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Analytification of mapping stacks

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Abstract

Derived mapping stacks are a fundamental source of examples of derived enhancements of classical moduli problems. For instance, they appear naturally in Gromov–Witten theory and in some branches of geometric representation theory. In this paper we show that in many cases the mapping stack construction commutes with the (complex or non-archimedean) derived analytification functor. Along the way, we establish several new foundational results in derived analytic geometry that are likely to be useful elsewhere. For instance, we provide a $\mathbb{C}$-analytic version of Kiehl's theorem for (derived) compact Stein spaces, and study some incarnations of analytic Tannaka duality. We apply these results to the study of the Riemann–Hilbert correspondence and the derived period domain.

1. Introduction

One of the main uses of derived algebraic geometry is to provide well-behaved derived enhancements of classical moduli problems: while the original moduli problem is often highly singular, its derived counterpart has controlled singularities, which often means that it is local complete intersection in the derived sense. This phenomenon is extremely useful in constructions that involve virtual fundamental classes. Examples can be found in Gromov–Witten theory [MR18, STV15, PY20a] and in geometric representation theory [Neg19, PS23, DPS23, DPS22]. It is often the case that these derived enhancements arise from mapping stack constructions, which are also one of the primary sources for interesting examples of derived schemes and stacks.

Recently, derived techniques have also become available in the (complex and non-archimedean) analytic setting. More precisely, let k denote either the field of complex numbers or a non-archimedean field equipped with a non-trivial valuation. In [Lur11c, Por19, PY18] the authors introduced ∞-categories dAn_k of *derived analytic spaces* and dAnSt_k of *derived analytic stacks* and showed that a significant number of the properties of derived schemes and stacks equally hold for their analytic counterparts. See Section 2 for a brief review of these notions. The motivations behind the early development of derived analytic geometry come from mirror

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symmetry and non-abelian Hodge theory. We refer to the introductions of [PY18, Por17b] for more details on these programs.

In [PY21] the authors extended the mapping stack construction in the analytic setting and established a representability criterion, that later allowed them in [PY20a] to construct the derived Deligne–Mumford analytic stack $\mathbb{R}\overline{\mathcal{M}}_{g,n}(X)$ of stable maps in a smooth and proper non-archimedean analytic variety X. The subsequent study of the geometric properties of $\mathbb{R}\overline{\mathcal{M}}_{g,n}(X)$ consolidated derived analytic techniques as a powerful framework to develop enumerative geometry in the non-archimedean setting. Together with [KY23, NXY19], this also strongly established the use of non-archimedean enumerative techniques in the study of mirror symmetry, as anticipated by Kontsevich–Soibelman in [KS06].

The current paper is more directly related to non-abelian Hodge theory. Nevertheless, we expect the main results of this work to be useful in a variety of other contexts. Our main goal is to study the extent to which the mapping stack construction commutes with analytification. We provide a general criterion guaranteeing that this is the case in a number of important geometric situations, and we later explore some consequences and applications. Let us formulate a more precise statement. As before, let k denote either the field of complex numbers or a non-archimedean field equipped with a non-trivial valuation. Let $\mathrm{dAff}_k^{\mathrm{afp}}$ denote the ∞-category of derived affine schemes almost of finite presentation[1] over k. Given derived stacks $X, Y \colon \big(\mathrm{dAff}_k^{\mathrm{afp}}\big)^{\mathrm{op}} \to \mathcal{S}$, we define $\mathbf{Map}(X,Y)$ as the derived stack

$$\begin{aligned} \mathbf{Map}(X,Y) \colon \big(\mathrm{dAff}_k^{\mathrm{afp}}\big) &\longrightarrow \mathcal{S} \\ T &\longmapsto \mathrm{Map}_{\mathrm{dSt}_k}(X \times T, Y) . \end{aligned}$$

Similarly, given derived analytic stacks $\mathscr{X}, \mathscr{Y} \colon \mathrm{dAn}_k^{\mathrm{op}} \to \mathcal{S}$, we define $\mathbf{AnMap}(\mathscr{X},\mathscr{Y})$ as the derived analytic stack

$$\begin{aligned} \mathbf{AnMap}(\mathscr{X},\mathscr{Y}) \colon \mathrm{dAn}_k^{\mathrm{op}} &\longrightarrow \mathcal{S} , \\ U &\longmapsto \mathrm{Map}_{\mathrm{dAnSt}_k}(\mathscr{X} \times U, \mathscr{Y}) . \end{aligned}$$

These functors parametrize families of maps of X (respectively, $\mathscr{X}$) into Y (respectively, $\mathscr{Y}$). Given derived stacks $X, Y \colon \big(\mathrm{dAff}_k^{\mathrm{afp}}\big)^{\mathrm{op}} \to \mathcal{S}$, there is a canonical map

$$\mathbf{Map}(X,Y)^{\mathrm{an}} \longrightarrow \mathbf{AnMap}\big(X^{\mathrm{an}}, Y^{\mathrm{an}}\big) , \tag{1.1}$$

where $(-)^{\mathrm{an}}$ denotes the derived analytification functor (see Section 3.1 for a review of its construction and properties).

The goal of this paper is to provide sufficiently general conditions on X and Y to guarantee that (1.1) is an equivalence. The case of first and foremost interest is the case where $Y = \mathbf{Perf}_k$ is the derived moduli stack of perfect complexes [TV07] and X is a proper scheme over k. In Section 4 we introduce an analytic counterpart of $\mathbf{Perf}_k$, denoted by $\mathbf{AnPerf}_k$. In terms of its functor of points, it simply sends a derived analytic space S to the stable ∞-category $\mathrm{Perf}(S)$ of perfect complexes on S (see Section 4.1 for the precise definition). In the algebraic setting the fact that $\mathbf{Perf}_k$ is a (locally) geometric stack[2] is a difficult theorem and the main result of [TV07]. To prove that $\mathbf{AnPerf}_k$ is a (locally) geometric analytic stack, one is bound to address

[1]A derived affine scheme $\mathrm{Spec}(A)$ is almost of finite presentation over k if $\pi_0(A)$ is finitely generated as a k-algebra and each $\pi_i(A)$ is finitely generated as $\pi_0(A)$-module. This condition ensures that we can consider its analytification.

[2]In this paper "geometric" stack is a synonym of "higher Artin" stack. See Remark 2.12 for a more detailed explanation.

the same issues dealt with in [TV07]. Rather than doing this directly, we prove the following result.

Theorem 1.2 (cf. Proposition 4.9 and Corollary 5.6). *Let k be either the field of complex numbers or a non-archimedean field equipped with a non-trivial valuation.*

(1) *There is a canonical equivalence*

$$\mathbf{Perf}_k^{\mathrm{an}} \simeq \mathbf{AnPerf}_k\,.$$

In particular, $\mathbf{AnPerf}_k$ is a derived locally geometric analytic stack.

(2) *Let X be a proper derived geometric stack locally almost of finite presentation over k. Assume that the stack $\mathbf{Map}(X, \mathbf{Perf}_k)$ of perfect complexes on X is locally geometric. Then the canonical map*

$$\mathbf{Map}(X, \mathbf{Perf}_k)^{\mathrm{an}} \longrightarrow \mathbf{AnMap}\big(X^{\mathrm{an}}, \mathbf{AnPerf}_k\big)$$

is an equivalence.

Remark 1.3. The assumption that the stack $\mathbf{Map}(X, \mathbf{Perf}_k)$ is locally geometric is automatically satisfied if X is a proper and underived scheme over k. See Remark 5.7.

