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Functoriality of HKR isomorphisms

Shengyuan Huang

Abstract

For a closed embedding of smooth schemes $X \hookrightarrow S$ with a fixed first-order splitting, one can construct Hochschild–Kostant–Rosenberg (HKR) isomorphisms between the derived scheme $X \times^R_S X$ and the total space of the shifted normal bundle $\mathbb{N}_{X/S}[-1]$, thanks to Arinkin–Căldăraru, Arinkin–Căldăraru–Hablicsek, and Grivaux. In this paper, we study the functoriality property of the HKR isomorphisms for a sequence of closed embeddings $X \hookrightarrow Y \hookrightarrow S$. The HKR isomorphism is functorial when a certain cohomology class, which we call the Bass–Quillen class, vanishes. We obtain Lie-theoretic interpretations for the HKR isomorphisms and for the Bass–Quillen class as well.

1. Introduction

1.1. Let X be a smooth algebraic variety over a field of characteristic zero. There is a Hochschild–Kostant–Rosenberg (HKR) isomorphism [Swa96] in the derived category of X

$$\Delta^*\Delta_*\mathscr{O}_X \cong \mathsf{Sym}_{\mathscr{O}_X}(\Omega_X[1])\,,$$

where Δ is the diagonal embedding of X in $X \times X$.

It was observed by Kapranov and Kontsevich [Kap99] that there is a Lie-theoretic interpretation of the HKR isomorphism. The derived loop space $LX = X \times^R_{X\times X} X$ has the structure of a derived group scheme over X, and the shifted tangent bundle $T_X[-1]$ is its Lie algebra [Kap99]. The HKR isomorphism can be thought of as a version of the exponential map $\mathbb{T}_X[-1] \to LX$, see [CR11, Mar09, Ram08], where $\mathbb{T}_X[-1]$ is the total space of the shifted tangent bundle.

1.2. In the case of classical Lie groups, we have a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle\exp}\uparrow & & \uparrow{\scriptstyle\exp} \\ \mathfrak{g} & \xrightarrow{df} & \mathfrak{h}\,. \end{array}$$

A map of schemes $X \to Y$ induces a map of derived group schemes $LX \to LY|_X$ over X. The statement for derived schemes analogous to the above Lie-theoretic statement is that the

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diagram

$$\begin{array}{ccc} LX & \longrightarrow & LY|_X \\ {\scriptstyle \text{HKR}}\uparrow & & \uparrow{\scriptstyle \text{HKR}} \\ \mathbb{T}_X[-1] & \longrightarrow & \mathbb{T}_Y|_X[-1] \end{array}$$

commutes. One can prove the commutativity of the diagram above easily using the methods in this paper.

1.3. We would like to consider the commutativity of this diagram in a more general setting. We replace the diagonal embedding with an arbitrary closed embedding $i\colon X \hookrightarrow S$ of smooth schemes.

The embedding i factors as

$$X \xhookrightarrow{\mu} X_S^{(1)} \xhookrightarrow{\nu} S\,,$$

where $X_S^{(1)}$ is the first-order neighborhood of X in S. We say that i splits to first order if and only if the map μ is split, that is, there exists a map of schemes $\varphi\colon X_S^{(1)} \to X$ such that $\varphi \circ \mu = \mathrm{id}$. There is a bijection between first-order splittings of i and splittings of the short exact sequence below [Gro64, Theorem 20.5.12(iv)]:

$$0 \longrightarrow T_X \longrightarrow T_S|_X \longrightarrow N_{X/S} \longrightarrow 0\,.$$

One can still get HKR isomorphisms if one replaces the diagonal embedding with an arbitrary closed embedding of schemes $i\colon X \hookrightarrow S$ with a fixed first-order splitting. Arinkin, Căldăraru, Hablicsek [AC12, ACH19], and Grivaux [Gri14] constructed HKR isomorphisms between the total space of the shifted normal bundle $\mathbb{N}_{X/S}[-1]$ and the derived self-intersection $X \times_S^R X$ from a fixed first-order splitting of i. In this paper, we study the functoriality of the HKR isomorphism constructed in [AC12].

1.4. We will be in the following setting from now on. Let $X \hookrightarrow Y \hookrightarrow S$ be a sequence of closed embeddings of smooth schemes. Assume that X is split to first order in Y, and similarly for X in S and Y in S. We want to understand if the two squares in the diagram

$$\begin{array}{ccccc} X \times_Y^R X & \longrightarrow & X \times_S^R X & \longrightarrow & Y \times_S^R X = Y \times_S^R Y|_X \\ \cong\uparrow & & \cong\uparrow & & \cong\uparrow \\ \mathbb{N}_{X/Y}[-1] & \longrightarrow & \mathbb{N}_{X/S}[-1] & \longrightarrow & \mathbb{N}_{Y/S}|_X[-1] \end{array}$$

are commutative.

1.5. There are two ways to construct isomorphisms between $\mathbb{N}_{X/S}[-1]$ and $X \times_S^R X$ from a fixed first-order splitting. The construction of HKR isomorphisms in [AC12] involves a differential graded $\mathscr{O}_{X_S^{(1)}}$-algebra resolution of $\mathscr{O}_X$. The construction of HKR isomorphisms in [ACH19, Gri14] reduces all HKR isomorphisms to HKR isomorphisms of diagonal embeddings constructed in [AC12]. For more details, see Section 2. It is unknown whether the first and the second constructions agree or not. This is an interesting question that needs to be solved in the future. In this paper, we only deal with the HKR isomorphisms constructed in [AC12].

1.6. We need to define a cohomology class before we state the main theorems of this paper. In the next two subsections, we define the cohomology class that we call the Bass–Quillen class and explain its relation with Bass–Quillen conjecture. We also explain a special case of it that will be used in our main theorems.

1.7. Definition of the Bass–Quillen class. Let t be the embedding $X \hookrightarrow X_Y^{(1)}$ and $s\colon X_Y^{(1)} \to X$ be a chosen first-order splitting. Consider the short exact sequence of $\mathscr{O}_{X_Y^{(1)}}$-modules

$$0 \longrightarrow t_* N_{X/Y}^\vee \longrightarrow \mathscr{O}_{X_Y^{(1)}} \longrightarrow t_*\mathscr{O}_X \longrightarrow 0\,.$$

For any vector bundle $\mathscr{G}$ on $X_Y^{(1)}$, tensor it with this short exact sequence. Then push the sequence forward onto X via s. Using the fact that $s \circ t = \mathrm{id}$ and the projection formula, we get a short exact sequence of vector bundles on X

$$0 \longrightarrow N_{X/Y}^\vee \otimes \mathscr{G}|_X \longrightarrow s_*\mathscr{G} \longrightarrow \mathscr{G}|_X \longrightarrow 0\,.$$

Dualizing, we get an extension class $\alpha_{s,\mathscr{G}}\colon N_{X/Y} \otimes \mathscr{G}^\vee|_X \to \mathscr{G}^\vee|_X[1]$. We call $\alpha_{s,\mathscr{G}}$ the Bass–Quillen class for the pair $(s,\mathscr{G})$ for the reason explained below.

1.8. Suppose that X is a smooth algebraic variety and Y is the total space of a vector bundle $\mathscr{E}$ on X. Let $\mathscr{F}$ be a vector bundle on Y. Then one can ask if $\mathscr{F}$ is isomorphic to the pull-back of some vector bundle on X. When X is affine, this is known as the Bass–Quillen problem and was answered affirmatively in [Lin81]. However, in the global case, the answer to this question is negative. In particular, there can be no vector bundle on X whose pull-back is $\mathscr{F}$ if the Bass–Quillen class

$$\alpha_{s,\mathscr{F}|_{X_Y^{(1)}}} \in \operatorname{Ext}^1\big(N_{X/Y} \otimes \mathscr{F}^\vee|_X, \mathscr{F}^\vee|_X\big)$$

is not zero because for any $\mathscr{G}$, the class $\alpha_{s,\mathscr{G}}$ vanishes if and only if $\mathscr{G}$ is isomorphic to $s^*\mathscr{G}|_X$; see [CG19, § 4.13].

Choose $\mathscr{G}$ to be $N_{Y/S}^\vee|_{X_Y^{(1)}}$, which gives a class $\alpha_{s,N_{Y/S}^\vee|_{X_Y^{(1)}}}$. For the same reason as above, the class $\alpha_{s,N_{Y/S}^\vee|_{X_Y^{(1)}}}$ vanishes if and only if $N_{Y/S}^\vee|_{X_Y^{(1)}} \cong s^*\big(N_{Y/S}^\vee|_X\big)$. This class will be used in Theorems A and B.

Here are the main results in this paper.

Theorem A. *Let $X \hookrightarrow Y \hookrightarrow S$ be a sequence of closed embeddings of smooth schemes. Further assume that there are compatible splittings on the tangent bundles*

$$T_X \longrightarrow T_Y|_X \longrightarrow T_S|_X\,, \qquad p\colon T_Y|_X \dashrightarrow T_X,\quad q|_X\colon T_S|_X \dashrightarrow T_Y|_X,\quad \rho\colon T_S|_X \dashrightarrow T_X\,.$$

Compatibility means that $p \circ q|_X = \rho$. Then the left square below is commutative. If the Bass–Quillen class of $N_{Y/S}^\vee|_{X_Y^{(1)}}$ is zero, then the right square below is also commutative.

$$\begin{array}{ccccc}
X \times_Y^R X & \longrightarrow & X \times_S^R X & \longrightarrow & Y \times_S^R X = Y \times_S^R Y|_X \\
\cong\uparrow & & \cong\uparrow & & \cong\uparrow \\
\mathbb{N}_{X/Y}[-1] & \longrightarrow & \mathbb{N}_{X/S}[-1] & \longrightarrow & \mathbb{N}_{Y/S}|_X[-1]\,.
\end{array}$$

The vertical maps are the HKR isomorphisms defined in [AC12]. The horizontal maps between normal bundles are the linear ones; that is, they are vector bundle maps.

1.9. Application to orbifolds. This paper is motivated by the study of Hochschild cohomology of an orbifold [CH21]. The main example we consider is the setting where S admits an action of a finite group G and X is the fixed locus of G and Y is the fixed locus of a subgroup $H \leqslant G$. There is a natural splitting on tangent bundles

$$0 \longrightarrow T_X \longrightarrow T_S|_X \longrightarrow N_{X/S} \longrightarrow 0\,,$$

where the dashed map is given by the formula $v \mapsto (1/|G|)\sum_{g\in G} g \cdot v$. The natural splitting on tangent bundles provides a first-order splitting of $X \hookrightarrow S$, and similarly one obtains first-order splittings for $X \hookrightarrow Y$ and $Y \hookrightarrow S$. One can check directly that the three splittings are compatible.

In [CH21], the authors define a natural product on the orbifold polyvector fields which is expected to be isomorphic to the orbifold Hochschild cohomology. In op. cit., this product is shown to be associative when the HKR isomorphisms are functorial. In particular, the product is associative when the Bass–Quillen classes vanish. It is known in many examples that the Bass–Quillen classes vanish and the product is associative; see [HX23].

1.10. There is a Lie-theoretic interpretation for Theorem A. Under certain extra assumptions, all derived self-intersections in Theorem A are groups in the derived category of differential graded (dg) schemes. The shifted normal bundles are their Lie algebras. One can check that the natural maps

$$X \times^R_Y X \longrightarrow X \times^R_S X \longrightarrow X \times^R_S Y = Y \times^R_S Y|_X$$

are maps of groups. The map $N_{X/Y}[-1] \to N_{Y/S}|_X[-1]$ respects the Lie structures in general. However, $N_{X/S}[-1] \to N_{Y/S}|_X[-1]$ may not preserve the Lie brackets.

