscheme $Z \subset X$ defined by the Jacobian ideal Jac_X. This fact is also valid for $k[t]$-schemes X of finite type dealt with in [NS22] and for formal $k[[t]]$-schemes of finite type dealt with in [Seb04]. However, it is not clear to us whether the same statement is valid for $k[t][[x_1, \dots, x_N]]$-schemes of finite type. To avoid this difficulty, many propositions in Section 5.1.3 are proved under stronger assumptions.

The paper is organized as follows. In Section 2, we discuss the theory of sheaves of special differentials introduced by de Fernex, Ein and Mustaţă in [dFEM11] and the theory of derivations. In Section 3, we review some definitions on log pairs. In Section 4, we discuss the theory of arc spaces of $k[[x_1, \dots, x_N]]$-schemes and see that the formulas in [EMY03] and [EM09] representing the minimal log discrepancies of k-varieties in terms of arc spaces can be generalized to the formal power series ring setting (Theorem 4.11). In Section 5, we discuss the theory of arc spaces for $k[t][[x_1, \dots, x_N]]$-schemes and affine formal $k[[t]]$-schemes. In Section 6, we review the theory of arc spaces of quotient varieties established by Denef and Loeser in [DL02] in the formal power series ring setting. In Section 7, we discuss the minimal log discrepancies of quotient singularities and describe them by the codimension of cylinders in arc spaces of the $k[t]$-schemes using the theories in Sections 5 and 6. In Section 8, we prove the PIA conjecture for quotient singularities, where the group action may be non-linear (Theorem 8.2). In Section 9, we prove the main theorems Corollary 9.1 and Theorem 9.2.

Notation

- We basically follow the notation and terminology in [Har77] and [Kol13].
- Throughout this paper, k is an algebraically closed field of characteristic zero. We say that X is a *variety over* k or a k*-variety* if X is an integral scheme that is separated and of finite type over k.

2. Sheaves of special differentials

Let R_0 be a ring. In this section, following [dFEM11, Appendix A], we define the sheaf Ω'_{X/R_0} of special differentials for a scheme X of finite type over $\operatorname{Spec} R_0[[x_1, \dots, x_N]]$. In [dFEM11, Appendix A], the sheaf Ω'_{X/R_0} of special differentials is defined for $R_0 = k$. This definition can be generalized to an arbitrary ring R_0. We are interested in the case where $R_0 = k$ or $R_0 = k[t]$ for later use.

Let $R = R_0[[x_1, \dots, x_N]]$.

Definition 2.1 (cf. [dFEM11, Appendix A]). (1) Let M be an R-module. Then an R_0-derivation $D\colon R \to M$ is called a *special* R_0*-derivation* if D satisfies

$$D(f) = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i} D(x_i)$$

for any $f \in R$.

(2) For an R-algebra A and an A-module M, an R_0-derivation $D\colon A \to M$ is called a *special* R_0*-derivation* if its restriction to R is a special R_0-derivation. We denote by $\mathrm{Der}'_{R_0}(A, M)$ the

set of all special R_0-derivations. Then $\mathrm{Der}'_{R_0}(A, M)$ is an A-submodule of $\mathrm{Der}_{R_0}(A, M)$.

Lemma 2.2. *Let M be an A-module that is separated for the $(x_1, \ldots, x_N)$-adic topology; that is, M satisfies $\bigcap_{\ell \geqslant 1}(x_1, \ldots, x_N)^\ell M = 0$. Then*

$$\mathrm{Der}'_{R_0}(A, M) = \mathrm{Der}_{R_0}(A, M)\,.$$

In particular, it holds in the following two cases:

(1) *when M is a finite R-module,*

(2) *when $M = A$ and A is an integral domain such that $(x_1, \ldots, x_N)A \neq A$.*

Proof. By the definition of special derivations, it is sufficient to show that $\mathrm{Der}'_{R_0}(R, M) = \mathrm{Der}_{R_0}(R, M)$. Let $D \in \mathrm{Der}_{R_0}(R, M)$. If we define $D' \colon R \to M$ by $D'(f) := \sum_{i=1}^{N}(\partial f/\partial x_i)D(x_i)$, then $D' \in \mathrm{Der}'_{R_0}(R, M)$. Hence it is sufficient to show that $D = 0$ if $D(x_i) = 0$ for all i. Let $f \in R$. For any $\ell \geqslant 0$, we may write $f = f_1 + f_2$ with $f_1 \in R_0[x_1, \ldots, x_n]$ and $f_2 \in (x_1, \ldots, x_n)^{\ell+1}R$. Then we have

$$D(f) = D(f_1) + D(f_2) = D(f_2) \in (x_1, \ldots, x_n)^\ell M.$$

This shows that $D(f) \in \bigcap_{\ell \geqslant 1}(x_1, \ldots, x_N)^\ell M = 0$.

In both cases (1) and (2), the A-module M is separated by [Mat89, Theorem 8.9]. □

Proposition 2.3 (cf. [dFEM11, Appendix A]). *For any R-algebra A, there exists an A-module Ω'_{A/R_0} with a special R_0-derivation $\mathrm{d}'_{A/R_0} \colon A \to \Omega'_{A/R_0}$ such that the induced map*

$$\mathrm{Hom}_A(\Omega'_{A/R_0}, M) \longrightarrow \mathrm{Der}'_{R_0}(A, M)\,, \quad f \longmapsto f \circ \mathrm{d}'_{A/R_0}$$

is an isomorphism for any A-module M.

Proof. The same proof as in [dFEM11, Lemmas A.1–A.3 and Corollary A.4] works. □

An A-module Ω'_{A/R_0} satisfying Proposition 2.3 is unique up to an isomorphism commuting with d'_{A/R_0}, and Ω'_{A/R_0} is called the *module of special differentials*. We sometimes abbreviate d'_{A/R_0} to d' when no confusion can arise. We note that Ω'_{A/R_0} depends on the choice of R. We list some properties on Ω'_{A/R_0} from [dFEM11, Appendix A].

Proposition 2.4 (cf. [dFEM11, Appendix A]). (1) *If $A = R[y_1, \ldots, y_m]$, then Ω'_{A/R_0} is a free A-module of rank $N + m$ with basis*

$$\mathrm{d}'_{A/R_0}(x_1), \ldots, \mathrm{d}'_{A/R_0}(x_N), \mathrm{d}'_{A/R_0}(y_1), \ldots, \mathrm{d}'_{A/R_0}(y_m)\,.$$

(2) *Let $f \colon A \to B$ be a homomorphism of R-algebras. Then we have an exact sequence*

$$\Omega'_{A/R_0} \otimes_A B \xrightarrow{\alpha} \Omega'_{B/R_0} \xrightarrow{\beta} \Omega_{B/A} \longrightarrow 0$$

of B-modules, where the maps α and β are defined by $\alpha\big(\mathrm{d}'_{A/R_0}(g) \otimes 1\big) = \mathrm{d}'_{B/R_0}(f(g))$ for $g \in A$ and $\beta\big(\mathrm{d}'_{B/R_0}(g)\big) = \mathrm{d}_{B/A}(g)$ for $g \in B$.

(3) *Let $f \colon A \to B$ be a surjective homomorphism of R-algebras with $I = \mathrm{Ker}\, f$. Then we have an exact sequence*

$$I/I^2 \xrightarrow{\delta} \Omega'_{A/R_0} \otimes_A B \xrightarrow{\alpha} \Omega'_{B/R_0} \longrightarrow 0$$

of B-modules, where the map δ is defined by $\delta(\overline{g}) = \mathrm{d}'_{A/R_0}(g) \otimes 1$ for $g \in I$.

(4) *If $S \subset A$ is a multiplicative system of A, then we have a canonical isomorphism*

$$\Omega'_{S^{-1}A/R_0} \simeq \Omega'_{A/R_0} \otimes_A S^{-1}A\,.$$

Proof. The same proofs as in [dFEM11, Lemmas A.1–3, A.6 and A.7] work. The exact sequences in assertions (2) and (3) are derived from the corresponding exact sequences

$$0 \longrightarrow \operatorname{Der}_A(B, M) \longrightarrow \operatorname{Der}'_{R_0}(B, M) \longrightarrow \operatorname{Der}'_{R_0}(A, M)\,,$$
$$0 \longrightarrow \operatorname{Der}'_{R_0}(B, M) \longrightarrow \operatorname{Der}'_{R_0}(A, M) \longrightarrow \operatorname{Hom}_B\bigl(I/I^2, M\bigr)$$

for any B-module M. The isomorphism in assertion (4) is derived from the isomorphisms

$$\begin{aligned}\operatorname{Hom}_{S^{-1}A}\bigl(\Omega'_{A/R_0} \otimes_A S^{-1}A, M\bigr) &\simeq \operatorname{Hom}_A(\Omega'_{A/R_0}, M)\\ &\simeq \operatorname{Der}'_{R_0}(A, M) \simeq \operatorname{Der}'_{R_0}\bigl(S^{-1}A, M\bigr)\end{aligned}$$

for any $S^{-1}A$-module M. □

Remark 2.5. (1) The usual module Ω_{A/R_0} of differentials is not a finite A-module in general when A is an R-algebra of finite type. However, the module Ω'_{A/R_0} of special differentials becomes a finite A-module and has similar properties to the module of differentials defined for R_0-algebras of finite type.

(2) The universal finite module of differentials is also a module defined with the same motivation (see [Kun86, Section 11]). For an R_0-algebra A, a finite A-module $\widetilde{\Omega}_{A/R_0}$ with an R_0-derivation $\widetilde{\mathrm{d}}_{A/R_0}\colon A \to \widetilde{\Omega}_{A/R_0}$ is called the *universal finite module of differentials* if it satisfies the following universal property:

- For any R_0-derivation $D\colon A \to M$ to a finite A-module M, there exists a unique homomorphism $\alpha\colon \widetilde{\Omega}_{A/R_0} \to M$ of A-modules satisfying $D = \alpha \circ \widetilde{\mathrm{d}}_{A/R_0}$.

In other words, $\widetilde{\Omega}_{A/R_0}$ and $\widetilde{\mathrm{d}}_{A/R_0}$ satisfy

$$\operatorname{Hom}_A\bigl(\widetilde{\Omega}_{A/R_0}, M\bigr) \xrightarrow{\simeq} \operatorname{Der}_{R_0}(A, M)\,, \quad \alpha \longmapsto \alpha \circ \widetilde{\mathrm{d}}_{A/R_0}$$

for any finite A-module M.

In contrast to the module Ω'_{A/R_0} of special differentials, the universal finite module $\widetilde{\Omega}_{A/R_0}$ of differentials does not necessarily exist.

(3) A ring A is called an *analytic R_0-algebra* if there exists a ring $R = R_0[[x_1, \dots, x_N]]$ for some $N \geqslant 0$ such that A is a finite R-algebra. If A is an analytic R_0-algebra, then the universal finite module $\widetilde{\Omega}_{A/R_0}$ of differentials exists. Furthermore, if A is a finite $R_0[[x_1, \dots, x_N]]$-algebra and if Ω'_{A/R_0} is the module of special differentials with respect to $R = R_0[[x_1, \dots, x_N]]$, then we have $\widetilde{\Omega}_{A/R_0} \simeq \Omega'_{A/R_0}$. This is because in this case, we have

$$\operatorname{Der}'_{R_0}(A, M) = \operatorname{Der}_{R_0}(A, M)$$

for any finite A-module M by Lemma 2.2. Therefore, we can also see that Ω'_{A/R_0} does not depend on the choice of R as long as A is finite as an R-module.

(4) Even if A is an algebra of finite type over $R = R_0[[x_1, \dots, x_N]]$, the module $\widetilde{\Omega}_{A/R_0}$ does not necessarily exist despite the fact that Ω'_{A/R_0} is a finite A-module. We will see that $\widetilde{\Omega}_{A/k}$ and $\widetilde{\Omega}_{B/k}$ do not exist for

$$A = k[[x]][y]\,, \quad B = k[[x]][y]/(1 - xy) \simeq k((x))\,.$$

In fact, since

$$\begin{aligned}\dim_{k((x))} \operatorname{Der}_k\bigl(k((x))\bigr) &= \dim_{k((x))} \operatorname{Hom}_{k((x))}\bigl(\Omega_{k((x))/k}, k((x))\bigr)\\ &= \operatorname{trdeg}_k k((x)) = \infty\,,\end{aligned}$$

$\mathrm{Der}_k(B)$ is not a finite B-module, and hence $\widetilde{\Omega}_{B/k}$ does not exist. Furthermore, since there is a natural injective map $\mathrm{Der}_k(B,B) \to \mathrm{Der}_k(A,B)$, the set $\mathrm{Der}_k(A,B)$ is not a finite A-module, and hence $\widetilde{\Omega}_{A/k}$ does not exist (cf. [Kun86, Corollary 11.10]).

Remark 2.6. (1) Let A be a ring. For a non-negative integer ℓ and subsets $F \subset A$ and $\Delta \subset \mathrm{Der}(A)$, we denote by $\mathcal{J}_\ell(F;\Delta)$ the ideal of A generated by the determinants $\det\big(D_i(f_j)\big)_{1\leqslant i,j\leqslant \ell}$ of all the matrices $\big(D_i(f_j)\big)_{1\leqslant i,j\leqslant \ell}$ of size ℓ with $D_i \in \Delta$ and $f_j \in F$.

If I is an ideal of A generated by $f_1,\dots,f_t$ and the A-submodule $A\Delta$ of $\mathrm{Der}(A)$ is generated by $D_1,\dots,D_s$ as an A-module, then we have

$$\mathcal{J}_\ell(I;\Delta) + I = \mathcal{J}_\ell(\{f_1,\dots,f_t\};\{D_1,\dots,D_s\}) + I\,.$$

(2) Let A be a regular ring, and let P be a prime ideal. For an ideal I of A such that $I \subset P$, the following hold (cf. [Mat89, Theorem 30.4]):

(a) The inclusion $\mathcal{J}_\ell(I;\mathrm{Der}(A)) \subset P$ holds for any $\ell > \mathrm{ht}(IA_P)$.

(b) The quotient A_P/IA_P is regular if $\mathcal{J}_\ell(I;\mathrm{Der}(A)) \not\subset P$ for $\ell = \mathrm{ht}(IA_P)$.

Some regular rings satisfy the inverse implication of assertion (b), and such rings are said to satisfy the *weak Jacobian condition* (WJ) (cf. [Mat89, Section 30]). Rings of finite type over k are classically known to satisfy (WJ); this is known as the Jacobian criterion for regularity. Matsumura proved in [Mat77] that R-algebras of finite type satisfy $(\mathrm{WJ})_k$ when $R = k[[x_1,\dots,x_N]]$ (see [Mat77, Theorem 9] for a more general result).

(c) Let $A = k[[x_1,\dots,x_N]][y_1,\dots,y_m]$, and let P and Q be prime ideals of A such that $Q \subset P$. Then A/Q is regular at P if and only if $\mathcal{J}_\ell(Q;\mathrm{Der}_k(A)) \not\subset P$ for $\ell = \mathrm{ht}\,Q$.

Note that $\mathrm{Der}_k(A) = \mathrm{Der}'_k(A)$ for $A = k[[x_1,\dots,x_N]][y_1,\dots,y_m]$ (cf. Lemma 2.2); this is a free A-module generated by the $\partial/\partial x_i$ and $\partial/\partial y_i$.

In [dFEM11, Proposition A.8], the local freeness of $\Omega'_{A/k}$ is proved for regular rings A when $R_0 = k$.

Proposition 2.7 ([dFEM11, Proposition A.8]). *Suppose $R_0 = k$. Let A be an R-algebra of finite type, and let $\mathfrak{q}$ be a prime ideal of A. If $A_\mathfrak{q}$ is regular, then $\Omega'_{A_\mathfrak{q}/k}$ is a free $A_\mathfrak{q}$-module of rank $\dim A_\mathfrak{q} + \dim_{k(\mathfrak{q})}\big(\Omega'_{k(\mathfrak{q})/k}\big)$.*

In the proof of [dFEM11, Proposition A.8], the following statement is proved.

Proposition 2.8 ([dFEM11, Proposition A.8]). *Suppose $R_0 = k$. Let $S := R[y_1,\dots,y_m]$, and let P and Q be prime ideals of S with $P \subset Q$. Let $A := S/P$, and let $\mathfrak{q}$ be the prime ideal of A corresponding to Q. If $A_\mathfrak{q}$ is regular, then the sequence*

$$0 \longrightarrow PS_Q/P^2S_Q \longrightarrow \Omega'_{S/k}\otimes_S A_\mathfrak{q} \longrightarrow \Omega'_{A_\mathfrak{q}/k} \longrightarrow 0$$

induced by Proposition 2.4(3) is exact and splits.

Remark 2.9. (1) By Propositions 2.7 and 2.8, we have

$$\dim S - \mathrm{ht}\,P = \dim A_\mathfrak{q} + \dim_{k(\mathfrak{q})}\big(\Omega'_{k(\mathfrak{q})/k}\big)\,.$$

This shows that $\mathrm{coht}\,P = \dim S - \mathrm{ht}\,P$ is independent of the choice of S. Lemma 2.11(1) below also proves this independence. Lemma 2.11(1) moreover gives a ring-theoretic interpretation of this value without using Ω'.

(2) In Proposition 2.7, we note that the rank of $\Omega'_{A_\mathfrak{q}/k}$ is not equal to $\dim A$ in general. If we set $S = k[[x]][y]$ and $P = Q = (xy-1)$ in Proposition 2.8, then we have $A_\mathfrak{q} = A = k((x))$. Part (1) shows that the rank of $\Omega'_{A_\mathfrak{q}/k}$ is equal to 1 even though $\dim A = 0$.

Definition 2.10. Suppose that R_0 is a Noetherian domain. Let X be an irreducible scheme of finite type over R, and let X_{red} be its underlying reduced subscheme. Let $X_{\mathrm{red}} \to \operatorname{Spec} R$ be the structure morphism, and let $\mathfrak{p} \in \operatorname{Spec} R$ be the image of the generic point of X_{red}. Then we define

$$\dim' X := \operatorname{trdeg}_{k(\mathfrak{p})} K(X_{\mathrm{red}}) + \dim R - \operatorname{ht} \mathfrak{p}\,,$$

where $k(\mathfrak{p}) := R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ and $K(X_{\mathrm{red}})$ is the function field of X_{red}. When $X = \operatorname{Spec} A$ is an affine scheme, we also write $\dim' A := \dim' X$.

Lemma 2.11. *Suppose that R_0 is a Noetherian domain. Let $X = \operatorname{Spec} A$ be an irreducible affine scheme of finite type over R. Then the following hold:*

(1) *If A is a domain, then $\dim S - \operatorname{ht} P = \dim' A$ for any representation $A \simeq S/P$ with $S = R[y_1, \ldots, y_m]$ and P a prime ideal of S.*
(2) *The inequality $\dim A \leqslant \dim' A$ holds.*
(3) *Suppose $R_0 = k$ and that A is a domain. Then $\dim' A = \dim_K(\Omega'_{K/k})$ for $K = \operatorname{Frac} A$.*
(4) *Suppose $R_0 = k$ or $R_0 = k[t]$. Then $\dim' A = \dim A$ if $A/\mathfrak{m} = k$ for some maximal ideal $\mathfrak{m}$ of A.*
(5) *Suppose that R is a universally catenary ring. If $I = (f_1, \ldots, f_c)$ is an ideal of A generated by c elements, then $\dim'(A/Q) \geqslant \dim' A - c$ for any minimal prime Q of I.*
(6) *Suppose that R is a universally catenary ring. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of A such that $\mathfrak{p} \subset \mathfrak{q}$. Let $B := A/\mathfrak{p}$ and $\overline{\mathfrak{q}} := \mathfrak{q}/\mathfrak{p} \in \operatorname{Spec} B$. Then we have $\operatorname{ht} \mathfrak{p} = \dim A_{\mathfrak{q}} - \dim B_{\overline{\mathfrak{q}}} = \dim' A - \dim' B$.*

Proof. (1) Let $\mathfrak{p}$ be the image of P by the map $\operatorname{Spec} S \to \operatorname{Spec} R$. Then by [Mat89, Theorem 15.5] (and the definition below in [Mat89]), it follows that

$$\operatorname{trdeg}_{\operatorname{Frac} R}(\operatorname{Frac} S) - \operatorname{ht} P = \operatorname{trdeg}_{k(\mathfrak{p})}(\operatorname{Frac}(S/P)) - \operatorname{ht} \mathfrak{p}\,.$$

Therefore, the assertion follows from $\operatorname{trdeg}_{\operatorname{Frac} R}(\operatorname{Frac} S) = m = \dim S - \dim R$.

(2) This follows from item (1) and the inequality $\dim(S/P) + \operatorname{ht} P \leqslant \dim S$. Item (3) follows from item (1) and Remark 2.9(1).

(4) We prove this for $R_0 = k[t]$ (the case when $R_0 = k$ is similar). We may assume that A is a domain. Take S and P as in item (1). Then it is sufficient to prove

$$\dim S - \operatorname{ht} P = \dim(S/P)\,.$$

Let M be the maximal ideal of S corresponding to $\mathfrak{m}$. Since $S/M = k$, the ideal M is of the form

$$M = (t - a, x_1, \ldots, x_N, y_1 - b_1, \ldots, y_m - b_m)$$

with $a, b_i \in k$. Therefore, we have $\dim S = N + m + 1 = \operatorname{ht} M$. Since S is a catenary ring, we also have $\dim(S/P) = \operatorname{ht} M - \operatorname{ht} P$, which proves $\dim S - \operatorname{ht} P = \dim(S/P)$.

(5) This follows from item (1) and Krull's height theorem. Item (6) also follows from item (1). □

Proposition 2.12. *Suppose $R_0 = k$. Let A be a domain of finite type over R, and let $\mathfrak{q}_1$ and $\mathfrak{q}_2$ be prime ideals of A such that $\mathfrak{q}_1 \subset \mathfrak{q}_2$. Let $B := A/\mathfrak{q}_1$ and $\overline{\mathfrak{q}}_2 := \mathfrak{q}_2/\mathfrak{q}_1 \in \operatorname{Spec} B$. If both $A_{\mathfrak{q}_2}$ and $B_{\overline{\mathfrak{q}}_2}$ are regular, then the sequence*

$$0 \longrightarrow \mathfrak{q}_1 A_{\mathfrak{q}_2}/\mathfrak{q}_1^2 A_{\mathfrak{q}_2} \longrightarrow \Omega'_{A_{\mathfrak{q}_2}/k} \otimes_{A_{\mathfrak{q}_2}} B_{\overline{\mathfrak{q}}_2} \longrightarrow \Omega'_{B_{\overline{\mathfrak{q}}_2}/k} \longrightarrow 0$$

induced by Proposition 2.8(3) is exact and splits.

Proof. By Proposition 2.8(3), the sequence

$$\mathfrak{q}_1 A_{\mathfrak{q}_2}/\mathfrak{q}_1^2 A_{\mathfrak{q}_2} \longrightarrow \Omega'_{A_{\mathfrak{q}_2}/k} \otimes_{A_{\mathfrak{q}_2}} B_{\overline{\mathfrak{q}}_2} \xrightarrow{\delta} \Omega'_{B_{\overline{\mathfrak{q}}_2}/k} \longrightarrow 0$$

is exact. Since $A_{\mathfrak{q}_2}$ and $B_{\overline{\mathfrak{q}}_2}$ are regular, it follows from Proposition 2.7 that

$$\Omega'_{A_{\mathfrak{q}_2}/k} \otimes_{A_{\mathfrak{q}_2}} B_{\overline{\mathfrak{q}}_2} \simeq B_{\overline{\mathfrak{q}}_2}^{\oplus \dim' A}, \quad \Omega'_{B_{\overline{\mathfrak{q}}_2}/k} \simeq B_{\overline{\mathfrak{q}}_2}^{\oplus \dim' B}.$$

Therefore, $\mathrm{Ker}(\delta)$ is a free $B_{\overline{\mathfrak{q}}_2}$-module of rank equal to $\dim' A - \dim' B$. On the other hand, since $A_{\mathfrak{q}_2}$ and $B_{\overline{\mathfrak{q}}_2}$ are regular, $\mathfrak{q}_1 A_{\mathfrak{q}_2}/\mathfrak{q}_1^2 A_{\mathfrak{q}_2}$ is also a free $B_{\overline{\mathfrak{q}}_2}$-module of rank equal to $\mathrm{ht}(\mathfrak{q}_1)$, which is equal to $\dim' A - \dim' B$ by Lemma 2.11(6). Hence, the induced surjective map $\mathfrak{q}_1 A_{\mathfrak{q}_2}/\mathfrak{q}_1^2 A_{\mathfrak{q}_2} \to \mathrm{Ker}(\delta)$ is an isomorphism. This completes the proof. □

DEFINITION 2.13. (1) Let X be a scheme over $\operatorname{Spec} R$. Then due to Proposition 2.4(4), there exists a quasi-coherent sheaf Ω'_{X/R_0} satisfying $\Omega'_{X/R_0}(U) \simeq \Omega'_{\mathcal{O}_X(U)/R_0}$ for any affine open subset $U \subset X$. Note that Ω'_{X/R_0} is coherent by Proposition 2.4(1),(3) when X is of finite type over $\operatorname{Spec} R$. The sheaf Ω'_{X/R_0} is called the *sheaf of special differentials*. We write ${\Omega'}^n_{X/R_0} := \bigwedge^n \Omega'_{X/R_0}$ for a non-negative integer n.

(2) Suppose $R_0 = k$ and that X is a scheme of finite type over $\operatorname{Spec} R$. Let n be a non-negative integer. Suppose that any irreducible component X_i of X satisfies $\dim' X_i = n$, where $\dim' X$ is defined in Definition 2.10. Then we write $\mathrm{Jac}'_{X/k} := \mathrm{Fitt}^n(\Omega'_{X/k})$; it is called the *special Jacobian ideal* of X (see [Eis95, Section 20.2] for the definition of the Fitting ideal). We note that $\dim' X_i = \dim X_i$ if X_i contains a k-point, by Lemma 2.11(4).

(3) Suppose $R_0 = k$ and that X is an integral normal scheme of finite type over $\operatorname{Spec} R$. Let $n = \dim' X$, and let $i\colon X_{\mathrm{reg}} \to X$ be the inclusion map from the regular locus X_{reg} of X. Then the *special canonical sheaf* $\omega'_{X/k}$ is defined by $\omega'_{X/k} = i_*\big({\Omega'}^n_{X_{\mathrm{reg}}/k}\big)$.

(4) Under the same setting as in part (3), a Weil divisor K_X on X satisfying $\mathcal{O}_X(K_X)|_{X_{\mathrm{reg}}} \simeq {\Omega'}^n_{X_{\mathrm{reg}}/k}$ is called the *canonical divisor* on X. The canonical divisor K_X is defined up to linear equivalence. Note that $\omega'_{X/k} \simeq \mathcal{O}_X(K_X)$ as usual. In fact, since $\mathrm{codim}_X(X \setminus X_{\mathrm{reg}}) \geqslant 2$ by the normality of X, it follows that

$$\Gamma\big(V, \mathcal{O}_X(K_X)\big) = \Gamma\big(V \cap X_{\mathrm{reg}}, \mathcal{O}_X(K_X)\big) = \Gamma\big(V, i_*i^*\mathcal{O}_X(K_X)\big)$$

for any open subset $V \subset X$.

(5) Under the same setting as in part (3), we say that X is $\mathbb{Q}$-*Gorenstein* if ${\omega'}^{[r]}_{X/k} := \big({\omega'}^{\otimes r}_{X/k}\big)^{**} \simeq \mathcal{O}_X(rK_X)$ is an invertible sheaf for some $r \in \mathbb{Z}_{>0}$. In this case, we have a canonical map

$$\eta_r\colon \big({\Omega'}^n_{X/k}\big)^{\otimes r} \longrightarrow {\omega'}^{[r]}_{X/k}.$$

Since ${\omega'}^{[r]}_{X/k}$ is an invertible sheaf, an ideal sheaf $\mathfrak{n}_{r,X} \subset \mathcal{O}_X$ is uniquely determined by the equality $\mathrm{Im}(\eta_r) = \mathfrak{n}_{r,X} \otimes_{\mathcal{O}_X} {\omega'}^{[r]}_{X/k}$. The ideal sheaf $\mathfrak{n}_{r,X}$ is called the rth *Nash ideal* of X.

Remark 2.14. As with the usual Jacobian ideals for varieties over k, the special Jacobian ideal can be locally described by the Jacobian matrix. Let $S = R[y_1, \dots, y_m]$ with $R = R_0[[x_1, \dots, x_N]]$, and let $A = S/I$ for some ideal $I = (f_1, \dots, f_r) \subset S$. Then by Proposition 2.4(3), we have an exact sequence

$$I/I^2 \longrightarrow \Omega'_{S/R_0} \otimes_S A \longrightarrow \Omega'_{A/R_0} \longrightarrow 0.$$

Here, we have $\Omega'_{S/R_0} \simeq S^{\oplus N+m}$ with basis the $\mathrm{d}'x_i$ and $\mathrm{d}'y_j$. Furthermore, for $f \in I$, we have

$$\mathrm{d}'f = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}\mathrm{d}'x_i + \sum_{j=1}^{m} \frac{\partial f}{\partial y_j}\mathrm{d}'y_j\,.$$

Therefore, we have

$$\mathrm{Fitt}^n\bigl(\Omega'_{A/R_0}\bigr) = \bigl(\mathcal{J}_{N+m-n}(I;\mathrm{Der}_{R_0}(S)) + I\bigr)/I\,.$$

Note that $\mathrm{Der}_{R_0}(S) = \mathrm{Der}'_{R_0}(S)$ is a free S-module generated by the $\partial/\partial x_i$ and $\partial/\partial y_j$.

This observation shows that if $R_0 = k$ and X is an integral scheme of finite type over R, then $\mathrm{Jac}'_{X/k}$ defines the singular locus of X by Remark 2.6(2)(c). Note that $\mathrm{ht}\, I = N + m - \dim' X$ by Lemma 2.11(1).

3. Log pairs

A *log pair* $(X, \mathfrak{a})$ consists of a normal $\mathbb{Q}$-Gorenstein k-variety X and an $\mathbb{R}$-ideal sheaf $\mathfrak{a}$ on X. Here, an $\mathbb{R}$-*ideal sheaf* $\mathfrak{a}$ on X is a formal product $\mathfrak{a} = \prod_{i=1}^{s} \mathfrak{a}_i^{r_i}$, where $\mathfrak{a}_1, \ldots, \mathfrak{a}_s$ are non-zero coherent ideal sheaves on X and $r_1, \ldots, r_s$ are positive real numbers. For a morphism $Y \to X$ and an $\mathbb{R}$-ideal sheaf $\mathfrak{a} = \prod_{i=1}^{s} \mathfrak{a}_i^{r_i}$ on X, we denote by $\mathfrak{a}\mathcal{O}_Y$ the $\mathbb{R}$-ideal sheaf $\prod_{i=1}^{s}(\mathfrak{a}_i\mathcal{O}_Y)^{r_i}$ on Y.

