



On K-moduli of quartic threefolds

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ABSTRACT

The family of smooth Fano 3-folds with Picard rank 1 and anticanonical volume 4 consists of quartic 3-folds and of double covers of the 3-dimensional quadric branched along an octic surface. They can all be parametrised as complete intersections of a quadric and a quartic in the weighted projective space $\mathbb{P}(1, 1, 1, 1, 1, 2)$, denoted by $X_{2,4} \subset \mathbb{P}(1^5, 2)$; all such smooth complete intersections are K-stable. With the aim of investigating the compactification of the moduli space of quartic 3-folds given by K-stability, we exhibit three phenomena: (i) there exist K-polystable complete intersection Fano 3-folds $X_{2,2,4} \subset \mathbb{P}(1^5, 2^2)$ which deform to quartic 3-folds and are neither quartic 3-folds nor double covers of quadric 3-folds – in other words, the closure of the locus parametrising complete intersections $X_{2,4} \subset \mathbb{P}(1^5, 2)$ in the K-moduli contains elements that are not of this type; (ii) any quasi-smooth $X_{2,2,4} \subset \mathbb{P}(1^5, 2^2)$ is K-polystable; (iii) the closure in the K-moduli space of the locus parametrising complete intersections $X_{2,2,4} \subset \mathbb{P}(1^5, 2^2)$ which are not complete intersections $X_{2,4} \subset \mathbb{P}(1^5, 2)$ contains only points which correspond to complete intersections $X_{2,2,4} \subset \mathbb{P}(1^5, 2^2)$.

1. Introduction

A *Fano variety* is a normal projective variety over \mathbb{C} for which the anticanonical divisor is \mathbb{Q} -Cartier and ample. There has been spectacular recent progress on constructing moduli spaces of Fano varieties using K-stability [ABHX20, Xu20, BLX22, Jia20, BX19, XZ20, BHLX21, LXZ22, LWX21]. For each positive integer n and for every positive rational number v , there exists an Artin stack $\mathcal{M}_{n,v}^{\text{Kss}}$, called the *K-moduli stack*, which is of finite type over \mathbb{C} and which parametrises K-semistable n -dimensional Fano varieties with anticanonical volume v . Moreover, this stack admits a good moduli space $M_{n,v}^{\text{Kps}}$, called the *K-moduli space*, which is projective over \mathbb{C} . The closed points of $M_{n,v}^{\text{Kps}}$ are in one-to-one correspondence with K-polystable n -dimensional Fano varieties with anticanonical volume v . We refer to [Xu21] for a survey on these topics.

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The next natural step is to investigate the geometry of K-moduli spaces, providing explicit descriptions when possible. K-moduli of smooth(able) 2-dimensional Fano varieties have been studied by Mabuchi–Mukai [MM93] and by Odaka–Spotti–Sun [OSS16]. More generally, one of the most well-known classes of Fano varieties consists of hypersurfaces of degree d in n -dimensional projective space \mathbb{P}^n , with $d \leq n$; all smooth Fano hypersurfaces are conjectured to be K-polystable [Xu21, Part 3]. Substantial progress has made towards clarifying this conjecture [AZ23, Theorem B]; however, very little is currently known about their K-moduli. One readily available compact moduli space for Fano hypersurfaces is the geometric invariant theory (GIT) moduli space, and Liu–Xu [LX19] showed that, for cubic 3-folds, the K-moduli space coincides with the GIT moduli space. An analogous result for cubic 4-folds was recently proved by Liu [Liu22]. There are additional results in dimension 2 and in other settings [ADL24, ADL23a, ADL23b, LP22]. A particular observation is the following conjecture, which is verified up to dimension 4.

CONJECTURE. *K-polystable limits of cubic hypersurfaces are cubic hypersurfaces.*

The simplest case in higher dimensions where the expected agreement between GIT moduli and K-moduli fails, or equivalently the conjecture above is invalid, is the moduli space of quartic 3-folds. Here it is known that their K-moduli space contains elements that are not in the GIT moduli.

1.1 K-moduli of quartic 3-folds

Let \mathcal{V}_4 denote the family of *smooth* Fano 3-folds with Picard rank 1 and anticanonical volume 4. Members of \mathcal{V}_4 are either:

- (i) smooth quartic hypersurfaces in \mathbb{P}^4 , simply called *smooth quartic 3-folds*; or
- (ii) double covers of the smooth quadric 3-fold with branch divisor of degree 8, often called *hyperelliptic*.

By [Che01, Fuj19a, Der16] each smooth Fano 3-fold in the family \mathcal{V}_4 is K-stable. It is not known how to characterise singular quartic 3-folds which are K-(poly/semi)stable. Since not every member of \mathcal{V}_4 is a quartic 3-fold, the connected component of the K-moduli space $M_{3,4}^{\text{Kps}}$ containing the members of \mathcal{V}_4 does not coincide with the GIT moduli space of quartic 3-folds. However, every smooth member of \mathcal{V}_4 is a complete intersection of type $(2, 4)$ in $\mathbb{P}(1^5, 2)$; this can be seen as follows (see also [Mor75, Example 4.3], the introduction of [OS20], and [ACC⁺23, Example 3.5.2]). Let x_0, \dots, x_4, y be the homogeneous coordinates of $\mathbb{P}(1^5, 2)$ with degrees $1, \dots, 1, 2$, respectively. A $(2, 4)$ -complete intersection X is given by the vanishing of

$$f_4(x_0, \dots, x_4) + f_2(x_0, \dots, x_4)y + ay^2 \quad \text{and} \quad g_2(x_0, \dots, x_4) + by,$$

where f_2 and g_2 are quadrics, f_4 is a quartic, and $a, b \in \mathbb{C}$. If X is smooth, then $a \neq 0$ or $b \neq 0$. If $b \neq 0$, from the second equation we can express y in terms of the x_i ; therefore, y is a redundant variable. This implies that X is a quartic hypersurface in \mathbb{P}^4 . If $b = 0$, then $a \neq 0$, so up to scaling, $a = 1$. By completing the square (that is, applying the automorphism of $\mathbb{P}(1^5, 2)$ given by $y \mapsto y + \frac{1}{2}f_2(x)$), we can assume that $f_2 = 0$; in this case X is the double cover of the quadric 3-fold $\{g_2(x) = 0\} \subset \mathbb{P}^4$ branched along the surface $\{f_4(x) = g_2(x) = 0\}$.

It is natural to ask whether the connected component of the K-moduli space $M_{3,4}^{\text{Kps}}$ containing the members of \mathcal{V}_4 parametrises $(2, 4)$ -complete intersections in $\mathbb{P}(1^5, 2)$. If this were true, it might suggest that this component of K-moduli can be constructed via some form of non-reductive GIT

by taking the moduli space of such complete intersections. We first prove that this is not the case.

THEOREM A (cf. Proposition 2.1, Theorems 3.1 and 4.1). *There exist K -polystable non-smooth Fano 3-folds that are smoothable to a quartic 3-fold and that are not $(2, 4)$ -complete intersections in $\mathbb{P}(1^5, 2)$.*

An immediate consequence is the following.

COROLLARY. *Let M be the connected component of the K -moduli space $M_{3,4}^{\text{Kps}}$ containing the members of \mathcal{V}_4 . Then in M there are points corresponding to K -polystable Fano 3-folds that are not $(2, 4)$ -complete intersections in $\mathbb{P}(1^5, 2)$.*

We provide two types of examples of Fano 3-folds which satisfy Theorem A: three toric examples that we study in Section 2, and an infinite family that we study in Section 3. These examples are all $(2, 2, 4)$ -complete intersections in the weighted projective 6-space $\mathbb{P}(1^5, 2^2)$.

We have seen that some K -polystable degenerations of smooth quartic 3-folds are hyperelliptics; in other words, K -moduli limits of quartics include hyperelliptics. In Theorem A we have seen that K -moduli limits of hyperelliptics include Fano 3-folds which are $(2, 2, 4)$ -complete intersections in $\mathbb{P}(1^5, 2^2)$ which are not $(2, 4)$ -complete intersections in $\mathbb{P}(1^5, 2)$. We call such 3-folds *pure* $(2, 2, 4)$ -complete intersections. It is natural to wonder whether pure $(2, 2, 4)$ -complete intersections degenerate further in the K -moduli space. Surprisingly, the K -moduli limits of pure $(2, 2, 4)$ -complete intersections are pure $(2, 2, 4)$ -complete intersections.

THEOREM B (cf. Proposition 4.4 and Corollary 4.5). *In the K -moduli space $M_{3,4}^{\text{Kps}}$, the locus of pure $(2, 2, 4)$ -complete intersections in $\mathbb{P}(1^5, 2^2)$ is closed.*

This is proved in Section 4 by showing that the K -moduli locus of pure $(2, 2, 4)$ -complete intersections in $\mathbb{P}(1^5, 2^2)$ is determined by the K -moduli space of pairs $(S, \frac{1}{16}\Delta)$, where S is a degree 4 del Pezzo surface and $\Delta \sim -4K_S$. In Section 5 we provide a complete study of the wall crossing for the pairs $(S, c\Delta)$, as c is a rational number in $(0, \frac{1}{16}]$.

1.2 Degenerations of quartic 3-folds inside $\mathbb{P}(1^5, 2^2)$

Denote the variables of the weighted projective space $\mathbb{P}(1^5, 2^2)$ by $x_0, \dots, x_4, y_0, y_1$, where y_0 and y_1 are the two variables of weight 2. Let $X \subset \mathbb{P}(1^5, 2^2)$ be a complete intersection of two quadrics and a quartic. If X is not a cone, then after a suitable change of coordinates, it is defined by equations

$$\begin{aligned} y_0 y_1 &= f(x_0, x_1, x_2, x_3, x_4), \\ g(x_0, x_1, x_2, x_3, x_4) &= a y_0 + b y_1, \\ h(x_0, x_1, x_2, x_3, x_4) &= c y_0 + d y_1, \end{aligned}$$

where f has degree 4, g and h have degree 2, and $a, b, c, d \in \mathbb{C}$. There are three possibilities depending on the rank of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If $\text{rk } A = 2$, then X is a quartic hypersurface in \mathbb{P}^4 . If $\text{rk } A = 1$, then X is hyperelliptic, that is, a $(2, 4)$ -complete intersection in $\mathbb{P}(1^5, 2)$. If A is the null matrix, then X has two singular points

at $p_0 = [0 : 0 : 0 : 0 : 0 : 1 : 0]$ and $p_1 = [0 : 0 : 0 : 0 : 0 : 0 : 1]$. Let us now assume that we are in the latter case, so that $X \subset \mathbb{P}(1^5, 2^2)$ is defined by

$$\begin{aligned} y_0 y_1 &= f(x_0, x_1, x_2, x_3, x_4), \\ g(x_0, x_1, x_2, x_3, x_4) &= 0, \\ h(x_0, x_1, x_2, x_3, x_4) &= 0, \end{aligned}$$

where f is a quartic and g and h are quadrics. In Section 2 we consider special binomials for f, g, h , and we study certain toric varieties X . In Section 3 we study the infinite family of varieties X obtained by picking general f, g, h . In Section 4 we first show that if the surface $\{g = h = 0\} \subset \mathbb{P}^4$ and the curve $\{f = g = h = 0\} \subset \mathbb{P}^4$ are smooth, then X is K-polystable. We then prove that K-polystable degenerations of these complete intersection $X_{2,2,4} \subset \mathbb{P}(1^5, 2^2)$ are also complete intersections of these types.

1.3 Overview of the proof

There are several methods available to verify K-(poly)stability of a given Fano variety. For the toric examples, we use the most natural tool: a toric variety is K-polystable if and only if the barycentre of its anticanonical polytope is the origin. This translates the algebro-geometric condition into a combinatorial one amenable to computer-assisted investigation. The toric examples which satisfy Theorem A were found via a computer search amongst Fano polytopes using techniques from mirror symmetry [ACC⁺16, CCGK16, CCG⁺13]. This is explained in Section 1.4; the examples are studied in Section 2.

For the infinite family of examples satisfying Theorem A, we first use estimates on stability thresholds. Two of the most useful tools in K-stability are the Fujita–Li valuative criterion [Li17, Fuj19b] and the stability threshold, also called the δ -invariant, introduced in [FO18]. It follows that a Fano variety X is K-stable if and only if $\delta(X) > 1$. Although $\delta(X)$ is extremely difficult to compute in general, a method to find lower bounds for $\delta(X)$ is described in [AZ22]. Roughly speaking, one chooses a flag over X and computes the refinement of the anticanonical linear system with respect to this flag [AZ22, §2]; the refinement provides a lower bound for $\delta(X)$. With a little care, one can choose the flag so that the stability threshold of the refinement is greater than 1, implying K-stability. We do this for the infinite family in Section 3. In Section 4 we use techniques of cyclic covers, cone construction, degeneration, and interpolation to obtain two results: we first prove that all complete intersections $X_{2,2,4}$ as described above are K-polystable as long as $\{g = h = 0\} \subset \mathbb{P}^4$ and $\{f = g = h = 0\} \subset \mathbb{P}^4$ are smooth; then we prove that the closure in the K-moduli space of the locus of complete intersections $X_{2,2,4} \subset \mathbb{P}(1^5, 2^2)$ which are not complete intersections $X_{2,4} \subset \mathbb{P}(1^5, 2)$ contains only points which correspond to complete intersections $X_{2,2,4} \subset \mathbb{P}(1^5, 2^2)$.

1.4 Connection to mirror symmetry

The three toric 3-folds presented in Proposition 2.1 and Remark 2.2 were found using a computer-assisted search guided by expectations arising from mirror symmetry for Fano varieties [ACC⁺16, CCGK16, CCG⁺13]. It is expected that deformation families of smooth (or maybe mildly singular, that is, with orbifold terminal singularities) Fano varieties of dimension n are in one-to-one correspondence with mutation-equivalence classes of certain ‘special’ Laurent polynomials in n variables. We need to specify: (i) the meaning of the word *mutation*, (ii) the meaning of the adjective ‘special’, and (iii) how the correspondence works.

- (i) By *mutation* we mean a generalisation, introduced in [ACGK12], of the notion of mutation in cluster algebra theory [FZ02]. We do not give the definition here; see [ACGK12, Definition 2] for details. It is sufficient to know that, under certain circumstances, a Laurent polynomial $f \in \mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ can be mutated to another Laurent polynomial $g \in \mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Two Laurent polynomials f and g are said to be mutation-equivalent if there exists a finite sequence of mutations transforming f to g .
- (ii) In place of ‘special’ we should have written *rigid maximally mutable*. The class of rigid maximally mutable Laurent polynomials (or *rigid MMLPs* for short) was introduced in [ACC⁺16] for dimension 2 and in [CKPT21] for any dimension. We remark that if $f \in \mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is a rigid MMLP in n variables, then its Newton polytope $P := \text{Newt}(f) \subset \mathbb{R}^n$ is an n -dimensional lattice polytope such that the origin lies in the interior of P , and each vertex of P is a primitive lattice vector: P is an example of a *Fano polytope*. One can consider the *spanning fan* (or *face fan*) whose cones are generated by the faces of P , and the (possibly singular) Fano toric variety X_P associated to this spanning fan.
- (iii) The correspondence between deformation families of Fano varieties and mutation-equivalence classes of rigid MMLPs is described in [ACC⁺16, CKPT21, CHK22]. Briefly, we expect it to work as follows: given a rigid MMLP f with Newton polytope P , we consider the Fano toric variety X_P and associate a (partial) smoothing of X_P . The reason why this assignment should be well defined is the following result due to Ilten [Ilt12]: if f and g are Laurent polynomials related via a mutation, then the Fano toric varieties X_P and X_Q are deformation-equivalent (here Q is the Newton polytope of g); that is, there exists a flat proper family $\mathcal{X} \rightarrow \mathbb{P}^1$ such that the fibre over $0 \in \mathbb{P}^1$ is X_P and the fibre over $\infty \in \mathbb{P}^1$ is X_Q . Notice that a singular Fano toric variety can have many different smoothings (see for example [KP21, Theorem 3.1] or [Pet20]); one needs to select the smoothing of X_P which is compatible with the mutations of f .

Now we explain how we found the toric examples. We start from the polytope P' whose vertices are the vectors in (2.2) in Remark 2.3. The toric variety $X_{P'}$ associated to the spanning fan of P' is the singular quartic hypersurface $\{x_1x_2x_3x_4 = x_0^4\} \subset \mathbb{P}^4$, which clearly deforms to members of \mathcal{V}_4 . There exists a unique rigid MMLP f such that $P' = \text{Newt}(f)$, namely

$$f = \frac{(1 + x + y + z)^4}{xyz} - 24.$$

We used the computer algebra system MAGMA [BCP97] to construct many Laurent polynomials g which are mutation-equivalent to f . Each of these polynomials g gives a Fano toric variety X_Q , where $Q := \text{Newt}(g)$, which, according to the mirror symmetry expectations described above, should be a degeneration of quartic 3-folds. We then filter for those g such that the polar of Q has barycentre at the origin; this is equivalent to requiring that the Fano toric variety X_Q is K-polystable. In this way we found three Laurent polynomials, one of which is given by

$$\begin{aligned} g = & x^3y^2z^4 + 2x^2y^3z^2 + 4x^2y^2z^2 + 8x^2yz^3 + 2x^2yz^2 + xy^4 + 4xy^3 + 8xy^2z + 6xy^2 \\ & + 16xyz + 4xy + 28xz^2 + 8xz + x + 12y + 56z/y + 12/y + 8/(xz) + 16/(xyz) + 70/(xy^2) \\ & + 8/(xy^2z) + 2/(x^2yz^2) + 4/(x^2y^2z^2) + 56/(x^2y^3z) + 2/(x^2y^3z^2) + 28/(x^3y^4z^2) \\ & + 8/(x^4y^5z^3) + 1/(x^5y^6z^4). \end{aligned}$$

The Newton polytope $P := \text{Newt}(g)$ is the Fano polytope given in Proposition 2.1. The Newton polytopes of the remaining two Laurent polynomials found using this method are described in Remark 2.2.

Notation and conventions

We work over an algebraically closed field of characteristic zero, denoted by \mathbb{C} . Every toric variety or toric singularity is assumed to be normal.

2. The toric examples

We begin by analysing a toric Fano 3-fold satisfying Theorem A.