With no assumptions other than the choice of a first-order splitting, the bundle $N_{X/S}[-1]$ carries an anti-symmetric bracket which may not satisfy the Jacobi identity. We call this structure a *bracket*. We will provide more details in Section 2.

Theorem B. *In the same setting as Theorem A, the vector bundle map $N_{X/S}[-1] \to N_{Y/S}|_X[-1]$ preserves the brackets if and only if the Bass–Quillen class*

$$\alpha_{s, N^\vee_{Y/S}|_{X^{(1)}_Y}} : N_{X/Y} \otimes N_{Y/S}|_X \longrightarrow N_{Y/S}|_X[1]$$

is zero.

Thus Theorems A and B provide a generalization of the original result for Lie groups to the setting of groups obtained as self-intersections.

1.11. Plan of the paper. Section 2 reviews background material. We recall what is known about the structure of arbitrary closed embeddings with further assumptions on (possibly higher-order) splittings, and how the HKR isomorphisms are constructed for a closed embedding with a fixed first-order splitting. We also provide details about the Lie-theoretic interpretation of the Bass–Quillen class and Theorem B.

Section 3 is devoted to the proof of Theorem A. To prove the commutativity of the diagram in Theorem A, it suffices to check whether the explicit resolutions of $\mathscr{O}_X$ and $\mathscr{O}_Y$ in [AC12] are compatible.

Section 4 is the proof of our Lie-theoretic interpretation of the Bass–Quillen class $\alpha_{s, N^\vee_{Y/S}|_{X^{(1)}_Y}}$. This class can be viewed as a Lie module structure map, where $N_{X/Y}[-1]$ is the Lie algebra and $N_{Y/S}|_X[-1]$ is the module. This interpretation will be used in Section 6.

Section 5 is about Theorem B for Lie algebras in the category of vector spaces. We consider an inclusion of Lie algebras $\mathfrak{h} \hookrightarrow \mathfrak{g}$ and the associated short exact sequence of $\mathfrak{h}$-modules

$$0 \longrightarrow \mathfrak{h} \xrightarrow{\alpha} \mathfrak{g} \xrightarrow{\beta} \mathfrak{n} = \mathfrak{g}/\mathfrak{h} \longrightarrow 0\,.$$

One can construct a bracket on $\mathfrak{n}$ which may not be compatible with the Lie bracket on $\mathfrak{g}$. Then we prove a result similar to Theorem B using methods that make sense in the derived category. This approach will be generalized to the derived category of schemes in Section 6.

Section 6 generalizes the proofs in Section 5 to the derived setting. We end this paper with an example where the Bass–Quillen class is not zero.

1.12. Conventions. All the schemes we considered in this paper are smooth over a field of characteristic zero or in positive characteristic greater than the dimension of the scheme S.

Let us explain the context of the theorems in this paper and in [AC12, ACH19, Swa96] for the experts. One can consider the HKR isomorphism

$$\Delta^*\Delta_*\mathscr{O}_X \cong \mathsf{Sym}_{\mathscr{O}_X}(\Omega_X[1])$$

and the statement in Theorem A in three different categories. The three categories are the derived category of coherent sheaves on X, the dg category of dg schemes, and the derived category of dg schemes. The dg category of dg schemes is an $(\infty, 1)$-category whose derived category is the derived category of dg schemes. Swan's result [Swa96] is stated in the derived category of coherent sheaves on X. The more general HKR isomorphism in [AC12] is stated in the derived category of coherent sheaves on X, but the two authors mentioned that their proofs work at the dg-level; that is, their main results hold in the derived category of dg schemes. Similarly, the results in [ACH19] are stated in the derived category of dg schemes.

In this paper, Theorem A is stated in the derived category of dg schemes, and Theorem B is stated in the derived category of coherent sheaves on X.

2. Background and Lie-theoretic interpretations

We first discuss what is known about the HKR isomorphism for the diagonal embedding. Then we recall the definition of the HKR isomorphisms for a general closed embedding $X \hookrightarrow S$ with a fixed first-order splitting. We briefly recall the Lie-theoretic interpretations of general embeddings with possibly higher-order splittings. We explain the Lie-theoretic interpretations for the Bass–Quillen class $\alpha_{s,N^\vee_{Y/S}|_{X^{(1)}_Y}}$ and Theorem B last.

2.1. The diagonal embedding. Let X be a smooth algebraic variety. There is an HKR isomorphism [Swa96]

$$\Delta^*\Delta_*\mathscr{O}_X \cong \mathsf{Sym}_{\mathscr{O}_X}(\Omega_X[1])\,,$$

where $\Delta\colon X \hookrightarrow X \times X$ is the diagonal embedding.

2.2. In the world of derived schemes, we consider the free loop space LX of X, defined as the derived self-intersection $X \times^R_{X\times X} X$. Its structure complex is $\Delta^*\Delta_*\mathscr{O}_X$, and the structure complex of the total space of the shifted tangent bundle $\mathbb{T}_X[-1] = \mathrm{Spec}_{\mathscr{O}_X}(\mathsf{Sym}(\Omega_X[1]))$ is $\mathsf{Sym}(\Omega_X[1])$. We can restate the HKR isomorphism as an isomorphism of derived schemes over X:

$$\mathbb{T}_X[-1] \xrightarrow{\cong} LX = X \times^R_{X\times X} X\,.$$

It can be viewed as the exponential map from the Lie algebra $\mathbb{T}_X[-1]$ to the group LX, as explained in the introduction.

2.3. General embeddings. There exist generalized HKR isomorphisms if we replace the diagonal embedding with an arbitrary closed embedding $i\colon X \hookrightarrow S$ of smooth schemes. Arinkin and Căldăraru [AC12] provided a necessary and sufficient condition for $X \times^R_S X$ to be isomorphic to $\mathbb{N}_{X/S}[-1]$ over X. In [ACH19], Arinkin, Căldăraru, and Hablicsek proved that the derived intersection $X \times^R_S X$ is isomorphic to $\mathbb{N}_{X/S}[-1]$ over $X \times X$ if and only if the embedding i splits to first order. Grivaux independently proved a similar result for complex manifolds in [Gri14].

2.4. Let us briefly recall how the HKR isomorphism $i^* i_* \mathscr{O}_X \cong \mathsf{Sym}\big(N^\vee_{X/S}[1]\big)$ was constructed in [AC12]. It is defined as the composite map

$$\mu^*\nu^*\nu_*\mu_*\mathscr{O}_X \longrightarrow \mu^*\mu_*\mathscr{O}_X \xrightarrow{\cong} \mathrm{T}^c\big(N^\vee_{X/S}[1]\big) \xrightarrow{\mathsf{exp}} \mathrm{T}\big(N^\vee_{X/S}[1]\big) \longrightarrow \mathsf{Sym}\big(N^\vee_{X/S}[1]\big)\,.$$

The leftmost map is given by the counit of the adjunction $\nu^* \dashv \nu_*$. The map exp is multiplication by $1/k!$ on the degree k piece, and the last map is the natural projection map. The term $\mathrm{T}^c\big(N^\vee_{X/S}[1]\big)$ is the free coalgebra on $N^\vee_{X/S}[1]$ with the shuffle product structure, and $\mathrm{T}\big(N^\vee_{X/S}[1]\big)$ is the tensor algebra on $N^\vee_{X/S}[1]$. The isomorphism $\mu^*\mu_*\mathscr{O}_X \cong \mathrm{T}^c\big(N^\vee_{X/S}[1]\big)$ in the middle is nontrivial and needs more explanation. With the splitting φ, one can build an explicit resolution of $\mu_*\mathscr{O}_X$ as an $\mathscr{O}_{X^{(1)}_S}$-algebra

$$\big(\mathrm{T}^c\big(\varphi^* N^\vee_{X/S}[1]\big), d\big) \longrightarrow \mu_*\mathscr{O}_X\,,$$

where $\big(\mathrm{T}^c\big(\varphi^* N^\vee_{X/S}[1]\big), d\big)$ is the free coalgebra on $\varphi^* N^\vee_{X/S}[1]$ with the shuffle product structure and a differential d. The differential is defined as follows. There is a short exact sequence on $X^{(1)}_S$

$$0 \longrightarrow \mu_* N^\vee_{X/S} \longrightarrow \mathscr{O}_{X^{(1)}_S} \longrightarrow \mu_*\mathscr{O}_X \longrightarrow 0\,.$$

Consider the composite map

$$\varphi^* N^\vee_{X/S} \longrightarrow \mu_*\mu^*\varphi^* N^\vee_{X/S} = \mu_* N^\vee_{X/S} \longrightarrow \mathscr{O}_{X^{(1)}_S}\,,$$

whose cokernel is $\mu_*\mathscr{O}_X$. Tensor the morphism above with $\big(\varphi^* N^\vee_{X/S}\big)^{\otimes(k-1)}$. We get the degree k piece of the differential, $d_k\colon (\varphi^* N^\vee_{X/S})^{\otimes k} \to (\varphi^* N^\vee_{X/S})^{\otimes(k-1)}$. The differential vanishes once we pull this resolution back on X via μ, so we get the desired isomorphism.

2.5. For any vector bundle $\mathscr{E}$ on X, we tensor the resolution above by $\varphi^*\mathscr{E}$. Using the projection formula and $\varphi \circ \mu = \mathrm{id}$, one can show that we get a resolution of $\mu_*\mathscr{E}$

$$\big(\mathrm{T}^c\big(\varphi^* N^\vee_{X/S}[1]\big) \otimes \varphi^*\mathscr{E}, d\big) \longrightarrow \mu_*\mathscr{E}\,.$$

The same argument shows that $i^* i_*(\mathscr{E}) \cong \mathscr{E} \otimes \mathsf{Sym}\big(N^\vee_{X/S}[1]\big)$, in other words, that $i^* i_*(-) \cong (-) \otimes \mathsf{Sym}\big(N^\vee_{X/S}[1]\big)$ as dg functors. This shows that $X \times^R_S X \cong \mathbb{N}_{X/S}[-1]$ over $X \times X$; see [ACH19].

2.6. Lie-theoretic interpretations for general self-intersections. Consider a closed embedding $i\colon X \hookrightarrow S$ of smooth schemes. The derived self-intersection $X \times^R_S X$ has an ∞-groupoid structure in the $(\infty, 1)$-category of dg schemes over X; see [GR17, TV08]. The associated L_∞-algebroid is $N_{X/S}[-1]$; see [CCT14]. Passing to the derived category, we get a groupoid in the derived category of dg schemes having X as the space of objects. The target and source maps are the two projections $\pi_1, \pi_2\colon X \times^R_S X \to X$. See [CCT14] for more details. When $S = X \times X$ and i is the diagonal embedding $\Delta\colon X \to X \times X$, there are two projections p_1 and $p_2\colon X \times X \to X$ such that $p_i \circ \Delta = \mathrm{id}$. This implies [ACH19] that the source map π_1 and the target map π_2 are equal in the derived category in this case, so $X \times^R_{X\times X} X$ becomes a group over X. A similar argument works if the inclusion from X to its formal neighborhood in S splits.