Let $\bigl(X, \mathfrak{a} = \prod_{i=1}^{s} \mathfrak{a}_i^{r_i}\bigr)$ be a log pair. Let $f\colon X' \to X$ be a proper birational morphism from a normal variety X', and let E be a prime divisor on X'. We denote by $K_{X'/X} := K_{X'} - f^*K_X$ the relative canonical divisor. Then the *log discrepancy* of $(X, \mathfrak{a})$ at E is defined as

$$a_E(X, \mathfrak{a}) := 1 + \mathrm{ord}_E(K_{X'/X}) - \mathrm{ord}_E \mathfrak{a}\,,$$

where we set $\mathrm{ord}_E \mathfrak{a} := \sum_{i=1}^{s} r_i \,\mathrm{ord}_E \mathfrak{a}_i$. The image $f(E)$ is called the *center* of E on X; we denote it by $c_X(E)$. For a closed point $x \in X$, we define *the minimal log discrepancy* at x as

$$\mathrm{mld}_x(X, \mathfrak{a}) := \inf_{c_X(E)=\{x\}} a_E(X, \mathfrak{a})$$

if $\dim X \geqslant 2$, where the infimum is taken over all prime divisors E over X with center $c_X(E) = \{x\}$. It is known that $\mathrm{mld}_x(X, \mathfrak{a}) \in \mathbb{R}_{\geqslant 0} \cup \{-\infty\}$ in this case (cf. [KM98, Corollary 2.31]). When $\dim X = 1$, we define $\mathrm{mld}_x(X, \mathfrak{a}) := \inf_{c_X(E)=\{x\}} a_E(X, \mathfrak{a})$ if the infimum is non-negative, and we define $\mathrm{mld}_x(X, \mathfrak{a}) := -\infty$ otherwise.

Let $R = k[[x_1, \ldots, x_N]]$. By [dFEM11, Appendix A], we can extend the definition above to normal R-schemes of finite type. Let X be an integral normal scheme of finite type over R. Then the canonical divisor K_X is defined in Definition 2.13(4). Suppose that X is $\mathbb{Q}$-Gorenstein, that is, rK_X is Cartier for some positive integer r. Let $f\colon Y \to X$ be a proper birational morphism over R from a regular scheme Y. Then the *relative canonical divisor* $K_{Y/X}$ of f is defined as the $\mathbb{Q}$-divisor supported on the exceptional locus of f such that $rK_Y - f^*(rK_X)$ and $rK_{Y/X}$ are linearly equivalent. We note that $K_{Y/X}$ is uniquely defined as a $\mathbb{Q}$-divisor (cf. [dFEM11, Lemma A.11(ii)]). Therefore, the log discrepancies and the minimal log discrepancies for k-varieties defined above can be extended to $\mathbb{Q}$-Gorenstein normal schemes of finite type over R, and we use the same notation.

Remark 3.1. Let X be a normal k-variety, and let $x \in X$ be a closed point. Let $\widehat{\mathcal{O}}_{X,x}$ be the completion of the local ring $\mathcal{O}_{X,x}$ at its maximal ideal. Let $\widehat{X} := \mathrm{Spec}\bigl(\widehat{\mathcal{O}}_{X,x}\bigr)$, and let $\widehat{x} \in \widehat{X}$ be the closed point. Then for the induced flat morphism $f\colon \widehat{X} \to X$, it follows that

$$f^*(K_X) = K_{\widehat{X}}\,, \quad f^*(\omega_X) = \omega'_{\widehat{X}/k}$$

by [dFEM11, Proposition A.14]. Furthermore, for an integer r, if rK_X is Cartier, then so is $rK_{\widehat{X}}$.

Suppose further that X is $\mathbb{Q}$-Gorenstein. Let $\mathfrak{a}$ be an $\mathbb{R}$-ideal sheaf on X. Then it follows from [Kaw21, Remark 2.6] (cf. [dFEM11, Proposition 2.11]) that

$$\mathrm{mld}_{\widehat{x}}\big(\widehat{X}, \widehat{\mathfrak{a}}\big) = \mathrm{mld}_x(X, \mathfrak{a})\,,$$

where $\widehat{\mathfrak{a}} := \mathfrak{a}\mathcal{O}_{\widehat{X}}$.

4. Arc spaces of $k[[x_1, \dots, x_N]]$-schemes

In this section, we suppose $R_0 = k$ and $R = k[[x_1, \dots, x_N]]$, and we discuss the jet schemes and the arc spaces of R-schemes of finite type. We refer the reader to [EM09] and [CNS18] for the theory of jet schemes and arc spaces of k-varieties. In this section, we see that the codimensions of cylinders of arc spaces can be defined in the same way as with k-varieties.

Let X be a scheme over k. Let (Sch/k) be the category of k-schemes and (Sets) the category of sets. For a non-negative integer m, we define a contravariant functor $F^X_m \colon (\mathsf{Sch}/k) \to (\mathsf{Sets})$ by

$$F^X_m(Y) = \mathrm{Hom}_k\big(Y \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[t]/\big(t^{m+1}\big), X\big)\,.$$

It is known that the functor F^X_m is always represented by a scheme X_m over k (cf. [CNS18, Chapter 3, Proposition 2.1.3]).

For $m \geqslant n \geqslant 0$, the canonical surjective ring homomorphism $k[t]/\big(t^{m+1}\big) \to k[t]/\big(t^{n+1}\big)$ induces a morphism $\pi^X_{mn} \colon X_m \to X_n$, which is called the *truncation morphism*. We have the projective limit and projections $X_\infty := \varprojlim_m X_m$ and $\psi^X_m \colon X_\infty \to X_m$; the projective limit X_∞ is called the *arc space* of X. There is a bijective map

$$\mathrm{Hom}_k(\mathrm{Spec}\, K, X_\infty) \simeq \mathrm{Hom}_k(\mathrm{Spec}\, K[[t]], X)$$

for any field K with $k \subset K$. For $m \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$, we denote by $\pi^X_m \colon X_m \to X$ the canonical truncation morphism. For $m \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ and a morphism $f \colon Y \to X$ of schemes over k, we denote by $f_m \colon Y_m \to X_m$ the morphism induced by f. We often abbreviate π^X_{mn}, π^X_m and ψ^X_m to π_{mn}, π_m and ψ_m, respectively, when no confusion can arise.

If X is a scheme of finite type over k, then so is X_m (cf. [EM09, Proposition 2.2]). In this paper, we deal with a scheme of finite type over $R = k[[x_1, \dots, x_N]]$.

PROPOSITION 4.1 (cf. [Ish09, Corollary 4.2]). *Let X be a scheme of finite type over the ring $R = k[[x_1, \dots, x_N]]$. Then the following hold:*

(1) *The scheme X_m is of finite type over R.*
(2) *For any $m \geqslant n \geqslant 0$, the truncation map $\pi_{mn} \colon X_m \to X_n$ is a morphism of finite type.*

Proof. We omit the proof because we will give a complete proof for Proposition 5.4, which deals with a more complicated case. See also Remark 4.2 below. □

Remark 4.2. The same arguments as in Lemmas 5.1 and 5.2 give a local description of X_m as follows:

(1) Let $S := k[[x_1, \dots, x_N]][y_1 \dots, y_M]$ and $A := \mathrm{Spec}\, S$. Then we have $A_m \simeq \mathrm{Spec}\, S_m$, where

$$S_m := k\big[\big[x^{(0)}_1, \dots, x^{(0)}_N\big]\big]\big[x^{(1)}_j, \dots, x^{(m)}_j, y^{(0)}_{j'}, \dots, y^{(m)}_{j'} \mid 1 \leqslant j \leqslant N,\, 1 \leqslant j' \leqslant M\big]\,.$$

Furthermore, for $m \geqslant n \geqslant 0$, the truncation map $\pi_{mn} \colon A_m \to A_n$ is induced by the ring inclusion $S_n \hookrightarrow S_m$.

(2) Let $X = \operatorname{Spec}(S/I)$ be the closed subscheme of A defined by an ideal $I = (f_1, \dots, f_r) \subset S$. For $1 \leqslant i \leqslant r$ and $0 \leqslant \ell \leqslant m$, we define $F_i^{(\ell)} \in S_m$ as follows:

$$f_i\Big(\sum_{\ell=0}^{m} x_1^{(\ell)} t^\ell, \dots, \sum_{\ell=0}^{m} x_N^{(\ell)} t^\ell, \sum_{\ell=0}^{m} y_1^{(\ell)} t^\ell, \dots, \sum_{\ell=0}^{m} y_M^{(\ell)} t^\ell\Big) \equiv \sum_{\ell=0}^{m} F_i^{(\ell)} t^\ell \pmod{t^{m+1}}.$$

Let

$$I_m := \big(F_i^{(s)} \mid 1 \leqslant i \leqslant r,\ 0 \leqslant s \leqslant m\big) \subset S_m$$

be the ideal of S_m generated by the $F_i^{(s)}$. Then we have $X_m \simeq \operatorname{Spec}(S_m/I_m)$. Furthermore, for $m \geqslant n \geqslant 0$, the truncation map $\pi_{mn} \colon X_m \to X_n$ is induced by the ring homomorphism $S_n/I_n \to S_m/I_m$.

A subset $C \subset X_\infty$ is called a *cylinder* if $C = \psi_m^{-1}(S)$ for some $m \geqslant 0$ and a constructible subset $S \subset X_m$. Typical examples of cylinders appearing in this paper are the *contact loci* $\operatorname{Cont}^m(\mathfrak{a})$ and $\operatorname{Cont}^{\geqslant m}(\mathfrak{a})$ defined below.

Definition 4.3. (1) For an arc $\gamma \in X_\infty$ and an ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$, the *order* of $\mathfrak{a}$ measured by γ is defined as follows:

$$\operatorname{ord}_\gamma(\mathfrak{a}) := \sup\{r \in \mathbb{Z}_{\geqslant 0} \mid \gamma^*(\mathfrak{a}) \subset (t^r)\},$$

where $\gamma^* \colon \mathcal{O}_X \to K[[t]]$ is the ring homomorphism induced by γ. Here K is a field extension of k.

(2) For $m \in \mathbb{Z}_{\geqslant 0}$, we define $\operatorname{Cont}^m(\mathfrak{a}), \operatorname{Cont}^{\geqslant m}(\mathfrak{a}) \subset X_\infty$ as follows:

$$\operatorname{Cont}^m(\mathfrak{a}) := \{\gamma \in X_\infty \mid \operatorname{ord}_\gamma(\mathfrak{a}) = m\},$$
$$\operatorname{Cont}^{\geqslant m}(\mathfrak{a}) := \{\gamma \in X_\infty \mid \operatorname{ord}_\gamma(\mathfrak{a}) \geqslant m\}.$$

By definition, we have

$$\operatorname{Cont}^{\geqslant m}(\mathfrak{a}) = \psi_{m-1}^{-1}(Z(\mathfrak{a})_{m-1}),$$

where $Z(\mathfrak{a})$ is the closed subscheme of X defined by the ideal sheaf $\mathfrak{a}$. Therefore, $\operatorname{Cont}^m(\mathfrak{a})$ and $\operatorname{Cont}^{\geqslant m}(\mathfrak{a})$ are cylinders.

For $m \leqslant n+1$, we define the subsets $\operatorname{Cont}^m(\mathfrak{a})_n$ and $\operatorname{Cont}^{\geqslant m}(\mathfrak{a})_n$ of X_n in the same way.

We denote by $\mathfrak{o}_X \subset \mathcal{O}_X$ the ideal sheaf

$$\mathfrak{o}_X := (x_1, \dots, x_N)\mathcal{O}_X \subset \mathcal{O}_X$$

generated by $x_1, \dots, x_N \in R$. In this paper, we are interested in arcs contained in the contact locus $\operatorname{Cont}^{\geqslant 1}(\mathfrak{o}_X)$. Due to the following lemma, the contact locus $\operatorname{Cont}^{\geqslant 1}(\mathfrak{o}_X)_m$ is a scheme of finite type over k.

Lemma 4.4. *Let X be a scheme of finite type over $R = k[[x_1, \dots, x_N]]$. Then for each $m \geqslant 0$, the contact locus $\operatorname{Cont}^{\geqslant 1}(\mathfrak{o}_X)_m \subset X_m$ is a scheme of finite type over k.*

Proof. The assertion follows from Proposition 4.1(2). □

For the proof of Lemma 4.7, we state Hensel's lemma in several variables.

Lemma 4.5. *Let K be a field. Let N, s and r be non-negative integers with $N + s \geqslant r$. Let $f_1, \dots, f_r \in K[[t]][[x_1, \dots, x_N]][x_{N+1}, \dots, x_{N+s}]$, and let $a_1, \dots, a_{N+s} \in K[[t]]$. Let $\mathcal{S} \subset \{1, \dots, N+s\}$ be a subset with cardinality $\#\mathcal{S} = r$. Let m and e be non-negative integers with $m \geqslant e$. Suppose that*

- $a_1, \dots, a_N \in (t)$,
- $f_i(a_1, \dots, a_{N+s}) \in \left(t^{m+e+1}\right)$ *for each* $1 \leqslant i \leqslant r$ *and*
- $\det\left(\partial f_i/\partial x_j(a_1, \dots, a_{N+s})\right)_{1\leqslant i\leqslant r,\ j\in\mathcal{S}} \notin \left(t^{e+1}\right)$.

Then the following hold:

(1) *There exist* $b_1, \dots, b_{N+s} \in K[[t]]$ *such that*
   - $f_i(b_1, \dots, b_{N+s}) = 0$ *for each* $1 \leqslant i \leqslant r$ *and*
   - $a_j - b_j \in \left(t^{m+1}\right)$ *for each* $1 \leqslant j \leqslant N+s$.

   Furthermore, for $b_1, \dots, b_{N+s} \in K[[t]]$ *and* $b'_1, \dots, b'_{N+s} \in K[[t]]$ *with the above two conditions, if*
   - $b_j - b'_j \in \left(t^{m+2}\right)$ *holds for each* $j \in \{1, \dots, N+s\} \setminus \mathcal{S}$,

   then
   - $b_j - b'_j \in \left(t^{m+2}\right)$ *also holds for each* $j \in \mathcal{S}$.

(2) *Moreover, for any sequence* $\left(a'_j \in K[[t]] \mid j \in \{1, \dots, N+s\}\setminus\mathcal{S}\right)$ *satisfying* $a'_j - a_j \in \left(t^{m+1}\right)$, *there exist* $b_1, \dots, b_{N+s} \in K[[t]]$ *satisfying the following conditions:*
   - $f_i(b_1, \dots, b_{N+s}) = 0$ *for each* $1 \leqslant i \leqslant r$,
   - $a'_j - b_j \in \left(t^{m+2}\right)$ *for each* $j \in \{1, \dots, N+s\} \setminus \mathcal{S}$ *and*
   - $a_j - b_j \in \left(t^{m+2}\right)$ *for each* $j \in \mathcal{S}$.

Proof. When $N = 0$ and $f_1, \dots, f_r \in K[x_1, \dots, x_s]$, the assertions follow from the proof of [DL99, Lemma 4.1] (cf. [EM09, Proposition 4.1]). The same proof works in our setting. $\square$

Remark 4.6. The same statement as in Lemma 4.5 holds even when we replace the ring $K[[t]][[x_1, \dots, x_N]][x_{N+1}, \dots, x_{N+s}]$ with $K[x_{N+1}, \dots, x_{N+s}][[t]]$. This version will be used in the proof of Proposition 5.31.

Lemma 4.7. *Let* N, s *and* r *be non-negative integers with* $N+s \geqslant r$. *Let* $R = k[[x_1, \dots, x_N]]$, *and let* $S = R[y_1, \dots, y_s]$. *Let* $I = (F_1, \dots, F_r)$ *be the ideal generated by elements* $F_1, \dots, F_r \in S$, *and let* $M = \operatorname{Spec}(S/I)$. *Let* $\mathfrak{o}_M \subset \mathcal{O}_M$ *be the ideal sheaf generated by* $x_1, \dots, x_N \in R$. *Let* $\overline{J} = \operatorname{Fitt}^{N+s-r}(\Omega'_{M/k})$. *Then, for non-negative integers* m *and* e *with* $m \geqslant e$, *the following hold:*

(1) *We have*

$$\psi_m\left(\operatorname{Cont}^e(\overline{J}) \cap \operatorname{Cont}^{\geqslant 1}(\mathfrak{o}_M)\right) = \pi_{m+e,m}\left(\operatorname{Cont}^e(\overline{J})_{m+e} \cap \operatorname{Cont}^{\geqslant 1}(\mathfrak{o}_M)_{m+e}\right).$$

(2) *The truncation map* $\pi_{m+1,m} \colon M_{m+1} \to M_m$ *induces a piecewise-trivial fibration*

$$\psi_{m+1}\left(\operatorname{Cont}^e(\overline{J}) \cap \operatorname{Cont}^{\geqslant 1}(\mathfrak{o}_M)\right) \longrightarrow \psi_m\left(\operatorname{Cont}^e(\overline{J}) \cap \operatorname{Cont}^{\geqslant 1}(\mathfrak{o}_M)\right)$$

   with fiber $\mathbb{A}^{N+s-r}$.

Proof. Let $J := \mathcal{J}_r\left(I; \operatorname{Der}_k(S)\right) \subset S$. Then we have $\overline{J} = (J+I)/I$ by Remark 2.14. Note that $\operatorname{Der}_k(S) = \operatorname{Der}'_k(S)$ is generated by the $\partial/\partial x_i$ and $\partial/\partial y_j$. Therefore, item (1) follows from the first assertion of Lemma 4.5(1). Furthermore, item (2) follows from Lemma 4.5(2) and the second assertion of Lemma 4.5(1). $\square$

The following proposition is a formal power series ring version of [EM09, Proposition 4.1].

Proposition 4.8. *Let* X *be an integral scheme of finite type over* $R = k[[x_1, \dots, x_N]]$ *with* $\dim X = n$. *Then there exists a positive integer* c *such that the following hold for non-negative integers* m *and* e *with* $m \geqslant ce$*:*

(1) *We have*

$$\psi_m\left(\mathrm{Cont}^e\left(\mathrm{Jac}'_{X/k}\right) \cap \mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_X)\right) = \pi_{m+e,m}\left(\mathrm{Cont}^e\left(\mathrm{Jac}'_{X/k}\right)_{m+e} \cap \mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_X)_{m+e}\right).$$

(2) *The truncation map $\pi_{m+1,m}\colon X_{m+1} \to X_m$ induces a piecewise-trivial fibration*

$$\psi_{m+1}\left(\mathrm{Cont}^e\left(\mathrm{Jac}'_{X/k}\right) \cap \mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_X)\right) \longrightarrow \psi_m\left(\mathrm{Cont}^e\left(\mathrm{Jac}'_{X/k}\right) \cap \mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_X)\right)$$

with fiber $\mathbb{A}^n$.

Proof. We omit the proof. See the proof of Proposition 5.9 for how it can be reduced to the complete intersection case proved in Lemma 4.7. Note that we may assume that X has a k-point, and hence we have $\dim' X = \dim X = n$ by Lemma 2.11(4). Otherwise, we have $\mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_X) = \emptyset$ (cf. Lemma 4.4), and the assertions are clear. □

Remark 4.9. Proposition 4.8 is a formal power series ring version of [EM09, Proposition 4.1]. Note that in [EM09, Proposition 4.1], they prove that $c = 1$ satisfies the statement. However, the weaker statement as in Proposition 4.8 using c is enough for our later use.

Definition 4.10. Let X be an integral scheme of finite type over $R = k[[x_1, \dots, x_N]]$, and let $C \subset X_\infty$ be a cylinder contained in $\mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_X)$.

(1) Assume $C \subset \mathrm{Cont}^e\left(\mathrm{Jac}'_{X/k}\right)$ for some $e \in \mathbb{Z}_{\geqslant 0}$. Then we define the codimension of C in X_∞ as

$$\mathrm{codim}(C) := (m+1)\dim X - \dim(\psi_m(C))$$

for any sufficiently large m. This is well defined by Proposition 4.8.

(2) In general, we define the codimension of C in X_∞ as

$$\mathrm{codim}(C) := \min_{e \in \mathbb{Z}_{\geqslant 0}} \mathrm{codim}\left(C \cap \mathrm{Cont}^e\left(\mathrm{Jac}'_{X/k}\right)\right).$$

By convention, $\mathrm{codim}(C) = \infty$ if $C \cap \mathrm{Cont}^e\left(\mathrm{Jac}'_{X/k}\right) = \emptyset$ for any $e \geqslant 0$.

The following theorem is a formal power series ring version of [EM09, Theorem 7.4].

Theorem 4.11. *Let X be a $\mathbb{Q}$-Gorenstein integral normal scheme of finite type over $R = k[[x_1, \dots, x_N]]$. Let x be a k-point of X, and let $\mathfrak{m}_x \subset \mathcal{O}_X$ be the corresponding maximal ideal sheaf. Let r be a positive integer such that rK_X is Cartier. Let $\mathfrak{a}$ be a non-zero ideal sheaf on X and δ a positive real number. Then we have*

$$\begin{aligned} \mathrm{mld}_x\left(X, \mathfrak{a}^\delta\right) &= \inf_{w,m \in \mathbb{Z}_{\geqslant 0}} \left\{\mathrm{codim}(C_{w,m}) - \frac{m}{r} - \delta w\right\} \\ &= \inf_{w,m \in \mathbb{Z}_{\geqslant 0}} \left\{\mathrm{codim}(C'_{w,m}) - \frac{m}{r} - \delta w\right\}, \end{aligned}$$

where

$$\begin{aligned} C_{w,m} &:= \mathrm{Cont}^w(\mathfrak{a}) \cap \mathrm{Cont}^m(\mathfrak{n}_{r,X}) \cap \mathrm{Cont}^{\geqslant 1}(\mathfrak{m}_x), \\ C'_{w,m} &:= \mathrm{Cont}^{\geqslant w}(\mathfrak{a}) \cap \mathrm{Cont}^m(\mathfrak{n}_{r,X}) \cap \mathrm{Cont}^{\geqslant 1}(\mathfrak{m}_x). \end{aligned}$$

Proof. The assertions are formal power series ring versions of [EM09, Theorem 7.4 and Remark 7.5], and their proofs also work in this setting by making the following modifications:

- Replace $\Omega_{-/k}$ with $\Omega'_{-/k}$ and $\mathrm{Jac}_{-/k}$ with $\mathrm{Jac}'_{-/k}$,
- Consider cylinders C contained in $\mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_X)$.

Theorem 7.4 in [EM09] is a consequence of [EM09, Lemma 7.3]. The key ingredients of the proof of that lemma are

- Theorem 6.2 and Corollary 6.4 in [EM09] and
- Proposition 5.11 in [EM09].

Theorem 6.2 and Corollary 6.4 in [EM09] give the codimension formula as in Proposition 5.43. They are formal consequences of [EM09, Proposition 4.4(i)], which is still valid in our setting when we replace Ω_X with $\Omega'_{X/k}$, due to Lemma 2.2 (see Lemma 5.13 for the detailed argument).

Proposition 5.11 in [EM09] is a proposition on codimensions like Proposition 5.36. It is a consequence of [EM09, Lemma 5.1 and Corollary 5.2] (Corollary 5.2 is a corollary of Lemma 5.1). The proof of [EM09, Lemma 5.1] still works in our setting when we replace $\mathrm{Jac}_{-/k}$ with $\mathrm{Jac}'_{-/k}$. The only important point is that the ideal $\mathrm{Jac}'_{X/k}$ defines the singular locus of X even when X is an integral scheme of finite type over R (cf. Remark 2.14).

Besides, [EM09, Lemma 6.1] is used in the proof of [EM09, Lemma 7.3], and [EM09, Proposition 3.2] is used in the proof of [EM09, Corollary 5.2]. Proposition 3.2 and Lemma 6.1 in [EM09] are formal consequences of the valuative criterion of properness, and their proofs work in our setting. □

5. Arc spaces of $k[t]$-schemes

In this section, we deal with the arc spaces of $k[t]$-schemes. Let X be a scheme over $k[t]$. For a non-negative integer m, we define a contravariant functor $F_m^X\colon (\mathsf{Sch}/k) \to (\mathsf{Sets})$ by

$$F_m^X(Y) = \mathrm{Hom}_{k[t]}\bigl(Y \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[t]/\bigl(t^{m+1}\bigr), X\bigr)\,.$$

By the same argument as in [CNS18, Chapter 4, Theorem 3.2.3], we can see that the functor F_m^X is always represented by a scheme X_m over k. We use the same symbols X_∞, π_{mn}, ψ_m and π_m as in Section 4 for this setting.

In this section, we deal with the following two categories of $k[t]$-schemes:

(1) The scheme X is of finite type over $k[t][[x_1, \dots, x_N]]$.

(2) The scheme X is affine, of the form $X = \mathrm{Spec}\bigl(k[x_1, \dots, x_N][[t]]/I\bigr)$.

We note that in [DL02], Denef and Loeser extend the theory of arc spaces of k-varieties to the case where

(3) the scheme X is of finite type over $k[t]$.

In [NS22], we dealt with the arc spaces of X in case (3). However, in this paper, we need to work on the arc spaces of X in cases (1) and (2). We also note that Sebag in [Seb04] deals with formal $k[[t]]$-schemes of finite type, and this theory can be applied to cases (2) and (3) (see also [CNS18] for this theory).

In Section 5.1, we discuss case (1). In Section 5.2, we discuss case (2), where we will not deal with formal $k[[t]]$-schemes in general but only deal with affine schemes in a minimal way.

5.1 Arc spaces of $k[t][[x_1, \dots, x_N]]$-schemes

In this subsection, we suppose $R_0 = k[t]$ and $R = k[t][[x_1, \dots, x_N]]$, and we discuss the arc spaces of R-schemes of finite type.

5.1.1 *Arc spaces.* First, we prove that if X is a scheme of finite type over R, then so is X_m.

Lemma 5.1. *Let $S := k[t][[x_1, \dots, x_N]][y_1, \dots, y_M]$, and let $A := \operatorname{Spec} S$. Then we have $A_m \simeq \operatorname{Spec} S_m$, where*

$$S_m := k\big[\big[x_1^{(0)}, \dots, x_N^{(0)}\big]\big]\big[x_j^{(1)}, \dots, x_j^{(m)}, y_{j'}^{(0)}, \dots, y_{j'}^{(m)} \mid 1 \leqslant j \leqslant N,\, 1 \leqslant j' \leqslant M\big]\,.$$

Furthermore, for $m \geqslant n \geqslant 0$, the truncation map $\pi_{mn} \colon A_m \to A_n$ is induced by the ring inclusion $S_n \hookrightarrow S_m$.

Proof. Let $Y := \operatorname{Spec} C$ be an affine scheme over k. We will give a natural bijective map

$$\Phi \colon \operatorname{Hom}_{k[t]}\big(S, C[t]/\big(t^{m+1}\big)\big) \longrightarrow \operatorname{Hom}_k(S_m, C)\,.$$

For each $0 \leqslant i \leqslant m$, we denote by p_i the projection

$$p_i \colon C[t]/\big(t^{m+1}\big) \longrightarrow C\,, \quad c_0 + c_1 t + \dots + c_m t^m \longmapsto c_i\,.$$

For $\alpha \in \operatorname{Hom}_{k[t]}\big(S, C[t]/\big(t^{m+1}\big)\big)$, we define $\Phi(\alpha) \in \operatorname{Hom}_k(S_m, C)$ as follows. First, we define the ring homomorphism $\alpha'_0 \colon S_0 \to C$ as the composition

$$S_0 \hookrightarrow S \xrightarrow{\alpha} C[t]/\big(t^{m+1}\big) \xrightarrow{p_0} C\,.$$

Then we define $\alpha' \colon S_m \to C$ as the ring homomorphism uniquely determined by the following conditions:

- $\alpha'(f) = \alpha'_0(f)$ for any $f \in S_0$,
- $\alpha'\big(x_j^{(s)}\big) = p_s\big(\alpha(x_j)\big)$ for all $1 \leqslant j \leqslant N$ and $1 \leqslant s \leqslant m$,
- $\alpha'\big(y_j^{(s)}\big) = p_s\big(\alpha(y_j)\big)$ for all $1 \leqslant j \leqslant M$ and $1 \leqslant s \leqslant m$.

Then we set $\Phi(\alpha) = \alpha'$.

Next, we define the inverse map

$$\Psi \colon \operatorname{Hom}_k(S_m, C) \longrightarrow \operatorname{Hom}_{k[t]}\big(S, C[t]/\big(t^{m+1}\big)\big)\,.$$

We set

$$S'_m := k\big[x_j^{(1)}, \dots, x_j^{(m)} \mid 1 \leqslant j \leqslant N\big]\big[\big[x_1^{(0)}, \dots, x_N^{(0)}\big]\big]\,,$$
$$S''_m := k\big[\big[x_1^{(0)}, \dots, x_N^{(0)}\big]\big]\big[x_j^{(1)}, \dots, x_j^{(m)} \mid 1 \leqslant j \leqslant N\big]\,.$$

We define a ring homomorphism $\Lambda_1 \colon k[t][x_1, \dots, x_N] \to S'_m[t]/\big(t^{m+1}\big)$ by

$$\Lambda_1(t) = t, \quad \Lambda_1(x_j) = x_j^{(0)} + x_j^{(1)} t + \dots + x_j^{(m)} t^m\,.$$

Since the inclusion $\Lambda_1\big((x_1, \dots, x_N)\big) \subset \big(x_1^{(0)}, \dots, x_N^{(0)}, t\big)$ holds, Λ_1 induces a ring homomorphism $\Lambda_2 \colon k[t][[x_1, \dots, x_N]] \to S'_m[t]/\big(t^{m+1}\big)$. Note that its image is contained in $S''_m[t]/\big(t^{m+1}\big)$. Furthermore, $S''_m[t]/\big(t^{m+1}\big)$ is a subring of $S_m[t]/\big(t^{m+1}\big)$. Therefore, we have a ring homomorphism $\Lambda_3 \colon k[t][[x_1, \dots, x_N]] \to S_m[t]/\big(t^{m+1}\big)$. Then we define $\Lambda \colon S \to S_m[t]/\big(t^{m+1}\big)$ as the ring homomorphism uniquely determined by the following conditions:

- $\Lambda(f) = \Lambda_3(f)$ for any $f \in k[t][[x_1, \dots, x_N]]$,
- $\Lambda(y_j) = y_j^{(0)} + y_j^{(1)} t + \dots + y_j^{(m)} t^m$ for each $1 \leqslant j \leqslant M$.

Then Λ is a $k[t]$-ring homomorphism.

For $\beta \in \mathrm{Hom}_k(S_m, C)$, we define $\Psi(\beta) \in \mathrm{Hom}_{k[t]}\big(S, C[t]/(t^{m+1})\big)$ as the composition

$$S \xrightarrow{\Lambda} S_m[t]/\big(t^{m+1}\big) \xrightarrow{\beta} C[t]/\big(t^{m+1}\big)\,,$$

where $S_m[t]/(t^{m+1}) \to C[t]/(t^{m+1})$ is the $k[t]$-ring homomorphism induced by $\beta\colon S_m \to C$.

By the construction of Φ and Ψ, if $\alpha' = \Phi(\alpha)$ and $\beta' = \Psi(\beta)$, then they satisfy the following:

- $p_s\big(\alpha(x_j)\big) = \alpha'\big(x_j^{(s)}\big)$ for all $1 \leqslant j \leqslant N$ and $0 \leqslant s \leqslant m$,
- $p_s\big(\alpha(y_j)\big) = \alpha'\big(y_j^{(s)}\big)$ for all $1 \leqslant j \leqslant M$ and $0 \leqslant s \leqslant m$,
- $p_s\big(\beta'(x_j)\big) = \beta\big(x_j^{(s)}\big)$ for all $1 \leqslant j \leqslant N$ and $0 \leqslant s \leqslant m$,
- $p_s\big(\beta'(y_j)\big) = \beta\big(y_j^{(s)}\big)$ for all $1 \leqslant j \leqslant M$ and $0 \leqslant s \leqslant m$.