Remark 1.4. (1) There are two main difficulties to overcome in trying to prove Theorem 1.2(1). The first one is that the analytification functor is defined by left Kan extension, and therefore it lacks a clear interpretation for the functor of points of $\mathbf{Perf}^{\mathrm{an}}$. We overcome this by providing a second, less obvious, universal property of $\mathbf{Perf}^{\mathrm{an}}$ and, more generally, of the analytification Y^{an} of a (locally) geometric derived stack Y; see Theorem 1.18 for an expository account and Theorem 3.13 for the precise result. Once this issue is solved, the non-archimedean case simply follows from the derived version of Kiehl's theorem, stating that for a (derived) affinoid space X with ring of global sections A, one has a canonical equivalence

$$\mathrm{Perf}(X) \simeq \mathrm{Perf}(A)\,.$$

See [PY21, Theorem 3.1]. In the $\mathbb{C}$-analytic setting replacing (derived) affinoid by (derived) Stein spaces, the analogous statement is *false*. To remedy this, an additional effort is required, which leads to some significant improvement of the known results in the literature. We will discuss this at greater length after the end of this remark.

(2) Concerning Theorem 1.2(2), considerations similar to above can be made. Having obtained a description of the functor of points of $\mathbf{Map}(X, \mathbf{Perf}_k)^{\mathrm{an}}$, one is reduced to proving a statement of *relative* GAGA type. More precisely, if S is a derived analytic space, we are naturally brought to consider the relative geometric stack $X \times S$ and compare relatively algebraic perfect complexes on it with analytic perfect complexes on $X^{\mathrm{an}} \times S$. Besides some technical issues in making precise sense of the object $X \times S$ and relatively algebraic perfect complexes on it, the main issue is that S must be allowed to be non-proper. In the non-archimedean case, the relative GAGA theorem proven in [PY16, Theorem 1.3] settles the issue. Nevertheless, in loc. cit. the question of having a relative GAGA theorem in the $\mathbb{C}$-analytic setting was left open. The difficulty is reducible to the fact that if S is a derived Stein space, its global sections are almost never noetherian. We will fix this issue at the same time as we explain how to adapt Kiehl's theorem to the $\mathbb{C}$-analytic setting.

(3) Bootstrapping on Theorem 1.2(2), one can obtain more examples of derived stacks F for which the canonical comparison map

$$\mathbf{Map}(F, \mathbf{Perf}_k)^{\mathrm{an}} \longrightarrow \mathbf{Map}\big(F^{\mathrm{an}}, \mathbf{AnPerf}_k\big)$$

is an equivalence. These examples are discussed at length in Section 5.2, and they include formal completions and Simpson's shapes X_{B}, X_{dR} and X_{Dol}, which play a crucial role in formulating the notions of local system, flat bundle and Higgs bundle in the realm of derived geometry.

As pointed out in the above remark, the $\mathbb{C}$-analytic case presents some additional difficulties. We overcome them using the classical notion of "compact Stein subsets", reinterpreted here as special pro-objects in the ∞-category $\mathrm{dAn}_{\mathbb{C}}$. More precisely, we prove the following. (If I is a filtered category and $F\colon I \to \mathcal{C}$ is a diagram, we let "colim"$_{i\in I} F(i)$ denote the associated ind-object in $\mathrm{Ind}(\mathcal{C})$.)

Theorem 1.5 (cf. Theorem 4.13 and Corollary 4.15). *Let $X \in \mathrm{dAn}_{\mathbb{C}}$ be a derived complex analytic space, and assume that it is Stein.* (*This means that its truncation $\mathrm{t}_0(X)$ is a Stein space. Equivalently, $\mathrm{H}^1(X;\mathcal{F}) = 0$ for every discrete coherent sheaf $\mathcal{F}$ on X.*) *Let $K \subset \mathrm{t}_0(X)$ be a compact subset admitting a fundamental system of open Stein neighbourhoods. For every Stein neighbourhood U of K inside X, write $A_U \coloneqq \Gamma\big(U;\mathcal{O}_U^{\mathrm{alg}}\big)$. Then there is a canonical equivalence in $\mathrm{Ind}\big(\mathrm{Cat}_\infty^{\mathrm{st},\otimes}\big)$*

$$\underset{K\subset U\subset X}{\text{“colim”}} \mathrm{Perf}(A_U) \simeq \underset{K\subset U\subset X}{\text{“colim”}} \mathrm{Perf}(U)\,, \tag{1.6}$$

where the ind-objects are parametrized by all the Stein open neighbourhoods of K. Furthermore, after realizing these ind-objects, we obtain an equivalence in $\mathrm{Cat}_\infty^{\mathrm{st},\otimes}$

$$\underset{K\subset U\subset X}{\mathrm{colim}} \mathrm{Perf}(U) \simeq \underset{K\subset U\subset X}{\mathrm{colim}} \mathrm{Perf}(A_U) \simeq \mathrm{Perf}(A_K)\,, \tag{1.7}$$

where

$$A_K \coloneqq \underset{K\subset U\subset X}{\mathrm{colim}} A_U$$

is the derived ring of germs of holomorphic functions on K.

Remark 1.8. (1) Although the equivalence (1.7) is easier both to state and to imagine, we want to draw the reader's attention to the fact that (1.6) contains strictly stronger information. For instance, it implies that (1.7) remains true after applying any functor $F\colon \mathrm{Cat}_\infty \to \mathcal{E}$, even when F does not itself commute with colimits. This simple consequence of Theorem 1.5 plays a major role in the proof of the main theorem of this paper.

(2) When this paper was first written, the closest known result to Theorem 1.5 was [Tay02, Proposition 11.9.2], asserting that, under the additional assumption that A_K is noetherian, there is an equivalence of abelian categories

$$\mathrm{Coh}^{\heartsuit}(K) \simeq \mathrm{Coh}^{\heartsuit}(A_K)\,,$$

where the left-hand side denotes the category of coherent sheaves on the locally ringed space $(K, \mathcal{O}_K)$, where $\mathcal{O}_K$ is the sheaf of *germs* of holomorphic functions of K inside X. Since then, the work of Clausen and Scholze on condensed mathematics established a much more closely related result; see [CS22, Theorem 9.15].

(3) Theorem 1.5 can also be seen as a vast generalization of [Tay02, Proposition 11.9.2]. First of all, considering K as a pro-object rather than as a locally ringed space allows us to drop the noetherianity assumption. Furthermore, it provides an equivalence of ind-categories, which is a much more powerful tool in practice, as observed above. Next, it lifts the abelian equivalence to an equivalence of stable ∞-categories of perfect complexes (and, in fact, in Theorem 4.13 the case of the stable ∞-category of unbounded coherent sheaves is also dealt with). Last but not least, Theorem 1.5 provides an extension to the derived setting as well, which is particularly

useful within the context of this paper. Observe that in the derived setting the stable ∞-category $\mathrm{Coh}(A_K)$ is quite different from the derived category of its heart $\mathrm{Coh}^{\heartsuit}(A_K)$. Therefore, the proof we provide of Theorem 1.5 is entirely independent of the one given in loc. cit. and works directly at the ∞-categorical level.

We now turn back to the question of determining under which conditions on X and Y the comparison map (1.1) is an equivalence. Having dealt with the case $Y = \mathbf{Perf}_k$, we invoke the Tannakian reconstruction formalism to deduce the statement for more general derived stacks Y. In first approximation, we say that Y is *Tannakian* if for every derived stack X, the natural map[3]

$$\mathrm{Map}_{\mathrm{dSt}_k}(X, Y) \longrightarrow \mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(X))$$

is fully faithful and its image has a precise characterization; see Definition 6.1 for the details. Our main theorem is then the following.

Theorem 1.9 (cf. Theorem 6.14). *Let $X, Y \colon \left(\mathrm{dAff}_k^{\mathrm{afp}}\right)^{\mathrm{op}} \to \mathcal{S}$ be derived stacks. Assume that*

1. *the derived stack X is a proper geometric derived stack locally almost of finite presentation over k;*
2. *the mapping stack $\mathbf{Map}(X, Y)$ is geometric;*
3. *Y is a geometric stack which is Tannakian and satisfies $\mathrm{QCoh}(Y) \simeq \mathrm{Ind}(\mathrm{Perf}(Y))$.*

Then the canonical morphism

$$\mathbf{Map}(X, Y)^{\mathrm{an}} \longrightarrow \mathbf{AnMap}\big(X^{\mathrm{an}}, Y^{\mathrm{an}}\big) \tag{1.10}$$

is an equivalence of derived analytic stacks.