Generally speaking, $N_{X/S}[-1]$ has an L_∞-algebroid structure in the $(\infty,1)$-category of dg quasi-coherent sheaves on X. However, the Lie bracket may not be $\mathscr{O}_X$-linear, and it may not satisfy the Jacobi identity when we pass to the derived category of X. Calaque, Căldăraru, and Tu proved that the induced bracket in the derived category is $\mathscr{O}_X$-linear if i splits to first order, and it satisfies the Jacobi identity if i splits to second order [CCT14]. As a consequence, $N_{X/S}[-1]$ admits a natural Lie algebra structure in the derived category if i splits to second order. Later, Calaque and Grivaux showed that $N_{X/S}[-1]$ has a natural Lie algebra structure if $X \hookrightarrow S$ is a tame quantized cycle, a weaker condition than splitting to second order, whose definition can be found in [CG19]. More precisely, for an embedding $i\colon X \hookrightarrow S$ with a chosen first-order splitting, they defined [CG19, § 4.3.1] the $\mathscr{O}_X$-linear bracket $N_{X/S} \otimes N_{X/S} \to \mathsf{Sym}^2 N_{X/S} \to N_{X/S}[1]$ explicitly by the extension class of the short exact sequence of vector bundles on X

$$0 \longrightarrow \mathsf{Sym}^2 N^\vee_{X/S} \cong \frac{I_X^2}{I_X^3} \longrightarrow \varphi_* \frac{I_X}{I_X^3} \longrightarrow \frac{I_X}{I_X^2} \cong N^\vee_{X/S} \longrightarrow 0\,,$$

where I_X is the ideal sheaf of X in S. This bracket satisfies the Jacobi identity under the tameness assumption [CG19, Theorem 5.8]. In the rest of this paper, we will only use embeddings that are split to first order without requiring the Jacobi identity to hold for this specific bracket.

2.7. We have finished the background material. The rest of this section is about the definition of the Bass–Quillen class and the Lie-theoretic interpretations of our results. From now on, we fix smooth subvarieties X and Y of S with closed embeddings i, j, and f

$$X \xhookrightarrow{\;f\;} Y \xhookrightarrow{\;j\;} S\,, \qquad i = j \circ f\colon X \hookrightarrow S.$$

We assume that f, j, and i are split to first order and that we have fixed first-order splittings of f, j, and i that will be denoted by s, π, and φ.

2.8. It is crucial to note that all the constructions of HKR isomorphisms depend on the choice of splitting. Therefore, we need to assume some compatibility on these splittings, namely, that $p \circ q|_X = \rho$

$$T_X \longrightarrow T_Y|_X \longrightarrow T_S|_X\,, \qquad p\colon T_Y|_X \dashrightarrow T_X,\quad q|_X\colon T_S|_X \dashrightarrow T_Y|_X,\quad \rho\colon T_S|_X \dashrightarrow T_X,$$

where p, q, and ρ are splittings on the tangent bundles corresponding to the first-order splittings s, π, and φ.

The Bass–Quillen class $\alpha_{s, N^\vee_{Y/S}|_{X^{(1)}_Y}}$ admits an interpretation as a Lie module structure map. We explain this interpretation in the rest of this section.

2.9. Lie-theoretic interpretations of the Bass–Quillen class and Theorem B. To make our Lie-theoretic interpretation clearer, let us assume in this section that all the three derived self-intersections $X \times^R_Y X$, $X \times^R_S X$, and $Y \times^R_S Y$ are groups. However, we will state and explain theorems and propositions later in Sections 3 through 6 which only assume the existence of first-order splittings. One can check that the natural maps

$$X \times^R_Y X \longrightarrow X \times^R_S X \longrightarrow X \times^R_S Y = Y \times^R_S Y|_X$$

are maps of groups. All the shifted normal bundles are Lie algebras under this assumption. In this section, we denote $N_{X/Y}[-1]$, $N_{X/S}[-1]$, and $N_{Y/S}|_X[-1]$ by $\mathfrak{h}$, $\mathfrak{g}$, and $\mathfrak{n}$, respectively. The functoriality of HKR isomorphisms can be viewed as the functoriality of the exponential maps from Lie algebras to Lie groups.

2.10. The map $\mathfrak{h} = N_{X/Y}[-1] \hookrightarrow \mathfrak{g} = N_{X/S}[-1]$ preserves the Lie brackets, so we are able to prove the commutativity of the left square in Theorem A with no difficulty. Moreover, the compatibility of the Lie brackets implies that $\mathfrak{g}$ is an $\mathfrak{h}$-module, and $\mathfrak{h} \hookrightarrow \mathfrak{g}$ is a map of $\mathfrak{h}$-modules. Therefore, $\mathfrak{n} = \mathfrak{g}/\mathfrak{h} = N_{Y/S}|_X[-1]$ has a natural $\mathfrak{h}$-module structure. We get a short exact sequence of $\mathfrak{h}$-modules

$$0 \longrightarrow \mathfrak{h} = N_{X/Y}[-1] \longrightarrow \mathfrak{g} = N_{X/S}[-1] \longrightarrow \mathfrak{n} = \mathfrak{g}/\mathfrak{h} = N_{Y/S}|_X[-1] \longrightarrow 0\,.$$

The $\mathfrak{h}$-module structure on $\mathfrak{g}/\mathfrak{h}\colon N_{X/Y} \otimes N_{Y/S}|_X \to N_{Y/S}|_X[1]$ is exactly the Bass–Quillen class $\alpha_{s,N^\vee_{Y/S}|_{X^{(1)}_Y}}$. We will prove this statement in Section 4.

On the other hand, the map $\mathfrak{g} = N_{X/S}[-1] \to \mathfrak{n} = N_{Y/S}|_X[-1]$ may not in general preserve the Lie brackets even if we assume that all the derived self-intersections are groups. This explains the difficulty for proving the functoriality of the exponential maps in the right square of Theorem A. In Section 6, we will show that $\mathfrak{h} = N_{X/Y}[-1]$ acts trivially on its module $\mathfrak{g}/\mathfrak{h} = N_{Y/S}|_X[-1]$ if and only if the Lie brackets are preserved, that is, $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h} = \mathfrak{n}$ is a map of Lie algebras. This is Theorem B. As a consequence, the right square in Theorem A commutes when the Lie brackets are preserved.

2.11. We restate Theorem B for Lie algebras in the category of vector spaces. Consider an injective morphism of Lie algebras $\mathfrak{h} \hookrightarrow \mathfrak{g}$. The quotient $\mathfrak{n} = \mathfrak{g}/\mathfrak{h}$ is naturally an $\mathfrak{h}$-module, so we get a short exact sequence of $\mathfrak{h}$-modules

$$0 \longrightarrow \mathfrak{h} \xrightarrow{\alpha} \mathfrak{g} \xrightarrow{\beta} \mathfrak{n} = \mathfrak{g}/\mathfrak{h} \longrightarrow 0\,.$$

There is a way to construct a bracket on $\mathfrak{n}$ once a splitting $\mathfrak{n} \dashrightarrow \mathfrak{g}$ of vector spaces is chosen [CG19]. The morphism β preserves the brackets if and only if $\mathfrak{h}$ acts trivially on $\mathfrak{n}$. The triviality of the $\mathfrak{h}$-module structure on $\mathfrak{n}$ implies that $\mathfrak{n}$ is actually a Lie algebra, the map β is a Lie algebra morphism, and the diagram

$$\begin{array}{ccc} G & \xrightarrow{\Phi} & N \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow{\beta = d\Phi} & \mathfrak{n} \end{array}$$

is commutative.

3. The proof of Theorem A

In this section, we prove Theorem A.

3.1. Suppose that $i\colon X \hookrightarrow S$ is a closed embedding of smooth schemes with a first-order splitting

$$X \underset{\mu}{\overset{\varphi}{\rightleftarrows}} X^{(1)}_S \overset{\nu}{\hookrightarrow} S.$$

In Section 2, we recalled the construction of the HKR isomorphism

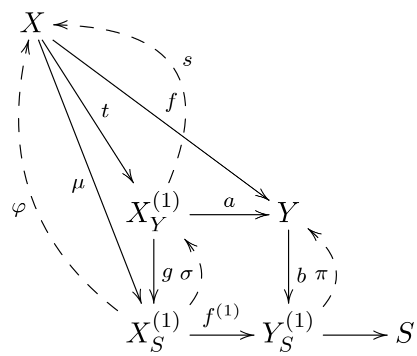
$$i^* i_* \mathscr{O}_X \cong \mathsf{Sym}\big(N^\vee_{X/S}[1]\big)$$

from [AC12]. It is defined as the composite map

$$\mu^*\nu^*\nu_*\mu_*\mathscr{O}_X \longrightarrow \mu^*\mu_*\mathscr{O}_X \xrightarrow{\cong} \mathrm{T}^c\big(N^\vee_{X/S}[1]\big) \xrightarrow{\mathsf{exp}} \mathrm{T}\big(N^\vee_{X/S}[1]\big) \longrightarrow \mathsf{Sym}\big(N^\vee_{X/S}[1]\big)\,.$$

It is easy to see that all the constructions are canonical except for the isomorphism $\mu^*\mu_*\mathscr{O}_X \cong \mathrm{T}^c\big(N^\vee_{X/S}[1]\big)$, which depends on the choice of the splitting φ.

3.2. We have a commutative diagram



under the assumptions in Theorem A. The solid arrows are the obvious ones. The dashed arrows π, s, and φ are the first-order splittings of the closed embeddings $j\colon Y \to S$, $f\colon X \to Y$, and $i\colon X \to S$, respectively. Notice that $X_Y^{(1)}$ is the fiber product of $X_S^{(1)}$ and Y over $Y_S^{(1)}$, so we can pull π back along the morphism $f^{(1)}$ to define σ. The compatibility condition on the splittings is equivalent to the condition that $s \circ \sigma = \varphi$.

Proof of Theorem A. To check the commutativity of the left square in Theorem A, it suffices to show that the diagram

$$\begin{array}{ccc} \mu^*\mu_*\mathscr{O}_X & \xrightarrow{\cong} & \mathrm{T}^c\big(N^\vee_{X/S}[1]\big) \\ \downarrow & & \downarrow \\ t^*t_*\mathscr{O}_X & \xrightarrow{\cong} & \mathrm{T}^c\big(N^\vee_{X/Y}[1]\big) \end{array}$$

is commutative since all the other constructions are canonical. The right vertical map is obtained from the natural vector bundle map $N^\vee_{X/S} \to N^\vee_{X/Y}$. The horizontal isomorphisms are constructed using the splittings from explicit resolutions of $\mu_*\mathscr{O}_X$ and $t_*\mathscr{O}_X$ on $X_S^{(1)}$ and $X_Y^{(1)}$, respectively. These resolutions are of the form $\big(\mathrm{T}^c\big(\varphi^*N^\vee_{X/S}[1]\big), d\big)$ and $\big(\mathrm{T}^c\big(s^*N^\vee_{X/Y}[1]\big), d\big)$, as explained in Section 2.4.

We have $g^*\varphi^*N^\vee_{X/S} = s^*N^\vee_{X/S}$ using the fact that $\varphi = s \circ \sigma$ and $\sigma \circ g = \mathrm{id}$. There is a natural map of vector bundles $g^*\varphi^*N^\vee_{X/S} = s^*N^\vee_{X/S} \to s^*N^\vee_{X/Y}$ that induces a map of complexes $g^*\big(\mathrm{T}^c\big(\varphi^*N^\vee_{X/S}[1]\big), d\big) \to \big(\mathrm{T}^c\big(s^*N^\vee_{X/Y}[1]\big), d\big)$. One can check carefully that the induced map is indeed a map of complexes; that is, the differentials are preserved. This proves that the diagram

$$\begin{array}{ccccc} g^*\big(\mathrm{T}^c\big(\varphi^*N^\vee_{X/S}[1]\big), d\big) & \longrightarrow & g^*\mu_*\mathscr{O}_X & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \big(\mathrm{T}^c\big(s^*N^\vee_{X/Y}[1]\big), d\big) & \longrightarrow & t_*\mathscr{O}_X & \longrightarrow & 0 \end{array}$$

that relates the two explicit resolutions of $\mathscr{O}_X$ as an $\mathscr{O}_{X_S^{(1)}}$-algebra and as an $\mathscr{O}_{X_Y^{(1)}}$-algebra is commutative. If we pull the natural map $g^*\varphi^*N^\vee_{X/S} = s^*N^\vee_{X/S} \to s^*N^\vee_{X/Y}$ back to X, we get the natural vector bundle map $N^\vee_{X/S} \to N^\vee_{X/Y}$. This proves that we get our desired commutative diagram at the beginning of the proof of Theorem A once we pull the commutative diagram above back to X.