Therefore, we have $\Psi \circ \Phi = \mathrm{id}$ and $\Phi \circ \Psi = \mathrm{id}$. Hence, F_m^A is represented by $\operatorname{Spec} S_m$. The second assertion follows from the construction of A_m. $\square$

LEMMA 5.2. *We use the notation of Lemma 5.1. Let $X = \operatorname{Spec}(S/I)$ be the closed subscheme of A defined by an ideal $I = (f_1, \dots, f_r) \subset S$. For $1 \leqslant i \leqslant r$ and $0 \leqslant \ell \leqslant m$, we define $F_i^{(\ell)} \in S_m$ as follows:*

$$f_i\Bigg(\sum_{\ell=0}^{m} x_1^{(\ell)} t^\ell, \dots, \sum_{\ell=0}^{m} x_N^{(\ell)} t^\ell, \sum_{\ell=0}^{m} y_1^{(\ell)} t^\ell, \dots, \sum_{\ell=0}^{m} y_M^{(\ell)} t^\ell\Bigg) \equiv \sum_{\ell=0}^{m} F_i^{(\ell)} t^\ell \pmod{t^{m+1}}.$$

Let

$$I_m := \big(F_i^{(s)} \mid 1 \leqslant i \leqslant r,\, 0 \leqslant s \leqslant m\big) \subset S_m$$

be the ideal of S_m generated by the $F_i^{(s)}$. Then we have $X_m \simeq \operatorname{Spec}(S_m/I_m)$. Furthermore, for $m \geqslant n \geqslant 0$, the truncation map $\pi_{mn}\colon X_m \to X_n$ is induced by the ring homomorphism $S_n/I_n \to S_m/I_m$.

Proof. Let $Y := \operatorname{Spec} C$ be an affine scheme over k. We can see that the bijective map Φ in the proof of Lemma 5.1 induces the bijective map $\mathrm{Hom}_{k[t]}\big(S/I, C[t]/(t^{m+1})\big) \to \mathrm{Hom}_k(S_m/I_m, C)$. Therefore, the functor F_m^X is represented by $\operatorname{Spec}(S_m/I_m)$. The second assertion follows from the construction of X_m. $\square$

Remark 5.3. More precisely,

$$f_i\Bigg(\sum_{\ell=0}^{m} x_1^{(\ell)} t^\ell, \dots, \sum_{\ell=0}^{m} x_N^{(\ell)} t^\ell, \sum_{\ell=0}^{m} y_1^{(\ell)} t^\ell, \dots, \sum_{\ell=0}^{m} y_M^{(\ell)} t^\ell\Bigg)$$

in Lemma 5.2 is defined as $\Lambda(f_i) \in S_m[t]/(t^{m+1})$, where Λ is defined in the proof of Lemma 5.1.

PROPOSITION 5.4. *Let X be a scheme of finite type over $R = k[t][[x_1, \dots, x_N]]$. Then the following hold:*

(1) *The scheme X_m is of finite type over R.*
(2) *For any $m \geqslant n \geqslant 0$, the truncation map $\pi_{mn}\colon X_m \to X_n$ is a morphism of finite type.*

Proof. Take an affine cover $X = U_1 \cup \cdots \cup U_s$. Then, F_m^X is represented by the scheme obtained by gluing the schemes $(U_i)_m$ constructed in Lemma 5.2 (cf. [EM09, Proposition 2.2]). Therefore, the assertions follow from Lemma 5.2. $\square$

In this setting, cylinders and the contact loci

$$\mathrm{Cont}^m(\mathfrak{a}), \mathrm{Cont}^{\geqslant m}(\mathfrak{a}) \subset X_\infty\,, \quad \mathrm{Cont}^m(\mathfrak{a})_n, \mathrm{Cont}^{\geqslant m}(\mathfrak{a})_n \subset X_n$$

are defined in the same way. We denote by $\mathfrak{o}_X \subset \mathcal{O}_X$ the ideal sheaf $\mathfrak{o}_X := (x_1, \dots, x_N)\mathcal{O}_X \subset \mathcal{O}_X$ generated by $x_1, \dots, x_N \in R$. From the next subsection, we basically work on arcs contained in the contact locus $\mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_X)$. Due to the following lemma, the contact locus $\mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_X)_m$ is a scheme of finite type over k.

Lemma 5.5. *Let X be a scheme of finite type over $R = k[t][[x_1, \dots, x_N]]$. Then the following hold:*

(1) *For each $m \geqslant 0$, the contact locus $\mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_X)_m \subset X_m$ is a scheme of finite type over k.*
(2) *Any k-arc of X is contained in $\mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_X)$.*

Proof. (1) This follows from Proposition 5.4.

(2) Let $\gamma \in X_\infty$ be a k-arc. We may assume that X is affine and write $X = \operatorname{Spec} A$ with $A = S/I$, where $S := k[t][[x_1, \dots, x_N]][y_1, \dots, y_m]$ and I is an ideal of S. Let $\gamma^*\colon A \to k[[t]]$ be the corresponding $k[t]$-ring homomorphism. Let M be the kernel of the composite map $S \to A \xrightarrow{\gamma^*} k[[t]] \to k$. Since $S/M = k$, the kernel M is of the form $(t, x_1, \dots, x_N, y_1 - a_1, \dots, y_m - a_m)$ for some $a_i \in k$. This shows that $\gamma^*(\mathfrak{o}_X) \subset (t)$ and hence $\gamma \in \mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_X)$. ☐

Lemma 5.6. *Let n be a non-negative integer, and let X be a scheme of finite type over $R = k[t][[x_1, \dots, x_N]]$. Suppose that each irreducible component X_i of X has $\dim' X_i \geqslant n+1$ (see Definition 2.10). Let $\gamma \in X_\infty$ be a k-arc with $\mathrm{ord}_\gamma\big(\mathrm{Fitt}^n\big(\Omega'_{X/k[t]}\big)\big) < \infty$. Then we have*

$$\gamma^*\Omega'_{X/k[t]}/T \simeq k[[t]]^{\oplus n}\,,$$

where T is the torsion part of $\gamma^*\Omega'_{X/k[t]}$.

Proof. We may assume that X is affine, and we may write $X = \operatorname{Spec} A$ with $A = S/I$, where

$$S := k[t][[x_1, \dots, x_N]][y_1, \dots, y_m]$$

and I is an ideal of S. If P is a minimal prime of I, then we have

$$\operatorname{ht} P = \dim S - \dim'(S/P) \leqslant (N + m + 1) - (n + 1) = N + m - n$$

by Lemma 2.11(1). Therefore, we have $\operatorname{ht} I \leqslant N + m - n$.

Let $\gamma^*\colon A \to k[[t]]$ be the corresponding $k[t]$-ring homomorphism, and let $\overline{\gamma}^*\colon A \to k((t))$ be its composition with $k[[t]] \to k((t))$. Let $\mathfrak{q} \subset A$ be the kernel of γ^* and $Q \subset S$ the corresponding prime ideal. Since $\overline{\gamma}^*$ factors through $A_\mathfrak{q}$, it is sufficient to show that $\Omega'_{A_\mathfrak{q}/k[t]} \otimes_{A_\mathfrak{q}} k((t))$ has dimension n as a $k((t))$-vector space.

Let $w_1, \dots, w_\ell \in I$ be generators of I. Let $M \in M_{N+m,\ell}(A_\mathfrak{q})$ be the Jacobian matrix with respect to $w_1, \dots, w_\ell \in I$ and the derivations $\partial/\partial x_i$ and $\partial/\partial y_i$. Then M defines a map $M\colon A_\mathfrak{q}^\ell \to A_\mathfrak{q}^{N+m}$, and its cokernel is isomorphic to $\Omega'_{A_\mathfrak{q}/k[t]}$ by Proposition 2.4(3),(4). Since $\mathrm{ord}_\gamma\big(\mathrm{Fitt}^n\big(\Omega'_{X/k[t]}\big)\big) < \infty$, the matrix M has an $(N+m-n)$-minor that is not contained in $\mathfrak{q}A_\mathfrak{q}$ (cf. Remark 2.14). Furthermore, since we have

$$\operatorname{ht}(IS_Q) \leqslant \operatorname{ht} I \leqslant N + m - n\,,$$

any $(N + m - n + 1)$-minor of M is contained in $\mathfrak{q}A_\mathfrak{q}$ by Remark 2.6(2)(a) (cf. [Mat89, Theorem 30.4]). Therefore, the image of M in $M_{N+m,\ell}(k((t)))$ has rank $N + m - n$, and it follows that $\Omega'_{A_\mathfrak{q}/k[t]} \otimes_{A_\mathfrak{q}} k((t))$ has dimension n as a $k((t))$-vector space. ☐

LEMMA 5.7. *Let n and e be non-negative integers, and let X be a scheme of finite type over $R = k[t][[x_1, \dots, x_N]]$. Suppose that each irreducible component X_i of X has $\dim' X_i \geqslant n+1$. Let $\gamma \in \mathrm{Cont}^e\bigl(\mathrm{Fitt}^n\bigl(\Omega'_{X/k[t]}\bigr)\bigr)$ be a k-arc. Then we have*

$$\gamma^* \Omega'_{X/k[t]} \simeq k[[t]]^{\oplus n} \oplus \bigoplus_i k[t]/(t^{e_i})$$

as $k[[t]]$-modules with $\sum_i e_i = e$.

Proof. The same proof as in [NS22, Lemma 2.13(1)] works due to Lemma 5.6. □

5.1.2 *Cylinders and codimension.* In this subsection, we define and discuss the codimensions of cylinders of the arc space of an R-scheme X of finite type. We define the codimension only for cylinders contained in the contact locus $\mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_X)$, where $\mathfrak{o}_X \subset \mathcal{O}_X$ is the ideal sheaf generated by $x_1, \dots, x_N \in R$. Due to Lemma 5.5, the contact locus $\mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_X)_m \subset X_m$ is a scheme of finite type over k, and hence cylinders contained in $\mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_X)$ are easier to handle than general cylinders.

First, we prove Proposition 5.9, which is necessary for defining the codimension of cylinders.

LEMMA 5.8. *Let N, s and r be non-negative integers with $N + s \geqslant r$. Let $R = k[t][[x_1, \dots, x_N]]$, and let $S = R[y_1, \dots, y_s]$. Let $I = (F_1, \dots, F_r)$ be the ideal generated by elements $F_1, \dots, F_r \in S$, and let $M = \mathrm{Spec}(S/I)$. Let $\mathfrak{o}_M \subset \mathcal{O}_M$ be the ideal sheaf generated by $x_1, \dots, x_N \in R$. Let $\overline{J} = \mathrm{Fitt}^{N+s-r}\bigl(\Omega'_{M/k[t]}\bigr)$. For non-negative integers m and e with $m \geqslant e$, the following hold:*

(1) *We have*

$$\psi_m\bigl(\mathrm{Cont}^e(\overline{J}) \cap \mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_M)\bigr) = \pi_{m+e,m}\bigl(\mathrm{Cont}^e(\overline{J})_{m+e} \cap \mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_M)_{m+e}\bigr).$$

(2) *The truncation map $\pi_{m+1,m} \colon M_{m+1} \to M_m$ induces a piecewise-trivial fibration*

$$\psi_{m+1}\bigl(\mathrm{Cont}^e(\overline{J}) \cap \mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_M)\bigr) \longrightarrow \psi_m\bigl(\mathrm{Cont}^e(\overline{J}) \cap \mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_M)\bigr)$$

with fiber $\mathbb{A}^{N+s-r}$.

Proof. Let $J := \mathcal{J}_r\bigl(I; \mathrm{Der}_{k[t]}(S)\bigr) \subset S$. Then we have $\overline{J} = (J+I)/I$ by Remark 2.14. Note that $\mathrm{Der}_{k[t]}(S) = \mathrm{Der}'_{k[t]}(S)$ is generated by the $\partial/\partial x_i$ and $\partial/\partial y_j$. Therefore, statement (1) follows from the first assertion of Lemma 4.5(1). Furthermore, statement (2) follows from Lemma 4.5(2) and the second assertion of Lemma 4.5(1). □

PROPOSITION 5.9. *Let n be a non-negative integer, and let X be a scheme of finite type over $R = k[t][[x_1, \dots, x_N]]$. Suppose that each irreducible component X_i of X has $\dim' X_i \geqslant n+1$. Then there exists a positive integer c such that the following hold for non-negative integers m and e with $m \geqslant ce$:*

(1) *We have*

$$\begin{aligned}\psi_m&\bigl(\mathrm{Cont}^e\bigl(\mathrm{Fitt}^n\bigl(\Omega'_{X/k[t]}\bigr)\bigr) \cap \mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_X)\bigr)\\ &= \pi_{m+e,m}\bigl(\mathrm{Cont}^e\bigl(\mathrm{Fitt}^n\bigl(\Omega'_{X/k[t]}\bigr)\bigr)_{m+e} \cap \mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_X)_{m+e}\bigr).\end{aligned}$$

(2) *The truncation map $\pi_{m+1,m} \colon X_{m+1} \to X_m$ induces a piecewise-trivial fibration*

$$\psi_{m+1}\bigl(\mathrm{Cont}^e\bigl(\mathrm{Fitt}^n\bigl(\Omega'_{X/k[t]}\bigr)\bigr) \cap \mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_X)\bigr) \longrightarrow \psi_m\bigl(\mathrm{Cont}^e\bigl(\mathrm{Fitt}^n\bigl(\Omega'_{X/k[t]}\bigr)\bigr) \cap \mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_X)\bigr)$$

with fiber $\mathbb{A}^n$.

Proof. The same proof as in [NS22, Proposition 2.17] works. We give a sketch of the proof.

We may assume that X is affine and write $X = \operatorname{Spec}(S/I_X)$, where

$$S := k[t][[x_1, \dots, x_N]][y_1, \dots, y_m]$$

and I_X is an ideal of S. By the assumption and Lemma 2.11(1), we have

$$\operatorname{ht} P = \dim S - \dim'(S/P) \leqslant (N+m+1) - (n+1) = N+m-n$$

for any minimal prime P of I_X. We set $r := N+m-n$.

Let $f_1, \dots, f_d$ be generators of I_X. For $1 \leqslant i \leqslant r$, we set $F_i := \sum_{j=1}^d a_{ij} f_j$ for general $a_{ij} \in k$. Let $M \subset \operatorname{Spec} S$ be the closed subscheme defined by the ideal $I_M := (F_1, \dots, F_r)$. We set

$$I_{X'} := (I_M : I_X) \subset S, \quad J := \mathcal{J}_r\big(I_M; \operatorname{Der}_{k[t]}(S)\big) \subset S .$$

Here, we claim that

$$J \subset \sqrt{I_X + I_{X'}} . \tag{5.1}$$

We note that if (5.1) is true, then the assertions for X can be reduced to those for M by the same argument as in [NS22, Proposition 2.17]. Therefore, the assertions follow from Lemma 5.8.

Let $\mathfrak{p}$ be a prime ideal satisfying $I_X + I_{X'} \subset \mathfrak{p}$. To prove (5.1), it is sufficient to show that S/I_M is not regular at $\mathfrak{p}$. Indeed, if S/I_M is not regular at $\mathfrak{p}$, then we have

$$J = \mathcal{J}_r\big(I_M; \operatorname{Der}_{k[t]}(S)\big) \subset \mathcal{J}_r\big(I_M; \operatorname{Der}_k(S)\big) \subset \mathfrak{p}$$

by $\operatorname{ht}(I_M S_{\mathfrak{p}}) \leqslant r$ and the Jacobian criterion of regularity (Remark 2.6(2)(a),(b)).

Suppose to the contrary that M is regular at $\mathfrak{p}$. Since any minimal prime P of I_X satisfies $\operatorname{ht} P \leqslant r$ and the $a_{ij} \in k$ are general, for any irreducible component X_0 of X, there exists an irreducible component M_0 of M such that $X_0 \subset M_0$ and $(X_0)_{\mathrm{red}} = (M_0)_{\mathrm{red}}$. Therefore, since $I_M \subset I_X \subset \mathfrak{p}$ and M is regular at $\mathfrak{p}$, we have $(I_M)_{\mathfrak{p}} = (I_X)_{\mathfrak{p}}$. This shows that

$$(I_{X'})_{\mathfrak{p}} = (I_M : I_X)_{\mathfrak{p}} = \big((I_M)_{\mathfrak{p}} : (I_X)_{\mathfrak{p}}\big) = S_{\mathfrak{p}} ,$$

which contradicts $I_{X'} \subset \mathfrak{p}$. This completes the proof of (5.1). $\square$

For an R-scheme X, a subset $C \subset X_\infty$ is called a *cylinder* if $C = \psi_m^{-1}(S)$ for some $m \geqslant 0$ and a constructible subset $S \subset X_m$.

Proposition 5.10. *Let n be a non-negative integer, and let X be a scheme of finite type over $R = k[t][[x_1, \dots, x_N]]$. Suppose that each irreducible component X_i of X has $\dim' X_i \geqslant n+1$. Let C be a cylinder in X_∞ which is contained in $\operatorname{Cont}^{\geqslant 1}(\mathfrak{o}_X) \cap \operatorname{Cont}^e\big(\operatorname{Fitt}^n(\Omega'_{X/k[t]})\big)$ for some $e \geqslant 0$. Then its image $\psi_m(C) \subset X_m$ is a constructible subset for any $m \geqslant 0$.*

Proof. Let $S \subset X_\ell$ be a constructible subset such that $\psi_\ell^{-1}(S) = C$. For $m \geqslant \ell$, we have

$$\pi_{m,\ell}^{-1}(S) \cap \psi_m(C) = \psi_m(C) = \pi_{m,\ell}^{-1}(S) \cap \psi_m(X_\infty) .$$

By the assumption $C \subset \operatorname{Cont}^{\geqslant 1}(\mathfrak{o}_X) \cap \operatorname{Cont}^e\big(\operatorname{Fitt}^n(\Omega'_{X/k[t]})\big)$, we also have

$$\psi_m(C) = \pi_{m,\ell}^{-1}(S) \cap \psi_m\Big(\operatorname{Cont}^{\geqslant 1}(\mathfrak{o}_X) \cap \operatorname{Cont}^e\big(\operatorname{Fitt}^n(\Omega'_{X/k[t]})\big)\Big) .$$

Let c be the positive integer appearing in Proposition 5.9. Then the constructibility of $\psi_m(C)$ follows from Proposition 5.9(1) if $m \geqslant \max\{ce, \ell\}$. When $m < \max\{ce, \ell\}$, the constructibility follows from that for $m = \max\{ce, \ell\}$ and Chevalley's theorem. $\square$

We define the codimensions of cylinders C when they satisfy $C \subset \operatorname{Cont}^{\geqslant 1}(\mathfrak{o}_X)$.

Definition 5.11. Let n be a non-negative integer, and let X be a scheme of finite type over $R = k[t][[x_1, \dots, x_N]]$. Suppose that each irreducible component X_i of X has $\dim' X_i \geqslant n+1$. Let $C \subset X_\infty$ be a cylinder contained in $\mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_X)$.

(1) Assume $C \subset \mathrm{Cont}^e\bigl(\mathrm{Fitt}^n\bigl(\Omega'_{X/k[t]}\bigr)\bigr)$ for some $e \in \mathbb{Z}_{\geqslant 0}$. Then we define the codimension of C in X_∞ as
$$\mathrm{codim}(C) := (m+1)n - \dim(\psi_m(C))$$
for any sufficiently large m. This is well defined by Proposition 5.9(2).

(2) In general, we define the codimension of C in X_∞ as follows:
$$\mathrm{codim}(C) := \min_{e \in \mathbb{Z}_{\geqslant 0}} \mathrm{codim}\bigl(C \cap \mathrm{Cont}^e\bigl(\mathrm{Fitt}^n\bigl(\Omega'_{X/k[t]}\bigr)\bigr)\bigr)\,.$$
By convention, $\mathrm{codim}(C) = \infty$ if $C \cap \mathrm{Cont}^e\bigl(\mathrm{Fitt}^n\bigl(\Omega'_{X/k[t]}\bigr)\bigr) = \emptyset$ for any $e \geqslant 0$.

Remark 5.12. The definition of the codimension above depends on the choice of n. In Section 5.1.3, we fix a non-negative integer n, and we use the codimension defined for this n.

Lemma 5.13. *Let X be a scheme of finite type over $R = k[t][[x_1, \dots, x_N]]$. Let p and m be non-negative integers with $2p+1 \geqslant m \geqslant p$. Let $\gamma \in X_p(k)$ be a jet. If $\pi_{m,p}^{-1}(\gamma) \neq \emptyset$, it follows that*
$$\pi_{m,p}^{-1}(\gamma) \simeq \mathrm{Hom}_{k[t]/(t^{p+1})}\bigl(\gamma^* \Omega'_{X/k[t]}, \bigl(t^{p+1}\bigr)/\bigl(t^{m+1}\bigr)\bigr)\,.$$

Proof. We may assume that X is affine, and we may write $X = \operatorname{Spec} A$ with an R-algebra A of finite type. Let $\gamma^* \colon A \to k[t]/\bigl(t^{p+1}\bigr)$ be the corresponding $k[t]$-ring homomorphism to γ. Take any $\alpha \in \pi_{m,p}^{-1}(\gamma)$. Let $\alpha^* \colon A \to k[t]/\bigl(t^{m+1}\bigr)$ be the corresponding $k[t]$-ring homomorphism. Then for the same reason as in the case of k-schemes (cf. [EM09, Proposition 4.4]), we have an isomorphism
$$\pi_{m,p}^{-1}(\gamma) \simeq \mathrm{Der}_{k[t]}\bigl(A, \bigl(t^{p+1}\bigr)/\bigl(t^{m+1}\bigr)\bigr)\,, \quad \beta \longmapsto \beta^* - \alpha^*\,.$$
Here, $\bigl(t^{p+1}\bigr)/\bigl(t^{m+1}\bigr)$ on the right-hand side has an A-module structure via γ^*. Then the assertion follows from the isomorphisms
$$\begin{aligned}\mathrm{Der}_{k[t]}\bigl(A, \bigl(t^{p+1}\bigr)/\bigl(t^{m+1}\bigr)\bigr) &= \mathrm{Der}'_{k[t]}\bigl(A, \bigl(t^{p+1}\bigr)/\bigl(t^{m+1}\bigr)\bigr) \\ &\simeq \mathrm{Hom}_A\bigl(\Omega'_{A/k[t]}, \bigl(t^{p+1}\bigr)/\bigl(t^{m+1}\bigr)\bigr)\,.\end{aligned}$$
The first equality follows from Lemma 2.2. □

5.1.3 *Thin and very thin cylinders.* We fix a non-negative integer n throughout this subsection.

Definition 5.14. Let X be a scheme of finite type over $R = k[t][[x_1, \dots, x_N]]$. Suppose that each irreducible component X_i of X has $\dim' X_i \geqslant n+1$. A subset $A \subset X_\infty$ is called *thin* if $A \subset Z_\infty$ for some closed subscheme Z of X with $\dim Z \leqslant n$. A subset A is called *very thin* if $A \subset Z_\infty$ for some closed subscheme Z of X with $\dim Z \leqslant n-1$.

The term "very thin" is used only in this paper. In Question 5.15(1) and Remark 5.16(1), we explain the motivation to introduce this terminology.

Question 5.15. Let R and X be as in Definition 5.14.

(1) Suppose that C is a thin cylinder of X_∞. Then, does $C \cap \mathrm{Cont}^e\bigl(\mathrm{Fitt}^n\bigl(\Omega'_{X/k[t]}\bigr)\bigr) = \emptyset$ hold for any $e \geqslant 0$?

(2) Suppose that X is an integral scheme and $Y \subset X$ is the closed subscheme defined by the ideal $\mathrm{Fitt}^n(\Omega'_{X/k[t]})$. Then, is Y_∞ a thin set of X_∞?

(3) Let $S = k[t][[x_1, \dots, x_N]][y_1, \dots, y_m]$, and let P be a prime ideal of S of height r. Suppose that $P \cap k[t] = (0)$. Then, does $\mathcal{J}_r\big(P; \mathrm{Der}_{k[t]}(S)\big) \not\subset P$ hold?

Remark 5.16. (1) Note that Question 5.15(1) is true for the arc spaces of k-varieties X:

- If C is a thin cylinder of X_∞, then $C \cap \mathrm{Cont}^e(\mathrm{Jac}_{X/k}) = \emptyset$ for any $e \geqslant 0$ (cf. [EM09, Lemma 5.1]).

The same statement is true for schemes X of finite type over $k[[x_1, \dots, x_N]]$, as can be seen by replacing $\mathrm{Jac}_{X/k}$ with $\mathrm{Jac}'_{X/k}$. Furthermore, Question 5.15(1) is also true for schemes X of finite type over $k[t]$ (cf. [NS22, Lemma 2.23]). However, the same proofs do not work for schemes X of finite type over $R = k[t][[x_1, \dots, x_N]]$, and hence it is not clear to us whether Question 5.15(1) is true for this setting (see also the discussion in item (3) below). This is why we introduce the term "very thin" and we will prove weaker statements instead in Lemma 5.18 for very thin cylinders and Proposition 5.23 for X with an additional assumption.

(2) Due to the proof of [EM09, Lemma 5.1], part (1) of Question 5.15 can be reduced to part (2) by Noetherian induction on the dimension. Furthermore, part (3) implies part (2).

(3) Question 5.15(3) is related to the weak Jacobian condition (WJ) explained in Remark 2.6(2). Indeed, if $N = 0$, then Question 5.15(3) can be proved using Remark 2.6(2)(c) as follows. We denote by

$$S' := (k[t] \setminus \{0\})^{-1} S = k(t)[y_1, \dots, y_m]$$

the localization. Then by the assumption $P \cap k[t] = (0)$, we have $PS' \neq S'$, and hence PS' is a prime ideal of height r. Since S' satisfies $(\mathrm{WJ})_{k(t)}$, we have

$$\mathcal{J}_r\big(P; \mathrm{Der}_{k[t]}(S)\big)S' + PS' = \mathcal{J}_r\big(PS'; \mathrm{Der}_{k(t)}(S')\big) + PS' \not\subset PS',$$

which proves $\mathcal{J}_r\big(P; \mathrm{Der}_{k[t]}(S)\big) \not\subset P$. Note that the same proof does not work when $N > 0$ because we have

$$S' := (k[t] \setminus \{0\})^{-1} S \neq k(t)[[x_1, \dots, x_N]][y_1, \dots, y_m],$$

and it is not clear whether $\mathcal{J}_r\big(PS'; \Delta\big) \not\subset PS'$ for $\Delta = \big\{\partial/\partial x_i, \partial/\partial y_j \mid i, j\big\}$.

(4) Question 5.15(1) is also true for the arc spaces (Greenberg schemes) of formal $k[[t]]$-schemes of finite type, which will be dealt with in Section 5.2 (see [CNS18, Chapter 6, Proposition 2.4.3]). Actually, Question 5.15(3) is true for this setting:

- Let $S = k[x_1, \dots, x_N][[t]]$, and let P be a prime ideal of S of height r. Suppose $P \cap k[[t]] = (0)$. Then we have $\mathcal{J}_r\big(P; \mathrm{Der}_{k[[t]]}(S)\big) \not\subset P$.

We denote by $S' := S_t = k[x_1, \dots, x_N]((t))$ the localization. We note that the assumption $P \cap k[[t]] = (0)$ is equivalent to $t \notin P$, and hence PS' is a prime ideal of height r. Since S' satisfies $(\mathrm{WJ})_{k((t))}$ by [Nag62, Theorem 46.3], the same proof as in item (3) above works, and we have $\mathcal{J}_r\big(P; \mathrm{Der}_{k[[t]]}(S)\big) \not\subset P$. Note that both $\mathrm{Der}_{k[[t]]}(S)$ and $\mathrm{Der}_{k((t))}(S')$ are generated by the $\partial/\partial x_i$.

In Lemmas 5.17 and 5.18 below, for a scheme X over $R = k[t][[x_1, \dots, x_N]]$, we also consider the jet schemes and the arc space in the sense of Section 4. To avoid confusion, we denote them by $\mathcal{L}_m(X)$ and $\mathcal{L}_\infty(X)$; that is, $\mathcal{L}_m(X)$ is the scheme representing the functor

$$F_m \colon (\mathsf{Sch}/k) \longrightarrow (\mathsf{Set}), \quad Y \longmapsto \mathrm{Hom}_k\big(Y \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[t]/(t^{m+1}), X\big),$$

and $\mathcal{L}_\infty(X) = \varprojlim_m \mathcal{L}_m(X)$ is the projective limit.

LEMMA 5.17. *Let X be a scheme of finite type over $R = k[t][[x_1, \dots, x_N]]$. Then the following hold:*

(1) *There exist natural closed immersions $X_m \to \mathcal{L}_m(X)$ for $m \geqslant 0$ that commute with the truncation morphisms.*

(2) *Let $x \in X$ be a k-point, and let $\widehat{\mathcal{O}}_{X,x}$ be the completion of the local ring $\mathcal{O}_{X,x}$ at its maximal ideal. Let $X' := \operatorname{Spec}(\widehat{\mathcal{O}}_{X,x})$, and let $x' \in X'$ be the corresponding k-point. Then for the truncation morphisms $\pi_m^X : \mathcal{L}_m(X) \to X$ and $\pi_m^{X'} : \mathcal{L}_m(X') \to X'$, we have $(\pi_m^X)^{-1}(x) \simeq (\pi_m^{X'})^{-1}(x')$.*

Proof. We may assume that X is affine, and we may write $X = \operatorname{Spec}(S/I)$ with

$$S := k[t][[x_1, \dots, x_N]][y_1, \dots, y_M]$$

and $I := (f_1, \dots, f_r)$ an ideal of S. We set $A := \operatorname{Spec} S$.

Then, by the same argument as in Lemma 5.1, we have $\mathcal{L}_m(A) \simeq \operatorname{Spec} T_m$, where

$$T_m := k\big[u^{(0)}\big]\big[\big[x_1^{(0)}, \dots, x_N^{(0)}\big]\big]\left[u^{(s)}, x_j^{(s)}, y_{j'}^{(s')} \;\middle|\; \begin{array}{l} 1 \leqslant j \leqslant N,\, 1 \leqslant j' \leqslant M, \\ 1 \leqslant s \leqslant m,\, 0 \leqslant s' \leqslant m \end{array}\right].$$

In the same way as in Lemma 5.1, we can define a ring homomorphism $\Lambda' : S \to T_m[t]/(t^{m+1})$ that satisfies

- $\Lambda'(t) = u^{(0)} + u^{(1)}t + \cdots + u^{(m)}t^m$,
- $\Lambda'(x_j) = x_j^{(0)} + x_j^{(1)}t + \cdots + x_j^{(m)}t^m$ for each $0 \leqslant j \leqslant N$ and
- $\Lambda'(y_j) = y_j^{(0)} + y_j^{(1)}t + \cdots + y_j^{(m)}t^m$ for each $0 \leqslant j \leqslant M$.

For $1 \leqslant i \leqslant r$ and $0 \leqslant \ell \leqslant m$, we define $G_i^{(\ell)} \in T_m$ by

$$\Lambda'(f_i) = \sum_{\ell=0}^{m} G_i^{(\ell)} t^\ell \pmod{t^{m+1}}.$$

Let

$$J_m := \big(G_i^{(s)} \mid 1 \leqslant i \leqslant r,\, 0 \leqslant s \leqslant m\big) \subset T_m$$

be the ideal generated by the $G_i^{(s)}$. Then by the same argument as in Lemma 5.2, we have $\mathcal{L}_m(X) \simeq \operatorname{Spec}(T_m/J_m)$.