Remark 1.11. (1) The assumption on X can be significantly weakened. In Theorem 6.14 we only assume X to satisfy the "universal GAGA property"; see Definition 5.1 for the precise formulation. In first approximation, the reader can imagine it as a slight strengthening of the condition that the canonical comparison map

$$\mathbf{Map}(X, \mathbf{Perf})^{\mathrm{an}} \longrightarrow \mathbf{AnMap}\big(X^{\mathrm{an}}, \mathbf{AnPerf}\big)$$

is an equivalence; see Proposition 5.2. The key point in working axiomatically on X is to include uniformly the case of Simpson's shapes X_{B}, X_{dR} and X_{Dol}, which are very far from being geometric stacks. See also Remark 1.4(3).

(2) When the characteristic of k is zero, the assumptions on Y are satisfied, for instance, when it is a quasi-compact quasi-separated Deligne–Mumford stack or it is the classifying stack of an affine group scheme of finite type (see the corollary to [HR17, Theorem B]). See also Example 6.3.

(3) No assumptions on the characteristic of k are required to prove Theorem 1.9. Nevertheless, the only known criteria to guarantee that the assumptions on Y are met require characteristic zero. See Remark 6.4.

Remark 1.12. The proof of Theorem 1.9 loosely follows the same strategy as Lurie's proof of the main theorem in [Lur04], but it is considerably more involved, and it also requires some new ideas. Furthermore, Theorem 1.9 is a threefold generalization of [Lur04]:

[3] The notation $\mathrm{Fun}^{\otimes}$ stands for symmetric monoidal k-*linear* functors.

(1) In [Lur04] it is only proven that (1.10) is an equivalence on the k-points. To prove this statement for the entire mapping stack requires some additional effort, notably a relative version of GAGA's theorem, which we discussed in Theorem 1.2. As further discussed in Remark 1.4(2), this result is far from being obvious, and in the $\mathbb{C}$-analytic setting it is entirely new in the generality discussed in the paper at hand.

(2) We remove the geometricity assumption on X, allowing us to consider Simpson's shapes X_{B}, X_{dR} and X_{Dol}. This has as a consequence the generalization of the Riemann–Hilbert correspondence described below, which is our main application.

(3) We allow our stacks X and Y to be derived.

Theorem 1.13 (cf. Corollary 7.6). *Let X be a smooth and proper scheme over $\mathbb{C}$. Let Y be a derived stack locally almost of finite presentation satisfying the same assumptions as in Theorem 1.9. Then there is a natural equivalence of derived analytic stacks*

$$\mathbf{Map}(X_{\mathrm{dR}}, Y)^{\mathrm{an}} \simeq \mathbf{Map}(X_{\mathrm{B}}, Y)^{\mathrm{an}},$$

which reduces to the Deligne Riemann–Hilbert correspondence for rank n vector bundles when $Y = \mathrm{BGL}_n$ obtained in [Del70].

Remark 1.14. (1) Theorem 1.13 is a vast generalization of the main result of [Por17b]. It is also an important stepping stone in the study of the non-abelian Hodge correspondence from the derived point of view, and it should be seen as part of an organic and ongoing research program on the irregular Riemann–Hilbert correspondence that has been initiated in [PT21, PT22, PT24].

(2) In [PS23, Theorem 1.5] the authors showed that when X is a smooth and proper surface over $\mathbb{C}$, Theorem 1.13 gives rise to an equivalence of categorified Hall algebras. Decategorifying, they obtained an equivalence of K-theoretical and cohomological Hall algebras.

Let us mention two more applications. In Proposition 7.3 we show that the derived analytification functor commutes with finite limits of geometric stacks. In Corollary 7.4 we revisit the derived period domain from [DH19] and construct it as a derived analytic moduli stack. More precisely, we have the following.

Theorem 1.15 (cf. Corollary 7.4). *Let (V, q) be a $2n$-shifted quadratic perfect complex, and assume that q is non-degenerate on the cohomology of V. The functor*

$$\mathbf{P}_n(V, q) \colon \mathrm{dStn}^{\mathrm{op}} \longrightarrow \mathcal{S}$$

which sends $S \in \mathrm{dStn}$ to the space of filtrations $\{F^*\}$ on $\pi_S^* V \in \mathrm{Perf}(S)$ which induce Hodge structures on the cohomology groups of V, with polarizations induced by q, is representable by a geometric derived analytic stack. Here $\pi_S \colon S \to \mathrm{Sp}(\mathbb{C})$ denotes the canonical map. Furthermore, $\mathbf{P}_n(V, q)$ coincides with the derived period domain considered in [DH19].

The original derived period domain in [DH19] was constructed in an ad hoc way by analytifying an algebraic moduli stack. In fact, the construction of the derived period map in [DH19] could be simplified using our Theorem 1.18 to bridge the algebraic and analytic aspects of the problem. We will not pursue this approach in this paper.

Before concluding this introduction, we want to emphasize two more technical results that, nevertheless, together with Theorem 1.5, provide a very significant extension of the currently available technology in derived analytic geometry. For these reasons, we believe these results to be of independent interest and likely to arise in other contexts, and we made an effort to make their statements independent of the rest of the paper. The first one is the following partial extension of Tannaka duality to the analytic setting.

Theorem 1.16 (cf. Lemma 6.10 and Propositions 6.5, 6.11 and 6.12). *Let $Y \in \mathrm{dSt}_k^{\mathrm{afp}}$ be a derived geometric stack locally almost of finite presentation, and let $X \in \mathrm{dAn}_k$. Assume that*

(1) *Y is Tannakian;*
(2) $\mathrm{QCoh}(Y) \simeq \mathrm{Ind}(\mathrm{Perf}(Y))$.

Then the assignment sending a morphism $f \colon X \to Y^{\mathrm{an}}$ to the composition

$$\mathrm{Perf}(Y) \longrightarrow \mathrm{Perf}\big(Y^{\mathrm{an}}\big) \xrightarrow{f^*} \mathrm{Perf}(X)$$

provides a fully faithful map

$$\mathrm{Map}_{\mathrm{dAnSt}_k}\big(X, Y^{\mathrm{an}}\big) \longrightarrow \mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(X))\,.$$

Furthermore, when X is a derived k-affinoid (respectively, Stein) space, we can identify the essential image of this functor with those k-linear, symmetric monoidal functors

$$F \colon \mathrm{QCoh}(Y) \longrightarrow \mathcal{O}_X\text{-Mod}$$

that commute with colimits, preserve perfect complexes and respect flat objects and connective objects.

The second result we would like to mention is our way of bypassing the problem raised in Remark 1.4(1). The issue at hand is to provide a manageable description of the functor of points of the analytification of a derived stack Y. When Y is a derived Deligne–Mumford stack locally almost of finite presentation over k, the *definition* of analytification implies that for every derived analytic space $X = (\mathcal{X}_X, \mathcal{O}_X)$, one has a canonical equivalence

$$\mathrm{Map}_{\mathrm{dAn}_k}\big(X, Y^{\mathrm{an}}\big) \simeq \mathrm{Map}_{{}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\text{ét}}(k))}\big(X^{\mathrm{alg}}, Y\big)\,. \tag{1.17}$$

Here $X^{\mathrm{alg}} = \big(\mathcal{X}_X, \mathcal{O}_X^{\mathrm{alg}}\big)$ is the derived analytic space X seen as a locally ringed space,[4] and ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\text{ét}}(k))$ denotes the ∞-category of those locally ringed spaces whose structure sheaf has strictly Henselian stalks. When Y is more generally a geometric stack, it can no longer be represented as an object in ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\text{ét}}(k))$ and Y^{an} is no longer an object in dAn_k. Therefore, the above equivalence loses its meaning. It is natural to replace dAn_k by dAnSt_k, but the right-hand side cannot be simply replaced by dSt_k because the construction $X \mapsto X^{\mathrm{alg}}$ is not sufficiently well behaved. We bypass this problem by proving the following.