Similarly, to prove the commutativity of the right square of Theorem A, it suffices to show that the diagram

$$\begin{array}{ccc} \mu^*\mu_*\mathscr{O}_X & \xrightarrow{\cong} & \mathrm{T}^c\big(N^\vee_{X/S}[1]\big) \\ \uparrow & & \uparrow \\ f^*b^*b_*\mathscr{O}_Y & \xrightarrow{\cong} & f^*\mathrm{T}^c\big(N^\vee_{Y/S}[1]\big) \end{array}$$

is commutative. The right vertical map is induced by the natural map of vector bundles $N^\vee_{Y/S}|_X \to N^\vee_{X/S}$.

If the Bass–Quillen class $\alpha_{s,N^\vee_{Y/S}|_{X^{(1)}_Y}}$ vanishes, then we have an isomorphism between $a^*N^\vee_{Y/S}$ and $s^*\big(N^\vee_{Y/S}|_X\big)$. The latter maps naturally to $s^*N^\vee_{X/S}$. Therefore, we get a map $\sigma^*a^*N^\vee_{Y/S} \cong \sigma^*s^*N^\vee_{Y/S}|_X \to \sigma^*s^*N^\vee_{X/S} = \varphi^*N^\vee_{X/S}$. Notice that $a\circ\sigma = \pi\circ f^{(1)}$ by the definition of σ, so we get a map $f^{(1)*}\pi^*N^\vee_{Y/S} = \sigma^*a^*N^\vee_{Y/S} \to \sigma^*s^*N^\vee_{X/S} = \varphi^*N^\vee_{X/S}$. This map induces a map of complexes $f^{(1)*}\big(\mathrm{T}^c\big(\pi^*N^\vee_{Y/S}[1]\big),d\big) \to \big(\mathrm{T}^c\big(\varphi^*N^\vee_{X/S}[1]\big),d\big)$. As a consequence, the diagram of resolutions of $\mu_*\mathscr{O}_X$ and $b_*\mathscr{O}_Y$

$$\begin{array}{ccccc} \big(\mathrm{T}^c\big(\varphi^*N^\vee_{X/S}[1]\big),d\big) & \longrightarrow & \mu_*\mathscr{O}_X & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \\ f^{(1)*}\big(\mathrm{T}^c\big(\pi^*N^\vee_{Y/S}[1]\big),d\big) & \longrightarrow & f^{(1)*}b_*\mathscr{O}_Y & \longrightarrow & 0 \end{array}$$

is commutative. We recover the map of vector bundles $N^\vee_{Y/S}|_X \to N^\vee_{X/S}$ if we pull the natural map $\sigma^*a^*N^\vee_{Y/S} = f^{(1)*}\pi^*N^\vee_{Y/S} \to \varphi^*N^\vee_{X/S} = \sigma^*s^*N^\vee_{X/S}$ back to X. This proves that we get our desired commutative diagram once we pull the commutative diagram above back to X. □

4. The Bass–Quillen class as a Lie module structure map

In this section, we prove the Lie-theoretic interpretation of the Bass–Quillen class that we explained in Section 2. We begin by stating the result under the assumption that closed embeddings split to first order only. Then we provide explanations in Lie-theoretic terms. We turn to the proof last.

4.1. We have a map $\Psi\colon N_{Y/S}|_X \dashrightarrow T_S|_X \to N_{X/S}$ because Y splits to first order in S. As a consequence, the following short exact sequence splits:

$$0 \longrightarrow N_{X/Y} \longrightarrow N_{X/S} \underset{\curvearrowleft\,\Psi}{\longrightarrow} N_{Y/S}|_X \longrightarrow 0\,.$$

Most of this section will be devoted to constructing a map $\kappa\colon N_{X/Y}\otimes N_{X/S} \to N_{X/S}[1]$. This map will be defined by the extension class of an explicit short exact sequence. Using this map, we will prove the following proposition.

Proposition 4.2. *In the same setting as Theorem A, the natural map $N_{X/Y}[-1] \to N_{X/S}[-1]$ preserves the brackets whose definition can be found in Section 2.6. There exists a map $\kappa\colon N_{X/Y}\otimes$*

$N_{X/S} \to N_{X/S}[1]$ *defined explicitly by the extension class of a short exact sequence. The diagram*

$$\begin{CD}
N_{X/Y} \otimes N_{Y/S}|_X @>>> N_{Y/S}|_X[1] \\
@AAA @AAA \\
N_{X/Y} \otimes N_{X/S} @>\kappa>> N_{X/S}[1] \\
@VVV @VVV \\
N_{X/S} \otimes N_{X/S} @>>> N_{X/S}[1]
\end{CD}$$

is commutative. Here all the vertical maps are the obvious maps of vector bundles. The top horizontal map is the Bass–Quillen class $\alpha_{s,N^\vee_{Y/S}|_{X^{(1)}_Y}}$*, and the bottom horizontal map is the bracket.*

4.3. To explain in Lie-theoretic terms the meaning of Proposition 4.2, assume for simplicity that the embedding $X \hookrightarrow S$ satisfies the additional conditions that make $\mathfrak{g} = N_{X/S}[-1]$ a Lie algebra. Then set $\mathfrak{h} = N_{X/Y}[-1]$. It is a subalgebra of $\mathfrak{g}$ because the map $\mathfrak{h} \hookrightarrow \mathfrak{g}$ preserves the brackets by Proposition 4.2. The bundle $N_{Y/S}|_X[-1]$ can be identified with $\mathfrak{g}/\mathfrak{h}$. Then the diagram in Proposition 4.2 becomes

$$\begin{CD}
\mathfrak{h} \otimes \mathfrak{g}/\mathfrak{h} @>>> \mathfrak{g}/\mathfrak{h} \\
@AAA @AAA \\
\mathfrak{h} \otimes \mathfrak{g} @>\kappa>> \mathfrak{g} \\
@VVV @VVV \\
\mathfrak{g} \otimes \mathfrak{g} @>>> \mathfrak{g}\,.
\end{CD}$$

The commutativity of

$$\begin{CD}
N_{X/Y} \otimes N_{X/S} @>\kappa>> N_{X/S}[1] \\
@VVV @VVV \\
N_{X/S} \otimes N_{X/S} @>>> N_{X/S}[1]\,,
\end{CD} \qquad
\begin{CD}
\mathfrak{h} \otimes \mathfrak{g} @>\kappa>> \mathfrak{g} \\
@VVV @VVV \\
\mathfrak{g} \otimes \mathfrak{g} @>>> \mathfrak{g}
\end{CD}$$

shows that the morphism κ is the structure map of the natural $\mathfrak{h}$-module structure on $\mathfrak{g} = N_{X/S}[-1]$, where $\mathfrak{h} = N_{X/Y}[-1]$ is the Lie algebra.

The commutativity of the diagram

$$\begin{CD}
N_{X/Y} \otimes N_{Y/S}|_X @>>> N_{Y/S}|_X[1] \\
@AAA @AAA \\
N_{X/Y} \otimes N_{X/S} @>\kappa>> N_{X/S}[1]\,,
\end{CD} \qquad
\begin{CD}
\mathfrak{h} \otimes \mathfrak{g}/\mathfrak{h} @>>> \mathfrak{g}/\mathfrak{h} \\
@AAA @AAA \\
\mathfrak{h} \otimes \mathfrak{g} @>\kappa>> \mathfrak{g}
\end{CD}$$

shows that the Bass–Quillen class $\alpha_{s,N^\vee_{Y/S}|_{X^{(1)}_Y}}$ at the top of the diagram is the structure map of the $\mathfrak{h}$-module structure on $\mathfrak{g}/\mathfrak{h} = N_{Y/S}|_X[-1]$.

4.4. Before we prove Proposition 4.2, we have to define the morphism $\kappa\colon N_{X/Y} \otimes N_{X/S} \to N_{X/S}[1]$ that appears in the middle of the diagram in Proposition 4.2. We plan to define the morphism κ explicitly by the extension class of a short exact sequence. There is a technical detail we need to deal with. As we can see, the bracket is defined as $\mathsf{Sym}^2 N_{X/S} \to N_{X/S}[1]$

instead of $N_{X/S} \otimes N_{X/S} \to N_{X/S}[1]$. It is easy to find the short exact sequence corresponding to $\mathsf{Sym}^2 N_{X/S} \to N_{X/S}[1]$. However, it is hard to find explicitly what short exact sequence the morphism $N_{X/S} \otimes N_{X/S} \to N_{X/S}[1]$ corresponds to. The same phenomenon appears when we try to define $N_{X/Y} \otimes N_{X/S} \to N_{X/S}[1]$. There is an anti-symmetric part $\wedge^2 N_{X/Y} \hookrightarrow N_{X/Y} \otimes N_{X/S}$. We can only define our desired Lie module structure map κ via the extension class of a short exact sequence after we kill this anti-symmetric part. Lemma 4.5 below is how we kill the anti-symmetric part of $N_{X/Y} \otimes N_{X/S} \cong (N_{X/Y} \otimes N_{X/Y}) \oplus (N_{X/Y} \otimes N_{Y/S}|_X)$ canonically.

Lemma 4.5. *The vector bundle $I_X^2/(I_X^3 + I_Y^2)$ on X is isomorphic to*

$$\mathsf{Sym}^2 N_{X/Y}^\vee \oplus \left(N_{X/Y}^\vee \otimes N_{Y/S}^\vee|_X\right),$$

where I_X and I_Y are the ideal sheaves of X and Y in S.

Proof. The cokernel of $\mathsf{Sym}^2 N_{Y/S}^\vee|_X \hookrightarrow \mathsf{Sym}^2 N_{X/S}^\vee$ is isomorphic to

$$\mathsf{Sym}^2 N_{X/Y}^\vee \oplus \left(N_{X/Y}^\vee \otimes N_{Y/S}^\vee|_X\right)$$

using the splitting in Section 4.1.

There is a commutative diagram on X

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \dfrac{I_Y^2}{I_Y^2 I_X} & \longrightarrow & \dfrac{I_X^2}{I_X^3} & \longrightarrow & \dfrac{I_X^2}{I_X^3 + I_Y^2} & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & & & \\ 0 & \longrightarrow & \mathsf{Sym}^2 N_{Y/S}^\vee|_X & \longrightarrow & \mathsf{Sym}^2 N_{X/S}^\vee & \longrightarrow & \mathsf{Sym}^2 N_{X/Y}^\vee \oplus \left(N_{X/Y}^\vee \otimes N_{Y/S}^\vee|_X\right) & \longrightarrow & 0\,. \end{array}$$

The two vertical maps are isomorphisms, so we can complete this diagram as an isomorphism of short exact sequences. This implies our desired isomorphism. □

Definition 4.6. Define the morphism $\kappa\colon N_{X/Y} \otimes N_{X/S} \to N_{X/S}[1]$ as follows:

$$N_{X/Y} \otimes N_{X/S} \longrightarrow \mathsf{Sym}^2 N_{X/Y} \oplus (N_{X/Y} \otimes N_{Y/S}|_X) \cong \left(\frac{I_X^2}{I_X^3 + I_Y^2}\right)^\vee \longrightarrow N_{X/S}[1],$$

where the map on the left is the obvious map under the identification $N_{X/S} \cong N_{X/Y} \oplus N_{Y/S}|_X$ in Section 4.1 and the map on the right is given by the extension class of the short exact sequence

$$0 \longrightarrow \frac{I_X^2}{I_X^3 + I_Y^2} \longrightarrow \varphi_* \frac{I_X}{I_X^3 + I_Y^2} \longrightarrow \frac{I_X}{I_X^2} \longrightarrow 0\,.$$

4.7. We will focus on the proof of Proposition 4.2. The result will follow from Lemma 4.9 and Propositions 4.11 and 4.12 below.

