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A stacky approach to identifying the semistable locus of bundles

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Abstract

We show that the semistable locus is the unique maximal open substack of the moduli stack of principal bundles over a curve that admits a schematic moduli space. For rank 2 vector bundles, it coincides with the unique maximal open substack that admits a separated moduli space, but for higher rank there exist other open substacks that admit separated moduli spaces.

1. Introduction

Semistable vector bundles over a curve form a proper moduli: the stability condition imposed on vector bundles "accidentally" coincides with the one arising from the action of a reductive group on a Quot scheme. This allows us to use the machinery of geometric invariant theory (GIT), whose outcome is always quasi-projective, to construct the moduli space. In other words, for vector bundles the stability, a priori, is a sufficient condition to obtain a proper moduli space, and there is no obvious reason why this condition is also necessary. Therefore, it would be interesting to either prove that it is indeed the case, or to find more proper moduli spaces of vector bundles.

Let us state this formally. Let C be a smooth projective connected curve, and let G be a connected reductive group. Denote by $\mathscr{B}un_G$ the moduli stack of principal G-bundles over C. The guiding problem in this paper is to single out all open substacks of $\mathscr{B}un_G$ that admit proper good moduli spaces (in the sense of [Alp13, Definition 4.1]).

By its very definition a good moduli space is only an algebraic space, not necessarily a scheme. If we require that an open substack of $\mathscr{B}un_G$ admits a schematic good moduli space (even without the properness assumption), then the classical stability condition appears.

Theorem A (Characteristic 0 version of Theorem 2.1). *Let C be a smooth projective connected curve over an algebraically closed field k of characteristic 0, and let G be a connected reductive group over k. Then the open substack $\mathscr{B}un_G^{\mathrm{ss}} \subseteq \mathscr{B}un_G$ of semistable principal G-bundles is the unique maximal open substack that admits a schematic good moduli space.*

This gives a stacky approach to identifying the semistable locus of principal bundles and hence provides an intrinsic way to introduce the classical stability condition. As a result the

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good moduli space of any open substack of $\mathscr{B}un_G$ containing unstable objects, if it exists, cannot be a scheme. In particular, the good moduli space of simple vector bundles is only an algebraic space, not a scheme (see Corollary 2.16). Furthermore, as GIT always produces schematic moduli spaces, Theorem A implies that the semistable locus is also the maximal locus to which GIT can be applied to construct a moduli space.

The idea for proving Theorem A is simple. A result due to Alper describes open substacks of a smooth stack that admit quasi-projective moduli spaces in terms of line bundles over it (see [Alp13, Theorem 11.14(ii)]). If G is almost simple and simply connected, then there is essentially only one line bundle over $\mathscr{B}un_G$ (see [Fal03, Theorem 17]), and this line bundle recovers the classical stability condition (see [Hei17, Corollary 1.16]). Via the structure theory of reductive groups employed in [BH10, § 5], we can reduce the general case to the almost simple and simply connected one.

In general a good moduli space need not be a scheme, and there could be more possibilities for open substacks admitting good moduli spaces. Here we consider the case $G = \mathrm{GL}_n$ and denote by $\mathscr{B}un_n^d$ the moduli stack of vector bundles of rank n and degree d over C.

In the rank 2 case, we again find the classical stability condition under the condition that the good moduli space is separated.

Theorem B (Characteristic 0 version of Theorem 4.1, [Zha22, Theorem A(3)]). *Let C be a smooth projective connected curve of genus $g_C > 1$ over an algebraically closed field k of characteristic 0. Then the open substack $\mathscr{B}un_2^{d,\mathrm{ss}} \subseteq \mathscr{B}un_2^d$ of semistable vector bundles is the unique maximal open substack that admits a separated good moduli space.*

In higher rank we construct an example showing that the conclusion in Theorem B no longer holds. To be precise, we have the following result.

Theorem C (Theorem 5.5, [Zha22, Theorem B]). *Let C be a smooth projective connected curve of genus $g_C > 2$ over an algebraically closed field k. Then there exists an open dense substack $\mathscr{U} \subseteq \mathscr{B}un_3^2$ that admits a separated non-proper good moduli space and is not contained in the open substack $\mathscr{B}un_3^{2,\mathrm{ss}} \subseteq \mathscr{B}un_3^2$ of semistable vector bundles.*

This example can be generalized to arbitrary higher rank, thus showing that there are many other open substacks of $\mathscr{B}un_n^d$ that admit separated non-proper good moduli spaces; that is, separatedness alone cannot help us identify the classical stability condition. Moreover, by Theorem A the good moduli space of $\mathscr{U} \subseteq \mathscr{B}un_3^2$ in Theorem C is only an algebraic space, not a scheme. In particular, this moduli space does not come from a GIT construction.

The key ingredient to prove Theorems B and C is the result [AHH23, Theorem A] characterizing algebraic stacks with separated or proper good moduli spaces. To apply this, we find simple translations of the notions of Θ-reductivity and S-completeness in the case of vector bundles.

Remark. We also observe that the maximality-type results Theorems A and B actually hold over an arbitrary base field; see Lemma 4.8.

Motivated by our result in rank 2 and the example in rank 3, we propose the following more specific question.

Question. Is $\mathscr{B}un_n^{d,\mathrm{ss}} \subseteq \mathscr{B}un_n^d$ (or more generally $\mathscr{B}un_G^{\mathrm{ss}} \subseteq \mathscr{B}un_G$) the unique maximal open substack that admits a proper good moduli space?

A positive answer to this question would yield another stacky approach to identifying the semistable locus of vector bundles. Nevertheless, Theorem A implies that the answer is yes if the condition "proper" is replaced by "quasi-projective". Theorem B answers this question affirmatively in rank 2, and Theorem C implies that the condition "proper" cannot be weakened to "separated".

The structure of this paper is as follows. In Section 2 we give the proof of Theorem A. To make the arguments also work in positive characteristic, we need to introduce a stability notion depending on a line bundle over a stack, as a plain generalization of the one defined by Alper in characteristic 0 (see [Alp13, Definition 11.1]). We also provide a direct argument for the vector bundles case in Section 2.2.3. In Section 3 we recall the existence criteria [AHH23, Theorem 5.4]. A first translation of these conditions for vector bundles, including a simple modular interpretation of S-completeness (see Proposition 3.17), is provided. As applications we re-prove that the good moduli space of semistable vector bundles is separated and that of simple vector bundles is not. In Section 4 we find more specific consequences of the existence criteria in rank 2, which are sufficient to prove Theorem B. We also give the argument generalizing the maximality-type results Theorems A and B to an arbitrary base field in Section 4.4. In Section 5 we construct the promised example in rank 3, therefore proving Theorem C.

Notation

Throughout the paper, unless stated otherwise, we work over an algebraically closed field k. By a reductive group we always mean a connected reductive group. Let C be a smooth projective connected curve, and let G be a reductive group over k. Denote by $\mathscr{B}un_G$ the moduli stack of principal G-bundles over C and by $\mathscr{B}un_n^d$ the moduli stack of vector bundles of rank n and degree d over C.

2. Schematic moduli spaces

In this section we prove the following result, valid in arbitrary characteristic. Recall that in characteristic 0 the notion of adequate moduli spaces (in the sense of [Alp14, Definition 5.1.1]) agrees with that of good moduli spaces.

Theorem 2.1. *Let C be a smooth projective connected curve over an algebraically closed field k, and let G be a reductive group over k. Then the open substack $\mathscr{B}un_G^{\mathrm{ss}} \subseteq \mathscr{B}un_G$ of semistable principal G-bundles is the unique maximal open substack that admits a schematic adequate moduli space.*

As mentioned before, the starting point is Alper's description for open substacks of a smooth stack that admit quasi-projective good moduli spaces in terms of line bundles in characteristic 0. A version of this also holds in positive characteristic if one replaces "good" with "adequate". To include this statement, we start by reviewing Alper's argument in this setting.

2.1 Adequate stability

Let $\mathscr{X}$ be an algebraic stack over a field k and $\mathcal{L}$ be a line bundle over $\mathscr{X}$.

Definition 2.2 (Analogue of [Alp13, Definition 11.1(b)]). A geometric point $x \in \mathscr{X}(\mathbf{K})$ is *adequately semistable with respect to* $\mathcal{L}$ if there exists a section $s \in \Gamma\big(\mathscr{X}, \mathcal{L}^{\otimes m}\big)$ for some integer $m > 0$ such that $s(x) \neq 0$ and $\mathscr{X}_s := \{y \in \mathscr{X} : s(y) \neq 0\} \to \mathrm{Spec}(k)$ is adequately affine (in the sense of [Alp14, Definition 4.1.1]). Denote by $(\mathscr{X})^{\text{a-ss}}_{\mathcal{L}} \subseteq \mathscr{X}$ the open substack of adequately semistable points with respect to $\mathcal{L}$.

Analogously to Alper's result, we have a description for open substacks of a smooth stack that admit quasi-projective adequate moduli spaces.

Theorem 2.3 (Analogue of [Alp13, Theorem 11.14(ii)]). *Let $\mathscr{X}$ be a smooth algebraic stack with affine diagonal over a field k. If $\mathscr{U} \subseteq \mathscr{X}$ is an open substack that admits a quasi-projective adequate moduli space, then there exists a line bundle $\mathcal{L}$ over $\mathscr{X}$ such that $\mathscr{U} \subseteq (\mathscr{X})^{\text{a-ss}}_{\mathcal{L}}$.*

Proof. Let $\pi\colon \mathscr{U} \to U$ denote the adequate moduli space. As U is quasi-projective, there exist an ample line bundle M on U together with global sections $m_1, \dots, m_r$ such that U is covered by the non-vanishing loci U_{m_i}. Let $\mathcal{M} = \pi^* M$ and $t_i = \pi^* m_i$. Then we obtain $\mathscr{U} = \bigcup_{i=1}^r \mathscr{U}_{t_i}$. To conclude we need to extend $\mathcal{M}$ together with the global sections t_i to $\mathscr{X}$ such that the non-vanishing loci remain unchanged.

We use Zorn's lemma to reduce to the case where $\mathscr{X}$ is Noetherian as follows. Consider the set Σ consisting of triples

$$(\mathscr{V}, \mathcal{L}, (s_1, \dots, s_r)),$$

where $\mathscr{V} \subseteq \mathscr{X}$ is an open substack containing $\mathscr{U}$, $\mathcal{L}$ is a line bundle over $\mathscr{V}$ extending $\mathcal{M}$, and s_i is a global section of $\mathcal{L}$ extending t_i such that $\mathscr{V}_{s_i} = \mathscr{U}_{t_i}$ for $i = 1, \dots, r$. There is a natural ordering $\leqslant$ on Σ, and it is easy to see that Zorn's lemma applies to Σ. Furthermore, if we can show that the above data extends in the Noetherian case, then a maximal element of Σ is the desired extension to $\mathscr{X}$, as $\mathscr{X}$ is locally Noetherian.

In this case the proof of [Alp13, Theorem 11.14(ii)] carries over if one adds the following input: adequate affineness is equivalent to affineness for any quasi-separated, quasi-compact morphism of algebraic spaces (see [Alp14, Theorem 4.3.1]). The idea is that – using smoothness – we can always extend a line bundle together with sections from an open substack to the entire stack. Adequate affineness shows that the non-vanishing loci of the extended sections stay the same. □

To get a better understanding of adequate stability, we compare it with some other popular stability notions which are also defined by line bundles.

Lemma 2.4. *For a geometric point $x \in \mathscr{X}(\mathbf{K})$, consider the following statements:*

(i) *The point x is cohomologically semistable with respect to $\mathcal{L}$ in the sense of* [Alp13, Definition 11.1]*; that is, there exist an integer $m > 0$ and a section $s \in \Gamma\big(\mathscr{X}, \mathcal{L}^{\otimes m}\big)$ such that $s(x) \neq 0$ and $\mathscr{X}_s \to \operatorname{Spec}(k)$ is cohomologically affine.*

(ii) *The point x is adequately semistable with respect to $\mathcal{L}$ in the sense of Definition 2.2; that is, there exist an integer $m > 0$ and a section $s \in \Gamma\big(\mathscr{X}, \mathcal{L}^{\otimes m}\big)$ such that $s(x) \neq 0$ and $\mathscr{X}_s \to \operatorname{Spec}(k)$ is adequately affine.*

(iii) *The point x is Θ-semistable with respect to $\mathcal{L}$ in the sense of* [Hei17, Definition 1.2] *or* [Hal22, Definition 0.0.4]*; that is, for all morphisms $f\colon \big[\mathbf{A}^1_{\mathbf{K}}/\mathbf{G}_{m,\mathbf{K}}\big] \to \mathscr{X}$ with $f(1) \cong x$ and $f(0) \ncong x$, we have $\operatorname{wt}_{\mathbf{G}_{m,\mathbf{K}}}(f^*\mathcal{L}) \leqslant 0$.*

Then we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii); that is, $(\mathscr{X})^{\text{c-ss}}_{\mathcal{L}} \subseteq (\mathscr{X})^{\text{a-ss}}_{\mathcal{L}} \subseteq (\mathscr{X})^{\Theta\text{-ss}}_{\mathcal{L}}$.

These notions remain unchanged if the line bundle $\mathcal{L}$ is replaced by a positive multiple.

Proof. A cohomologically affine morphism is adequately affine by [Alp14, Proposition 4.2.1(2)], so we have (i) $\Rightarrow$ (ii). To see (ii) $\Rightarrow$ (iii), let $x \in \mathscr{X}(\mathbf{K})$ be a geometric point such that there exists a section $s \in \Gamma(\mathscr{X}, \mathcal{L}^{\otimes m})$ for some integer $m > 0$ satisfying $s(x) \neq 0$. For any morphism $f\colon [\mathbf{A}^1_{\mathbf{K}}/\mathbf{G}_{m,\mathbf{K}}] \to \mathscr{X}$ with $f(1) \cong x$, the pulled-back section $f^*s \in \Gamma([\mathbf{A}^1_{\mathbf{K}}/\mathbf{G}_{m,\mathbf{K}}], f^*\mathcal{L}^{\otimes m})$ does not vanish at 1. In particular, $\Gamma([\mathbf{A}^1_{\mathbf{K}}/\mathbf{G}_{m,\mathbf{K}}], f^*\mathcal{L}^{\otimes m}) \neq 0$. For any line bundle $\mathcal{N}$ over $[\mathbf{A}^1_{\mathbf{K}}/\mathbf{G}_{m,\mathbf{K}}]$, we have (see [Hei17, § 1.A.a])

$$\Gamma\big([\mathbf{A}^1_{\mathbf{K}}/\mathbf{G}_{m,\mathbf{K}}], \mathcal{N}\big) = \Gamma\big(\mathbf{A}^1_{\mathbf{K}}, \mathcal{N}\big)^{\mathbf{G}_{m,\mathbf{K}}} = \begin{cases} \mathbf{K}\cdot a^{-d}e & \text{if } \mathrm{wt}_{\mathbf{G}_{m,\mathbf{K}}}(\mathcal{N}) = d \leqslant 0\,, \\ 0 & \text{if } \mathrm{wt}_{\mathbf{G}_{m,\mathbf{K}}}(\mathcal{N}) = d > 0\,, \end{cases}$$

where $\Gamma(\mathbf{A}^1_{\mathbf{K}}, \mathcal{N}) = \mathbf{K}[a]\cdot e$, that is, e is the unique non-vanishing section (up to a scalar) of the line bundle $\mathcal{N}$ over $\mathbf{A}^1_{\mathbf{K}}$. This implies that $m \cdot \mathrm{wt}_{\mathbf{G}_{m,\mathbf{K}}}(f^*\mathcal{L}) = \mathrm{wt}_{\mathbf{G}_{m,\mathbf{K}}}(f^*\mathcal{L}^{\otimes m}) \leqslant 0$. ☐

To close this subsection, we note that Θ-stability is not particularly interesting for algebraic spaces as there is no testing morphism at each point, so all points are Θ-semistable with respect to any line bundle. Here we can prove a slightly more general result which we need later: line bundles from algebraic spaces do not affect the Θ-semistable locus.

LEMMA 2.5. *Let $g\colon \mathscr{X} \to X$ be a morphism from an algebraic stack to an algebraic space. Given two line bundles $\mathcal{L}$, $\mathcal{L}'$ over $\mathscr{X}$ such that $\mathcal{L}$ descends to X, we have*

$$(\mathscr{X})^{\Theta\text{-ss}}_{\mathcal{L}\otimes\mathcal{L}'} = (\mathscr{X})^{\Theta\text{-ss}}_{\mathcal{L}'}\,.$$

In particular, we have $(\mathscr{X})^{\Theta\text{-ss}}_{\mathcal{L}} = \mathscr{X}$.

Proof. Let $\mathcal{L} = g^*\mathcal{L}_0$ for some line bundle $\mathcal{L}_0$ over X. For any field $\mathbf{K}/k$ and any morphism $f\colon [\mathbf{A}^1_{\mathbf{K}}/\mathbf{G}_{m,\mathbf{K}}] \to \mathscr{X}$, we compute that

$$\begin{aligned}\mathrm{wt}_{\mathbf{G}_{m,\mathbf{K}}}\big(f^*(\mathcal{L}\otimes\mathcal{L}')\big) &= \mathrm{wt}_{\mathbf{G}_{m,\mathbf{K}}}\big((g\circ f)^*\mathcal{L}_0 \otimes f^*\mathcal{L}'\big) = \mathrm{wt}_{\mathbf{G}_{m,\mathbf{K}}}((g\circ f)^*\mathcal{L}_0) + \mathrm{wt}_{\mathbf{G}_{m,\mathbf{K}}}\big(f^*\mathcal{L}'\big) \\ &= \mathrm{wt}_{\mathbf{G}_{m,\mathbf{K}}}\big(f^*\mathcal{L}'\big)\end{aligned}$$

since the morphism $[\mathbf{A}^1_{\mathbf{K}}/\mathbf{G}_{m,\mathbf{K}}] \xrightarrow{f} \mathscr{X} \xrightarrow{g} X$ factors via the good moduli space morphism $[\mathbf{A}^1_{\mathbf{K}}/\mathbf{G}_{m,\mathbf{K}}] \to \mathrm{Spec}(\mathbf{K})$. In particular, we see that a geometric point $x \in \mathscr{X}(\mathbf{K})$ is Θ-semistable with respect to $\mathcal{L}\otimes\mathcal{L}'$ if and only if it is Θ-semistable with respect to $\mathcal{L}'$. ☐

2.2 Proof of Theorem 2.1

The connected components of $\mathscr{B}un_G$ are indexed by elements in $\pi_1(G)$ (see [Hof10b, Theorem 5.8]), so we may assume that $\mathscr{U} \subseteq \mathscr{B}un_G^d$ for some $d \in \pi_1(G)$. Note that $\mathscr{B}un_G^{d,\mathrm{ss}}$ admits a schematic adequate moduli space (see for example [HS10, Theorem 3.2.3(i)] or [GLSS08, Corollary 3.4.3]).

Let $\mathscr{U} \subseteq \mathscr{B}un_G^d$ be an open substack that admits a schematic adequate moduli space $\mathscr{U} \to U$. To conclude we need to show that $\mathscr{U} \subseteq \mathscr{B}un_G^{d,\mathrm{ss}}$. Choose an affine open cover $\{U_i\}_{i\in I}$ of U. Then $\{\mathscr{U}_i\}_{i\in I} := \{\mathscr{U} \times_U U_i\}_{i\in I}$ is an open cover of $\mathscr{U}$, and each base change $\mathscr{U}_i \to U_i$ is again an adequate moduli space by [Alp14, Proposition 5.2.9(1)]. Note that U_i is quasi-projective as $\mathscr{U}_i$ is of finite type; see [Alp14, Theorem 6.3.3]. Since $\mathscr{B}un_G^d$ is smooth (see [Hof10b, Proposition 4.1]), by Theorem 2.3 we have

$$\mathscr{U}_i \subseteq \big(\mathscr{B}un_G^d\big)^{\text{a-ss}}_{\mathcal{L}_i} \quad \text{for some line bundle } \mathcal{L}_i \text{ over } \mathscr{B}un_G^d\,.$$

There is a natural line bundle $\mathcal{L}_{\det}$ over $\mathscr{B}un_G^d$, that is, the determinant line bundle of cohomology (see for example [Hei17, § 1.F.a]). Via the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathrm{Lie(G)})$, which defines for any G-bundle $\mathcal{P}$ over C its adjoint bundle $\mathrm{Ad}_*\mathcal{P} := \mathcal{P} \times^G \mathrm{Lie}(G)$, we set

$$\mathcal{L}_{\det}|_{\mathcal{P}} := \det H^*(C, \mathrm{Ad}_*\mathcal{P})^{-1}\,.$$

In particular, if $G = \mathrm{GL}_n$, then for any vector bundle $\mathcal{E}$ over C, we have

$$\mathcal{L}_{\det}|_{\mathcal{E}} := \det H^*(C, \mathcal{E}nd(\mathcal{E}))^{-1}\,.$$

Since the Θ-semistable locus determined by $\mathcal{L}_{\det}$ is $\mathscr{B}un_G^{d,\mathrm{ss}}$ (see [Hei17, §§ 1.E and 1.F]), the idea for determining $\left(\mathscr{B}un_G^d\right)_{\mathcal{L}_i}^{\text{a-ss}}$ is to compare $\mathcal{L}_i$ with $\mathcal{L}_{\det}$.