Theorem 1.18 (cf. Theorem 3.13). *Let $X \in \mathrm{dAn}_k$, and let $\mathcal{X}_X$ be its underlying ∞-topos of sheaves on the étale site of X. Let $\mathbf{1}_X$ denote the final object of $\mathcal{X}_X$. There exist functors*

$$F_X^s \colon \mathrm{dAnSt}_k \longrightarrow \mathcal{X}_X\,, \quad G_X^s \colon \mathrm{dSt}_k^{\mathrm{afp}} \longrightarrow \mathcal{X}_X$$

and a natural transformation $\alpha \colon G_X^s \to F_X^s \circ (-)^{\mathrm{an}}$ satisfying the following conditions:

(1) *Given $Y \in \mathrm{dSt}_k^{\mathrm{afp}}$, there is a natural equivalence*

$$\mathrm{Map}_{\mathrm{dAnSt}_k}\big(X, Y^{\mathrm{an}}\big) \simeq F_X^s\big(Y^{\mathrm{an}}\big)(\mathbf{1}_X)\,.$$

[4] For simplicity, in the introduction we use the word space. For technical reasons, in the main body of the paper we rather work with locally ringed ∞-topoi.

(2) *Given* $Y \in \mathrm{dSt}_k^{\mathrm{afp}}$*, the functor* $G_X^{\mathrm{s}}(Y)$ *is the sheafification of the functor sending an étale morphism* $U \to X$ *to*

$$\mathrm{Map}_{\mathrm{dSt}_k}\big(\mathrm{Spec}\big(\Gamma\big(U; \mathcal{O}_U^{\mathrm{alg}}\big)\big), Y\big) .$$

(3) *If* Y *is a geometric stack, the natural transformation* α *is an equivalence.*

Although more complicated than the adjunction available for derived Deligne–Mumford stacks, Theorem 1.18 is equally useful in practice because it gives a way of describing morphisms *into* Y^{an} in terms of (sheaves of) maps into Y. The entire Section 3 is devoted to formulating and proving this theorem.

Notation and conventions

In this paper we freely use the language of ∞-categories. Although the discussion is often independent of the chosen model for ∞-categories, whenever needed we identify them with quasi-categories and refer to [Lur09] for the necessary foundational material.

The notation $\mathcal{S}$ and Cat_∞ is reserved to denote the ∞-categories of spaces and of ∞-categories, respectively. If $\mathcal{C} \in \mathrm{Cat}_\infty$, we denote by $\mathcal{C}^\simeq$ the maximal ∞-groupoid contained in $\mathcal{C}$. We let $\mathrm{Cat}_\infty^{\mathrm{st}}$ denote the ∞-category of stable ∞-categories with exact functors between them. We also let $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ denote the ∞-category of presentable ∞-categories with left adjoints between them. Similarly, we let $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}$ denote the ∞-categories of stably presentable ∞-categories with left adjoints between them. Finally, we set

$$\mathrm{Cat}_\infty^{\mathrm{st},\otimes} \coloneqq \mathrm{CAlg}\big(\mathrm{Cat}_\infty^{\mathrm{st}}\big) , \quad \mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}_{\mathrm{st}} \coloneqq \mathrm{CAlg}\big(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}\big) .$$

Given an ∞-category $\mathcal{C}$, we denote by $\mathrm{PSh}(\mathcal{C})$ the ∞-category of $\mathcal{S}$-valued presheaves. We follow the conventions introduced in [PY16, § 2.4] for ∞-categories of sheaves on an ∞-site.

For a field k, we reserve the notation CAlg_k for the ∞-category of simplicial commutative rings over k. We often refer to objects in CAlg_k simply as *derived commutative rings*. We denote its opposite by dAff_k, and we refer to it as the ∞-category of *derived affine schemes*. We say that a derived ring $A \in \mathrm{CAlg}_k$ is *almost of finite presentation* if $\pi_0(A)$ is of finite presentation over k and $\pi_i(A)$ is a finitely presented $\pi_0(A)$-module.[5] We denote by $\mathrm{dAff}_k^{\mathrm{afp}}$ the full subcategory of dAff_k spanned by derived affine schemes $\mathrm{Spec}(A)$ such that A is almost of finite presentation. When k either is a non-archimedean field equipped with a non-trivial valuation or is the field of complex numbers, we let An_k denote the category of analytic spaces over k. We denote by $\mathrm{Sp}(k)$ the analytic space associated with k.

Throughout the paper we need to consider stacks both with values in $\mathcal{S}$ and with values in Cat_∞. We use the following convention: if $(\mathcal{C}, \tau)$ is an ∞-site and $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is a Cat_∞-valued stack, we denote by $\mathbf{F}$ the $\mathcal{S}$-valued stack defined by

$$\mathbf{F}(X) \coloneqq F(X)^\simeq , \quad X \in \mathcal{C} .$$

Given stacks $T \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ and $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$, we let $\mathbf{Map}(T, F)$ denote the Cat_∞-valued stack defined by

$$\mathbf{Map}(T, F)(X) \coloneqq F(T \times X) .$$

Here we are implicitly extending F to a functor $\mathrm{St}(\mathcal{C}, \tau)^{\mathrm{op}} \to \mathrm{Cat}_\infty$. Notice that

$$\big(\mathbf{Map}(T, F)(X)\big)^\simeq \simeq \mathbf{Map}(T, \mathbf{F})(X) , \quad X \in \mathcal{C} ,$$

[5] Equivalently, A is almost of finite presentation if $\pi_0(A)$ is of finite presentation and the cotangent complex $\mathbb{L}_{A/k}$ is an almost perfect complex over A.

where the **Map** on the right-hand side now denotes the internal hom in $\mathrm{St}(\mathcal{C}, \tau)$.

Finally, in this paper we are concerned with ind- and (to a lesser extent) pro-objects. Given an ∞-category $\mathcal{C}$, we let $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C})$ denote the ∞-categories of ind- and pro-objects in $\mathcal{C}$, respectively. If I is a filtered category and $F \colon I \to \mathcal{C}$ is a diagram, we let "colim"$_{i\in I} F(i)$ denote the associated ind-object in $\mathrm{Ind}(\mathcal{C})$. We use the notation "lim" for pro-objects.

2. Review of derived analytic geometry

We start by reviewing the basic notions and facts about derived analytic geometry. We refer the reader to the papers [Lur11c, PY18, Por19, PY20b] for more extensive discussions of the foundations.

2.1 Definitions and basic facts

We let k denote either the field $\mathbb{C}$ of complex numbers or a complete non-archimedean field with non-trivial valuation. We refer to [Lur11a, §§ 1, 3] and [Lur11c, §§ 11–12] for a more thorough explanation of the ideas we briefly recall below. We also refer the reader to [Por19, § 2] and [PY18, § 2] for more detailed reviews.

Notation 2.1. (1) Let $\mathcal{T}_{\mathrm{disc}}(k)$ denote the full subcategory of k-schemes spanned by affine spaces $\mathbb{A}^n_k$. A morphism in $\mathcal{T}_{\mathrm{disc}}(k)$ is said to be *admissible* if it is an isomorphism. We endow $\mathcal{T}_{\mathrm{disc}}(k)$ with the trivial Grothendieck topology.

(2) Let $\mathcal{T}_{\mathrm{ét}}(k)$ denote the category of smooth k-schemes. A morphism in $\mathcal{T}_{\mathrm{ét}}(k)$ is said to be *admissible* if it is an étale morphism. We endow $\mathcal{T}_{\mathrm{ét}}(k)$ with the étale topology $\tau_{\mathrm{ét}}$.