Lemma 4.8. *The vector bundle $\mathsf{Sym}^2 N_{X/Y}^\vee$ is isomorphic to $I_X^2/(I_X^3 + I_X I_Y)$.*

Proof. The ideal sheaf of X in Y is $I_X/I_Y \subset \mathcal{O}_Y = \mathcal{O}_S/I_Y$. Note that $I_X^n/I_Y^n \neq (I_X/I_Y)^n \subset \mathcal{O}_S/I_Y$. It is easy to show that $(I_X/I_Y)^n \cong (I_X^n + I_Y)/I_Y \subset \mathcal{O}_Y = \mathcal{O}_S/I_Y$. Therefore,

$$\mathsf{Sym}^2 N_{X/Y}^\vee \cong \frac{\left(I_X/I_Y\right)^2}{\left(I_X/I_Y\right)^3} \cong \frac{I_X^2 + I_Y}{I_X^3 + I_Y} \cong \frac{I_X^2}{I_X^2 \cap \left(I_X^3 + I_Y\right)}\,.$$

We have $I_X^2 \cap \left(I_X^3 + I_Y\right) = \left(I_X^2 \cap I_X^3\right) + \left(I_X^2 \cap I_Y\right)$ because $I_X^3 \subset I_X^2$. The equality $I_X^2 \cap I_Y = I_X I_Y$ is due to the injective map

$$N_{Y/S}^\vee|_X = \frac{I_Y}{I_Y I_X} \hookrightarrow N_{X/S}^\vee = \frac{I_X}{I_X^2}\,.$$ □

Lemma 4.9. *The map of vector bundles $N_{X/Y}[-1] \to N_{X/S}[-1]$ preserves the brackets.*

Proof. One can check that the two short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \dfrac{I_X^2}{I_X^3} & \longrightarrow & \varphi_* \dfrac{I_X}{I_X^3} & \longrightarrow & \dfrac{I_X}{I_X^2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \dfrac{I_X^2}{I_X^3 + I_X I_Y} & \longrightarrow & s_* \dfrac{I_X}{I_X^3 + I_Y} & \longrightarrow & \dfrac{I_X}{I_X^2 + I_Y} & \longrightarrow & 0
\end{array}$$

are compatible. □

4.10. On the other hand, the map $N_{X/S}[-1] \to N_{Y/S}|_X[-1]$ may not preserve the brackets because there is no map $\big(\pi_*(I_Y/I_Y^3)\big)|_X \to \varphi_*\big(I_X/I_X^3\big)$ generally.

We now prove the commutativity of the two diagrams in Proposition 4.2. It is divided into two propositions below.

Proposition 4.11. *The map in Definition 4.6 is compatible with the bracket of $N_{X/S}[-1]$. This is equivalent to saying that the diagram*

$$\begin{array}{ccc}
N_{X/Y} \otimes N_{X/S} & \longrightarrow & N_{X/S}[1] \\
\downarrow & & \downarrow \\
N_{X/S} \otimes N_{X/S} & \longrightarrow & N_{X/S}[1]
\end{array}$$

is commutative.

Proof. We need to show that the three squares in the diagram

$$\begin{array}{ccccccc}
N_{X/Y} \otimes N_{X/S} & \longrightarrow & \mathsf{Sym}^2 N_{X/Y} \oplus (N_{X/Y} \otimes N_{Y/S}|_X) & \xrightarrow{\cong} & \left(\dfrac{I_X^2}{I_X^3 + I_Y^2}\right)^{\vee} & \longrightarrow & N_{X/S}[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
N_{X/S} \otimes N_{X/S} & \longrightarrow & \mathsf{Sym}^2 N_{X/S} & \xrightarrow{\cong} & \left(\dfrac{I_X^2}{I_X^3}\right)^{\vee} & \longrightarrow & N_{X/S}[1]
\end{array}$$

are commutative. Clearly, the one on the left is commutative. The commutativity of the isomorphism in the middle follows from the compatibility of the two short exact sequences in Lemma 4.5. The square on the right commutes because the two short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \dfrac{I_X^2}{I_X^3 + I_Y^2} & \longrightarrow & \varphi_* \dfrac{I_X}{I_X^3 + I_Y^2} & \longrightarrow & \dfrac{I_X}{I_X^2} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \dfrac{I_X^2}{I_X^3} & \longrightarrow & \varphi_* \dfrac{I_X}{I_X^3} & \longrightarrow & \dfrac{I_X}{I_X^2} & \longrightarrow & 0
\end{array}$$

are compatible. □

Proposition 4.12. *The diagram*

$$\begin{array}{ccc} N_{X/Y}\otimes N_{X/S} & \longrightarrow & N_{X/Y}\otimes N_{Y/S}|_X \\ \downarrow & & \downarrow \\ N_{X/S}[1] & \longrightarrow & N_{Y/S}[1] \end{array}$$

is commutative, where the left vertical map is from Definition 4.6 and the right vertical map is the Bass–Quillen class $\alpha_{s,N^\vee_{Y/S}|_{X^{(1)}_Y}}$.

Proof. We need to prove that the three squares in the diagram (4.1)

$$\begin{array}{ccc} N_{X/Y}\otimes N_{X/S} & \longrightarrow & N_{X/Y}\otimes N_{Y/S}|_X \\ \downarrow & & \downarrow \\ \mathsf{Sym}^2 N_{X/Y}\oplus (N_{X/Y}\otimes N_{Y/S}|_X) & \longrightarrow & N_{X/Y}\otimes N_{Y/S}|_X \\ \downarrow\cong & & \downarrow\cong \\ \left(\dfrac{I_X^2}{I_X^3+I_Y^2}\right)^\vee & \longrightarrow & \left(\dfrac{I_X}{I_X^2+I_Y}\right)^\vee\otimes\left(\dfrac{I_Y}{I_YI_X}\right)^\vee \\ \downarrow & & \downarrow \\ N_{X/S}[1] & \longrightarrow & N_{Y/S}[1] \end{array} \tag{4.1}$$

are commutative. Obviously, the one on the top is commutative.

To prove that the isomorphism in the middle is compatible, we construct a commutative diagram

$$\begin{array}{ccccccc} N^\vee_{X/Y}\otimes N^\vee_{Y/S}|_X & \dashrightarrow & N^\vee_{X/S}\otimes N^\vee_{X/S} & \longrightarrow & \mathsf{Sym}^2 N^\vee_{X/S} & \longrightarrow & \mathsf{Sym}^2 N^\vee_{X/Y}\oplus\left(N^\vee_{X/Y}\otimes N^\vee_{Y/S}|_X\right) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ \dfrac{I_X}{I_X^2+I_Y}\otimes\dfrac{I_Y}{I_YI_X} & \dashrightarrow & \dfrac{I_X}{I_X^2}\otimes\dfrac{I_X}{I_X^2} & \longrightarrow & \dfrac{I_X^2}{I_X^3} & \longrightarrow & \dfrac{I_X^2}{I_X^3+I_Y^2}, \end{array}$$

where the dashed arrows are defined by the splitting in Section 4.1 and the right square commutes as mentioned in the proof of Proposition 4.11. Clearly, the left and middle squares are commutative. We intend to prove that this big commutative diagram is exactly dual to the one in the middle of (4.1). It suffices to show that the map

$$\frac{I_X}{I_X^2+I_Y}\otimes\frac{I_Y}{I_YI_X}\dashrightarrow\frac{I_X}{I_X^2}\otimes\frac{I_X}{I_X^2}\longrightarrow\frac{I_X^2}{I_X^3}\longrightarrow\frac{I_X^2}{I_X^3+I_Y^2}$$

defined using the splitting is equal to the natural map

$$\frac{I_X}{I_X^2+I_Y}\otimes\frac{I_Y}{I_YI_X}\longrightarrow\frac{I_X^2}{I_X^3+I_Y^2},\quad (a\otimes b)\longmapsto ab\quad\text{for }\ a\in\frac{I_X}{I_X^2+I_Y}\text{ and }b\in\frac{I_Y}{I_YI_X}.$$

One can check this easily.

Let us focus on the commutativity of the bottom square in (4.1):

$$\begin{array}{ccc} \left(\dfrac{I_X^2}{I_X^3+I_Y^2}\right)^\vee & \longrightarrow & N_{X/S}[1] \\ \downarrow & & \downarrow \\ \left(\dfrac{I_X}{I_X^2+I_Y}\right)^\vee \otimes \left(\dfrac{I_Y}{I_Y I_X}\right)^\vee & \longrightarrow & N_{Y/S}[1]\,. \end{array}$$

The bottom horizontal map is defined by a short exact sequence

$$0 \longrightarrow N_{X/Y}^\vee \otimes N_{Y/S}^\vee|_X \longrightarrow s_* a^* N_{Y/S}^\vee \longrightarrow N_{Y/S}^\vee \longrightarrow 0\,.$$

Moreover, we have a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_{X/Y}^\vee \otimes N_{Y/S}^\vee|_X & \longrightarrow & s_* a^* N_{Y/S}^\vee & \longrightarrow & N_{Y/S}^\vee|_X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & \dfrac{I_X I_Y}{I_Y\left(I_X^2+I_Y\right)} & \longrightarrow & s_* \dfrac{I_Y}{I_Y\left(I_X^2+I_Y\right)} & \longrightarrow & \dfrac{I_Y}{I_Y I_X} & \longrightarrow & 0\,. \end{array}$$

This implies that $N_{X/Y}^\vee \otimes N_{Y/S}^\vee \cong I_X I_Y / I_Y(I_X^2+I_Y)$. It suffices to show that the short exact sequence

$$0 \longrightarrow \frac{I_X I_Y}{I_Y(I_X^2+I_Y)} \longrightarrow s_* \frac{I_Y}{I_Y(I_X^2+I_Y)} \longrightarrow \frac{I_Y}{I_Y I_X} \longrightarrow 0$$

is compatible with the short exact sequence

$$0 \longrightarrow \frac{I_X^2}{I_X^3+I_Y^2} \longrightarrow \varphi_* \frac{I_X}{I_X^3+I_Y^2} \longrightarrow \frac{I_X}{I_X^2} \longrightarrow 0\,.$$

There is a natural map of sheaves $g_*(I_Y/I_Y(I_X^2+I_Y)) \to I_X/(I_X^3+I_Y^2)$. We get the map $s_*(I_Y/I_Y(I_X^2+I_Y)) \to \varphi_*(I_X/(I_X^3+I_Y^2))$ by applying φ_* on both sides, so the two short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \dfrac{I_X I_Y}{I_Y\left(I_X^2+I_Y\right)} & \longrightarrow & s_* \dfrac{I_Y}{I_Y\left(I_X^2+I_Y\right)} & \longrightarrow & \dfrac{I_Y}{I_Y I_X} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \dfrac{I_X^2}{I_X^3+I_Y^2} & \longrightarrow & \varphi_* \dfrac{I_X}{I_X^3+I_Y^2} & \longrightarrow & \dfrac{I_X}{I_X^2} & \longrightarrow & 0 \end{array}$$

are compatible. □

5. Theorem B in classical Lie theory

As a warm-up to proving Theorem B, we present here an analogous result in Lie theory. We give a proof of this result using techniques that can be adapted to the derived setting of Theorem B.