2.2.1 *The case where G is almost simple and simply connected.* In this case $\mathscr{B}un_G$ is connected, and by [Fal03, Theorem 17] we have

$$\mathrm{Pic}(\mathscr{B}un_G) \cong \mathbf{Z}\,.$$

To determine the adequately semistable locus $(\mathscr{B}un_G)_{\mathcal{L}}^{\text{a-ss}}$ for a line bundle $\mathcal{L}$ over $\mathscr{B}un_G$, we may assume that $\mathcal{L} \cong \mathcal{L}_{\det}^{\otimes e}$ for some integer e. Then we compute that

$$\begin{aligned}
(\mathscr{B}un_G)_{\mathcal{L}}^{\text{a-ss}} &= \begin{cases} (\mathscr{B}un_G)_{\mathcal{L}_{\det}}^{\text{a-ss}} & \text{if } e > 0\,,\\ \emptyset & \text{if } e = 0\,,\\ (\mathscr{B}un_G)_{\mathcal{L}_{\det}^{-1}}^{\text{a-ss}} & \text{if } e < 0 \end{cases}\\
&\subseteq \begin{cases} (\mathscr{B}un_G)_{\mathcal{L}_{\det}}^{\Theta\text{-ss}} & \text{if } e > 0\,,\\ \emptyset & \text{if } e = 0\,,\\ (\mathscr{B}un_G)_{\mathcal{L}_{\det}^{-1}}^{\Theta\text{-ss}} & \text{if } e < 0 \end{cases} \quad \text{(by Lemma 2.4)}\\
&= \begin{cases} \mathscr{B}un_G^{\mathrm{ss}} & \text{if } e > 0\,, \quad \text{(by [Hei17, Corollary 1.16])}\\ \emptyset & \text{if } e \leqslant 0\,. \end{cases}
\end{aligned}$$

Indeed, if $e = 0$, then we have $\Gamma(\mathscr{B}un_G, \mathcal{O}_{\mathscr{B}un_G}) = k$ by [BH10, Theorem 5.3.1(i)]. Thus, for any non-zero section $s \in \Gamma(\mathscr{B}un_G, \mathcal{O}_{\mathscr{B}un_G})$, the morphism $(\mathscr{B}un_G)_s = \mathscr{B}un_G \to \mathrm{Spec}(k)$ is not adequately affine since it is not quasi-compact. This shows that $(\mathscr{B}un_G)_{\mathcal{O}_{\mathscr{B}un_G}}^{\text{a-ss}} = \emptyset$. If $e < 0$, we again claim that $(\mathscr{B}un_G)_{\mathcal{L}_{\det}^{-1}}^{\Theta\text{-ss}} = \emptyset$, that is, for any point $\mathcal{P} \in \mathscr{B}un_G(\mathbf{K})$ for some field $\mathbf{K}/k$, there exists a morphism $f\colon \left[\mathbf{A}^1_{\mathbf{K}}/\mathbf{G}_{m,\mathbf{K}}\right] \to \mathscr{B}un_G$ such that $f(1) \cong \mathcal{P}$ and

$$\mathrm{wt}_{\mathbf{G}_{m,\mathbf{K}}}\left(f^*\mathcal{L}_{\det}^{-1}\right) = -\mathrm{wt}_{\mathbf{G}_{m,\mathbf{K}}}(f^*\mathcal{L}_{\det}) > 0.$$

By [Hei17, § 1.F.b] cocharacters of G and reductions of $\mathcal{P}$ to the corresponding parabolic subgroups define testing morphisms for $\mathcal{P}$. To construct the desired reduction, we fix $T \subseteq B \subseteq G$, a maximal torus and a Borel subgroup. Let Φ be the set of roots of G with respect to T. Let $\Phi^+ \subseteq \Phi$ be the positive system of roots determined by B and $\Delta \subseteq \Phi^+$ be the set of simple roots relative to Φ^+. For each $\alpha \in \Delta$ we have

- a rational cocharacter $\omega_\alpha \in X_*(T)_{\mathbf{Q}}$ defined by $\langle \omega_\alpha, \beta\rangle = \delta_{\alpha\beta}$ for any $\beta \in \Delta$,
- a decomposition $\mathrm{Lie}(G) = \oplus_i \mathrm{Lie}(G)_{\alpha,i}$, where $\mathrm{Lie}(G)_{\alpha,i} \subseteq \mathrm{Lie}(G)$ is the subspace on which ω_α acts with weight i. Note that all such weights are integers and each of these spaces is a representation of T.

Choose $n > 0$ such that $\lambda := n\sum_{\alpha\in\Delta}\omega_\alpha \in X_*(T)$. Then the parabolic subgroup $P_\lambda \subseteq G$ defined by λ is B. By [DS95, Proposition 3] there is a reduction $\mathcal{P}_B$ of $\mathcal{P}$ to a B-bundle with

$$\deg\left(\mathcal{P}_B/R_u(B) \times^T \mathrm{Lie}(G)_\alpha\right) < 0 \quad \text{for each } \alpha \in \Phi^+\,, \tag{2.1}$$

as every element in Φ^+ is a non-negative linear combination of simple roots. By [Hei17, § 1.F.b] the cocharacter λ and the reduction $\mathcal{P}_B$ of $\mathcal{P}$ define a morphism $f\colon [\mathbf{A}^1_{\mathbf{K}}/\mathbf{G}_{m,\mathbf{K}}] \to \mathscr{B}un_G$ with $f(1) \cong \mathcal{P}$, and the computation in [Hei17, § 1.F.c] yields

$$\begin{aligned}\mathrm{wt}_{\mathbf{G}_{m,\mathbf{K}}}(f^*\mathcal{L}_{\det}) &= 2n \sum_{\alpha\in\Delta}\sum_{i>0} i\cdot \deg\bigl(\mathcal{P}_B/R_u(B)\times^T \mathrm{Lie}(G)_{\alpha,i}\bigr)\\ &= 2n \sum_{\alpha\in\Delta}\sum_{i>0}\sum_{\langle\omega_\alpha,\beta\rangle=i} i\cdot \deg\bigl(\mathcal{P}_B/R_u(B)\times^T \mathrm{Lie}(G)_{\beta}\bigr) < 0\,.\end{aligned}$$

Indeed, any root $\beta\in\Phi$ with $\langle\omega_\alpha,\beta\rangle = i$ is a positive root since when we express β in terms of simple roots, one of its coefficients (the coefficient of α) is positive. We conclude by (2.1).

Since $\mathscr{U}_i$ is non-empty, it follows that $\mathcal{L}_i \cong \mathcal{L}_{\det}^{\otimes e_i}$ for some integer $e_i > 0$ and thus $\mathscr{U}_i \subseteq \mathscr{B}un_G^{\mathrm{ss}}$ for each i. Hence, $\mathscr{U} \subseteq \mathscr{B}un_G^{\mathrm{ss}}$.

2.2.2 *The general case.* Up to central isogeny, a reductive group is a product of almost simple, simply connected groups and a torus. This turns out to be useful as we observe that admitting a (schematic) moduli space is well behaved under the morphism induced by a central isogeny. Indeed, for any central isogeny of reductive groups

$$1 \longrightarrow \mu \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$$

mapping $e \in \pi_1(\tilde{G})$ to $d \in \pi_1(G)$, the induced morphism $\mathscr{B}un^e_{\tilde{G}} \to \mathscr{B}un^d_G$ is a $\mathscr{B}un_\mu$-torsor (see [BH10, Example 5.1.4]).

Claim 2.6. *The structure morphism $\mathscr{B}un_\mu \to \mathrm{Spec}(k)$ is adequately affine.*

Proof. Since μ is of multiplicative type over an algebraically closed field, we can write $\mu = \prod_i \mu_{n_i}$ for some tuple of positive integers (n_i). Then we have $\mathscr{B}un_\mu \cong \prod_i \mathscr{B}un_{\mu_{n_i}}$. Applying [BH10, Lemma 2.2.1] to the short exact sequence $1 \to \mu_{n_i} \to \mathbf{G}_m \xrightarrow{(-)^{n_i}} \mathbf{G}_m \to 1$ yields an isomorphism $\mathscr{B}un_{\mu_{n_i}} \cong \mathscr{P}ic^0[n_i]$, where $\mathscr{P}ic^0[n_i]$ denotes the n_i-torsion substack of the Picard stack $\mathscr{P}ic^0$. In particular, $\mathscr{B}un_{\mu_{n_i}}$ admits a good moduli space $\mathrm{Pic}^0[n_i]$, the n_i-torsion subgroup of the Picard variety Pic^0. Then the structure morphism $\mathscr{B}un_\mu \to \mathrm{Spec}(k)$ decomposes as

$$\mathscr{B}un_\mu \cong \prod_i \mathscr{B}un_{\mu_{n_i}} \xrightarrow{\mathrm{gms}} \prod_i \mathrm{Pic}^0[n_i] \longrightarrow \mathrm{Spec}(k)\,,$$

where the first morphism is by definition cohomologically affine (in particular, adequately affine) and the second morphism is finite (in particular, affine) since each $\mathrm{Pic}^0[n_i]$ is finite over k (see [Sta24, Tag 03RP(5)]). ☐

Thus, we can show that if an open substack of $\mathscr{B}un^d_G$ admits a (schematic) adequate moduli space, then so does its preimage in $\mathscr{B}un^e_{\tilde{G}}$. To be precise, we have the following. For the definition of a $\mathscr{G}$-torsor, see [BH10, Definition 5.1.3].

Lemma 2.7. *Let $\mathscr{G}$ be a group stack over an algebraically closed field k. Let $\varphi\colon \mathscr{Y} \to \mathscr{X}$ be a $\mathscr{G}$-torsor. If $\mathscr{G} \to \mathrm{Spec}(k)$ is adequately affine, then φ is adequately affine. In particular, if an open substack $\mathscr{U} \subseteq \mathscr{X}$ admits a (schematic) adequate moduli space, then so does $\varphi^{-1}(\mathscr{U}) \subseteq \mathscr{Y}$.*

Proof. By [Alp14, Proposition 4.2.1(4)] the adequate affineness can be checked after a faithfully flat base change. As a $\mathscr{G}$-torsor is faithfully flat and

$$(\mathrm{act}, \mathrm{pr}_2)\colon \mathscr{G}\times_k \mathscr{Y} \longrightarrow \mathscr{Y}\times_{\mathscr{X}} \mathscr{Y}$$

is an isomorphism by definition, the adequate affineness of φ follows from the assumption on $\mathscr{G}$ and [Alp14, Proposition 4.2.1(6)] applied to

$$\begin{tikzcd} \mathscr{G} \times_k \mathscr{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathscr{G} \arrow[d] \\ \mathscr{Y} \arrow[r] & \operatorname{Spec}(k)\,. \end{tikzcd}$$

Moreover, if $\mathscr{U} \subseteq \mathscr{X}$ admits an adequate moduli space $\mathscr{U} \to U$, then by the above argument the $\mathscr{G}$-torsor $\varphi^{-1}(\mathscr{U}) \to \mathscr{U}$ is adequately affine and so is the composition $\psi\colon \varphi^{-1}(\mathscr{U}) \to \mathscr{U} \to U$. Thus, the adequate moduli space of $\varphi^{-1}(\mathscr{U})$ is given by $\varphi^{-1}(\mathscr{U}) \to V := \operatorname{Spec}(\psi_* \mathcal{O}_{\varphi^{-1}(\mathscr{U})})$, see [Alp14, Remark 5.1.2], which is affine over U. Then we conclude that V is a scheme if U is. □

Remark 2.8. For morphisms between moduli stacks of principal bundles (over a general base) induced by a central isogeny of reductive groups, this result can alternatively be proven using [HH23, Lemma A.4], and the induced morphism between moduli spaces is even finite in this case. For the general question in the introduction of whether $\mathscr{B}un_G^{\mathrm{ss}} \subseteq \mathscr{B}un_G$ is the unique maximal open substack that admits a proper moduli space, this stronger assertion can help reduce it to the case where G is almost simple and simply connected.

We fix some notation for the rest of the subsection. Let G be a reductive group, $\tilde{G}$ be the universal cover of the derived subgroup $[G,G]$ of G, and Z^0 be the reduced identity component of the center of G. By [BH10, Lemma 5.3.2] and the discussion following it, there exist an extension of reductive groups

$$1 \longrightarrow \tilde{G} \longrightarrow \hat{G} \xrightarrow{\mathrm{dt}} \mathbf{G}_m \longrightarrow 1$$

and an extension $\hat{\pi}\colon \hat{G} \to G$ of $\tilde{G} \to G$ such that the induced map $\pi_1(\hat{G}) \to \pi_1(G)$ maps $1 \in \mathbf{Z} = \pi_1(\mathbf{G}_m) = \pi_1(\hat{G})$ to $d \in \pi_1(G)$. Moreover, we have a central isogeny

$$1 \longrightarrow \mu \longrightarrow Z^0 \times \hat{G} \xrightarrow{(z^0,\hat{g}) \mapsto (z^0 \cdot \hat{\pi}(\hat{g}), \mathrm{dt}(\hat{g}))} G \times \mathbf{G}_m \longrightarrow 1$$

such that the induced morphism

$$\varphi\colon\ \mathscr{B}un^0_{Z^0} \times \mathscr{B}un^1_{\hat{G}} \longrightarrow \mathscr{B}un^d_G \times \mathscr{P}ic^1$$

is a $\mathscr{B}un_\mu$-torsor. For any line bundle L of degree 1 over C, we denote by $\mathscr{B}un_{\hat{G},L}$ the stack defined by the fiber product

$$\begin{tikzcd} \mathscr{B}un_{\hat{G},L} \arrow[r, "\lambda_L"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathscr{B}un^1_{\hat{G}} \arrow[d, "\mathrm{dt}_*"] \\ \operatorname{Spec}(k) \arrow[r, "L"'] & \mathscr{P}ic^1\,. \end{tikzcd}$$

By base change we again obtain a $\mathscr{B}un_\mu$-torsor

$$\varphi_L\colon\ \mathscr{B}un^0_{Z^0} \times \mathscr{B}un_{\hat{G},L} \longrightarrow \mathscr{B}un^d_G\,.$$

Thus, Theorem A for $\mathscr{B}un^d_G$ follows from the analogous statement for $\mathscr{B}un^0_{Z^0} \times \mathscr{B}un_{\hat{G},L}$ and the fact that the morphism φ_L maps semistable objects to semistable objects. To this end we need a suitable stability condition on $\mathscr{B}un^0_{Z^0} \times \mathscr{B}un_{\hat{G},L}$, in particular on $\mathscr{B}un_{\hat{G},L}$.

DEFINITION 2.9. A geometric point $x \in \mathscr{B}un_{\hat{G},L}(\mathbf{K})$ is *semistable* if $\lambda_L(x) \in \mathscr{B}un^{1,\mathrm{ss}}_{\hat{G}}(\mathbf{K})$, that is,

$$\mathscr{B}un^{\mathrm{ss}}_{\hat{G},L} := \lambda_L^{-1}\big(\mathscr{B}un^{1,\mathrm{ss}}_{\hat{G}}\big) \subseteq \mathscr{B}un_{\hat{G},L}\,.$$

Similarly, we define

$$\big(\mathscr{B}un^0_{Z^0} \times \mathscr{B}un_{\hat{G},L}\big)^{\mathrm{ss}} := \mathscr{B}un^{0,\mathrm{ss}}_{Z^0} \times \mathscr{B}un^{\mathrm{ss}}_{\hat{G},L}\,.$$

Remark 2.10. Consequently, the morphism φ_L maps semistable objects to semistable objects as φ does so, and we have the following commutative diagram:

$$\begin{tikzcd}
\mathscr{B}un^0_{Z^0} \times \mathscr{B}un_{\hat{G},L} \arrow[r, "\varphi_L"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathscr{B}un^d_G \times \mathrm{Spec}(k) \arrow[d, "\mathrm{id}\times L"] \\
\mathscr{B}un^0_{Z^0} \times \mathscr{B}un^1_{\hat{G}} \arrow[r, "\varphi"'] & \mathscr{B}un^d_G \times \mathscr{P}ic^1\,.
\end{tikzcd}$$

The stability condition on $\mathscr{B}un_{\hat{G},L}$ can be described using line bundles. If $\mathcal{L}_{\mathrm{det}}$ is the determinant line bundle of cohomology over $\mathscr{B}un_{\hat{G}}$ and $\mathcal{L}_{\mathrm{det},L}$ is its pullback to $\mathscr{B}un_{\hat{G},L}$, then by [Hei17, Corollary 1.16] we have $\mathscr{B}un^{1,\mathrm{ss}}_{\hat{G}} = \big(\mathscr{B}un^1_{\hat{G}}\big)^{\Theta\text{-ss}}_{\mathcal{L}_{\mathrm{det}}}$, and we claim that

$$\mathscr{B}un^{\mathrm{ss}}_{\hat{G},L} = (\mathscr{B}un_{\hat{G},L})^{\Theta\text{-ss}}_{\mathcal{L}_{\mathrm{det},L}}\,. \tag{2.2}$$

This is a consequence of the following result.

Lemma 2.11. *Let $\mathcal{N}$ be a line bundle over $\mathscr{B}un^1_{\hat{G}}$ such that the reduced identity component $\hat{Z}^0$ of the center of $\hat{G}$ acts trivially. Then the Θ-semistable locus of $\mathscr{B}un^1_{\hat{G}}$ with respect to $\mathcal{N}$ pulls back to the Θ-semistable locus of $\mathscr{B}un_{\hat{G},L}$ with respect to $\lambda_L^*\mathcal{N}$; that is,*

$$(\mathscr{B}un_{\hat{G},L})^{\Theta\text{-ss}}_{\lambda_L^*\mathcal{N}} = \lambda_L^{-1}\big((\mathscr{B}un^1_{\hat{G}})^{\Theta\text{-ss}}_{\mathcal{N}}\big)\,.$$

Proof. Let $\Theta := [\mathbf{A}^1/\mathbf{G}_m]$ be the quotient stack. By definition

$$\lambda_L^{-1}\big((\mathscr{B}un^1_{\hat{G}})^{\Theta\text{-ss}}_{\mathcal{N}}\big) \subseteq (\mathscr{B}un_{\hat{G},L})^{\Theta\text{-ss}}_{\lambda_L^*\mathcal{N}}$$

as any Θ-testing morphism of a point in $\mathscr{B}un_{\hat{G},L}$ with respect to $\lambda_L^*\mathcal{N}$ is a Θ-testing morphism of its image in $\mathscr{B}un_{\hat{G}}$ with respect to $\mathcal{N}$. So it remains to prove the converse. The Cartesian diagram

$$\begin{tikzcd}
\lambda_L\colon\ \mathscr{B}un_{\hat{G},L} \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathscr{B}un_{\hat{G},\cong L} \arrow[r, hook, "\lambda_{\cong L}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathscr{B}un^1_{\hat{G}} \arrow[d, "\mathrm{dt}_*"] \\
L\colon \mathrm{Spec}(k) \arrow[r] & \mathrm{B}\mathbf{G}_m \arrow[r, hook] & \mathscr{P}ic^1
\end{tikzcd}$$

factorizes the morphism λ_L as a closed immersion $\lambda_{\cong L}\colon \mathscr{B}un_{\hat{G},\cong L} \to \mathscr{B}un^1_{\hat{G}}$ followed by a $\mathbf{G}_m$-torsor $g\colon \mathscr{B}un_{\hat{G},L} \to \mathscr{B}un_{\hat{G},\cong L}$. Since any morphism $\Theta \to \mathscr{B}un^1_{\hat{G}}$ mapping 1 into $\mathscr{B}un_{\hat{G},\cong L}$ factors through $\lambda_{\cong L}$, the Θ-semistable locus of $\mathcal{N}$ pulls back to that of $\lambda_{\cong L}^*\mathcal{N}$. It suffices to consider the $\mathbf{G}_m$-torsor part; that is, if $x \in \mathscr{B}un_{\hat{G},L}(\mathbf{K})$ is Θ-semistable with respect to $\lambda_L^*\mathcal{N}$, then we show that $g(x) \in \mathscr{B}un_{\hat{G},\cong L}(\mathbf{K})$ is Θ-semistable with respect to $\lambda_{\cong L}^*\mathcal{N}$. That is, for any morphism $f\colon \Theta \to \mathscr{B}un_{\hat{G},\cong L}$ mapping 1 to $g(x)$, we show that

$$\mathrm{wt}_{\mathbf{G}_m}(f^*\lambda_{\cong L}^*\mathcal{N}) \leqslant 0\,.$$

Clearly, this holds if f lifts to $\mathscr{B}un_{\hat{G},L}$, and we claim that we can assume this by modifying f, with the sign of the weight unchanged. Note that a morphism $f\colon \Theta \to \mathscr{B}un_{\hat{G},\cong L}$ corresponds to two morphisms $\Theta \to \mathscr{B}un^1_{\hat{G}}$ and $\Theta \to \mathrm{B}\mathbf{G}_m$ together with an isomorphism when further mapped to $\mathscr{P}ic^1$, that is, a $\hat{G}$-bundle $\mathcal{P}$ over $C \times \Theta$ and a line bundle $\mathcal{M}$ over Θ together with an isomorphism

$$\mathrm{dt}_*(\mathcal{P}) \cong \mathrm{pr}_C^* L \otimes \mathrm{pr}_\Theta^* \mathcal{M} \quad \text{over } C \times \Theta\,.$$

Then f lifts to $\mathscr{B}un_{\hat{G},L}$ if and only if $\mathcal{M}$ is trivial.

For the first morphism $\Theta \to \mathscr{B}un^1_{\hat{G}}$, there is a modification arising from $\hat{Z}^0$. Choose a cocharacter $\sigma\colon \mathbf{G}_m \to \hat{Z}^0$ such that the composition $\mathbf{G}_m \xrightarrow{\sigma} \hat{Z}^0 \hookrightarrow \hat{G} \xrightarrow{\mathrm{dt}} \mathbf{G}_m$ corresponds to some negative integer $\ell < 0$ (this is possible since $\hat{Z}^0$ is a split torus and $\hat{G}$ is generated by $\hat{Z}^0$ and the derived subgroup $[\hat{G},\hat{G}]$). This yields a morphism $\Theta \to \mathrm{B}\mathbf{G}_m \xrightarrow{\sigma_*} \mathrm{B}\hat{Z}^0 \to \mathscr{B}un^0_{\hat{Z}^0}$, where the morphism $\mathrm{B}\hat{Z}^0 \to \mathscr{B}un^0_{\hat{Z}^0}$ is given by pullback along the projection $C \times S \to S$ for any test scheme S. Then we can modify the original $\Theta \to \mathscr{B}un^1_{\hat{G}}$ by coupling it with the above morphism $\Theta \to \mathscr{B}un^0_{\hat{Z}^0}$ via the morphism $\mathscr{B}un^0_{\hat{Z}^0} \times \mathscr{B}un^1_{\hat{G}} \to \mathscr{B}un^1_{\hat{G}}$ induced by multiplication $\hat{Z}^0 \times \hat{G} \to \hat{G}$. The image $f(1)$ does not change under this modification. This yields a $\hat{G}$-bundle $\mathcal{P}'$ over $C \times \Theta$ with

$$\mathrm{dt}_*(\mathcal{P}') \cong \mathrm{dt}_*(\mathcal{P}) \otimes \mathrm{pr}_\Theta^* \mathcal{M}^{\otimes \ell}\,.$$

As $\hat{Z}^0$ acts trivially on $\mathcal{N}$, we claim that this modification does not change the weight. Indeed, the composition $\hat{Z}^0 \hookrightarrow \hat{G} \xrightarrow{\mathrm{dt}} \mathbf{G}_m$ is surjective as dt is. Then $\mathrm{dt}_*\colon \mathscr{B}un^1_{\hat{G}} \to \mathscr{P}ic^1$ descends to the rigidifications of $\mathscr{B}un^1_{\hat{G}}$ and $\mathscr{P}ic^1$ divided by their central automorphisms such that the following diagram commutes (see [ACV03, Theorem 5.1.5]):

$$\begin{tikzcd}
\mathscr{B}un^1_{\hat{G}} \arrow[r, "\mathrm{dt}_*"] \arrow[d] & \mathscr{P}ic^1 \arrow[d] \\
\left(\mathscr{B}un^1_{\hat{G}}\right)^{\hat{Z}^0} \arrow[r] & \left(\mathscr{P}ic^1\right)^{\mathbf{G}_m} \cong \mathrm{Pic}^1\,.
\end{tikzcd}$$

Note that as $\hat{Z}^0$ is a split torus, $\mathscr{B}un^1_{\hat{G}} \to \left(\mathscr{B}un^1_{\hat{G}}\right)^{\hat{Z}^0}$ is a relative good moduli space. Since $\hat{Z}^0$ acts trivially on $\mathcal{N}$, it descends to the rigidification $\left(\mathscr{B}un^1_{\hat{G}}\right)^{\hat{Z}^0}$ by the analogue of [Alp13, Theorem 10.3], and we are done.