PROPOSITION 2.1. *Let P be the polytope with vertices*

$$\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -5 \\ -6 \\ -4 \end{pmatrix} \quad (2.1)$$

in the lattice $N = \mathbb{Z}^3$, and let X be the toric variety associated to the spanning fan of P . Then:

- (1) *The variety X is a \mathbb{Q} -factorial K -polystable Fano 3-fold.*
- (2) *We have $\text{Pic}(X) \simeq \mathbb{Z}$ and $\text{Cl}(X) \simeq \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$.*
- (3) *The variety X is the quotient $\mathbb{P}^3/(\mu_2 \times \mu_8)$, where μ_2 acts linearly with weights $(0, 1, 0, 1)$ and μ_8 acts linearly with weights $(0, 5, 1, 6)$.*
- (4) *The singular locus of X consists of six rational curves generically along which X has transverse A_1 , A_3 , or A_7 singularities.*
- (5) *There are exactly two non-Gorenstein points on X , and both of them have Gorenstein index 2 and are not canonical.*
- (6) *The variety X is not a $(2, 4)$ -complete intersection in $\mathbb{P}(1^5, 2)$.*
- (7) *The variety X is the $(2, 2, 4)$ -complete intersection in $\mathbb{P}(1^5, 2^2)$ given by the equations*

$$x_0^4 - y_0 y_1 = 0, \quad x_2^2 - x_0 x_3 = 0, \quad x_3^2 - x_1 x_4 = 0,$$

where $x_0, \dots, x_4, y_0, y_1$ are the homogeneous coordinates of $\mathbb{P}(1^5, 2^2)$ with degrees $1, \dots, 1, 2, 2$, respectively.

- (8) *The variety X deforms to a quartic 3-fold.*

Proof. (1) Let Σ be the spanning fan of P . It is clear that X is a Fano 3-fold. Since P is a tetrahedron, each cone of Σ is simplicial; therefore, X is \mathbb{Q} -factorial. Let M be the lattice dual to N , and let $\langle \cdot, \cdot \rangle: M \times N \rightarrow \mathbb{Z}$ be the dual pairing. We denote by the same symbol its extension to the associated real vector spaces, that is, $M_{\mathbb{R}} \times N_{\mathbb{R}} \rightarrow \mathbb{R}$. Consider the polar P° of P :

$$P^\circ := \{u \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq -1 \text{ for all } v \in P\}.$$

This is the polytope associated to the toric boundary of X , which is an anticanonical divisor. One can show that P° is the rational polytope in $M_{\mathbb{R}}$ with vertices $(-1, 0, \frac{3}{2})$, $(-1, 1, 0)$, $(3, -1, -2)$, $(-1, 0, \frac{1}{2})$. Since the barycentre of P° is the origin, X is K -polystable by [Ber16].

(2)–(3) Consider the linear map $\rho: \mathbb{Z}^4 \rightarrow N = \mathbb{Z}^3$ which maps the i th standard basis vector to the i th vertex of P in (2.1). Consider the transpose ${}^t\rho: M = \mathbb{Z}^3 \rightarrow \mathbb{Z}^4$. This is injective, and its cokernel is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ via the homomorphism $\mathbb{Z}^4 \rightarrow \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ given by the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ \bar{0}^2 & \bar{1}^2 & \bar{0}^2 & \bar{1}^2 \\ \bar{0}^8 & \bar{5}^8 & \bar{1}^8 & \bar{6}^8 \end{pmatrix},$$

where \cdot^2 (respectively, \cdot^8) denotes the reduction modulo 2 (respectively, 8). The short exact sequence

$$0 \longrightarrow M \xrightarrow{t\rho} \mathbb{Z}^4 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \longrightarrow 0$$

is the divisor sequence [CLS11, Theorem 4.1.3]; hence the divisor class group of X is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$. By [CLS11, Proposition 4.2.5] the Picard group of X is free; moreover, since X is \mathbb{Q} -factorial, the Picard rank of X coincides with the rank of $\text{Cl}(X)$, which is 1. The presentation of X as a finite abelian quotient of \mathbb{P}^3 follows from [CLS11, Exercise 5.1.13].

(4) Let $\rho_1, \rho_2, \rho_3, \rho_4$ be the elements in N which appear in (2.1). For $1 \leq i < j \leq 4$, let $\sigma_{ij} \in \Sigma$ be the 2-dimensional cone with rays $\mathbb{R}_{\geq 0}\rho_i$ and $\mathbb{R}_{\geq 0}\rho_j$. Let C_{ij} be the closure of the torus orbit on X associated to the cone σ_{ij} : it is a smooth rational curve. The shape of the cone σ_{ij} determines the singularities of X generically along C_{ij} . We write down the analysis for C_{12} , and we omit the other cases.

We consider the 2-dimensional lattice $N_{12} = N \cap (\mathbb{R}\rho_1 + \mathbb{R}\rho_2)$. This is the orthogonal to $(8, -2, -5) \in M$. The vectors ρ_1 and ρ_2 form an \mathbb{R} -basis of $\mathbb{R}\rho_1 + \mathbb{R}\rho_2 = N_{12} \otimes_{\mathbb{Z}} \mathbb{R}$, but not a \mathbb{Z} -basis of N_{12} . The finite abelian group $N_{12}/(\mathbb{Z}\rho_1 + \mathbb{Z}\rho_2)$ has order 2, and its generator is the class of

$$\frac{1}{2}\rho_1 + \frac{1}{2}\rho_2 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \in N_{12}.$$

This implies that the toric variety associated to the cone σ_{12} in the lattice N_{12} is the A_1 surface singularity $\text{Spec } \mathbb{C}[x, y, z]/(xy - z^2)$. This implies that X has transverse A_1 singularities generically along C_{12} .

(5) By part (4) the non-Gorenstein locus of X is contained in the set of the four torus-fixed points: $p_{123}, p_{124}, p_{134}$, and p_{234} . Here p_{ijk} is the torus-fixed point on X corresponding to the 3-dimensional cone σ_{ijk} with rays $\mathbb{R}_{\geq 0}\rho_i, \mathbb{R}_{\geq 0}\rho_j$, and $\mathbb{R}_{\geq 0}\rho_k$. We need to analyse the singularities of X at these points.

Let us start from p_{123} . The three vectors ρ_1, ρ_2 , and ρ_3 lie on the affine plane

$$H_{(2,0,-1),2} := \{v \in N_{\mathbb{R}} \mid \langle (2, 0, -1), v \rangle = 2\}.$$

Since the lattice vector $(2, 0, -1) \in M$ is primitive, we get that the Gorenstein index of X at p_{123} is 2. Moreover, the singularity $p_{123} \in X$ is not canonical because the lattice vector

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{4}\rho_1 + \frac{1}{8}\rho_2 + \frac{1}{8}\rho_3$$

lies in the interior of the polytope with vertices $0, \rho_1, \rho_2, \rho_3$ (see [CLS11, Proposition 11.4.12b]).

Now consider p_{124} . The three vectors ρ_1, ρ_2 , and ρ_4 lie on the affine plane

$$H_{(-3,1,2),1} := \{v \in N_{\mathbb{R}} \mid \langle (-3, 1, 2), v \rangle = 1\}.$$

Since the lattice vector $(-3, 1, 2) \in M$ is primitive, we get that the Gorenstein index of X at p_{124} is 1. By [CLS11, Proposition 11.4.11] the singularity $p_{124} \in X$ is canonical.

In an analogous way we can prove that p_{134} is a Gorenstein canonical singularity, whereas p_{234} is a non-canonical singularity with Gorenstein index 2.

(6) For brevity, set $\mathbb{P} := \mathbb{P}(1^5, 2)$. The singular locus of \mathbb{P} consists of a single point $p = [0 : 0 : 0 : 0 : 0 : 1]$. Towards a contradiction assume that X is a $(2, 4)$ -complete intersection in \mathbb{P} ,

and consider the corresponding closed embedding $X \hookrightarrow \mathbb{P}$. We have that $X \setminus \{p\} \hookrightarrow \mathbb{P} \setminus \{p\}$ is a regular closed embedding because it is locally defined by the vanishing of the dehomogenisations of the quartic and the quadric that define X inside \mathbb{P} . This implies that $X \setminus \{p\}$ is a local complete intersection variety, and in particular Gorenstein. Therefore, the non-Gorenstein locus of X is contained in $\{p\}$, so it either is empty or consists of one point. This contradicts part (5).

(7) In the lattice $M \oplus \mathbb{Z}$ we consider the cone τ whose apex is at the origin and which is spanned by $P^\circ \times \{1\}$. In other words, τ is the cone over P° placed at height 1. The primitive generators of τ are

$$y_1 = (-2, 0, 3, 2), \quad x_1 = (-1, 1, 0, 1), \quad x_4 = (3, -1, -2, 1), \quad y_0 = (-2, 0, 1, 2).$$

The Hilbert basis (that is, the minimal set of generators) of the monoid $\tau \cap (M \oplus \mathbb{Z})$ is made up of $y_1, x_1, x_4, y_0, x_0 = (-1, 0, 1, 1), x_2 = (0, 0, 0, 1), x_3 = (1, 0, -1, 1)$.

Since P° is the moment polytope of the toric boundary of X , which is anticanonical and hence ample, we have that $X = \text{Proj } \mathbb{C}[\tau \cap (M \oplus \mathbb{Z})]$, where the \mathbb{N} -grading is given by the projection $M \oplus \mathbb{Z} \rightarrow \mathbb{Z}$. This shows that X is a closed subvariety of the weighted projective space $\mathbb{P}(1^5, 2^2)$, equipped with homogeneous coordinates $x_0, \dots, x_4, y_0, y_1$. It is easy to see that the equations of X in $\mathbb{P}(1^5, 2^2)$ are

$$x_0^4 - y_0 y_1 = 0, \quad x_2^2 - x_0 x_3 = 0, \quad x_3^2 - x_1 x_4 = 0.$$

Thus X is a $(2, 2, 4)$ -complete intersection in $\mathbb{P}(1^5, 2^2)$.

(8) By using an argument similar to the one that appears in the introduction, one can see that the general $(2, 2, 4)$ -complete intersection in $\mathbb{P}(1^5, 2^2)$ is a quartic 3-fold. \square

Remark 2.2. In addition to the example presented in Proposition 2.1, we found two additional Fano toric 3-folds which satisfy Theorem A.

- (1) Let X be the toric 3-fold associated to the spanning fan of the polytope in \mathbb{Z}^3 with vertices $(1, 3, 2), (1, 3, 0), (1, 0, 2), (1, 0, 0), (-1, -1, 2), (-1, -1, -4), (-1, -2, 2), (-1, -2, -4)$. There are isomorphisms $\text{Cl}(X) \simeq \mathbb{Z}^5 \oplus \mathbb{Z}/2\mathbb{Z}$ and $\text{Pic}(X) \simeq \mathbb{Z}$. One can prove that X is the closed subvariety of $\mathbb{P}(1^5, 2^2)$ defined by the equations

$$x_0^3 x_3 - y_0 y_1 = 0, \quad x_2^2 - x_0 x_3 = 0, \quad x_2 x_3 - x_1 x_4 = 0.$$

- (2) Let X be the toric 3-fold associated to the spanning fan of the polytope in \mathbb{Z}^3 with vertices $(3, 4, 4), (3, 2, 4), (1, 2, 0), (1, 0, 0), (-1, 0, 0), (-1, -2, 0), (-3, -2, -4), (-3, -4, -4)$. There are isomorphisms $\text{Cl}(X) \simeq \mathbb{Z}^5 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and $\text{Pic}(X) \simeq \mathbb{Z}$. One can prove that X is the closed subvariety of $\mathbb{P}(1^5, 2^2)$ defined by the equations

$$x_0^2 x_4^2 - y_0 y_1 = 0, \quad x_2^2 - x_0 x_4 = 0, \quad x_1 x_3 - x_0 x_4 = 0.$$

Each of these examples is a $(2, 2, 4)$ -complete intersection in $\mathbb{P}(1^5, 2^2)$ and has two singular points with Gorenstein index 2.

Remark 2.3. In addition to the three examples presented in Proposition 2.1 and in Remark 2.2, we know a further K-polystable Fano toric 3-fold which deforms to quartic 3-folds: this is the toric variety associated to the spanning fan of the polytope with vertices

$$\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \quad (2.2)$$

in $N = \mathbb{Z}^3$, and it is the \mathbb{Q} -factorial hypersurface $\{x_1x_2x_3x_4 - x_0^4 = 0\}$ in \mathbb{P}^4 . We conjecture that these four toric varieties are the only K-polystable Fano toric 3-folds that deform to quartic 3-folds.

Remark 2.4. A Fano polytope P is called *symmetric* if the only point which is fixed by every automorphism of P is the origin. The examples in Remark 2.2 are Fano polytopes which are not simplices and are such that their polar has barycentre at the origin.

Remark 2.5. According to [Hal24] it is expected that in every mutation-equivalence class of 2-dimensional Fano polytopes, there is at most one polytope P such that its polar P° has barycentre at the origin. This is not true in higher dimension; for instance, the four polytopes presented in Proposition 2.1 and Remarks 2.2 and 2.3 are mutation-equivalent, and the polar of each of them has barycentre at the origin.

3. The infinite family

3.1 Statement

The infinite family of examples satisfying Theorem A is given by the following.

THEOREM 3.1. *Let $x_0, \dots, x_4, y_0, y_1$ be the homogeneous coordinates of the weighted projective space $\mathbb{P}(1^5, 2^2)$ with degrees $1, \dots, 1, 2, 2$, respectively. Consider x_0, \dots, x_4 as the homogeneous coordinates of \mathbb{P}^4 as well. Let f be a quartic in x_0, \dots, x_4 , and let g and h be two quadrics in x_0, \dots, x_4 . Consider the zero-loci*

$$\begin{aligned}\Delta &= \{f = 0, g = h = 0\} \subset \mathbb{P}^4, \\ S &= \{g = h = 0\} \subset \mathbb{P}^4, \\ X &= \{y_0y_1 - f = 0, g = h = 0\} \subset \mathbb{P}(1^5, 2^2).\end{aligned}\tag{3.1}$$

Assume that Δ is a smooth curve and that S is a smooth surface. Then the following statements hold:

- (1) *The surface S is a smooth del Pezzo of degree 4 and contains exactly 16 lines; their intersection points form a finite subset $\Sigma \subset S$ consisting of 40 points.*
- (2) *The variety X is a Kawamata log terminal (klt) Fano 3-fold, and the group $G = \mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$ acts faithfully on X .*
- (3) *If $\Sigma \cap \Delta = \emptyset$, then X is K-polystable.*

Remark 3.2. The last condition $\Sigma \cap \Delta = \emptyset$ is satisfied for general choices of f, g, h . Therefore, a general X in Theorem 3.1 is K-polystable. We believe that the condition $\Sigma \cap \Delta = \emptyset$ can be removed from Theorem 3.1(3).

Remark 3.3. Note that if $X \subset \mathbb{P}(1^5, 2)$ is a hyperelliptic cone, that is, the term y^2 does not appear in the degree 4 equation, or if $X \subset \mathbb{P}(1^5, 2^2)$ is a cone, that is, the quadratic part in the variables y_0, y_1 of the degree 4 equation f does not have rank 2, then X is unstable. This can be checked rather easily using [ZZ22, Theorem 1.4] by setting $r = 1/2$, $c = 0$, and $n = 3$ to obtain $\delta(X) \leq 5/12$.

The remainder of the section is devoted to giving a proof of Theorem 3.1; hence we always work in the setting of Theorem 3.1. We begin with an explanation of the construction of the models, their symmetries, and the generality condition on them. We then present the proof of the

main claim about their K-stability; the proof uses the theory of refinements introduced in [AZ22], which provides a lower bound for stability thresholds. Indeed, some formulae are readily available in [ACC⁺23, § 1.7] which compute the refinement for a flag if the flag is geometrically realised on a Mori dream space birational model of the Fano variety. We will use them in the proof of Theorem 3.1(3).

3.2 The models and their symmetries

Proof of Theorem 3.1(1). All of this is very classical. \square

Conversely, if S is a smooth del Pezzo surface of degree 4, then we can view S as a complete intersection $\{g = h = 0\} \subset \mathbb{P}^4$, where g and h are quadrics in x_0, \dots, x_4 . We denote by Σ the set consisting of the 40 intersection points of the lines on S .

Remark 3.4. Let p be a point in S .

- (1) If p is not contained in any line in S , then there are exactly ten smooth conics in S that contain p ; moreover, five of these ten conics can be chosen to intersect pairwise transversally at p .
- (2) If p is contained in a line in S and $p \notin \Sigma$, then there are exactly five smooth conics in S that contain p , and any two of them intersect transversally at p .
- (3) If $p \in \Sigma$, then it is contained in a unique smooth conic in S .

Proof of Theorem 3.1(2). We study the singularities of X by looking at the affine charts of $\mathbb{P}(1^5, 2^2)$.

Let us consider the chart $x_0 \neq 0$; this is isomorphic to \mathbb{A}^6 with affine coordinates $x_1, \dots, x_4, y_0, y_1$. Inside this \mathbb{A}^6 the variety X is given by the equations

$$y_0 y_1 - \bar{f} = 0, \quad \bar{g} = 0, \quad \bar{h} = 0,$$

where $\bar{f} = f|_{x_0=1}$ and similarly for \bar{g} and \bar{h} . The Jacobian matrix of the equations of X in \mathbb{A}^6 is

$$J = \begin{pmatrix} -\partial_{x_1} \bar{f} & -\partial_{x_2} \bar{f} & -\partial_{x_3} \bar{f} & -\partial_{x_4} \bar{f} & y_1 & y_0 \\ \partial_{x_1} \bar{g} & \partial_{x_2} \bar{g} & \partial_{x_3} \bar{g} & \partial_{x_4} \bar{g} & 0 & 0 \\ \partial_{x_1} \bar{h} & \partial_{x_2} \bar{h} & \partial_{x_3} \bar{h} & \partial_{x_4} \bar{h} & 0 & 0 \end{pmatrix}.$$

Since $S \subset \mathbb{P}^4$ is a smooth surface, the bottom-left 2×4 submatrix of J has rank 2 in all points of X . Therefore, it is clear that J has rank 3 at all points of X which satisfy $y_0 \neq 0$ or $y_1 \neq 0$. The points of X which satisfy $y_0 = y_1 = 0$ also satisfy $f = 0$, and from the smoothness of Δ , it follows that the rank of J is 3 at these points. Therefore, the intersection of X with the chart $x_0 \neq 0$ is smooth. Similarly, this also holds for the other charts $x_i \neq 0$ for $i \in \{0, \dots, 4\}$. In other words, we have proven that the singular points of X must satisfy $x_0 = \dots = x_4 = 0$. There are exactly two such points:

$$p_0 = [0 : 0 : 0 : 0 : 0 : 1 : 0] \quad \text{and} \quad p_1 = [0 : 0 : 0 : 0 : 0 : 0 : 1].$$

We need to study the singularity types of these two points on X .