5.1. Consider an injective map of Lie algebras in vector spaces $\alpha\colon \mathfrak{h} \hookrightarrow \mathfrak{g}$. There is a short exact sequence of $\mathfrak{h}$-modules

$$0 \longrightarrow \mathfrak{h} \xrightarrow{\alpha} \mathfrak{g} \xrightarrow{\beta} \mathfrak{n} = \mathfrak{g}/\mathfrak{h} \longrightarrow 0\,.$$

Given a vector-space map $\gamma\colon \mathfrak{n} \dashrightarrow \mathfrak{g}$ splitting β, we define a bracket on $\mathfrak{n}$ by the formula $[x,y]_{\mathfrak{n}} = \beta([\gamma(x),\gamma(y)]_{\mathfrak{g}})$ for any $x,y \in \mathfrak{n}$. In general, the bracket on $\mathfrak{n}$ may not be a Lie bracket, and the map β may not respect the brackets.

We define a map $\mathfrak{g}\otimes\mathfrak{n} \to \mathfrak{n}$ by $\sum_i x_i \otimes y_i \mapsto \sum_i \beta([x_i,\gamma(y_i)])$ for $x_i \in \mathfrak{g}$ and $y_i \in \mathfrak{n}$. This map may not define a $\mathfrak{g}$-module structure on $\mathfrak{n}$ if β is not a morphism of Lie algebras. We state a proposition which is important in Section 6. Its proof is left to the reader.

PROPOSITION 5.2. *In general, one can check that the diagram*

$$\begin{array}{ccc} \mathfrak{g}\otimes\mathfrak{g} & \longrightarrow & \mathfrak{g} \\ \downarrow{\scriptstyle \mathrm{id}\otimes\beta} & & \downarrow{\scriptstyle \beta} \\ \mathfrak{g}\otimes\mathfrak{n} & \longrightarrow & \mathfrak{n} \end{array}$$

is not commutative. However, the diagram

$$\begin{array}{ccc} \wedge^2\mathfrak{g} = \wedge^2\mathfrak{h}\oplus\wedge^2\mathfrak{n}\oplus(\mathfrak{h}\otimes\mathfrak{n}) & \longrightarrow & \mathfrak{g} \\ \downarrow & & \downarrow{\scriptstyle \beta} \\ (\mathfrak{h}\otimes\mathfrak{n})\oplus\wedge^2\mathfrak{n} & \longrightarrow & \mathfrak{n} \end{array}$$

is commutative if we identify $\mathfrak{g}$ with $\mathfrak{h}\oplus\mathfrak{n}$ as a direct sum of vector spaces via γ.

Therefore, the right square of the diagram

$$\begin{array}{ccccc} \mathfrak{g}\otimes\mathfrak{g} & \longrightarrow & \wedge^2\mathfrak{g} = \wedge^2\mathfrak{h}\oplus\wedge^2\mathfrak{n}\oplus(\mathfrak{h}\otimes\mathfrak{n}) & \longrightarrow & \mathfrak{g} \\ \downarrow{\scriptstyle \mathrm{id}\otimes\beta} & & \downarrow & & \downarrow{\scriptstyle \beta} \\ \mathfrak{g}\otimes\mathfrak{n} & \longrightarrow & (\mathfrak{h}\otimes\mathfrak{n})\oplus\wedge^2\mathfrak{n} & \longrightarrow & \mathfrak{n} \end{array}$$

is commutative, but the square on the left is not.

Here is the theorem analogous to Theorem B.

THEOREM 5.3. *The map β preserves the brackets of $\mathfrak{g}$ and $\mathfrak{n}$ if and only if $\mathfrak{h}$ acts trivially on $\mathfrak{n}$. In this case, the bracket on $\mathfrak{n}$ is a Lie bracket, and β is a morphism of Lie algebras.*

Proof. It is easy to see that the map $\mathfrak{g}\otimes\mathfrak{n}\to\mathfrak{n}$ has the following properties:

(I) It is compatible with the $\mathfrak{h}$-module structure on $\mathfrak{n}$. Equivalently, the diagram

$$\begin{array}{ccc} \mathfrak{h}\otimes\mathfrak{n} & \longrightarrow & \mathfrak{n} \\ \downarrow{\scriptstyle \alpha\otimes\mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathfrak{g}\otimes\mathfrak{n} & \longrightarrow & \mathfrak{n} \end{array}$$

is commutative. This follows from the observation that the $\mathfrak{h}$-module structure on $\mathfrak{n}$ can be defined as $x\otimes y\mapsto\beta([\alpha(x),\gamma(y)])$ for any $x\in\mathfrak{h}$ and $y\in\mathfrak{n}$.

(II) It defines a $\mathfrak{g}$-module structure on $\mathfrak{n}$ if β is a morphism of Lie algebras. Equivalently, the diagram

$$\begin{array}{ccc} \mathfrak{g}\otimes\mathfrak{n} & \longrightarrow & \mathfrak{n} \\ \downarrow{\scriptstyle \beta\otimes\mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathfrak{n}\otimes\mathfrak{n} & \longrightarrow & \mathfrak{n} \end{array}$$

is commutative if β is a morphism of Lie algebras.

(III) The diagram

$$\begin{array}{ccccc}
\mathfrak{n}\otimes\mathfrak{n} & \longrightarrow & \wedge^2\mathfrak{n} & \longrightarrow & \mathfrak{n} \\
{\scriptstyle \gamma\otimes\mathrm{id}}\downarrow & & \downarrow & & \uparrow \\
\mathfrak{g}\otimes\mathfrak{n} & \longrightarrow & (\mathfrak{h}\otimes\mathfrak{n})\oplus\wedge^2\mathfrak{n} & \longrightarrow & \mathfrak{n}
\end{array}$$

is commutative.

We first prove that $\mathfrak{h}$ acts trivially on $\mathfrak{n}$ if β preserves the brackets. Compose the two commutative diagrams in (I) and (II):

$$\begin{array}{ccc}
\mathfrak{h}\otimes\mathfrak{n} & \longrightarrow & \mathfrak{n} \\
{\scriptstyle \alpha\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathfrak{g}\otimes\mathfrak{n} & \longrightarrow & \mathfrak{n} \\
{\scriptstyle \beta\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathfrak{n}\otimes\mathfrak{n} & \longrightarrow & \mathfrak{n}\,.
\end{array}$$

The map $\mathfrak{h}\otimes\mathfrak{n}\to\mathfrak{n}$ is zero because $\beta\circ\alpha=0$.

Let us turn to proving that β preserves the brackets if the Lie module structure map $\mathfrak{h}\otimes\mathfrak{n}\to\mathfrak{n}$ is zero. Identify $\mathfrak{g}$ with $\mathfrak{h}\oplus\mathfrak{n}$ via γ. With properties (I) and (III), one can conclude that the map $\mathfrak{g}\otimes\mathfrak{n}=(\mathfrak{h}\otimes\mathfrak{n})\oplus(\mathfrak{n}\otimes\mathfrak{n})\to\mathfrak{n}$ that we defined at the beginning of this section is the Lie module structure map $\mathfrak{h}\otimes\mathfrak{n}\to\mathfrak{n}$ plus the bracket on $\mathfrak{n}$: $\mathfrak{n}\otimes\mathfrak{n}\to\mathfrak{n}$.

If $\mathfrak{h}\otimes\mathfrak{n}\to\mathfrak{n}$ is zero, then the diagram

$$\begin{array}{ccccc}
\mathfrak{g}\otimes\mathfrak{n} & \longrightarrow & (\mathfrak{h}\otimes\mathfrak{n})\oplus\wedge^2\mathfrak{n} & \longrightarrow & \mathfrak{n} \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{n}\otimes\mathfrak{n} & \longrightarrow & \wedge^2\mathfrak{n} & \longrightarrow & \mathfrak{n}
\end{array}$$

is commutative. Put the commutative diagram in Proposition 5.2 and the diagram above together:

$$\begin{array}{ccc}
\wedge^2\mathfrak{g} & \longrightarrow & \mathfrak{g} \\
\downarrow & & \downarrow \\
(\mathfrak{h}\otimes\mathfrak{n})\oplus\wedge^2\mathfrak{n} & \longrightarrow & \mathfrak{n} \\
\downarrow & & \downarrow \\
\wedge^2\mathfrak{n} & \longrightarrow & \mathfrak{n}\,.
\end{array}$$

We conclude that β is a map of Lie algebras. □

6. The proof of Theorem B

We generalize the proof given in Section 5 to prove Theorem B. We first define a morphism $N_{X/S}\otimes N_{Y/S}|_X\to N_{Y/S}|_X[1]$ that is analogous to the map $\mathfrak{g}\otimes\mathfrak{n}\to\mathfrak{n}$ in Section 5. Then we prove statements similar to properties (I) and (II), and Proposition 5.2 in Section 5. We prove Theorem B last.

6.1. The short exact sequence

$$0\longrightarrow N_{X/Y}\longrightarrow N_{X/S}\longrightarrow N_{Y/S}|_X\longrightarrow 0$$

shifted by -1 is analogous to the short exact sequence in (2.11). We need to define a map $N_{X/S} \otimes N_{Y/S}|_X \to N_{Y/S}|_X[1]$ that is the analog of the map $\mathfrak{g} \otimes \mathfrak{n} \to \mathfrak{n}$ from Section 5. Before we define this map, we need to deal with the same technical issue that appears in Section 4. Using the splitting in Section 4.1, we see that there is an anti-symmetric part $\wedge^2 N_{Y/S}|_X$ in $N_{X/S} \otimes N_{Y/S}|_X = (N_{X/Y} \oplus N_{Y/S}|_X) \otimes N_{Y/S}|_X$. We need to kill this anti-symmetric part canonically. Lemmas 6.2 and 6.3 show how we do this.

Lemma 6.2. *We have* $I_Y^2 \cap I_X^2 I_Y = I_Y^2 I_X$.

Proof. We have $I_Y^2 \cap I_X^3 = I_Y^2 I_X$ because the map

$$\mathsf{Sym}^2 N_{Y/S}^\vee|_X = \frac{I_Y^2}{I_Y^2 I_X} \longrightarrow \mathsf{Sym}^2 N_{X/S}^\vee = \frac{I_X^2}{I_X^3}$$

is injective. Then we have $I_Y^2 I_X \subset I_Y^2 \cap I_X^2 I_Y \subset I_Y^2 \cap I_X^3 = I_Y^2 I_X$. □

Lemma 6.3. *There is an isomorphism of vector bundles on* X

$$\left(\frac{I_X I_Y}{I_X^2 I_Y}\right)^\vee \cong \mathsf{Sym}^2 N_{Y/S}|_X \oplus (N_{X/Y} \otimes N_{Y/S}|_X)\,.$$

Proof. There is a morphism of short exact sequences

$$\begin{array}{ccccccccc}
 & & & & \overset{\bar\tau}{\longleftarrow\!-\!-} & & & & \\
0 & \longrightarrow & \frac{I_Y}{I_Y I_X} \otimes \frac{I_Y}{I_Y I_X} & \longrightarrow & \frac{I_X}{I_X^2} \otimes \frac{I_Y}{I_Y I_X} & \longrightarrow & \frac{I_X}{I_X^2 + I_Y} \otimes \frac{I_Y}{I_Y I_X} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow v & & \downarrow u & & \\
0 & \longrightarrow & \frac{I_Y^2}{I_Y^2 I_X} & \longrightarrow & \frac{I_X I_Y}{I_X^2 I_Y} & \longrightarrow & \frac{I_X I_Y}{I_Y\left(I_X^2 + I_Y\right)} & \longrightarrow & 0\,.
\end{array}$$

Everything above is clear except for the injectivity of $I_Y^2/I_Y^2 I_X \to I_X I_Y/I_X^2 I_Y$. This is due to Lemma 6.2.