(3) Let $\mathcal{T}_{\mathrm{an}}(k)$ denote the category of smooth k-analytic spaces.[6] A morphism in $\mathcal{T}_{\mathrm{an}}(k)$ is said to be *admissible* if it is an étale morphism. We endow $\mathcal{T}_{\mathrm{an}}(k)$ with the quasi-étale topology [PY18, Construction 2.2] or [PY20b, § 2].

Definition 2.2 (cf. [Lur11a, Definition 3.1.4]). Let $\mathcal{X}$ be an ∞-topos. A $\mathcal{T}_{\mathrm{an}}(k)$-*structure* is a functor $\mathcal{O} \colon \mathcal{T}_{\mathrm{an}}(k) \to \mathcal{X}$ which commutes with products and pullbacks along admissible morphisms and takes admissible covers to effective epimorphisms. We denote by $\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$ the full subcategory of $\mathrm{Fun}(\mathcal{T}_{\mathrm{an}}(k), \mathcal{X})$ spanned by $\mathcal{T}_{\mathrm{an}}(k)$-structures.

Definition 2.3 (cf. [Lur11a, Definition 3.1.4]). Let $\mathcal{X}$ be an ∞-topos. A $\mathcal{T}_{\mathrm{an}}(k)$-structure $\mathcal{O}$ is said to be *local* if it takes $\tau_{\mathrm{ét}}$-covers to effective epimorphisms. A morphism of $\mathcal{T}_{\mathrm{an}}(k)$-structures $\mathcal{O} \to \mathcal{O}'$ is said to be *local* if for every admissible morphism $U \to V$, the square

$$\begin{array}{ccc} \mathcal{O}(U) & \longrightarrow & \mathcal{O}(V) \\ \downarrow & & \downarrow \\ \mathcal{O}'(U) & \longrightarrow & \mathcal{O}'(V) \end{array}$$

is a pullback square in $\mathcal{X}$. We denote by $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$ the (non-full) subcategory of $\mathrm{Str}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$ spanned by local structures and local morphisms between them.

Remark 2.4. One can give similar definitions for $\mathcal{T}_{\mathrm{ét}}(k)$ and $\mathcal{T}_{\mathrm{disc}}(k)$. Notice that a $\mathcal{T}_{\mathrm{disc}}(k)$-structure is simply a product-preserving functor $\mathcal{O} \colon \mathcal{T}_{\mathrm{disc}}(k) \to \mathcal{X}$. For this reason, we can canonically identify $\mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k)}(\mathcal{X})$ with the ∞-category of derived commutative rings $\mathrm{CAlg}_k(\mathcal{X})$ in $\mathcal{X}$.

[6] In the non-archimedean setting smoothness is understood in the sense of Berkovich. See [Ber94].

When $\mathcal{X} = \mathcal{S}$ is the ∞-topos of spaces, $\mathrm{CAlg}_k(\mathcal{X})$ coincides with the underlying ∞-category of the model category of simplicial commutative k-algebras.

Example 2.5. (1) Let X be a $\mathbb{C}$-analytic space, and let X^{top} denote its underlying topological space. Let $\mathcal{X} \coloneqq \mathrm{Sh}(X^{\mathrm{top}})$ be the ∞-topos of sheaves on X. We define a $\mathcal{T}_{\mathrm{an}}(\mathbb{C})$-structure $\mathcal{O}$ on $\mathcal{X}$ as the functor sending an object $U \in \mathcal{T}_{\mathrm{an}}(\mathbb{C})$ to the sheaf $\mathcal{O}(U) \in \mathcal{X}$ defined by

$$\mathrm{Op}(X^{\mathrm{top}}) \ni V \longmapsto \mathcal{O}(U)(V) \coloneqq \mathrm{Hom}_{\mathrm{An}_{\mathbb{C}}}(V, U)\,,$$

where $\mathrm{Op}(X^{\mathrm{top}})$ denotes the poset of open subsets of X^{top}. Notice that $\mathcal{O}(\mathbf{A}^1_{\mathbb{C}})$ coincides with the usual sheaf of holomorphic functions on X.

(2) Let X be a rigid analytic space, and let $X_{\text{ét}}$ denote the small étale site of X. Let $\mathcal{X} \coloneqq \mathrm{Sh}(X_{\text{ét}}, \tau_{\text{ét}})^\wedge$ be the hypercompletion of the ∞-topos of sheaves on $X_{\text{ét}}$. Then we can define a $\mathcal{T}_{\mathrm{an}}(k)$-structure $\mathcal{O}$ on $\mathcal{X}$ as the functor sending $U \in \mathcal{T}_{\mathrm{an}}(k)$ to the sheaf $\mathcal{O}(U) \in \mathcal{X}$ defined by

$$X_{\text{ét}} \ni V \longmapsto \mathcal{O}(U)(V) \coloneqq \mathrm{Hom}_{\mathrm{An}_k}(V, U)\,.$$

Once again, $\mathcal{O}(\mathbf{A}^1_k)$ coincides with the usual sheaf of analytic functions on X.

The analytification functor introduced in the $\mathbb{C}$-analytic case in [Gro63, Exposé XII] and in the k-analytic case in [Ber90] restricts to a functor

$$(-)^{\mathrm{an}} \colon \mathcal{T}_{\mathrm{disc}}(k) \longrightarrow \mathcal{T}_{\mathrm{an}}(k)\,.$$

Precomposition with $(-)^{\mathrm{an}}$ provides for every ∞-topos $\mathcal{X}$ a functor

$$(-)^{\mathrm{alg}} \colon \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X}) \longrightarrow \mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k)}(\mathcal{X}) \simeq \mathrm{CAlg}_k(\mathcal{X})\,.$$

We refer to this functor as the *underlying algebra functor*.

Definition 2.6. A *derived k-analytic space* is a pair $(\mathcal{X}, \mathcal{O}_X)$, where $\mathcal{X}$ is a hypercomplete ∞-topos and $\mathcal{O}_X$ is a $\mathcal{T}_{\mathrm{an}}(k)$-structure on $\mathcal{X}$ such that

(1) locally on $\mathcal{X}$, the analytic space $(\mathcal{X}, \pi_0\mathcal{O}_X)$ is equivalent to a $\mathcal{T}_{\mathrm{an}}(k)$-structured topos arising from the construction of Example 2.5;

(2) the sheaves $\pi_i(\mathcal{O}_X^{\mathrm{alg}})$ are coherent sheaves of $\pi_0(\mathcal{O}_X^{\mathrm{alg}})$-modules.

Theorem 2.7 ([Lur11c, PY18]). *Derived k-analytic spaces assemble into an ∞-category* dAn_k *that has the following properties:*

(1) *Fibre products in* dAn_k *exist.*

(2) *The construction of Example 2.5 provides a fully faithful embedding of the category of ordinary k-analytic spaces* An_k *in* dAn_k.

One of the difficult points of Theorem 2.7 is to actually construct dAn_k as an ∞-category. This is achieved by the general methods of [Lur11a], realizing dAn_k as a full subcategory of the ∞-category of $\mathcal{T}_{\mathrm{an}}(k)$-structured ∞-topoi ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$. More generally, one can define ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T})$ whenever $\mathcal{T}$ is a pregeometry. We refer the reader to [Lur11a, Definition 1.4.8] for a detailed construction.

Remark 2.8. Theorem 2.7 gives a first hint that the notion of derived analytic space introduced in [Lur11c, PY18] is a solid one. Since the appearance of these papers, the theory has been greatly developed. We mention a version of the GAGA theorem in the derived setting, that has been obtained in [Por19], and a detailed analysis of (derived) deformation theory in [PY20b] that led to an analytic version of Lurie's representability theorem. On the applications side, we mention derived versions of the Riemann–Hilbert correspondence [Por17b] and of the Griffiths period map [DH19].

2.2 Derived affinoid, Stein and compact Stein spaces

To any derived analytic space $X = (\mathcal{X}, \mathcal{O}_X)$, we can canonically attach an analytic space

$$\mathrm{t}_0(X) \coloneqq (\mathcal{X}, \pi_0(\mathcal{O}_X)) .$$

We refer to $\mathrm{t}_0(X)$ as the truncation of X. The truncation allows us to define the derived counterparts of Stein and k-affinoid spaces.