Let S_m be the ring defined in Lemma 5.1. Let $\Xi : T_m \to S_m$ be a surjective ring homomorphism defined by

- $\Xi\big(u^{(1)}\big) = 1$ and $\Xi(u^{(s)}) = 0$ for each $s = 0$ and $2 \leqslant s \leqslant m$,
- $\Xi\big(x_j^{(s)}\big) = x_j^{(s)}$ for each $1 \leqslant j \leqslant N$ and $0 \leqslant s \leqslant m$,
- $\Xi\big(y_j^{(s)}\big) = y_j^{(s)}$ for each $1 \leqslant j \leqslant M$ and $0 \leqslant s \leqslant m$.

We note that the ring homomorphism $\Lambda : S \to S_m[t]/(t^{m+1})$ defined in the proof of Lemma 5.1 coincides with the composition

$$S \xrightarrow{\Lambda'} T_m[t]/\big(t^{m+1}\big) \xrightarrow{\Xi} S_m[t]/\big(t^{m+1}\big),$$

where $T_m[t]/(t^{m+1}) \to S_m[t]/(t^{m+1})$ is the $k[t]$-ring homomorphism induced by Ξ. Therefore, $F_i^{(s)}$ in Lemma 5.2 coincides with $\Xi\big(G_i^{(s)}\big)$ for each i and s.

Let $I_m \subset S_m$ be the ideal defined in Lemma 5.2. Then Ξ induces a surjective ring homomorphism $T_m/J_m \to S_m/I_m$. It gives a closed immersion

$$X_m \simeq \operatorname{Spec}(S_m/I_m) \hookrightarrow \operatorname{Spec}(T_m/J_m) \simeq \mathcal{L}_m(X)\,,$$

which completes the proof of assertion (1).

Since $x \in X$ is a k-point, the corresponding maximal ideal of S is of the form

$$\big(t-a, x_1, \dots, x_N, y_1 - b_1, \dots, y_M - b_M\big)$$

with $a, b_1, \dots, b_M \in k$. For simplicity, assume $a = b_1 = \cdots = b_M = 0$. Then $\big(\pi_m^X\big)^{-1}(x) \subset \mathcal{L}_m(X)$ is isomorphic to the closed subscheme of $\operatorname{Spec} T_m$ defined by

$$J_m + \big(u^{(0)}, x_1^{(0)}, \dots, x_N^{(0)}, y_1^{(0)}, \dots, y_M^{(0)}\big)\,.$$

On the other hand, we have

$$\widehat{\mathcal{O}}_{A,x} \simeq k[[t, x_1, \dots, x_N, y_1, \dots, y_M]]\,.$$

We set $A' := \operatorname{Spec}\big(\widehat{\mathcal{O}}_{A,x}\big)$ and

$$T'_m := k\big[\big[u^{(0)}, x_1^{(0)}, \dots, x_N^{(0)}, y_1^{(0)}, \dots, y_M^{(0)}\big]\big] \left[u^{(s)}, x_j^{(s)}, y_{j'}^{(s)} \;\middle|\; \begin{array}{c} 1 \leqslant j \leqslant N, \\ 1 \leqslant j' \leqslant M, \\ 1 \leqslant s \leqslant m \end{array} \right].$$

Then by [Ish09, Proposition 4.1], we have $\mathcal{L}_m(A') \simeq \operatorname{Spec} T'_m$. Furthermore, by [Ish09, Corollary 4.2], we have $\mathcal{L}_m(X') \simeq \operatorname{Spec}(T'_m/J_m T'_m)$. Therefore, $\big(\pi_m^{X'}\big)^{-1}(x') \subset \mathcal{L}_m(X')$ is isomorphic to the closed subscheme of $\operatorname{Spec} T'_m$ defined by

$$J_m T'_m + \big(u^{(0)}, x_1^{(0)}, \dots, x_N^{(0)}, y_1^{(0)}, \dots, y_M^{(0)}\big)\,.$$

Therefore, we have $\big(\pi_m^X\big)^{-1}(x) \simeq \big(\pi_m^{X'}\big)^{-1}(x')$, which completes the proof of assertion (2). $\square$

Lemma 5.18. *Let X be a scheme of finite type over $R = k[t][[x_1, \dots, x_N]]$. Suppose that each irreducible component X_i of X has $\dim' X_i \geqslant n+1$. Let $C \subset X_\infty$ be a cylinder contained in $\operatorname{Cont}^{\geqslant 1}(\mathfrak{o}_X)$. If C is very thin, then $C \cap \operatorname{Cont}^e\big(\operatorname{Fitt}^n\big(\Omega'_{X/k[t]}\big)\big) = \emptyset$ for any $e \geqslant 0$.*

Proof. Suppose to the contrary that $C \cap \operatorname{Cont}^e\big(\operatorname{Fitt}^n\big(\Omega'_{X/k[t]}\big)\big) \neq \emptyset$ for some $e \geqslant 0$. By replacing C with $C \cap \operatorname{Cont}^e\big(\operatorname{Fitt}^n\big(\Omega'_{X/k[t]}\big)\big)$, we may assume $\emptyset \neq C \subset \operatorname{Cont}^e\big(\operatorname{Fitt}^n\big(\Omega'_{X/k[t]}\big)\big)$. Pick a k-arc $\gamma \in C$. Let $x := \pi_\infty^X(\gamma) \in X$ be the k-point of X. Then, by replacing C with $C \cap \big(\pi_\infty^X\big)^{-1}(x)$, we may assume $C \subset \big(\pi_\infty^X\big)^{-1}(x)$.

Since C is a very thin set, there exists a closed subscheme $Z \subset X$ such that $C \subset Z_\infty$ and $\dim Z \leqslant n-1$. Since $\gamma \in Z_\infty$, it follows that $x \in Z$. Let $\widehat{\mathcal{O}}_{Z,x}$ be the completion of the local ring $\mathcal{O}_{Z,x}$ at its maximal ideal. Let $Z' := \operatorname{Spec}\big(\widehat{\mathcal{O}}_{Z,x}\big)$, and let $x' \in Z'$ be the corresponding k-point. Then, since $C \subset \big(\pi_\infty^Z\big)^{-1}(x)$, we may identify C with a subset of $\mathcal{L}_\infty(Z')$ by Lemma 5.17(1),(2). Note that $C \subset \mathcal{L}_\infty(Z')$ is not necessarily a cylinder of $\mathcal{L}_\infty(Z')$ under this identification.

Let $\mathcal{S}$ be the set of the closed subschemes Y' of Z' with the following condition:

- There exists a cylinder C' of X_∞ such that $\emptyset \neq C' \subset C$ and $C' \subset \mathcal{L}_\infty(Y')$.

Here, the inclusion $C' \subset \mathcal{L}_\infty(Y')$ is considered by the identifications $\mathcal{L}_\infty(Y') \subset \mathcal{L}_\infty(Z')$ and $C' \subset C \subset \mathcal{L}_\infty(Z')$. Let Y' be a minimal element of $\mathcal{S}$, and let C' be a corresponding cylinder of X_∞. Then Y' is reduced by the minimality.

We now prove that Y' is irreducible. Suppose to the contrary that $Y' = Y'_1 \cup \cdots \cup Y'_\ell$ is the irreducible decomposition with $\ell \geqslant 2$. By the minimality of Y', it follows that $C' \not\subset \mathcal{L}_\infty(Y'_1)$, and hence we have

$$C'' := C' \cap \bigl(\bigl(\psi_q^{Z'}\bigr)^{-1}(\mathcal{L}_q(Y'_1)) \setminus \bigl(\psi_{q+1}^{Z'}\bigr)^{-1}(\mathcal{L}_{q+1}(Y'_1)) \bigr) \neq \emptyset$$

for some $q \geqslant -1$, where we set $\bigl(\psi_q^{Z'}\bigr)^{-1}(\mathcal{L}_q(Y'_1)) = \mathcal{L}_\infty(Z')$ for $q = -1$ by abuse of notation. Here, we have taken the intersection in the space $\mathcal{L}_\infty(Z')$. Since $C'' \cap \mathcal{L}_\infty(Y'_1) = \emptyset$, we have

$$C'' \subset C \setminus \mathcal{L}_\infty(Y'_1) \subset \mathcal{L}_\infty(Y') \setminus \mathcal{L}_\infty(Y'_1) \subset \mathcal{L}_\infty(Y'_2 \cup \cdots \cup Y'_\ell)\,.$$

To get a contradiction by the minimality of Y', it is sufficient to show that C'' is a cylinder of X_∞. For this purpose, we show that $C'_q := C' \cap \bigl(\psi_q^{Z'}\bigr)^{-1}(\mathcal{L}_q(Y'_1))$ is a cylinder of X_∞. Under the identifications

$$\begin{tikzcd}
 & X_\infty \cap \bigl(\pi_\infty^X\bigr)^{-1}(x) \arrow[r, "\psi_q^X"] & X_q \cap \bigl(\pi_q^X\bigr)^{-1}(x) \\
 & \cup & \cup \\
 & Z_\infty \cap \bigl(\pi_\infty^Z\bigr)^{-1}(x) \arrow[r] & Z_q \cap \bigl(\pi_q^Z\bigr)^{-1}(x) \\
 & \cap & \cap \\
 & \mathcal{L}_\infty(Z) \cap \bigl(\pi_\infty^Z\bigr)^{-1}(x) \arrow[r] & \mathcal{L}_q(Z) \cap \bigl(\pi_q^Z\bigr)^{-1}(x) \\
C' \arrow[r, phantom, "\subset"] & \mathcal{L}_\infty(Z') \cap \bigl(\pi_\infty^{Z'}\bigr)^{-1}(x') \arrow[u, "\simeq"] \arrow[r, "\psi_q^{Z'}"] & \mathcal{L}_q(Z') \cap \bigl(\pi_q^{Z'}\bigr)^{-1}(x') \arrow[u, "\simeq"] \\
 & \cup & \cup \\
 & \mathcal{L}_\infty(Y'_1) \cap \bigl(\pi_\infty^{Y'_1}\bigr)^{-1}(x') \arrow[r] & \mathcal{L}_q(Y'_1) \cap \bigl(\pi_q^{Y'_1}\bigr)^{-1}(x')\,,
\end{tikzcd}$$

we can consider the intersection $F = \mathcal{L}_q(Y'_1) \cap Z_q \cap \bigl(\pi_q^Z\bigr)^{-1}(x)$; it can be identified with a closed subset of $X_q \cap \bigl(\pi_q^X\bigr)^{-1}(x)$. Then we have the equality $C'_q = C' \cap \bigl(\psi_q^X\bigr)^{-1}(F)$ since $C' \subset Z_\infty$. Therefore, C'_q turns out to be a cylinder of X_∞, and hence so is $C'' = C'_q \setminus C'_{q+1}$. We have proved that Y' is integral.

Let $Y'' \subset Y'$ be the subscheme defined by $\mathrm{Jac}'_{Y'/k}$. Since Y' is reduced, we have $Y'' \subsetneq Y'$ by the Jacobian criterion of regularity (cf. Remark 2.6(2)(c)). By the minimality of Y', we have $C' \not\subset Y''_\infty$, and hence $C' \cap \mathrm{Cont}^{e'}\bigl(\mathrm{Jac}'_{Y'/k}\bigr) \neq \emptyset$ for some $e' \geqslant 0$. Take a k-arc $\beta \in C' \cap \mathrm{Cont}^{e'}\bigl(\mathrm{Jac}'_{Y'/k}\bigr)$. For $m \geqslant 0$, we set

$$D_{m,\beta} := \bigl(\psi_m^{Y'}\bigr)^{-1}\bigl(\psi_m^{Y'}(\beta)\bigr) \subset \mathcal{L}_\infty(Y')\,, \quad E_{m,\beta} := \bigl(\psi_m^X\bigr)^{-1}\bigl(\psi_m^X(\beta)\bigr) \subset X_\infty\,.$$

Then by applying Proposition 4.8(2) to the map

$$\psi_{m+1}^{Y'}(D_{m,\beta}) \longrightarrow \psi_m^{Y'}(D_{m,\beta}) = \bigl\{\psi_m^{Y'}(\beta)\bigr\}\,,$$

we have $\dim\bigl(\psi_{m+1}^{Y'}(D_{m,\beta})\bigr) = \dim Y'$ for sufficiently large m. On the other hand, by applying Proposition 5.9(2) to the map

$$\psi_{m+1}^X(E_{m,\beta}) \longrightarrow \psi_m^X(E_{m,\beta}) = \bigl\{\psi_m^X(\beta)\bigr\}\,,$$

we have $\dim\bigl(\psi_{m+1}^X(E_{m,\beta}))\bigr) = n$ for sufficiently large m.

Since $\dim Y' \leqslant \dim Z \leqslant n-1$, to get a contradiction, it is enough to show that $E_{m,\beta} \subset D_{m,\beta}$

for sufficiently large m. Since C' is a cylinder of X_∞, there exists a constructible subset $V \subset X_p$ for some $p \geqslant 0$ such that $C' = (\psi_p^X)^{-1}(V)$. We will prove the inclusion $E_{m,\beta} \subset D_{m,\beta}$ for any $m \geqslant p$.

Let $X' := \operatorname{Spec}(\widehat{\mathcal{O}}_{X,x})$. Then by Lemma 5.17(1),(2), we have the following diagram:

$$\begin{array}{ccccc} X_\infty \cap \bigl(\pi_\infty^X\bigr)^{-1}(x) & \subset & \mathcal{L}_\infty(X') \cap \bigl(\pi_\infty^{X'}\bigr)^{-1}(x') & \supset & \mathcal{L}_\infty(Y') \cap \bigl(\pi_\infty^{Y'}\bigr)^{-1}(x') \\ {\scriptstyle \psi_m^X}\Big\downarrow & & \Big\downarrow{\scriptstyle \psi_m^{X'}} & & \Big\downarrow{\scriptstyle \psi_m^{Y'}} \\ X_m \cap \bigl(\pi_m^X\bigr)^{-1}(x) & \subset & \mathcal{L}_m(X') \cap \bigl(\pi_m^{X'}\bigr)^{-1}(x') & \supset & \mathcal{L}_m(Y') \cap \bigl(\pi_m^{Y'}\bigr)^{-1}(x') . \end{array}$$

Let $\beta_m := \psi_m^X(\beta)$. For $m \geqslant p$, we have

$$E_{m,\beta} = \bigl(\psi_m^X\bigr)^{-1}(\beta_m) = \bigl(\psi_m^{X'}\bigr)^{-1}(\beta_m) \cap X_\infty \cap \bigl(\pi_\infty^X\bigr)^{-1}(x) = \bigl(\psi_m^{X'}\bigr)^{-1}(\beta_m) \cap C' .$$

On the other hand, we have

$$D_{m,\beta} = \bigl(\psi_m^{Y'}\bigr)^{-1}(\beta_m) = \bigl(\psi_m^{X'}\bigr)^{-1}(\beta_m) \cap \mathcal{L}_\infty(Y') \cap \bigl(\pi_m^{Y'}\bigr)^{-1}(x') .$$

Since $C' \subset \mathcal{L}_\infty(Y') \cap \bigl(\pi_m^{Y'}\bigr)^{-1}(x')$, we have $E_{m,\beta} \subset D_{m,\beta}$ for $m \geqslant p$. This completes the proof. ☐

We prove a much weaker version of [NS22, Lemma 2.26(1)].

Lemma 5.19. *Let X be a scheme of finite type over $R = k[t][[x_1, \dots, x_N]]$. For each $a \in k$, we denote by X_a the closed subscheme of X defined by $(t-a)\mathcal{O}_X$. Suppose that for any $a \in k^\times$, the scheme X_a is integral regular and has $\dim' X_a = n$. Then there exists a positive integer ℓ such that $\operatorname{ord}_\gamma\bigl(\mathfrak{o}_X + \operatorname{Fitt}^n\bigl(\Omega'_{X/k[t]}\bigr)\bigr) \leqslant \ell$ for any k-arc $\gamma \in X_\infty$. In particular, if γ satisfies $\operatorname{ord}_\gamma(\mathfrak{o}_X) = \infty$, then we have $\operatorname{ord}_\gamma\bigl(\operatorname{Fitt}^n\bigl(\Omega'_{X/k[t]}\bigr)\bigr) \leqslant \ell$.*

Proof. We may assume that X is affine, and we may write $X = \operatorname{Spec}(S/I)$, where

$$S := k[t][[x_1, \dots, x_N]][y_1, \dots, y_m]$$

and I is an ideal of S. We set $I_a := (I + (t-a))/(t-a)$, which is an ideal of the ring $S/(t-a) \simeq k[[x_1, \dots, x_N]][y_1, \dots, y_m]$. We have $\operatorname{ht}(I_a) = N + m - n$ for any $a \in k^\times$ since $\dim' X_a = n$.

We set

$$J := \mathcal{J}_{N+m-n}\bigl(I; \operatorname{Der}_{k[t]}(S)\bigr) \subset S .$$

Note that $\operatorname{Der}_{k[t]}(S) = \operatorname{Der}'_{k[t]}(S)$ is generated by the $\partial/\partial x_i$ and $\partial/\partial y_i$. We have $\operatorname{Fitt}^n\bigl(\Omega'_{X/k[t]}\bigr) = (J+I)/I$ by Remark 2.14. Let $a \in k^\times$. Since $\operatorname{ht}(I_a) = N + m - n$, we have

$$\operatorname{Jac}'_{X_a/k} = (J + I + (t-a))/(I + (t-a)) .$$

Since X_a is regular, we have $J + I + (t-a) = S$ by the Jacobian criterion of regularity (cf. Remark 2.6(2)(c)). Therefore, for any $a \in k^\times$, we have $(J + I + (t-a))S' = S'$, where $S' := S/(x_1, \dots, x_N) \simeq k[t][y_1, \dots, y_m]$. Then we have $t^\ell \in J + I + (x_1, \dots, x_N)$ for some $\ell \geqslant 0$, by Hilbert's nullstellensatz. This proves the assertions. ☐

Lemma 5.20. *Let P be a prime ideal of $S = k[[x_1, \dots, x_N]]$ of height r, and let I be an ideal of S satisfying $P \subsetneq I$. If S/P is regular, then $\mathcal{J}_{r+1}(I; \operatorname{Der}_k(S)) \not\subset I$.*

Proof. Note that $\operatorname{Der}_k(S) = \operatorname{Der}'_k(S)$ is generated by the $\partial/\partial x_i$. First, we prove the assertion when $r = 0$. Let $f \in I \setminus \{0\}$ be an element with the minimum order a. Suppose that x_i appears in the lowest-order term of f. Then it follows from the minimality of a that $\partial f/\partial x_i \notin I$, which proves the assertion when $r = 0$.

Suppose $r > 0$. Since the quotient S/P is regular, by the Jacobian criterion of regularity (cf. Remark 2.6(2)(c)), there exist $D_1, \ldots, D_r \in \mathrm{Der}_k(S)$ and $f_1, \ldots, f_r \in P$ such that

$$u := \det(D_i(f_j))_{1 \leqslant i,j \leqslant r} \notin (x_1, \ldots, x_N)\,.$$

Since S/P is a complete regular local ring with coefficient field k, the ring S/P is isomorphic to $k[[y_1, \ldots, y_{N-r}]]$. Therefore, by what we have already proved, there exist $D' \in \mathrm{Der}_k(S/P)$ and $f' \in I/P$ such that $D'(f') \notin I/P$. Let $f_{r+1} \in I$ be a lift of f'. By [Mat89, Theorem 30.8], there exists a lift $D_{r+1} \in \mathrm{Der}_k(S)$ of D' too. Since $D_{r+1}(P) = 0$, we have

$$\det(D_i(f_j))_{1 \leqslant i,j \leqslant r+1} = u D_{r+1}(f_{r+1}) \notin I\,,$$

which shows that $\mathcal{J}_{r+1}(I; \mathrm{Der}_k(S)) \not\subset I$. □

Remark 5.21. We are interested in the case where $\operatorname{ht} I = r+1$. If I is a prime ideal, then it is true more generally that $\mathcal{J}_\ell(I; \mathrm{Der}_k(S)) \not\subset I$ for $\ell = \operatorname{ht} I$. It is true because S satisfies the weak Jacobian condition $(\mathrm{WJ})_k$ (cf. Remark 2.6(2)). If I is not a prime ideal, then $\mathcal{J}_\ell(I; \mathrm{Der}_k(S)) \not\subset I$ does not hold in general.

Lemma 5.22. *Let P be a prime ideal of $S = k[[x_1, \ldots, x_N]][y_1, \ldots, y_m]$ of height r, and let I be an ideal of S satisfying $P \subsetneq I$. Suppose that S/P is regular and $I + (x_1, \ldots, x_N) \neq S$. Then it follows that $\mathcal{J}_{r+1}(I; \mathrm{Der}_k(S)) \not\subset I$.*

Proof. Note that $\mathrm{Der}_k(S) = \mathrm{Der}'_k(S)$ is generated by the $\partial/\partial x_i$ and $\partial/\partial y_i$. Since we have $I + (x_1, \ldots, x_N) \neq S$, there exists a maximal ideal $\mathfrak{m}$ containing I of the form

$$\mathfrak{m} = (x_1, \ldots, x_N, y_1 - a_1, \ldots, y_m - a_m)\,,$$

where $a_i \in k$. Let $\widehat{S}$ be the completion of S at $\mathfrak{m}$. Let $Y_i \in \widehat{S}$ be the image of $y_i - a_i$. Then we have $\widehat{S} \simeq k[[x_1, \ldots, x_N, Y_1, \ldots, Y_m]]$, and $\mathrm{Der}_k\big(\widehat{S}\big)$ is generated by the $\partial/\partial x_i$ and $\partial/\partial Y_i$. Therefore, we have

$$\mathcal{J}_{r+1}\big(I\widehat{S}; \mathrm{Der}_k\big(\widehat{S}\big)\big) + I\widehat{S} = \mathcal{J}_{r+1}(I; \mathrm{Der}_k(S))\widehat{S} + I\widehat{S}\,.$$

We also note that $\widehat{S}/P\widehat{S}$ is regular and $P\widehat{S} \subsetneq I\widehat{S}$. Therefore, we have $\mathcal{J}_{r+1}\big(I\widehat{S}; \mathrm{Der}_k\big(\widehat{S}\big)\big) \not\subset I\widehat{S}$ by Lemma 5.20; this shows the assertion $\mathcal{J}_{r+1}(I; \mathrm{Der}_k(S)) \not\subset I$. □

Proposition 5.23. *Let X be a scheme of finite type over $R = k[t][[x_1, \ldots, x_N]]$. Suppose that each irreducible component X_i of X has $\dim' X_i \geqslant n+1$. For each $a \in k$, we denote by X_a the closed subscheme of X defined by $(t-a)\mathcal{O}_X$. Suppose that for any $a \in k^\times$, the scheme X_a is integral regular and has $\dim' X_a = n$. Then, there is no thin cylinder C of X_∞ containing a k-arc γ with $\mathrm{ord}_\gamma(\mathfrak{o}_X) = \infty$.*

Proof. We may assume that X is affine, and we may write $X = \operatorname{Spec} A$ with $A = S/I$, where $S := k[t][[x_1, \ldots, x_N]][y_1, \ldots, y_m]$ and I is an ideal of S.

Suppose to the contrary that there exists a thin cylinder C containing a k-arc γ with $\mathrm{ord}_\gamma(\mathfrak{o}_X) = \infty$. Replacing C with $C \cap \mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_X)$, we may assume $C \subset \mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_X)$. It follows by Lemma 5.19 that $e := \mathrm{ord}_\gamma\big(\mathrm{Fitt}^n\big(\Omega'_{X/k[t]}\big)\big) < \infty$. By replacing C with the intersection $C \cap \mathrm{Cont}^e\big(\mathrm{Fitt}^n\big(\Omega'_{X/k[t]}\big)\big)$, we may assume $\emptyset \neq C \subset \mathrm{Cont}^e\big(\mathrm{Fitt}^n\big(\Omega'_{X/k[t]}\big)\big)$.

Let $\mathcal{S}$ be the set of the closed subschemes W of X with the following condition:

- There exists a cylinder C' of X_∞ such that $\gamma \in C' \subset C$ and $C' \subset W_\infty$.

Let W be a minimal element of $\mathcal{S}$, and let C' be a corresponding cylinder of X. Then W is reduced by the minimality. Let $W = W_1 \cup \cdots \cup W_\ell$ be its irreducible decomposition. Since C is thin, we may assume $\dim W \leqslant n$. Here, we claim the following.

Claim 5.24. (1) *We have* $\gamma \in (W_i)_\infty$ *for each* $1 \leqslant i \leqslant \ell$.

(2) *We have* $\dim W_i = n$ *for each* $1 \leqslant i \leqslant \ell$.

(3) *Denote by* $Z_i \subset W_i$ *the closed subscheme defined by* $\mathrm{Fitt}^{n-1}(\Omega'_{W_i/k[t]})$. *Then* $Z_i \subsetneq W_i$ *for each* $1 \leqslant i \leqslant \ell$.

First, we assume this claim and finish the proof. By Claim 5.24(3) and the minimality of W, we have

$$C' \not\subset (Z_1 \cup \cdots \cup Z_\ell)_\infty = (Z_1)_\infty \cup \cdots \cup (Z_\ell)_\infty .$$

Take a k-arc $\beta \in C' \setminus \big((Z_1)_\infty \cup \cdots \cup (Z_\ell)_\infty\big)$. For each i, we denote by $I_{Z_i} \subset A$ the ideal corresponding to Z_i, and we set $q_i := \mathrm{ord}_\beta(I_{Z_i}) < \infty$. Then

$$C'' := C' \cap \bigcap_i \mathrm{Cont}^{q_i}(I_{Z_i})$$

is a non-empty cylinder of X_∞. By applying Proposition 5.9(2) to W_i and its cylinder

$$C'' \cap (W_i)_\infty \subset \mathrm{Cont}^{q_i}\big(\mathrm{Fitt}^{n-1}(\Omega'_{W_i/k[t]})\big) ,$$

it follows that the truncation map $\psi_{m+1}\big(C''\cap(W_i)_\infty\big) \to \psi_m\big(C''\cap(W_i)_\infty\big)$ has $(n-1)$-dimensional fibers for sufficiently large m. Therefore,

$$\psi_{m+1}(C'') = \bigcup_i \psi_{m+1}\big(C'' \cap (W_i)_\infty\big) \longrightarrow \psi_m(C'') = \bigcup_i \psi_m\big(C'' \cap (W_i)_\infty\big)$$

also has $(n-1)$-dimensional fibers for sufficiently large m. However, by Proposition 5.9(2), it should have n-dimensional fibers because $\emptyset \neq C'' \subset \mathrm{Cont}^e\big(\mathrm{Fitt}^n(\Omega'_{X/k[t]})\big)$. We get a contradiction. Therefore, it is sufficient to prove Claim 5.24(3). □

Proof of Claim 5.24. (1) Suppose to the contrary that $\gamma \notin (W_1)_\infty$. Let $I_{W_1} \subset A$ be the ideal corresponding to W_1, and let $q := \mathrm{ord}_\gamma(I_{W_1}) < \infty$. Then the cylinder $C'' := C' \cap \mathrm{Cont}^q(I_{W_1})$ contains γ and satisfies

$$C'' \subset W_\infty \setminus (W_1)_\infty \subset (W_2 \cup \cdots \cup W_\ell)_\infty ,$$

which contradicts the minimality of W.

(2) Suppose to the contrary that $\dim W_1 \leqslant n-1$. Let $W' := W_2 \cup \cdots \cup W_\ell$, and let $I_{W'} \subset A$ be the ideal corresponding to W'. By the minimality of W, it follows that $C' \not\subset W'_\infty$. Therefore, we have $C'' := C' \cap \mathrm{Cont}^q(I_{W'}) \neq \emptyset$ for some $q \geqslant 0$. Since $C'' \cap W'_\infty = \emptyset$, we have $C'' \subset (W_1)_\infty$, and C'' turns out to be a very thin cylinder of X_∞. This contradicts $\emptyset \neq C'' \subset \mathrm{Cont}^e\big(\mathrm{Fitt}^n(\Omega'_{X/k[t]})\big)$ by Lemma 5.18.

(3) Let H be one of the W_i. Let Q be the prime ideal of S corresponding to H. Since H_∞ contains a k-arc, H contains a k-point. Therefore, by Lemma 2.11(4), we have

$$\mathrm{ht}\, Q = \dim S - \dim' H = \dim S - \dim H = N + m - n + 1 .$$

First, we prove that

$$Q + (x_1, \ldots, x_N) + (t-a) \neq S \tag{5.2}$$

for some $a \in k^\times$. Suppose to the contrary that $Q + (x_1, \ldots, x_N) + (t-a) = S$ for any $a \in k^\times$. Then by Hilbert's nullstellensatz, it follows that $t^\ell \in Q + (x_1, \ldots, x_N)$ for some $\ell \geqslant 0$. This contradicts $\gamma \in H_\infty$ and $\mathrm{ord}_\gamma(\mathfrak{o}_X) = \infty$, and we get (5.2) for some $a \in k^\times$.

The inequality (5.2) implies $Q + (t-a) \neq S$. Furthermore, we have $t - a \notin Q$ because H_∞ contains a k-arc. Therefore, we have $\mathrm{ht}(Q + (t-a)) = \mathrm{ht}\, Q + 1 = N + m - n + 2$.

We set

$$
\begin{aligned}
S_a &:= S/(t-a) \simeq k[[x_1,\dots,x_N]][y_1,\dots,y_m]\,,\\
I_a &:= (I+(t-a))/(t-a)\,,\quad Q_a := (Q+(t-a))/(t-a)\,.
\end{aligned}
$$

Then we have $\mathrm{ht}(Q_a) = N+m-n+1$. Furthermore, we have $\mathrm{ht}(I_a) = N+m-n$ by the assumption $\dim' X_a = n$. Therefore, it follows that $Q_a \supsetneq I_a$. Let $J := \mathcal{J}_{N+m-n+1}\big(Q;\mathrm{Der}_{k[t]}(S)\big)$. Then the ideal $(J+(t-a))/(t-a)$ of S_a coincides with $\mathcal{J}_{N+m-n+1}\big(Q_a;\mathrm{Der}_k(S_a)\big)$. Note that S_a/I_a is regular, $\mathrm{ht}(I_a) = N+m-n$ and $Q_a \supsetneq I_a$. Therefore, by Lemma 5.22, we have

$$
(J+(t-a))/(t-a) = \mathcal{J}_{N+m-n+1}\big(Q_a;\mathrm{Der}_k(S_a)\big) \not\subset Q_a = (Q+(t-a))/(t-a)\,.
$$

In particular, we have $J \not\subset Q$.

Since $\mathrm{Fitt}^{n-1}\big(\Omega'_{H/k[t]}\big) = (J+Q)/Q$, this completes the proof of assertion (3). □

Lemma 5.25. *Let $R = k[t][[x_1,\dots,x_N]]$. Let $e_1,\dots,e_N$ and d be integers satisfying $0 < e_i \leqslant d$ for each i. For each $c \in k^\times$, let $T_c\colon R \to R$ be the ring isomorphism defined by $T_c(t) = c^{-d}t$ and $T_c(x_i) = c^{e_i}x_i$. Let I be an ideal of R that is T_c-invariant (that is, $T_c(I) = I$) for any $c \in k^\times$. Let P be a minimal prime of I. Then P satisfies one of the following conditions:*

(1) *$P \cap k[t] \neq (0)$ and $t \in P$;*
(2) *$P \cap k[t] = (0)$ and $P + (t-a) \neq R$ for any $a \in k^\times$;*
(3) *$P \cap k[t] = (0)$, and there exists an $f \in P$ such that*

$$
f - t^\ell \in \big(t^{\ell+1}x_1,\dots,t^{\ell+1}x_N\big) + (x_1,\dots,x_N)^{\ell+1}
$$

for some $\ell \geqslant 0$.