The chart $y_0 \neq 0$ on $\mathbb{P}(1^5, 2^2)$ gives the quotient singularity $\frac{1}{2}(1, 1, 1, 1, 1, 0)$ with orbifold coordinates x_0, \dots, x_4, y_1 . It is easy to see that $X \cap \{y_0 \neq 0\}$ is isomorphic to the complete intersection $\{g = h = 0\} \subset \frac{1}{2}(1, 1, 1, 1, 1)$. Since $\frac{1}{2}(1, 1, 1, 1, 1)$ is the affine cone over $(\mathbb{P}^4, \mathcal{O}(2))$, we have that $\{g = h = 0\} \subset \frac{1}{2}(1, 1, 1, 1, 1)$ is the affine cone over $(S, -2K_S)$, which is a klt singularity. This shows that p_0 and p_1 are klt singularities of X .

By adjunction $-K_X = \mathcal{O}_{\mathbb{P}(1^5, 2^2)}(1)|_X$, so X is a klt Fano 3-fold. Its degree is $(-K_X)^3 = (1^3 \cdot 4 \cdot 2 \cdot 2)/2^2 = 4$.

Now we need to construct an effective action of $G = \mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$ on X . Consider the \mathbb{C}^* -action given by

$$[x_0 : x_1 : x_2 : x_3 : x_4 : y_0 : y_1] \mapsto [x_0 : x_1 : x_2 : x_3 : x_4 : \lambda y_0 : \lambda^{-1} y_1]$$

for $\lambda \in \mathbb{C}^*$. Furthermore, the group $\text{Aut}(X)$ also contains an involution σ that is given by

$$[x_0 : x_1 : x_2 : x_3 : x_4 : y_0 : y_1] \mapsto [x_0 : x_1 : x_2 : x_3 : x_4 : y_1 : y_0].$$

Together, they generate a subgroup $G \subset \text{Aut}(X)$ that is isomorphic to $\mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$. Note that the two singular points p_0 and p_1 on X are swapped by the action of σ and fixed by the action of \mathbb{C}^* . This concludes the proof of Theorem 3.1(2). \square

Let $\rho: X \dashrightarrow S$ be the rational map given by

$$[x_0 : x_1 : x_2 : x_3 : x_4 : y_0 : y_1] \mapsto [x_0 : x_1 : x_2 : x_3 : x_4].$$

Then ρ is undefined precisely at the points p_0 and p_1 , resulting in the following G -equivariant commutative diagram:

$$\begin{array}{ccc} & \tilde{X} & \\ \pi \swarrow & & \searrow \eta \\ X & \xrightarrow{\rho} & S, \end{array} \quad (3.2)$$

where \tilde{X} is a smooth projective 3-fold, π is a birational morphism that contracts two irreducible smooth surfaces E_0 and E_1 to the points p_0 and p_1 , respectively, and η is a G -equivariant conic bundle. Furthermore, the surfaces E_0 and E_1 are sections of the conic bundle η , so that η induces isomorphisms $E_0 \cong S$ and $E_1 \cong S$. We have $E_0|_{E_0} \sim -2K_{E_0}$ and $E_1|_{E_1} \sim -2K_{E_1}$, which gives

$$-K_{\tilde{X}} \sim_{\mathbb{Q}} \pi^*(-K_X) + \frac{1}{2}(E_0 + E_1) \sim_{\mathbb{Q}} \eta^*(-K_S) + E_0 + E_1.$$

COROLLARY 3.5. *The 3-fold \tilde{X} is a Mori dream space.*

Proof. Let ε be a sufficiently small positive rational number. Then $(\tilde{X}, \frac{1}{2}(1 + \varepsilon)(E_0 + E_1))$ has klt singularities, and $-(K_{\tilde{X}} + \frac{1}{2}(1 + \varepsilon)(E_0 + E_1))$ is ample. The claim follows from [BCHM10, Corollary 1.3.1]. \square

Let R_0 and R_1 be the surfaces in X that are cut out by $y_0 = f = 0$ and $y_1 = f = 0$, respectively. Denote by \tilde{R}_0 and \tilde{R}_1 the strict transforms of R_0 and R_1 on \tilde{X} , respectively. Then $\eta^*(\Delta) = \tilde{R}_0 + \tilde{R}_1$, and Δ is the discriminant curve of the conic bundle η . We have the following non- G -equivariant commutative diagram:

$$\begin{array}{ccccc} & U & \xleftarrow{\eta_0} & \tilde{X} & \xrightarrow{\eta_1} & U \\ & \psi \swarrow & & \downarrow \eta & \searrow \psi & \\ V & & \phi \searrow & & \swarrow \phi & V \\ & & & S & & \\ & \tau \swarrow & & & \searrow \tau & \\ & & & & & \end{array} \quad (3.3)$$

where $U = \mathbb{P}(\mathcal{O}_S(2K_S) \oplus \mathcal{O}_S)$, ϕ is the \mathbb{P}^1 -bundle given by the projection $\mathbb{P}(\mathcal{O}_S(2K_S) \oplus \mathcal{O}_S) \rightarrow S$, the morphism ψ is a contraction of the negative section of the \mathbb{P}^1 -bundle ϕ , the variety V is a

cone over S , the map τ is the projection from the vertex of the cone, and the morphisms η_0 and η_1 are birational contractions of the surfaces \tilde{R}_0 and \tilde{R}_1 , respectively.

Remark 3.6. On the left-hand side of (3.3), we have $\phi^*(\Delta) = \eta_0(R_1)$, the morphism ψ contracts $\eta_0(E_0)$, and $\psi \circ \eta_0(E_1)$ is a smooth hyperplane section of the cone $V \subset \mathbb{P}^{13}$. On the right-hand side of (3.3), we have $\phi^*(\Delta) = \eta_1(R_0)$, ψ contracts $\eta_1(E_1)$, and $\psi \circ \eta_1(E_0)$ is a hyperplane section of the cone V . The involution σ swaps the left- and right-hand sides of the diagram (3.3).

Set $Z = R_0 \cap R_1$ and $\tilde{Z} = \tilde{R}_0 \cap \tilde{R}_1$. Then $Z = \pi(\tilde{Z})$, $Z \cong \tilde{Z} \cong \Delta$, and $Z = \{y_0 = y_1 = f = 0\} \subset X$. Observe that $p_0 \notin Z$, $p_1 \notin Z$, $\tilde{Z} \cap E_0 = \tilde{Z} \cap E_1 = \emptyset$, and every point in Z and \tilde{Z} is G -invariant.

LEMMA 3.7. *Let q be a G -invariant point in \tilde{X} , let C be a G -invariant irreducible curve in \tilde{X} , and let B be a G -invariant irreducible surface in \tilde{X} . Then the following assertions hold:*

- (1) *We have $q \in \tilde{Z}$.*
- (2) *Either $C = \tilde{Z}$, or C is a smooth fibre of the conic bundle η .*
- (3) *We have $B = \eta^*(\mathcal{C})$ for some irreducible curve $\mathcal{C} \subset S$.*

In particular, we have that $B \cap \tilde{Z} \neq \emptyset$.

Proof. Left to the reader. □

3.3 K-stability of the general models

The proof of Theorem 3.1(3) is quite involved. However, it is less difficult to produce specific examples of X with large symmetries and prove that they are K-polystable by taking advantage of the group of symmetries. Here, for illustration, we present one such case. Then we proceed by proving that a general X is K-polystable (see Theorem 3.1(3)).

Proof of a special case of Theorem 3.1(3): diagonal models. Suppose that

$$\begin{aligned} f &= \alpha_0 x_0^4 + \alpha_1 x_1^4 + \alpha_2 x_2^4 + \alpha_3 x_3^4 + \alpha_4 x_4^4, \\ g &= x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2, \\ h &= \epsilon_0 x_0^2 + \epsilon_1 x_1^2 + \epsilon_2 x_2^2 + \epsilon_3 x_3^2 + \epsilon_4 x_4^2, \end{aligned}$$

where $\alpha_0, \dots, \alpha_4$ are sufficiently general complex numbers and $\epsilon_0, \dots, \epsilon_4$ are pairwise distinct complex numbers. Then the group $\text{Aut}(X)$ is larger than the group G in Theorem 3.1(2) because $\text{Aut}(X)$ also contains 15 additional involutions given by

$$[x_0 : x_1 : x_2 : x_3 : x_4 : y_0 : y_1] \mapsto [x_0 : (-1)^a x_1 : (-1)^b x_2 : (-1)^c x_3 : (-1)^d x_4 : y_0 : y_1]$$

for $a, b, c, d \in \{0, 1\}$. Let Γ be the subgroup in $\text{Aut}(X)$ generated by G and these 15 involutions. Then $\Gamma \cong \mathbb{C}^* \rtimes (\mathbb{Z}/2\mathbb{Z})^5$, and (3.2) is Γ -equivariant. Note that X does not contain Γ -invariant points. Furthermore, using the diagram (3.2) and [DI09, Theorem 6.9], we conclude that $\text{Cl}^G(X) = \mathbb{Z}[-K_X]$. Now, arguing as in the proof of [ACC⁺23, Theorem 1.52], we conclude that

$$\alpha_\Gamma(X) \geq 1,$$

where $\alpha_\Gamma(X)$ is the Γ -invariant α -invariant of Tian for the 3-fold X ; see [Tia87] for the analytic definition and [CS11] for an algebraic definition of α . It follows that X is K-polystable by [Tia87] as

$$\alpha_\Gamma(X) > \frac{3}{4} = \frac{\dim X}{1 + \dim X}.$$

This concludes the claim that X is K-polystable. \square

Proof of Theorem 3.1(3). Suppose that X is not K-polystable. By [Zhu21, Corollary 4.14] there exists a G -invariant prime divisor \mathbf{F} over X for which $\beta(\mathbf{F}) \leq 0$. For the definition of $\beta(\mathbf{F})$, see [Fuj19b, Li17]. Let \mathfrak{C} be the centre of the divisor \mathbf{F} on X . Then, for every point $P \in \mathfrak{C}$, we must have $\delta_P(X) \leq 1$, where $\delta_P(X)$ is the local stability threshold of X at p . For the precise definition of $\delta_P(X)$, see [AZ22, Definition 2.5] and [ACC⁺23, § 1.5]. On the other hand, by Lemma 3.7, we have the following four possibilities:

- (1) \mathfrak{C} is a G -invariant point in Z .
- (2) \mathfrak{C} is the curve Z .
- (3) \mathfrak{C} is an irreducible fibre of the rational map $\rho: X \dashrightarrow S$.
- (4) \mathfrak{C} is a G -invariant surface and $\mathfrak{C} \cap Z \neq \emptyset$.

Hence, we see that either \mathfrak{C} is an irreducible curve that is a fibre of the rational map $\rho: X \dashrightarrow S$, or the curve Z contains a point x such that $\delta_x(X) \leq 1$. We will show that both cases are impossible, by first lifting the problem to the level of \tilde{X} and then using the results obtained in [AZ22] and [ACC⁺23, § 1.7] to derive a contradiction.

Since \tilde{X} is smooth and it is a Mori dream space, it is more convenient to work on \tilde{X} than on X . Namely, we let $L = \pi^*(-K_X)$, and we let $\tilde{\mathfrak{C}}$ be the centre of the divisor \mathbf{F} on the 3-fold \tilde{X} . Then $\tilde{\mathfrak{C}}$ is the strict transform of the centre \mathfrak{C} , and we have the following four possibilities:

- (1) $\tilde{\mathfrak{C}}$ is a G -invariant point in \tilde{Z} .
- (2) $\tilde{\mathfrak{C}}$ is the curve \tilde{Z} .
- (3) $\tilde{\mathfrak{C}}$ is a smooth fibre of the conic bundle $\eta: \tilde{X} \rightarrow S$.
- (4) $\tilde{\mathfrak{C}}$ is a G -invariant surface and $\tilde{\mathfrak{C}} \cap \tilde{Z} \neq \emptyset$.

Note that $\delta_x(\tilde{X}, L) = \delta_{\pi(x)}(X)$ for every point $x \in \tilde{Z}$. For the definition of $\delta_x(\tilde{X}, L)$, see [AZ22, Definition 2.5] and [ACC⁺23, § 1.5]. Therefore, we conclude that one of the following two cases hold:

- (\diamond) $\tilde{\mathfrak{C}}$ a smooth fibre of the conic bundle $\eta: \tilde{X} \rightarrow S$.
- (\heartsuit) There exists a point $x \in \tilde{Z}$ such that $\delta_x(\tilde{X}, L) \leq 1$.

In both cases, let us introduce a new curve \mathcal{C} in the 3-fold \tilde{X} as follows:

- In case (\diamond) we let $\mathcal{C} = \tilde{\mathfrak{C}}$.
- In case (\heartsuit) we let \mathcal{C} be the (singular) fibre of the conic bundle η that contains x .

Let $p = \eta(\mathcal{C})$. It follows from Remark 3.4 that S contains a smooth conic \mathcal{C} such that $p \in \mathcal{C}$. Moreover, if $p \in \Delta$, it follows from Remark 3.4 that we can also choose \mathcal{C} such that it intersects the curve Δ transversally at the point p because $\Sigma \cap \Delta = \emptyset$ by assumption.

Let $B = \eta^*(\mathcal{C})$, and let $B' = \eta^*(Z)$ for a general conic $Z \in |-K_S - \mathcal{C}|$. Then B is normal, it has at most Du Val singularities of type \mathbb{A} , it is smooth along the curve \mathcal{C} , and B' is smooth.

Let us compute $\beta(B)$. We have $\beta(B) = A_X(B) - S_L(B) = 1 - S_L(B)$, where

$$S_L(B) = \frac{1}{L^3} \int_0^\infty \text{vol}(L - uB) du.$$

For $u \in \mathbb{R}_{\geq 0}$, the divisor $L - uB$ is pseudo-effective if and only if $u \leq 1$ because

$$L - uB \sim_{\mathbb{Q}} (1 - u)B + B' + \frac{1}{2}(E_0 + E_1).$$

For $u \in [0, 1]$, let $P(u)$ be the positive part of the Zariski decomposition of the divisor $L - uB$, and let $N(u)$ be its negative part. Then

$$P(u) = L - uB - \frac{u}{2}(E_0 + E_1)$$

and $N(u) = \frac{1}{2}u(E_0 + E_1)$ for every $u \in [0, 1]$. This gives

$$S_L(B) = \frac{1}{L^3} \int_0^\infty \text{vol}(L - uB) du = \frac{1}{4} \int_0^1 (P(u))^3 du = \frac{1}{4} \int_0^1 (2u^3 - 6u + 4) du = \frac{3}{8},$$

which implies that $\beta(B) = \frac{5}{8}$. Here, we used the following intersections on the 3-fold \tilde{X} :

$$\begin{aligned} B^3 &= 0, (B')^3 = 0, E_0^3 = 16, E_1^3 = 16, E_0^2 \cdot E_1 = 0, E_0 \cdot E_1^2 = 0, E_0 \cdot B \cdot E_1 = 0, E_0 \cdot B' \cdot E_1 = 0, \\ E_0 \cdot B \cdot B' &= 2, B \cdot B' \cdot E_1 = 2, E_0 \cdot B^2 = 0, B^2 \cdot B' = 0, E_0 \cdot (B')^2 = 0, B \cdot (B')^2 = 0, \\ B^2 \cdot E_1 &= 0, E_0^2 \cdot B = -4, B \cdot E_1^2 = -4, E_0^2 \cdot B' = -4, B' \cdot E_1^2 = -4, (B')^2 \cdot E_1 = 0. \end{aligned}$$

We set $\mathbf{e}_0 = E_0|_B$ and $\mathbf{e}_1 = E_1|_B$. Then \mathbf{e}_0 and \mathbf{e}_1 are smooth irreducible rational disjoint curves, and B is smooth along \mathbf{e}_0 and \mathbf{e}_1 . On B , we have $\mathbf{e}_0^2 = \mathbf{e}_1^2 = -4$, $\mathbf{e}_0 \cdot \mathbf{e}_1 = 0$, $\mathbf{e}_0 \cdot \mathcal{C} = \mathbf{e}_1 \cdot \mathcal{C} = 1$.

Suppose that case (\diamond) holds. Then $\tilde{\mathcal{C}}$ is a smooth fibre of the conic bundle η , and $\mathcal{C} = \tilde{\mathcal{C}}$. Set

$$S(W_{\bullet, \bullet}^B; \mathcal{C}) = \frac{3}{L^3} \int_0^1 \int_0^\infty \text{vol}(P(u)|_B - v\mathcal{C}) dv du.$$

Using [ACC⁺23, Corollary 1.7.26], we get $S(W_{\bullet, \bullet}^B; \mathcal{C}) \geq 1$ since $\beta(\mathbf{F}) \leq 0$, $\beta(B) > 0$, and $\mathcal{C} \not\subset \text{Supp}(N(u))$. On the other hand, it is easy to compute $S(W_{\bullet, \bullet}^B; \mathcal{C})$. Indeed, take $v \in \mathbb{R}_{\geq 0}$. Then

$$P(u)|_B - v\mathcal{C} \sim_{\mathbb{R}} (2 - v)\mathcal{C} + \frac{1 - u}{2}(\mathbf{e}_0 + \mathbf{e}_1).$$

Therefore, the divisor $P(u)|_B - v\mathcal{C}$ is nef for $v \leq 2u$, and it is not pseudo-effective for $v > 2$. Moreover, if $v \in [2u, 2]$, the positive part of its Zariski decomposition is $\frac{1}{4}(2 - v)(4\mathcal{C} + \mathbf{e}_0 + \mathbf{e}_1)$. Then

$$\text{vol}(P(u)|_B - v\mathcal{C}) = \begin{cases} 2 - 2u^2 + 2uv - 2v & \text{if } 0 \leq v \leq 2u, \\ \frac{1}{2}(v - 2)^2 & \text{if } 2u \leq v \leq 2, \\ 0 & \text{if } v \geq 2. \end{cases}$$

Integrating, we get $S(W_{\bullet, \bullet}^B; \mathcal{C}) = \frac{3}{4}$, which gives a contradiction. Thus case (\diamond) does not hold.