The short exact sequence on the top is the dual of the sequence of the normal bundles tensored with $N_{Y/S}^\vee|_X$, so it splits naturally. The map u is an isomorphism, as mentioned in the proof of Proposition 4.12. One can construct a splitting $v \circ \tau \circ u^{-1}$ for the short exact sequence on the bottom. Therefore,

$$\frac{I_X I_Y}{I_X^2 I_Y} \cong \frac{I_Y^2}{I_Y^2 I_X} \oplus \frac{I_X I_Y}{I_Y(I_X^2 + I_Y)} \cong \mathsf{Sym}^2 N_{Y/S}^\vee|_X \oplus \left(N_{X/Y}^\vee \otimes N_{Y/S}^\vee|_X\right).$$

The diagram of two short exact sequences above says that there is a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & \overset{\bar\tau}{\longleftarrow\!-\!-} & & & & \\
0 & \longrightarrow & N_{Y/S}^\vee|_X \otimes N_{Y/S}^\vee|_X & \longrightarrow & N_{X/S}^\vee \otimes N_{Y/S}^\vee|_X & \longrightarrow & N_{X/Y}^\vee \otimes N_{Y/S}^\vee|_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow v & & \downarrow \mathrm{id} & & \\
0 & \longrightarrow & \mathsf{Sym}^2 N_{Y/S}^\vee|_X & \longrightarrow & \mathsf{Sym}^2 N_{Y/S}^\vee|_X \oplus \left(N_{Y/S}^\vee|_X \otimes N_{X/Y}^\vee\right) & \longrightarrow & N_{X/Y}^\vee \otimes N_{Y/S}^\vee|_X & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & \frac{I_Y^2}{I_Y^2 I_X} & \longrightarrow & \frac{I_X I_Y}{I_X^2 I_Y} & \longrightarrow & \frac{I_X I_Y}{\left(I_X^2 + I_Y\right) I_Y} & \longrightarrow & 0\,.
\end{array}$$

□

Definition 6.4. Define the morphism $N_{X/S} \otimes N_{Y/S}|_X \to N_{Y/S}[1]$ as follows:

$$N_{X/S} \otimes N_{Y/S}|_X \longrightarrow \mathsf{Sym}^2(N_{Y/S}|_X) \oplus (N_{X/Y} \otimes N_{Y/S}|_X) \cong \left(\frac{I_X I_Y}{I_X^2 I_Y}\right)^\vee \longrightarrow N_{Y/S}|_X[1]\,,$$

where the first map is the obvious one under the identification $N_{X/S} \cong N_{X/Y} \oplus N_{Y/S}|_X$ and the second map is given by the extension class of the short exact sequence

$$0 \longrightarrow \frac{I_X I_Y}{I_X^2 I_Y} \longrightarrow \varphi_* \frac{I_Y}{I_Y I_X^2} \longrightarrow \frac{I_Y}{I_Y I_X} \longrightarrow 0\,.$$

The following proposition is analogous to property (I) in Section 5.

Proposition 6.5. *There is a commutative diagram*

$$\begin{array}{ccc} N_{X/S} \otimes N_{Y/S}|_X & \longrightarrow & N_{Y/S}|_X[1] \\ \uparrow & & \uparrow \\ N_{X/Y} \otimes N_{Y/S}|_X & \longrightarrow & N_{Y/S}|_X[1]\,, \end{array}$$

where the horizontal map at the top is from Definition 6.4, and the horizontal map at the bottom is the Bass–Quillen class $\alpha_{s,N^\vee_{Y/S}|_{X^{(1)}_Y}}$.

Proof. The isomorphism $N^\vee_{X/Y} \otimes N^\vee_{Y/S}|_X \cong I_X I_Y/(I_X^2 + I_Y)I_Y$ is mentioned in the proof of Proposition 4.12. It suffices to prove that the three squares in

$$\begin{array}{ccccccc} N_{X/S} \otimes N_{Y/S}|_X & \longrightarrow & \mathsf{Sym}^2 N_{Y/S}|_X \oplus (N_{X/Y} \otimes N_{Y/S}|_X) & \xrightarrow{\cong} & \left(\frac{I_X I_Y}{I_X^2 I_Y}\right)^\vee & \longrightarrow & N_{Y/S}|_X[1] \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ N_{X/Y} \otimes N_{Y/S}|_X & \longrightarrow & N_{X/Y} \otimes N_{Y/S}|_X & \xrightarrow{\cong} & \left(\frac{I_X I_Y}{(I_X^2 + I_Y)I_Y}\right)^\vee & \longrightarrow & N_{Y/S}|_X[1] \end{array}$$

are commutative. Obviously, the left one is commutative. The commutativity of the one in the middle is due to the compatibility of the short exact sequence in Lemma 6.3. The rest of our proof is devoted to the commutativity of the square on the right.

There is a natural map $I_Y/I_Y I_X^2 \to g_*(I_Y/(I_X^2 + I_Y)I_Y)$. We get a map $\varphi_*(I_Y/I_Y I_X^2) \to s_*(I_Y/(I_X^2 + I_Y)I_Y)$ by applying φ_* on both sides. This gives the two compatible short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{I_X I_Y}{I_X^2 I_Y} & \longrightarrow & \varphi_* \frac{I_Y}{I_Y I_X^2} & \longrightarrow & \frac{I_Y}{I_Y I_X} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \frac{I_X I_Y}{(I_X^2 + I_Y)I_Y} & \longrightarrow & s_* \frac{I_Y}{(I_X^2 + I_Y)I_Y} & \longrightarrow & \frac{I_Y}{I_Y I_X} & \longrightarrow & 0\,, \end{array}$$

which prove that the diagram on the right in the proposition is commutative. □

The following proposition is similar to Proposition 5.2.

PROPOSITION 6.6. *There is a commutative diagram*

$$\begin{array}{ccc}
\mathsf{Sym}^2 N_{X/S} & \longrightarrow & N_{X/S}[1] \\
\downarrow & & \downarrow \\
\mathsf{Sym}^2 N_{Y/S}|_X \oplus (N_{X/Y} \otimes N_{Y/S}|_X) \cong \left(\dfrac{I_X I_Y}{I_X^2 I_Y}\right)^{\vee} & \longrightarrow & N_{Y/S}|_X[1]\,,
\end{array}$$

where the bottom horizontal map is from Definition 6.4.

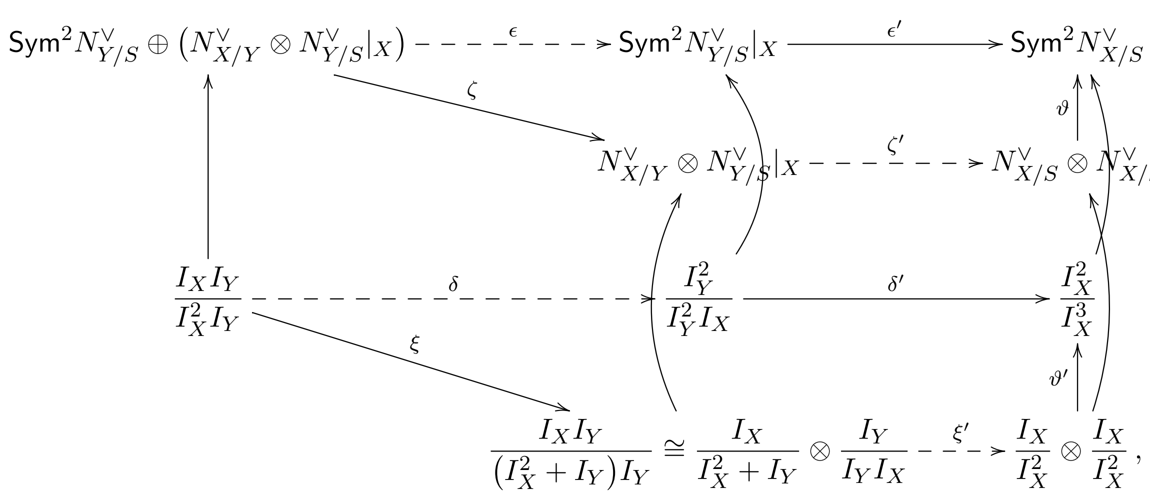
Proof. We need to prove that the two squares in diagram (6.1)

$$\begin{array}{ccccc}
\mathsf{Sym}^2 N_{X/S} & \xrightarrow{\cong} & \left(\dfrac{I_X^2}{I_X^3}\right)^{\vee} & \longrightarrow & N_{X/S}[1] \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{Sym}^2 N_{Y/S}|_X \oplus (N_{X/Y} \otimes N_{Y/S}|_X) & \xrightarrow{\cong} & \left(\dfrac{I_X I_Y}{I_X^2 I_Y}\right)^{\vee} & \longrightarrow & N_{Y/S}|_X[1]
\end{array} \tag{6.1}$$

are commutative. The square on the right of (6.1) commutes because the two short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \dfrac{I_X I_Y}{I_X^2 I_Y} & \longrightarrow & \varphi_* \dfrac{I_Y}{I_Y I_X^2} & \longrightarrow & \dfrac{I_Y}{I_Y I_X} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \dfrac{I_X^2}{I_X^3} & \longrightarrow & \varphi_* \dfrac{I_X}{I_X^3} & \longrightarrow & \dfrac{I_X}{I_X^2} & \longrightarrow & 0
\end{array}$$

are compatible. Let us prove that the left diagram in (6.1) commutes. We construct a diagram



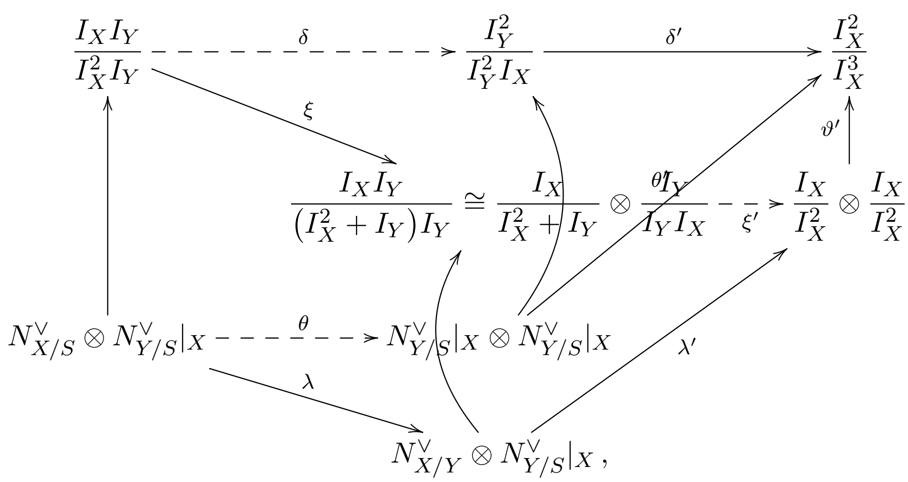
where all the vertical maps are natural isomorphisms. The dashed arrows are constructed by splittings in the proof of Lemma 6.3, and the solid arrows are the obvious ones that also appear in the proof of Lemma 6.3. The two short exact sequences and their splittings in the proof of Lemma 6.3 are compatible, so the diagram above is commutative. Taking the direct sum and direct product of the maps above, we get a commutative diagram

$$\begin{array}{ccccc}
\mathsf{Sym}^2 N_{Y/S}^\vee \oplus \left(N_{X/Y}^\vee \otimes N_{Y/S}^\vee|_X\right) & \xrightarrow{\epsilon\times\zeta} & \mathsf{Sym}^2 N_{Y/S}^\vee \oplus \left(N_{X/Y}^\vee \otimes N_{Y/S}^\vee|_X\right) & \xrightarrow{\epsilon'\oplus(\vartheta\circ\zeta')} & \mathsf{Sym}^2 N_{X/S}^\vee \\
\uparrow\cong & & \uparrow\cong & & \uparrow\cong \\
\dfrac{I_X I_Y}{I_X^2 I_Y} & \xrightarrow{\xi\times\delta} & \dfrac{I_Y^2}{I_Y^2 I_X} \oplus \dfrac{I_X I_Y}{I_X^2 + I_Y} & \xrightarrow{(\vartheta'\circ\xi')\oplus\delta'} & \dfrac{I_X^2}{I_X^3}\,.
\end{array}$$

We intend to prove that the diagram above is dual to the left square in (6.1). This says that we need to prove the following statement.