For the second morphism $\Theta \to \mathrm{B}\mathbf{G}_m$, we can precompose it with $(-)^n\colon \Theta \to \Theta$ for any positive integer $n > 0$. Again this modification does not change the sign of the weight and gives a $\hat{G}$-bundle $\mathcal{P}''$ over $C \times \Theta$ with

$$\mathrm{dt}_*(\mathcal{P}'') \cong \mathrm{dt}_*(\mathcal{P}') \otimes \mathrm{pr}_\Theta^* \mathcal{M}^{\otimes n}\,.$$

Therefore, we can take $n = -(\ell + 1)$ to kill the factor $\mathcal{M}$. □

To conclude the proof of Theorem 2.1, it suffices to show the following.

Proposition 2.12. *The semistable locus* $\left(\mathscr{B}un^0_{Z^0} \times \mathscr{B}un_{\hat{G},L}\right)^{\mathrm{ss}} \subseteq \mathscr{B}un^0_{Z^0} \times \mathscr{B}un_{\hat{G},L}$ *is the unique maximal open substack that admits a schematic adequate moduli space.*

Indeed, if $\mathscr{U} \subseteq \mathscr{B}un^d_G$ is an open substack that admits a schematic adequate moduli space, then by Lemma 2.7 so does the preimage $\varphi_L^{-1}(\mathscr{U}) \subseteq \mathscr{B}un^0_{Z^0} \times \mathscr{B}un_{\hat{G},L}$. By Proposition 2.12

$$\varphi_L^{-1}(\mathscr{U}) \subseteq \left(\mathscr{B}un^0_{Z^0} \times \mathscr{B}un_{\hat{G},L}\right)^{\mathrm{ss}},$$

and hence

$$\mathscr{U} = \varphi_L\big(\varphi_L^{-1}(\mathscr{U})\big) \subseteq \varphi_L\big(\left(\mathscr{B}un^0_{Z^0} \times \mathscr{B}un_{\hat{G},L}\right)^{\mathrm{ss}}\big) \subseteq \mathscr{B}un_G^{d,\mathrm{ss}},$$

where the first equality holds since φ_L is a torsor and the final inclusion holds by Remark 2.10.

Proof of Proposition 2.12. If $\mathscr{U} \subseteq \mathscr{B}un^0_{Z^0} \times \mathscr{B}un_{\hat{G},L}$ is an open substack that admits a schematic adequate moduli space, then we show that $\mathscr{U} \subseteq \left(\mathscr{B}un^0_{Z^0} \times \mathscr{B}un_{\hat{G},L}\right)^{\mathrm{ss}}$. As before we may assume that the adequate moduli space is quasi-projective. Since $\mathscr{B}un^0_{Z^0} \times \mathscr{B}un_{\hat{G},L}$ is smooth (see [Hof10b, Proposition 4.1 and Corollary 4.2]), by Theorem 2.3 there exists a line bundle $\mathcal{L}$ over $\mathscr{B}un^0_{Z^0} \times \mathscr{B}un_{\hat{G},L}$ such that

$$\mathscr{U} \subseteq \left(\mathscr{B}un^0_{Z^0} \times \mathscr{B}un_{\hat{G},L}\right)^{\text{a-ss}}_{\mathcal{L}} \subseteq \left(\mathscr{B}un^0_{Z^0} \times \mathscr{B}un_{\hat{G},L}\right)^{\Theta\text{-ss}}_{\mathcal{L}}.$$

To proceed we need some information about $\mathcal{L}$. By [BH10, § 4.4, (11)] we have an isomorphism

$$\mathscr{B}un_{\hat{G},L} \cong \prod_{i=1}^r \mathscr{B}un_{\hat{G}_i,L},$$

where $\hat{G}_i$ is an extension $1 \to \tilde{G}_i \to \hat{G}_i \to \mathbf{G}_m \to 1$ and $\tilde{G} \cong \prod_{i=1}^r \tilde{G}_i$ is the decomposition into almost simple and simply connected factors. By [BH10, Lemma 2.1.4, Proposition 4.4.7(i), Corollary 4.4.5, and Theorem 4.2.1(i)], we can describe the Picard group as follows:

$$\begin{aligned}\operatorname{Pic}\big(\mathscr{B}un^0_{Z^0} \times \mathscr{B}un_{\hat{G},L}\big) &\cong \operatorname{Pic}\big(\mathscr{B}un^0_{Z^0}\big) \oplus \operatorname{Pic}(\mathscr{B}un_{\hat{G},L})\\ &\cong \operatorname{Pic}\big(\mathscr{B}un^0_{Z^0}\big) \oplus \bigoplus_{i=1}^r \operatorname{Pic}(\mathscr{B}un_{\hat{G}_i,L})\\ &\cong \operatorname{Pic}\big(\mathscr{B}un^0_{Z^0}\big) \oplus \bigoplus_{i=1}^r \mathbf{Z}.\end{aligned}$$

Let $\mathcal{L}_{\det,i}$ be the determinant line bundle of cohomology over $\mathscr{B}un_{\hat{G}_i}$ and $\mathcal{L}_{\det,i,L}$ be its pullback to $\mathscr{B}un_{\hat{G}_i,L}$. Recall that both the Θ-semistable and the adequately semistable locus of a line bundle remain unchanged after replacing the line bundle by a positive tensor power. After such a replacement for $\mathcal{L}$ and $\mathcal{L}_{\det,L}$, we have

$$\mathcal{L}_{\det,L} = \boxtimes_{i=1}^r \mathcal{L}^{\otimes c_i}_{\det,i,L} \quad \text{and} \quad \mathcal{L} = \mathcal{L}_0 \boxtimes \boxtimes_{i=1}^r \mathcal{L}^{\otimes e_i}_{\det,i,L} \tag{2.3}$$

for some line bundle $\mathcal{L}_0$ over $\mathscr{B}un^0_{Z^0}$ and integers c_i, e_i. Note that the Θ-semistable locus of $\mathcal{L}$ is the product of the Θ-semistable loci of $\mathcal{L}_0$ and $\mathcal{L}^{\otimes e_i}_{\det,i,L}$ as we can destabilize separately using a filtration on one component and the trivial filtration on the other components. This means that

$$\left(\mathscr{B}un^0_{Z^0} \times \mathscr{B}un_{\hat{G},L}\right)^{\Theta\text{-ss}}_{\mathcal{L}} = \left(\mathscr{B}un^0_{Z^0}\right)^{\Theta\text{-ss}}_{\mathcal{L}_0} \times \prod_{i=1}^r \left(\mathscr{B}un_{\hat{G}_i,L}\right)^{\Theta\text{-ss}}_{\mathcal{L}^{\otimes e_i}_{\det,i,L}}.$$

To obtain a non-empty adequately semistable locus, there is some restriction on $\mathcal{L}_0$.

Lemma 2.13. *Let $\mathbf{T}$ be a split torus. Let $\mathscr{X}$ be a connected algebraic stack that is a $\mathbf{T}$-gerbe over some stack $\overline{\mathscr{X}}$ and $\mathcal{L}$ be a line bundle over $\mathscr{X}$. If $(\mathscr{X})^{\text{a-ss}}_{\mathcal{L}} \neq \emptyset$, then the central $\mathbf{T}$-automorphisms act trivially on $\mathcal{L}$.*

Proof. Let $x \in (\mathscr{X})^{\text{a-ss}}_{\mathcal{L}}(\mathbf{K})$ be a geometric point; that is, there exists a section $s \in \Gamma\big(\mathscr{X}, \mathcal{L}^{\otimes m}\big)$ for some integer $m > 0$ with $s(x) \neq 0$. Consider the morphism $f\colon \mathrm{B}\mathbf{T}_{\mathbf{K}} \hookrightarrow \mathrm{BAut}_{\mathscr{X}(\mathbf{K})}(x) \xrightarrow{x} \mathscr{X}$. Then we have $f^*s \in \Gamma\big(\mathrm{B}\mathbf{T}_{\mathbf{K}}, f^*\mathcal{L}^{\otimes m}\big) \neq 0$. However, for any line bundle $\mathcal{N}$ over $\mathrm{B}\mathbf{T}_{\mathbf{K}}$,

$$\Gamma(\mathrm{B}\mathbf{T}_{\mathbf{K}}, \mathcal{N}) = \Gamma(\mathrm{Spec}(\mathbf{K}), \mathcal{N})^{\mathbf{T}_{\mathbf{K}}} = \begin{cases} \mathbf{K}\cdot e & \text{if } \mathrm{wt}_{\mathbf{T}_{\mathbf{K}}}(\mathcal{N}) \text{ is trivial}\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where $\Gamma(\mathrm{Spec}(\mathbf{K}), \mathcal{N}) = \mathbf{K}\cdot e$ and e is the unique non-vanishing section (up to scalar) of $\mathcal{N}$. In particular, we see that $\mathrm{wt}_{\mathbf{T}_{\mathbf{K}}}(f^*\mathcal{L}^{\otimes m})$ is trivial; that is, the central $\mathbf{T}$-automorphisms act trivially on the fiber $\mathcal{L}_x$. Since $\mathscr{X}$ is connected and the $\mathbf{T}$-weight is locally constant in a family, we are done. □

Claim 2.14. *We have* $\big(\mathscr{B}un^0_{Z^0}\big)^{\Theta\text{-ss}}_{\mathcal{L}_0} = \mathscr{B}un^0_{Z^0} = \mathscr{B}un^{0,\mathrm{ss}}_{Z^0}$.

Proof. We fix an isomorphism $Z^0 \cong \mathbf{G}_m^s$. Then $\mathscr{B}un^0_{Z^0} \cong \prod_{i=1}^s \mathscr{P}ic^0$ is a trivial $\mathbf{G}_m^s$-gerbe over the product $\prod_{i=1}^s \mathrm{Pic}^0$ of Picard varieties. In particular,

$$\mathscr{B}un^{0,\mathrm{ss}}_{Z^0} = \mathscr{B}un^0_{Z^0}\ .$$

As the adequately semistable locus of $\mathcal{L}$ is non-empty, the central Z^0-automorphisms act trivially on $\mathcal{L}$ by Lemma 2.13. By definition the same holds for each $\mathcal{L}_{\det,i,L}$. Thus, the central Z^0-automorphisms act trivially on $\mathcal{L}_0$ as well. By [Alp13, Theorem 10.3] the line bundle $\mathcal{L}_0$ descends to the good moduli space $\mathscr{B}un^0_{Z^0} \to \prod_{i=1}^s \mathrm{Pic}^0$. By Lemma 2.5 such a line bundle does not contribute to the Θ-semistable locus. □

Note that $(\mathscr{B}un_{\hat{G}_i,L})^{\Theta\text{-ss}}_{\mathcal{L}^{\otimes e_i}_{\det,i,L}} \neq \emptyset$ for each i as $\big(\mathscr{B}un^0_{Z^0} \times \mathscr{B}un_{\hat{G},L}\big)^{\Theta\text{-ss}}_{\mathcal{L}} \neq \emptyset$ and by (2.3). Furthermore, by Lemma 2.11 we have

$$\big(\mathscr{B}un_{\hat{G}_i,L}\big)^{\Theta\text{-ss}}_{\mathcal{L}_{\det,i,L}} = \lambda_L^{-1}\big((\mathscr{B}un_{\hat{G}_i})^{\Theta\text{-ss}}_{\mathcal{L}_{\det,i}}\big)\,.$$

As in the almost simple and simply connected case, we have $e_i > 0$ for each i. Similarly, we also have $c_i > 0$ for each i as $(\mathscr{B}un_{\hat{G},L})^{\Theta\text{-ss}}_{\mathcal{L}_{\det,L}} \neq \emptyset$. Finally, we compute

$$\begin{aligned}
\big(\mathscr{B}un^0_{Z^0} \times \mathscr{B}un_{\hat{G},L}\big)^{\Theta\text{-ss}}_{\mathcal{L}} &= \big(\mathscr{B}un^0_{Z^0}\big)^{\Theta\text{-ss}}_{\mathcal{L}_0} \times \prod_{i=1}^r (\mathscr{B}un_{\hat{G}_i,L})^{\Theta\text{-ss}}_{\mathcal{L}^{\otimes e_i}_{\det,i,L}} \\
&= \mathscr{B}un^{0,\mathrm{ss}}_{Z^0} \times \prod_{i=1}^r (\mathscr{B}un_{\hat{G}_i,L})^{\Theta\text{-ss}}_{\mathcal{L}_{\det,i,L}} \quad (\text{Claim } 2.14) \\
&= \mathscr{B}un^{0,\mathrm{ss}}_{Z^0} \times (\mathscr{B}un_{\hat{G},L})^{\Theta\text{-ss}}_{\mathcal{L}_{\det,L}} \quad (\text{by } (2.3)) \\
&= \mathscr{B}un^{0,\mathrm{ss}}_{Z^0} \times \mathscr{B}un^{\mathrm{ss}}_{\hat{G},L} \quad (\text{by } (2.2)) \\
&= \big(\mathscr{B}un^0_{Z^0} \times \mathscr{B}un_{\hat{G},L}\big)^{\mathrm{ss}} \quad (\text{by definition})\,.
\end{aligned}$$

This shows that $\mathscr{U} \subseteq \big(\mathscr{B}un^0_{Z^0} \times \mathscr{B}un_{\hat{G},L}\big)^{\mathrm{ss}}$, and we are done. □

2.2.3 *The vector bundles case.* For readers interested in vector bundles, that is, $G = \mathrm{GL}_n$, there is an alternative argument. As before, it suffices to show that any line bundle $\mathcal{L}$ over $\mathscr{B}un^d_n$ such that $\big(\mathscr{B}un^d_n\big)^{\text{a-ss}}_{\mathcal{L}} \neq \emptyset$ satisfies $\big(\mathscr{B}un^d_n\big)^{\text{a-ss}}_{\mathcal{L}} \subseteq \mathscr{B}un^{d,\mathrm{ss}}_n$. The claim is clear for $n = 1$. Assume that $n \geqslant 2$ in the following. By [BH10, Theorem 5.3.1(iv)] the morphism $\det\colon \mathrm{GL}_n \to \mathbf{G}_m$ induces a commutative diagram of commutative groups

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(\pi_1(\mathbf{G}_m), J_C) \arrow[r] \arrow[d, "\det_*"'] & \mathrm{Pic}\big(\mathscr{P}ic^d\big) \arrow[r] \arrow[d, "\det^*"] & \mathrm{NS}\big(\mathscr{P}ic^d\big) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Hom}(\pi_1(\mathrm{GL}_n), J_C) \arrow[r] & \mathrm{Pic}\big(\mathscr{B}un_n^d\big) \arrow[r] \arrow[d] & \mathrm{NS}\big(\mathscr{B}un_n^d\big) \arrow[r] & 0 \\ & & \mathrm{Pic}(\mathscr{B}un_{n,L}) \arrow[d, equal, "{\text{[BH10, Proposition 4.2.3 and Theorem 4.2.1(i)]}}"] & & \\ & & \mathbf{Z}\,, & & \end{tikzcd}$$

where

- the left-most column is an isomorphism induced by $\det_*\colon \pi_1(\mathrm{GL}_n) \xrightarrow{\sim} \pi_1(\mathbf{G}_m)$, where J_C denotes the Jacobian of C;
- the middle column is exact, induced by the Cartesian diagram

$$\begin{tikzcd} \mathscr{B}un_{n,L} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathscr{B}un_n^d \arrow[d, "\det"] \\ \mathrm{Spec}(k) \arrow[r, "L"'] & \mathscr{P}\mathrm{ic}^d\,, \end{tikzcd}$$

  where L is a fixed line bundle of degree d over C; the morphism $\mathrm{Pic}\big(\mathscr{B}un_n^d\big) \to \mathrm{Pic}(\mathscr{B}un_{n,L})$ is surjective by [Hof12, Theorem 3.1];
- the right-most column is injective by [BH10, Proposition 5.2.11], where $\mathrm{NS}(-)$ denotes the Néron–Severi group.

Applying the snake lemma yields a short exact sequence

$$0 \longrightarrow \mathrm{Pic}\big(\mathscr{P}ic^d\big) \xrightarrow{\det^*} \mathrm{Pic}\big(\mathscr{B}un_n^d\big) \longrightarrow \mathrm{Pic}(\mathscr{B}un_{n,L}) \longrightarrow 0\,.$$

Further, it is split since the morphism $\det\colon \mathscr{B}un_n^d \to \mathscr{P}ic^d$ has a section $\mathcal{N} \mapsto \mathcal{N} \oplus \mathcal{O}_C^{\oplus n-1}$. Then for any line bundle $\mathcal{L}$ over $\mathscr{B}un_n^d$, replacing it with some positive multiple, we may assume that

$$\mathcal{L} \cong \det{}^*\mathcal{L}_0 \otimes \mathcal{L}_{\det}^{\otimes e}$$

for some line bundle $\mathcal{L}_0$ over $\mathscr{P}ic^d$ and integer e. Note that the stack $\mathscr{B}un_n^d$ is a $\mathbf{G}_m$-gerbe over the rigidified stack $\overline{\mathscr{B}un}_n^d := \big[\mathscr{B}un_n^d\,/\mathrm{B}\mathbf{G}_m\big]$ obtained by dividing all automorphisms by $\mathbf{G}_m$ (see [ACV03, Theorem 5.1.5]). By Lemma 2.13 the central $\mathbf{G}_m$-automorphisms act trivially on $\mathcal{L}$. Since in the definition of $\mathcal{L}_{\det}$, we use the determinant of $H^*(C, \mathcal{E}nd(\mathcal{E}))$ instead of $H^*(C, \mathcal{E})$, the central $\mathbf{G}_m$-automorphisms act trivially on $\mathcal{L}_{\det}$ and hence trivially on $\mathcal{L}_0$. Then $\mathcal{L}_0$ descends to the good moduli space $g\colon \mathscr{P}ic^d \to \mathrm{Pic}^d$ (see [Alp13, Theorem 10.3]). Finally, we obtain

$$\mathcal{L} \cong (g \circ \det)^*\mathcal{N} \otimes \mathcal{L}_{\det}^{\otimes e}$$

for some line bundle $\mathcal{N}$ over Pic^d. First we exclude the case $e = 0$.

Claim 2.15. *If* $\big(\mathscr{B}un_n^d\big)_{\mathcal{L}}^{\text{a-ss}} \neq \emptyset$, *then* $e \neq 0$.

Proof. If $e = 0$, then $\mathcal{L} \cong (g \circ \det)^*\mathcal{N}$. Let $L \in \mathrm{Pic}^d(k)$ be a closed point such that the fiber product $\mathscr{X} := \left(\mathscr{B}un_n^d\right)_{\mathcal{L}}^{\text{a-ss}} \times_{\mathrm{Pic}^d} \mathrm{Spec}(k)$ is non-empty, that is,

$$\begin{tikzcd}
\mathscr{X} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & \mathscr{B}un_{n,\cong L} \arrow[rr] \arrow[d, hook] \arrow[drr, phantom, "\lrcorner"] & & \mathrm{Spec}(k) \arrow[d, hook, "L"] \\
\left(\mathscr{B}un_n^d\right)_{\mathcal{L}}^{\text{a-ss}} \arrow[r] & \mathscr{B}un_n^d \arrow[r, "\det"'] & \mathscr{P}ic^d \arrow[r, "g"'] & \mathrm{Pic}^d .
\end{tikzcd}$$

By [Alp14, Lemma 5.2.11] we have

$$\emptyset \neq \mathscr{X} \subseteq (\mathscr{B}un_{n,\cong L})_{\mathcal{L}'}^{\text{a-ss}} \quad \text{for } \mathcal{L}' := \mathcal{L}|_{\mathscr{B}un_{n,\cong L}}.$$

Since $\mathcal{L}$ is pulled back from Pic^d, it follows that $\mathcal{L}' \cong \mathcal{O}_{\mathscr{B}un_{n,\cong L}}$. Note that since $\mathscr{B}un_{n,\cong L}$ is connected, reduced and satisfies the existence part of the valuative criterion for properness, every global section of $\mathcal{O}_{\mathscr{B}un_{n,\cong L}}$ is constant (see also for example [BH10, Proposition 4.4.7(i)]). This implies that $(\mathscr{B}un_{n,\cong L})_{\mathcal{L}'}^{\text{a-ss}} = \emptyset$ since $\mathscr{B}un_{n,\cong L}$ is not quasi-compact, so we have a contradiction. $\square$

Hereafter we assume that $e \neq 0$. By Lemmas 2.4 and 2.5 we have

$$\left(\mathscr{B}un_n^d\right)_{\mathcal{L}}^{\text{a-ss}} \subseteq \left(\mathscr{B}un_n^d\right)_{\mathcal{L}}^{\Theta\text{-ss}} = \left(\mathscr{B}un_n^d\right)_{\mathcal{L}_{\det}^{\otimes e}}^{\Theta\text{-ss}}.$$

Further, we can compute that

$$\begin{aligned}
\left(\mathscr{B}un_n^d\right)_{\mathcal{L}}^{\text{a-ss}} \subseteq \left(\mathscr{B}un_n^d\right)_{\mathcal{L}}^{\Theta\text{-ss}} = \left(\mathscr{B}un_n^d\right)_{\mathcal{L}_{\det}^{\otimes e}}^{\Theta\text{-ss}} &= \begin{cases} \left(\mathscr{B}un_n^d\right)_{\mathcal{L}_{\det}}^{\Theta\text{-ss}} & \text{if } e > 0\,, \\ \left(\mathscr{B}un_n^d\right)_{\mathcal{L}_{\det}^{-1}}^{\Theta\text{-ss}} & \text{if } e < 0 \end{cases} \\
&= \begin{cases} \mathscr{B}un_n^{d,\text{ss}} & \text{if } e > 0 \quad (\text{by [Hei17, Lemma 1.11]})\,, \\ \emptyset & \text{if } e < 0\,. \end{cases}
\end{aligned}$$

Indeed, any vector bundle $\mathcal{E}$ of rank $n \geqslant 2$ admits a sub-line bundle $\mathcal{L}$ with $\mu(\mathcal{L}) < \mu(\mathcal{E})$. By [Hei17, §§ 1.E.b and 1.E.c] the two-step filtration $0 \subseteq \mathcal{L} \subseteq \mathcal{E}$ defines a testing morphism $f\colon [\mathbf{A}^1/\mathbf{G}_m] \to \mathscr{B}un_n^d$ with $f(1) \cong \mathcal{E}$, and it is computed that

$$\mathrm{wt}_{\mathbf{G}_m}\left(f^*\mathcal{L}_{\det}^{-1}\right) = -\mathrm{wt}_{\mathbf{G}_m}(f^*\mathcal{L}_{\det}) = -2(n-1)(\mu(\mathcal{L}) - \mu(\mathcal{E}/\mathcal{L})) > 0\,.$$

This shows that no vector bundle of rank $n \geqslant 2$ is Θ-semistable with respect to $\mathcal{L}_{\det}^{-1}$.