Now, we assume that the case (\heartsuit) holds and derive a contradiction. Recall that this means that $\delta_x(\tilde{X}, L) \leq 1$ for some point $x \in \tilde{Z}$. In this case the curve \mathcal{C} is singular. Namely, we have $\mathcal{C} = C_0 + C_1$, where C_0 and C_1 are smooth irreducible rational curves that intersect each other transversally at x . Without loss of generality, we may assume that $C_0 \cap E_0 \neq \emptyset$. Moreover, since the surface B is smooth along \mathcal{C} , the numerical intersections of the curves C_0 , C_1 , \mathbf{e}_0 , \mathbf{e}_1 on the surface B are given in the following table:

| | C_0 | C_1 | \mathbf{e}_0 | \mathbf{e}_1 |
|----------------|-------|-------|----------------|----------------|
| C_0 | -1 | 1 | 1 | 0 |
| C_1 | 1 | -1 | 0 | 1 |
| \mathbf{e}_0 | 1 | 0 | -4 | 0 |
| \mathbf{e}_1 | 0 | 1 | 0 | -4 |

Let us estimate $\delta_x(\tilde{X}, L)$ using [ACC⁺23, Theorem 1.7.30]. We have $N(u)|_B = \frac{1}{2}u(\mathbf{e}_0 + \mathbf{e}_1)$ and $x \notin \mathbf{e}_0 \cup \mathbf{e}_1$. For every $u \in [0, 1]$, we let

$$t(u) = \inf \{ v \in \mathbb{R}_{\geq 0} \mid \text{the divisor } P(u)|_B - vC_0 \text{ is pseudo-effective} \}.$$

For $v \in [0, t(u)]$, we let $P(u, v)$ be the positive part of the Zariski decomposition of $P(u)|_B - vC_0$, and we let $N(u, v)$ be its negative part. Then we set

$$S(W_{\bullet, \bullet}^B; C_0) = \frac{3}{L^3} \int_0^1 \int_0^{t(u)} \text{vol}(P(u)|_B - vC_0) dv du.$$

Note that $C_0 \not\subset \text{Supp}(N(u, v))$ for every $u \in [0, 1)$ and $v \in (0, t(u))$. Thus, we can let

$$F_x(W_{\bullet, \bullet, \bullet}^{B, C_0}) = \frac{6}{L^3} \int_0^1 \int_0^{t(u)} (P(u, v) \cdot C_0) \cdot \text{ord}_x(N(u, v)|_{C_0}) dv du.$$

Finally, we let

$$S(W_{\bullet, \bullet, \bullet}^{B, C_0}; x) = \frac{3}{L^3} \int_0^1 \int_0^{t(u)} (P(u, v) \cdot C_0)^2 dv du + F_x(W_{\bullet, \bullet, \bullet}^{B, C_0}).$$

Then, since $C_0 \not\subset \text{Supp}(N(u))$, it follows from [ACC⁺23, Theorem 1.7.30] that

$$1 \geq \delta_x(\tilde{X}, L) \geq \min \left\{ \frac{1}{S(W_{\bullet, \bullet, \bullet}^{B, C_0}; x)}, \frac{1}{S(W_{\bullet, \bullet}^B; C_0)}, \frac{1}{S_L(B)} \right\}.$$

Recall that $S_L(B) = \frac{3}{8}$. Thus, either $S(W_{\bullet, \bullet, \bullet}^{B, C_0}; x) \geq 1$ or $S(W_{\bullet, \bullet}^B; C_0) \geq 1$ (or both).

Let us compute $S(W_{\bullet, \bullet, \bullet}^{B, C_0}; x)$ and $S(W_{\bullet, \bullet}^B; C_0)$. As above, take $v \in \mathbb{R}_{\geq 0}$. Then

$$P(u)|_B - vC_0 \sim_{\mathbb{R}} (2 - v)C_0 + 2C_1 + \frac{1 - u}{2}(\mathbf{e}_0 + \mathbf{e}_1).$$

Therefore, since the intersection form of the curves $C_1, \mathbf{e}_0, \mathbf{e}_1$ on the surface B is negative-definite, we see that $P(u)|_B - vC_0$ is pseudo-effective if and only if $v \leq 2$, so $t(u) = 2$. Moreover, if $0 \leq u \leq \frac{1}{5}$, then

$$P(u, v) = \begin{cases} (2 - v)C_0 + 2C_1 + \frac{1}{2}(1 - u)(\mathbf{e}_0 + \mathbf{e}_1) & \text{if } 0 \leq v \leq 2u, \\ \frac{1}{4}(2 - v)(4C_0 + \mathbf{e}_0) + 2C_1 + \frac{1}{2}(1 - u)\mathbf{e}_1 & \text{if } 2u \leq v \leq \frac{1}{2}(1 - u), \\ \frac{1}{4}(2 - v)(4C_0 + \mathbf{e}_0) + \frac{1}{2}(5 - 2v - u)C_1 + \frac{1}{2}(1 - u)\mathbf{e}_1 & \text{if } 1 - u \leq 2v \leq 1 + 3u, \\ \frac{1}{12}(2 - v)(12C_0 + 16C_1 + 3\mathbf{e}_0 + 4\mathbf{e}_1) & \text{if } \frac{1}{2}(1 + 3u) \leq v \leq 2, \end{cases}$$

and

$$N(u, v) = \begin{cases} 0 & \text{if } 0 \leq v \leq 2u, \\ \frac{1}{4}(v - 2u)\mathbf{e}_0 & \text{if } 2u \leq v \leq \frac{1}{2}(1 - u), \\ \frac{1}{4}(v - 2u)\mathbf{e}_0 + \frac{1}{2}(2v + u - 1)C_1 & \text{if } 1 - u \leq 2v \leq 1 + 3u, \\ \frac{1}{4}(v - 2u)\mathbf{e}_0 + \frac{1}{3}(4v - 2)C_1 + \frac{1}{6}(2v - 3u - 1)\mathbf{e}_1 & \text{if } \frac{1}{2}(1 + 3u) \leq v \leq 2, \end{cases}$$

so computation gives

$$P(u, v) \cdot C_0 = \begin{cases} \frac{1}{2}(1 - u + 2v) & \text{if } 0 \leq v \leq 2u, \\ \frac{1}{4}(2 + 3v) & \text{if } 2u \leq v \leq \frac{1}{2}(1 - u), \\ \frac{1}{4}(4 - 2u - v) & \text{if } 1 - u \leq 2v \leq 1 + 3u, \\ \frac{1}{12}(14 - 7v) & \text{if } \frac{1}{2}(1 + 3u) \leq v \leq 2, \end{cases}$$

$$\text{vol}(P(u)|_B - vC_0) = \begin{cases} 2 - 2u^2 + uv - v^2 - v & \text{if } 0 \leq v \leq 2u, \\ 2 - v - u^2 - \frac{3}{4}v^2 & \text{if } 2u \leq v \leq \frac{1}{2}(1 - u), \\ \frac{1}{4}(9 - 3u^2 + 4uv + v^2 - 2u - 8v) & \text{if } 1 - u \leq 2v \leq 1 + 3u, \\ \frac{7}{12}(2 - v)^2 & \text{if } \frac{1}{2}(1 + 3u) \leq v \leq 2. \end{cases}$$

Similarly, if $\frac{1}{5} \leq u \leq 1$, then

$$P(u, v) = \begin{cases} (2 - v)C_0 + 2C_1 + \frac{1}{2}(1 - u)(\mathbf{e}_0 + \mathbf{e}_1) & \text{if } 0 \leq v \leq \frac{1}{2}(1 - u), \\ (2 - v)C_0 + \frac{1}{2}(5 - 2v - u)C_1 + \frac{1}{2}(1 - u)(\mathbf{e}_0 + \mathbf{e}_1) & \text{if } \frac{1}{2}(1 - u) \leq v \leq 2u, \\ \frac{1}{4}(2 - v)(4C_0 + \mathbf{e}_0) + \frac{1}{2}(5 - 2v - u)C_1 + \frac{1}{2}(1 - u)\mathbf{e}_1 & \text{if } 1 - u \leq 2v \leq 1 + 3u, \\ \frac{1}{12}(2 - v)(12C_0 + 16C_1 + 3\mathbf{e}_0 + 4\mathbf{e}_1) & \text{if } \frac{1}{2}(1 + 3u) \leq v \leq 2, \end{cases}$$

$$N(u, v) = \begin{cases} 0 & \text{if } 0 \leq v \leq \frac{1}{2}(1 - u), \\ \frac{1}{2}(2v + u - 1)C_1 & \text{if } \frac{1}{2}(1 - u) \leq v \leq 2u, \\ \frac{1}{4}(v - 2u)\mathbf{e}_0 + \frac{1}{2}(2v + u - 1)C_1 & \text{if } 1 - u \leq 2v \leq 1 + 3u, \\ \frac{1}{4}(v - 2u)\mathbf{e}_0 + \frac{1}{3}(4v - 2)C_1 + \frac{1}{6}(2v - 3u - 1)\mathbf{e}_1 & \text{if } \frac{1}{2}(1 + 3u) \leq v \leq 2, \end{cases}$$

$$P(u, v) \cdot C_0 = \begin{cases} \frac{1}{2}(1 - u + 2v) & \text{if } 0 \leq v \leq \frac{1}{2}(1 - u), \\ 1 - u & \text{if } \frac{1}{2}(1 - u) \leq v \leq 2u, \\ \frac{1}{4}(4 - 2u - v) & \text{if } 1 - u \leq 2v \leq 1 + 3u, \\ \frac{1}{12}(14 - 7v) & \text{if } \frac{1}{2}(1 + 3u) \leq v \leq 2, \end{cases}$$

$$\text{vol}(P(u)|_B - vC_0) = \begin{cases} 2 - 2u^2 + uv - v^2 - v & \text{if } 0 \leq v \leq \frac{1}{2}(1 - u), \\ \frac{1}{4}(1 - u)(7u - 8v + 9) & \text{if } \frac{1}{2}(1 - u) \leq v \leq 2u, \\ \frac{1}{4}(9 - 3u^2 + 4uv + v^2 - 2u - 8v) & \text{if } 1 - u \leq 2v \leq 1 + 3u, \\ \frac{7}{12}(2 - v)^2 & \text{if } \frac{1}{2}(1 + 3u) \leq v \leq 2. \end{cases}$$

Now, integrating $\text{vol}(P(u)|_B - vC_0)$ and $(P(u, v) \cdot C_0)^2$, we obtain $S(W_{\bullet, \bullet}^{B, C_0}; C_0) = \frac{13}{16}$ and

$$S(W_{\bullet, \bullet}^{B, C_0}; x) = \frac{77}{320} + F_x(W_{\bullet, \bullet}^{B, C_0}).$$

Furthermore, since $x \notin \mathbf{e}_0 \cup \mathbf{e}_1$ and the curves C_0 and C_1 intersect transversally at x , we have

$$\begin{aligned} F_x(W_{\bullet, \bullet}^{B, C_0}) &= \frac{3}{2} \int_0^1 \int_0^2 \text{ord}_x(N(u, v)|_{C_0}) \times (P(u, v) \cdot C_0) dv du \\ &= \frac{3}{2} \int_0^{\frac{1}{5}} \int_{\frac{1-u}{2}}^{\frac{1+3u}{2}} \frac{2v + u - 1}{2} \times (P(u, v) \cdot C_0) dv du + \frac{3}{2} \int_0^{\frac{1}{5}} \int_{\frac{1+3u}{2}}^2 \frac{4v - 2}{3} \times (P(u, v) \cdot C_0) dv du \end{aligned}$$

$$\begin{aligned}
 & + \frac{3}{2} \int_{\frac{1}{5}}^1 \int_{\frac{1-u}{2}}^{\frac{1+3u}{2}} \frac{2v+u-1}{2} \times (P(u,v) \cdot C_0) dvdu + \frac{3}{2} \int_{\frac{1}{5}}^1 \int_{\frac{1+3u}{2}}^2 \frac{4v-2}{3} \times (P(u,v) \cdot C_0) dvdu \\
 & = \frac{3}{2} \int_0^{\frac{1}{5}} \int_{\frac{1-u}{2}}^{\frac{1+3u}{2}} \frac{2v+u-1}{2} \times \frac{4-2u-v}{4} dvdu + \frac{3}{2} \int_0^{\frac{1}{5}} \int_{\frac{1+3u}{2}}^2 \frac{4v-2}{3} \times \frac{14-7v}{12} dvdu \\
 & + \frac{3}{2} \int_{\frac{1}{5}}^1 \int_{\frac{1-u}{2}}^{2u} \frac{2v+u-1}{2} \times (1-u) dvdu + \frac{3}{2} \int_{\frac{1}{5}}^1 \int_{2u}^{\frac{1+3u}{2}} \frac{2v+u-1}{2} \times \frac{4-2u-v}{4} dvdu \\
 & + \frac{3}{2} \int_{\frac{1}{5}}^1 \int_{\frac{1+3u}{2}}^2 \frac{4v-2}{3} \times \frac{14-7v}{12} dvdu = \frac{183}{320},
 \end{aligned}$$

which gives $S(W_{\bullet,\bullet,\bullet}^{B,C_0}; x) = \frac{13}{16}$. Thus, we see that $S(W_{\bullet,\bullet,\bullet}^{B,C_0}; x) = S(W_{\bullet,\bullet,\bullet}^B; C_0) = \frac{13}{16} < 1$, which gives a contradiction. This completes the proof of Theorem 3.1. \square

4. K-moduli compactification

As in previous sections, let X be a pure complete intersection $X_{2,2,4} \subset \mathbb{P}(1^5, 2^2)$, that is, a $(2, 2, 4)$ -complete intersection in $\mathbb{P}(1^5, 2^2)$ which is not a $(2, 4)$ -complete intersection in $\mathbb{P}(1^5, 2)$. For convenience, assume at no cost that X is given by $y_1^2 = y_0^2 - f(x_0, \dots, x_4)$, $g(x_0, \dots, x_4) = 0$, and $h(x_0, \dots, x_4) = 0$, where f , g , and h have degrees 4, 2, and 2, respectively. First, we make a slight improvement on the previous results using techniques of cyclic covers, cone construction, degeneration, and interpolation.

THEOREM 4.1. *Let $\Delta = \{f = 0, g = h = 0\} \subset \mathbb{P}^4$ and $S = \{g = h = 0\} \subset \mathbb{P}^4$. Assume that Δ is a smooth curve and S is a smooth surface. Then X is K-polystable.*

Proof. By forgetting y_1 in the defining equation of X , we obtain a double cover $X \rightarrow (Y, \frac{1}{2}D)$, where $Y = \{g = h = 0\} \subset \mathbb{P}(1^5, 2)$ and $D = \{y_0^2 = f\}|_Y$. In other words, $Y \cong C_p(S, \mathcal{O}_S(2))$. By rescaling the y_0 -coordinate only, it is easy to see that (Y, D) admits an isotrivial degeneration to (Y, D_0) , where $D_0 = \{y_0^2 = 0\}|_Y$ is twice the section at infinity, which we denote by $S_\infty = \frac{1}{2}D_0$. Since S is a smooth del Pezzo surface of degree 4, we know that S is K-stable. Hence by the cone construction [LL19, LX20, ZZ22], we know that $(Y, (1 - \frac{1}{3}r)S_\infty)$ is K-polystable for $\mathcal{O}_S(2) = -r^{-1}K_S$; that is, $r = \frac{1}{2}$. Hence $(Y, \frac{5}{12}D_0)$ is K-polystable, which implies that $(Y, \frac{5}{12}D)$ is K-semistable by openness. Since $D \sim \mathcal{O}_Y(4)$ and $-K_Y \sim \mathcal{O}_Y(3)$, we know that $(Y, \frac{3}{4}D)$ is a klt log Calabi–Yau pair as D is smooth and contained in the smooth locus of Y . Therefore, by interpolation of K-stability [ADL24, Proposition 2.13], we know that (Y, cD) is K-stable for every $c \in (\frac{5}{12}, \frac{3}{4})$. In particular, taking $c = \frac{1}{2}$ yields the K-stability of $(Y, \frac{1}{2}D)$, which implies the K-polystability of X by the cyclic cover result on K-polystability [Der16, LZ22, Zhu21]. \square

Next we study the K-moduli compactification of pure $(2, 2, 4)$ -complete intersections in $\mathbb{P}(1^5, 2^2)$. The following result is inspired by [ADL23b, Theorem 5.2].

PROPOSITION 4.2. *Let S be a log del Pezzo surface. Let $\Delta \sim -4K_S$ be an effective \mathbb{Q} -Cartier Weil divisor given by $\Delta = \{f = 0\}$ for $f \in H^0(S, -4K_S)$. Let $Y = C_p(S, -2K_S)$ be the projective cone. Let $D = \{y^2 = f\}$ be a divisor in Y , where $\{y = 0\}$ is the section at infinity of Y . Let $c \in (0, \frac{1}{4}) \cap \mathbb{Q}$. If $(S, c\Delta)$ is K-semistable (respectively, K-polystable), then $(Y, \frac{1}{12}(5 + 16c)D)$ is also K-semistable (respectively, K-polystable).*

Proof. The proof is similar to [ADL23b, proof of Theorem 5.2]. Write $a := \frac{1}{12}(5 + 16c)$ and $b := \frac{1}{12}(11 + 4c)$. Denote by $\tau: Y \rightarrow Y$ the involution preserving each ruling given by $\tau^*y = -y$. Let

$Z = C_p(S, -4K_S)$ be a new projective cone, where $S_1 = \{z = 0\}$ is the section at infinity. Then the quotient map of τ yields a double cover $\pi: Y \rightarrow Z$ branched along S_1 . Moreover, it is clear that D is a τ -invariant divisor whose quotient $S_2 = D/\tau$ is a section of Z such that $S_2|_{S_1} = \Delta$. Thus we have a crepant finite Galois morphism $\pi: (Y, aD) \rightarrow (Z, \frac{1}{2}S_1 + aS_2)$. By [LZ22, Zhu21], it suffices to show K-semistability (respectively, K-polystability) for $(Z, \frac{1}{2}S_1 + aS_2)$.

We first treat K-semistability. The natural \mathbb{G}_m -action on (Z, S_1) degenerates S_2 to $S_{2,0}$ as the cone over Δ . Let $r \in \mathbb{Q}_{>0}$ be chosen such that $-4K_S \sim_{\mathbb{Q}} r^{-1}(-K_S - c\Delta)$; that is, $r = \frac{1}{4}(1 - 4c)$. It follows from the cone construction [LX20, Proposition 5.3] (see also [LL19]) that $(Z, (1 - \frac{1}{3}r)S_1 + cS_{2,0})$ is K-semistable. Since $1 - \frac{1}{3}r = \frac{1}{12}(11 + 4c) = b$, we know that $(Z, bS_1 + cS_2)$ is K-semistable by the openness of K-semistability [BLX22, Xu20]. By the symmetry between S_1 and S_2 , we know that $(Z, cS_1 + bS_2)$ is also K-semistable. Since $(\frac{1}{2}, a)$ is a convex combination of (c, b) and (b, c) , we know that $(Z, \frac{1}{2}S_1 + aS_2)$ is K-semistable.