Proof. First, we prove that P is also T_c-invariant for any $c \in k^\times$. Let $P_1,\dots,P_m$ be the minimal primes of I. Since T_c is an isomorphism, it induces a permutation on $P_1,\dots,P_m$. Let $p\colon k^\times \to \mathfrak{S}_m$ be the induced group homomorphism, where $\mathfrak{S}_m$ is the symmetric group of degree m. For any $c \in k^\times$, we can take a $b \in k^\times$ such that $c = b^{m!}$. Therefore, we have $p(c) = p(b^{m!}) = (p(b))^{m!} = 1$. This shows that $T_c(P_i) = P_i$ for any $c \in k^\times$ and $1 \leqslant i \leqslant m$.

Suppose $P \cap k[t] \neq (0)$. Then $t - a \in P$ for some $a \in k$. Since P is T_c-invariant for any $c \in k^\times$, it follows that $a = 0$. Therefore, P satisfies condition (1).

Suppose $P \cap k[t] = (0)$ and $P + (t-a) = R$ for some $a \in k^\times$. We will prove that P satisfies condition (3). Since P is T_c-invariant for any $c \in k^\times$, it follows that $P + (t-a) = R$ for any $a \in k^\times$. Then by Hilbert's nullstellensatz, it follows that

- $t^\ell \in P + (x_1,\dots,x_N)$ for some $\ell \geqslant 0$.

Therefore, there exists a $g \in P$ such that $g - t^\ell \in (x_1,\dots,x_N)$.

We set $M := (x_1,\dots,x_N) \subset k[t][[x_1,\dots,x_N]]$. Since M is T_c-invariant, T_c induces an automorphism on $k[t][[x_1,\dots,x_N]]/M^{\ell+1}$. Hence, $k[t][[x_1,\dots,x_N]]/M^{\ell+1}$ has a graded ring structure satisfying $\deg t = -d$ and $\deg x_i = e_i$. Then $\big(P + M^{\ell+1}\big)/M^{\ell+1}$ is a homogeneous ideal. Therefore, the term $g_{-d\ell}$ of g with degree $-d\ell$ is contained in $P + M^{\ell+1}$. We may write $g_{-d\ell} = f - h$ with $f \in P$ and $h \in M^{\ell+1}$. On the other hand, since $g_{-d\ell} - t^\ell \in M$, we see that $g_{-d\ell} - t^\ell \in (t^{\ell+1}x_1,\dots,t^{\ell+1}x_N)$ by looking at the degrees of its terms. Therefore, condition (3) holds for this f. □

Remark 5.26. Let I and P be as in Lemma 5.25. Then the following hold for $Y := \mathrm{Spec}(R/P)$:

- $Y_\infty = \emptyset$ if P is of the form (1),
- $Y_\infty \cap \mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_Y) = \emptyset$ if P is of the form (3).

5.2 Arc spaces of affine formal $k[[t]]$-schemes

In this subsection, we discuss the arc space of X of the form $X = \operatorname{Spec}\bigl(k[x_1, \dots, x_N][[t]]/I\bigr)$. As we will mention in Remark 5.27, the arc space of X can be seen as the Greenberg scheme of the corresponding affine formal scheme. In this subsection, we do not deal with general formal $k[[t]]$-schemes.

Remark 5.27. Sebag in [Seb04] investigates the theory of arc spaces of formal $k[[t]]$-schemes with k a perfect field, and the theory can be applied to $X = \operatorname{Spec}\bigl(k[x_1, \dots, x_N][[t]]/I\bigr)$, which is dealt with in this subsection. The reader is also referred to [CNS18] for this theory.

With a scheme $X = \operatorname{Spec}\bigl(k[x_1, \dots, x_N][[t]]/I\bigr)$, we can associate the formal affine scheme $\mathcal{X} = \operatorname{Spf}\bigl(k[x_1, \dots, x_N][[t]]/I\bigr)$. Then the Greenberg schemes $\mathrm{Gr}_m(\mathcal{X})$ and $\mathrm{Gr}(\mathcal{X})$ defined in [Seb04] are isomorphic to X_m and X_∞, respectively. Therefore, the theory of Greenberg schemes developed in [Seb04] and [CNS18] can be applied to the arc space X_∞ of X.

Definition 5.28 (cf. [CNS18, Appendix 3.3]). Let I be an ideal of $S = k[x_1, \dots, x_N][[t]]$, and let $A := S/I$. Then we denote by $\widehat{\Omega}_{A/k[[t]]}$ the completion of the A-module $\Omega_{A/k[[t]]}$ with respect to the (t)-adic topology; that is,

$$\widehat{\Omega}_{A/k[[t]]} := \varprojlim_n \bigl(\Omega_{A/k[[t]]}/(t^n)\Omega_{A/k[[t]]}\bigr)\,.$$

The canonical derivation $\mathrm{d}_{A/k[[t]]}\colon A \to \Omega_{A/k[[t]]}$ induces a derivation

$$\widehat{\mathrm{d}}_{A/k[[t]]}\colon A \longrightarrow \widehat{\Omega}_{A/k[[t]]}\,.$$

We sometimes abbreviate $\widehat{\mathrm{d}}_{A/k[[t]]}$ to $\widehat{\mathrm{d}}$.

When $X = \operatorname{Spec} A$, we denote by $\widehat{\Omega}_{X/k[[t]]}$ the sheaf on X associated with the A-module $\widehat{\Omega}_{A/k[[t]]}$.

Remark 5.29 (cf. [CNS18, Example 3.3.5 in the appendix]). (1) $\widehat{\Omega}_{S/k[[t]]}$ is a free S-module of rank N with basis $\widehat{\mathrm{d}}_{S/k[[t]]}(x_1), \dots, \widehat{\mathrm{d}}_{S/k[[t]]}(x_N)$. Furthermore, we have an exact sequence

$$I/I^2 \xrightarrow{\delta} \widehat{\Omega}_{S/k[[t]]} \otimes_S A \xrightarrow{\alpha} \widehat{\Omega}_{A/k[[t]]} \longrightarrow 0$$

of A-modules, where α is the map satisfying $\alpha\bigl(\widehat{\mathrm{d}}_{S/k[[t]]}(g) \otimes 1\bigr) = \widehat{\mathrm{d}}_{A/k[[t]]}(\overline{g})$ for $g \in S$, and δ is the map satisfying $\delta(\overline{g}) = \widehat{\mathrm{d}}_{S/k[[t]]}(g) \otimes 1$ for $g \in I$. In particular, $\widehat{\Omega}_{A/k[[t]]}$ is a finite A-module.

(2) The canonical derivation $\widehat{\mathrm{d}}\colon A \to \widehat{\Omega}_{A/k[[t]]}$ has the following universal property:

(2-1) The induced map

$$\operatorname{Hom}_A\bigl(\widehat{\Omega}_{A/k[[t]]}, M\bigr) \longrightarrow \operatorname{Der}_{k[[t]]}(A, M)\,, \quad f \longmapsto f \circ \widehat{\mathrm{d}}$$

is an isomorphism for any A-module M that is complete with respect to the (t)-adic topology. In particular, this map is an isomorphism for any finite A-module (cf. [Mat89, Theorem 8.7]).

This follows from the following general fact from [Gro64, (20.4.8.2)]:

(2-2) Let B be a topological ring and C a topological B-algebra. Let N be a topological C-module. Then we have an isomorphism

$$\operatorname{Hom}^{\mathrm{c}}_C(\Omega_{C/B}, N) \xrightarrow{\simeq} \operatorname{Der}^{\mathrm{c}}_B(C, N)\,, \quad f \longmapsto f \circ \mathrm{d}_{C/B}\,.$$

Here, $\operatorname{Hom}^{\mathrm{c}}_C(\Omega_{C/B}, N)$ denotes the set of the continuous homomorphisms $\Omega_{C/B} \to N$ of C-modules, and $\operatorname{Der}^{\mathrm{c}}_B(C, N)$ denotes the set of the continuous B-derivations $C \to N$. The topology on $\Omega_{C/B} = \mathcal{I}/\mathcal{I}^2$ is defined as the quotient topology, where $\mathcal{I}$ is the kernel of the augmentation map $C \otimes_B C \to C$, $a \otimes b \mapsto ab$.

In our case, the topology on $\Omega_{A/k[[t]]}$ coincides with the (t)-adic topology (cf. [Gro64, Proposition 20.4.5]). Therefore, we have

$$\mathrm{Hom}^{\mathrm{c}}_A\big(\Omega_{A/k[[t]]}, M\big) = \mathrm{Hom}_A\big(\Omega_{A/k[[t]]}, M\big)\,, \quad \mathrm{Der}^{\mathrm{c}}_{k[[t]]}(A, M) = \mathrm{Der}_{k[[t]]}(A, M)$$

for any A-module M with the (t)-adic topology. Hence, by assertion (2-2), we have an isomorphism

$$\mathrm{Hom}_A\big(\Omega_{A/k[[t]]}, M\big) \xrightarrow{\simeq} \mathrm{Der}_{k[[t]]}(A, M)$$

for any A-module M. Moreover, if M is complete with respect to the (t)-adic topology, then we have an isomorphism

$$\mathrm{Hom}_A\big(\widehat{\Omega}_{A/k[[t]]}, M\big) \xrightarrow{\simeq} \mathrm{Hom}_A\big(\Omega_{A/k[[t]]}, M\big)\,,$$

which shows property (2-1).

(3) By part (2), it follows that $\mathrm{Der}_{k[[t]]}(S)$ is a free S-module of rank N generated by the $\partial/\partial x_i$. Therefore, by the exact sequence in part (1), we have

$$\mathrm{Fitt}^n\big(\widehat{\Omega}_{A/k[[t]]}\big) = \big(\mathcal{J}_{N-n}\big(I; \mathrm{Der}_{k[[t]]}(S)\big) + I\big)/I\,.$$

LEMMA 5.30. *Let n and e be non-negative integers, and let X be an affine scheme of the form $X = \mathrm{Spec}\big(k[x_1, \ldots, x_N][[t]]/I\big)$. Suppose that each irreducible component X_i of X has $\dim X_i \geqslant n+1$. Let $\gamma \in \mathrm{Cont}^e\big(\mathrm{Fitt}^n\big(\widehat{\Omega}_{X/k[[t]]}\big)\big)$ be a k-arc. Then we have*

$$\gamma^* \widehat{\Omega}_{X/k[[t]]} \simeq k[[t]]^{\oplus n} \oplus \bigoplus_i k[t]/(t^{e_i})$$

as $k[[t]]$-modules with $\sum_i e_i = e$.

Proof. The same proofs as in Lemmas 5.6 and 5.7 work. Note that any minimal prime P of I satisfies $\mathrm{ht}\, P \leqslant N - n$. This is because

$$\mathrm{ht}\, P = \dim S - \dim(S/P) \leqslant (N+1) - (n+1) = N - n\,,$$

where we set $S := k[x_1, \ldots, x_N][[t]]$. The first equality follows from the facts that any maximal ideal M of S has $\mathrm{ht}\, M = N+1$ and S is a catenary ring. □

PROPOSITION 5.31. *Let n be a non-negative integer, and let X be an affine scheme of the form $X = \mathrm{Spec}\big(k[x_1, \ldots, x_N][[t]]/I\big)$. Suppose that each irreducible component X_i of X has $\dim X_i \geqslant n+1$. Then, there exists a positive integer c such that the following hold for non-negative integers m and e with $m \geqslant ce$:*

(1) *We have*

$$\psi_m\big(\mathrm{Cont}^e\big(\mathrm{Fitt}^n\big(\widehat{\Omega}_{X/k[[t]]}\big)\big)\big) = \pi_{m+e,m}\big(\mathrm{Cont}^e\big(\mathrm{Fitt}^n\big(\widehat{\Omega}_{X/k[[t]]}\big)\big)_{m+e}\big)\,.$$

(2) *The truncation map $\pi_{m+1,m}\colon X_{m+1} \to X_m$ induces a piecewise-trivial fibration*

$$\psi_{m+1}\big(\mathrm{Cont}^e\big(\mathrm{Fitt}^n\big(\widehat{\Omega}_{X/k[[t]]}\big)\big)\big) \longrightarrow \psi_m\big(\mathrm{Cont}^e\big(\mathrm{Fitt}^n\big(\widehat{\Omega}_{X/k[[t]]}\big)\big)\big)$$

with fiber $\mathbb{A}^n$.

Proof. The same proof as for Proposition 5.9 works (cf. Remark 4.6). □

Remark 5.32. When X is flat over $k[[t]]$, Proposition 5.31(1) is proved in [CNS18, Chapter 5, Proposition 2.3.4], and Proposition 5.31(2) is proved in [Seb04, Lemme 4.5.4] (cf. [CNS18, Chapter 5, Theorem 2.3.11]). We also note that Proposition 5.31(2) can be reduced to the flat case by the argument in [NS22, Remark 2.14(3)].

We define cylinders and their codimensions.

Definition 5.33. Let n be a non-negative integer, and let X be an affine scheme of the form $X = \operatorname{Spec}\bigl(k[x_1, \ldots, x_N][[t]]/I\bigr)$. Suppose that each irreducible component X_i of X has $\dim X_i \geqslant n+1$. A subset $C \subset X_\infty$ is called a *cylinder* if $C = \psi_m^{-1}(S)$ for some $m \geqslant 0$ and a constructible subset $S \subset X_m$. We define the codimension of C as follows:

(1) Assume $C \subset \operatorname{Cont}^e\bigl(\operatorname{Fitt}^n\bigl(\widehat{\Omega}_{X/k[[t]]}\bigr)\bigr)$ for some $e \in \mathbb{Z}_{\geqslant 0}$. Then we define the codimension of C in X_∞ as
$$\operatorname{codim}(C) := (m+1)n - \dim(\psi_m(C))$$
for any sufficiently large m. This is well defined by Proposition 5.31.

(2) In general, we define the codimension of C in X_∞ as follows:
$$\operatorname{codim}(C) := \min_{e \in \mathbb{Z}_{\geqslant 0}} \operatorname{codim}\Bigl(C \cap \operatorname{Cont}^e\bigl(\operatorname{Fitt}^n\bigl(\widehat{\Omega}_{X/k[[t]]}\bigr)\bigr)\Bigr).$$
By convention, $\operatorname{codim}(C) = \infty$ if $C \cap \operatorname{Cont}^e\bigl(\operatorname{Fitt}^n\bigl(\widehat{\Omega}_{X/k[[t]]}\bigr)\bigr) = \emptyset$ for any $e \geqslant 0$.

Remark 5.34. As in Remark 5.12, the definition of the codimension above depends on the choice of n.

Definition 5.35. Let n be a non-negative integer, and let X be an affine scheme of the form $X = \operatorname{Spec}\bigl(k[x_1, \ldots, x_N][[t]]/I\bigr)$. Suppose that each irreducible component X_i of X has $\dim X_i \geqslant n+1$. A subset $A \subset X_\infty$ is called *thin* if $A \subset Z_\infty$ for some closed subscheme Z of X with $\dim Z \leqslant n$.

Proposition 5.36 (cf. [Seb04, Théorème 6.3.5]). *Let n be a non-negative integer, and let X be an affine scheme of the form $X = \operatorname{Spec}\bigl(k[x_1, \ldots, x_N][[t]]/I\bigr)$. Suppose that each irreducible component X_i of X has $\dim X_i \geqslant n+1$. Let C be a cylinder in X_∞. Let $\{C_\lambda\}_{\lambda \in \Lambda}$ be a set of countably many disjoint subcylinders $C_\lambda \subset C$. If $C \setminus (\bigsqcup_{\lambda \in \Lambda} C_\lambda) \subset X_\infty$ is a thin set, then we have*
$$\operatorname{codim}(C) = \min_{\lambda \in \Lambda} \operatorname{codim}(C_\lambda).$$

Proof. This follows from [CNS18, Chapter 6, Lemma 3.4.1 and Example 3.5.2]. □

Lemma 5.37 (cf. [CNS18, Chapter 5, Proposition 2.2.6]). *Let X be an affine scheme of the form $X = \operatorname{Spec}\bigl(k[x_1, \ldots, x_N][[t]]/I\bigr)$. Let p and m be non-negative integers with $2p+1 \geqslant m \geqslant p$. Let $\gamma \in X_p(k)$ be a jet with $\pi_{m,p}^{-1}(\gamma) \neq \emptyset$. Then we have*
$$\pi_{m,p}^{-1}(\gamma) \simeq \operatorname{Hom}_{k[t]/(t^{p+1})}\bigl(\gamma^*\widehat{\Omega}_{X/k[[t]]}, (t^{p+1})/(t^{m+1})\bigr).$$

Proof. We set $A := k[x_1, \ldots, x_N][[t]]/I$. For the same reason as in the proof of Lemma 5.13, we have $\pi_{m,p}^{-1}(\gamma) \simeq \operatorname{Der}_{k[[t]]}\bigl(A, (t^{p+1})/(t^{m+1})\bigr)$. Furthermore, by the universal property of $\widehat{\Omega}_{A/k[[t]]}$ (cf. Remark 5.29(2)), we have
$$\operatorname{Der}_{k[[t]]}\bigl(A, (t^{p+1})/(t^{m+1})\bigr) \simeq \operatorname{Hom}_A\bigl(\widehat{\Omega}_{A/k[[t]]}, (t^{p+1})/(t^{m+1})\bigr),$$
which proves the assertion. □

5.3 Codimension formulae

In this subsection, we discuss a $k[t]$-morphism $f\colon X \to Y$ of affine $k[t]$-schemes in the following two cases:

(a) We have $X = \operatorname{Spec}\bigl(k[x_1, \dots, x_N][[t]]/I\bigr)$ and $Y = \operatorname{Spec}\bigl(k[x_1, \dots, x_M][[t]]/J\bigr)$.

(b) We have $X = \operatorname{Spec}\bigl(k[x_1, \dots, x_M][[t]]/I\bigr)$ and $Y = \operatorname{Spec}\bigl(k[t][[x_1, \dots, x_L]]/J\bigr)$. Furthermore, f satisfies $(x_1, \dots, x_L)\mathcal{O}_X \subset (t)$.

Lemma 5.38. *In case (a) above, the canonical map $f^*\widehat{\Omega}_{Y/k[[t]]} \to \widehat{\Omega}_{X/k[[t]]}$ is induced. In case (b), the canonical map $f^*\Omega'_{Y/k[t]} \to \widehat{\Omega}_{X/k[[t]]}$ is induced.*

Proof. Let $A = \mathcal{O}_X$ and $B = \mathcal{O}_Y$ be the corresponding rings and $g\colon B \to A$ the corresponding $k[t]$-ring homomorphism.

First, we deal with case (a). Since $g\colon B \to A$ is a $k[t]$-ring homomorphism, $\widehat{\Omega}_{A/k[[t]]}$ is a complete B-module with respect to the (t)-adic topology. Therefore, by the universal property of $\widehat{\Omega}_{B/k[[t]]}$ (cf. Remark 5.29(2)), the derivation $\widehat{\mathrm{d}}_{A/k[[t]]} \circ g\colon B \to \widehat{\Omega}_{A/k[[t]]}$ factors through $\widehat{\Omega}_{B/k[[t]]}$. This completes the proof in case (a).

Next, we deal with case (b). By the universal property of $\Omega'_{B/k[t]}$, it is sufficient to show that the composition

$$B \xrightarrow{g} A \xrightarrow{\widehat{\mathrm{d}}_{A/k[[t]]}} \widehat{\Omega}_{A/k[[t]]}$$

is a special B-derivation. Note that $\widehat{\Omega}_{A/k[[t]]}$ is a complete A-module with respect to the (t)-adic topology. Since $g\bigl((x_1, \dots, x_L)\bigr) \subset (t)$ by assumption, $\widehat{\Omega}_{A/k[[t]]}$ is a separated B-module with respect to the $(x_1, \dots, x_L)$-adic topology. Therefore $\widehat{\mathrm{d}}_{A/k[[t]]} \circ g$ is a special B-derivation by Lemma 2.2. $\square$

We define the order of the Jacobian for a morphism.

Definition 5.39. (1) Let $f\colon X \to Y$ be a morphism of affine $k[t]$-schemes of the form (a) above. Then f induces a homomorphism $f^*\widehat{\Omega}_{Y/k[[t]]} \to \widehat{\Omega}_{X/k[[t]]}$ by Lemma 5.38. Let $\gamma \in X_\infty$ be a k-arc, let $\gamma' := f_\infty(\gamma)$. Finally, let S be the torsion part of $\gamma^*\widehat{\Omega}_{X/k[[t]]}$. Then we define the *order* $\operatorname{ord}_\gamma\bigl(\mathrm{jac}_f\bigr)$ *of the Jacobian* of f at γ as the length of the $k[[t]]$-module

$$\operatorname{Coker}\bigl(\gamma'^*\widehat{\Omega}_{Y/k[[t]]} \longrightarrow \gamma^*\widehat{\Omega}_{X/k[[t]]}/S\bigr)\,.$$

In particular, if $\operatorname{ord}_\gamma\bigl(\mathrm{jac}_f\bigr) < \infty$, then we have

$$\operatorname{Coker}\bigl(\gamma'^*\widehat{\Omega}_{Y/k[[t]]} \longrightarrow \gamma^*\widehat{\Omega}_{X/k[[t]]}/S\bigr) \simeq \bigoplus_i k[t]/(t^{e_i})$$

as $k[[t]]$-modules with some positive integers e_i satisfying $\sum_i e_i = \operatorname{ord}_\gamma\bigl(\mathrm{jac}_f\bigr)$.

(2) Let $f\colon X \to Y$ be a morphism of affine $k[t]$-schemes of the form (b) above. Then f induces a homomorphism $f^*\Omega'_{Y/k[t]} \to \widehat{\Omega}_{X/k[[t]]}$ by Lemma 5.38. Let $\gamma \in X_\infty$ be a k-arc, and let $\gamma' := f_\infty(\gamma)$. Finally, let S be the torsion part of $\gamma^*\widehat{\Omega}_{X/k[[t]]}$. Then we also define the *order* $\operatorname{ord}_\gamma\bigl(\mathrm{jac}_f\bigr)$ *of the Jacobian* of f at γ as the length of the $k[[t]]$-module

$$\operatorname{Coker}\bigl(\gamma'^*\Omega'_{Y/k[t]} \longrightarrow \gamma^*\widehat{\Omega}_{X/k[[t]]}/S\bigr)\,.$$

(3) By abuse of notation, for $e \geqslant 0$, we define

$$\operatorname{Cont}^e\bigl(\mathrm{jac}_f\bigr) := \{\gamma \in X_\infty \mid \operatorname{ord}_\gamma\bigl(\mathrm{jac}_f\bigr) = e\}\,.$$

Lemma 5.40. *Let n be a non-negative integer. Let $f\colon X \to Y$ be a morphism of affine $k[t]$-schemes of the form (a) above. Let $g\colon Y \to Z$ be a morphism of affine $k[t]$-schemes of the form (b). Suppose that each irreducible component W_i of X, Y and Z has $\dim W_i \geqslant n+1$. Let $\gamma \in X_\infty$ be a k-arc, and let $\gamma' := f_\infty(\gamma)$. Suppose*

$$\mathrm{ord}_\gamma\big(\mathrm{Fitt}^n\big(\widehat{\Omega}_{X/k[[t]]}\big)\big) < \infty\,, \quad \mathrm{ord}_{\gamma'}\big(\mathrm{Fitt}^n\big(\widehat{\Omega}_{Y/k[[t]]}\big)\big) < \infty\,.$$

Then we have

$$\mathrm{ord}_\gamma\big(\mathrm{jac}_{g\circ f}\big) = \mathrm{ord}_\gamma\big(\mathrm{jac}_f\big) + \mathrm{ord}_{\gamma'}\big(\mathrm{jac}_g\big)\,.$$

Proof. The same proof as in [NS22, Lemma 2.10] works due to Lemma 5.30. □

Remark 5.41. The same statement holds if g is of the form (a).

Remark 5.42. We note in this remark that all propositions (Proposition 2.29, Lemmas 2.31 and 2.32, and Proposition 2.33) in [NS22, Section 2.6] can be seen to be true for a $k[t]$-morphism f of the form (a) or (b) by making the following modifications:

- Replace the conditions $(\star)_n$ and $(\star\star)_n$ in [NS22] on X with the following condition:
  - "Each irreducible component X_i of X has $\dim X_i \geqslant n+1$."
- Replace $\Omega_{-/k[t]}$ with $\widehat{\Omega}_{-/k[[t]]}$ and $\Omega'_{-/k[t]}$, and replacing $\mathrm{Jac}_{-/k[t]}$ with $\mathrm{Fitt}^n\big(\widehat{\Omega}_{-/k[[t]]}\big)$ and $\mathrm{Fitt}^n\big(\Omega'_{-/k[t]}\big)$.

We note that [NS22, Proposition 2.29(2), Lemmas 2.31 and 2.32, and Proposition 2.33] are formal consequences of [NS22, Proposition 2.29(1), Lemma 2.13(1) and Proposition 2.17]. The formal power series ring versions of [NS22, Proposition 2.29(1), Lemma 2.13(1) and Proposition 2.17] are proved in Lemmas 5.13 and 5.37, Lemmas 5.7 and 5.30, and Propositions 5.9 and 5.31 in this paper.

Furthermore, [NS22, Lemma 2.34] is also true in the formal power series ring setting as can be seen by replacing $A = \mathrm{Spec}\, k[t][x_1, \dots, x_N]$ with $\mathrm{Spec}\, k[x_1, \dots, x_N][[t]]$. Indeed, the proof of [NS22, Lemma 2.34] works in this setting.

Proposition 5.43. *Let n be a non-negative integer. Let $f\colon X \to Y$ be a morphism of affine $k[t]$-schemes of the form (b) above. Suppose that each irreducible component W_i of X and Y has $\dim W_i \geqslant n+1$. Let $e, e', e'' \in \mathbb{Z}_{\geqslant 0}$. Let $A \subset X_\infty$ be a cylinder, and let $B = f_\infty(A)$. Assume*

$$A \subset \mathrm{Cont}^{e''}\big(\mathrm{Fitt}^n\big(\widehat{\Omega}_{X/k[[t]]}\big)\big) \cap \mathrm{Cont}^{e}\big(\mathrm{jac}_f\big)\,, \quad B \subset \mathrm{Cont}^{e'}\big(\mathrm{Fitt}^n\big(\Omega'_{Y/k[t]}\big)\big)\,.$$

Then, B is a cylinder of Y_∞ contained in $\mathrm{Cont}^{\geqslant 1}(\mathfrak{o}_Y)$, where $\mathfrak{o}_Y \subset \mathcal{O}_Y$ is the ideal sheaf generated by $x_1, \dots, x_L \in \mathcal{O}_Y$. Moreover, if $f_\infty|_A$ is injective, then it follows that $\mathrm{codim}(A) + e = \mathrm{codim}(B)$.

Proof. By Remark 5.42, the same proofs as in [NS22, Section 2.6] work by making the following modifications:

- Replace $\Omega_{X/k[t]}$ with $\widehat{\Omega}_{X/k[[t]]}$, and $\Omega_{Y/k[t]}$ with $\Omega'_{Y/k[t]}$.
- Replace $\mathrm{Jac}_{X/k[t]}$ with $\mathrm{Fitt}^n\big(\widehat{\Omega}_{X/k[[t]]}\big)$, and $\mathrm{Jac}_{Y/k[t]}$ with $\mathrm{Fitt}^n\big(\Omega'_{Y/k[t]}\big)$. □

Proposition 5.44. *Suppose that a finite group G acts on the ring $k[x_1, \dots, x_N]$ over k. Let $I \subset k[x_1, \dots, x_N]^G[[t]]$ be an ideal, and let $I' \subset k[x_1, \dots, x_N][[t]]$ be the ideal generated by I. Set*

$$X := \mathrm{Spec}\big(k[x_1, \dots, x_N][[t]]/I'\big) \quad \textit{and} \quad Y := \mathrm{Spec}\big(k[x_1, \dots, x_N]^G[[t]]/I\big),$$

and let $f\colon X \to X/G = Y$ be the quotient morphism. Suppose that each irreducible component W_i of X and Y has $\dim W_i \geqslant n+1$. Let $A \subset X_\infty$ be a G-invariant cylinder, and let $B = f_\infty(A)$. Let $e, e', e'' \in \mathbb{Z}_{\geqslant 0}$. Assume

$$A \subset \mathrm{Cont}^{e''}\big(\mathrm{Fitt}^n\big(\widehat{\Omega}_{X/k[[t]]}\big)\big) \cap \mathrm{Cont}^{e}\big(\mathrm{jac}_f\big)\,, \quad B \subset \mathrm{Cont}^{e'}\big(\mathrm{Fitt}^n\big(\widehat{\Omega}_{Y/k[[t]]}\big)\big)\,.$$

Then B is a cylinder of Y_∞ with $\mathrm{codim}(A) + e = \mathrm{codim}(B)$.

Proof. Note that f is a morphism of the form (a). By Remark 5.42, the proof of [NS22, Proposition 2.35] works. $\square$

Remark 5.45. (1) Proposition 5.43 is also true for $k[t]$-morphisms of the form (a). Indeed, this is known for morphisms $f\colon X \to Y$ of formal $k[[t]]$-schemes (not necessarily affine). When X is smooth over $k[[t]]$, this is proved in [Seb04, Lemme 7.1.3] (cf. [CNS18, Chapter 5, Theorem 3.2.2]). The general case is proved in [Yas24, Lemmas 10.19 and 10.20]. We also note that Yasuda proves it in the more general setting, for formal Deligne–Mumford stacks of arbitrary characteristics.

(2) Proposition 5.43 is also true for a $k[t]$-morphism $f\colon X \to Y$ of $k[t][[x_1, \dots, x_N]]$-schemes of finite type (not necessarily affine) by making the following modification:

- Replace the condition "each irreducible component W_i of X and Y has $\dim W_i \geqslant n+1$" in Proposition 5.43 with "each irreducible component W_i of X and Y has $\dim' W_i \geqslant n+1$."

6. The theory of Denef and Loeser for quotient singularities

In this section, we review the theory of arc spaces of quotient varieties established by Denef and Loeser [DL02] (cf. [Yas16], [NS22, Section 3]). We explain their theory in the formal power series ring setting.

Let d be a positive integer and $\xi \in k$ a primitive dth root of unity. Let $G \subset \mathrm{GL}_N(k)$ be a finite subgroup with order d that acts linearly on $\overline{A} := \widehat{\mathbb{A}}_k^N := \mathrm{Spec}\, k[[x_1, \dots, x_N]]$. Let $\overline{X} \subset \overline{A}$ be a G-invariant subscheme. We denote by $A := \overline{A}/G$ and $X := \overline{X}/G$ the quotient schemes. Let $Z \subset A$ be the minimum closed subset such that $\overline{A} \to A$ is étale outside Z.