2.3 Applications

Recall that the moduli stack of simple vector bundles admits an (in general non-separated) good moduli space (see [NS65, Theorem 1] or via rigidification by the central $\mathbf{G}_m$ automorphisms as in [ACV03, Theorem 5.1.5]). It is known to experts that this moduli space is not a scheme, and such a fact can be deduced directly from Theorem 2.1.

Corollary 2.16. *If there exist simple unstable vector bundles of rank $n \geqslant 2$ and degree d (for example if $g_C \geqslant 4$), then the good moduli space of simple vector bundles of rank n and degree d is only an algebraic space, not a scheme.*

Remark 2.17. Simple unstable vector bundles of rank $n \geqslant 2$ and degree d are quite abundant. Here we sketch a construction generalizing [NS65, Remark 12.3]. Twisting by a degree 1 line bundle, we may assume that $0 < d \leqslant n$. Choose a stable vector bundle $\mathcal{V}$ of rank $n-1$ and degree d such that $H^0(C, \mathcal{V}) = 0$ and $H^1(C, \mathcal{V}) \neq 0$ (such a vector bundle exists whenever $g_C \geqslant 2$ and $d < (n-1)(g_C - 1)$). For any non-trivial extension $0 \to \mathcal{V} \to \mathcal{E} \to \mathcal{O}_C \to 0$, we claim that $\mathcal{E}$ is unstable and simple. Indeed, observe that any non-zero endomorphism ψ of $\mathcal{E}$ induces an endomorphism of $\mathcal{V}$ since $\mathcal{V}$ is stable of positive slope. As $\mathcal{V}$ is stable, this endomorphism is given by a scalar, say λ. Then the endomorphism $\psi - \lambda \cdot \mathrm{id}_{\mathcal{E}}$ of $\mathcal{E}$ yields a global section of $\mathcal{V}$; otherwise, the defining sequence splits. By assumption $\mathcal{V}$ has no non-zero global sections. This shows that ψ itself is a scalar and hence $\mathcal{E}$ is simple. The conditions $0 < d \leqslant n \pmod{n}$, $g_C \geqslant 2$, and $d < (n-1)(g_C - 1)$ are satisfied in the following cases:

- $g_C \geqslant 4$.
- $g_C = 3$ and $n \geqslant 3$, or $n = 2$ and d is odd.
- $g_C = 2$ and neither of d, $d+1$ is a multiple of n.

This construction can also be used to show that the good moduli space of simple vector bundles is non-separated; see Corollary 3.19.

Finally, we remark that Theorem 2.1 provides another example that the local structure theorem in [AHR20, Theorem 1.1] is optimal. To be precise, we have the following.

Corollary 2.18. *The stack $\mathscr{B}un_G$ is only étale-locally a quotient stack around unstable bundles with linearly reductive stabilizer groups, not Zariski-locally.*

3. Existence criteria for vector bundles

In this section we translate the existence criteria for algebraic stacks to admit separated adequate moduli spaces stated in [AHH23] for vector bundles.

Theorem 3.1 ([AHH23, Theorem 5.4]). *Let $\mathscr{X}$ be a quasi-compact algebraic stack, locally of finite type and with affine diagonal over a field k. Suppose that $\mathscr{X}$ is locally reductive. Then $\mathscr{X}$ admits a separated adequate moduli space X if and only if*

- *$\mathscr{X}$ is Θ-reductive and*
- *$\mathscr{X}$ is S-complete.*

The separated adequate moduli space X is proper if and only if $\mathscr{X}$ in addition satisfies the existence part of the valuative criterion for properness.

3.1 Θ-reductivity

Denote by

- $\mathrm{B}\mathbf{G}_m := [\mathrm{Spec}(\mathbf{Z})/\mathbf{G}_m]$ the classifying stack of the multiplicative group $\mathbf{G}_m$,
- $\Theta := \left[\mathbf{A}^1/\mathbf{G}_m\right]$ the quotient stack defined by the standard contracting action of $\mathbf{G}_m$ on the affine line $\mathbf{A}^1 = \mathrm{Spec}(\mathbf{Z}[t])$.

Both stacks are defined over $\mathrm{Spec}(\mathbf{Z})$ and thus pull back to any base. For any discrete valuation ring (DVR) R with fraction field K and residue field κ, we define $\Theta_R := \Theta \times \mathrm{Spec}(R)$, and let $0 := [0/\mathbf{G}_m] \times \mathrm{Spec}(\kappa)$ be its unique closed point.

Definition 3.2 ([AHH23, Definition 3.10]). A locally Noetherian algebraic stack $\mathscr{X}$ over a field k

is Θ-*reductive* if for every DVR R any commutative diagram

$$\begin{tikzcd} \Theta_R - \{0\} \arrow[r] \arrow[d, hook] & \mathscr{X} \arrow[d] \\ \Theta_R \arrow[r] \arrow[ur, dashed, "\exists!"'] & \mathrm{Spec}(k) \end{tikzcd}$$

of solid arrows can be uniquely filled in.

Remark 3.3. Observe that $\Theta_R - \{0\}$ is covered by the two open substacks

$$\left[\mathbf{A}^1 - \{0\}/\mathbf{G}_m\right] \times \mathrm{Spec}(R) \cong \mathrm{Spec}(R) \quad \text{and} \quad \left[\mathbf{A}^1/\mathbf{G}_m\right] \times \mathrm{Spec}(K) = \Theta_K$$

glued along $\mathrm{Spec}(K)$. Thus, a morphism $\Theta_R - \{0\} \to \mathscr{X}$ corresponds to the data of morphisms $\mathrm{Spec}(R) \to \mathscr{X}$ and $\Theta_K \to \mathscr{X}$ together with an isomorphism of their restrictions to $\mathrm{Spec}(K)$.

To investigate Θ-reductivity for open substacks of $\mathscr{B}un_n^d$, we enlarge the target and classify morphisms $\Theta \to \mathscr{C}oh_n^d$ to the stack of coherent sheaves. By definition these morphisms correspond to coherent sheaves over $C \times \mathbf{A}^1$ flat over $\mathbf{A}^1$ and $\mathbf{G}_m$-equivariant for the action defined on the coordinate parameter, which can be further characterized using the Rees construction.

Lemma 3.4 ([Hei17, Lemma 1.10]). *Let $\mathscr{U} \subseteq \mathscr{C}oh_n^d$ be an open substack. Let T be a locally Noetherian scheme over k. A morphism $\Theta \times T \to \mathscr{U}$ is equivalent to the data of*

- *a coherent sheaf $\mathcal{E}_T$ over $C \times T$, flat over T, such that for any point $t \in T$ the fiber $\mathcal{E}_t$ is coherent of rank n and degree d;*
- *a finite $\mathbf{Z}$-graded ascending filtration $\mathcal{E}_T^\bullet$ of $\mathcal{E}_T$ such that each $\mathcal{E}_T^i/\mathcal{E}_T^{i-1}$ is flat over T and the associated graded sheaf satisfies $\mathrm{gr}(\mathcal{E}_T^\bullet) \in \mathscr{U}(T)$.*

Corollary 3.5 ([ABB+22, Proposition 3.8]). *The stack $\mathscr{C}oh_n^d$ is Θ-reductive.*

Proof. Given a DVR R over k with fraction field K, according to Remark 3.3 and Lemma 3.4, a morphism $\Theta_R - \{0\} \to \mathscr{C}oh_n^d$ is equivalent to a coherent sheaf $\mathcal{E}_R$ over $C \times \mathrm{Spec}(R)$, flat over R, and a finite $\mathbf{Z}$-graded ascending filtration $\mathcal{E}_K^\bullet$ of the generic fiber $\mathcal{E}_K$. Regard $\mathcal{E}_R$ as a subsheaf of $\mathcal{E}_K$, and define $\mathcal{E}_R^i := \mathcal{E}_K^i \cap \mathcal{E}_R$, where the intersection is taken inside $\mathcal{E}_K$. Since $\mathcal{E}_R^i/\mathcal{E}_R^{i-1}$ is a subsheaf of $\mathcal{E}_K^i/\mathcal{E}_K^{i-1}$, it is torsion-free and hence flat over R. By Lemma 3.4 the finite $\mathbf{Z}$-graded ascending filtration $\mathcal{E}_R^\bullet$ of $\mathcal{E}_R$ defines a morphism $\Theta_R \to \mathscr{C}oh_n^d$ extending $\Theta_R - \{0\} \to \mathscr{C}oh_n^d$. The uniqueness of such an extension follows as $\mathcal{E}_R^i \subseteq \mathcal{E}_R$ is the unique subsheaf with generic fiber $\mathcal{E}_K^i$ such that $\mathcal{E}_R^i/\mathcal{E}_R^{i-1}$ is flat over R. $\square$

The fact that $\mathscr{C}oh_n^d$ is Θ-reductive simplifies the verification of Θ-reductivity for its open substacks. If we complete the setup of Θ-reductivity for $\mathscr{U} \subseteq \mathscr{C}oh_n^d$ as

$$\begin{tikzcd} \Theta_R - \{0\} \arrow[r] \arrow[d, hook] \arrow[dr] & \mathscr{U} \arrow[d, hook] \\ \Theta_R \arrow[r, dashed, "\exists!"'] & \mathscr{C}oh_n^d, \end{tikzcd}$$

then the dotted arrow can be uniquely filled in. To check whether $\mathscr{U}$ is Θ-reductive, it suffices to check whether the image of $0 \in \Theta_R$ lies in $\mathscr{U}$. Together with Remark 3.3 and Lemma 3.4, we summarize this as follows.

Proposition 3.6. *An open substack $\mathscr{U} \subseteq \mathscr{C}oh_n^d$ is Θ-reductive if and only if for*

– *every DVR R over k with fraction field K and residue field κ,*
– *every family $\mathcal{E}_R \in \mathscr{U}(R)$, and*
– *every filtration $\mathcal{E}_K^\bullet$ of the generic fiber $\mathcal{E}_K$ with $\mathrm{gr}(\mathcal{E}_K^\bullet) \in \mathscr{U}(K)$,*

the uniquely induced filtration $\mathcal{E}_\kappa^\bullet$ of the special fiber $\mathcal{E}_\kappa$ satisfies $\mathrm{gr}(\mathcal{E}_\kappa^\bullet) \in \mathscr{U}(\kappa)$.

For vector bundles Proposition 3.6 provides a first obstruction to Θ-reductivity as the associated graded sheaf $\mathrm{gr}(\mathcal{E}_\kappa^\bullet)$ might have torsion. Indeed, if an open substack of vector bundles "supports" a family for which the generic fiber admits a subbundle that extends only to a subsheaf of the special fiber, then it cannot be Θ-reductive. Such families are prototypes of the so-called Θ-testing families. To start, we prove a technical lemma that given two families of coherent sheaves, any extension of their special fibers comes from a global one.

Lemma 3.7 (Extension family). *Let R be a DVR over k with fraction field K and residue field κ. Given*

– *two families $\mathcal{G}_{1,R} \in \mathscr{C}oh_{n_1}^{d_1}(R)$ and $\mathcal{G}_{2,R} \in \mathscr{C}oh_{n_2}^{d_2}(R)$, and*
– *an element $[\varrho] \in \mathrm{Ext}^1(\mathcal{G}_{2,\kappa}, \mathcal{G}_{1,\kappa})$,*

there exists an extension family $\mathcal{G}_R \in \mathscr{C}oh_n^d(R)$ of $\mathcal{G}_{2,R}$ by $\mathcal{G}_{1,R}$ with special fiber $[\varrho]$; that is

– *the generic fiber $\mathcal{G}_K$ fits into a short exact sequence*
$$0 \longrightarrow \mathcal{G}_{1,K} \longrightarrow \mathcal{G}_K \longrightarrow \mathcal{G}_{2,K} \longrightarrow 0\,;$$
– *the special fiber $\mathcal{G}_\kappa$ fits into the short exact sequence corresponding to*
$$[\varrho]\colon 0 \longrightarrow \mathcal{G}_{1,\kappa} \longrightarrow \mathcal{G}_\kappa \longrightarrow \mathcal{G}_{2,\kappa} \longrightarrow 0\,.$$

Proof. The desired extension family is constructed via the $\mathscr{E}xt$-stack (see for example [Hof10a, Appendix A]). Consider the stack $\mathscr{E}xt(t_2, t_1)$ parametrizing extensions
$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0\,,$$
where $\mathcal{E}_i$ is a coherent sheaf over C of type $t_i := (n_i, d_i)$ for $i = 1, 2$. It comes with two natural morphisms. Define a morphism $\mathrm{pr}_2\colon \mathscr{E}xt(t_2, t_1) \to \mathscr{C}oh_n^d$ via $[0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0] \mapsto \mathcal{E}$. Furthermore, the assignment $[0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0] \mapsto (\mathcal{E}_1, \mathcal{E}_2)$ defines a morphism
$$\mathrm{pr}_{13}\colon \mathscr{E}xt(t_2, t_1) \longrightarrow \mathscr{C}oh_{n_1}^{d_1} \times \mathscr{C}oh_{n_2}^{d_2}$$
which is a generalized vector bundle (see [Hof10a, Proposition A.3(ii)] or [GHS14, Corollary 3.2]). This implies that in the Cartesian diagram
$$\begin{tikzcd} \mathscr{E}xt(\mathcal{G}_{2,R}, \mathcal{G}_{1,R}) \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner"] & & \mathscr{E}xt(t_2, t_1) \arrow[d, "\mathrm{pr}_{13}"] \\ \mathrm{Spec}(R) \arrow[rr, "(\mathcal{G}_{1,R}, \mathcal{G}_{2,R})"'] \arrow[u, dashed, bend left, "{[s]}"] & & \mathscr{C}oh_{n_1}^{d_1} \times \mathscr{C}oh_{n_2}^{d_2}\,, \end{tikzcd}$$
there exists a 2-term complex $[V_0 \to V_1]$ of vector bundles over $\mathrm{Spec}(R)$ such that
$$\mathscr{E}xt(\mathcal{G}_{2,R}, \mathcal{G}_{1,R}) \cong [V_1/V_0]\,.$$
If there is a section $[s]\colon \mathrm{Spec}(R) \to \mathscr{E}xt(\mathcal{G}_{2,R}, \mathcal{G}_{1,R})$ mapping $\mathrm{Spec}(\kappa)$ to $[\varrho]$, then the composition
$$\mathrm{Spec}(R) \xrightarrow{[s]} \mathscr{E}xt(\mathcal{G}_{2,R}, \mathcal{G}_{1,R}) \longrightarrow \mathscr{E}xt(t_2, t_1) \xrightarrow{\mathrm{pr}_2} \mathscr{C}oh_n^d$$

gives rise to the desired family $\mathcal{G}_R \in \mathscr{C}oh_n^d(R)$. To show the existence of such a section, we lift $[\varrho]$ to $\varrho \in V_{1,\kappa}$, that is,

$$\begin{tikzcd}
\varrho \in V_{1,\kappa} \arrow[r, hook] \arrow[d, two heads] & V_1 \arrow[d, two heads, "\sigma"] \\
{[\varrho] \in \mathscr{E}xt(\mathcal{G}_{2,\kappa}, \mathcal{G}_{1,\kappa})} \arrow[r, hook] \arrow[d] & \mathscr{E}xt(\mathcal{G}_{2,R}, \mathcal{G}_{1,R}) \cong [V_1/V_0] \arrow[d] \\
\operatorname{Spec}(\kappa) \arrow[r, hook] & \operatorname{Spec}(R)\,.
\end{tikzcd}$$

Since $\operatorname{Spec}(R)$ is affine, any element of the special fiber of a quasi-coherent sheaf lifts to a global section over $\operatorname{Spec}(R)$. Thus, there is a section $s\colon \operatorname{Spec}(R) \to V_1$ mapping $\operatorname{Spec}(\kappa)$ to ϱ. Then we take $[s] := \sigma \circ s\colon \operatorname{Spec}(R) \to \mathscr{E}xt(\mathcal{G}_{2,R}, \mathcal{G}_{1,R})$. □

Now we can state the construction of Θ-testing families.

Proposition 3.8 (Θ-testing families). *Let $\mathscr{U} \subseteq \mathscr{B}un_n^d$ be an open substack. Let R be a DVR over k with fraction field K and residue field κ. If there exists a triple*

$$(\mathcal{G}_{1,R}, \mathcal{G}_{2,R}, [\varrho]) \in \mathscr{C}oh_{n_1}^{d_1}(R) \times \mathscr{C}oh_{n_2}^{d_2}(R) \times \operatorname{Ext}^1(\mathcal{G}_{2,\kappa}, \mathcal{G}_{1,\kappa})$$

such that $\mathcal{G}_{1,K} \oplus \mathcal{G}_{2,K} \in \mathscr{U}(K)$, $\mathcal{G}_{1,\kappa} \oplus \mathcal{G}_{2,\kappa} \notin \mathscr{U}(\kappa)$, and $\mathcal{E}([\varrho]) \in \mathscr{U}(\kappa)$, where $\mathcal{E}([\varrho])$ is the extension vector bundle associated to $[\varrho]$, then $\mathscr{U}$ is not Θ-reductive.

Proof. Choose an extension family $\mathcal{G}_R \in \mathscr{C}oh_n^d(R)$ of $\mathcal{G}_{2,R}$ by $\mathcal{G}_{1,R}$ with special fiber $[\varrho]$ using Lemma 3.7 (any such extension family will be called a Θ-testing family). Since $\mathscr{U} \subseteq \mathscr{B}un_n^d$ is open, $\mathcal{G}_\kappa = \mathcal{E}([\varrho]) \in \mathscr{U}(\kappa)$ implies that $\mathcal{G}_R \in \mathscr{U}(R)$. By construction the filtration $\mathcal{G}_K^\bullet$ of $\mathcal{G}_K$ satisfies $\operatorname{gr}(\mathcal{G}_K^\bullet) = \mathcal{G}_{1,K} \oplus \mathcal{G}_{2,K} \in \mathscr{U}(K)$, while the uniquely induced filtration $\mathcal{G}_\kappa^\bullet$ of $\mathcal{G}_\kappa$ does not. Thus by Proposition 3.6 the stack $\mathscr{U}$ is not Θ-reductive. □

As a special case of Θ-testing families, we let the first family be a constant family and the second family be given by an elementary modification of vector bundles (see for example [HL10, Definition 5.2.1]). In this case Proposition 3.8 takes the following simple form.

Corollary 3.9. *Let $\mathscr{U} \subseteq \mathscr{B}un_n^d$ be an open substack. If there exists a quadruple*

$$(\mathcal{E}_1, \mathcal{E}_2, D, [\varrho]) \in \mathscr{B}un_{n_1}^{d_1}(k) \times \mathscr{B}un_{n_2}^{d_2}(k) \times \operatorname{Div}^{\text{eff}}(C) \times \operatorname{Ext}^1\left(\mathcal{E}_2^D \oplus \mathcal{O}_D, \mathcal{E}_1\right),$$

where $\mathcal{E}_2^D \subseteq \mathcal{E}_2$ is an elementary modification of $\mathcal{E}_2$ along the effective divisor D, such that $\mathcal{E}_1 \oplus \mathcal{E}_2 \in \mathscr{U}(k)$ and $\mathcal{E}([\varrho]) \in \mathscr{U}(k)$, where $\mathcal{E}([\varrho])$ is the extension vector bundle associated to $[\varrho]$, then $\mathscr{U}$ is not Θ-reductive.

Proof. Let R be a DVR over k. Let $\mathcal{G}_{1,R} \in \mathscr{C}oh_{n_1}^{d_1}(R)$ be the constant family given by $\mathcal{E}_1$. There is a surjective morphism $\mathcal{E}_2 \to \mathcal{O}_D$ that induces a short exact sequence $0 \to \mathcal{E}_2^D \to \mathcal{E}_2 \to \mathcal{O}_D \to 0$. Let $\mathcal{G}_{2,R} \in \mathscr{C}oh_{n_2}^{d_2}(R)$ be the degeneration family given by the Rees construction applied to the filtration $0 \subseteq \mathcal{E}_2^D \subseteq \mathcal{E}_2$. Then the triple $(\mathcal{G}_{1,R}, \mathcal{G}_{2,R}, [\varrho])$ satisfies the condition of Proposition 3.8, and hence $\mathscr{U}$ is not Θ-reductive. □

3.2 S-completeness

For any DVR R with fraction field K, residue field κ, and uniformizer $\pi \in R$, as in [Hei17, § 2.B], the "separatedness test space" is defined as the quotient stack

$$\overline{\mathrm{ST}}_R := [\operatorname{Spec}(R[x,y]/xy - \pi)/\mathbf{G}_m]\,,$$

where x, y have $\mathbf{G}_m$-weights 1, -1, respectively. Denote by $0 := \mathrm{B}\mathbf{G}_{m,\kappa} = [\mathrm{Spec}(\kappa)/\mathbf{G}_m]$ its unique closed point defined by the vanishing of both x and y.

Definition 3.10 ([AHH23, Definition 3.38]). A locally Noetherian algebraic stack $\mathscr{X}$ over a field k is *S-complete* if, for every DVR R, any commutative diagram

$$\begin{tikzcd} \overline{\mathrm{ST}}_R - \{0\} \arrow[r] \arrow[d, hook] & \mathscr{X} \arrow[d] \\ \overline{\mathrm{ST}}_R \arrow[r] \arrow[ur, dashed, "\exists!"'] & \mathrm{Spec}(k) \end{tikzcd}$$

of solid arrows can be uniquely filled in.