The K-polystable part follows from a similar argument to [ADL23b, proof of Theorem 5.2]. We include a proof here for the readers' convenience. By [LZ22, Zhu21] it suffices to show the K-polystability of $(Z, \frac{1}{2}S_1 + aS_2)$. By [LWX21] there exists a special test configuration $(\mathcal{Z}, \frac{1}{2}S_1 + aS_2)$ of $(Z, \frac{1}{2}S_1 + aS_2)$ whose central fibre $(Z', \frac{1}{2}S'_1 + aS'_2)$ is K-polystable. Therefore, $\text{Fut}(\mathcal{Z}, \frac{1}{2}S_1 + aS_2) = 0$. By the linearity of Futaki invariants and the K-semistability of $(Z, bS_1 + cS_2)$ and $(Z, cS_1 + bS_2)$, we know that

$$\text{Fut}(Z, bS_1 + cS_2) = \text{Fut}(Z, cS_1 + bS_2) = 0.$$

By [LWX21, Lemma 3.1] we know that both $(Z', bS'_1 + cS'_2)$ and $(Z', cS'_1 + bS'_2)$ are K-semistable. Since $(S, c\Delta)$ is K-polystable, we know that $(Z, bS_1 + cS_{2,0})$ is the K-polystable degeneration of $(Z, bS_1 + cS_2)$ by [LL19, LX20, LXZ22]. Thus by [LWX21] we have a sequence of special degenerations of K-semistable log Fano pairs

$$(Z, bS_1 + cS_2) \rightsquigarrow (Z', bS'_1 + cS'_2) \rightsquigarrow (Z, bS_1 + cS_{2,0}).$$

This implies that $(Z, S_1) \cong (Z', S'_1)$. By the symmetry between S_1 and S_2 , we also have $(Z, S_2) \cong (Z', S'_2)$. Moreover, restricting the above degeneration sequence to S_1 yields that $(S_1, S_2|_{S_1}) \cong (S'_1, S'_2|_{S'_1})$. Since S_1 and S_2 are two sections of the projective cone Z , we conclude that $(Z, S_1 + S_2) \cong (Z, S_1 + S_2)$, which implies that $(Z, \frac{1}{2}S_1 + aS_2)$ is K-polystable. \square

From Proposition 4.2, we see that if $c = \frac{1}{16}$, then $\frac{1}{12}(5 + 16c) = \frac{1}{2}$, which precisely gives the K-moduli of the double cover X of $(Y, \frac{1}{2}D)$. In other words, let \mathfrak{M}_c be the K-moduli space of $(S, c\Delta)$, where S is a \mathbb{Q} -Gorenstein smoothable del Pezzo surface of degree 4 and $\Delta \sim -4K_S$. In the convention of [ADL24], the K-moduli space is defined by $\mathfrak{M}_c := KM_{\chi_0, 4, c}$, where χ_0 is the Hilbert polynomial of an anti-canonically polarised smooth del Pezzo surface of degree 4. From the following result we see that there exists a finite injective morphism $\psi: \mathfrak{M}_{1/16} \rightarrow M_{3,4}^{\text{Kps}}$ whose image is precisely the closure of the locus parametrising pure $(2, 2, 4)$ -complete intersections $X_{2,2,4} \subset \mathbb{P}(1^5, 2^2)$. Thus to describe the compactification of the K-moduli of pure $(2, 2, 4)$ -complete intersections in $\mathbb{P}(1^5, 2^2)$, it suffices to describe the K-moduli space $\mathfrak{M}_{1/16}$ of $(S, \frac{1}{16}\Delta)$ consisting of a del Pezzo surface S of degree 4 and $\Delta \sim -4K_S$. As a consequence, the converse of Proposition 4.2 also holds.

THEOREM 4.3. *There exists a finite injective morphism $\psi: \mathfrak{M}_{1/16} \rightarrow M_{3,4}^{\text{Kps}}$ whose image is the closure of the locus parametrising pure $(2, 2, 4)$ -complete intersections $X_{2,2,4} \subset \mathbb{P}(1^5, 2^2)$.*

Proof. Let $\mathcal{M}_c := \mathcal{KM}_{\chi_0, 4, c}$ be the K-moduli stack defined in [ADL24] whose good moduli space is \mathfrak{M}_c . We fix $c = \frac{1}{16}$ throughout the proof. Let $\pi: (\mathcal{S}, c\Delta_{\mathcal{S}}) \rightarrow \mathcal{M}_c$ be the universal family of

K-semistable pairs. We will construct a family of K-semistable \mathbb{Q} -Fano 3-folds of volume 4 over an algebraic stack \mathcal{W} which is a composition of two μ_2 -gerbes over \mathcal{M}_c . This gives us a morphism of stacks $\Psi: \mathcal{W} \rightarrow \mathcal{M}_{3,4}^{\text{Kss}}$, where the target denotes the K-moduli stack of K-semistable \mathbb{Q} -Fano 3-folds of volume 4 (see for example [Xu21]). Then ψ is obtained by descent to good moduli spaces. The construction of such a family has three steps.

Firstly, we construct the projective cone $\mathcal{Z} \rightarrow \mathcal{M}_c$. Let $\mathcal{L} := \mathcal{O}_S(\Delta_S)$. Then we know that \mathcal{L} is a line bundle by Proposition 4.4. Let $\mathcal{Z} := \text{Proj}_{\mathcal{M}_c} \mathcal{R}[t]$, where $\mathcal{R} = \bigoplus_{m=0}^{\infty} \pi_* \mathcal{L}^{\otimes m}$. Here $\pi_* \mathcal{L}^{\otimes m}$ and t have degrees m and 1, respectively. Let $\mathcal{S}_1 := \{t = 0\}$ and $\mathcal{S}_2 := \{t - s = 0\}$ be two Cartier divisors in \mathcal{Z} , where $s \in H^0(\mathcal{M}_c, \pi_* \mathcal{L})$ is a section defining Δ_S .

Secondly, we construct a double cover \mathcal{Y} of \mathcal{Z} branched along \mathcal{S}_1 after base change to a μ_2 -gerbe. Our argument is similar to [ADL23b, proof of Proposition 6.12]. Let $\mathcal{G} := K_{\mathcal{Z}/\mathcal{M}_c} + \mathcal{S}_1$. Then we know that \mathcal{G} is a \mathbb{Q} -Cartier Weil divisor such that $-4\mathcal{G}_t \sim (\mathcal{S}_1)_t$ for every $t \in |\mathcal{M}_c|$. Hence $\mathcal{O}_{\mathcal{Z}}(4\mathcal{G} + \mathcal{S}_1)$ is a trivial line bundle along each fibre \mathcal{Z}_t , which implies that it descends to a line bundle \mathcal{F} over \mathcal{M}_c . Let $\phi_{\mathcal{Y}}: \mathcal{Y} \rightarrow \mathcal{M}_c$ be the μ_2 -gerbe obtained as the second root stack of \mathcal{F} . Hence there exists a line bundle $\mathcal{F}'_{\mathcal{Y}}$ on \mathcal{Y} such that $\mathcal{F}'_{\mathcal{Y}}^{\otimes 2} \cong \phi_{\mathcal{Y}}^* \mathcal{F}$. Let $\pi_{\mathcal{Y}}: (\tilde{\mathcal{Z}}, \tilde{\mathcal{S}}_1 + \tilde{\mathcal{S}}_2) \rightarrow \mathcal{Y}$ be the base change of $(\mathcal{Z}, \mathcal{S}_1 + \mathcal{S}_2) \rightarrow \mathcal{M}_c$ to \mathcal{Y} . Denote by $\tilde{\mathcal{G}}$ the pullback of \mathcal{G} to $\tilde{\mathcal{Z}}$. Then $\mathcal{N} := \pi_{\mathcal{Y}}^* \mathcal{F}'_{\mathcal{Y}} \otimes \mathcal{O}_{\tilde{\mathcal{Z}}}(-2\tilde{\mathcal{G}})$ is a \mathbb{Q} -Cartier divisorial sheaf such that $\mathcal{N}^{[2]} \cong \mathcal{O}_{\tilde{\mathcal{Z}}}(\tilde{\mathcal{S}}_1)$. We take the double cover of $\tilde{\mathcal{Z}}$ branched along $\tilde{\mathcal{S}}_1$,

$$\mathcal{Y} := \text{Spec}_{\tilde{\mathcal{Z}}} \mathcal{O}_{\tilde{\mathcal{Z}}} \oplus \mathcal{N}^{[-1]},$$

where the $\mathcal{O}_{\tilde{\mathcal{Z}}}$ -algebra structure is induced by $\mathcal{N}^{[-2]} \xrightarrow{\tilde{s}} \mathcal{O}_{\tilde{\mathcal{Z}}}$, where \tilde{s} is a section of $\mathcal{N}^{[2]}$ such that $\{\tilde{s} = 0\} = \tilde{\mathcal{S}}_1$. Let \mathcal{D} be the pullback of \mathcal{S}_2 to \mathcal{Y} .

Lastly, we construct a double cover \mathcal{X} of \mathcal{Y} branched along \mathcal{D} after base change to a μ_2 -gerbe. The construction is similar to the previous step, so we omit the details. In the end, we obtain a μ_2 -gerbe $\mathcal{W} \rightarrow \mathcal{Y}$ such that we can take a double cover \mathcal{X} of \mathcal{Y} branched along \mathcal{D} , where $(\mathcal{Y}, \mathcal{D}) = (\mathcal{Y}, \mathcal{D}) \times_{\mathcal{Y}} \mathcal{W}$. By Kawamata–Viehweg vanishing, the above construction commutes with base change, and hence the family $\mathcal{X} \rightarrow \mathcal{W}$ has K-semistable fibres by Proposition 4.2, where a general fibre is a pure $(2, 2, 4)$ -complete intersection in $\mathbb{P}(1^5, 2^2)$. Thus we obtain a morphism $\Psi: \mathcal{W} \rightarrow \mathcal{M}_{3,4}^{\text{Kss}}$. Moreover, \mathcal{W} admits a good moduli space isomorphic to \mathfrak{M}_c by [Alp13] (see also [ADL23b, proof of Proposition 6.12]). Hence Ψ descends to a morphism between good moduli spaces $\psi: \mathfrak{M}_{1/16} \rightarrow M_{3,4}^{\text{Kps}}$ by [Alp13, Theorem 6.6]. It remains to show that ψ is a finite injective morphism. Since $\mathfrak{M}_{1/16}$ is proper, it suffices to show that ψ is injective on \mathbb{C} -points.

Suppose $\psi(S, \Delta) \cong \psi(S', \Delta')$. To show that ψ is injective, it suffices to show that $(S, \Delta) \cong (S', \Delta')$. By Proposition 4.4, we know that both S and S' are del Pezzo surfaces of degree 4 with Du Val singularities. Thus S and S' are both $(2, 2)$ -complete intersections in \mathbb{P}^4 , and we may write

$$(S, \Delta) = (\{g = h = 0\}, \{f = 0\}|_S) \quad \text{and} \quad (S', \Delta') = (\{g' = h' = 0\}, \{f' = 0\}|_{S'}),$$

where g, h, g', h' (respectively, f, f') are homogeneous polynomials of degree 2 (respectively, degree 4) in $(x_0, x_1, x_2, x_3, x_4)$. Let $X = \psi(S, \Delta)$ and $X' = \psi(S', \Delta')$. Then we have $X \cong X'$ by assumption. Moreover, we know that $X = \{g = h = y_0 y_1 - f = 0\}$ and $X' = \{g' = h' = y_0 y_1 - f' = 0\}$ are pure $(2, 2, 4)$ -complete intersections in $\mathbb{P}(1^5, 2^2)$. Since the embedding $X \hookrightarrow \mathbb{P}(1^5, 2^2)$ is induced by the linear systems $|-K_X|$ and $|-2K_X|$, the coordinates (x_0, \dots, x_4) form a basis of $H^0(X, -K_X)$, and the coordinates (y_0, y_1) are liftings of a basis of $\text{coker}(\text{Sym}^2 H^0(X, -K_X) \rightarrow H^0(X, -2K_X))$, we know that X and X' are projectively equivalent as weighted complete intersections. Thus under a change of coordinates $\sigma \in \text{Aut}(\mathbb{P}(1^5, 2^2))$,

we have $X' = \sigma^* X$. We may decompose $\sigma = \sigma_x \circ \sigma_y$, where σ_x is linear in x_i and fixes y_0 and y_1 , and σ_y fixes each x_i . Then a simple analysis of the transformation of equations under σ^* shows that $(S', \Delta') = \sigma_x^*(S, \Delta)$, where we treat σ_x as an element in $\mathrm{PGL}_5(\mathbb{C})$. Thus the proof is finished. \square

The next result bounds the singularities of surfaces appearing in \mathfrak{M}_c .

PROPOSITION 4.4. *Let $[(S, c\Delta)] \in \mathfrak{M}_c$ be a K -polystable pair for some $c \in (0, \frac{1}{16}]$. Then either S is smooth, or S has only A_1 -, A_2 -, or A_3 -singularities. If in addition S has an A_3 -singularity, then we have*

$$c = \frac{1}{16}, \quad S \cong \{y_3^2 = y_2 y_4\} \subset \mathbb{P}(1, 2, 3, 4)_{[y_1, y_2, y_3, y_4]}, \quad \text{and} \quad \Delta = \{y_4^4 = 0\}.$$

Remark. Note that the surface S in Proposition 4.4 is №25 from [CP21, Big Table] (see also Table 1).

Proof. We know that

$$(-K_S - c\Delta)^2 = (1 - 4c)^2(-K_S)^2 = 4(1 - 4c)^2.$$

Let $x \in S$ be a singular point with local orbifold group G_x . The local-global volume comparison [LL19] implies

$$\frac{4}{|G_x|} = \widehat{\mathrm{vol}}(x, S) \geq \widehat{\mathrm{vol}}(x, S, c\Delta) \geq \frac{4}{9}(-K_S - c\Delta)^2 = \frac{16}{9}(1 - 4c)^2 \geq 1. \quad (4.1)$$

Thus we have that $|G_x| \leq 4$. This, together with the T -singularity condition (see [KS88]), implies that $x \in S$ is A_1 , A_2 , A_3 , or of type $\frac{1}{4}(1, 1)$.

Next, we rule out the case of type $\frac{1}{4}(1, 1)$. Assume to the contrary that $x \in S$ has type $\frac{1}{4}(1, 1)$. Since $|G_x| = 4$, every inequality from (4.1) is an equality. Thus we have $x \notin \mathrm{Supp}(\Delta)$, $c = \frac{1}{16}$, and the inequality in the local-global volume comparison from [Liu18, LL19] is an equality. By [Liu18, Remark 38(1)] we have that $S \cong \mathbb{P}^2/\mu_4$, where the μ_4 -action on \mathbb{P}^2 has weight $(1, 1, 0)$ and $\frac{1}{16}\Delta$ corresponds to the orbifold divisor of \mathbb{P}^2/μ_4 . In particular, we have $S \cong \mathbb{P}(1, 1, 4)$, which implies that $(K_S^2) = 9 \neq 4$, so we have a contradiction. Thus singularities of type $\frac{1}{4}(1, 1)$ cannot appear.

Finally, we consider the case where $x \in S$ has type A_3 . Similarly to the previous case of type $\frac{1}{4}(1, 1)$, we have $x \notin \mathrm{Supp}(\Delta)$, $c = \frac{1}{16}$. Moreover, $S \cong \mathbb{P}^2/\mu_4$, where the μ_4 -action on \mathbb{P}^2 has weight $(1, 3, 0)$ and $\frac{1}{16}\Delta$ corresponds to the orbifold divisor of \mathbb{P}^2/μ_4 . Suppose given a generator g of μ_4 acting on $\mathbb{P}_{[x, y, z]}^2$ as

$$g \cdot [x, y, z] = [ix, -iy, z] = [-x, y, iz].$$

Denote by $R = \bigoplus_{m=0}^{\infty} H^0(\mathbb{P}^2, \mathcal{O}(m))$ the graded section ring of $(\mathbb{P}^2, \mathcal{O}(1))$. If we pick the latter μ_4 -linearisation on $(\mathbb{P}^2, \mathcal{O}(1))$ of weight $(2, 0, 1)$, then the μ_4 -invariant subring R^{μ_4} is generated by y, x^2, xz^2, z^4 . Thus we have

$$\mathbb{P}^2/\mu_4 \cong \mathrm{Proj} R^{\mu_4} \cong (\{y_3^2 = y_2 y_4\} \subset \mathbb{P}(1, 2, 3, 4)),$$

where the generators correspond to $y_1 = y, y_2 = x^2, y_3 = xz^2$, and $y_4 = z^4$. Since the μ_4 -action has one orbifold curve given by $(z = 0)$ of order 2, whose reduced image in \mathbb{P}^2/μ_4 corresponds to $\frac{1}{2}(y_4 = 0)$, we have $\frac{1}{16}\Delta = \frac{1}{2} \cdot \frac{1}{2}(y_4 = 0)$, which implies that $\Delta = \{y_4^4 = 0\}$. It is easy to check that $(K_S^2) = 4$ and $\Delta \sim -4K_S$; hence $[(S, \frac{1}{16}\Delta)] \in \mathfrak{M}_{1/16}$. The proof is finished. \square

COROLLARY 4.5. *The K -moduli compactification of pure $(2, 2, 4)$ -complete intersections $\mathbb{P}(1^5, 2^2)$ only consists of complete intersections of the same type.*

Proof. By Proposition 4.4 we see that every pair (S, Δ) in $\mathfrak{M}_{1/16}$ satisfies that S has Du Val singularities. Hence $S = \{g = h = 0\}$ is always a complete intersection of two quadrics in \mathbb{P}^4 and $\Delta = \{f = 0\}|_S$, which implies that $Y = C_p(S, -2K_S) = \{g = h = 0\}$ is a complete intersection in $\mathbb{P}(1^5, 2)$. Hence $\psi(S, \Delta)$ is a complete intersection $X_{2,2,4} \subset \mathbb{P}(1^5, 2^2)$ given by $\{y_0 y_1 = f\}$ and $\{g = h = 0\}$. Thus the statement follows from Theorem 4.3. \square

5. Wall crossing

In this section we study the explicit wall crossings for the K -moduli spaces \mathfrak{M}_c where $c \in (0, \frac{1}{16}]$. Similar problems were studied in [ADL24, ADL23a, ADL23b, Pap22, Zha24].

Let us recall the convention from [ADL24]. A *wall* $c = c_i$ of the K -moduli spaces \mathfrak{M}_c is a rational value such that K -polystability changes from $c = c_i - \epsilon$ to $c = c_i + \epsilon$ for $0 < \epsilon \ll 1$. By [ADL24, Theorems 1.1 and 1.2] there are finitely many walls $0 < c_1 < c_2 < \dots < c_k \leq \frac{1}{16}$ for the K -moduli spaces \mathfrak{M}_c with $c \in (0, \frac{1}{16}]$. Moreover, we have wall crossing diagrams of birational morphisms

$$\mathfrak{M}_{c_i - \epsilon} \longrightarrow \mathfrak{M}_{c_i} \longleftarrow \mathfrak{M}_{c_i + \epsilon}.$$

A K -polystable pair $(S, c_i \Delta)$ in \mathfrak{M}_{c_i} is called a *new K -polystable pair on the wall c_i* if $(S, c \Delta)$ is K -unstable whenever $c \neq c_i$. By [ADL24] we know that every new K -polystable pair on a wall admits an effective \mathbb{G}_m -action, hence is not K -stable; on the other hand, every wall admits a new K -polystable pair.