Definition 2.9. A derived analytic space $X \in \mathrm{dAn}_k$ is said to be a derived Stein space (in the $\mathbb{C}$-analytic case) or a derived k-affinoid space (in the non-archimedean case) if its truncation $\mathrm{t}_0(X)$ is a Stein or k-affinoid space, respectively. We denote by $\mathrm{dStn}_{\mathbb{C}}$ (respectively, dAfd_k) the full subcategory of $\mathrm{dAn}_{\mathbb{C}}$ (respectively, dAn_k) spanned by derived Stein spaces (respectively, derived k-affinoid spaces).

Warning 2.10. Let k be a non-archimedean field. In the underived setting if A and B are k-affinoid algebras, every ring homomorphism $A \to B$ is automatically continuous. This allows us to embed fully faithfully the category of k-affinoid algebras in the category of commutative rings. In the derived setting this is entirely false: indeed, let $X = \mathrm{Sp}(A)$ be an (underived) k-affinoid space, and let $M \in \mathrm{Coh}^{\heartsuit}(A)$ be a discrete coherent A-module. Via [PY21, Theorem 3.1], we review M as a coherent sheaf $\mathcal{F}$ on X. Consider the split square-zero extension $X[\mathcal{F}[1]]$; see [PY20b, Definition 5.14]. By definition, it is a derived k-affinoid space, and its global sections are $A \oplus M[1]$. A morphism $A \to A \oplus M[1]$ splitting the natural projection $A \oplus M[1] \to A$ classifies an *algebraic* derivation $\mathbb{L}_A \to M[1]$. On the other hand, a morphism of derived k-analytic spaces $X[\mathcal{F}[1]] \to X$ splitting the natural map $X \to X[\mathcal{F}[1]]$ corresponds to an *analytic* derivation $\mathbb{L}^{\mathrm{an}}_X \to \mathcal{F}[1]$; see [PY20b, Definition 5.4]. The algebraic and the analytic cotangent complexes are very different: the former is typically infinite-dimensional (see [GR03, Lemma 7.3.30]), while the latter is connective and has coherent cohomology (see [PY20b, Corollary 5.40]). Similar considerations hold in the $\mathbb{C}$-analytic setting.

Notation 2.11. In this paper we made an effort to present as far as possible statements that are equally true in the complex and non-archimedean analytic case. In particular, following the convention of [PY16], we say "analytic" whenever the statement applies to both settings. When k is not specified and can be either $\mathbb{C}$ or a non-archimedean field, we also use the notation dAfd_k to denote $\mathrm{dStn}_{\mathbb{C}}$.

In [Por19, § 3.1] and in [PY18, § 7.1], it is shown that the étale topology defines a Grothendieck topology on dAfd_k. We let

$$\mathrm{dAnSt}_k \coloneqq \mathrm{Sh}(\mathrm{dAfd}_k, \tau_{\text{ét}})^{\wedge}$$

be the ∞-category of hypercomplete sheaves on $(\mathrm{dAfd}_k, \tau_{\text{ét}})$. We refer to this ∞-category as the ∞-category of derived analytic stacks. Moreover, let $\mathbf{P}_{\mathrm{sm}}$ denote the collection of smooth morphisms in dAfd_k (cf. [PY20b, Definition 5.46]). Then $(\mathrm{dAfd}_k, \tau_{\text{ét}}, \mathbf{P}_{\mathrm{sm}})$ is a geometric context in the sense of [PY16, Definition 2.2]. In particular, the notion of derived analytic geometric stack is defined; see [PY16, Definition 2.8].

Remark 2.12. Notice that here geometric stack is understood in the sense of Simpson [Sim96]. More precisely, we say that a stack X is geometric if it is n-geometric for some $n \geqslant -1$: when

$n = -1$, this means that they are representable by derived k-affinoid (respectively, Stein) spaces. Inductively, a stack X is n-geometric if there exists a smooth effective epimorphism $u\colon U \to X$, where U is a disjoint union of derived k-affinoid (respectively, Stein) spaces, u is representable by $(n-1)$-geometric stacks and the diagonal $X \to X \times X$ is representable by $(n-1)$-geometric stacks as well. In the non-archimedean case the condition on the diagonal is superfluous (see [PY18, Corollary 8.6]).

With this terminology, 1-geometric stacks are closely related to Artin stacks. Furthermore, we avoid the terminology *algebraic stack* because it would give rise to word conflicts in the context of this paper: for us, a derived analytic stack is simply a hypercomplete sheaf on $(\mathrm{dAfd}_k, \tau_{\text{ét}})$.

In dealing with (derived) $\mathbb{C}$-analytic geometry, a frequent difficulty one encounters is that we cannot identify coherent sheaves on a Stein space with modules of finite presentation over the global sections. The classical solution to this problem, as can be found in [Tay02, Proposition 11.9.2], is to work with compact Stein spaces. In loc. cit. a compact Stein space K is a locally ringed space which can be realized as a compact subset of a Stein space U, admitting a fundamental system of Stein open neighbourhoods. The sheaf of functions on K is the sheaf of overconvergent functions on K. However, considering K as an actual locally ringed space has several disadvantages: first of all, it is difficult to generalize to the derived setting, and second it often requires an extra noetherianity hypothesis on K. We will circumvent these issues by considering a compact Stein as a pro-object in $\mathrm{dAn}_{\mathbb{C}}$ (see in particular Theorem 4.8).

Construction 2.13. Let $X \in \mathrm{dAn}_{\mathbb{C}}$ and let $K \subset \mathrm{t}_0(X)$ be a compact subset of $\mathrm{t}_0(X)$. If $U \subset X$ is an open immersion of derived analytic spaces, we write $K \subset U$ to mean that $K \subset \mathrm{t}_0(U)$. Now suppose that K admits a fundamental system of Stein open neighbourhoods inside $\mathrm{t}_0(X)$. Using the equivalence of sites $\mathrm{t}_0(X)_{\text{ét}} \simeq X_{\text{ét}}$ provided by [PY18, Lemma 7.16], we can interpret any open neighbourhood of K inside $\mathrm{t}_0(X)$ as a derived analytic space which is open inside X. We therefore define

$$(K)_X \coloneqq \underset{K \subset U \subset X}{\text{“}\lim\text{”}} U \in \mathrm{Pro}(\mathrm{dAnSt}_{\mathbb{C}}) \,,$$

where the diagram ranges over all the open Stein neighbourhoods of K inside X.

Definition 2.14. A derived compact Stein space is a pro-object which is equivalent to the pro-object $(K)_X$ arising from Construction 2.13.

Compact Stein spaces are especially useful in virtue of the following theorem.

Theorem 2.15 (cf. [Lur09, Corollary 7.3.4.10]). *Let X be a locally compact topological space, and let $\mathcal{C}$ be a presentable ∞-category in which filtered colimits are left exact. Then there is an equivalence of ∞-categories*

$$\mathrm{Sh}(X; \mathcal{C}) \simeq \mathrm{Sh}_{\mathcal{K}}(X; \mathcal{C}) \,,$$

where the right-hand side denotes the sheaves on compact subsets of X, in the sense of [Lur09, Definition 7.3.4.1].