Remark 3.11. Observe that $\overline{\mathrm{ST}}_R - \{0\}$ is covered by the two open substacks

$$\big(\overline{\mathrm{ST}}_R - \{0\}\big)|_{x\neq 0} \cong \mathrm{Spec}(R) \quad \text{and} \quad \big(\overline{\mathrm{ST}}_R - \{0\}\big)|_{y\neq 0} \cong \mathrm{Spec}(R)$$

glued along $\mathrm{Spec}(K)$, that is,

$$\overline{\mathrm{ST}}_R - \{0\} \cong \mathrm{Spec}(R) \bigcup_{\mathrm{Spec}(K)} \mathrm{Spec}(R)\,.$$

Therefore, a morphism $\overline{\mathrm{ST}}_R - \{0\} \to \mathscr{X}$ is the data of two morphisms $\mathrm{Spec}(R) \to \mathscr{X}$ together with an isomorphism of their restrictions to $\mathrm{Spec}(K)$.

To investigate S-completeness for open substacks of $\mathscr{B}un_n^d$, as before we enlarge the target and classify morphisms $\overline{\mathrm{ST}}_R \to \mathscr{C}oh_n^d$. By definition these morphisms correspond to coherent sheaves over $C \times \overline{\mathrm{ST}}_R$ flat over $\overline{\mathrm{ST}}_R$, which can be further characterized as follows.

Lemma 3.12 ([AHH23, Corollary 7.14]). *Let R be a DVR over k with uniformizer π. Then a quasi-coherent sheaf $\mathcal{G}$ over $C\times\overline{\mathrm{ST}}_R$ corresponds to a $\mathbf{Z}$-graded coherent sheaf $\bigoplus_{i\in\mathbf{Z}} \mathcal{G}_i$ over $C\times\mathrm{Spec}(R)$ together with a diagram*

$$\begin{tikzcd} \cdots \arrow[r, bend right, "x"'] & \mathcal{G}_{i-1} \arrow[l, bend right, "y"'] \arrow[r, bend right, "x"'] & \mathcal{G}_i \arrow[l, bend right, "y"'] \arrow[r, bend right, "x"'] & \mathcal{G}_{i+1} \arrow[l, bend right, "y"'] \arrow[r, bend right, "x"'] & \cdots \arrow[l, bend right, "y"'] \end{tikzcd}$$

such that $xy = yx = \pi$. Moreover,

(i) *$\mathcal{G}$ is coherent if each $\mathcal{G}_i$ is coherent, $x\colon \mathcal{G}_{i-1} \to \mathcal{G}_i$ is an isomorphism for $i \gg 0$, and $y\colon \mathcal{G}_i \to \mathcal{G}_{i-1}$ is an isomorphism for $i \ll 0$;*

(ii) *$\mathcal{G}$ is flat over $\overline{\mathrm{ST}}_R$ if and only if the maps x, y, and the induced map $x\colon \mathcal{G}_{i-1}/y\mathcal{G}_i \to \mathcal{G}_i/y\mathcal{G}_{i+1}$ are injective.*

Lemma 3.13. *The stack $\mathscr{C}oh_n^d$ is S-complete.*

Proof. For every DVR R over k and any commutative diagram

$$\begin{tikzcd} \overline{\mathrm{ST}}_R - \{0\} \arrow[r] \arrow[d, hook, "j"'] & \mathscr{C}oh_n^d \arrow[d] \\ \overline{\mathrm{ST}}_R \arrow[r] \arrow[ur, dashed, "\exists!"'] & \mathrm{Spec}(k) \end{tikzcd}$$

of solid arrows, we show that there exists a unique dotted arrow filling it in.

Uniqueness. Let $\mathcal{E}$ be the $\big(\overline{\mathrm{ST}}_R - \{0\}\big)$-flat coherent sheaf over $C \times \big(\overline{\mathrm{ST}}_R - \{0\}\big)$ defined by the morphism $\overline{\mathrm{ST}}_R - \{0\} \to \mathscr{C}oh_n^d$. If a $\overline{\mathrm{ST}}_R$-flat coherent extension exists, then it must be isomorphic to the quasi-coherent sheaf $(\mathrm{id}_C \times j)_*\mathcal{E}$ over $C \times \overline{\mathrm{ST}}_R$ via the adjunction map, as the closed point $0 \in \overline{\mathrm{ST}}_R$ has codimension 2 and $C \times \overline{\mathrm{ST}}_R$ is normal.

Existence. Let K be the fraction field of R and $\pi \in R$ be a uniformizer. The morphism $\overline{\mathrm{ST}}_R - \{0\} \to \mathscr{C}oh_n^d$ is the same as two families $\mathcal{E}_R, \mathcal{E}'_R \in \mathscr{C}oh_n^d(R)$ together with an isomorphism $\mathcal{E}_K \cong \mathcal{E}'_K$ by Remark 3.11. Let $a \in \mathbf{N}$ (respectively, $b \in \mathbf{N}$) be the minimal integer such that $\mathcal{E}_R \subseteq \pi^{-a}\mathcal{E}'_R$ (respectively, $\mathcal{E}'_R \subseteq \pi^{-b}\mathcal{E}_R$). Then we can define a $\mathbf{Z}$-graded coherent sheaf $\oplus_{i\in\mathbf{Z}}\mathcal{G}_i$ over $C \times \mathrm{Spec}(R)$ as follows:

$$\mathcal{G}_i := \begin{cases} \mathcal{E}'_R & \text{if } i \leqslant -b\,, \\ \pi^i\mathcal{E}_R \cap \mathcal{E}'_R & \text{if } -b < i < 0\,, \\ \mathcal{E}_R \cap \pi^{-i}\mathcal{E}'_R & \text{if } 0 \leqslant i < a\,, \\ \mathcal{E}_R & \text{if } i \geqslant a\,. \end{cases}$$

Let $x_i \colon \mathcal{G}_i \to \mathcal{G}_{i+1}$ and $y_i \colon \mathcal{G}_{i+1} \to \mathcal{G}_i$ act via

$$\begin{tikzcd}
\cdots \arrow[r, bend right, "\pi"'] & \mathcal{E}'_R \arrow[l, bend right, "\mathrm{id}"'] \arrow[r, bend right, "\pi"'] & \mathcal{G}_{-(b-1)} \arrow[l, bend right, "\mathrm{icl}"'] \arrow[r, bend right, "\pi"'] & \cdots \arrow[l, bend right, "\mathrm{icl}"'] \arrow[r, bend right, "\pi"'] & \mathcal{E}_R \cap \mathcal{E}'_R \arrow[l, bend right, "\mathrm{icl}"'] \arrow[r, bend right, "\mathrm{icl}"'] & \cdots \arrow[l, bend right, "\pi"'] \arrow[r, bend right, "\mathrm{icl}"'] & \mathcal{G}_{a-1} \arrow[l, bend right, "\pi"'] \arrow[r, bend right, "\mathrm{icl}"'] & \mathcal{E}_R \arrow[l, bend right, "\pi"'] \arrow[r, bend right, "\mathrm{id}"'] & \cdots \arrow[l, bend right, "\pi"'] \\
\cdots & -b & -(b-1) & \cdots & 0 & \cdots & a-1 & a & \cdots
\end{tikzcd}$$

By Lemma 3.12 this diagram defines a coherent sheaf over $C \times \overline{\mathrm{ST}}_R$ flat over $\overline{\mathrm{ST}}_R$, that is, a morphism $\overline{\mathrm{ST}}_R \to \mathscr{C}oh_n^d$ which extends $\overline{\mathrm{ST}}_R - \{0\} \to \mathscr{C}oh_n^d$ by construction. Indeed, it remains to check that the induced map $x_i \colon \mathcal{G}_i/y_i\mathcal{G}_{i+1} \to \mathcal{G}_{i+1}/y_{i+1}\mathcal{G}_{i+2}$ is injective for all $i \in \mathbf{Z}$, that is,

$$\cdots \longrightarrow 0 \longrightarrow \frac{\mathcal{E}'_R}{\pi^{-(b-1)}\mathcal{E}_R \cap \mathcal{E}'_R} \xrightarrow{\pi} \frac{\pi^{-(b-1)}\mathcal{E}_R \cap \mathcal{E}'_R}{\pi^{-(b-2)}\mathcal{E}_R \cap \mathcal{E}'_R} \xrightarrow{\pi} \cdots \xrightarrow{\pi} \frac{\pi^{-1}\mathcal{E}_R \cap \mathcal{E}'_R}{\mathcal{E}_R \cap \mathcal{E}'_R} \xrightarrow{\pi} \frac{\mathcal{E}_R \cap \mathcal{E}'_R}{\pi\mathcal{E}_R \cap \mathcal{E}'_R}$$
$$\xrightarrow{\mathrm{icl}} \frac{\mathcal{E}_R \cap \pi^{-1}\mathcal{E}'_R}{\pi\mathcal{E}_R \cap \pi^{-1}\mathcal{E}'_R} \xrightarrow{\mathrm{icl}} \cdots \xrightarrow{\mathrm{icl}} \frac{\mathcal{E}_R \cap \pi^{-(a-1)}\mathcal{E}'_R}{\pi\mathcal{E}_R} \xrightarrow{\mathrm{icl}} \mathcal{E}_k \xrightarrow{\mathrm{id}} \cdots$$

is injective in each degree. If $i < 0$, then the induced map $x_i \colon \mathcal{G}_i/y_i\mathcal{G}_{i+1} \to \mathcal{G}_{i+1}/y_{i+1}\mathcal{G}_{i+2}$ is injective since in the following commutative diagram the top square is Cartesian:

$$\begin{tikzcd}
\mathcal{G}_{i+1} = \pi^{-(i+1)}\mathcal{E}_R \cap \mathcal{E}'_R \arrow[r, hook, "x_{i+1}=\pi"] \arrow[d, hook, "y_i=\mathrm{icl}"'] \arrow[dr, phantom, "\lrcorner"] & \pi^{-(i+2)}\mathcal{E}_R \cap \mathcal{E}'_R = \mathcal{G}_{i+2} \arrow[d, hook, "y_{i+1}=\mathrm{icl}"] \\
\mathcal{G}_i = \pi^{-i}\mathcal{E}_R \cap \mathcal{E}'_R \arrow[r, hook, "x_i=\pi"'] \arrow[d, two heads] & \pi^{-(i+1)}\mathcal{E}_R \cap \mathcal{E}'_R = \mathcal{G}_{i+1} \arrow[d, two heads] \\
\dfrac{\mathcal{G}_i}{\mathcal{G}_{i+1}} = \dfrac{\pi^{-i}\mathcal{E}_R \cap \mathcal{E}'_R}{\pi^{-(i+1)}\mathcal{E}_R \cap \mathcal{E}'_R} \arrow[r, hook, dashed, "x_i=\pi"'] & \dfrac{\pi^{-(i+1)}\mathcal{E}_R \cap \mathcal{E}'_R}{\pi^{-(i+2)}\mathcal{E}_R \cap \mathcal{E}'_R} = \dfrac{\mathcal{G}_{i+1}}{\mathcal{G}_{i+2}}\,.
\end{tikzcd}$$

A similar argument works for the case $i > 0$. ☐

The fact that $\mathscr{C}oh_n^d$ is S-complete simplifies the verification of S-completeness for its open substacks. If we complete the setup of S-completeness for $\mathscr{U} \subseteq \mathscr{C}oh_n^d$ as

$$\begin{tikzcd} \overline{\mathrm{ST}}_R - \{0\} \arrow[r] \arrow[d, hook] \arrow[dr] & \mathscr{U} \arrow[d, hook] \\ \overline{\mathrm{ST}}_R \arrow[r, dashed, "\exists!"'] & \mathscr{C}oh_n^d \, , \end{tikzcd}$$

then the dotted arrow can be uniquely filled in. To check whether $\mathscr{U}$ is S-complete, it suffices to check whether the image of $0 \in \overline{\mathrm{ST}}_R$ lies in $\mathscr{U}$.

Remark 3.14. The Θ-reductivity and S-completeness of the stack $\mathscr{C}oh(X)$ of coherent sheaves over a proper scheme X over k can be shown directly and uniformly as follows. Let R be a DVR, and let $\mathscr{X}$ be Θ_R or $\overline{\mathrm{ST}}_R$. The unique closed point $0 \in \mathscr{X}$ has codimension 2, and we denote by $\mathscr{U} := \mathscr{X} - \{0\}$ its open complement and by $j \colon X \times \mathscr{U} \hookrightarrow X \times \mathscr{X}$ the open immersion. For any $\mathscr{U}$-flat coherent sheaf $\mathcal{F}$ over $X \times \mathscr{U}$, we claim that $j_*\mathcal{F}$ is coherent and $\mathscr{X}$-flat. For the coherence, note that $\mathcal{F}$ is $\mathscr{U}$-flat, and all of its associated points lie in the generic fiber of the projection $X \times \mathscr{U} \to \mathscr{U}$ by [Sta24, Tag 05DB], so we can apply [Sta24, Tag 0AWA] to conclude. For the $\mathscr{X}$-flatness, we just need to check it around 0. As in the proof of [AHH23, Lemma 7.15], by the local criterion for flatness, it suffices to show that the local Koszul complex

$$0 \longrightarrow j_*\mathcal{F} \xrightarrow{x \oplus (-y)} j_*\mathcal{F} \oplus j_*\mathcal{F} \xrightarrow{y \oplus x} j_*\mathcal{F}$$

is exact. Here x, y will be replaced by the regular sequence π, t at 0 if $\mathscr{X} = \Theta_R$. By $\mathscr{U}$-flatness the sequence over $X \times \mathscr{U}$

$$0 \longrightarrow \mathcal{F} \xrightarrow{x \oplus (-y)} \mathcal{F} \oplus \mathcal{F} \xrightarrow{y \oplus x} \mathcal{F} \longrightarrow 0$$

is exact. Then we conclude by taking j_*, which is left exact.

Moreover, if $\mathcal{E}$ is another $\mathscr{X}$-flat coherent extension of $\mathcal{F}$, then the adjunction map $\mathcal{E} \to j_*\mathcal{F}$ is an isomorphism by [Sta24, Tag 0AVM]. This proves the uniqueness.

By deformation theory any morphism $\overline{\mathrm{ST}}_R \to \mathscr{C}oh_n^d$ is determined by its restriction to the coordinate cross $xy = 0$ as the target is smooth. Since $\overline{\mathrm{ST}}_R|_{xy=0} = \Theta_\kappa \cup_{\mathrm{B}\mathbf{G}_{m,\kappa}} \Theta_\kappa$ (where κ is the residue field of R) and a morphism $\Theta \cup_{\mathrm{B}\mathbf{G}_m} \Theta \to \mathscr{C}oh_n^d$ corresponds to two coherent sheaves with opposite filtrations by Lemma 3.4, we first make this into a definition.

Definition 3.15. Two points $\mathcal{E} \ncong \mathcal{E}' \in \mathscr{C}oh_n^d(\mathbf{K})$ for some field $\mathbf{K}/k$ are said to have *opposite filtrations* if there exist two finite $\mathbf{Z}$-graded filtrations of subsheaves

$$\begin{aligned} \mathcal{E}^\bullet &\colon 0 \subseteq \cdots \subseteq \mathcal{E}^{i-1} \subseteq \mathcal{E}^i \subseteq \mathcal{E}^{i+1} \subseteq \cdots \subseteq \mathcal{E} \, , \\ \mathcal{E}'_\bullet &\colon \mathcal{E}' \supseteq \cdots \supseteq \mathcal{E}'_{i-1} \supseteq \mathcal{E}'_i \supseteq \mathcal{E}'_{i+1} \supseteq \cdots \supseteq 0 \end{aligned}$$

such that $\mathcal{E}^i/\mathcal{E}^{i-1} \cong \mathcal{E}'_i/\mathcal{E}'_{i+1}$ for all $i \in \mathbf{Z}$.

Here are some basic properties of coherent sheaves with opposite filtrations.

Lemma 3.16. *For any field $\mathbf{K}/k$ and any two points $\mathcal{E} \ncong \mathcal{E}' \in \mathscr{C}oh_n^d(\mathbf{K})$ with opposite filtrations $\mathcal{E}^\bullet$, $\mathcal{E}'_\bullet$, there exist*

(i) *isomorphisms $\mathrm{gr}(\mathcal{E}^\bullet) \cong \mathrm{gr}(\mathcal{E}'_\bullet)$ and $\det(\mathcal{E}) \cong \det(\mathcal{E}')$,*

(ii) *non-zero morphisms $h \colon \mathcal{E} \to \mathcal{E}'$ and $h' \colon \mathcal{E}' \to \mathcal{E}$ such that $h \circ h' = h' \circ h = 0$.*

Proof. Part (i) is clear from the definition. For part (ii) let $a \in \mathbf{Z}$ be the minimal integer such that $\mathcal{E}^a = \mathcal{E}$, which is also the minimal integer such that $\mathcal{E}'_{a+1} = 0$. Let $b \in \mathbf{Z}$ be the maximal integer such that $\mathcal{E}'_b = \mathcal{E}'$, which is also the maximal integer such that $\mathcal{E}^{b-1} = 0$. Then we take

$$\begin{aligned} h \colon \mathcal{E} \twoheadrightarrow \mathcal{E}/\mathcal{E}^{a-1} = \mathcal{E}^a/\mathcal{E}^{a-1} \cong \mathcal{E}'_a/\mathcal{E}'_{a+1} = \mathcal{E}'_a \hookrightarrow \mathcal{E}' \,, \\ h' \colon \mathcal{E}' \twoheadrightarrow \mathcal{E}'/\mathcal{E}'_{b+1} = \mathcal{E}'_b/\mathcal{E}'_{b+1} \cong \mathcal{E}^b/\mathcal{E}^{b-1} = \mathcal{E}^b \hookrightarrow \mathcal{E} \,. \end{aligned}$$ □

Now we can prove a modular characterization of S-completeness for coherent sheaves in terms of points with opposite filtrations, where a powerful tool – coherent completeness – is employed.

PROPOSITION 3.17. *An open substack $\mathscr{U} \subseteq \mathscr{C}oh_n^d$ is S-complete if and only if for any field $\mathbf{K}/k$ and $\mathcal{E} \not\cong \mathcal{E}' \in \mathscr{U}(\mathbf{K})$ with opposite filtrations $\mathcal{E}^\bullet$, $\mathcal{E}'_\bullet$, we have $\operatorname{gr}(\mathcal{E}^\bullet) \cong \operatorname{gr}(\mathcal{E}'_\bullet) \in \mathscr{U}(\mathbf{K})$.*

Proof. If part. Let R be a DVR over k with residue field κ. For any commutative diagram

$$\begin{tikzcd} \overline{\mathrm{ST}}_R - \{0\} \arrow[r] \arrow[d, hook] & \mathscr{U} \arrow[d, hook] \\ \overline{\mathrm{ST}}_R \arrow[r, "\exists ! f"'] \arrow[ur, dashed] & \mathscr{C}oh_n^d \end{tikzcd}$$

of solid arrows, where f is the unique morphism given by Lemma 3.13, we show that there exists a dotted arrow filling it in, that is, $f(0) \in \mathscr{U}$. Keep the notation from the proof of Lemma 3.13. By [AHH23, Corollary 7.14] the restriction $f|_{x=0}$ (respectively, $f|_{y=0}$) gives rise to a filtration $\mathcal{E}_\kappa^\bullet$ (respectively, $(\mathcal{E}'_\kappa)_\bullet$) of $\mathcal{E}_\kappa$ (respectively, $\mathcal{E}'_\kappa$)

$$\mathcal{E}_\kappa^\bullet \colon \; \cdots \longrightarrow 0 \longrightarrow \frac{\mathcal{E}'_R}{\pi^{-(b-1)}\mathcal{E}_R \cap \mathcal{E}'_R} \longrightarrow \cdots \longrightarrow \frac{\mathcal{E}_R \cap \pi^{-(a-1)}\mathcal{E}'_R}{\pi \mathcal{E}_R} \longrightarrow \mathcal{E}_\kappa \longrightarrow \cdots$$
$$\left(\text{respectively, } \cdots \leftarrow \mathcal{E}'_\kappa \leftarrow \frac{\pi^{-(b-1)}\mathcal{E}_R \cap \mathcal{E}'_R}{\pi \mathcal{E}'_R} \leftarrow \cdots \leftarrow \frac{\mathcal{E}_R}{\mathcal{E}_R \cap \pi^{-(a-1)}\mathcal{E}'_R} \leftarrow 0 \leftarrow \cdots : (\mathcal{E}'_\kappa)_\bullet \right).$$

Furthermore, we have $f(0) = \operatorname{gr}(\mathcal{E}_\kappa^\bullet) \cong \operatorname{gr}((\mathcal{E}'_\kappa)_\bullet)$. By construction there exists an isomorphism

$$\mathcal{E}_\kappa^i/\mathcal{E}_\kappa^{i-1} \cong (\mathcal{E}'_\kappa)_i/(\mathcal{E}'_\kappa)_{i+1} \quad \text{for all } i \in \mathbf{Z}\,;$$

that is, the two filtrations $\mathcal{E}_\kappa^\bullet$ and $(\mathcal{E}'_\kappa)_\bullet$ are opposite. Then by assumption we have an isomorphism $\operatorname{gr}(\mathcal{E}_\kappa^\bullet) \cong \operatorname{gr}((\mathcal{E}'_\kappa)_\bullet) \in \mathscr{U}(\kappa)$, that is, $f(0) \in \mathscr{U}(\kappa)$, and we are done.