Claim. *The map* $((\vartheta' \circ \xi') \oplus \delta') \circ (\xi \times \delta)$ *we constructed above is equal to the natural map* $I_X I_Y / I_X^2 I_Y \to I_X^2 / I_X^3$.

Consider the following diagram:



where the dashed arrows are the splittings in the proof of Lemma 6.3. The diagram above is commutative because the two short exact sequences and their splittings in the proof of Lemma 6.3 are compatible.

Notice that $N_{X/S}^\vee \otimes N_{Y/S}^\vee|_X \to I_X I_Y / I_X^2 I_Y$ is surjective. To prove the claim, it suffices to show that the map

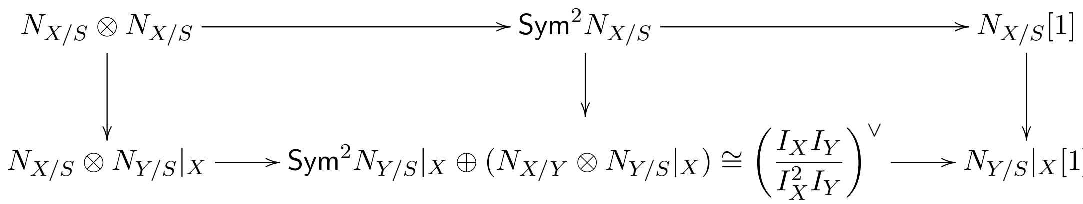
$$(\theta' \oplus (\vartheta' \circ \lambda')) \circ (\theta \times \lambda) \colon N_{X/S}^\vee \otimes N_{Y/S}^\vee|_X = \frac{I_X}{I_X^2} \otimes \frac{I_Y}{I_Y I_X} \longrightarrow \mathsf{Sym}^2 N_{X/S}^\vee = \frac{I_X^2}{I_X^3}$$

constructed via the splittings is equal to the natural map

$$\frac{I_X}{I_X^2} \otimes \frac{I_Y}{I_Y I_X} \longrightarrow \frac{I_X^2}{I_X^3}\,, \quad (a \otimes b) \longmapsto ab \quad \text{for } a \in \frac{I_X}{I_X^2} \text{ and } b \in \frac{I_Y}{I_Y I_X}\,.$$

One can verify this easily. ∎

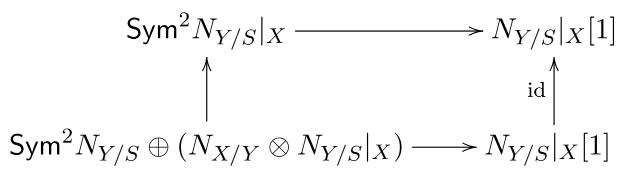
Remark 6.7. However, the big diagram



is generally not commutative. One can check that the left square is not commutative.

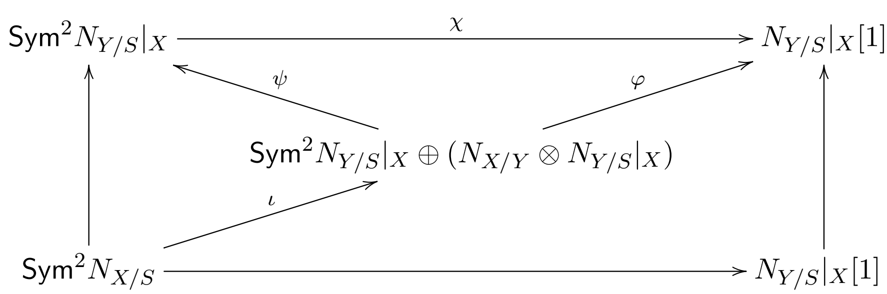
The following lemma is analogous to property (II) in Section 5.

Lemma 6.8. *If $N_{X/S}[-1] \to N_{Y/S}|_X[-1]$ preserves the brackets, then there is a commutative diagram*



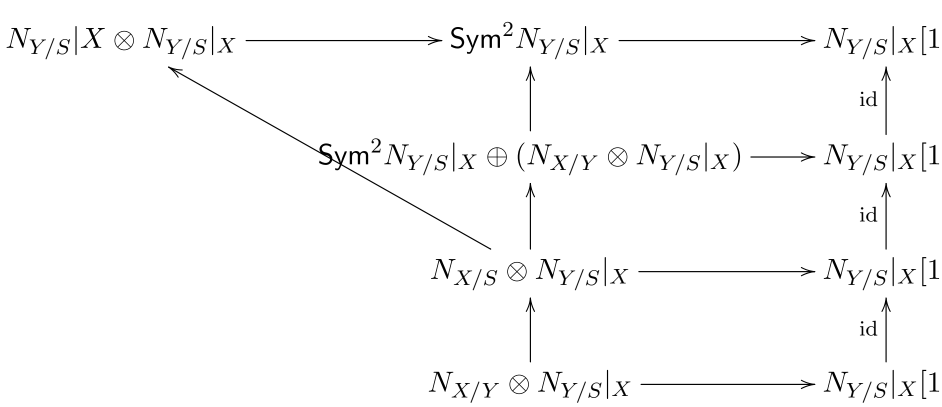
where the bottom horizontal map is from Definition 6.4.

Proof. Put what we want to prove into a larger diagram:



The outer square commutes because we assume that the brackets are preserved. We want to show that $\chi \circ \psi = \varphi$. The commutativity of the outer square and Proposition 6.6 show that $\chi \circ \psi \circ \iota = \varphi \circ \iota$. The map ι splits naturally, so we have our desired result. □

Proof of Theorem B. We first prove that the Bass–Quillen Lie module map $\alpha_{s,N^\vee_{Y/S}|_{X^{(1)}_Y}}$ is zero if the brackets are preserved. There is a commutative diagram



due to Lemma 6.8 and Proposition 6.5. Since composition $N_{X/Y} \to N_{X/S} \to N_{Y/S}|_X$ is zero, the Bass–Quillen class $\alpha_{s,N^\vee_{Y/S}|_{X^{(1)}_Y}}$ that appears at the bottom of the diagram above vanishes.

We prove that the vector bundle map $N_{X/S}[-1] \to N_{Y/S}|_X[-1]$ preserves the brackets if the Bass–Quillen class $\alpha_{s,N^\vee_{Y/S}|_{X^{(1)}_Y}}$ is zero in what follows. In [CG19], Calaque and Grivaux showed that the brackets on $N_{X/S}[-1]$ and $N_{Y/S}|_X[-1]$ defined by the extension classes of short exact sequences can be also defined as follows:

$$\mathsf{Sym}^2 N_{X/S} \dashrightarrow \mathsf{Sym}^2 T_S|_X \longrightarrow T_S|_X[1] \longrightarrow N_{X/S}[1]\,,$$

$$\mathsf{Sym}^2 N_{Y/S}|_X \dashrightarrow \mathsf{Sym}^2 T_S|_X \longrightarrow T_S|_X[1] \longrightarrow N_{Y/S}|_X[1]\,.$$

The dashed arrow is due the fact that $f\colon X \hookrightarrow S$ and $j\colon Y \hookrightarrow S$ split to first order. The map in the middle is the Atiyah class.

Using the compatibility condition on splittings of tangent bundles and the fact above, we conclude that the diagram

$$\begin{array}{ccccccc}
\mathsf{Sym}^2 N_{Y/S}|_X & \dashrightarrow & \mathsf{Sym}^2 T_S|_X & \longrightarrow & T_S|_X[1] & \longrightarrow & N_{Y/S}|_X[1] \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \uparrow \\
\mathsf{Sym}^2 N_{X/S} & \dashrightarrow & \mathsf{Sym}^2 T_S|_X & \longrightarrow & T_S|_X[1] & \longrightarrow & N_{X/S}[1]
\end{array}$$

is commutative. The diagram above and Proposition 6.6 show that we have a commutative diagram

$$\begin{array}{ccc}
\mathsf{Sym}^2 N_{Y/S}|_X & \longrightarrow & N_{Y/S}|_X[1] \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathsf{Sym}^2 N_{Y/S}|_X \oplus (N_{Y/S}|_X \otimes N_{X/Y}) & \longrightarrow & N_{Y/S}|_X[1]\,,
\end{array}$$

which is analogous to property (III) in Section 5. Based on the diagram above and Proposition 6.5, it is clear that the diagram

$$\begin{array}{ccc}
\mathsf{Sym}^2 N_{Y/S}|_X \oplus (N_{Y/S}|_X \otimes N_{X/Y}) & \longrightarrow & N_{Y/S}|_X[1] \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathsf{Sym}^2 N_{Y/S}|_X & \longrightarrow & N_{Y/S}|_X[1]
\end{array}$$

is commutative if the Bass–Quillen class $\alpha_{s,N^\vee_{Y/S}|_{X^{(1)}_Y}}$ is zero. Compose the diagram above with the one in Proposition 6.6. We get a commutative diagram

$$\begin{array}{ccc}
\mathsf{Sym}^2 N_{X/S} & \longrightarrow & N_{X/S}[1] \\
\downarrow & & \downarrow \\
\mathsf{Sym}^2 N_{Y/S}|_X \oplus (N_{Y/S}|_X \otimes N_{X/Y}) & \longrightarrow & N_{Y/S}|_X[1] \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathsf{Sym}^2 N_{Y/S}|_X & \longrightarrow & N_{Y/S}|_X[1]\,,
\end{array}$$

so we conclude that the brackets are preserved if the Bass–Quillen class $\alpha_{s,N^\vee_{Y/S}|_{X^{(1)}_Y}}$ is zero. □

6.9. We end this section by providing an example where the Bass–Quillen class is not zero. Consider the embeddings

$$X \underset{\overset{p_1}{\dashleftarrow}}{\xrightarrow{\Delta_X}} X \times X = Y \underset{\overset{\pi_1}{\dashleftarrow}}{\xrightarrow{\Delta_{X\times X}}} S = X \times X \times X \times X$$

for a smooth scheme X, where all the inclusions are the diagonal embeddings and all the splittings are the projections to the first factor. The normal bundle $N_{Y/S}$ is $p_1^* T_X \oplus p_2^* T_X$ in this case, where p_1 and p_2 are the two projections $X \times X \to X$. It is clear that the Bass–Quillen class $\alpha_{p_1, N^\vee_{Y/S}|_{X^{(1)}_Y}} : T_X \otimes (T_X \oplus T_X) \to T_X[1]$ is zero plus the Atiyah class: $T_X[-1] \otimes T_X[-1] \to T_X[-1]$. It is not zero in general.

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Shengyuan Huang s.huang.5@bham.ac.uk
School of Mathematics, University of Birmingham, Edgbaston, Birmingham, B15 2TT, UK