In addition to walls in $(0, \frac{1}{16}]$, there is an initial wall crossing at $c_0 = 0$, where we will have a fibre-type contraction $\mathfrak{M}_\epsilon \rightarrow \mathfrak{M}_0$. Here \mathfrak{M}_0 is the K -moduli space of \mathbb{Q} -Gorenstein smoothable del Pezzo surfaces of degree 4, which is literally the K -moduli space of $(S, 0\Delta)$. By [MM93, OSS16] we know that \mathfrak{M}_0 is isomorphic to the GIT moduli space of complete intersections of two quadric hypersurfaces in \mathbb{P}^4 . Note that this initial wall crossing was not explicitly stated in [ADL24], although the techniques therein are sufficient to deduce such a result.

THEOREM 5.1. *There are four walls for $c \in (0, \frac{1}{16}]$ for the K -moduli spaces \mathfrak{M}_c : $c_1 = \frac{1}{28}$, $c_2 = \frac{1}{22}$, $c_3 = \frac{1}{19}$, $c_4 = \frac{1}{16}$. The new K -polystable pairs $(S, c_i \Delta)$ on the walls are completely classified in Proposition 4.4 and in Theorems 5.2 and 5.8, where S is isomorphic to $\mathbb{N}^\circ 25$, 29, and 33 in Table 1, respectively. Moreover, the initial wall crossing at $c_0 = 0$ gives a surjective morphism $\mathfrak{M}_\epsilon \rightarrow \mathfrak{M}_0$ with connected fibres whose general fibre is isomorphic to \mathbb{P}^{40}/μ_2^4 .*

To find all walls for the K -moduli spaces \mathfrak{M}_c , where $c \in (0, \frac{1}{16}]$, we need to classify new K -polystable pairs $(S, c \Delta)$, that is, the pairs that are K -polystable but not K -stable, and where S is K -unstable. This way we can obtain a complete list of walls and new K -polystable pairs on the walls.

By Proposition 4.4 we know that either S is a K -unstable del Pezzo surface of degree 4 with at worst A_2 singularities that admits a \mathbb{G}_m -action, or S is isomorphic to $\mathbb{N}^\circ 25$. The following table from [CP21, Big Table] classifies del Pezzo surfaces of degree 4 with at worst A_2 -singularities that admit a \mathbb{G}_m -action, with an additional row of $\mathbb{N}^\circ 25$ which shows up in $\mathfrak{M}_{1/16}$ only.

Here \mathbb{B}_2 represents the Borel subgroup of $\mathrm{PGL}_2(\mathbb{C})$.

From Table 1 and [MM93, OSS16], we see that $\mathbb{N}^\circ 25$, 29, 33 are K -unstable, while $\mathbb{N}^\circ 30$, 35 are K -polystable, and $\mathbb{N}^\circ 34$ is K -semistable but not K -polystable.

TABLE 1. Del Pezzo surfaces of degree 4 with \mathbb{G}_m -actions and at worst A_2 -singularities, together with №25.

| № | ρ | Sing. | Aut^0 | Equation and ambient space |
|----|--------|------------|------------------------------------|---|
| 29 | 2 | $A_2 2A_1$ | \mathbb{G}_m^2 | $\{y_2 y'_2 = y_1^3 y'_1\} \subset \mathbb{P}(1, 1, 2, 2)$ |
| 30 | 2 | $4A_1$ | \mathbb{G}_m^2 | $\{y_2 y'_2 = y_1^2 y_1'^2\} \subset \mathbb{P}(1, 1, 2, 2)$ |
| 33 | 3 | $A_2 A_1$ | \mathbb{G}_m | $\{x_0 x_1 - x_2 x_3 = x_1 x_2 + x_2 x_4 + x_3 x_4 = 0\} \subset \mathbb{P}^4$ |
| 34 | 3 | $3A_1$ | \mathbb{G}_m | $\{y_2 y'_2 = y_1^2 y'_1 (y_1 + y'_1)\} \subset \mathbb{P}(1, 1, 2, 2)$ |
| 35 | 4 | $2A_1$ | \mathbb{G}_m | $\{y_2 y'_2 = y_1 y'_1 (y'_1 - y_1)(y'_1 - \lambda y_1)\} \subset \mathbb{P}(1, 1, 2, 2)$ for $\lambda \in \mathbb{C} \setminus \{0, 1\}$ |
| 25 | 1 | $A_3 2A_1$ | $\mathbb{B}_2 \times \mathbb{G}_m$ | $\{y_3^2 = y_2 y_4\} \subset \mathbb{P}(1, 2, 3, 4)$ |

From now on, we study the walls from \mathfrak{M}_ϵ to $\mathfrak{M}_{1/16}$ which involve only №25, 29, or 33. Since there is only one wall at $c = \frac{1}{16}$ involving №25 by Proposition 4.4, we will focus on walls involving №29 or 33.

5.1 Surface №29

Here $S = \{y_2 y'_2 = y_1^3 y'_1\} \subset \mathbb{P}(1, 1, 2, 2)_{[y_1, y'_1, y_2, y'_2]}$.

THEOREM 5.2. *There are precisely four walls in $(0, \frac{1}{16}]$ involving S from №29 such that $(S, c_i \Delta_i)$ is a new K -polystable pair for the following choices of c_i and Δ_i :*

- (1) $c_1 = \frac{1}{28}$, $\Delta_1 = \{y_1'^8 = 0\}$;
- (2) $c_2 = \frac{1}{22}$, $\Delta_2 = \{y_1 y_1'^7 = 0\}$;
- (3) $c_3 = \frac{1}{19}$, $\Delta_3 = \{(y_2 + y'_2) y_1'^6 = 0\}$;
- (4) $c_4 = \frac{1}{16}$, $\Delta_4 = \Delta_{4,\mathbf{a}}$ or $\Delta_{4,\infty}$, where $\Delta_{4,\mathbf{a}} = \{y_2 y_1'^6 + a_1 y_1'^2 y_1'^6 + a_2 y_1 y_2' y_1'^5 + a_3 y_2'^2 y_1'^4 = 0\}$ for $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{C} \times (\mathbb{C}^2 \setminus \{(0, 0)\})$ and $\Delta_{4,\infty} = \{y_1^2 y_1'^6 = 0\}$.

We first show that these pairs are K -polystable. The fact that S is toric follows from

$$(s, t) \cdot [y_1, y'_1, y_2, y'_2] = [s y_1, s^{-3} y'_1, t y_2, t^{-1} y'_2].$$

It is clear that Δ_i for $i \leq 2$ is toric. Thus it suffices to show that the Futaki character satisfies $\text{Fut}_{(S, c_i \Delta_i)} = 0$. In addition, S admits an involution $\tau: S \rightarrow S$ given by $\tau([y_1, y'_1, y_2, y'_2]) = [y_1, y'_1, y'_2, y_2]$.

Set $C = \{y'_1 = y_2 = 0\}$, $C' = \{y'_1 = y'_2 = 0\}$, $B = \{y_1 = y_2 = 0\}$, $B' = \{y_1 = y'_2 = 0\}$. The intersection numbers of these curves are summarised in Table 2.

LEMMA 5.3. *The Mori cone $\overline{\text{NE}}(S)$ is generated by $[C]$ and $[C']$.*

Proof. From the torus action on S , we see that there are precisely four torus-invariant divisors: C , C' , B , and B' . Thus $\overline{\text{NE}}(S)$ is generated by these four curves. From Table 2 we see that both B and B' are nef and big, while both C and C' are extremal. Thus the proof is finished. \square

We first compute the S -invariants of these curves.

TABLE 2. Intersection numbers of the surface №29.

| • | C | C' | B | B' |
|------|----------------|----------------|---------------|---------------|
| C | $-\frac{1}{2}$ | 1 | $\frac{1}{2}$ | 0 |
| C' | 1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ |
| B | $\frac{1}{2}$ | 0 | $\frac{1}{6}$ | $\frac{1}{3}$ |
| B' | 0 | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{6}$ |

PROPOSITION 5.4. *Under the above notation, we have*

$$S_S(C) = S_S(C') = \frac{5}{6}, \quad S_S(B) = S_S(B') = \frac{7}{6}.$$

Proof. First of all, the involution τ satisfies $\tau(C) = C'$ and $\tau(B) = B'$. Thus we have equalities $S_S(C) = S_S(C')$ and $S_S(B) = S_S(B')$.

Next, we compute $S_S(C)$. It is clear that $-K_S = \mathcal{O}_S(2) = 2(C + C')$. This gives

$$S_S(C) = \frac{1}{4} \int_0^\infty \text{vol}_S((2-t)C + 2C') dt.$$

It follows from Table 2 and Lemma 5.3 that $(2-t)C + 2C'$ is nef if $0 \leq t \leq 1$ and not big if $t \geq 2$. For $1 \leq t \leq 2$ the Zariski decomposition is

$$(2-t)C + 2C' = P + N = (2-t)(C + 2C') + (2t-2)C'.$$

Thus computation shows that

$$\text{vol}((2-t)C + 2C') = \begin{cases} 4 - 2t - \frac{t^2}{2} & \text{if } 0 \leq t \leq 1, \\ \frac{3}{2}(2-t)^2 & \text{if } 1 \leq t \leq 2. \end{cases}$$

Thus

$$S_S(C) = \frac{1}{4} \left(\int_0^1 \left(4 - 2t - \frac{t^2}{2} \right) dt + \int_1^2 \frac{3}{2}(2-t)^2 dt \right) = \frac{5}{6}.$$

Finally, we compute $S_S(B)$. It is clear that $-K_S = \mathcal{O}_S(2) = 2(B + B')$. This gives

$$S_S(B) = \frac{1}{4} \int_0^\infty \text{vol}_S((2-t)B + 2B') dt.$$

Moreover, by Table 2 and Lemma 5.3, we know that $(2-t)B + 2B'$ is nef if $0 \leq t \leq 2$ and not big if $t \geq 3$. For $2 \leq t \leq 3$ the Zariski decomposition is

$$(2-t)B + 2B' = P + N = (6-2t)B' + (t-2)C.$$

Thus computation shows that

$$\text{vol}((2-t)B + 2B') = \begin{cases} 4 - 2t + \frac{1}{6}t^2 & \text{if } 0 \leq t \leq 2, \\ \frac{2}{3}(3-t)^2 & \text{if } 2 \leq t \leq 3. \end{cases}$$

Thus

$$S_S(B) = \frac{1}{4} \left(\int_0^2 \left(4 - 2t + \frac{t^2}{6} \right) dt + \int_2^3 \frac{2}{3}(3-t)^2 dt \right) = \frac{7}{6}. \quad \square$$

PROPOSITION 5.5. *The pairs $(S, c_i \Delta_i)$ from Theorem 5.2 are K -polystable and not K -stable.*

Proof. We first look at cases (1) and (2). For these two cases $(S, c_i \Delta_i)$ is toric hence not K-stable. Thus we just need to show that the Futaki character satisfies $\text{Fut}_{(S, c_i \Delta_i)} = 0$. Since the one-parameter subgroup induced by ord_C , $\text{ord}_{C'}$, ord_B , and ord'_B generates the torus, it suffices to show that $\beta_{(S, c_i \Delta_i)}(v) = 0$, where v is one of the four valuations. By the symmetry of τ , it suffices to show that $\beta_{(S, c_i \Delta_i)}(C) = \beta_{(S, c_i \Delta_i)}(B) = 0$.

Since $\Delta_i \equiv -4K_S$, we know that $A_{(S, c_i \Delta_i)}(v) = A_S(v) - c_i v(\Delta_i)$ and that $S_{(S, c_i \Delta_i)}(v) = (1 - 4c_i)S_S(v)$. Thus

$$\beta_{(S, c_i \Delta_i)}(v) = A_S(v) - c_i v(\Delta_i) - (1 - 4c_i)S_S(v).$$

(1) Since $c_1 = \frac{1}{28}$ and $\Delta_1 = (y_1'^8 = 0)$, we have

$$\begin{aligned} \beta_{(S, c_1 \Delta_1)}(C) &= 1 - \frac{8}{28} - \left(1 - \frac{4}{28}\right) \frac{5}{6} = 0, \\ \beta_{(S, c_1 \Delta_1)}(B) &= 1 - \left(1 - \frac{4}{28}\right) \frac{7}{6} = 0. \end{aligned}$$

(1) Since $c_2 = \frac{1}{22}$ and $\Delta_2 = (y_1 y_1'^7 = 0)$, we have

$$\begin{aligned} \beta_{(S, c_2 \Delta_2)}(C) &= 1 - \frac{7}{22} - \left(1 - \frac{4}{22}\right) \frac{5}{6} = 0, \\ \beta_{(S, c_2 \Delta_2)}(B) &= 1 - \frac{1}{22} - \left(1 - \frac{4}{22}\right) \frac{7}{6} = 0. \end{aligned}$$

Next, we look at case (3). In this case, (S, Δ_3) admits a \mathbb{G}_m -action σ given by

$$s \cdot [y_1, y_1', y_2, y_2'] = [sy_1, s^{-3}y_1', y_2, y_2'].$$

Thus $(S, c_3 \Delta_3)$ is not K-stable. In the affine chart $U := \{y_1 = 1\} \subset S$, we have an isomorphism $U \cong \mathbb{A}^2$ given by $[1, y_1', y_2, y_2'] \mapsto (y_2, y_2')$, where $C|_U$ and $C'|_U$ correspond to $(y_2 = 0)$ and $(y_2' = 0)$, respectively. If we write $v_0 = \text{ord}_{[1, 0, 0, 0]}$, then v_0 is the quasi-monomial combination of ord_C and $\text{ord}_{C'}$ of weight $(1, 1)$ (see for example [JM12] for a definition). Thus we have

$$\beta_S(v_0) = \beta_S(C) + \beta_S(C') = 2\beta_S(C) = 2(1 - S_S(C)) = \frac{1}{3}.$$

It is clear that v_0 induces the \mathbb{G}_m -action σ . Thus by [Liu23, Theorem 3.2] (cf. [ACC⁺23, Theorem 1.3.9] and [IS17]), it suffices to show that $\beta_{(S, c_3 \Delta_3)}(v_0) = 0$ and that $\beta_{(S, c_3 \Delta_3)}(D) > 0$ for every vertical prime divisor D on S .

We first compute the β -invariant of v_0 . Since $\beta_S(v_0) = \frac{1}{3}$ and $A_S(v_0) = 2$, we know that $S_S(v_0) = \frac{5}{3}$. Thus

$$\beta_{(S, c_3 \Delta_3)}(v_0) = 2 - c_3 v_0(\Delta_3) - (1 - 4c_3)S_S(v_0) = 2 - \frac{13}{19} - \left(1 - \frac{4}{19}\right) \frac{5}{3} = 0.$$

Next we compute the β -invariants of the vertical divisors D . A simple analysis of the σ -action on S shows that D is one of the following:

- (i) C or C' ,
- (ii) B or B' ,
- (iii) $D_a = \{y_2 + ay_2' = 0\}$ for $a \neq 0$.

Next, we split into these three cases:

(i) Since Δ_3 is invariant under the involution τ , we have

$$\beta_{(S, c_3 \Delta_3)}(C) = \beta_{(S, c_3 \Delta_3)}(C') = 1 - \frac{6}{19} - \left(1 - \frac{4}{19}\right) \frac{5}{6} = \frac{1}{38} > 0.$$

(ii) Similarly, we have

$$\beta_{(S, c_3 \Delta_3)}(B) = \beta_{(S, c_3 \Delta_3)}(B') = 1 - \left(1 - \frac{4}{19}\right) \frac{7}{6} = \frac{3}{38} > 0.$$

(iii) We know that $\text{ord}_{D_a}(\Delta_3) \leq 1$. Moreover, we know that $-K_S - tD_a = \mathcal{O}_S(2 - 2t)$, which is nef if $0 \leq t \leq 1$ and not big if $t \geq 1$. Hence

$$S_S(D_a) = \frac{1}{4} \int_0^1 (2 - 2t)^2 dt = \frac{1}{3}.$$

Thus

$$\beta_{(S, c_3 \Delta_3)}(D_a) \geq 1 - \frac{1}{19} - \left(1 - \frac{4}{19}\right) \frac{1}{3} = \frac{13}{19} > 0.$$

Finally, we look at case (4). We first treat the case of $\Delta_{4, \infty}$. It is clear that $\Delta_{4, \infty} = \{y_1^2 y_1' = 0\}$ is toric. Hence $(S, c_4 \Delta_{4, \infty})$ is not K-stable. Since $c_4 = \frac{1}{16}$, we have

$$\begin{aligned} \beta_{(S, c_4 \Delta_{4, \infty})}(C) &= 1 - \frac{6}{16} - \left(1 - \frac{4}{16}\right) \frac{5}{6} = 0, \\ \beta_{(S, c_4 \Delta_{4, \infty})}(B) &= 1 - \frac{2}{16} - \left(1 - \frac{4}{16}\right) \frac{7}{6} = 0. \end{aligned}$$

This shows that $(S, c_4 \Delta_{4, \infty})$ is K-polystable.

Next, we treat the case of $\Delta_{4, \mathbf{a}}$ for $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{C} \times (\mathbb{C}^2 \setminus \{(0, 0)\})$. Since $\Delta_{4, \mathbf{a}} = \{y_2 y_1'^6 + a_1 y_1^2 y_1'^6 + a_2 y_1 y_2' y_1'^5 + a_3 y_2'^2 y_1'^4 = 0\}$, we know that $(S, \Delta_{4, \mathbf{a}})$ admits a \mathbb{G}_m -action σ' given by

$$t \cdot [y_1, y_1', y_2, y_2'] = [ty_1, y_1', t^2 y_2, ty_2'].$$

Thus $(S, c_4 \Delta_{4, \mathbf{a}})$ is not K-stable. In the affine chart $U' := \{y_1' = 1\} \subset S$, we have an isomorphism $U' \cong \{y_1^3 = y_2 y_2'\} \subset \mathbb{A}_{(y_1, y_2, y_2')}^3$. Let v_1 be the monomial valuation on U' of weight $(1, 2, 1)$ in (y_1, y_2, y_2') that is centred at the point $[0, 1, 0, 0]$. Then v_1 is a quasi-monomial combination of ord_B and $\text{ord}_{B'}$ of weight $(\frac{2}{3}, \frac{1}{3})$. Thus we have

$$\beta_S(v_1) = \frac{2}{3} \beta_S(B) + \frac{1}{3} \beta_S(B') = \beta_S(B) = -\frac{1}{6}.$$

It is clear that v_1 induces the \mathbb{G}_m -action σ' . Thus by [Liu23, Theorem 3.2] (cf. [ACC⁺23, Theorem 1.3.9] and [IS17]), it suffices to show that $\beta_{(S, c_4 \Delta_{4, \mathbf{a}})}(v_1) = 0$ and that $\beta_{(S, c_4 \Delta_{4, \mathbf{a}})}(D) > 0$ for every vertical prime divisor D on S .