Fix $\gamma \in G$. Since G is a finite group, γ can be diagonalized in some new basis $x_1^{(\gamma)}, \dots, x_N^{(\gamma)}$. Let $\mathrm{diag}(\xi^{e_1}, \dots, \xi^{e_N})$ be the diagonal matrix with $0 < e_i \leqslant d$. Then we define a $k[t]$-ring homomorphism

$$\lambda_\gamma^*\colon k[t][[x_1, \dots, x_N]]^G \longrightarrow k[x_1, \dots, x_N]^{C_\gamma}[[t]]\,, \quad x_i^{(\gamma)} \longmapsto t^{e_i/d} x_i^{(\gamma)}\,,$$

where C_γ is the centralizer of γ in G. Let $i\colon k[x_1, \dots, x_N]^{C_\gamma}[[t]] \to k[x_1, \dots, x_N][[t]]$ be the inclusion map, and let $\overline{\lambda}_\gamma^* = i \circ \lambda_\gamma^*$ be the composite map.

Let $I_X \subset k[[x_1, \dots, x_N]]^G$ be the defining ideal of X in A. Let $I'_X \subset k[t][[x_1, \dots, x_N]]^G$ be the ideal generated by I_X. Let

$$\widetilde{I}_X^{(\gamma)} \subset k[x_1, \dots, x_N]^{C_\gamma}[[t]]\,, \quad \overline{I}_X^{(\gamma)} \subset k[x_1, \dots, x_N][[t]]$$

be the ideals generated by $\lambda_\gamma^*(I'_X)$ and $\overline{\lambda}_\gamma^*(I'_X)$, respectively. Then we have the following diagram:

$$\begin{tikzcd}
k[t][[x_1,\dots,x_N]]^G \arrow[r, "\lambda_\gamma^*", "x_i^{(\gamma)} \mapsto t^{e_i/d} x_i^{(\gamma)}"'] \arrow[rr, bend left, "\overline{\lambda}_\gamma^*"] \arrow[d, two heads] & k[x_1,\dots,x_N]^{C_\gamma}[[t]] \arrow[r] \arrow[d, two heads] & k[x_1,\dots,x_N][[t]] \arrow[d, two heads] \\
k[t][[x_1,\dots,x_N]]^G/I'_X \arrow[r, "\mu_\gamma^*"] \arrow[rr, bend right, "\overline{\mu}_\gamma^*"'] & k[x_1,\dots,x_N]^{C_\gamma}[[t]]/\widetilde{I}_X^{(\gamma)} \arrow[r] & k[x_1,\dots,x_N][[t]]/\overline{I}_X^{(\gamma)} .
\end{tikzcd}$$

We define $k[t]$-schemes $\overline{A}^{(\gamma)}$, $\widetilde{X}^{(\gamma)}$ and $\overline{X}^{(\gamma)}$ as follows:

$$\begin{aligned}
\overline{A}^{(\gamma)} &:= \operatorname{Spec} k[x_1,\dots,x_N][[t]], \\
\widetilde{X}^{(\gamma)} &:= \operatorname{Spec}\bigl(k[x_1,\dots,x_N]^{C_\gamma}[[t]]/\widetilde{I}_X^{(\gamma)}\bigr)\,, \\
\overline{X}^{(\gamma)} &:= \operatorname{Spec}\bigl(k[x_1,\dots,x_N][[t]]/\overline{I}_X^{(\gamma)}\bigr)\,.
\end{aligned}$$

Let $\overline{A}_\infty^{(\gamma)}$, $\widetilde{X}_\infty^{(\gamma)}$ and $\overline{X}_\infty^{(\gamma)}$ be their arc spaces as $k[t]$-schemes, defined in Section 5. Then we have the following diagram of arc spaces:

$$\begin{tikzcd}
A_\infty & \bigl(\overline{A}^{(\gamma)}/C_\gamma\bigr)_\infty \arrow[l, "\lambda_\gamma"'] & \overline{A}_\infty^{(\gamma)} \arrow[l] \arrow[ll, bend right, "\overline{\lambda}_\gamma"'] \\
X_\infty \arrow[u, hook] & \widetilde{X}_\infty^{(\gamma)} \arrow[l, "\mu_\gamma"] \arrow[u, hook] & \overline{X}_\infty^{(\gamma)} . \arrow[l] \arrow[u, hook] \arrow[ll, bend left, "\overline{\mu}_\gamma"]
\end{tikzcd}$$

We denote by μ_γ and $\overline{\mu}_\gamma$ the restrictions of λ_γ and $\overline{\lambda}_\gamma$ to $\widetilde{X}_\infty^{(\gamma)}$ and $\overline{X}_\infty^{(\gamma)}$, respectively.

Remark 6.1. (1) Here, we have used the fact that the arc spaces of

$$\operatorname{Spec}\bigl(k[t][[x_1,\dots,x_N]]^G\bigr)\,, \quad \operatorname{Spec}\bigl(k[t][[x_1,\dots,x_N]]^G/I'_X\bigr)$$

as $k[t]$-schemes (defined in Section 5) are isomorphic to the arc spaces of A and X as k-schemes (defined in Section 4).

(2) Furthermore, the vertical arrows are closed immersions. Under these identifications, we have

$$\lambda_\gamma^{-1}(X_\infty) = \widetilde{X}_\infty^{(\gamma)}\,, \quad \overline{\lambda}_\gamma^{-1}(X_\infty) = \overline{X}_\infty^{(\gamma)}\,.$$

Proposition 6.2 ([DL02, Section 2], cf. [NS22, Sections 3.1 and 3.2]). *The ring homomorphism λ_γ^* induces the maps $\lambda_\gamma \colon (\overline{A}^{(\gamma)}/C_\gamma)_\infty \to A_\infty$ and $\overline{\lambda}_\gamma \colon \overline{A}_\infty^{(\gamma)} \to A_\infty$, and the following hold:*

1. *There is a natural inclusion $\overline{A}_\infty^{(\gamma)}/C_\gamma \hookrightarrow \bigl(\overline{A}^{(\gamma)}/C_\gamma\bigr)_\infty$.*
2. *The composite map $\overline{A}_\infty^{(\gamma)}/C_\gamma \hookrightarrow \bigl(\overline{A}^{(\gamma)}/C_\gamma\bigr)_\infty \xrightarrow{\lambda_\gamma} A_\infty$ is injective outside Z_∞.*
3. *We have $\bigsqcup_{\langle\gamma\rangle \in \mathrm{Conj}(G)} \bigl(\overline{\lambda}_\gamma\bigl(\overline{A}_\infty^{(\gamma)}\bigr) \setminus Z_\infty\bigr) = A_\infty \setminus Z_\infty$, where $\mathrm{Conj}(G)$ denotes the set of the conjugacy classes of G.*

Proof. In [DL02] and [NS22], the assertions are proved for the polynomial ring $k[t][x_1, \ldots, x_N]$. Their proofs also work in the formal power series ring setting. $\square$

By Remark 6.1(2), we can deduce the same statement for X.

Proposition 6.3 (cf. [NS22, Section 3.3]). *The ring homomorphism λ_γ^* induces the maps $\mu_\gamma \colon \widetilde{X}_\infty^{(\gamma)} \to X_\infty$ and $\overline{\mu}_\gamma \colon \overline{X}_\infty^{(\gamma)} \to X_\infty$, and the following hold:*

(1) *There is a natural inclusion $\overline{X}_\infty^{(\gamma)}/C_\gamma \hookrightarrow \widetilde{X}_\infty^{(\gamma)}$.*

(2) *The composite map $\overline{X}_\infty^{(\gamma)}/C_\gamma \hookrightarrow \widetilde{X}_\infty^{(\gamma)} \xrightarrow{\mu_\gamma} X_\infty$ is injective outside Z_∞.*

(3) *We have $\bigsqcup_{\langle\gamma\rangle \in \mathrm{Conj}(G)} \big(\overline{\mu}_\gamma(\overline{X}_\infty^{(\gamma)}) \setminus Z_\infty\big) = X_\infty \setminus Z_\infty$.*

Remark 6.4. (1) In [NS22], the integer e_i is taken to satisfy $0 \leqslant e_i \leqslant d-1$. Note that the ring homomorphism λ_γ^* cannot be defined in this way in our formal power series ring setting.

(2) It is also natural to define a $k[t]$-ring homomorphism

$$\overline{\lambda}_\gamma'^{*} \colon k[t][[x_1, \ldots, x_N]]^G \longrightarrow k[t][[x_1, \ldots, x_N]], \quad x_i^{(\gamma)} \longmapsto t^{e_i/d} x_i^{(\gamma)}$$

and schemes

$$\overline{A}'^{(\gamma)} := \operatorname{Spec} k[t][[x_1, \ldots, x_N]], \quad \overline{X}'^{(\gamma)} := \operatorname{Spec}\big(k[t][[x_1, \ldots, x_N]]/\overline{I}_X'^{(\gamma)}\big),$$

where $\overline{I}_X'^{(\gamma)}$ is the ideal of $k[t][[x_1, \ldots, x_N]]$ generated by $\overline{\lambda}_\gamma'^{*}(I_X')$. Then by the same argument as in this section, $\overline{\lambda}_\gamma'^{*}$ induces maps

$$\overline{\lambda}_\gamma' \colon \overline{A}_\infty'^{(\gamma)} \longrightarrow A_\infty, \quad \overline{\mu}_\gamma' \colon \overline{X}_\infty'^{(\gamma)} \longrightarrow X_\infty.$$

However, as we can see in the discussion below, if $\overline{A}^{(\gamma)}$ and $\overline{X}^{(\gamma)}$ are replaced by $\overline{A}'^{(\gamma)}$ and $\overline{X}'^{(\gamma)}$, then Propositions 6.2 and 6.3 are no longer valid.

First, we note that

$$\overline{A}_m'^{(\gamma)} \simeq \operatorname{Spec}\big(k\big[\big[x_1^{(0)}, \ldots, x_N^{(0)}\big]\big]\big[x_1^{(s)}, \ldots, x_N^{(s)} \mid 1 \leqslant s \leqslant m\big]\big),$$
$$\overline{A}_m^{(\gamma)} \simeq \operatorname{Spec}\big(k\big[x_1^{(s)}, \ldots, x_N^{(s)} \mid 0 \leqslant s \leqslant m\big]\big),$$

and we have a natural morphism $\overline{A}_m'^{(\gamma)} \to \overline{A}_m^{(\gamma)}$ induced by the ring inclusion

$$k\big[x_1^{(s)}, \ldots, x_N^{(s)} \mid 0 \leqslant s \leqslant m\big] \longhookrightarrow k\big[\big[x_1^{(0)}, \ldots, x_N^{(0)}\big]\big]\big[x_1^{(s)}, \ldots, x_N^{(s)} \mid 1 \leqslant s \leqslant m\big].$$

Since the morphisms $\overline{A}_m'^{(\gamma)} \to \overline{A}_m^{(\gamma)}$ are compatible with the truncation maps, they induce a map $\overline{A}_\infty'^{(\gamma)} \to \overline{A}_\infty^{(\gamma)}$. The map $\overline{X}_\infty'^{(\gamma)} \to \overline{X}_\infty^{(\gamma)}$ is also induced, and we have the following commutative diagrams:

$$\begin{tikzcd} A_\infty & \overline{A}_\infty^{(\gamma)} \arrow[l, "\overline{\lambda}_\gamma"'] \\ & \overline{A}_\infty'^{(\gamma)}, \arrow[ul, "\overline{\lambda}_\gamma'"] \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} X_\infty & \overline{X}_\infty^{(\gamma)} \arrow[l, "\overline{\mu}_\gamma"'] \\ & \overline{X}_\infty'^{(\gamma)}. \arrow[ul, "\overline{\mu}_\gamma'"] \arrow[u] \end{tikzcd}$$

Furthermore, by construction, $\overline{A}_\infty'^{(\gamma)} \to \overline{A}_\infty^{(\gamma)}$ induces isomorphisms

$$\overline{A}_\infty'^{(\gamma)} \cap \mathrm{Cont}^{\geqslant 1}\big((x_1, \ldots, x_N)\big) \simeq \overline{A}_\infty^{(\gamma)} \cap \mathrm{Cont}^{\geqslant 1}\big((x_1, \ldots, x_N)\big),$$
$$\overline{X}_\infty'^{(\gamma)} \cap \mathrm{Cont}^{\geqslant 1}\big((x_1, \ldots, x_N)\big) \simeq \overline{X}_\infty^{(\gamma)} \cap \mathrm{Cont}^{\geqslant 1}\big((x_1, \ldots, x_N)\big).$$

We also note that $\overline{A}'^{(\gamma)}_\infty \to \overline{A}^{(\gamma)}_\infty$ induces injective maps

$$\overline{A}'^{(\gamma)}_\infty(k) \hookrightarrow \overline{A}^{(\gamma)}_\infty(k)\,, \quad \overline{X}'^{(\gamma)}_\infty(k) \hookrightarrow \overline{X}^{(\gamma)}_\infty(k)$$

on k-points. However, these two maps are not surjective in general (see item (3) below).

(3) Suppose $N = 2$ and $G = \langle\gamma\rangle$, where $\gamma\colon k[[x_1, x_2]] \to k[[x_1, x_2]]$ is the involution defined by $\gamma(x_i) = -x_i$ for $i \in \{1, 2\}$. Then we have $d = 2$, $e_1 = e_2 = 1$ and $k[[x_1, x_2]]^G = k\big[\big[x_1^2, x_1x_2, x_2^2\big]\big]$. We denote by $\alpha \in A_\infty$ the k-arc corresponding to the $k[t]$-ring homomorphism

$$\alpha^*\colon k[t]\big[\big[x_1^2, x_1x_2, x_2^2\big]\big] \longrightarrow k[[t]]\,, \quad x_1^2 \longmapsto t\,, \quad x_1x_2 \longmapsto t\,, \quad x_2^2 \longmapsto t\,.$$

Then, α is contained in the image of $\overline{\lambda}_\gamma\colon \overline{A}^{(\gamma)}_\infty \to A_\infty$. Indeed, if $\beta \in \overline{A}^{(\gamma)}_\infty$ is the k-arc defined by

$$\beta^*\colon k[x_1, x_2][[t]] \longrightarrow k[[t]]\,, \quad x_1 \longmapsto 1\,, \quad x_2 \longmapsto 1\,,$$

then we have $\alpha^* = \beta^* \circ \overline{\lambda}^*_\gamma$ and hence $\alpha = \overline{\lambda}_\gamma(\beta)$.

On the other hand, α is not contained in the image of $\overline{\lambda}'_\gamma\colon \overline{A}'^{(\gamma)}_\infty \to A_\infty$ because there is no $k[t]$-ring homomorphism $\beta^*\colon k[t][[x_1, x_2]] \to k[[t]]$ satisfying $\beta^*(x_1) = \beta^*(x_2) = 1$.

7. Arc spaces of hyperquotient singularities

In this section, we prove in Theorem 7.9 that [NS22, Theorem 4.8] is still valid in the formal power series ring setting.

7.1 Minimal log discrepancies of hyperquotient singularities

Let d be a positive integer, and let $\xi \in k$ be a primitive dth root of unity. Let $G \subset \mathrm{GL}_N(k)$ be a finite group with order d that acts linearly on $\overline{A} := \widehat{\mathbb{A}}^N_k = \operatorname{Spec} k[[x_1, \dots, x_N]]$. We denote by $A := \overline{A}/G$ the quotient scheme. Let $Z \subset A$ be the minimum closed subset such that $\overline{A} \to A$ is étale outside Z. We assume $\operatorname{codim} Z \geqslant 2$, and hence the quotient morphism $\overline{A} \to A$ is étale in codimension 1. We note that A is $\mathbb{Q}$-Gorenstein (cf. Remark 3.1). We fix a positive integer r such that $\omega'^{[r]}_{A/k}$ is invertible.

We fix $\gamma \in G$. Let C_γ be the centralizer of γ in G. Since G is a finite group, γ can be diagonalized in a suitable basis $x_1, \dots, x_N$. Let $\operatorname{diag}(\xi^{e_1}, \dots, \xi^{e_N})$ be the diagonal matrix with $0 < e_i \leqslant d$. We define a $k[t]$-ring homomorphism

$$\lambda^*_\gamma\colon k[t][[x_1, \dots, x_N]]^G \longrightarrow k[x_1, \dots, x_N]^{C_\gamma}[[t]]\,, \quad x_i \longmapsto t^{e_i/d}x_i\,,$$

and define $\overline{\lambda}^*_\gamma\colon k[t][[x_1, \dots, x_N]]^G \to k[x_1, \dots, x_N][[t]]$ as the composition of λ^*_γ and the inclusion $i\colon k[x_1, \dots, x_N]^{C_\gamma}[[t]] \to k[x_1, \dots, x_N][[t]]$.

Let $f_1, \dots, f_c \in k[[x_1, \dots, x_N]]^G$ be a regular sequence. We set

$$B := \operatorname{Spec}\big(k[[x_1, \dots, x_N]]^G/(f_1, \dots, f_c)\big)\,,$$
$$\overline{B} := \operatorname{Spec}\big(k[[x_1, \dots, x_N]]/(f_1, \dots, f_c)\big)\,.$$

We denote by $n := N - c$ their dimensions.

Suppose that B is normal. Then it follows that $\overline{B} \to B$ is also étale in codimension 1, and $\overline{B}$ is also normal. Indeed, since $\operatorname{codim}_A Z \geqslant 2$, we have $A_{\mathrm{sing}} = Z$ by the purity of the branch locus (cf. [Nag59]). Therefore, we have $B \cap Z \subset B_{\mathrm{sing}}$ since $f_1, \dots, f_c$ is a regular sequence (cf. [Sta24, Tag 00NU]). Then it follows that $\operatorname{codim}_B(Z \cap B) \geqslant \operatorname{codim}_B(B_{\mathrm{sing}}) \geqslant 2$ by the normality of B. Therefore, $\overline{B} \to B$ is also étale in codimension 1, and hence $\overline{B}$ is also normal by

Serre's criterion of normality.

Note that $\omega'^{[r]}_{B/k}$ is invertible. Indeed, we have the adjunction formula

$$\omega'_{B/k} \simeq \det{}^{-1}\big(I/I^2\big) \otimes_{\mathcal{O}_A} \omega'_{A/k}\,,$$

where $I := (f_1, \dots, f_c) \subset k[[x_1, \dots, x_N]]^G$, since the sequence

$$0 \longrightarrow I/I^2 \longrightarrow \Omega'_{A/k} \otimes_{\mathcal{O}_A} \mathcal{O}_B \longrightarrow \Omega'_{B/k} \longrightarrow 0$$

is exact at a regular point of B by Proposition 2.12.

We define ideals I', $\widetilde{I}^{(\gamma)}$ and $\overline{I}^{(\gamma)}$ by

$$\begin{aligned} I' &:= (f_1, \dots, f_c) \subset k[t][[x_1, \dots, x_N]]^G\,, \\ \widetilde{I}^{(\gamma)} &:= \big(\lambda_\gamma^*(f_1), \dots, \lambda_\gamma^*(f_c)\big) \subset k[x_1, \dots, x_N]^{C_\gamma}[[t]]\,, \\ \overline{I}^{(\gamma)} &:= \big(\overline{\lambda}_\gamma^*(f_1), \dots, \overline{\lambda}_\gamma^*(f_c)\big) \subset k[x_1, \dots, x_N][[t]]\,. \end{aligned}$$

Then we have the following diagram:

$$\begin{tikzcd} k[t][[x_1, \dots, x_N]]^G \arrow[r, "\lambda_\gamma^*"'] \arrow[d, two heads] \arrow[rr, bend left, "\overline{\lambda}_\gamma^*"] & k[x_1, \dots, x_N]^{C_\gamma}[[t]] \arrow[r, "i"'] \arrow[d, two heads] & k[x_1, \dots, x_N][[t]] \arrow[d, two heads] \\ k[t][[x_1, \dots, x_N]]^G/I' \arrow[r, "\mu_\gamma^*"] \arrow[rr, bend right, "\overline{\mu}_\gamma^*"'] & k[x_1, \dots, x_N]^{C_\gamma}[[t]]/\widetilde{I}^{(\gamma)} \arrow[r, "j"] & k[x_1, \dots, x_N][[t]]/\overline{I}^{(\gamma)}\,. \end{tikzcd}$$

We define $k[t]$-schemes A', $\widetilde{A}^{(\gamma)}$, $\overline{A}^{(\gamma)}$, B', $\widetilde{B}^{(\gamma)}$ and $\overline{B}^{(\gamma)}$ as follows:

$$\begin{aligned} A' &:= \operatorname{Spec} k[t][[x_1, \dots, x_N]]^G\,, & B' &:= \operatorname{Spec}\big(k[t][[x_1, \dots, x_N]]^G/I'\big)\,, \\ \widetilde{A}^{(\gamma)} &:= \operatorname{Spec} k[x_1, \dots, x_N]^{C_\gamma}[[t]]\,, & \widetilde{B}^{(\gamma)} &:= \operatorname{Spec}\big(k[x_1, \dots, x_N]^{C_\gamma}[[t]]/\widetilde{I}^{(\gamma)}\big)\,, \\ \overline{A}^{(\gamma)} &:= \operatorname{Spec} k[x_1, \dots, x_N][[t]]\,, & \overline{B}^{(\gamma)} &:= \operatorname{Spec}\big(k[x_1, \dots, x_N][[t]]/\overline{I}^{(\gamma)}\big)\,. \end{aligned}$$

Then we have the following morphisms between the corresponding $k[t]$-schemes:

$$\begin{tikzcd} A' & \widetilde{A}^{(\gamma)} \arrow[l, "\lambda_\gamma"'] & \overline{A}^{(\gamma)} \arrow[l, "q"'] \arrow[ll, bend right, "\overline{\lambda}_\gamma"'] \\ B' \arrow[u, hook, "\sigma"] & \widetilde{B}^{(\gamma)} \arrow[l, "\mu_\gamma"] \arrow[u, hook] & \overline{B}^{(\gamma)}\,. \arrow[l, "p"] \arrow[u, hook, "\tau"'] \arrow[ll, bend left, "\overline{\mu}_\gamma"] \end{tikzcd}$$

Remark 7.1. (1) Note that

- A' and B' are affine schemes of the form $\operatorname{Spec}\big(k[t][[x_1, \dots, x_M]]/J\big)$, and
- $\widetilde{A}^{(\gamma)}$, $\overline{A}^{(\gamma)}$, $\widetilde{B}^{(\gamma)}$ and $\overline{B}^{(\gamma)}$ are of the form $\operatorname{Spec}\big(k[x_1, \dots, x_M][[t]]/J\big)$.

We will use the notion of the sheaf $\Omega'_{X/k[t]}$ of special differentials for $X = A', B'$ defined in Section 2 and use the notion of the sheaf $\widehat{\Omega}_{X/k[[t]]}$ for $X = \widetilde{A}^{(\gamma)}, \overline{A}^{(\gamma)}, \widetilde{B}^{(\gamma)}, \overline{B}^{(\gamma)}$ defined in Section 5.2.

Since I', $\widetilde{I}^{(\gamma)}$ and $\overline{I}^{(\gamma)}$ are generated by c elements, each irreducible component W_i of B', $\widetilde{B}^{(\gamma)}$ and $\overline{B}^{(\gamma)}$ has $\dim W_i \geqslant n+1$. Therefore, we can apply lemmas and propositions from Sections 5.1 and 5.2 to their arc spaces.

(2) Lemma 7.3(3),(4) below shows that $\overline{B}^{(\gamma)}$ has only one irreducible component that is flat over $k[[t]]$. Furthermore, the component V has $\dim V = n+1$.

Lemma 7.2. *Let $f\colon C_1 \to C_2$ be a flat ring homomorphism of Noetherian rings. Then the following hold:*

(1) *Let P be a prime ideal of C_1 such that $PC_2 \neq C_2$. Then $\operatorname{ht} P = \operatorname{ht}(PC_2)$.*

(2) *Suppose that f is faithfully flat. If $I \subsetneq C_1$ is a proper ideal of C_1, then $\operatorname{ht} I = \operatorname{ht}(IC_2)$.*

Proof. (1) Let Q be a minimal prime of PC_2. Then by the going-down theorem, we have $P = Q \cap C_1$. Therefore, we have $\operatorname{ht} Q = \operatorname{ht} P$ by [Mat89, Theorem 15.1], which proves part (1).

(2) Note that $IC_2 \neq C_2$ by the faithfully flatness. First, the inequality $\operatorname{ht} I \leqslant \operatorname{ht}(IC_2)$ follows from the going-down theorem. We now prove the opposite inequality. Take a minimal prime P of I such that $\operatorname{ht} I = \operatorname{ht} P$. Then by part (1), it follows that $\operatorname{ht} I = \operatorname{ht} P = \operatorname{ht}(PC_2) \geqslant \operatorname{ht}(IC_2)$, which completes the proof. ☐

Lemma 7.3. *We set $F_i := \overline{\lambda}^*_\gamma(f_i)$ for each $1 \leqslant i \leqslant c$. Consider the following ring diagram:*

$$\begin{tikzcd}
S_1 := k[x_1,\dots,x_N][[t]] \arrow[r, two heads] \arrow[d, hook, "h_1"'] & C_1 := k[x_1,\dots,x_N][[t]]/(F_1,\dots,F_c) \arrow[d, "g_1"'] \\
S_2 := k[x_1,\dots,x_N]((t)) \arrow[r, two heads] \arrow[d, hook, "h_2"'] & C_2 := k[x_1,\dots,x_N]((t))/(F_1,\dots,F_c) \arrow[d, "g_2"'] \\
S_3 := k[x_1,\dots,x_N]\big(\big(t^{1/d}\big)\big) \arrow[r, two heads] & C_3 := k[x_1,\dots,x_N]\big(\big(t^{1/d}\big)\big)/(F_1,\dots,F_c) \\
S_4 := k[[x_1,\dots,x_N]] \arrow[r, two heads] \arrow[u, hook, "h_3"', "x_i \mapsto t^{e_i/d}x_i"] & C_4 := k[[x_1,\dots,x_N]]/(f_1,\dots,f_c)\,. \arrow[u, "g_3"', "x_i \mapsto t^{e_i/d}x_i"]
\end{tikzcd}$$

We set $I_i := (F_1,\dots,F_c) \subset S_i$ for $i \in \{1,2,3\}$ and $I_4 := (f_1,\dots,f_c) \subset S_4$. Then the following hold:

(1) *The morphisms h_1, h_2 and h_3 are regular, and hence so are g_1, g_2 and g_3.*

(2) *The morphisms h_2 and h_3 are faithfully flat, and hence so are g_2 and g_3.*

(3) *The rings C_2 and C_3 are normal domains. In particular, I_2 and I_3 are prime ideals.*

(4) *We have $\operatorname{ht}(I_2) = \operatorname{ht}(I_3) = c$.*

Proof. (1),(2) Since h_1 is the localization by $t \in S_1$, it is regular. Note that the inclusion map

$$h_2'\colon S_2' := k[x_1,\dots,x_N]\big[t,t^{-1}\big] \longrightarrow S_3' := k[x_1,\dots,x_N]\big[t^{1/d},t^{-1/d}\big]$$

is étale and faithfully flat. Since h_2 is the base change $-\otimes_{S_2'} S_2$ of h_2', it follows that h_2 is étale (in particular, regular) and faithfully flat.

Let P be a prime ideal of S_4, and let Q be a prime ideal of S_3 such that $Q \cap S_4 = P$. To see that h_3 is regular, it is sufficient to show the following two conditions:

- We have $\operatorname{ht}(P(S_3)_Q) = \operatorname{ht} P$.
- The quotient S_3/PS_3 is regular at Q.

Note that the first condition is equivalent to the flatness by [Mat89, Theorem 23.1] since S_4 is regular and S_3 is Cohen–Macaulay. We also note that the inequality $\mathrm{ht}(P(S_3)_Q) \leqslant \mathrm{ht}\, P$ always holds by [Mat89, Theorem 15.1].

Let $r := \mathrm{ht}\, P$. Since S_4/P is regular at P, by the Jacobian criterion of regularity, there exist $D_1, \dots, D_r \in \mathrm{Der}_k(S_4)$ and $a_1, \dots, a_r \in P$ such that $c := \det(D_i(a_j))_{ij} \notin P$. Since $\mathrm{Der}_k(S_4)$ is generated by the $\partial/\partial x_i$, we may assume that $D_i = \partial/\partial x_i$ holds for $1 \leqslant i \leqslant r$ by changing the indices of x_i. We note that $\partial/\partial x_i$ defines a derivation on S_3; we denote it by $D'_i \in \mathrm{Der}_k(S_3)$. Then we have

$$D'_i(h_3(a_j)) = t^{e_i/d} h_3(D_i(a_j))\,,$$

and hence we have

$$\det\bigl(D'_i(h_3(a_j))\bigr)_{ij} = t^{e'/d} h_3(c)\,,$$

where $e' := \sum_{i=1}^{r} e_i$. Here, we have $t^{e'/d} h_3(c) \notin Q$ since $P = Q \cap S_4$ and $c \notin P$. Therefore, it follows that $\mathrm{ht}(P(S_3)_Q) \geqslant r$ by [Mat89, Theorem 30.4(1)]. Hence, we have $\mathrm{ht}(P(S_3)_Q) = r$, and S_3/PS_3 is regular at Q by [Mat89, Theorem 30.4(2)]. We have proved the regularity of h_3.

To see the faithful flatness of h_3, it is sufficient to show that $\mathfrak{m}S_3 \neq S_3$ for the maximal ideal $\mathfrak{m} = (x_1, \dots, x_N)$ of S_4. This is clear because $\mathfrak{m}S_3 \subset (x_1, \dots, x_N) \neq S_3$.

(3) Note that C_4 is normal by our assumption. Since the normality is preserved under faithfully flat regular ring homomorphisms (cf. [Mat89, Theorem 32.2]), the rings C_2 and C_3 are normal. Therefore, it is sufficient to show that C_2 and C_3 are domains. In what follows, we only prove that C_2 is a domain since the same proof works for C_3. Suppose to the contrary that I_2 has minimal primes P_1 and P_2 with $P_1 \neq P_2$. Then by the normality of C_2, we have $P_1 + P_2 = S_2$ (cf. [Eis95, Proposition 2.20]). Set $Q_1 := P_1 \cap S_1$ and $Q_2 := P_2 \cap S_1$. Since $P_1 + P_2 = S_2$, it follows that $t^s \in Q_1 + Q_2$ for some $s \geqslant 0$. To get a contradiction, we prove that $Q_1, Q_2 \subset \bigl(x_1, \dots, x_N, t^{s+1}\bigr)$.

Suppose to the contrary that there exist $g \in k[[t]]$ and $h \in (x_1, \dots, x_N)$ such that $g + h \in Q_1$ and $g \notin \bigl(t^{s+1}\bigr)$. Let $0 \leqslant a \leqslant s$ be the minimal a such that $g \notin \bigl(t^{a+1}\bigr)$. Then $g - bt^a \in \bigl(t^{a+1}\bigr)$ for some $b \in k^\times$. We may assume $b = 1$ by replacing g and h with $b^{-1}g$ and $b^{-1}h$. For $c \in k^\times$, we denote by $T_c \colon S_1 \to S_1$ the ring isomorphism

$$T_c \colon k[x_1, \dots, x_N][[t]] \longrightarrow k[x_1, \dots, x_N][[t]]\,, \quad t \longmapsto c^{-d} t\,, \quad x_i \longmapsto c^{e_i} x_i\,.$$

Since I_1 is T_c-invariant, so is its minimal prime Q_1. Since the ideal $\bigl(t^{s+1}\bigr)$ of S_1 is also T_c-invariant, T_c induces a ring isomorphism $T'_c : S_1/\bigl(t^{s+1}\bigr) \to S_1/\bigl(t^{s+1}\bigr)$. Hence, $S_1/\bigl(t^{s+1}\bigr)$ has a graded ring structure satisfying $\deg t = -d$ and $\deg x_i = e_i$. Then $\bigl(Q_1 + \bigl(t^{s+1}\bigr)\bigr)/\bigl(t^{s+1}\bigr)$ is a homogeneous ideal. Therefore, the term f_{-da} of $f := g + h$ with degree $-da$ is contained in $Q_1 + \bigl(t^{s+1}\bigr)$. Since $f_{-da} - t^a \in (x_1, \dots, x_N)$, we see that $f_{-da} - t^a \in \bigl(t^{a+1}\bigr)$ by looking at the degrees of its terms. Therefore, $f_{-da} = t^a(1 + f')$ for some $f' \in (t)$. Since $1 + f' \in S_1^\times$, we have $t^a \in Q_1 + \bigl(t^{s+1}\bigr)$, and hence $t^a \in Q_1$. Therefore, we have $P_1 = S_2$, and we get a contradiction.