Only if part. Let $\mathcal{E}, \mathcal{E}' \in \mathscr{U}(\mathbf{K})$ be two points (for some field $\mathbf{K}/k$) with opposite filtrations $\mathcal{E}^\bullet$, $\mathcal{E}'_\bullet$. We show that $\operatorname{gr}(\mathcal{E}^\bullet) \cong \operatorname{gr}(\mathcal{E}'_\bullet) \in \mathscr{U}(\mathbf{K})$. For this we define $R := \mathbf{K}[[\pi]]$ and $A := R[x,y]/xy - \pi$. Let $\mathbf{G}_m$ act on A such that x and y have $\mathbf{G}_m$-weights 1 and -1, respectively. The two opposite filtrations $\mathcal{E}^\bullet$, $\mathcal{E}'_\bullet$ define a family of coherent sheaves over the coordinate cross $\operatorname{Spec}(\mathbf{K}[x,y]/xy) \cong \operatorname{Spec}(A/\pi)$. Indeed, applying the Rees construction to $\mathcal{E}^\bullet$ (respectively. $\mathcal{E}'_\bullet$) gives rise to a $\mathbf{G}_m$-invariant family over $\operatorname{Spec}(\mathbf{K}[x]) = \operatorname{Spec}(A/(\pi, y))$ (respectively, $\operatorname{Spec}(\mathbf{K}[y]) = \operatorname{Spec}(A/(\pi, x))$). Using the isomorphisms $\mathcal{E}^i/\mathcal{E}^{i-1} \cong \mathcal{E}'_i/\mathcal{E}'_{i+1}$ to glue these two families at the origin $\operatorname{Spec}(\mathbf{K}) = \operatorname{Spec}(A/(\pi, x, y))$, we obtain a $\mathbf{G}_m$-invariant family over

$$\operatorname{Spec}(\mathbf{K}[x]) \cup_{\operatorname{Spec}(\mathbf{K})} \operatorname{Spec}(\mathbf{K}[y]) \cong \operatorname{Spec}(\mathbf{K}[x,y]/xy) \cong \operatorname{Spec}(A/\pi)\,.$$

This defines a morphism $f_1 \colon [\operatorname{Spec}(A/\pi)/\mathbf{G}_m] \to \mathscr{C}oh_n^d$. Then we want to find compatible lifts

$f_i\colon [\mathrm{Spec}(A/\pi^i)/\mathbf{G}_m] \to \mathscr{C}oh_n^d$ of f_1 for all $i \geqslant 1$, that is, a commutative diagram

$$\begin{tikzcd} \big[\mathrm{Spec}\big(A/\pi^{i-1}\big)/\mathbf{G}_m\big] \arrow[r, "f_{i-1}"] \arrow[d, hook] & \mathscr{C}oh_n^d \arrow[d] \\ \big[\mathrm{Spec}\big(A/\pi^{i}\big)/\mathbf{G}_m\big] \arrow[r] \arrow[ur, dashed, "f_i"] & \mathrm{Spec}(k)\,. \end{tikzcd}$$

Since $\mathscr{C}oh_n^d$ is smooth over k, any morphism $\mathrm{Spec}(A/\pi) \to \mathscr{C}oh_n^d$ can be lifted infinitesimally to $\mathrm{Spec}(A/\pi^i)$ for all $i \geqslant 1$. The obstruction to the existence of a $\mathbf{G}_m$-equivariant lifting lies in $H^1\big(\mathrm{B}\mathbf{G}_m, f_{i-1}^*\mathbf{T}_{\mathscr{C}oh_n^d/k}|_{\mathrm{B}\mathbf{G}_m} \otimes \big(\pi^{i-1}\big)/\big(\pi^i\big)\big) = 0$. Thus there exists a compatible family of morphisms $\big[\mathrm{Spec}\big(A/\pi^i\big)/\mathbf{G}_m\big] \to \mathscr{C}oh_n^d$. Finally, by coherent completeness (see [HR19, Corollary 1.5]), this yields a morphism

$$f\colon \overline{\mathrm{ST}}_R = [\mathrm{Spec}(A)/\mathbf{G}_m] \longrightarrow \mathscr{C}oh_n^d$$

such that $f(0) = \mathrm{gr}(\mathcal{E}^\bullet) \cong \mathrm{gr}(\mathcal{E}'_\bullet)$ and $\overline{\mathrm{ST}}_R - \{0\}$ maps to $\mathscr{U} \subseteq \mathscr{C}oh_n^d$. Since $\mathscr{U}$ is S-complete, $f(0) \in \mathscr{U}$; that is, $\mathrm{gr}(\mathcal{E}^\bullet) \cong \mathrm{gr}(\mathcal{E}'_\bullet) \in \mathscr{U}(\mathbf{K})$. □

To verify S-completeness using Proposition 3.17, we need to know when two coherent sheaves have opposite filtrations. In some situations this can be fully understood; for example the stack $\mathscr{B}un_n^{d,s}$ of stable vector bundles has no points with opposite filtrations by Lemma 3.16(ii) and thus is S-complete. To illustrate this characterization a little bit more, we present one example and one non-example. First we give a quick proof that $\mathscr{B}un_n^{d,\mathrm{ss}}$ is S-complete. For a different proof in a more general setup, see [AHH23, Lemma 8.4].

Corollary 3.18. *The stack $\mathscr{B}un_n^{d,\mathrm{ss}}$ of semistable vector bundles is S-complete; that is, the adequate moduli space of semistable vector bundles of rank n and degree d is separated.*

Proof. By Proposition 3.17 it suffices to prove that if $\mathcal{E} \ncong \mathcal{E}' \in \mathscr{B}un_n^{d,\mathrm{ss}}(\mathbf{K})$ for some field $\mathbf{K}/k$ have opposite filtrations $\mathcal{E}^\bullet, \mathcal{E}'_\bullet$, then any $\mathcal{E}^i/\mathcal{E}^{i-1} \neq 0$ is semistable of slope $\mu := d/n$, as this will imply that $\mathrm{gr}(\mathcal{E}^\bullet) = \mathrm{gr}(\mathcal{E}'_\bullet) \in \mathscr{B}un_n^{d,\mathrm{ss}}(\mathbf{K})$. A simple computation about slopes yields that it is equivalent to show that any $\mathcal{E}^i \neq 0$ has slope μ (and hence is automatically semistable). Let $b \in \mathbf{Z}$ be the maximal integer such that $\mathcal{E}^{b-1} = 0$. We prove by induction that

$$\mu(\mathcal{E}^i) = \mu = \mu(\mathcal{E}'_{i+1}) \quad \text{for } i \geqslant b\,.$$

Consider the morphism $\mathcal{E}' \twoheadrightarrow \mathcal{E}'/\mathcal{E}'_{b+1} = \mathcal{E}'_b/\mathcal{E}'_{b+1} \cong \mathcal{E}^b/\mathcal{E}^{b-1} = \mathcal{E}^b \hookrightarrow \mathcal{E}$. The semistability of $\mathcal{E}$ and $\mathcal{E}'$ yields that $\mu = \mu(\mathcal{E}') \leqslant \mu(\mathcal{E}'/\mathcal{E}'_{b+1}) = \mu\big(\mathcal{E}^b\big) \leqslant \mu(\mathcal{E}) = \mu$ and hence $\mu\big(\mathcal{E}^b\big) = \mu = \mu(\mathcal{E}'_{b+1})$. Suppose that $\mu\big(\mathcal{E}^{i-1}\big) = \mu = \mu(\mathcal{E}'_i)$; then $\mu\big(\mathcal{E}^i\big) = \mu = \mu(\mathcal{E}'_{i+1})$ follows from the inequality

$$\mu = \mu\big(\mathcal{E}^{i-1}\big) \geqslant \mu\big(\mathcal{E}^i\big) \geqslant \mu\big(\mathcal{E}^i/\mathcal{E}^{i-1}\big) = \mu(\mathcal{E}'_i/\mathcal{E}'_{i+1}) \geqslant \mu(\mathcal{E}'_i) = \mu\,.$$

The separatedness of the adequate moduli space follows from [AHH23, Proposition 3.48(2)]. □

Next we show that the good moduli space of simple vector bundles is in general non-separated. This is certainly well known (for example, in the coprime case it follows from a straightforward generalization of [NS65, Remark 12.3]), and we can give a direct proof here.

Corollary 3.19. *Let $n \geqslant 2$ and assume that the triple (g_C, n, d) satisfies one of the conditions in Remark 2.17 (for example $g_C \geqslant 4$). Then the stack $\mathscr{B}un_n^{d,\mathrm{simple}}$ of simple vector bundles is not S-complete; that is, the good moduli space of simple vector bundles of rank n and degree d is non-separated.*

Proof. Twisting by a degree 1 line bundle, we may assume that $0 < d \leqslant n$. By Proposition 3.17 it suffices to find two simple vector bundles with opposite filtrations. Let $\mathcal{V}$ be a stable vector bundle of rank $n-1$ and degree d such that $H^0(C, \mathcal{V}) = 0$ and $H^1(C, \mathcal{V}) \neq 0$. Such a vector bundle exists by our assumption on (g_C, n, d). Then $H^0\big(C, \mathcal{V}^{-1}\big) = 0$ as $\mathcal{V}^{-1}$ is stable of negative slope and $H^1(C, \mathcal{V}^{-1}) \neq 0$ by the Riemann–Roch theorem. A similar argument to the one given in Remark 2.17 yields that any non-split extensions of the form $0 \to \mathcal{V} \to \mathcal{E} \to \mathcal{O}_C \to 0$ and $0 \to \mathcal{O}_C \to \mathcal{E}' \to \mathcal{V} \to 0$ are simple. Furthermore, they have opposite filtrations by construction, and we conclude. The non-separatedness of the good moduli space follows from [AHH23, Proposition 3.48(2)]. □

There is a sufficient condition for vector bundles to have opposite filtrations, which can be seen as a partial inverse to Lemma 3.16.

Proposition 3.20. *Suppose given two points $\mathcal{E} \ncong \mathcal{E}' \in \mathscr{B}un_n^d(\mathbf{K})$ for some field $\mathbf{K}/k$. If there exist*

(i) *an isomorphism $\lambda \colon \det(\mathcal{E}) \xrightarrow{\sim} \det(\mathcal{E}')$ and*

(ii) *morphisms $h \colon \mathcal{E} \to \mathcal{E}'$ of rank r and $h' \colon \mathcal{E}' \to \mathcal{E}$ of rank $n-r$ such that $h \circ h' = h' \circ h = 0$ and the diagram*

$$\begin{tikzcd} \wedge^r(\mathcal{E}) \arrow[r, "\wedge^r(h)"] \arrow[d, "\varphi_{\mathcal{E}}"', "\cong"] & \wedge^r(\mathcal{E}') \arrow[d, "\cong"', "\varphi_{\mathcal{E}'}"] \\ \wedge^{n-r}(\mathcal{E})^{-1} \otimes \det(\mathcal{E}) \arrow[r, "\wedge^{n-r}(h')^{-1} \otimes \lambda"'] & \wedge^{n-r}(\mathcal{E}')^{-1} \otimes \det(\mathcal{E}') \end{tikzcd} \tag{3.1}$$

commutes, where $\varphi_{\mathcal{E}}$ and $\varphi_{\mathcal{E}'}$ are the canonical isomorphisms,

then $\mathcal{E}$ and $\mathcal{E}'$ have opposite filtrations.

Proof. Suppose that there exist morphisms $h \colon \mathcal{E} \to \mathcal{E}'$ of rank r and $h' \colon \mathcal{E}' \to \mathcal{E}$ of rank $n-r$ satisfying the conditions in part (ii). Let $\mathcal{E}_1' := \operatorname{Im}(h') \subseteq \mathcal{E}$ (respectively, $\mathcal{E}_1 := \operatorname{Im}(h) \subseteq \mathcal{E}'$) and $\mathcal{E}_2 \subseteq \mathcal{E}$ (respectively, $\mathcal{E}_2' \subseteq \mathcal{E}'$) be the subbundle generated by $\mathcal{E}_1'$ (respectively, $\mathcal{E}_1$). The conditions $h \circ h' = 0 = h' \circ h$ yield the following commutative diagram with exact rows and columns:

$$\begin{tikzcd} & 0 & & & \\ & \mathcal{G}_1 \arrow[u] & & 0 \arrow[d] & \\ 0 \arrow[r] & \mathcal{E}_2 \arrow[r] \arrow[u] & \mathcal{E} \arrow[r] \arrow[d, shift left, "h"] & \mathcal{E}_1 \arrow[r] \arrow[d, "i"] & 0 \\ 0 & \mathcal{E}_1' \arrow[l] \arrow[u, "i'"] & \mathcal{E}' \arrow[l] \arrow[u, shift left, "h'"] & \mathcal{E}_2' \arrow[l] \arrow[d] & 0 \arrow[l] \\ & 0 \arrow[u] & & \mathcal{G}_2 \arrow[d] & \\ & & & 0\,, & \end{tikzcd} \tag{3.2}$$

where $\mathcal{G}_1 := \mathcal{E}_2/\mathcal{E}_1'$ and $\mathcal{G}_2 := \mathcal{E}_2'/\mathcal{E}_1$ are torsion sheaves of rank 0. By part (i) and the commutativity of (3.1), it follows that $\mathcal{G}_1 \cong \mathcal{G}_2$, and therefore $0 \subseteq \mathcal{E}_1' \subseteq \mathcal{E}_2 \subseteq \mathcal{E}$ and $\mathcal{E}' \supseteq \mathcal{E}_2' \supseteq \mathcal{E}_1 \supseteq 0$ are opposite filtrations of $\mathcal{E}$ and $\mathcal{E}'$. Indeed, an easy computation shows that

$$\mathcal{G}_2 \cong \operatorname{coker}\big(\det(\mathcal{E}_1) \xrightarrow{\det(i)} \det(\mathcal{E}_2')\big)\,,$$
$$\mathcal{G}_1 \cong \operatorname{coker}\big(\det(\mathcal{E}_2)^{-1} \otimes \det(\mathcal{E}) \xrightarrow{\det(i')^{-1}\otimes\lambda} \det(\mathcal{E}_1')^{-1} \otimes \det(\mathcal{E}')\big)\,.$$

Furthermore, using (3.2) we can decompose (3.1) as

$$\begin{tikzcd}
 & \det(\mathcal{E}_1) \arrow[rr, hook, "\det(i)"] \arrow[dd, "\cong"] & & \det(\mathcal{E}_2') \arrow[dl] \arrow[dd, "\cong"] \\
\wedge^r(\mathcal{E}) \arrow[ur] \arrow[rr] \arrow[dd, "\cong"'] & & \wedge^r(\mathcal{E}') \arrow[dd, "\cong"] & \\
 & \det(\mathcal{E}_2)^{-1}\otimes\det(\mathcal{E}) \arrow[rr, hook, "\det(i')^{-1}\otimes\lambda"] & & \det(\mathcal{E}_1')^{-1}\otimes\det(\mathcal{E}') \arrow[dl] \\
\wedge^{n-r}(\mathcal{E})^{-1}\otimes\det(\mathcal{E}) \arrow[ur] \arrow[rr] & & \wedge^{n-r}(\mathcal{E}')^{-1}\otimes\det(\mathcal{E}')\,. &
\end{tikzcd}$$

This gives rise to the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \det(\mathcal{E}_1) \arrow[r, "\det(i)"] \arrow[d, "\cong"'] & \det(\mathcal{E}_2') \arrow[r] \arrow[d, "\cong"] & \mathcal{G}_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \det(\mathcal{E}_2)^{-1}\otimes\det(\mathcal{E}) \arrow[r, "\det(i')^{-1}\otimes\lambda"] \arrow[d, equal] & \det(\mathcal{E}_1')^{-1}\otimes\det(\mathcal{E}') \arrow[r] & \mathcal{G} \arrow[r] & 0 \\
0 \arrow[r] & \det(\mathcal{E}_2)^{-1}\otimes\det(\mathcal{E}) \arrow[r, "\det(i')^{-1}\otimes\mathrm{id}"'] & \det(\mathcal{E}_1')^{-1}\otimes\det(\mathcal{E}) \arrow[r] \arrow[u, "\mathrm{id}\otimes\lambda", "\cong"'] & \mathcal{G}_1 \arrow[r] \arrow[u] & 0\,.
\end{tikzcd}$$

Hence, $\mathcal{G}_1 \cong \mathcal{G} \cong \mathcal{G}_2$. □

Corollary 3.21. *Let $\mathbf{K}/k$ be a field. Two points $\mathcal{E} \ncong \mathcal{E}' \in \mathscr{B}un_n^d(\mathbf{K})$ have opposite filtrations if there exist an isomorphism $\det(\mathcal{E}) \xrightarrow{\sim} \det(\mathcal{E}')$ and a morphism $\mathcal{E} \to \mathcal{E}'$ of rank $n-1$.*

Proof. Let $h\colon \mathcal{E} \to \mathcal{E}'$ be a morphism of rank $n-1$. Using the condition that (3.1) should commute, we can define a rank 1 morphism $h'\colon \mathcal{E}' \to \mathcal{E}$ which satisfies $h \circ h' = h' \circ h = 0$, and we are done by Proposition 3.20. □

Another piece of information from the proof of Proposition 3.17 is that coherent sheaves with opposite filtrations have a common generalization. This can be seen a non-separation phenomenon in the topological sense; we formalize an algebraic analogue of it.

Definition 3.22. Two points $\mathcal{E} \ncong \mathcal{E}' \in \mathscr{C}oh_n^d(\mathbf{K})$ for some field $\mathbf{K}/k$ are said to be *non-separated* if there exist

(i) a DVR R over k with fraction field K and residue field $\mathbf{K}$, and
(ii) two families $\mathcal{G}_R, \mathcal{G}_R' \in \mathscr{C}oh_n^d(R)$ with $\mathcal{G}_K \cong \mathcal{G}_K'$ such that

$$\mathcal{G}_{\mathbf{K}} \cong \mathcal{E} \text{ and } \mathcal{G}_{\mathbf{K}}' \cong \mathcal{E}'.$$

In this case we say that the non-separation of $\mathcal{E}$ and $\mathcal{E}'$ is realized by $\mathcal{G}_R$, $\mathcal{G}'_R$.

It turns out that for coherent sheaves, having opposite filtration and being non-separated are equivalent. This gives a geometric description of coherent sheaves with opposite filtrations.

Corollary 3.23. *Let $\mathbf{K}/k$ be a field. For any two points $\mathcal{E} \not\cong \mathcal{E}' \in \mathscr{C}oh_n^d(\mathbf{K})$, the following are equivalent:*

(i) *The points $\mathcal{E}$ and $\mathcal{E}'$ have opposite filtrations.*

(ii) *The points $\mathcal{E}$ and $\mathcal{E}'$ are non-separated.*

Proof. (i) $\Rightarrow$ (ii) Suppose that $\mathcal{E} \not\cong \mathcal{E}' \in \mathscr{C}oh_n^d(\mathbf{K})$ have opposite filtrations $\mathcal{E}^\bullet$, $\mathcal{E}'_\bullet$. Consider the morphism $f\colon \overline{\mathrm{ST}}_R \to \mathscr{C}oh_n^d$ constructed in the proof of the "only if" part in Proposition 3.17 using these two opposite filtrations. The non-separatedness of $\mathcal{E}$ and $\mathcal{E}'$ is realized by the families corresponding to $f|_{x\neq 0}$ and $f|_{y\neq 0}$.

(ii) $\Rightarrow$ (i) Suppose that the non-separatedness of $\mathcal{E}$ and $\mathcal{E}'$ is realized by $\mathcal{G}_R$, $\mathcal{G}'_R$. This defines a morphism $\overline{\mathrm{ST}}_R - \{0\} \to \mathscr{C}oh_n^d$ extending to $f\colon \overline{\mathrm{ST}}_R \to \mathscr{C}oh_n^d$. By the proof of the "if" part in Proposition 3.17, the restrictions $f|_{x=0}$ and $f|_{y=0}$ define opposite filtrations of $\mathcal{E}$ and $\mathcal{E}'$. □

Remark 3.24. The equivalence established in Corollary 3.23 relates to several non-separation results in the literature. For example, Lemma 3.16 gives an algebraic proof of [Nor79, Proposition 2 and Corollary] or [LO87, Proposition 2.9], and Proposition 3.20 (respectively, Corollary 3.21) gives an algebraic proof of [Nor78, Proposition 2] (respectively, [Nor78, Corollary 1]).

3.3 Local reductivity

In positive characteristic the existence criterion of moduli spaces requires an additional condition called local reductivity, which we recall here.

Definition 3.25 ([AHH23, Definition 2.1]). If $\mathscr{X}$ is an algebraic stack and $x \in |\mathscr{X}|$ is a point, an *étale quotient presentation* around x is a pointed étale morphism $f\colon (\mathscr{W}, w) \to (\mathscr{X}, x)$ of algebraic stacks such that (i) $\mathscr{W} \cong [\mathrm{Spec}(A)/\mathrm{GL}_n]$ for some n and (ii) f induces an isomorphism of stabilizer groups at w; that is, $\mathrm{Aut}_{\mathscr{W}}(p) = \mathrm{Aut}_{\mathscr{X}}(f(p))$ for any point $p \in \mathscr{W}(\mathbf{K})$ representing w, where $\mathbf{K}/k$ is a field.

Remark 3.26. The notion of étale quotient presentation remains unchanged if we require in part (i) that $\mathscr{W} \cong [\mathrm{Spec}(A)/G]$ for some reductive group G. Indeed, for any closed embedding $G \to \mathrm{GL}_n$ we have $[\mathrm{Spec}(A)/G] \cong [((\mathrm{Spec}(A) \times \mathrm{GL}_n)/G)/\mathrm{GL}_n]$ and $(\mathrm{Spec}(A) \times \mathrm{GL}_n)/G$ is affine since G is reductive.

Definition 3.27 ([AHH23, Definition 2.5]). A quasi-separated algebraic stack $\mathscr{X}$ with affine stabilizers is *locally reductive* if every point of $\mathscr{X}$ specializes to a closed point and every closed point admits an étale quotient presentation.

Recall that in characteristic 0, having a separated moduli space implies (actually is equivalent to) Θ-reductivity and S-completeness (see [AHH23, Theorem A]). For principal bundles this implication remains valid in positive characteristic.

Proposition 3.28. *Let C be a smooth projective connected curve over an algebraically closed field k, and let G be a reductive group over k. Then a quasi-compact open substack $\mathscr{U} \subseteq \mathscr{B}un_G$ admits a separated adequate moduli space if and only if it is locally reductive, Θ-reductive, and S-complete.*

Proof. Without loss of generality we may assume that $\mathscr{U} \subseteq \mathscr{B}un_G^d$ for some $d \in \pi_1(G)$. By Theorem 3.1 it remains to prove that if $\mathscr{U} \subseteq \mathscr{B}un_G^d$ admits an adequate moduli space (not necessarily separated) $\mathscr{U} \to U$, then $\mathscr{U}$ is locally reductive. This follows if $\mathscr{U}$ is a quotient stack for a reductive group action. Indeed, if $\mathscr{U} = [X/H]$ for a reductive group H acting on an algebraic space X, then it suffices to show that étale-locally around any point $x \in X$ with closed H-orbit, there is an H-invariant open affine neighbourhood. Let $u \in U$ be the image of x under the composition $X \to \mathscr{U} \to U$, and let $V \to U$ be an étale affine neighbourhood of u. Note that the morphism $X \to \mathscr{U}$ is affine. Then the composition $X \to \mathscr{U} \to U$ is adequately affine and hence affine (see [Alp14, Theorem 4.3.1]). Thus, $X \times_U V \to X$ is an H-invariant étale affine neighbourhood of x, and we are done by Remark 3.26.