We first compute the β -invariant of v_1 . Since $A_S(v_1) = \frac{2}{3} A_S(B) + \frac{1}{3} A_S(B') = 1$, we know that $S_S(v_1) = A_S(v_1) - \beta_S(v_1) = \frac{7}{6}$. Moreover, we have $\Delta_{4, \mathbf{a}}|_{U'} = \{y_2 + a_1 y_1^2 + a_2 y_1 y_2' + a_3 y_2'^2 = 0\}$, which implies $v_1(\Delta_{4, \mathbf{a}}) = 2$. Thus

$$\beta_{(S, c_4 \Delta_{4, \mathbf{a}})}(v_1) = 1 - c_4 v_1(\Delta_{4, \mathbf{a}}) - (1 - 4c_4) S_S(v_1) = 1 - \frac{2}{16} - \left(1 - \frac{4}{16}\right) \frac{7}{6} = 0.$$

We also observe that C' is the other horizontal divisor of σ' , whose β -invariant is automatically zero as $\beta = \text{Fut}$ is linear on the space of cocharacters.

Next we compute the β -invariants of the vertical divisors D . A simple analysis of the σ' -action on S shows that D is one of the following:

- (i) C ,
- (ii) B or B' ,
- (iii) $D_b = (y_2 + by_1^2 = 0) - 2B$ for $b \neq 0$.

Next, we split into these three cases:

- (i) Since $\text{ord}_C(\Delta_{4,\mathbf{a}}) = 4$ or 5 , we have

$$\beta_{(S, c_4 \Delta_{4,\mathbf{a}})}(C) \geq 1 - \frac{5}{16} - \left(1 - \frac{4}{16}\right) \frac{5}{6} = \frac{1}{16} > 0.$$

- (ii) Since $\text{ord}_{B'}(\Delta_{4,\mathbf{a}}) = 0$ and $\text{ord}_B(\Delta_{4,\mathbf{a}})$ is 0 or 1, we have

$$\beta_{(S, c_4 \Delta_{4,\mathbf{a}})}(B') \geq \beta_{(S, c_4 \Delta_{4,\mathbf{a}})}(B) \geq 1 - \frac{1}{16} - \left(1 - \frac{4}{16}\right) \frac{7}{6} = \frac{1}{16} > 0.$$

- (iii) In the affine chart $U = \{y_1 = 1\} \cong \mathbb{A}_{(y_2, y_2')}^2$, we have

$$\Delta_{4,\mathbf{a}}|_U = \{y_2^4 y_2'^6 (y_2^3 + a_1 y_2^2 + a_2 y_2 + a_3) = 0\} \quad \text{and} \quad D_b|_U = \{y_2 + b = 0\}.$$

Hence $\text{ord}_{D_b}(\Delta_{4,\mathbf{a}}) \leq 3$ as $b \neq 0$. Moreover, we know that $D_b \equiv \mathcal{O}_S(2) - 2B \equiv 2B'$. Hence

$$S_S(D_b) = \frac{1}{4} \int_0^\infty \text{vol}_S(-K_S - 2tB') dt = \frac{1}{2} S_S(B') = \frac{7}{12}.$$

Thus

$$\beta_{(S, c_4 \Delta_{4,\mathbf{a}})}(D_b) \geq 1 - \frac{3}{16} - \left(1 - \frac{4}{16}\right) \frac{7}{12} = \frac{3}{8} > 0. \quad \square$$

PROPOSITION 5.6. *Let S be the surface $\mathbb{N}29$ from Table 1. Suppose that $\Delta \in |-4K_S|$ and $c \in (0, \frac{1}{16}]$ satisfy that $(S, c\Delta)$ is K -polystable but not K -stable. Then $(S, c\Delta) \cong (S, c_i \Delta_i)$ for some $1 \leq i \leq 4$ and c_i, Δ_i described in Theorem 5.2.*

Proof. Let $f(y_1, y_1', y_2, y_2')$ be the defining polynomial of Δ . Since $S = \{y_2 y_2' = y_1^3 y_1'\} \subset \mathbb{P}(1, 1, 2, 2)$, we may assume that no monomial in f is divisible by $y_2 y_2'$. We split into the following cases:

(i) The term $y_1'^8$ appears in f . Then Δ admits a special degeneration to $\{y_1'^8 = 0\} = \Delta_1$ under the \mathbb{G}_m -action σ . Thus $(S, c_1 \Delta)$ is K -semistable, and its K -polystable degeneration is $(S, c_1 \Delta_1)$. This shows that $(S, \Delta) \cong (S, \Delta_1)$ and $c = c_1$ by [ADL24, Proposition 3.18].

(ii) The term $y_1'^8$ does not appear in f , while $y_1 y_1'^7$ appears in f . Then Δ admits a special degeneration to $\{y_1 y_1'^7 = 0\} = \Delta_2$ under the \mathbb{G}_m -action σ . By similar argument to that in case (i), we conclude that $(S, \Delta) \cong (S, \Delta_2)$ and $c = c_2$.

(iii) Both $y_1'^8$ and $y_1 y_1'^7$ do not appear in f , while $y_1'^6$ divides some monomial in f . Let $(a_1 y_1^2 + a_0 y_2 + a_0' y_2') y_1'^6$ be the sum of all terms in f divisible by $y_1'^6$.

If $a_0 \neq 0$ and $a_0' \neq 0$, then Δ admits a special degeneration to $\Delta' := \{(a_0 y_2 + a_0' y_2') y_1'^6 = 0\}$ under the \mathbb{G}_m -action σ . After rescaling we have $(S, \Delta') \cong (S, \Delta_3)$. By similar argument as in case (i), we conclude that $(S, \Delta) \cong (S, \Delta_3)$ and $c = c_3$.

If either $a_0 = 0$ or $a_0' = 0$ but not both, we will show that $c = c_4$ and $(S, \Delta) \cong (S, \Delta_{4,\mathbf{a}})$ for some $\mathbf{a} \in \mathbb{C} \times (\mathbb{C}^2 \setminus \{(0, 0)\})$. By symmetry we may assume that $a_0 \neq 0$ and $a_0' = 0$. Hence after rescaling we may assume that $a_0 = 1$. Under the \mathbb{G}_m -action σ' induced by the valuation v_1 ,

we then see that $v_1(\Delta) = 2$ and Δ admits a special degeneration to $\Delta' = \{y_2 y_1'^6 + a_1 y_1^2 y_1'^6 + a_2 y_1 y_2' y_1'^5 + a_3 y_2'^2 y_1'^4 = 0\}$. Moreover, we compute

$$\beta_{(S, c\Delta)}(v_1) = 1 - cv_1(\Delta) - (1 - 4c)S_S(v_1) = 1 - 2c - (1 - 4c)\frac{7}{6}.$$

Since $(S, c\Delta)$ is K-polystable, we have $\beta_{(S, c\Delta)}(v_1) \geq 0$, which implies that $c \geq \frac{1}{16}$. Thus we have $c = c_4 = \frac{1}{16}$ and $\beta_{(S, c\Delta)}(v_1) = 0$. By K-polystability, we have $(S, c\Delta) \cong (S, c_4\Delta')$. If $(a_2, a_3) \neq (0, 0)$, then $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{C} \times (\mathbb{C}^2 \setminus \{(0, 0)\})$ and $(S, \Delta) \cong (S, \Delta') \cong (S, \Delta_{4, \mathbf{a}})$. If $a_2 = a_3 = 0$ and $a_1 \neq 0$, then clearly Δ' admits a special degeneration to $\{y_1^2 y_1'^6 = 0\} = \Delta_{4, \infty}$ under σ^{-1} , which contradicts the K-polystability. If $a_2 = a_3 = a_1 = 0$, then $\Delta' = \{y_2 y_1'^6 = 0\}$ is toric. Then we have

$$\beta_{(S, c\Delta')}(B) = 1 - \frac{3}{16} - \left(1 - \frac{4}{16}\right)\frac{7}{6} = -\frac{1}{16} < 0.$$

This implies that $(S, c\Delta')$ is K-unstable, which contradicts our assumption that $(S, c\Delta') \cong (S, c\Delta)$ is K-polystable.

If $a_0 = a'_0 = 0$, then $a_1 \neq 0$. Hence Δ admits a special degeneration to $\{y_1^2 y_1'^6 = 0\} = \Delta_{4, \infty}$ under the \mathbb{G}_m -action σ . By a similar argument to that in case (i), we conclude that $(S, \Delta) \cong (S, \Delta_{4, \infty})$ and $c = c_4$.

(iv) No monomial in f is divisible by $y_1'^6$. Then we will show that $(S, c\Delta)$ is K-unstable for every $c \in (0, \frac{1}{16}]$. Recall that the affine chart $U' = \{y_1' = 1\}$ is isomorphic to $\{y_1^3 = y_2 y_2'\} \subset \mathbb{A}^3$. Let v_2 be the monomial valuation on U of weight $(2, 3, 3)$ in (y_1, y_2, y_2') centred at $[0, 1, 0, 0]$. Then v_2 is a quasi-monomial combination of ord_B and $\text{ord}_{B'}$ of weight $(1, 1)$. Thus we have

$$\beta_S(v_2) = \beta_S(B) + \beta_S(B') = 2\beta_S(B) = -\frac{1}{3}.$$

Moreover, $A_S(v_2) = A_S(B) + A_S(B') = 2$ and hence $S_S(v_2) = A_S(v_2) - \beta_S(v_2) = \frac{7}{3}$. Since no monomial in f is divisible by $y_1'^6$, we have

$$v_2(\Delta) \geq \min\{v_2(y_1^3), v_2(y_1 y_2), v_2(y_1 y_2'), v_2(y_2^2), v_2(y_2'^2)\} = 5.$$

Thus

$$\beta_{(S, c\Delta)}(v_2) = 2 - cv_2(\Delta) - (1 - 4c)\frac{7}{3} \leq 2 - 5c - (1 - 4c)\frac{7}{3} = \frac{13c - 1}{3} < 0$$

as $c \leq \frac{1}{16} < \frac{1}{13}$. This implies that $(S, c\Delta)$ is K-unstable for every $c \in (0, \frac{1}{16}]$. \square

Proof of Theorem 5.2. This follows directly from Propositions 5.5 and 5.6. \square

5.2 Surface №33

Here $S = \{x_0 x_1 - x_2 x_3 = x_1 x_2 + x_2 x_4 + x_3 x_4 = 0\} \subset \mathbb{P}^4$.

LEMMA 5.7. *Let S be the surface №33. Then S is isomorphic to the ordinary blow-up of $\mathbb{P}(1, 2, 3)$ at two smooth points $p_1 = [1, 0, 0]$ and $p_2 = [1, 1, 0]$.*

Proof. It follows from [CP21, Appendix A] or from [CT88] that the surface S is a blow-up of a del Pezzo surface of degree 6 with the same singularities as S at two smooth points. It is well known that such a sextic del Pezzo surface is isomorphic to $\mathbb{P}(1, 2, 3)$ and its automorphism group is the Borel subgroup of the group $\text{PGL}_3(\mathbb{C})$; see [CP21, Big Table]. Since the surface S is also unique [CP21, Big Table], we see that S can be obtained from $\mathbb{P}(1, 2, 3)$ by blowing up the points $[1 : 0 : 0]$ and $[1 : 1 : 0]$ – up to the action of $\text{Aut}(\mathbb{P}(1, 2, 3))$, this choice of blow-up points is unique. \square

We denote by $\pi: S \rightarrow \mathbb{P}(1, 2, 3)$ the blow-up map. Let σ be the \mathbb{G}_m -action on S that is a lifting of the following \mathbb{G}_m -action on $\mathbb{P}(1, 2, 3)_{[x,y,z]}$:

$$t \cdot [x, y, z] = [tx, t^2y, z].$$

Let E_1 and E_2 be π -exceptional curves over p_1 and p_2 , respectively. Let $C_1 := \pi_*^{-1}(y = 0)$ and $C_2 := \pi_*^{-1}(y = x^2)$. Let $F := \pi^*(x = 0)$ and $B := \pi_*^{-1}(z = 0)$. Let $Q_a := \pi^*(y = ax^2)$ for $a \in \mathbb{C} \setminus \{0, 1\}$. By analysing the σ -action on S , it is clear that B is the only horizontal divisor on S , while C_1, C_2, E_1, E_2, F , and Q_a give all vertical divisors on S .

The intersection numbers of these curves are summarised in Table 3.

TABLE 3. Intersection numbers of the surface №33.

| • | C_1 | C_2 | E_1 | E_2 | B |
|-------|----------------|----------------|-------|-------|----------------|
| C_1 | $-\frac{1}{3}$ | $\frac{2}{3}$ | 1 | 0 | 0 |
| C_2 | $\frac{2}{3}$ | $-\frac{1}{3}$ | 0 | 1 | 0 |
| E_1 | 1 | 0 | -1 | 0 | 1 |
| E_2 | 0 | 1 | 0 | -1 | 1 |
| B | 0 | 0 | 1 | 1 | $-\frac{1}{2}$ |

THEOREM 5.8. *There is precisely one wall in $(0, \frac{1}{16}]$ involving S from №33 such that $(S, c\Delta)$ is a new K -polystable pair: $c = \frac{1}{16}$ and $\Delta = 6B + 2E_1 + 2E_2 + \pi^*\Gamma$, where $\Gamma = \{y^3 + b_1x^2y^2 + b_2x^4y + b_3x^6 = 0\}$ for $(b_1, b_2, b_3) \in \mathbb{C}^3$ in the projective coordinates $[x, y, z]$ of $\mathbb{P}(1, 2, 3)$.*

LEMMA 5.9. *The Mori cone $\overline{\text{NE}}(S)$ is generated by $[C_1]$, $[C_2]$, $[E_1]$, $[E_2]$, and $[B]$.*

Proof. From the \mathbb{G}_m -action σ on S , we see that the list of σ -invariant divisors is: $C_1, C_2, E_1, E_2, F, Q_a$, and B . Thus $\overline{\text{NE}}(S)$ is generated by these curves. From Table 3 we see that C_1, C_2, E_1, E_2 , and B are all extremal. Besides, $F \sim \pi^*\mathcal{O}(1)$ and $Q_a \sim \pi^*\mathcal{O}(2)$ are both nef and big. Thus the proof is finished. \square

PROPOSITION 5.10. *Under the above notation, we have*

$$S_S(C_1) = S_S(C_2) = \frac{7}{8}, \quad S_S(E_1) = S_S(E_2) = \frac{7}{24}, \quad S_S(B) = \frac{5}{6}, \quad S_S(F) = \frac{7}{6}, \quad S_S(Q_a) = \frac{7}{12}.$$

Proof. First of all, consider the involution $\tau: S \rightarrow S$ induced by $[x, y, z] \mapsto [x, x^2 - y, z]$ on $\mathbb{P}(1, 2, 3)$. It is clear that $\tau(C_1) = C_2$ and $\tau(E_1) = E_2$. Thus we have equalities $S_S(C_1) = S_S(C_2)$ and $S_S(E_1) = S_S(E_2)$. Besides, since $Q_a \sim 2F$, we have $S_S(Q_a) = \frac{1}{2}S_S(F)$.

Next, we compute $S_S(C_1)$. It is clear that $-K_S = \pi^*\mathcal{O}(6) - E_1 - E_2$. It follows from Table 3 and Lemma 5.9 that $-K_S - tC_1$ is nef if $0 \leq t \leq 1$ and not big if $t \geq 2$. For $1 \leq t \leq 2$, the Zariski decomposition $-K_S - tC_1 = P(t) + N(t)$ gives the nef part

$$P(t) = \begin{cases} \pi^*\mathcal{O}(6 - 2t) - E_2 & \text{if } 1 \leq t \leq \frac{3}{2}, \\ (4 - 2t)(\pi^*\mathcal{O}(3) - E_2) & \text{if } \frac{3}{2} \leq t \leq 2. \end{cases}$$

Thus computation shows that

$$\text{vol}(-K_S - tC_1) = \begin{cases} 4 - 2t - \frac{1}{3}t^2 & \text{if } 0 \leq t \leq 1, \\ 5 - 4t + \frac{1}{3}(2t^2) & \text{if } 1 \leq t \leq \frac{3}{2}, \\ 2(2 - t)^2 & \text{if } \frac{3}{2} \leq t \leq 2. \end{cases}$$

Thus

$$S_S(C_1) = \frac{1}{4} \left(\int_0^1 \left(4 - 2t - \frac{t^2}{3} \right) dt + \int_1^{\frac{3}{2}} \left(5 - 4t + \frac{2t^2}{3} \right) dt + \int_{\frac{3}{2}}^2 2(2-t)^2 dt \right) = \frac{7}{8}.$$

Next, we compute $S_S(E_1)$. It follows from Table 3 and Lemma 5.9 that $-K_S - tE_1$ is nef if $0 \leq t \leq 1$ and not big if $t \geq \frac{3}{2}$. For $1 \leq t \leq \frac{3}{2}$, the Zariski decomposition $-K_S - tE_1 = P(t) + N(t)$ gives the nef part

$$P(t) = (3 - 2t)(\pi^*\mathcal{O}(6) - 2E_1 - E_2).$$

Thus computation shows that

$$\text{vol}(-K_S - tE_1) = \begin{cases} 4 - 2t - \frac{t^2}{3} & \text{if } 0 \leq t \leq 1, \\ (3 - 2t)^2 & \text{if } 1 \leq t \leq \frac{3}{2}. \end{cases}$$

Thus

$$S_S(E_1) = \frac{1}{4} \left(\int_0^1 (4 - 2t - \frac{t^2}{3}) dt + \int_1^{\frac{3}{2}} (3 - 2t)^2 dt \right) = \frac{17}{24}.$$

Next, we compute $S_S(B)$. It follows from Table 3 and Lemma 5.9 that $-K_S - tB$ is nef if $0 \leq t \leq 1$ and not big if $t \geq 2$. For $1 \leq t \leq 2$, the Zariski decomposition $-K_S - tB = P(t) + N(t)$ gives the nef part

$$P(t) = \pi^*\mathcal{O}(6 - 3t).$$

Thus computation shows that

$$\text{vol}(-K_S - tB) = \begin{cases} 4 - 2t - \frac{t^2}{2} & \text{if } 0 \leq t \leq 1, \\ \frac{3}{2}(2 - t)^2 & \text{if } 1 \leq t \leq 2. \end{cases}$$

Thus

$$S_S(B) = \frac{1}{4} \left(\int_0^1 \left(4 - 2t - \frac{t^2}{2} \right) dt + \int_1^2 \frac{3}{2}(2 - t)^2 dt \right) = \frac{5}{6}.$$

Finally, we compute $S_S(F)$. It follows from Table 3 and Lemma 5.9 that $-K_S - tF$ is nef if $0 \leq t \leq 2$ and not big if $t \geq 3$. For $2 \leq t \leq 3$ the Zariski decomposition $-K_S - tF = P(t) + N(t)$ gives the nef part

$$P(t) = (3 - t)(\pi^*\mathcal{O}(4) - E_1 - E_2).$$

Thus computation shows that

$$\text{vol}(-K_S - tF) = \begin{cases} 4 - 2t + \frac{t^2}{6} & \text{if } 0 \leq t \leq 2, \\ \frac{2}{3}(3 - t)^2 & \text{if } 2 \leq t \leq 3. \end{cases}$$

Thus

$$S_S(F) = \frac{1}{4} \left(\int_0^2 \left(4 - 2t + \frac{t^2}{6} \right) dt + \int_2^3 \frac{2}{3}(3 - t)^2 dt \right) = \frac{7}{6}. \quad \square$$

PROPOSITION 5.11. *Let S be the surface \mathbb{N}^3 from Table 1. Suppose that $\Delta \in |-4K_S|$ and $c \in (0, \frac{1}{16}]$ satisfy that $(S, c\Delta)$ is K -polystable but not K -stable. Then $c = \frac{1}{16}$ and $\text{ord}_B(\Delta) = 6$.*

Proof. Assume that $(S, c\Delta)$ is K -polystable but not K -stable. Since S admits a \mathbb{G}_m -action σ but is not toric, we may assume that Δ is \mathbb{G}_m -invariant. It is clear that $-K_S - 2B \sim E_i + E_j$ is at the boundary of $\overline{\text{NE}}(S)$. Thus $\text{ord}_B(\Delta) \leq 8$.