In applications, it is important to know explicitly how the above equivalence works. Therefore, let X be a locally compact space. Let us denote by $\mathcal{K}(X)$ the set of compact subsets of X and by $\mathcal{U}(X)$ the set of open subsets of X. We order both $\mathcal{K}(X)$, $\mathcal{U}(X)$ and their union $\mathcal{K}(X) \cup \mathcal{U}(X)$ by inclusion. Let

$$\kappa\colon \mathcal{K}(X) \hookrightarrow \mathcal{K}(X) \cup \mathcal{U}(X) \,, \quad u\colon \mathcal{U}(X) \hookrightarrow \mathcal{K}(X) \cup \mathcal{U}(X)$$

be the natural inclusions. Then [Lur09, Theorem 7.3.4.9] shows that the fully faithful functors induced by left and right Kan extensions

$$\mathrm{Ran}_\kappa \colon \mathrm{Sh}_{\mathcal{K}}(X;\mathcal{C}) \longrightarrow \mathrm{Fun}((\mathcal{K}(X)\cup\mathcal{U}(X))^{\mathrm{op}},\mathcal{C}) \longleftarrow \mathrm{Sh}(X;\mathcal{C}) : \mathrm{Lan}_u$$

have the same essential image. Now let $\mathcal{F} \in \mathrm{PSh}(X;\mathcal{C}) \coloneqq \mathrm{Fun}(\mathcal{U}(X)^{\mathrm{op}},\mathcal{C})$. Let $\widetilde{\mathcal{F}} \coloneqq \mathrm{Lan}_u(\mathcal{F})$. There is a natural transformation

$$\eta \colon \widetilde{\mathcal{F}} \longrightarrow \mathrm{Ran}_\kappa\big(\widetilde{\mathcal{F}}|_{\mathcal{K}(X)}\big)\,.$$

By restricting to $\mathcal{U}(X)$ and using the full faithfulness of Lan_u, we obtain a natural transformation

$$\eta_{\mathcal{U}} \colon \mathcal{F} \longrightarrow \mathrm{Ran}_\kappa\big(\widetilde{\mathcal{F}}|_{\mathcal{K}(X)}\big)\big|_{\mathcal{U}(X)}\,.$$

Using [Lur09, Theorem 7.3.4.9], we immediately obtain the following result.

Lemma 2.16. *With the above notation, suppose furthermore that $\widetilde{\mathcal{F}}|_{\mathcal{K}(X)}$ belongs to $\mathrm{Sh}_{\mathcal{K}}(X;\mathcal{C})$. Then $\eta_{\mathcal{U}}$ exhibits $\mathrm{Ran}_\kappa\big(\widetilde{\mathcal{F}}|_{\mathcal{K}(X)}\big)\big|_{\mathcal{U}(X)}$ as the sheafification of $\mathcal{F}$.*

Proof. Let $\mathcal{G} \in \mathrm{Sh}(X;\mathcal{C})$. Then we have

$$\mathrm{Map}_{\mathrm{PSh}(X;\mathcal{C})}(\mathcal{F},\mathcal{G}) \simeq \mathrm{Map}_{\mathrm{Fun}((\mathcal{K}(X)\cup\mathcal{U}(X))^{\mathrm{op}},\mathcal{C})}(\mathrm{Lan}_u(\mathcal{F}),\mathrm{Lan}_u(\mathcal{G}))\,.$$

Let $\widetilde{\mathcal{G}} \coloneqq \mathrm{Lan}_u(\mathcal{G})$. Then [Lur09, Theorem 7.3.4.9] implies that $\widetilde{\mathcal{G}} \simeq \mathrm{Ran}_\kappa(\widetilde{\mathcal{G}}|_{\mathcal{K}(X)})$. In particular, η induces an equivalence

$$\mathrm{Map}_{\mathrm{Fun}((\mathcal{K}(X)\cup\mathcal{U}(X))^{\mathrm{op}},\mathcal{C})}(\mathrm{Lan}_u(\mathcal{F}),\mathrm{Lan}_u(\mathcal{G})) \simeq \mathrm{Map}_{\mathrm{Sh}_{\mathcal{K}}(X;\mathcal{C})}\big(\widetilde{\mathcal{F}}|_{\mathcal{K}(X)},\widetilde{\mathcal{G}}|_{\mathcal{K}(X)}\big)\,.$$

Applying right Kan extension along κ again and restricting to $\mathcal{U}(X)$ finally shows that $\eta_{\mathcal{U}}$ induces an equivalence

$$\mathrm{Map}_{\mathrm{PSh}(X;\mathcal{C})}(\mathcal{F},\mathcal{G}) \simeq \mathrm{Map}_{\mathrm{Sh}(X;\mathcal{C})}\big(\mathrm{Ran}_\kappa\big(\widetilde{\mathcal{F}}|_{\mathcal{K}(X)}\big)\big|_{\mathcal{U}(X)},\mathcal{G}\big)\,.$$

The proof is therefore complete. □

3. Analytification of geometric stacks

3.1 The derived analytification functor

The analytification functor

$$(-)^{\mathrm{an}} \colon \mathcal{T}_{\mathrm{\acute{e}t}}(k) \longrightarrow \mathcal{T}_{\mathrm{an}}(k)$$

respects the classes of admissible morphisms and the coverings, and so it is a transformation of pregeometries. As a consequence, [Lur11a, Theorem 2.1.1] shows that it gives rise to an adjunction of ∞-categories

$$(-)^{\mathrm{alg}} \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k)) \leftrightarrows {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{\acute{e}t}}(k)) : (-)^{\mathrm{an}}\,.$$

The functor $(-)^{\mathrm{alg}}$ can be informally described as the functor mapping a $\mathcal{T}_{\mathrm{an}}(k)$-structured topos $(\mathcal{X},\mathcal{O}_X)$ to the $\mathcal{T}_{\mathrm{\acute{e}t}}(k)$-structured topos $(\mathcal{X},\mathcal{O}_X^{\mathrm{alg}})$. We refer to the right adjoint

$$(-)^{\mathrm{an}} \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{\acute{e}t}}(k)) \longrightarrow {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$$

as the *derived analytification functor*. Using [Por17a, Theorem 1.7], we see that ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{\acute{e}t}}(k))$ contains fully faithfully the ∞-category of derived Deligne–Mumford stacks, as defined more classically in [TV08b]. We can summarize the main properties of this functor in the following theorem.

Theorem 3.1 (cf. [Por19, PY20b]). *Let $X = (\mathcal{X},\mathcal{O}_X)$ be a derived Deligne–Mumford stack locally almost of finite presentation. Then:*

(1) *The object* $X^{\mathrm{an}} \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ *is a derived analytic space.*
(2) *The canonical map* $\varepsilon_X \colon (X^{\mathrm{an}})^{\mathrm{alg}} \to X$ *in* ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\text{ét}}(k))$ *is flat.*
(3) *If furthermore* X *is an underived scheme, then under the fully faithful embedding* $\mathrm{An}_k \hookrightarrow \mathrm{dAn}_k$*, the analytification* X^{an} *introduced in* [Ber90, Gro63] *coincides with the derived analytification of* X*.*

In many situations of geometric interest, Deligne–Mumford stacks are too restrictive and need to be replaced by geometric stacks (also known as Artin stacks). Having defined the derived analytification functor at the level of derived Deligne–Mumford stacks locally almost of finite presentation, it is straightforward to extend it to arbitrary derived stacks locally almost of finite presentation by left Kan extension (cf. [PY16, § 6.1] and [TV08a]). This procedure is also implicitly used in [PY20b, Por19]. In this paper we need a slightly more general procedure that allows us to define the analytification of arbitrary derived stacks (not necessarily locally almost of finite presentation). The construction is as follows.

The functor $(-)^{\mathrm{an}} \colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\text{ét}}(k)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ restricts to

$$(-)^{\mathrm{an}} \colon \mathrm{dAff}_k^{\mathrm{afp}} \longrightarrow \mathrm{dAn}_k \, .$$

This is a continuous morphism of sites, and therefore it induces a functor between the associated ∞-categories of hypercomplete sheaves (thanks to [PY16, Proposition 2.22])

$$(-)^{\mathrm{an,afp}} \colon \mathrm{dSt}_k^{\mathrm{afp}} \coloneqq \mathrm{Sh}(\mathrm{dAff}_k^{\mathrm{afp}}, \tau_{\text{ét}})^\wedge \longrightarrow \mathrm{dAnSt}_k \, .$$

On the other hand, let

$$j \colon \mathrm{dAff}_k^{\mathrm{afp}} \hookrightarrow \mathrm{dAff}_k \tag{3.2}$$

be the natural inclusion. The right Kan extension of $(-)^{\mathrm{an,afp}}$ along j provides a functor

$$(-)^{\mathrm{an}} \colon \mathrm{dAff}_k \longrightarrow \mathrm{dAnSt}_k \, .$$

Example 3.3. Let $\mathrm{Spec}(A) \in \mathrm{dAff}_k$ be a derived affine k-scheme, not necessarily almost of finite type. Then

$$\mathrm{Spec}(A)^{\mathrm{an}} \xrightarrow{\sim} \lim_{B \to A} \mathrm{Spec}(B)^{\mathrm{an}} \, ,$$

where the limit ranges over all of the morphisms $B \to A$ with B almost of finite presentation.