Therefore, it remains to show that $\mathscr{U}$ is a quotient stack for a reductive group action. Choose a closed embedding $\phi\colon G \to \mathrm{GL}_n$. This induces a representable morphism $\phi_*\colon \mathscr{B}un_G^d \to \mathscr{B}un_n^e$ (see [Hof10b, Fact 2.3]), where $e = \phi_*(d) \in \mathbf{Z}$. Since $\mathscr{U} \subseteq \mathscr{B}un_G^d$ is quasi-compact and $\mathscr{B}un_n^e$ has an exhaustive filtration given by quotient stacks for GL_N-actions (see for example [Wan11, Lemmas 4.1.5 and 4.1.11]), the restriction $\phi_*|_{\mathscr{U}}\colon \mathscr{U} \to \mathscr{B}un_n^e$ factors through such a quotient stack. As any algebraic stack admitting a representable morphism to a quotient stack is also a quotient stack for an action of the same group, we are done. □

Remark 3.29. The Θ-reductivity and S-completeness of $\mathscr{U}$ can be verified directly and uniformly. Let R be a DVR over k, and let $\mathscr{X}$ be Θ_R or $\overline{\mathrm{ST}}_R$. For any morphism $\mathscr{X} - \{0\} \to \mathscr{U}$, by the universal property of the adequate moduli space (see [AHR19, Theorem 3.12]), the composition $\mathscr{X} - \{0\} \to \mathscr{U} \to U$ factors through $\mathscr{X} - \{0\} \to \mathrm{Spec}(R) \to U$ (the same would hold for the composition of any extension $\mathscr{X} \to \mathscr{U} \to U$) since U is separated. Then we can base change to $\mathrm{Spec}(R)$ and instead prove the Θ-reductivity and S-completeness of the fiber product $\mathscr{U} \times_U \mathrm{Spec}(R) \to \mathrm{Spec}(R)$. In the proof of Proposition 3.28, we saw that $\mathscr{U} \cong [X/\mathrm{GL}_N]$ is a quotient stack and hence $\mathscr{U} \times_U \mathrm{Spec}(R) \cong [X_R/\mathrm{GL}_{N,R}]$. Since X_R is an affine scheme and $\mathrm{GL}_{N,R}$ is a reductive group scheme over R, we conclude by [AHH23, Propositions 3.21(2) and 3.44(2)].

4. Rank 2

Both Θ-reductivity and S-completeness have simple consequences in rank 2. This allows us to apply Proposition 3.28 and obtain the following.

Theorem 4.1. *Let C be a smooth projective connected curve of genus $g_C > 1$ over an algebraically closed field k. Then the open substack $\mathscr{B}un_2^{d,\mathrm{ss}} \subseteq \mathscr{B}un_2^d$ of semistable vector bundles is the unique maximal open substack that admits a separated adequate moduli space.*

The road map is: If $\mathscr{U} \subseteq \mathscr{B}un_2^d$ is an open substack, then

- if $\mathscr{U}$ is Θ-reductive, then it cannot contain any direct sum of line bundles of different degrees (Proposition 4.2);
- if $\mathscr{U}$ is S-complete, then it contains some direct sum of line bundles of different degrees, provided that it contains an unstable vector bundle (Proposition 4.6 and Lemma 4.5).

Thus, to obtain a separated adequate moduli space, we must have $\mathscr{U} \subseteq \mathscr{B}un_2^{d,\mathrm{ss}}$.

4.1 More on Θ-reductivity

Recall that any open substack of $\mathscr{B}un_n^d$ supporting a Θ-testing family cannot be Θ-reductive (see Corollary 3.9). This fact has a simple implication in rank 2.

PROPOSITION 4.2. *Let $\mathscr{U} \subseteq \mathscr{B}un_2^d$ be a Θ-reductive open substack. Then $\mathscr{U}$ cannot contain any direct sum of line bundles of different degrees.*

To prove this we need the following technical lemma.

LEMMA 4.3. *Let $\mathscr{U} \subseteq \mathscr{B}un_2^d$ be an open substack. If $\mathcal{V}_1 \oplus \mathcal{V}_2 \in \mathscr{U}(k)$ for some line bundles $\mathcal{V}_1$, $\mathcal{V}_2$ of degrees $d_1 > d_2$, respectively, then for any effective divisor D on C of degree $d_1 - d_2$, there exist line bundles $\mathcal{L}_1$, $\mathcal{L}_2$ of degrees d_1, d_2, respectively, such that*

$$\mathcal{L}_1 \oplus \mathcal{L}_2 \in \mathscr{U}(k) \quad \textit{and} \quad \mathcal{L}_2(D) \oplus \mathcal{L}_1(-D) \in \mathscr{U}(k)\,.$$

Proof. To start we note that any effective divisor D on C of degree $d_1 - d_2$ induces an automorphism of $\operatorname{Pic}^{d_1} \times \operatorname{Pic}^{d_2}$:

$$\lambda_D \colon \operatorname{Pic}^{d_1} \times \operatorname{Pic}^{d_2} \xrightarrow{\sim} \operatorname{Pic}^{d_1} \times \operatorname{Pic}^{d_2}\,, \quad (\mathcal{L}_1, \mathcal{L}_2) \longmapsto (\mathcal{L}_2(D), \mathcal{L}_1(-D))\,.$$

Let $\oplus \colon \operatorname{Pic}^{d_1} \times \operatorname{Pic}^{d_2} \to \mathscr{B}un_2^d$ denote the direct sum morphism. Then we find that the open subset $\Omega := \oplus^{-1}(\mathscr{U}) \subseteq \operatorname{Pic}^{d_1} \times \operatorname{Pic}^{d_2}$ is non-empty as it contains $(\mathcal{V}_1, \mathcal{V}_2)$, which implies that $\lambda_D^{-1}(\Omega) \cap \Omega \neq \emptyset$. Then any point $(\mathcal{L}_1, \mathcal{L}_2)$ in this intersection satisfies the required conditions. □

Proof of Proposition 4.2. Suppose that $\mathcal{V}_1 \oplus \mathcal{V}_2 \in \mathscr{U}(k)$ for some line bundles $\mathcal{V}_1$, $\mathcal{V}_2$ of degrees $d_1 > d_2$, respectively. Fix an effective divisor D on C of degree $d_1 - d_2$. By Lemma 4.3 there exist line bundles $\mathcal{L}_1$, $\mathcal{L}_2$ of degrees d_1, d_2, respectively such that

$$\mathcal{L}_1 \oplus \mathcal{L}_2 \in \mathscr{U}(k) \quad \text{and} \quad \mathcal{L}_2(D) \oplus \mathcal{L}_1(-D) \in \mathscr{U}(k)\,.$$

Let $[\varrho] \in \operatorname{Ext}^1(\mathcal{O}_D \oplus \mathcal{L}_2, \mathcal{L}_1(-D))$ be the extension class represented by

$$0 \longrightarrow \mathcal{L}_1(-D) \xrightarrow{(\text{incl},0)} \mathcal{L}_1 \oplus \mathcal{L}_2 \longrightarrow \mathcal{O}_D \oplus \mathcal{L}_2 \longrightarrow 0\,.$$

By definition the quadruple

$$(\mathcal{L}_2(D), \mathcal{L}_1(-D), D, [\varrho]) \in \operatorname{Pic}^{d_1}(k) \times \operatorname{Pic}^{d_2}(k) \times \operatorname{Div}^{\text{eff}}(C) \times \operatorname{Ext}^1(\mathcal{O}_D \oplus \mathcal{L}_2, \mathcal{L}_1(-D))$$

satisfies $\mathcal{L}_2(D) \oplus \mathcal{L}_1(-D) \in \mathscr{U}(k)$ and $\mathcal{E}([\varrho]) = \mathcal{L}_1 \oplus \mathcal{L}_2 \in \mathscr{U}(k)$. Then $\mathscr{U}$ is not Θ-reductive by Corollary 3.9, so we have a contradiction. □

4.2 More on S-completeness

Recall that S-completeness can be described in terms of vector bundles with opposite filtrations (see Proposition 3.17), which in rank 2 are simply the ones with the same determinant and non-zero morphisms between them.

LEMMA 4.4. *Let $\mathbf{K}/k$ be a field. Two points $\mathcal{E} \not\cong \mathcal{E}' \in \mathscr{B}un_2^d(\mathbf{K})$ have opposite filtrations if and only if there exist an isomorphism $\det(\mathcal{E}) \xrightarrow{\sim} \det(\mathcal{E}')$ and a non-zero morphism $\mathcal{E} \to \mathcal{E}'$.*

Proof. Only if part. This follows from Lemma 3.16. *If part.* This follows from Corollary 3.21. □

Similarly, S-completeness also has a simple implication in rank 2.

Lemma 4.5. *Let $\mathscr{U} \subseteq \mathscr{B}un_2^d$ be an S-complete open substack. If $\mathbf{K}/k$ is a field extension and $\mathcal{E} \ncong \mathcal{E}' \in \mathscr{U}(\mathbf{K})$ have opposite filtrations with $\mathcal{E}$ unstable, then $\mathscr{U}$ contains a direct sum of line bundles of different degrees.*

Proof. If $\mathcal{E}^\bullet$, $\mathcal{E}'_\bullet$ are opposite filtrations of $\mathcal{E}$ and $\mathcal{E}'$, then we have $\mathrm{gr}(\mathcal{E}^\bullet) \in \mathscr{U}(\mathbf{K})$ by Proposition 3.17; furthermore, $\mathrm{gr}(\mathcal{E}^\bullet)$ is unstable since $\mathcal{E}$ is. □

4.3 Proof of Theorem 4.1

Theorem 4.1 can be deduced from the following result.

Proposition 4.6. *Let $\mathscr{U} \subseteq \mathscr{B}un_2^d$ be an open substack containing an unstable vector bundle. Then there exist two points in $\mathscr{U}$ with opposite filtrations such that one of them is unstable.*

The argument is essentially from the proof of [Nor78, Proposition 4]; that is, we can produce two such points via flipping the Harder–Narasimhan filtration of the unstable vector bundle. To achieve this we need the following technical lemma, motivated by [MS09, § 2].

Lemma 4.7. *Let $\mathscr{U} \subseteq \mathscr{B}un_2^d$ be an open substack containing an unstable vector bundle $\mathcal{E}$ with Harder–Narasimhan filtration $0 \to \mathcal{L}_1 \to \mathcal{E} \to \mathcal{L}_2 \to 0$. Define*

$$\Delta := \{\mathcal{L} \in \mathrm{Pic}^0 : \exists [e] \in \mathrm{Ext}^1\big(\mathcal{L}^{-1} \otimes \mathcal{L}_2, \mathcal{L} \otimes \mathcal{L}_1\big) \text{ such that } \mathcal{E}([e]) \in \mathscr{U}\}\,,$$

where $\mathcal{E}([e])$ is the extension vector bundle associated to $[e]$. Then $\Delta \subseteq \mathrm{Pic}^0$ is open and dense.

Proof. By assumption Δ is non-empty as it contains $\mathcal{O}_C$. It remains to show that $\Delta \subseteq \mathrm{Pic}^0$ is open.

Let $d_i := \deg(\mathcal{L}_i)$ for $i = 1, 2$, and let $\vartheta_1 \colon \mathrm{Pic}^0 \to \mathrm{Pic}^{d_1}$ (respectively, $\vartheta_2 \colon \mathrm{Pic}^0 \to \mathrm{Pic}^{d_2}$) be the morphism given by $\mathcal{L} \mapsto \mathcal{L} \otimes \mathcal{L}_1$ (respectively, $\mathcal{L} \mapsto \mathcal{L}^{-1} \otimes \mathcal{L}_2$). Denote by $\mathcal{P}_i \to C \times \mathrm{Pic}^{d_i}$ the universal family and by $\mathcal{Q}_i := (\mathrm{id}_C \times \vartheta_i)^*\mathcal{P}_i$ its pullback to $C \times \mathrm{Pic}^0$. The two families define a Cartesian diagram (see Lemma 3.7)

$$\begin{tikzcd} \mathscr{E}xt(\mathcal{Q}_2, \mathcal{Q}_1) \arrow[r, "\psi"] \arrow[d, "\mathrm{pr}_{13}"'] \arrow[dr, phantom, "\lrcorner"] & \mathscr{E}xt(d_2, d_1) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_{13}"] & \mathscr{C}oh_2^d \\ \mathrm{Pic}^0 \arrow[r, "(\mathcal{Q}_1, \mathcal{Q}_2)"'] & \mathscr{C}oh_1^{d_1} \times \mathscr{C}oh_1^{d_2}\,. & \end{tikzcd}$$

Then $\mathrm{pr}_{13}^{-1}(\mathcal{L}) = \mathscr{E}xt(\mathcal{Q}_2, \mathcal{Q}_1)|_{\mathcal{L}} = \big[\mathrm{Ext}^1\big(\mathcal{L}^{-1} \otimes \mathcal{L}_2, \mathcal{L} \otimes \mathcal{L}_1\big)/\mathrm{Hom}\big(\mathcal{L}^{-1} \otimes \mathcal{L}_2, \mathcal{L} \otimes \mathcal{L}_1\big)\big]$ for any $\mathcal{L} \in \mathrm{Pic}^0$. By definition $\Delta = \mathrm{pr}_{13} \circ (\mathrm{pr}_2 \circ \psi)^{-1}(\mathscr{U})$, which is open in Pic^0 since pr_{13} is smooth. □

Proof of Proposition 4.6. Let $\mathcal{E} \in \mathscr{U}(k)$ be an unstable vector bundle with Harder–Narasimhan filtration $0 \to \mathcal{L}_1 \to \mathcal{E} \to \mathcal{L}_2 \to 0$. Fix a line bundle $\beta \in \mathrm{Pic}^{d_1 - d_2}$ admitting a non-zero global section, where $d_i := \deg(\mathcal{L}_i)$. Consider the open dense subset $\Delta \subseteq \mathrm{Pic}^0$ defined in Lemma 4.7. Then the intersection $\Delta \cap \big(\mathcal{L}_1 \otimes \mathcal{L}_2^{-1} \otimes \beta^{-1} \otimes \Delta\big)^{-1}$ is non-empty. For any point $\mathcal{L}$ in this intersection, that is, $\mathcal{L} \in \Delta$ and $\mathcal{L}^{-1} \otimes \mathcal{L}_1^{-1} \otimes \mathcal{L}_2 \otimes \beta \in \Delta$, by definition there exist $[x] \in \mathrm{Ext}^1\big(\mathcal{L}^{-1} \otimes \mathcal{L}_2, \mathcal{L} \otimes \mathcal{L}_1\big)$ and $[y] \in \mathrm{Ext}^1\big(\mathcal{L} \otimes \mathcal{L}_1 \otimes \beta^{-1}, \mathcal{L}^{-1} \otimes \mathcal{L}_2 \otimes \beta\big)$ such that both $\mathcal{E}([x])$ and $\mathcal{E}([y])$ lie in $\mathscr{U}$, that is,

$$0 \longrightarrow \mathcal{L} \otimes \mathcal{L}_1 \longrightarrow \mathcal{E}([x]) \longrightarrow \mathcal{L}^{-1} \otimes \mathcal{L}_2 \longrightarrow 0\,,$$
$$0 \longrightarrow \mathcal{L}^{-1} \otimes \mathcal{L}_2 \otimes \beta \longrightarrow \mathcal{E}([y]) \longrightarrow \mathcal{L} \otimes \mathcal{L}_1 \otimes \beta^{-1} \longrightarrow 0\,.$$

Then $\mathcal{E}([x])$ and $\mathcal{E}([y])$ are both unstable with the same determinant, and there exist non-zero morphisms between them. By Lemma 4.4 they have opposite filtrations, and we are done. □

4.4 Generalizing to arbitrary base field

The maximality-type results Theorems A and B hold over an arbitrary field k since we can reduce to the case where the base field is algebraically closed. To indicate the base curve, we temporarily use $\mathscr{B}un_G(C)$ instead of $\mathscr{B}un_G$. Fix an algebraic closure $\bar{k}$ of k.

LEMMA 4.8. *Let C be a smooth projective geometrically connected curve over a field k, and let G be a geometrically connected reductive group over k. Then we have the following: if the open substack $\mathscr{B}un^{\mathrm{ss}}_{G_{\bar{k}}}(C_{\bar{k}}) \subseteq \mathscr{B}un_{G_{\bar{k}}}(C_{\bar{k}})$ of semistable principal $G_{\bar{k}}$-bundles over $C_{\bar{k}}$ is the unique maximal open substack that admits a (schematic or separated) adequate moduli space, then so is the open substack $\mathscr{B}un^{\mathrm{ss}}_G(C) \subseteq \mathscr{B}un_G(C)$ of semistable principal G-bundles over C.*

Proof. If $\mathscr{U} \subseteq \mathscr{B}un_G(C)$ is an open substack that admits a (schematic or separated) adequate moduli space, then so is the base change $\mathscr{U}_{\bar{k}} \subseteq \mathscr{B}un_G(C)_{\bar{k}} = \mathscr{B}un_{G_{\bar{k}}}(C_{\bar{k}})$ by [Alp14, Proposition 5.2.9(1)]. In a diagram, we have

$$\begin{tikzcd}
\mathscr{U}_{\bar{k}} \arrow[r] \arrow[d, "\mathrm{ams}"'] \arrow[dr, phantom, "\lrcorner"] & \mathscr{U} \arrow[d, "\mathrm{ams}"] \\
U_{\bar{k}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & U \arrow[d] \\
\mathrm{Spec}(\bar{k}) \arrow[r] & \mathrm{Spec}(k)\,.
\end{tikzcd}$$

By assumption we obtain $\mathscr{U}_{\bar{k}} \subseteq \mathscr{B}un^{\mathrm{ss}}_{G_{\bar{k}}}(C_{\bar{k}})$. As a G-bundle $\mathcal{P}$ over C is semistable if the base change $\mathcal{P}_{\bar{k}}$ over $C_{\bar{k}}$ is semistable, this shows that $\mathscr{U} \subseteq \mathscr{B}un^{\mathrm{ss}}_G(C)$. $\square$

5. Higher rank

In this section we construct an open substack $\mathscr{U} \subseteq \mathscr{B}un^2_3$ that admits a separated non-proper adequate moduli space and is not contained in $\mathscr{B}un^{2,\mathrm{ss}}_3$.

Throughout this section we assume that $g_C > 2$. This guarantees the existence of $(1,0)$-stable vector bundles over C of arbitrary rank and degree (see [NR78, Proposition 5.4(i)]). Recall [NR78, Definition 5.1] that for any pair of non-negative integers (i,j), a vector bundle $\mathcal{E}$ over C is said to be (i,j)-*stable* if for any non-zero proper subbundle $0 \neq \mathcal{F} \subsetneq \mathcal{E}$, we have

$$\frac{\deg(\mathcal{F}) + i}{\mathrm{rk}(\mathcal{F})} < \frac{\deg(\mathcal{E}) + i - j}{\mathrm{rk}(\mathcal{E})}\,.$$

5.1 Construction of $\mathscr{U} \subseteq \mathscr{B}un^2_3$

Let λ be the polygon in the rank-degree plane consisting of vertices $\{(0,0),(1,1),(3,2)\}$, and denote by $\mathscr{B}un^{2,\leqslant\lambda}_3 \subseteq \mathscr{B}un^2_3$ the open substack consisting of vector bundles whose Harder–Narasimhan polygons lie below λ.

LEMMA 5.1. *Let $\mathcal{E} \in \mathscr{B}un^{2,\leqslant\lambda}_3(\mathbf{K})$ for some field $\mathbf{K}/k$ and $\mathcal{F} \subseteq \mathcal{E}$ be a subsheaf of rank at most 2. Then $\deg(\mathcal{F}) \leqslant 1$. In particular, if $\deg(\mathcal{F}) = 1$, then $\mathcal{F} \subseteq \mathcal{E}$ is a subbundle.*

Proof. If $\deg(\mathcal{F}) > 1$, then the Harder–Narasimhan polygon of $\mathcal{E}$ lies strictly above λ. $\square$

The desired open substack $\mathscr{U} \subseteq \mathscr{B}un^2_3$ will be an open substack of $\mathscr{B}un^{2,\leqslant\lambda}_3$. Since we want it to be essentially different from $\mathscr{B}un^{2,\mathrm{ss}}_3$, we need to include some unstable vector bundles.

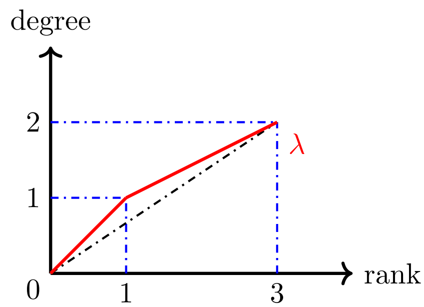

FIGURE 1. The polygon λ in the rank-degree plane.

However, not every unstable vector bundle in $\mathscr{B}un_3^{2,\leqslant\lambda}$ is allowed. The S-completeness gives a first restriction: those points $\mathcal{E} \in \mathscr{B}un_3^{2,\leqslant\lambda}$ with Harder–Narasimhan filtration $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$ such that $\mathrm{Hom}(\mathcal{F}, \mathcal{L}) \neq 0$ should be removed. Then Θ-reductivity also removes those points $\mathcal{E} \in \mathscr{B}un_3^{2,\leqslant\lambda}$ fitting into a short exact sequence $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{L} \to 0$ such that $\mathrm{Hom}(\mathcal{F}, \mathcal{L}) \neq 0$. To retain Θ-reductivity in this new substack, it suffices to remove, in both situations, those points $\mathcal{E} \in \mathscr{B}un_3^{2,\leqslant\lambda}$ such that $\mathcal{F}$ is not $(1,0)$-stable. Note that $\mathrm{Hom}(\mathcal{F}, \mathcal{L}) \neq 0$ implies that $\mathcal{F}$ is not $(1,0)$-stable as any non-zero morphism from $\mathcal{F}$ to $\mathcal{L}$ would yield a sub-line bundle of degree at least 0. In summary we should cut out the subset

$$\mathscr{A} := \left\langle \mathcal{E} \in \left|\mathscr{B}un_3^{2,\leqslant\lambda}\right| : \begin{array}{c} \mathcal{E} \text{ fits into } 0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0 \text{ or} \\ 0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0 \\ \text{for some } \mathcal{L} \in \mathscr{B}un_1^1 \text{ and } \mathcal{F} \in \mathscr{B}un_2^1 - \mathscr{B}un_2^{1,s}(1,0) \end{array} \right\rangle,$$

where $\mathscr{B}un_2^{1,s}(1,0) \subseteq \mathscr{B}un_2^1$ denotes the open substack of $(1,0)$-stable vector bundles. Note that a rank 2, degree 1 vector bundle is $(1,0)$-stable if and only if all of its sub-line bundles have degrees less than 0. To start we show that $\mathscr{A}$ is a closed subset.