Note that $\mathrm{ht}(I_4) = c$ by our assumption. Then, assertion (4) follows from assertion (2) and Lemma 7.2. □

LEMMA 7.4. *Let S_i and h_i be as in Lemma 7.3. Let $\mathfrak{c} \subset S_4^G$ be an ideal of S_4^G. Let $\mathfrak{c}_1$ be the ideal of S_1 generated by the image of $\mathfrak{c}$ by the ring homomorphism $S_4^G \to S_1$, $x_i \mapsto t^{e_i/d} x_i$. We set $\mathfrak{c}_2 := \mathfrak{c}_1 S_2$. Then we have $\mathrm{ht}(\mathfrak{c}_2) = \mathrm{ht}\, \mathfrak{c}$.*

Proof. We define $\mathfrak{c}_3 := \mathfrak{c}_2 S_3$ and $\mathfrak{c}_4 := \mathfrak{c} S_4$. Then we have $\mathfrak{c}_3 = \mathfrak{c}_4 S_3$. Since $S_4^G \hookrightarrow S_4$ is an integral extension, we have $\mathrm{ht}\, \mathfrak{c} = \mathrm{ht}(\mathfrak{c} S_4) = \mathrm{ht}(\mathfrak{c}_4)$. Furthermore, we have $\mathrm{ht}(\mathfrak{c}_3) = \mathrm{ht}(\mathfrak{c}_4 S_3) = \mathrm{ht}(\mathfrak{c}_4)$ and $\mathrm{ht}(\mathfrak{c}_3) = \mathrm{ht}(\mathfrak{c}_2 S_3) = \mathrm{ht}(\mathfrak{c}_2)$ by Lemmas 7.3(2) and 7.2. This completes the proof. □

Definition 7.5. (1) We define sheaves $\omega'_{A'/k[t]}$ on A' and $\omega'_{B'/k[t]}$ on B' using the special canonical sheaves $\omega'_{A/k}$ and $\omega'_{B/k}$ defined in Section 2. Let $A' \to A$ and $B' \to B$ be the morphisms induced by the natural ring inclusions

$$k[[x_1, \dots, x_N]]^G \hookrightarrow k[t][[x_1, \dots, x_N]]^G ,$$
$$k[[x_1, \dots, x_N]]^G / I \hookrightarrow k[t][[x_1, \dots, x_N]]^G / I' .$$

Then, we define

$$\omega'_{A'/k[t]} := \omega'_{A/k} \otimes_{\mathcal{O}_A} \mathcal{O}_{A'} , \quad \omega'_{B'/k[t]} := \omega'_{B/k} \otimes_{\mathcal{O}_B} \mathcal{O}_{B'} .$$

Note that they satisfy

$$\omega'_{B'/k[t]} \simeq \det{}^{-1}\bigl(I'/I'^2\bigr) \otimes_{\mathcal{O}_{B'}} \sigma^* \omega'_{A'/k[t]} .$$

Here, $\det^{-1}\bigl(I'/I'^2\bigr) := \bigl(\bigwedge^c \bigl(I'/I'^2\bigr)\bigr)^*$ is an invertible sheaf on B'.

(2) The canonical map

$$\Omega'^{n}_{B'/k[t]} \longrightarrow \omega'_{B'/k[t]}$$

is induced by $\Omega'^{n}_{B/k} \to \omega'_{B/k}$ and the isomorphism $\Omega'_{B'/k[t]} \simeq \Omega'_{B/k} \otimes_{\mathcal{O}_B} \mathcal{O}_{B'}$. The canonical map $\Omega'^{n}_{A'/k[t]} \to \omega'_{A'/k[t]}$ is also defined.

We define an ideal sheaf $\mathfrak{n}_{r,B'} \subset \mathcal{O}_{B'}$ by

$$\mathrm{Im}\bigl(\bigl(\Omega'^{n}_{B'/k[t]}\bigr)^{\otimes r} \longrightarrow \omega'^{[r]}_{B'/k[t]}\bigr) = \mathfrak{n}_{r,B'} \otimes_{\mathcal{O}_{B'}} \omega'^{[r]}_{B'/k[t]} .$$

It satisfies $\mathfrak{n}_{r,B'} = \mathfrak{n}_{r,B} \mathcal{O}_{B'}$.

(3) We define ideal sheaves $\mathrm{Jac}'_{B'/k[t]}$, $\mathrm{Jac}'_{\widetilde{B}^{(\gamma)}/k[[t]]}$ and $\mathrm{Jac}'_{\overline{B}^{(\gamma)}/k[[t]]}$ by

$$\mathrm{Jac}'_{B'/k[t]} := \mathrm{Fitt}^n\bigl(\Omega'_{B'/k[t]}\bigr) \subset \mathcal{O}_{B'} ,$$
$$\mathrm{Jac}'_{\widetilde{B}^{(\gamma)}/k[[t]]} := \mathrm{Fitt}^n\bigl(\widehat{\Omega}_{\widetilde{B}^{(\gamma)}/k[[t]]}\bigr) \subset \mathcal{O}_{\widetilde{B}^{(\gamma)}} ,$$
$$\mathrm{Jac}'_{\overline{B}^{(\gamma)}/k[[t]]} := \mathrm{Fitt}^n\bigl(\widehat{\Omega}_{\overline{B}^{(\gamma)}/k[[t]]}\bigr) \subset \mathcal{O}_{\overline{B}^{(\gamma)}} .$$

Here, we note that $\widetilde{B}^{(\gamma)}$ and $\overline{B}^{(\gamma)}$ are not necessarily equidimensional.

(4) We define an invertible sheaf $L_{\overline{B}^{(\gamma)}}$ on $\overline{B}^{(\gamma)}$ by

$$L_{\overline{B}^{(\gamma)}} := \overline{\mu}_\gamma^*\bigl(\det{}^{-1}\bigl(I'/I'^2\bigr)\bigr) \otimes_{\mathcal{O}_{\overline{B}^{(\gamma)}}} \tau^* \widehat{\Omega}^N_{\overline{A}^{(\gamma)}/k[[t]]}.$$

Then there exist canonical homomorphisms

$$\widehat{\Omega}^n_{\overline{B}^{(\gamma)}/k[[t]]} \longrightarrow L_{\overline{B}^{(\gamma)}} , \quad \overline{\mu}_\gamma^* \omega'^{[r]}_{B'/k[t]} \longrightarrow L^{[r]}_{\overline{B}^{(\gamma)}}$$

such that the following diagram commutes (cf. [NS22, Lemma 4.5(2)]):

$$\begin{tikzcd}
\overline{\mu}_\gamma^*\bigl(\Omega'^{n}_{B'/k[t]}\bigr)^{\otimes r} \arrow[r] \arrow[d] & \bigl(\widehat{\Omega}^n_{\overline{B}^{(\gamma)}/k[[t]]}\bigr)^{\otimes r} \arrow[d] \\
\overline{\mu}_\gamma^* \omega'^{[r]}_{B'/k[t]} \arrow[r] & L^{[r]}_{\overline{B}^{(\gamma)}} .
\end{tikzcd}$$

Furthermore, by the same argument as in [NS22, Lemma 4.5(1)], we have

$$\mathrm{Im}\bigl(\widehat{\Omega}^n_{\overline{B}^{(\gamma)}/k[[t]]} \longrightarrow L_{\overline{B}^{(\gamma)}}\bigr) = \mathrm{Jac}'_{\overline{B}^{(\gamma)}/k[[t]]} \otimes_{\mathcal{O}_{\overline{B}^{(\gamma)}}} L_{\overline{B}^{(\gamma)}} .$$

(5) We define ideal sheaves $\mathfrak{n}'_{1,p}$ and $\mathfrak{n}'_{1,\overline{\mu}_\gamma}$ on $\overline{B}^{(\gamma)}$ by

$$\operatorname{Im}\bigl(p^*\widehat{\Omega}^n_{\widetilde{B}^{(\gamma)}/k[[t]]} \longrightarrow L_{\overline{B}^{(\gamma)}}\bigr) = \mathfrak{n}'_{1,p} \otimes_{\mathcal{O}_{\overline{B}^{(\gamma)}}} L_{\overline{B}^{(\gamma)}} ,$$
$$\operatorname{Im}\bigl(\overline{\mu}^*_\gamma \Omega'^{\,n}_{B'/k[t]} \longrightarrow L_{\overline{B}^{(\gamma)}}\bigr) = \mathfrak{n}'_{1,\overline{\mu}_\gamma} \otimes_{\mathcal{O}_{\overline{B}^{(\gamma)}}} L_{\overline{B}^{(\gamma)}} .$$

(6) We define $\operatorname{age}'(\gamma) := \sum_{i=1}^N e_i/d$. Note that we took e_i to satisfy $0 < e_i \leqslant d$. The age of γ is usually defined by $\operatorname{age}(\gamma) = \operatorname{age}'(\gamma) - \#\{1 \leqslant i \leqslant N \mid e_i = d\}$.

LEMMA 7.6. *Let $\alpha \in \overline{B}^{(\gamma)}_\infty$ be a k-arc with $\operatorname{ord}_\alpha\bigl(\operatorname{Jac}'_{\overline{B}^{(\gamma)}/k[[t]]}\bigr) < \infty$. Then the following hold:*

(1) $\operatorname{ord}_\alpha\bigl(\operatorname{jac}_p\bigr) + \operatorname{ord}_\alpha\bigl(\operatorname{Jac}'_{\overline{B}^{(\gamma)}/k[[t]]}\bigr) = \operatorname{ord}_\alpha\bigl(\mathfrak{n}'_{1,p}\bigr)$,

(2) $\operatorname{ord}_\alpha\bigl(\operatorname{jac}_{\overline{\mu}_\gamma}\bigr) + \operatorname{ord}_\alpha\bigl(\operatorname{Jac}'_{\overline{B}^{(\gamma)}/k[[t]]}\bigr) = \operatorname{ord}_\alpha\bigl(\mathfrak{n}'_{1,\overline{\mu}_\gamma}\bigr)$.

Proof. The same proof as in [NS22, Lemma 4.6] works due to Lemma 5.30. □

LEMMA 7.7. *Let $\alpha \in \overline{B}^{(\gamma)}_\infty$ be a k-arc. Set $\alpha' := \overline{\mu}_{\gamma\infty}(\alpha)$. Suppose $\alpha' \notin Z_\infty$. Then it follows that*

$$\operatorname{ord}_\alpha\bigl(\mathfrak{n}'_{1,\overline{\mu}_\gamma}\bigr) = \frac{1}{r}\operatorname{ord}_{\alpha'}(\mathfrak{n}_{r,B'}) + \operatorname{age}'(\gamma) .$$

Proof. The same proof as in [NS22, Lemma 4.7] works. □

LEMMA 7.8. *Let $I_Z \subset \mathcal{O}_A$ be the ideal sheaf defining $Z \subset A$. Let J be one of the following ideal sheaves on $\overline{B}^{(\gamma)}$:*

$$I_Z\mathcal{O}_{\overline{B}^{(\gamma)}} , \quad \operatorname{Jac}'_{\overline{B}^{(\gamma)}/k[[t]]} , \quad \operatorname{Jac}'_{\widetilde{B}^{(\gamma)}/k[[t]]}\mathcal{O}_{\overline{B}^{(\gamma)}} ,$$
$$\operatorname{Jac}'_{B'/k[t]}\mathcal{O}_{\overline{B}^{(\gamma)}} , \quad \mathfrak{n}'_{1,p} , \quad \mathfrak{n}'_{1,\overline{\mu}_\gamma} , \quad \mathfrak{n}_{r,B'}\mathcal{O}_{\overline{B}^{(\gamma)}} .$$

Let $W \subset \overline{B}^{(\gamma)}$ be the closed subscheme defined by J. Then W_∞ is a thin subset of $\overline{B}^{(\gamma)}_\infty$.

Proof. We set

$$J_1 := I_Z\mathcal{O}_{\overline{B}^{(\gamma)}} , \quad J_2 := \operatorname{Jac}'_{\overline{B}^{(\gamma)}/k[[t]]} , \quad J_3 := \operatorname{Jac}'_{\widetilde{B}^{(\gamma)}/k[[t]]}\mathcal{O}_{\overline{B}^{(\gamma)}} ,$$
$$J_4 := \operatorname{Jac}'_{B'/k[t]}\mathcal{O}_{\overline{B}^{(\gamma)}} , \quad J_5 := \mathfrak{n}'_{1,p} , \quad J_6 := \mathfrak{n}'_{1,\overline{\mu}_\gamma} , \quad J_7 := \mathfrak{n}_{r,B'}\mathcal{O}_{\overline{B}^{(\gamma)}} ,$$

and we denote by $W_i \subset \overline{B}^{(\gamma)}$ the closed subscheme defined by J_i.

Since $B \cap Z \subset B_{\text{sing}}$, we have $(W_1)_{\text{red}} \subset (W_4)_{\text{red}}$. Since the map $\eta_r \colon (\Omega'^{\,n}_{B/k})^{\otimes r} \to \omega'^{[r]}_{B/k}$ in Definition 2.13(5) is an isomorphism on the regular locus B_{reg}, we have $(W_7)_{\text{red}} \subset (W_4)_{\text{red}}$. By Lemma 7.7, we have

$$(W_6)_\infty \cup (W_1)_\infty = (W_7)_\infty \cup (W_1)_\infty .$$

By Lemmas 7.6 and 5.40, we have

$$(W_5)_\infty \subset (W_2)_\infty \cup (W_3)_\infty \cup (W_6)_\infty .$$

Therefore, it is sufficient to show the assertion for

$$J_2 = \operatorname{Jac}'_{\overline{B}^{(\gamma)}/k[[t]]} , \quad J_3 = \operatorname{Jac}'_{\widetilde{B}^{(\gamma)}/k[[t]]}\mathcal{O}_{\overline{B}^{(\gamma)}} , \quad J_4 = \operatorname{Jac}'_{B'/k[t]}\mathcal{O}_{\overline{B}^{(\gamma)}} .$$

We set

$$S_1 := k[x_1, \dots, x_N][[t]] , \quad S_2 := k[x_1, \dots, x_N]((t)) ,$$
$$T_1 := k[x_1, \dots, x_N]^{C_\gamma}[[t]] , \quad T_2 := k[x_1, \dots, x_N]^{C_\gamma}((t)) .$$

The ideal $J_2 = \mathrm{Jac}'_{\overline{B}^{(\gamma)}/k[[t]]}$ corresponds to the ideal

$$\mathfrak{r} := \mathcal{J}_c\big(\overline{I}^{(\gamma)}; \mathrm{Der}_{k[[t]]}(S_1)\big) + \overline{I}^{(\gamma)} \subset S_1$$

of S_1. To show that $(W_2)_\infty$ is a thin set, it is sufficient to show that $\mathrm{ht}(\mathfrak{r}S_2) \geqslant c+1$. Since $\overline{I}^{(\gamma)}S_2$ is a prime ideal of height c by Lemma 7.3(3),(4), it is sufficient to show that $\mathfrak{r}S_2 \not\subset \overline{I}^{(\gamma)}S_2$. Since S_2 satisfies the weak Jacobian condition $(\mathrm{WJ})_{k((t))}$ over $k((t))$ by [Nag62, Theorem 46.3], we have

$$\mathcal{J}_c\big(\overline{I}^{(\gamma)}; \mathrm{Der}_{k[[t]]}(S_1)\big)S_2 + \overline{I}^{(\gamma)}S_2 = \mathcal{J}_c\big(\overline{I}^{(\gamma)}S_2; \mathrm{Der}_{k((t))}(S_2)\big) + \overline{I}^{(\gamma)}S_2 \not\subset \overline{I}^{(\gamma)}S_2\,.$$

Here, the first equality follows from the fact that both $\mathrm{Der}_{k[[t]]}(S_1)$ and $\mathrm{Der}_{k((t))}(S_2)$ are generated by the derivations $\partial/\partial x_i$. This completes the proof of the assertion for J_2.

Let $\mathfrak{r} \subset T_1$ be the ideal of T_1 corresponding to $\mathrm{Jac}'_{\widetilde{B}^{(\gamma)}/k[[t]]} \subset T_1/\widetilde{I}^{(\gamma)}$. Then it is sufficient to show that $\mathrm{ht}(\mathfrak{r}S_2) \geqslant c+1$. Note that $\widetilde{I}^{(\gamma)}T_2 = \overline{I}^{(\gamma)}S_2 \cap T_2$. Therefore, $\widetilde{I}^{(\gamma)}T_2$ is a prime ideal of height c. By the same argument as above, we have $\mathfrak{r}T_2 \not\subset \widetilde{I}^{(\gamma)}T_2$ and hence $\mathrm{ht}(\mathfrak{r}T_2) \geqslant c+1$. Since $T_2 \subset S_2$ is an integral extension, we have $\mathrm{ht}(\mathfrak{r}S_2) = \mathrm{ht}(\mathfrak{r}T_2) \geqslant c+1$, which completes the proof of the assertion for J_3.

Let $\mathfrak{r} \subset k[[x_1, \dots, x_N]]^G$ be the ideal corresponding to $\mathrm{Jac}'_{B/k} \subset k[[x_1, \dots, x_N]]^G/I$. Since $J_4 = \mathfrak{r}\mathcal{O}_{\overline{B}^{(\gamma)}}$, it is sufficient to show $\mathrm{ht}(\mathfrak{r}S_2) \geqslant c+1$. We have $\mathrm{ht}\,\mathfrak{r} \geqslant c+2$ by the normality of B. Therefore, we have $\mathrm{ht}(\mathfrak{r}S_2) = \mathrm{ht}\,\mathfrak{r} \geqslant c+2$ by Lemma 7.4. This completes the proof of the assertion for J_4. □

Theorem 7.9. *Let $x = 0 \in B$ be the origin, and let $\mathfrak{m}_x \subset \mathcal{O}_B$ be the corresponding maximal ideal. Let $\mathfrak{a} \subset \mathcal{O}_B$ be a non-zero ideal sheaf and δ a positive real number. Then*

$$\begin{aligned}\mathrm{mld}_x(B, \mathfrak{a}^\delta) &= \inf_{w, b_1 \in \mathbb{Z}_{\geqslant 0}, \gamma \in G} \big\{\mathrm{codim}(C_{w,\gamma,b_1}) + \mathrm{age}'(\gamma) - b_1 - \delta w\big\} \\ &= \inf_{w, b_1 \in \mathbb{Z}_{\geqslant 0}, \gamma \in G} \big\{\mathrm{codim}(C'_{w,\gamma,b_1}) + \mathrm{age}'(\gamma) - b_1 - \delta w\big\}\end{aligned}$$

for

$$\begin{aligned} C_{w,\gamma,b_1} &:= \mathrm{Cont}^w\big(\mathfrak{a}\mathcal{O}_{\overline{B}^{(\gamma)}}\big) \cap \mathrm{Cont}^{\geqslant 1}\big(\mathfrak{m}_x\mathcal{O}_{\overline{B}^{(\gamma)}}\big) \cap \mathrm{Cont}^{b_1}\big(\mathrm{Jac}'_{\overline{B}^{(\gamma)}/k[[t]]}\big)\,, \\ C'_{w,\gamma,b_1} &:= \mathrm{Cont}^{\geqslant w}\big(\mathfrak{a}\mathcal{O}_{\overline{B}^{(\gamma)}}\big) \cap \mathrm{Cont}^{\geqslant 1}\big(\mathfrak{m}_x\mathcal{O}_{\overline{B}^{(\gamma)}}\big) \cap \mathrm{Cont}^{b_1}\big(\mathrm{Jac}'_{\overline{B}^{(\gamma)}/k[[t]]}\big)\,. \end{aligned}$$

Proof. The formula for C_{w,γ,b_1} is the formal power series ring version of [NS22, Theorem 4.8]. The same proof as in [NS22, Theorem 4.8] works. First, [EM09, Theorem 7.4] plays an important role in the proof of [NS22, Theorem 4.8]; it can be replaced by Theorem 4.11. Furthermore, [NS22, Propositions 2.25, 2.33, 2.35, 3.4 and 3.8, and Lemmas 2.10, 4.6 and 4.7], which are also the key ingredients of the proof of [NS22, Theorem 4.8], are replaced by Propositions 5.36, 5.43, 5.44 and 6.3, and Lemmas 5.40, 7.6 and 7.7 in this paper.

We also note that Proposition 5.36 is applied to Z_∞ and W_∞, where W is the closed subscheme of $\overline{B}^{(\gamma)}$ corresponding to one of the following ideals:

$$\mathrm{Jac}'_{\overline{B}^{(\gamma)}/k[[t]]}\,, \quad \mathrm{Jac}'_{\widetilde{B}^{(\gamma)}/k[[t]]}\mathcal{O}_{\overline{B}^{(\gamma)}}\,, \quad \mathrm{Jac}'_{B'/k[t]}\mathcal{O}_{\overline{B}^{(\gamma)}}\,, \quad \mathfrak{n}'_{1,p}\,, \quad \mathfrak{n}'_{1,\overline{\mu}_\gamma}\,, \quad \mathfrak{n}_{r,B'}\mathcal{O}_{\overline{B}^{(\gamma)}}\,.$$

By Lemma 7.8, they are actually thin sets.

The formula for C'_{w,γ,b_1} is the formal power series ring version of [NS22, Corollary 4.9], and the same proof works. □

Remark 7.10. Theorem 7.9 can be easily extended to $\mathbb{R}$-ideals $\mathfrak{a} = \prod_{i=1}^{r} \mathfrak{a}_i^{\delta_i}$, where $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ are non-zero ideal sheaves on B and $\delta_1, \ldots, \delta_r$ are positive real numbers. In this setting, we have

$$\begin{aligned}\operatorname{mld}_x\bigl(B, \prod_{i=1}^{r} \mathfrak{a}_i^{\delta_i}\bigr) &= \inf_{w_1,\ldots,w_r,b_1 \in \mathbb{Z}_{\geqslant 0}, \gamma \in G} \Bigl\{ \operatorname{codim}(C_{w_1,\ldots,w_r,\gamma,b_1}) + \operatorname{age}'(\gamma) - b_1 - \sum_{i=1}^{r} \delta_i w_i \Bigr\} \\ &= \inf_{w_1,\ldots,w_r,b_1 \in \mathbb{Z}_{\geqslant 0}, \gamma \in G} \Bigl\{ \operatorname{codim}(C'_{w_1,\ldots,w_r,\gamma,b_1}) + \operatorname{age}'(\gamma) - b_1 - \sum_{i=1}^{r} \delta_i w_i \Bigr\}\end{aligned}$$

for

$$\begin{aligned}C_{w_1,\ldots,w_r,\gamma,b_1} &:= \Bigl(\bigcap_{i=1}^{r} \operatorname{Cont}^{w_i}\bigl(\mathfrak{a}_i \mathcal{O}_{\overline{B}^{(\gamma)}}\bigr)\Bigr) \cap \operatorname{Cont}^{\geqslant 1}\bigl(\mathfrak{m}_x \mathcal{O}_{\overline{B}^{(\gamma)}}\bigr) \cap \operatorname{Cont}^{b_1}\bigl(\operatorname{Jac}'_{\overline{B}^{(\gamma)}/k[[t]]}\bigr), \\ C'_{w_1,\ldots,w_r,\gamma,b_1} &:= \Bigl(\bigcap_{i=1}^{r} \operatorname{Cont}^{\geqslant w_i}\bigl(\mathfrak{a}_i \mathcal{O}_{\overline{B}^{(\gamma)}}\bigr)\Bigr) \cap \operatorname{Cont}^{\geqslant 1}\bigl(\mathfrak{m}_x \mathcal{O}_{\overline{B}^{(\gamma)}}\bigr) \cap \operatorname{Cont}^{b_1}\bigl(\operatorname{Jac}'_{\overline{B}^{(\gamma)}/k[[t]]}\bigr).\end{aligned}$$

7.2 Properties of $\overline{B}'^{(\gamma)}$

In the remainder of this section, we define a scheme $\overline{B}'^{(\gamma)}$ and investigate its properties, which will be used in Section 8.

We denote by $\overline{\lambda}'^{*}_{\gamma}$ the $k[t]$-ring homomorphism

$$\overline{\lambda}'^{*}_{\gamma} : k[t][[x_1, \ldots, x_N]]^G \longrightarrow k[t][[x_1, \ldots, x_N]], \quad x_i \longmapsto t^{e_i/d} x_i .$$

We set

$$\overline{I}'^{(\gamma)} := \bigl(\overline{\lambda}'^{*}_{\gamma}(f_1), \ldots, \overline{\lambda}'^{*}_{\gamma}(f_c)\bigr) \subset k[t][[x_1, \ldots, x_N]], \quad \overline{B}'^{(\gamma)} := \operatorname{Spec}\bigl(k[t][[x_1, \ldots, x_N]]/\overline{I}'^{(\gamma)}\bigr).$$

Then $\overline{B}'^{(\gamma)}$ is a scheme of finite type over $R = k[t][[x_1, \ldots, x_N]]$. Let $\Omega'_{\overline{B}'^{(\gamma)}/k[t]}$ be the sheaf of special differentials defined in Definition 2.13(1) with respect to R and $R_0 = k[t]$. We set

$$\operatorname{Jac}'_{\overline{B}'^{(\gamma)}/k[t]} := \operatorname{Fitt}^n\bigl(\Omega'_{\overline{B}'^{(\gamma)}/k[t]}\bigr).$$

First, we study the dimensions of the irreducible components of $\overline{B}'^{(\gamma)}$.

Lemma 7.11. *We set $F_i := \overline{\lambda}'^{*}_{\gamma}(f_i)$ for each $1 \leqslant i \leqslant c$. Consider the following diagram of rings:*

$$\begin{tikzcd}
S_1 := k[t][[x_1, \ldots, x_N]] \arrow[r, two heads] \arrow[d, hook, "h_1"'] & C_1 := k[t][[x_1, \ldots, x_N]]/(F_1, \ldots, F_c) \arrow[d, "g_1"'] \\
S_2 := k[t, t^{-1}][[x_1, \ldots, x_N]] \arrow[r, two heads] \arrow[d, hook, "h_2"'] & C_2 := k[t, t^{-1}][[x_1, \ldots, x_N]]/(F_1, \ldots, F_c) \arrow[d, "g_2"'] \\
S_3 := k\bigl[t^{1/d}, t^{-1/d}\bigr][[x_1, \ldots, x_N]] \arrow[r, two heads] & C_3 := k\bigl[t^{1/d}, t^{-1/d}\bigr][[x_1, \ldots, x_N]]/(F_1, \ldots, F_c) \\
S_4 := k\bigl[t^{1/d}, t^{-1/d}\bigr][[x_1, \ldots, x_N]] \arrow[r, two heads] \arrow[u, "\simeq"', "x_i \mapsto t^{e_i/d} x_i"] & C_4 := k\bigl[t^{1/d}, t^{-1/d}\bigr][[x_1, \ldots, x_N]]/(f_1, \ldots, f_c) \arrow[u, "\simeq"', "x_i \mapsto t^{e_i/d} x_i"] \\
S_5 := k[[x_1, \ldots, x_N]] \arrow[r, two heads] \arrow[u, hook, "h_3"] & C_5 := k[[x_1, \ldots, x_N]]/(f_1, \ldots, f_c). \arrow[u, "g_3"']
\end{tikzcd}$$

We set $I_i := (F_1, \ldots, F_c) \subset S_i$ for $i \in \{1,2,3\}$ and $I_i := (f_1, \ldots, f_c) \subset S_i$ for $i \in \{4,5\}$. Then the following hold:

(1) *The morphisms h_1, h_2 and h_3 are regular, and hence so are g_1, g_2 and g_3.*

(2) *The morphisms h_2 and h_3 are faithfully flat, and hence so are g_2 and g_3.*

(3) *The rings C_2, C_3 and C_4 are normal domains. In particular, I_2, I_3 and I_4 are prime ideals.*

(4) *We have $\mathrm{ht}(I_2) = \mathrm{ht}(I_3) = \mathrm{ht}(I_4) = c$.*

(5) *There exists only one minimal prime P of I_1 of the form (2) in Lemma 5.25. Furthermore, it satisfies $\mathrm{ht}\, P = c$ and $P = I_2 \cap S_1$.*

Proof. (1) We only show the regularity of h_2 since the other two can be proved more easily. Since the ring inclusion $k[t, t^{-1}] \to k\left[t^{1/d}, t^{-1/d}\right]$ is étale, so is its base change

$$k[t, t^{-1}][[x_1, \ldots, x_N]] \longrightarrow k[t, t^{-1}][[x_1, \ldots, x_N]]\left[t^{1/d}, t^{-1/d}\right].$$

Furthermore, the ring inclusion

$$k[t, t^{-1}][[x_1, \ldots, x_N]]\left[t^{1/d}, t^{-1/d}\right] \longrightarrow k\left[t^{1/d}, t^{-1/d}\right][[x_1, \ldots, x_N]]$$

is regular since it can be seen as the completion at the prime ideal $(x_1, \ldots, x_N)$ and the ring on the left side is an excellent ring, in particular a G-ring (cf. [Mat80, Theorem 79]). Therefore, their composition h_2 turns out to be regular.

(2) Any maximal ideal M of S_2 is of the form $M = (t - a, x_1, \ldots, x_N)$, where $a \in k^\times$. Therefore, we have $MS_3 \neq S_3$, and hence h_2 is faithfully flat. The same proof works for h_3.

(3) Note that C_5 is normal by our assumption. Therefore, the normality of C_2, C_3 and C_4 follows from assertions (1) and (2) and the fact that the normality is preserved under faithfully flat regular ring homomorphisms (cf. [Mat89, Theorem 32.2]). In what follows, we prove that C_2, C_3 and C_4 are domains. Since h_2 is faithfully flat, g_2 is injective (cf. [Mat89, Theorem 7.5]). Therefore, it is sufficient to show that C_4 is a domain. Let $P_1, \ldots, P_\ell$ be the minimal primes of I_4. Suppose to the contrary that $\ell \geqslant 2$. Since C_4 is normal, we have $P_1 + P_2 = S_4$ (cf. [Eis95, Proposition 2.20]). Take maximal ideals M_1 and M_2 of S_4 such that $P_i \subset M_i$ for each $i \in \{1,2\}$. We may write $M_i = \left(t^{1/d} - a_i, x_1, \ldots, x_N\right)$ with $a_i \in k^\times$. For each $c \in k^\times$, we denote by T_c the ring isomorphism

$$T_c \colon S_4 \longrightarrow S_4\,, \quad t^{1/d} \longmapsto ct^{1/d}\,, \quad x_i \longmapsto x_i\,.$$

Then, I_4 is T_c-invariant for any $c \in k^\times$. Therefore, its minimal primes P_1 and P_2 are also T_c-invariant for any $c \in k^\times$. Therefore, $P_1 \subset M_2$, and hence $P_1 + P_2 \subset M_2$, so we have a contradiction.