First, we show that $\text{ord}_B(\Delta) \neq 8$. Suppose not; then $-4K_S - 8B \sim 4(E_1 + E_2)$ is exceptional over $\mathbb{P}(1, 2, 3)$, which implies that $\Delta = 8B + 4E_1 + 4E_2$. By K-polystability and Proposition 5.10, we have

$$0 = \beta_{(S, c\Delta)}(B) = 1 - 8c - (1 - 4c)\frac{5}{6} = 0,$$

which is equivalent to $c = \frac{1}{28}$. On the other hand,

$$0 < \beta_{(S, c\Delta)}(F) = 1 - (1 - 4c)\frac{7}{6} = 0,$$

so we have a contradiction.

Next, we show that $\text{ord}_B(\Delta) \neq 7$. Suppose not; then $((-4K_S - 7B) \cdot E_i) = -3$, which implies that $3(E_1 + E_2)$ is in the base component of $|-4K_S - 7B|$. Thus we may write $\Delta = 7B + 3E_1 + 3E_2 + \pi^*\Gamma_1$, where $\pi^*\Gamma_1 \sim -4K_S - 7B - 3E_1 - 3E_2 \sim \pi^*\mathcal{O}(3)$. Since Γ_1 is \mathbb{G}_m -invariant and does not contain $\pi_*B = (z = 0)$, it must contain $\pi_*F = (x = 0)$, which implies that $\text{ord}_F(\pi^*\Gamma_1) \geq 1$. By K-polystability and Proposition 5.10, we have

$$0 = \beta_{(S, c\Delta)}(B) = 1 - 7c - (1 - 4c)\frac{5}{6} = 0,$$

which is equivalent to $c = \frac{1}{22}$. On the other hand,

$$0 < \beta_{(S, c\Delta)}(F) \leq 1 - c - (1 - 4c)\frac{7}{6} = 0,$$

so we have a contradiction.

Next, we show that if $\text{ord}_B(\Delta) = 6$, then $c = \frac{1}{16}$. By K-polystability and Proposition 5.10, we have

$$0 = \beta_{(S, c\Delta)}(B) = 1 - 6c - (1 - 4c)\frac{5}{6} = 0,$$

which is equivalent to $c = \frac{1}{16}$.

Finally, we show that $\text{ord}_B(\Delta) > 5$. Suppose not; then we have

$$0 = \beta_{(S, c\Delta)}(B) \geq 1 - 5c - (1 - 4c)\frac{5}{6},$$

which implies that $c \geq \frac{1}{10}$, so we have a contradiction. \square

Proof of Theorem 5.8. Assume that $(S, c\Delta)$ is K-polystable but not K-stable, where S is the surface №33 from Table 1 and where $\Delta \in |-4K_S|$ and $c \in (0, \frac{1}{16}]$. By Proposition 5.11, we must have $c = \frac{1}{16}$ and $\text{ord}_B(\Delta) = 6$. Since $((-4K_S - 6B) \cdot E_i) = -2$, we know that $2(E_1 + E_2)$ is in the base component of $|-4K_S - 6B|$. Thus we may write $\Delta = 6B + 2E_1 + 2E_2 + \pi^*\Gamma$, where $\pi^*\Gamma \sim -4K_S - 6B - 2E_1 - 2E_2 \sim \pi^*\mathcal{O}(6)$ is σ -invariant and does not contain B in its support.

Next, we classify all possible Γ such that $(S, \frac{1}{16}\Delta)$ is K-polystable and not K-stable. Since this pair is invariant under the \mathbb{G}_m -action σ but not toric, by [Liu23, Theorem 3.2] (cf. [ACC⁺23, Theorem 1.3.9] and [IS17]), the pair being K-polystable is equivalent to $\beta_{(S, c\Delta)}(D) > 0$ for every vertical divisor D on S . Here we use the fact that $\beta_{(S, c\Delta)}(B) = 0$ by the proof of Proposition 5.11 and the fact that B is horizontal. A simple analysis of the σ -action on S shows that D is one of the following:

- (i) F ,
- (ii) E_1 or E_2 ,
- (iii) C_1 or C_2 ,

(iv) Q_a for $a \in \mathbb{C} \setminus \{0, 1\}$.

Next, we split into these four cases.

(i) Computation gives

$$\beta_{(S, c\Delta)}(F) = 1 - \frac{1}{16} \text{ord}_F(\pi^* \Gamma) - \left(1 - \frac{4}{16}\right) \frac{7}{6}.$$

Thus $\beta_{(S, c\Delta)}(F) > 0$ is equivalent $\text{ord}_F(\pi^* \Gamma) < 2$. Therefore, we have $\text{ord}_F(\pi^* \Gamma) = 0$ as it is always even.

(ii) Computation gives

$$\beta_{(S, c\Delta)}(E_i) = 1 - \frac{1}{16} (2 + \text{ord}_{E_i}(\pi^* \Gamma)) - \left(1 - \frac{4}{16}\right) \frac{7}{24}.$$

Thus $\beta_{(S, c\Delta)}(E_i) > 0$ is equivalent to $\text{ord}_{E_i}(\pi^* \Gamma) < \frac{21}{2}$. This is always true as $\text{ord}_{E_i}(\pi^* \Gamma) \leq 3$.

(iii) Computation gives

$$\beta_{(S, c\Delta)}(C_i) = 1 - \frac{1}{16} \text{ord}_{C_i}(\pi^* \Gamma) - \left(1 - \frac{4}{16}\right) \frac{7}{8}.$$

Thus $\beta_{(S, c\Delta)}(C_i) > 0$ is equivalent to $\text{ord}_{C_i}(\pi^* \Gamma) < \frac{11}{2}$. This is always true as $\text{ord}_{C_i}(\pi^* \Gamma) \leq 3$.

(iv) Computation gives

$$\beta_{(S, c\Delta)}(Q_a) = 1 - \frac{1}{16} \text{ord}_{Q_a}(\pi^* \Gamma) - \left(1 - \frac{4}{16}\right) \frac{7}{12}.$$

Thus $\beta_{(S, c\Delta)}(Q_a) > 0$ is equivalent to $\text{ord}_{Q_a}(\pi^* \Gamma) < \frac{11}{2}$. This is always true as $\text{ord}_{Q_a}(\pi^* \Gamma) \leq 3$.

To summarise, $(S, \frac{1}{16}\Delta)$ is K-polystable and not K-stable if and only if Γ is \mathbb{G}_m -invariant and does not contain $\pi_* B = \{z = 0\}$ or $\pi_* F = \{x = 0\}$ in its support. This is the same as saying that $\Gamma = \{y^3 + b_1 x^2 y^2 + b_2 x^4 y + b_3 x^6 = 0\}$ for $(b_1, b_2, b_3) \in \mathbb{C}^3$. \square

Proof of Theorem 5.1. Let $(S, c\Delta)$ be a new K-polystable pair on a wall $c \in (0, \frac{1}{16}]$. In particular, $(S, c\Delta)$ is K-polystable with an effective \mathbb{G}_m -action, and S is K-unstable. By Proposition 4.4 we know that S has at worst A_3 -singularities.

If S has an A_3 -singularity, the classification of walls and new K-polystable pairs is completed in Proposition 4.4, where we take $c = \frac{1}{16}$ and $(S, \Delta) \cong \{(y_3^2 = y_2 y_4) \subset \mathbb{P}(1, 2, 3, 4), \{y_4^4 = 0\}\}$. In particular, S is isomorphic to the surface $\mathbb{N}25$.

If S has no A_3 -singularities, then Proposition 4.4 implies that S is a K-unstable del Pezzo surface of degree 4 with at worst A_2 -singularities admitting an effective \mathbb{G}_m -action. Thus by Table 1 we know that S is isomorphic to either $\mathbb{N}29$ or $\mathbb{N}33$. The classifications of walls and new K-polystable pairs for these two surfaces are completed in Theorems 5.2 and 5.8.

It remains to prove the statement on the initial wall $c_0 = 0$. Let \mathcal{M}_ϵ and \mathcal{M}_0 be the corresponding K-moduli stacks whose good moduli spaces are \mathfrak{M}_ϵ and \mathfrak{M}_0 , respectively. Since K-semistability for coefficients is a closed condition, we know that every $[(S, \Delta)] \in \mathcal{M}_\epsilon$ satisfies that S is K-semistable. In particular, the forgetful map $[(S, \Delta)] \mapsto [S]$ gives a morphism between Artin stacks $\mathcal{M}_\epsilon \rightarrow \mathcal{M}_0$. Since good moduli spaces are initial among maps to algebraic spaces [Alp13, Theorem 6.6], the forgetful map induces a morphism $\phi_0: \mathfrak{M}_\epsilon \rightarrow \mathfrak{M}_0$ between K-moduli spaces.

Finally, we analyse the map ϕ_0 . Let $[S] \in \mathfrak{M}_0$ be a general point; then we may assume that S is a general K-stable smooth del Pezzo surface of degree 4. Moreover, we may assume

that $\text{Aut}(S) \cong \mu_2^4$ by [DI09, Section 6.4]. By [Zho24] we know that there exists an $\epsilon_0 > 0$ such that $(S, \epsilon\Delta)$ is K-stable for every $\Delta \in |-4K_S|$ and every $\epsilon \in (0, \epsilon_0)$. This implies that the finite quotient $\mathbb{P}(H^0(S, -4K_S))/\text{Aut}(S)$ admits a closed immersion into the fibre $\phi_0^{-1}([S])$. On the other hand, if $[(S', \Delta')] \in \phi_0^{-1}([S])$, then S' and S are S -equivalent K-semistable Fano varieties. Since S is K-stable, by [LWX21, BX19] we know that $S' \cong S$, which implies that $[(S', \Delta')]$ is contained in the image of $\mathbb{P}(H^0(S, -4K_S))/\text{Aut}(S)$. Thus we have $\mathbb{P}(H^0(S, -4K_S))/\text{Aut}(S) \cong \phi_0^{-1}[S]$. By a Riemann–Roch computation, we have $h^0(S, -4K_S) = 41$, which implies that $\mathbb{P}(H^0(S, -4K_S))/\text{Aut}(S) \cong \mathbb{P}^{40}/\mu_2^4$. Since both \mathfrak{M}_c and \mathfrak{M}_0 are normal by [ADL23a, Theorem 2.21], the statement follows by Zariski’s main theorem. \square

Remark 5.12. In Section 2, we found three K-polystable toric Fano 3-folds of the form $X_{2,2,4} \subset \mathbb{P}(1^5, 2^2)$ defined by $y_0y_1 = f(x_0, \dots, x_4)$, $g(x_0, \dots, x_4) = 0$, and $h(x_0, \dots, x_4) = 0$, where f , g , and h are as follows:

- (1) $f = x_0^4$, $g = x_2^2 - x_0x_3$, and $h = x_3^2 - x_1x_4$;
- (2) $f = x_0^3x_3$, $g = x_2^2 - x_0x_3$, and $h = x_2x_3 - x_1x_4$;
- (3) $f = x_0^2x_4^2$, $g = x_2^2 - x_0x_4$, and $h = x_1x_3 - x_0x_4$.

By Proposition 4.2 and Corollary 4.5, each of the above toric Fano 3-folds corresponds to a K-polystable pair $(S, \frac{1}{16}\Delta)$ in $\mathfrak{M}_{1/16}$, where $S = \{g = h = 0\} \subset \mathbb{P}^4$ and $\Delta = \{f = 0\}$. It turns out that these three surface pairs are precisely the toric ones in $\mathfrak{M}_{1/16}$:

Case (1) corresponds to the surface №25 and $\Delta = \{y_4^4 = 0\}$ via

$$[x_0, x_1, x_2, x_3, x_4] = [y_4, y_2^2, y_1y_3, y_1^2y_2, y_1^4].$$

Case (2) corresponds to the surface №29 and $\Delta = \Delta_{4,\infty}$ via

$$[x_0, x_1, x_2, x_3, x_4] = [y_1'^2, y_2, y_1y_1', y_1^2, y_2'].$$

Case (3) corresponds to the surface №30 and $\Delta = \{y_1^4y_1'^4 = 0\}$ via

$$[x_0, x_1, x_2, x_3, x_4] = [y_1'^2, y_2, y_1y_1', y_2', y_1'^2].$$

Here the K-polystability of $(S, \frac{1}{2}\Delta)$ from case (1) or (2) follows from Theorem 5.1, while that for case (3) follows by interpolation [ADL24, Proposition 2.13] as №30 is K-polystable and $\frac{1}{4}\Delta$ is the reduced toric boundary.

5.3 Further discussions

Let M be the K-moduli component of quartic 3-folds. While our results provide a decisive statement about the global nature of objects in M , several questions remain open. For instance, it would be interesting to determine all of the pairs parametrised by $\mathfrak{M}_{1/16}$ via wall crossings. Although Theorem 5.1 gives a complete description of walls and new K-polystable pairs; one would need to describe the exceptional loci for K-moduli wall crossings, which is similar to the variation of GIT analysis on basins and attractions. We leave this for future studies.

It would also be handy to have an explicit description of GIT-stable quartic 3-folds. Indeed, we think that the locus in M parametrisng pure $(2, 2, 4)$ -complete intersections $X_{2,2,4} \subset \mathbb{P}(1^5, 2^2)$ should correspond to the locus of the union of two quadric hypersurfaces $\{gh = 0\} \subset \mathbb{P}^4$ in the GIT moduli space of quartic 3-folds via wall crossing. On the one hand, $\{gh = 0\} \subset \mathbb{P}^4$ degenerates to $\{y_0y_1 = 0, g = h = 0\} \subset \mathbb{P}(1^5, 2^2)$ via iterated degeneration to the normal cone. On the other hand, $X_{2,2,4} = \{y_0y_1 = f, g = h = 0\} \subset \mathbb{P}(1^5, 2^2)$ also degenerates to $\{y_0y_1 = 0, g = h = 0\} \subset \mathbb{P}(1^5, 2^2)$ via a suitable one-parameter subgroup. Note that $\dim M = 45$, and the

dimension of the locus parametrising pure $(2, 2, 4)$ -complete intersections in $\mathbb{P}(1^5, 2^2)$ is equal to $\dim \mathfrak{M}_{1/16}$, which is 42.

In a different direction, the following example from [ADL23b, Remark 6.13] gives another closed locus in M .

Example 5.13. The K-moduli compactification $\overline{\mathfrak{M}}_{3/4}^K$ of $(\mathbb{P}^3, \frac{3}{4}D)$, where D is a quartic surface, was studied in [ADL23b]. If $(X, \frac{3}{4}D)$ is a K-polystable 3-fold pair in $\overline{\mathfrak{M}}_{3/4}^K$, then we can take the quadruple cyclic cover $Y \rightarrow X$ branched along D . By [LZ22, Zhu21] we know that Y is K-polystable. Since a quadruple cyclic cover of \mathbb{P}^3 branched along a quartic surface given by $\{f(x_0, x_1, x_2, x_3) = 0\}$ is precisely the quartic 3-fold $\{x_4^4 = f(x_0, x_1, x_2, x_3)\} \subset \mathbb{P}^4$, we know that Y is a K-polystable limit of a family of smooth quartic 3-folds. This cyclic cover construction can be made fibrewise (see [ADL23b, § 6.3] for the case of quartic double solids), which yields a finite morphism $\iota: \overline{\mathfrak{M}}_{3/4}^K \rightarrow M$. It is clear that the image of ι has dimension 19 and is a K-moduli compactification of smooth quartic 3-folds of the form $\{x_4^4 = f(x_0, x_1, x_2, x_3)\} \subset \mathbb{P}^4$. By [ADL23b, Theorem 5.16] we know that there are four of the Fano 3-folds X appearing in $\overline{\mathfrak{M}}_{3/4}^K$: \mathbb{P}^3 , X_h , $\mathbb{P}(1, 1, 2, 4)$, and X_u . Here $X_h = \{x_0x_1 = x_2x_3\} \subset \mathbb{P}(1^4, 2)$, which is isomorphic to the projective cone over $\mathbb{P}^1 \times \mathbb{P}^1$ with polarisation $\mathcal{O}(2, 2)$, and X_u is a 2-step birational modification of a $\mathbb{P}(1, 2, 3)$ -bundle over \mathbb{P}^1 constructed in [ADL23b, § 4.2]. One can check that the quadruple cyclic cover of X_h is a $(2, 4)$ -complete intersection of $\mathbb{P}(1^5, 2)$, while the quadruple cyclic cover of $\mathbb{P}(1, 1, 2, 4)$ is a weighted hypersurface of degree 8 in $\mathbb{P}(1, 1, 2, 2, 4)$. However, we expect that the quadruple cyclic cover of X_u will not be a weighted complete intersection. Such quadruple cyclic covers of X_u form an 18-dimensional closed locus in the K-moduli space M .

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