LEMMA 5.2. *The subset* $\mathscr{A} \subseteq \left|\mathscr{B}un_3^{2,\leqslant\lambda}\right|$ *is closed.*

Proof. The subset $\mathscr{A} \subseteq \left|\mathscr{B}un_3^{2,\leqslant\lambda}\right|$ is constructible; to conclude we claim that it is stable under specialization. Let R be a DVR over k with fraction field K and residue field κ. For any family $\mathcal{E}_R \in \mathscr{B}un_3^{2,\leqslant\lambda}(R)$ with $\mathcal{E}_K \in \mathscr{A}(K)$, we show that $\mathcal{E}_\kappa \in \mathscr{A}(\kappa)$. If $0 \to \mathcal{F}_K \to \mathcal{E}_K \to \mathcal{L}_K \to 0$ is a defining sequence of $\mathcal{E}_K \in \mathscr{A}(K)$, we consider its degeneration $0 \to \mathcal{F}_\kappa \to \mathcal{E}_\kappa \to \mathcal{L}_\kappa \to 0$, and then $\mathcal{F}_\kappa \subseteq \mathcal{E}_\kappa$ is a subbundle by Lemma 5.1. Since $\mathcal{F}_K$ is not (1,0)-stable, neither is $\mathcal{F}_\kappa$. This shows that $\mathcal{E}_\kappa \in \mathscr{A}(\kappa)$. The same argument applies if $\mathcal{E}_K \in \mathscr{A}(K)$ is realized by a sequence of the form $0 \to \mathcal{L}_K \to \mathcal{E}_K \to \mathcal{F}_K \to 0$. $\square$

Then we equip the closed subset $\mathscr{A} \subseteq \left|\mathscr{B}un_3^{2,\leqslant\lambda}\right|$ with the reduced induced algebraic stack structure so that it becomes a closed substack of $\mathscr{B}un_3^{2,\leqslant\lambda}$. Let

$$\mathscr{U} := \mathscr{B}un_3^{2,\leqslant\lambda} - \mathscr{A} \subseteq \mathscr{B}un_3^{2,\leqslant\lambda} \subseteq \mathscr{B}un_3^2,$$

that is,

$$\mathscr{U} = \left\langle \mathcal{E} \in \mathscr{B}un_3^{2,\leqslant\lambda} : \begin{array}{c} \text{If } \mathcal{E} \text{ fits into } 0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0 \text{ or} \\ 0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0 \\ \text{for some } \mathcal{L} \in \mathscr{B}un_1^1 \text{ and } \mathcal{F} \in \mathscr{B}un_2^1, \text{ then } \mathcal{F} \text{ is } (1,0)\text{-stable} \end{array} \right\rangle;$$

it is an open substack. To close we show that $\mathscr{U}$ contains unstable vector bundles.

LEMMA 5.3. *Suppose $\mathcal{L} \in \mathscr{B}un_1^1(\mathbf{K})$ and $\mathcal{F} \in \mathscr{B}un_2^1(\mathbf{K})$ for some field $\mathbf{K}/k$. Then $\mathcal{L} \oplus \mathcal{F} \in \mathscr{U}(\mathbf{K})$ if and only if $\mathcal{F}$ is $(1,0)$-stable. This exhausts all decomposable vector bundles in $\mathscr{U}$. In particular, the open substack $\mathscr{U}$ is non-empty and $\mathscr{U} \nsubseteq \mathscr{B}un_3^{2,\mathrm{ss}}$.*

Proof. As $\mathscr{B}un_3^{2,\leqslant\lambda}$ does not contain direct sums of line bundles, direct sums of the form $\mathcal{L} \oplus \mathcal{F}$ exhaust all decomposable vector bundles in $\mathscr{U}$ by Lemma 5.1.

Only if part. This is clear from the definition of $\mathscr{U}$.

If part. By the uniqueness of the Harder–Narasimhan filtration, $\mathcal{L} \oplus \mathcal{F}$ cannot be realized as a point in $\mathscr{A}$ via a sequence of the form $0 \to \mathcal{L}' \to \mathcal{L} \oplus \mathcal{F} \to \mathcal{F}' \to 0$ for some $\mathcal{L}' \in \mathscr{B}un_1^1(\mathbf{K})$ and $\mathcal{F}' \in \mathscr{B}un_2^1(\mathbf{K})$. If $\mathcal{L} \oplus \mathcal{F}$ fits into a sequence $0 \to \mathcal{F}' \to \mathcal{L} \oplus \mathcal{F} \to \mathcal{L}' \to 0$, then we consider the composition $\mathcal{L} \hookrightarrow \mathcal{L} \oplus \mathcal{F} \twoheadrightarrow \mathcal{L}'$.

If it is non-zero, then $\mathcal{L} \cong \mathcal{L}'$ and hence $\mathcal{F}' \cong \mathcal{F}$. This sequence does not realize $\mathcal{L} \oplus \mathcal{F}$ as a point in $\mathscr{A}$. If it is zero, then it factors through $\mathcal{L} \hookrightarrow \mathcal{F}'$ and hence $\mathcal{F}' \cong \mathcal{L} \oplus \mathcal{L}_0$ for some degree 0 sub-line bundle $\mathcal{L}_0 \subseteq \mathcal{F}$, so we have a contradiction. This shows that $\mathcal{L} \oplus \mathcal{F} \notin \mathscr{A}(\mathbf{K})$. □

COROLLARY 5.4. *Suppose that $\mathcal{E} \in \mathscr{B}un_3^{2,\leqslant\lambda}(\mathbf{K})$ for some field $\mathbf{K}/k$ fits into a short exact sequence $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$ or $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{L} \to 0$ for some $\mathcal{L} \in \mathscr{B}un_1^1(\mathbf{K})$ and $\mathcal{F} \in \mathscr{B}un_2^1(\mathbf{K})$. Then $\mathcal{E} \in \mathscr{U}(\mathbf{K})$ if and only if $\mathcal{L} \oplus \mathcal{F} \in \mathscr{U}(\mathbf{K})$, which holds if and only if $\mathcal{F}$ is $(1,0)$-stable.*

Proof. If part. This is clear since $\mathscr{U}$ is open. *Only if part.* If $\mathcal{L} \oplus \mathcal{F} \notin \mathscr{U}(\mathbf{K})$, then $\mathcal{F}$ is not $(1,0)$-stable by Lemma 5.3. This shows that $\mathcal{E} \notin \mathscr{U}(\mathbf{K})$. □

The main result of this section is the following.

THEOREM 5.5. *The open dense substack $\mathscr{U} \subseteq \mathscr{B}un_3^2$ admits a separated non-proper good moduli space. Furthermore, it is not contained in $\mathscr{B}un_3^{2,\mathrm{ss}}$.*

5.2 Proof of Theorem 5.5

To show that $\mathscr{U}$ admits a separated non-proper adequate moduli space, we check the conditions of Theorem 3.1.

LEMMA 5.6. *The open substack $\mathscr{U}$ is locally reductive.*

Proof. By [AHH23, Remark 2.6] it suffices to show that closed points of $\mathscr{U}$ have linearly reductive stabilizers. If $\mathcal{E} \in \mathscr{U}(\mathbf{K})$ is a closed point, then

- if $\mathcal{E}$ is stable, then its stabilizer is $\mathbf{G}_m$;
- if $\mathcal{E}$ is unstable, then it must be decomposable by Corollary 5.4 (as it is closed), that is, of the form $\mathcal{L} \oplus \mathcal{F}$ such that $\mathcal{L} \in \mathscr{B}un_1^1(\mathbf{K})$ and $\mathcal{F} \in \mathscr{B}un_2^1(\mathbf{K})$ is $(1,0)$-stable by Lemma 5.3. In this case the stabilizer is $\mathbf{G}_m^2$ since there is no morphism between $\mathcal{L}$ and $\mathcal{F}$.

In either case, the stabilizer of $\mathcal{E}$ is linearly reductive. □

To verify Θ-reductivity we need to understand the decomposable vector bundles in $\mathscr{U}$. They have been determined in Lemma 5.3.

LEMMA 5.7. *The open substack $\mathscr{U}$ is Θ-reductive.*

Proof. Use Proposition 3.6. Let R be a DVR over k with fraction field K and residue field κ. For any family $\mathcal{E}_R \in \mathscr{U}(R)$ and any (non-trivial) filtration $\mathcal{E}_K^\bullet$ of $\mathcal{E}_K$ with $\mathrm{gr}(\mathcal{E}_K^\bullet) \in \mathscr{U}(K)$, by

Lemma 5.3 we have $\mathrm{gr}(\mathcal{E}_K^\bullet) = \mathcal{L}_K \oplus \mathcal{F}_K \in \mathscr{U}(K)$ for some $\mathcal{L}_K \in \mathscr{B}un_1^1(K)$ and $\mathcal{F}_K \in \mathscr{B}un_2^1(K)$. Then the filtration $\mathcal{E}_K^\bullet$ is of the form

$$0 \longrightarrow \mathcal{L}_K \longrightarrow \mathcal{E}_K \longrightarrow \mathcal{F}_K \longrightarrow 0 \quad \text{or} \quad 0 \longrightarrow \mathcal{F}_K \longrightarrow \mathcal{E}_K \longrightarrow \mathcal{L}_K \longrightarrow 0\,.$$

In the first case the uniquely induced filtration $\mathcal{E}_\kappa^\bullet$ of $\mathcal{E}_\kappa$ is of the form

$$0 \longrightarrow \mathcal{L}_\kappa \longrightarrow \mathcal{E}_\kappa \longrightarrow \mathcal{F}_\kappa \longrightarrow 0\,,$$

and we need to show that $\mathrm{gr}(\mathcal{E}_\kappa^\bullet) = \mathcal{L}_\kappa \oplus \mathcal{F}_\kappa \in \mathscr{U}(\kappa)$. Note that $\mathcal{L}_\kappa \in \mathscr{B}un_1^1(\kappa)$; it is a subbundle of $\mathcal{E}_\kappa$ by Lemma 5.1; that is, $\mathcal{F}_\kappa \in \mathscr{B}un_2^1(\kappa)$. Since $\mathcal{E}_\kappa \in \mathscr{U}(\kappa)$, we are done by Corollary 5.4. The second case is similar. □

To verify S-completeness we need to know points in $\mathscr{U}$ with opposite filtrations, which can be completely determined as follows.

Lemma 5.8. *Let $\mathcal{E} \ncong \mathcal{E}' \in \mathscr{U}(\mathbf{K})$ for some field $\mathbf{K}/k$ be two points with opposite filtrations $\mathcal{E}^\bullet$, $\mathcal{E}'_\bullet$. Assume that $\mathcal{E}$ is unstable with Harder–Narasimhan filtration $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$. Then the underlying unweighted filtrations of $\mathcal{E}^\bullet$, $\mathcal{E}'_\bullet$ take one of the following forms:*

(i) $\mathcal{E}^\bullet\colon 0 \subseteq \mathcal{L} \subseteq \mathcal{E}$ *and* $\mathcal{E}'_\bullet\colon \mathcal{E}' \supseteq \mathcal{F} \supseteq 0$,

(ii) $\mathcal{E}^\bullet\colon 0 \subseteq \mathcal{F} \subseteq \mathcal{E}$ *and* $\mathcal{E}'_\bullet\colon \mathcal{E}' \supseteq \mathcal{L} \supseteq 0$.

Proof. Let $a \in \mathbf{Z}$ be the minimal integer such that $\mathcal{E}^a = \mathcal{E}$, which is also the minimal integer such that $\mathcal{E}'_{a+1} = 0$. Let $b \in \mathbf{Z}$ be the maximal integer such that $\mathcal{E}'_b = \mathcal{E}'$, which is also the maximal integer such that $\mathcal{E}^{b-1} = 0$. Assume that $\mathcal{E}^{i-1} \neq \mathcal{E}^i$ for $a \geqslant i \geqslant b$. By definition we have $\mathrm{rk}\big(\mathcal{E}/\mathcal{E}^{a-1}\big) = \mathrm{rk}(\mathcal{E}'_a) \geqslant 1$ and $\mathrm{rk}\big(\mathcal{E}'/\mathcal{E}'_{b+1}\big) = \mathrm{rk}\big(\mathcal{E}^b\big) \geqslant 1$. Thus,

$$2 \geqslant \mathrm{rk}\big(\mathcal{E}^{a-1}\big) \geqslant \cdots \geqslant \mathrm{rk}\big(\mathcal{E}^b\big) \geqslant 1 \quad \text{and} \quad 2 \geqslant \mathrm{rk}(\mathcal{E}'_{b+1}) \geqslant \cdots \geqslant \mathrm{rk}(\mathcal{E}'_a) \geqslant 1\,.$$

Applying Lemma 5.1 to $\mathcal{E}$ and $\mathcal{E}'$ yields

$$\deg\big(\mathcal{E}^i\big) \leqslant 1 \quad \text{and} \quad \deg(\mathcal{E}'_{i+1}) \leqslant 1 \quad \text{for } a-1 \geqslant i \geqslant b\,.$$

However, the isomorphism $\mathcal{E}^i/\mathcal{E}^{i-1} \cong \mathcal{E}'_i/\mathcal{E}'_{i+1}$ implies

$$\deg\big(\mathcal{E}^i\big) + \deg(\mathcal{E}'_{i+1}) = 2 \quad \text{for } a-1 \geqslant i \geqslant b\,.$$

Then we obtain $\deg(\mathcal{E}^i) = \deg(\mathcal{E}'_{i+1}) = 1$. By Lemma 5.1 this means that each $\mathcal{E}^i$ (respectively $\mathcal{E}'_{i+1}$) is a subbundle of $\mathcal{E}$ (respectively, $\mathcal{E}'$) and thus

$$2 \geqslant \mathrm{rk}\big(\mathcal{E}^{a-1}\big) > \cdots > \mathrm{rk}\big(\mathcal{E}^b\big) \geqslant 1 \quad \text{and} \quad 2 \geqslant \mathrm{rk}(\mathcal{E}'_{b+1}) > \cdots > \mathrm{rk}(\mathcal{E}'_a) \geqslant 1\,.$$

In particular, $2 \geqslant a-b \geqslant 1$. But $a-b=2$ is impossible as in any filtration $\mathcal{E}^\bullet\colon 0 \subseteq \mathcal{E}^b \subseteq \mathcal{E}^{a-1} \subseteq \mathcal{E}$ of length 3, we have that $\mathcal{E}^{a-1}$ is $(1,0)$-stable by Corollary 5.4. This gives a contradiction since $\mathcal{E}^b \subseteq \mathcal{E}^{a-1}$ is a degree 1 sub-line bundle. Then $a-b=1$; that is,

$$\mathcal{E}^\bullet\colon 0 \subseteq \mathcal{E}^b \subseteq \mathcal{E} \quad \text{and} \quad \mathcal{E}'_\bullet\colon \mathcal{E}' \supseteq \mathcal{E}'_{b+1} \supseteq 0\,.$$

There are two cases: either $\mathrm{rk}(\mathcal{E}^b) = 1$ or $\mathrm{rk}(\mathcal{E}^b) = 2$. If $\mathrm{rk}(\mathcal{E}^b) = 1$, then we have $\mathcal{E}^b \cong \mathcal{L}$ by the uniqueness of Harder–Narasimhan filtration and $\mathcal{E}'_{b+1} \cong \mathcal{E}/\mathcal{E}^b \cong \mathcal{E}/\mathcal{L} = \mathcal{F}$. This is case (i). If $\mathrm{rk}(\mathcal{E}^b) = 2$, then we know that $\mathcal{E}^b$ is stable by Corollary 5.4 and the non-zero composition $\mathcal{E}^b \hookrightarrow \mathcal{E} \twoheadrightarrow \mathcal{F}$ is an isomorphism, splitting the sequence $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$. Thus, we conclude that $\mathcal{E}'_{b+1} \cong \mathcal{E}/\mathcal{E}^b \cong \mathcal{E}/\mathcal{F} = \mathcal{L}$. This is case (ii). □

Corollary 5.9. *The open substack $\mathscr{U}$ is S-complete.*

Proof. Use Proposition 3.17. For any field $\mathbf{K}/k$ and any two points $\mathcal{E} \not\cong \mathcal{E}' \in \mathscr{U}(\mathbf{K})$ with opposite filtrations $\mathcal{E}^\bullet$, $\mathcal{E}'_\bullet$, we may assume that $\mathcal{E}$ is unstable with Harder–Narasimhan filtration $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$. By Lemma 5.8 the underlying unweighted filtrations of such $\mathcal{E}^\bullet$, $\mathcal{E}'_\bullet$ take the form

$$\begin{cases} \mathcal{E}^\bullet\colon & 0 \subseteq \mathcal{F} \subseteq \mathcal{E}\,, \\ \mathcal{E}'_\bullet\colon & 0 \subseteq \mathcal{L} \subseteq \mathcal{E}'\,, \end{cases} \quad \text{or} \quad \begin{cases} \mathcal{E}^\bullet\colon & 0 \subseteq \mathcal{L} \subseteq \mathcal{E}\,, \\ \mathcal{E}'_\bullet\colon & 0 \subseteq \mathcal{F} \subseteq \mathcal{E}'\,. \end{cases}$$

In either case, we have $\mathcal{L} \oplus \mathcal{F} \in \mathscr{U}(\mathbf{K})$ by Corollary 5.4. This finishes the proof. □

Finally, the open substack $\mathscr{U}$ does not satisfy the existence part of the valuative criterion for properness since there are unextendable families.

LEMMA 5.10. *There exists a family $\mathcal{E}_R \in \mathscr{B}un_3^{2,\mathrm{ss}}(R)$ for some DVR R over k with fraction field K and residue field κ such that*

(i) *the generic fiber $\mathcal{E}_K$ fits into a short exact sequence*

$$0 \longrightarrow \mathcal{F}_K \longrightarrow \mathcal{E}_K \longrightarrow \mathcal{L}_K \longrightarrow 0$$

for some $\mathcal{L}_K \in \mathscr{B}un_1^1(K)$ and $\mathcal{F}_K \in \mathscr{B}un_2^1(K)$ with $\mathcal{F}_K$ being $(1,0)$-stable; in particular, $\mathcal{E}_K \in \mathscr{U}(K)$;

(ii) *the special fiber $\mathcal{E}_\kappa$ fits into a short exact sequence*

$$0 \longrightarrow \mathcal{F}_\kappa \longrightarrow \mathcal{E}_\kappa \longrightarrow \mathcal{L}_\kappa \longrightarrow 0$$

for some $\mathcal{L}_\kappa \in \mathscr{B}un_1^1(\kappa)$ and $\mathcal{F}_\kappa \in \mathscr{B}un_2^1(\kappa)$ with $\mathcal{F}_\kappa$ being stable but not $(1,0)$-stable; in particular, $\mathcal{E}_\kappa \notin \mathscr{U}(\kappa)$.

Proof. Since $\mathscr{B}un_2^{1,s}$ is irreducible, there exists a family $\mathcal{F}_R \in \mathscr{B}un_2^{1,s}(R)$ for some DVR R over k with fraction field K and residue field κ such that

$$\mathcal{F}_K \in \mathscr{B}un_2^{1,s}(1,0)(K) \quad \text{and} \quad \mathcal{F}_\kappa \in \big(\mathscr{B}un_2^{1,s} - \mathscr{B}un_2^{1,s}(1,0)\big)(\kappa)\,.$$

Fix a family $\mathcal{L}_R \in \mathscr{B}un_1^1(R)$ and a non-split extension $[\varrho]\colon 0 \to \mathcal{F}_\kappa \to \mathcal{E}_\kappa \to \mathcal{L}_\kappa \to 0$. The same argument as in the proof of Lemma 5.3 shows that $\mathcal{E}_\kappa$ is stable. Let $\mathcal{E}_R \in \mathscr{C}oh_3^2(R)$ be an extension family of $\mathcal{L}_R$ by $\mathcal{F}_R$ with special fiber $[\varrho]$ (see Lemma 3.7). Since $\mathcal{E}_\kappa \in \mathscr{B}un_3^{2,\mathrm{ss}}(\kappa)$, by the openness of stability, we have $\mathcal{E}_R \in \mathscr{B}un_3^{2,\mathrm{ss}}(R)$. □

COROLLARY 5.11. *The open substack $\mathscr{U}$ does not satisfy the existence part of the valuative criterion for properness.*

Proof. Any family $\mathcal{E}_R \in \mathscr{B}un_3^{2,\mathrm{ss}}(R)$ as in Lemma 5.10 defines a diagram

$$\begin{tikzcd} \operatorname{Spec}(K) \arrow[r, "\mathcal{E}_K"] \arrow[d, hook] & \mathscr{U} \arrow[d] \\ \operatorname{Spec}(R) \arrow[r] \arrow[ur, dashed, "\times"'] & \operatorname{Spec}(k) \end{tikzcd}$$

of solid arrows, and we claim that there is no dotted arrow filling it in. Assume to the contrary that there exists a family $\mathcal{E}'_R \in \mathscr{U}(R)$ such that $\mathcal{E}'_K \cong \mathcal{E}_K$. Then we consider the short exact sequences $0 \to \mathcal{F}'_\kappa \to \mathcal{E}'_\kappa \to \mathcal{L}'_\kappa \to 0$ induced by that on the generic fiber $0 \to \mathcal{F}_K \to \mathcal{E}_K \to \mathcal{L}_K \to 0$ in Lemma 5.10. Since $\mathcal{E}'_\kappa \in \mathscr{U}(\kappa)$, we have that $\mathcal{F}'_\kappa$ is $(1,0)$-stable by Corollary 5.4. Both $\mathcal{F}_\kappa$ and $\mathcal{F}'_\kappa$ are stable limits of $\mathcal{F}_K$, so they are isomorphic by Langton's theorem A (see [Lan75]). But $\mathcal{F}'_\kappa$ is $(1,0)$-stable while $\mathcal{F}_\kappa$ is not, so we have a contradiction. □

Proof of Theorem 5.5. The open substack $\mathscr{U} \subseteq \mathscr{B}un_3^2$ is non-empty and not contained in $\mathscr{B}un_3^{2,\mathrm{ss}}$ by Lemma 5.3. It is locally reductive (Lemma 5.6), Θ-reductive (Lemma 5.7), S-complete (Corollary 5.9), and does not satisfy the existence part of the valuative criterion (Corollary 5.11). By Theorem 3.1 the substack $\mathscr{U}$ admits a separated non-proper adequate moduli space. Furthermore, since every closed point of $\mathscr{U}$ has linearly reductive stabilizer by the proof of Lemma 5.6, this adequate moduli space is actually a good moduli space by [AHR19, Corollary 6.11]. □

Remark 5.12. It is also interesting to find a compactification of $\mathscr{U} \subseteq \mathscr{B}un_3^2$, that is, an open substack $\overline{\mathscr{U}} \subseteq \mathscr{B}un_3^2$ containing $\mathscr{U}$ that admits a proper good moduli space. The naïve candidate $\mathscr{U} \cup \mathscr{B}un_3^{2,\mathrm{ss}}$, while S-complete, is not Θ-reductive.

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