(4) Note that $\mathrm{ht}(I_5) = c$ by our assumption. Therefore, assertion (4) follows from assertion (2) and Lemma 7.2.

(5) Let $P_1, \ldots, P_\ell$ be the minimal primes of I_1. Then the inclusion $P_1^{a_1} \cap \cdots \cap P_\ell^{a_\ell} \subset I_1$ holds for some $a_1, \ldots, a_\ell \geqslant 1$. Since h_1 is flat, we have

$$I_2 = I_1 S_2 \supset \left(P_1^{a_1} \cap \cdots \cap P_\ell^{a_\ell}\right) S_2 = P_1^{a_1} S_2 \cap \cdots \cap P_\ell^{a_\ell} S_2$$

by [Mat89, Theorem 7.4]. If P_i is of the form (1) or (3) in Lemma 5.25, then we have $P_i S_2 = S_2$. Since $I_2 \neq S_2$, some P_i has to be of the form (2) in Lemma 5.25.

Suppose that P is a minimal prime of I_1 of the form (2) in Lemma 5.25. Then $P + (t - a) \neq S_1$ for any $a \in k^\times$, and hence we have $PS_2 \neq S_2$. Since we have $c = \mathrm{ht}\, I_2 \leqslant \mathrm{ht}(PS_2) = \mathrm{ht}\, P \leqslant c$ by Lemma 7.2 and Krull's height theorem, it follows that $\mathrm{ht}\, P = c$ and $I_2 = PS_2$. Since we have

$c = \operatorname{ht} P \leqslant \operatorname{ht}(I_2 \cap S_1) \leqslant \operatorname{ht} I_2 = c$ by the going-down theorem, we have $P = I_2 \cap S_1$, which also shows the uniqueness of P. □

Remark 7.12. Let $\overline{B}'^{(\gamma)} = V_1 \cup \cdots \cup V_\ell$ be the irreducible decomposition. By Lemma 7.11(5), the scheme $\overline{B}'^{(\gamma)}$ has a unique irreducible component V of the form (2) in Lemma 5.25, which satisfies $\dim V = \dim' V = n+1$. Furthermore, any irreducible component V' other than V satisfies

$$V'_\infty \cap \operatorname{Cont}^{\geqslant 1}\big(\mathfrak{o}_{\overline{B}'^{(\gamma)}}\big) = \emptyset$$

by Remark 5.26. Here, $\mathfrak{o}_{\overline{B}'^{(\gamma)}} \subset \mathcal{O}_{\overline{B}'^{(\gamma)}}$ denotes the ideal sheaf generated by $x_1, \dots, x_N$.

Next, we show the relationship between $\overline{B}^{(\gamma)}_\infty$ and $\overline{B}'^{(\gamma)}_\infty$.

Lemma 7.13. *Let $\mathfrak{o}_{\overline{B}^{(\gamma)}} \subset \mathcal{O}_{\overline{B}^{(\gamma)}}$ and $\mathfrak{o}_{\overline{B}'^{(\gamma)}} \subset \mathcal{O}_{\overline{B}'^{(\gamma)}}$ be the ideal sheaves generated by $x_1, \dots, x_N$. Then the following hold:*

(1) *For $m \geqslant 0$, there exist canonical morphisms $\overline{B}'^{(\gamma)}_m \to \overline{B}^{(\gamma)}_m$ that commute with the truncation morphisms. In particular, they induce $\overline{B}'^{(\gamma)}_\infty \to \overline{B}^{(\gamma)}_\infty$.*

(2) *The map $\overline{B}'^{(\gamma)}_\infty \to \overline{B}^{(\gamma)}_\infty$ induces an isomorphism*
$$\overline{B}'^{(\gamma)}_\infty \cap \operatorname{Cont}^{\geqslant 1}\big(\mathfrak{o}_{\overline{B}'^{(\gamma)}}\big) \xrightarrow{\simeq} \overline{B}^{(\gamma)}_\infty \cap \operatorname{Cont}^{\geqslant 1}\big(\mathfrak{o}_{\overline{B}^{(\gamma)}}\big).$$

(3) *For a k-arc $\gamma \in \overline{B}'^{(\gamma)}_\infty \cap \operatorname{Cont}^{\geqslant 1}\big(\mathfrak{o}_{\overline{B}'^{(\gamma)}}\big)$ and the corresponding arc $\gamma' \in \overline{B}^{(\gamma)}_\infty \cap \operatorname{Cont}^{\geqslant 1}\big(\mathfrak{o}_{\overline{B}^{(\gamma)}}\big)$, it follows that*
$$\operatorname{ord}_\gamma\big(\operatorname{Jac}'_{\overline{B}'^{(\gamma)}/k[t]}\big) = \operatorname{ord}_{\gamma'}\big(\operatorname{Jac}'_{\overline{B}^{(\gamma)}/k[[t]]}\big).$$

Proof. Assertions (1) and (2) follow from the discussion in Remark 6.4(2). Assertion (3) follows from Remarks 2.14 and 5.29(3). □

Lemma 7.14. *Let $W \subset \overline{B}'^{(\gamma)}$ be the closed subscheme defined by $\operatorname{Jac}'_{\overline{B}'^{(\gamma)}/k[t]}$. Then the intersection $W_\infty \cap \operatorname{Cont}^{\geqslant 1}\big(\mathfrak{o}_{\overline{B}'^{(\gamma)}}\big)$ is a thin subset of $\overline{B}'^{(\gamma)}_\infty$.*

Proof. Let $T_c\colon k[t][[x_1, \dots, x_N]] \to k[t][[x_1, \dots, x_N]]$ be the ring isomorphism defined in Lemma 5.25. Set

$$J := \operatorname{Jac}'_{\overline{B}'^{(\gamma)}/k[t]},$$

and let $\overline{J} \subset k[t][[x_1, \dots, x_N]]$ be the corresponding ideal. Since $\overline{I}'^{(\gamma)}$ is T_c-invariant, $\overline{J}$ is also T_c-invariant for each $c \in k^\times$. Therefore, by Lemma 5.25, each minimal prime P of $\overline{J}$ satisfies one of the conditions in Lemma 5.25. By Remark 5.26, it is sufficient to show that $\operatorname{ht} P \geqslant c+1 = N-n+1$ for P satisfying condition (2) in Lemma 5.25. Since $P + (t-1) \neq k[t][[x_1, \dots, x_N]]$ in this case, it suffices to show that $\operatorname{ht}(P + (t-1)) \geqslant N-n+2$. Under the identification $k[t][[x_1, \dots, x_N]]/(t-1) \simeq k[[x_1, \dots, x_N]]$, the ideal $\big(\overline{J} + (t-1)\big)/(t-1)$ corresponds to $\operatorname{Jac}'_{\overline{B}/k}$. Since $\overline{B}$ is normal and hence regular in codimension 1, it follows that $\operatorname{ht}\big(\overline{J} + (t-1)\big) \geqslant N-n+3$ by the Jacobian criterion of regularity. Therefore, we get the desired inequality for P. □

8. PIA formula for quotient singularities of non-linear action

In this section, we generalize [NS22, Theorem 5.1] to non-linear group actions (Theorem 8.2). First, we clarify the definition of quotient singularities in this paper.

Definition 8.1. Let X be a variety over k and $x \in X$ a closed point. We say that X has a *quotient singularity* at x if there exist a quasi-projective variety $\overline{M}$ over k, a finite subgroup $G \subset \operatorname{Aut}(\overline{M})$ and a smooth closed point $\overline{y} \in \overline{M}$ such that $\mathcal{O}_{X,x} \simeq \mathcal{O}_{M,y}$, where $M := \overline{M}/G$ is the quotient variety and $y \in M$ is the image of $\overline{y}$.

Theorem 8.2. *Suppose that a variety X has a quotient singularity at a closed point $x \in X$. Let Y be a subvariety of X of codimension c that is locally defined by c equations $h_1, \dots, h_c \in \mathfrak{m}_{X,x}$ at x. Suppose that Y is klt at x. Let $\mathfrak{a} \subset \mathcal{O}_X$ be an ideal sheaf, and let δ be a positive real number. Suppose $\mathfrak{b} := \mathfrak{a}\mathcal{O}_Y \neq 0$. Then we have*

$$\operatorname{mld}_x\bigl(X, (h_1 \cdots h_c)\mathfrak{a}^\delta\bigr) = \operatorname{mld}_x\bigl(Y, \mathfrak{b}^\delta\bigr).$$

Proof. Since X has a quotient singularity at x, there exist a variety $\overline{M}$ with a smooth closed point $\overline{y} \in \overline{M}$ and a finite subgroup $G' \subset \operatorname{Aut}(\overline{M})$ such that $\mathcal{O}_{X,x} \simeq \mathcal{O}_{M',y'}$, where $M' := \overline{M}/G'$ and $y' \in M'$ is the image of $\overline{y}$.

We denote by $G := \{g \in G' \mid g(\overline{y}) = \overline{y}\}$ the stabilizer group of $\overline{y}$, by $M := \overline{M}/G$ the quotient variety, and by $y \in M$ the image of $\overline{y}$. Then we note that $M = \overline{M}/G \to M' = \overline{M}/G'$ is étale at y (cf. [Kol13, Section 3.17]). Furthermore, G acts on $\mathfrak{m}^i_{\overline{M},\overline{y}}$ for each $i \geqslant 0$, and hence the projection $s\colon \mathfrak{m}_{\overline{M},\overline{y}} \to \mathfrak{m}_{\overline{M},\overline{y}}/\mathfrak{m}^2_{\overline{M},\overline{y}}$ becomes a G-equivariant k-linear map. Let u be any k-linear section of s. Then the map $u' := |G|^{-1}\sum_{g \in G} g^{-1} \circ u \circ g$ gives a G-equivariant k-linear section $u'\colon \mathfrak{m}_{\overline{M},\overline{y}}/\mathfrak{m}^2_{\overline{M},\overline{y}} \to \mathfrak{m}_{\overline{M},\overline{y}}$ of s. Let $N := \dim X$. Then u' induces a ring homomorphism

$$k[x_1, \dots, x_N] \longrightarrow \mathcal{O}_{\overline{M},\overline{y}}$$

that is étale. Furthermore, G acts linearly on $k[x_1, \dots, x_N]$, and the ring homomorphism above becomes G-equivariant. Since the ring homomorphism above is étale, we get an isomorphism $k[[x_1, \dots, x_N]] \xrightarrow{\simeq} \widehat{\mathcal{O}}_{\overline{M},\overline{y}}$. Note that $k[[x_1, \dots, x_N]]^{G_{\mathrm{pr}}} \simeq k[[x_1, \dots, x_N]]$ for $G_{\mathrm{pr}} \subset G$, where G_{pr} is the subgroup generated by the pseudo-reflections (cf. [Kol13, Section 3.18]). Hence $\overline{M}/G_{\mathrm{pr}}$ is smooth at the image of $\overline{y}$. Therefore, by replacing G with G/G_{pr} and $\overline{M}$ with $\overline{M}/G_{\mathrm{pr}}$, we may assume that G does not contain a pseudo-reflection. Then we have the following ring diagram:

$$\begin{array}{ccccc}
\mathcal{O}_{X,x} & & & & \\
\wr\! \mid & & & & \\
\mathcal{O}_{M',y'} & & & & \\
\Big\downarrow \text{étale} & & & & \\
\mathcal{O}_{M,y} & \subset & \widehat{\mathcal{O}}_{M,y} & & \\
\| & & \| & & \\
\mathcal{O}^G_{\overline{M},\overline{y}} & \subset & \widehat{\mathcal{O}}^G_{\overline{M},\overline{y}} & \simeq & k[[x_1, \dots, x_N]]^G \\
\cap & & \cap & & \cap \\
\mathcal{O}_{\overline{M},\overline{y}} & \subset & \widehat{\mathcal{O}}_{\overline{M},\overline{y}} & \simeq & k[[x_1, \dots, x_N]] .
\end{array}$$

We denote by $\bigl(\overline{N}, \overline{y}\bigr) \subset \bigl(\overline{M}, \overline{y}\bigr)$ the germ defined by the images of $h_1, \dots, h_c \in \mathfrak{m}_{X,x}$ in $\mathcal{O}_{\overline{M},\overline{y}}$. Let $f_1, \dots, f_c \in k[[x_1, \dots, x_N]]^G$ be the images of $h_1, \dots, h_c \in \mathfrak{m}_{X,x}$. Then we have an isomorphism $\widehat{\mathcal{O}}_{\overline{N},\overline{y}} \simeq k[[x_1,\dots,x_N]]/(f_1,\dots,f_c)$. We set

$$A := \operatorname{Spec} k[[x_1,\dots,x_N]]^G, \quad B := \operatorname{Spec}\bigl(k[[x_1,\dots,x_N]]^G/(f_1,\dots,f_c)\bigr)$$

and let $x' \in A$ be the origin. Let $\mathfrak{a}' \subset \mathcal{O}_A$ and $\mathfrak{b}' \subset \mathcal{O}_B$ be the ideal sheaves corresponding to $\mathfrak{a} \subset \mathcal{O}_X$ and $\mathfrak{b} \subset \mathcal{O}_Y$. Since

$$\operatorname{mld}_x\bigl(X,(h_1\cdots h_c)\mathfrak{a}^\delta\bigr) = \operatorname{mld}_{x'}\bigl(A,(f_1\cdots f_c)\mathfrak{a}'^\delta\bigr), \quad \operatorname{mld}_x\bigl(Y,\mathfrak{b}^\delta\bigr) = \operatorname{mld}_{x'}\bigl(B,\mathfrak{b}'^\delta\bigr)$$

(cf. Remark 3.1), it is sufficient to show that

$$\operatorname{mld}_{x'}\bigl(A,(f_1\cdots f_c)\mathfrak{a}'^\delta\bigr) = \operatorname{mld}_{x'}\bigl(B,\mathfrak{b}'^\delta\bigr). \tag{8.1}$$

Note that G, A and B satisfy all assumptions in Section 7. Therefore, we use the notation of that section. Since Y is klt at x, so is B. Since $\overline{B} \to B$ is étale in codimension 1, it follows that the germ $(\overline{N},\overline{y})$ is also klt. This fact will be used in the proof of Claim 8.3.

The equality (8.1) is proved for k-varieties in [NS22, Theorem 5.1]. The key ingredients of the proof of [NS22, Theorem 5.1] are [NS22, Corollary 4.9, Lemma 2.34 and Claim 5.2]. First, [NS22, Corollary 4.9] can be replaced by Theorem 7.9 in the formal power series ring setting. Second, [NS22, Lemma 2.34] is still true in our setting as can be seen by replacing $k[t][x_1,\dots,x_N]$ by $k[x_1,\dots,x_N][[t]]$ (cf. Remark 5.42). On the other hand, the proof of [NS22, Claim 5.2] does not work directly because they use the result on the rational connectedness proved by Hacon and McKernan [HM07], which is not clear for the formal power series ring setting. We also note the lack of [NS22, Lemma 2.27] in our setting. It will be replaced by Proposition 5.23.

In what follows, we only prove [NS22, Claim 5.2] in the formal power series ring setting.

CLAIM 8.3 (cf. [NS22, Claim 5.2]). *Let $C \subset \overline{A}^{(\gamma)}_\infty$ be a cylinder that is the intersection of finitely many cylinders of the form $\operatorname{Cont}^{\geqslant \ell}\bigl(\mathfrak{c}\mathcal{O}_{\overline{A}^{(\gamma)}}\bigr)$, where $\mathfrak{c} \subset \mathcal{O}_A$ is an ideal sheaf on A and ℓ is a non-negative integer. Let C' be an irreducible component of C. Then $C' \cap \overline{B}^{(\gamma)}_\infty$ contains a k-arc δ such that $\operatorname{ord}_\delta\bigl(\operatorname{Jac}'_{\overline{B}^{(\gamma)}/k[[t]]}\bigr) < \infty$.*

Proof of Claim 8.3. First, we introduce a k-action on the arc space $\overline{A}^{(\gamma)}_\infty$ as follows. For a k-arc $\alpha \in \overline{A}^{(\gamma)}_\infty$, we set $g_i^\alpha := \alpha^*(x_i) \in k[[t]]$, where $\alpha^*\colon k[x_1,\dots,x_N][[t]] \to k[[t]]$ is the corresponding $k[t]$-ring homomorphism. For $a \in k$, we define $a \cdot \alpha \in \overline{A}^{(\gamma)}_\infty$ by $g_i^{a\cdot\alpha}(t) := a^{e_i} g_i^\alpha\bigl(a^d t\bigr)$. Then for $f \in k[[x_1,\dots,x_N]]^G$, we have $v(t) = u(a^d t)$ for

$$u(t) := \alpha^*\bigl(\overline{\lambda}^*_\gamma(f)\bigr), \quad v(t) := (a\cdot\alpha)^*\bigl(\overline{\lambda}^*_\gamma(f)\bigr) \in k[[t]].$$

Therefore, we have $\operatorname{ord}_\alpha\bigl(\overline{\lambda}^*_\gamma(f)\bigr) = \operatorname{ord}_{a\cdot\alpha}\bigl(\overline{\lambda}^*_\gamma(f)\bigr)$ if $a \in k^\times$. Hence, any cylinder of the form $\operatorname{Cont}^{\geqslant\ell}\bigl(\mathfrak{c}\mathcal{O}_{\overline{A}^{(\gamma)}}\bigr)$ with an ideal $\mathfrak{c} \subset \mathcal{O}_A = k[[x_1,\dots,x_N]]^G$ is invariant under the k-action. Therefore, C in the statement and its irreducible component C' are also invariant under the k-action.

We denote by $\beta \in \overline{A}^{(\gamma)}_\infty$ the trivial arc determined by $g_i^\beta = 0$ for each i. Note that $\beta = 0\cdot\alpha$ for any k-arc $\alpha \in \overline{A}^{(\gamma)}_\infty$. Therefore, we have $\beta \in C'$ and hence

$$\beta \in C' \cap \operatorname{Cont}^{\geqslant 1}\bigl(\mathfrak{o}_{\overline{B}^{(\gamma)}}\bigr) \neq \emptyset.$$

By Lemma 7.13(1),(2), there exists a cylinder $D \subset \overline{B}'^{(\gamma)}_\infty$ isomorphic to $C' \cap \operatorname{Cont}^{\geqslant 1}\bigl(\mathfrak{o}_{\overline{B}^{(\gamma)}}\bigr)$ under the map in Lemma 7.13(2). Then by Lemma 7.13(3), it is sufficient to show that D contains a k-arc δ such that $\operatorname{ord}_\delta\bigl(\operatorname{Jac}'_{\overline{B}'^{(\gamma)}/k[t]}\bigr) < \infty$. Therefore, Claim 8.3 follows from Claim 8.4 below and Lemma 7.14. $\square$

Claim 8.4. *Let $D \subset \overline{B}'^{(\gamma)}_\infty$ be a cylinder contained in* $\mathrm{Cont}^{\geqslant 1}\big(\mathfrak{o}_{\overline{B}'^{(\gamma)}}\big)$. *If D contains the trivial arc β, then D is not a thin subset of $\overline{B}'^{(\gamma)}_\infty$.*

Proof of Claim 8.4. Let $T \subset \overline{B}'^{(\gamma)}$ be the closed subscheme defined by $\mathfrak{o}_{\overline{B}'^{(\gamma)}}$. First, we prove that $\overline{B}'^{(\gamma)}$ has a desingularization $r\colon W \to \overline{B}'^{(\gamma)}$ with the following conditions:

(1) For each $a \in k^\times$, the closed subscheme $W_a \subset W$ defined by the ideal $(t-a)\mathcal{O}_W \subset \mathcal{O}_W$ is an integral regular scheme with $\dim' W_a = n$.

(2) The restriction $r|_{T'}\colon T' \to T$ has a section, where $T' := r^{-1}(T)$.

We set $F_i := \overline{\lambda}'^*_\gamma(f_i) = f_i\big(t^{e_1/d}x_1, \dots, t^{e_N/d}x_N\big)$. Then, for

$$
\begin{aligned}
V_1 &:= \overline{B}'^{(\gamma)} = \mathrm{Spec}\big(k[t][[x_1,\dots,x_N]]/(F_1,\dots,F_c)\big)\,,\\
V_2 &:= \mathrm{Spec}\big(k\big[t,t^{-1}\big][[x_1,\dots,x_N]]/(F_1,\dots,F_c)\big)\,,\\
V_3 &:= \mathrm{Spec}\big(k\big[t^{1/d},t^{-1/d}\big][[x_1,\dots,x_N]]/(f_1,\dots,f_c)\big)\,,\\
V_4 &:= \mathrm{Spec}\big(\mathcal{O}_{\overline{N},\overline{y}}\big[t^{1/d},t^{-1/d}\big]\big)\,,\\
V_5 &:= \mathrm{Spec}\big(\mathcal{O}_{\overline{N},\overline{y}}\big)\,,
\end{aligned}
$$

we have following natural morphisms:

$$V_1 \longleftarrow V_2 \xleftarrow{\big[x_i \mapsto t^{-e_i/d}x_i\big]} V_3 \longrightarrow V_4 \longrightarrow V_5\,.$$

Note that these four morphisms are regular (cf. Lemma 7.11(1)). Hence by the functorial desingularization by Temkin [Tem12, Theorem 1.2.1], there exist desingularizations $r_i\colon W_i \to V_i$ with the following Cartesian diagram:

$$
\begin{tikzcd}
W_1 \arrow[r, "r_1"] & V_1 \\
W_2 \arrow[r, "r_2"] \arrow[u] & V_2 \arrow[u] \\
W_3 \arrow[r, "r_3"] \arrow[u] \arrow[d] & V_3 \arrow[u] \arrow[d] \\
W_4 \simeq W_5 \times (\mathbb{A}^1_k \setminus \{0\}) \arrow[r, "r_4"] \arrow[d] & V_4 = V_5 \times \big(\mathbb{A}^1_k \setminus \{0\}\big) \arrow[d] \\
W_5 \arrow[r, "r_5"] & V_5\,.
\end{tikzcd}
$$

We will prove that r_1 satisfies conditions (1) and (2) above.

(1) For each $i \in \{1,2\}$ and $a \in k^\times$, we denote by $(W_i)_a \subset W_i$ and $(V_i)_a \subset V_i$ the closed subschemes defined by $(t-a)\mathcal{O}_{W_i} \subset \mathcal{O}_{W_i}$ and $(t-a)\mathcal{O}_{V_i} \subset \mathcal{O}_{V_i}$, respectively. Similarly, for each $i \in \{3,4\}$ and $a \in k^\times$, we denote by $(W_i)_a \subset W_i$ and $(V_i)_a \subset V_i$ the closed subschemes defined by $\big(t^{1/d}-a\big)\mathcal{O}_{W_i} \subset \mathcal{O}_{W_i}$ and $\big(t^{1/d}-a\big)\mathcal{O}_{V_i} \subset \mathcal{O}_{V_i}$, respectively. Then, the Cartesian diagram

above induces the following Cartesian diagram for each $a \in k^{\times}$:

$$
\begin{tikzcd}
(W_1)_a \arrow[r] & (V_1)_a \\
(W_2)_a \arrow[r] \arrow[u] & (V_2)_a \arrow[u] \\
\bigsqcup_{b^d=a}(W_3)_b \arrow[r] \arrow[u] \arrow[d] & \bigsqcup_{b^d=a}(V_3)_b \arrow[u] \arrow[d] \\
\bigsqcup_{b^d=a}(W_4)_b \arrow[r] & \bigsqcup_{b^d=a}(V_4)_b \,.
\end{tikzcd}
$$

Here, by construction, we have $(V_1)_a \simeq (V_2)_a \simeq (V_3)_b$ for any $a, b \in k^{\times}$ with $b^d = a$. Therefore, we also have $(W_1)_a \simeq (W_2)_a \simeq (W_3)_b$. Since $W_4 \simeq W_5 \times \left(\mathbb{A}^1_k \setminus \{0\}\right)$, the scheme $(W_4)_b$ turns out to be regular for any $b \in k^{\times}$. Since $(W_3)_b \to (W_4)_b$ is a regular morpshism, $(W_3)_b$ is also regular (cf. [Mat89, Theorem 32.2]). Note that the morphism $(V_3)_b \to (V_4)_b \simeq V_5$ is isomorphic to the completion map

$$
\overline{B} = \operatorname{Spec}\bigl(\widehat{\mathcal{O}}_{\overline{N},\overline{y}}\bigr) \longrightarrow \operatorname{Spec}\bigl(\mathcal{O}_{\overline{N},\overline{y}}\bigr) = V_5 \,.
$$

Since $W_5 \to V_5$ is a birational map, so is $(W_3)_b \to (V_3)_b$. Since $(V_3)_b \simeq \overline{B}$ is integral, so is $(W_3)_b$. Furthermore, we have $\dim'(W_3)_b = \dim'(V_3)_b = \dim(V_3)_b = n$. Therefore, for any $a \in k^{\times}$, the scheme $(W_1)_a$ is integral regular with $\dim'(W_1)_a = n$.

(2) Let $T_i \subset V_i$ be the closed subschemes defined by the ideals $(x_1, \ldots, x_N)\mathcal{O}_{V_i}$ for $i \in \{1,2,3\}$ and by the ideal $\mathfrak{m}_{\overline{N},\overline{y}}\mathcal{O}_{V_4}$ for $i = 4$. Let $T'_i := r_i^{-1}(T_i)$. Then, we have the following Cartesian diagram:

$$
\begin{tikzcd}
T'_1 \arrow[r] & T_1 \simeq \mathbb{A}^1_k \\
T'_2 \arrow[r] \arrow[u] & T_2 \simeq \mathbb{A}^1_k \setminus \{0\} \arrow[u, hook] \\
T'_3 \arrow[r] \arrow[u] \arrow[d] & T_3 \simeq \mathbb{A}^1_k \setminus \{0\} \arrow[u, two heads] \arrow[d, "\simeq"] \\
T'_4 \arrow[r] & T_4 \simeq \mathbb{A}^1_k \setminus \{0\} \,.
\end{tikzcd}
$$

Since this diagram is Cartesian, any closed fiber of $T'_2 \to T_2$ is isomorphic to some fiber of $T'_4 \to T_4$. Since the germ $\left(\overline{N}, \overline{y}\right)$ is klt, so is V_4. Therefore, $T'_4 \to T_4$ has rationally connected fibers by [HM07, Corollary 1.7(1)], and so does $T'_2 \to T_2$. Consequently, $T'_2 \to T_2$ has a section by [GHS03]. Hence by the properness of r_1, the morphism $T'_1 \to T_1$ also has a section. We have proved that conditions (1) and (2) hold.

Note that $T_\infty = \{\beta\}$ consists of only one arc β. By the above, there exists a $\beta' \in T'_\infty \subset W_\infty$ such that $r_\infty(\beta') = \beta$. Suppose to the contrary that D is a thin subset of $\overline{B}'^{(\gamma)}_\infty$. Since $\beta' \in r_\infty^{-1}(D)$ satisfies $\operatorname{ord}_{\beta'}(\mathfrak{o}_W) = \infty$, to get a contradiction by Proposition 5.23, it is sufficient to show that $r_\infty^{-1}(D)$ is also a thin subset of W_∞.

By Lemma 7.11(5), there exists a unique irreducible component U of $\overline{B}'^{(\gamma)}$ of the form (2) in Lemma 5.25. We also note that $U'_\infty \cap D = \emptyset$ for any other irreducible component U' of $\overline{B}'^{(\gamma)}$ since $D \subset \mathrm{Cont}^{\geqslant 1}\big(\mathfrak{o}_{\overline{B}'^{(\gamma)}}\big)$ (cf. Remark 5.26). Therefore, since D is a thin set, there exists a closed subscheme $F \subset U$ such that $D \subset F_\infty$ and $\dim F \leqslant n$. Let W' be the connected component of W that dominates U. We note that $\dim W' = \dim U = n+1$ by Lemma 7.11(5). We set $F' := r^{-1}(F) \cap W'$. Then we have $r_\infty^{-1}(D) \subset F'_\infty$ and $\dim F' \leqslant n$, and hence $r_\infty^{-1}(D)$ is also a thin set. This completes the proof of Claim 8.4. □

This complete the proof of Theorem 8.2. □

Remark 8.5. Theorem 8.2 can be generalized to $\mathbb{R}$-ideals due to Remark 7.10. We have the equality $\mathrm{mld}_x\big(X, (h_1 \cdots h_c)\mathfrak{a}\big) = \mathrm{mld}_x(Y, \mathfrak{b})$ for an $\mathbb{R}$-ideal $\mathfrak{a}$ on X and $\mathfrak{b} := \mathfrak{a}\mathcal{O}_Y$.

9. Proof of the main theorems

As a corollary of Theorem 8.2, we prove the PIA conjecture for quotients of locally complete intersection singularities.

Corollary 9.1. *Suppose that a variety X has a quotient singularity at a closed point $x \in X$. Let Y be a subvariety of X of codimension c that is locally defined by c equations at x. Suppose that Y is klt at x. Let $\mathfrak{a}$ be an $\mathbb{R}$-ideal sheaf on Y. Let D be a prime divisor on Y through x that is klt and Cartier at x. Suppose that D is not contained in the cosupport of the $\mathbb{R}$-ideal sheaf $\mathfrak{a}$. Then we have*

$$\mathrm{mld}_x\big(Y, \mathfrak{a}\mathcal{O}_Y(-D)\big) = \mathrm{mld}_x(D, \mathfrak{a}\mathcal{O}_D)\,.$$

Proof. Take an $\mathbb{R}$-ideal sheaf $\mathfrak{b}$ on X such that $\mathfrak{a} = \mathfrak{b}\mathcal{O}_Y$ and local equations $h_1, \ldots, h_c \in \mathcal{O}_{X,x}$ of Y in X at x. Furthermore, take $g \in \mathcal{O}_{X,x}$ such that its image $\overline{g} \in \mathcal{O}_{Y,x}$ defines D. Then we have

$$\mathrm{mld}_x\big(Y, \mathfrak{a}\mathcal{O}_Y(-D)\big) = \mathrm{mld}_x\big(X, (h_1 \cdots h_c \cdot g)\mathfrak{b}\big) = \mathrm{mld}_x(D, \mathfrak{a}\mathcal{O}_D)$$

by applying Theorem 8.2 twice. □

Theorem 9.2. *Let X be a variety with only quotient singularities. Let Y be a klt subvariety of X of codimension c that is locally defined by c equations in X. Let $\mathfrak{a}$ be an $\mathbb{R}$-ideal sheaf on Y. Then the function*

$$|Y| \longrightarrow \mathbb{R}_{\geqslant 0} \cup \{-\infty\}\,, \quad x \longmapsto \mathrm{mld}_x(Y, \mathfrak{a})$$

is lower semi-continuous, where we denote by $|Y|$ the set of all closed points of Y for the Zariski topology.

Proof. We use the notation of the proof of Corollary 9.1. Then by Theorem 8.2, we have $\mathrm{mld}_x(Y, \mathfrak{a}) = \mathrm{mld}_x\big(X, (h_1 \cdots h_c)\mathfrak{b}\big)$. The assertion follows from the fact proved in [Nak16, Corollary 1.3] that the lower semi-continuity holds for the variety X with only quotient singularities. □

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Yusuke Nakamura y.nakamura@math.nagoya-u.ac.jp
Graduate School of Mathematics, Nagoya University, Furo-cho, Chikusa-ku, Nagoya, 464-8602, Japan

Kohsuke Shibata shibata.kohsuke@mail.dendai.ac.jp
Department of Mathematics, School of Engineering, Tokyo Denki University, Adachi-ku, Tokyo, 120-8551, Japan