Lemma 3.4. *Let* $U \to V$ *be an étale covering in* dAff_k *and* $U_\bullet$ *its Čech nerve. The canonical map*

$$|U_\bullet^{\mathrm{an}}| \longrightarrow V^{\mathrm{an}}$$

is an equivalence in dAnSt_k*.*

Proof. Observe that the category of derived affines of finite type $\mathrm{dAff}_k^{\mathrm{fp}}$, equipped with étale maps and the étale topology, forms a finitary geometry in the sense of [Lur11a, Remark 2.2.8]. Moreover, $\mathrm{dAff}_k \simeq \mathrm{Pro}(\mathrm{dAff}_k^{\mathrm{ft}})$, and therefore it follows from that remark that we can assume, without loss of generality, the existence of $V' \in \mathrm{dAff}_k^{\mathrm{fp}}$ and an étale covering $U' \to V'$ such that

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ U' & \longrightarrow & V' \end{array}$$

is a pullback square. Let $U'_\bullet$ be the Čech nerve of $U' \to V'$. Since

$$(-)^{\mathrm{an}} \colon \mathrm{dAff}_k^{\mathrm{afp}} \longrightarrow \mathrm{dAn}_k$$

is a morphism of sites, we see that

$$\left|(U'_\bullet)^{\mathrm{an}}\right| \longrightarrow (V')^{\mathrm{an}}$$

is an equivalence in dAnSt_k. Since colimits are stable under pullbacks in dAnSt_k, the diagram

$$\begin{array}{ccc} \left|U_\bullet^{\mathrm{an}}\right| & \longrightarrow & V \\ \downarrow & & \downarrow \\ \left|(U'_\bullet)^{\mathrm{an}}\right| & \longrightarrow & (V')^{\mathrm{an}} \end{array}$$

is a pullback square. It follows that the top horizontal morphism is an equivalence, completing the proof. □

Lemma 3.4 guarantees that the functor $(-)^{\mathrm{an}} \colon \mathrm{dAff}_k \to \mathrm{dAnSt}_k$ extends uniquely to a colimit-preserving functor

$$(-)^{\mathrm{an}} \colon \mathrm{dSt}_k \longrightarrow \mathrm{dAnSt}_k\,.$$

We refer to this functor as the *derived analytification functor*. The following lemma guarantees that there is no ambiguity when analytifying a derived stack locally almost of finite presentation.

Lemma 3.5. *For any* $F \in \mathrm{dSt}_k^{\mathrm{afp}}$ *there is a canonical equivalence*

$$F^{\mathrm{an,afp}} \simeq (j_s(F))^{\mathrm{an}}\,,$$

where j_s *is defined by left Kan extension along the inclusion* $j \colon \mathrm{dAff}_k^{\mathrm{afp}} \to \mathrm{dAff}_k$.

Proof. As both $j_s \colon \mathrm{dSt}_k^{\mathrm{afp}} \to \mathrm{dSt}_k$ and $(-)^{\mathrm{an}} \colon \mathrm{dSt}_k \to \mathrm{dAnSt}_k$ commute with colimits, it is enough to check that the two constructions agree when $F \in \mathrm{dAff}_k^{\mathrm{afp}}$. In this case, the result follows from the full faithfulness of the functors $\mathrm{dAff}_k^{\mathrm{afp}} \hookrightarrow \mathrm{dAff}_k$ and $\mathrm{dAff}_k^{\mathrm{afp}} \to \mathrm{dSt}_k^{\mathrm{afp}}$. □

Remark 3.6. One can adapt the definition of analytification for stacks with values in a general presentable ∞-category $\mathcal{T}$ (extending the case $\mathcal{T} = \mathcal{S}$ considered above). Indeed, the continuous morphism of sites $(-)^{\mathrm{an,afp}} \colon \mathrm{dAff}_k^{\mathrm{afp}} \to \mathrm{dAn}_k$ induces an analytification functor

$$(-)^{\mathrm{an,afp}} \colon \mathrm{Sh}_{\mathcal{T}}\big(\mathrm{dAff}_k^{\mathrm{afp}}, \tau_{\text{ét}}\big)^{\wedge} \longrightarrow \mathrm{Sh}_{\mathcal{T}}\big(\mathrm{dAn}_k, \tau_{\text{ét}}\big)\,.$$

The other extension steps carry over in the exact same way, yielding an analytification functor

$$(-)^{\mathrm{an}} \colon \mathrm{Sh}_{\mathcal{T}}\big(\mathrm{dAff}_k, \tau_{\text{ét}}\big)^{\wedge} \longrightarrow \mathrm{Sh}_{\mathcal{T}}\big(\mathrm{dAn}_k, \tau_{\text{ét}}\big)^{\wedge}\,.$$

Other than the case $\mathcal{T} = \mathcal{S}$, we are mostly interested in the case where $\mathcal{T} = \mathrm{Cat}_\infty$.

3.2 A universal property of analytification

When Y is a derived Deligne–Mumford stack locally almost of finite type and X is a derived analytic space, the very definition of $(-)^{\mathrm{an}}$ implies that the canonical map

$$\mathrm{Map}_{\mathrm{dAn}_k}\big(X, Y^{\mathrm{an}}\big) \longrightarrow \mathrm{Map}_{{}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\text{ét}}(k))}\big(X^{\mathrm{alg}}, Y\big)$$

is an equivalence. However, it is unreasonable to expect to be able to lift the above adjunction to the level of the categories dAnSt_k and dSt_k, even when restricting to geometric stacks on both sides. The reason is that there is a significant difference between the object X^{alg} and its restricted functor of points

$$\mathrm{Map}_{{}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\text{ét}}(k))}\big(-, X^{\mathrm{alg}}\big) \colon \mathrm{dAff}_k \longrightarrow \mathcal{S}\,.$$

For instance, the global sections of these two objects differ. In order to bypass this difficulty, we adapt the method first introduced in [Lur04], which consists in providing an alternative description of both $\mathrm{Map}_{\mathrm{dAn}_k}\big(X, Y^{\mathrm{an}}\big)$ and $\mathrm{Map}_{\mathrm{R}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{\acute{e}t}}(k))}\big(X^{\mathrm{alg}}, Y\big)$ as sheaves on X.

Let $X \in \mathrm{dAnSt}_k$. Define

$$\mathrm{dAfd}_X \coloneqq \mathrm{dAfd}_k \times_{\mathrm{dAnSt}_k} (\mathrm{dAnSt}_k)_{/X} \,.$$

We endow dAfd_X with the étale topology, that we still denote by $\tau_{\mathrm{\acute{e}t}}$. We denote by $\mathcal{X}_X$ the corresponding ∞-topos:

$$\mathcal{X}_X \coloneqq \mathrm{St}(\mathrm{dAfd}_X, \tau_{\mathrm{\acute{e}t}}) \,.$$

Consider the forgetful functor

$$F_X \colon \mathrm{dAfd}_X \longrightarrow \mathrm{dAfd}_k \,,$$

that sends a morphism $U \to X$ to the source U. Then F_X is both a continuous and a cocontinuous morphism of sites. As in [PY16] we let $F_X^p \colon \mathrm{PSh}(\mathrm{dAfd}_k) \to \mathrm{PSh}(\mathrm{dAfd}_X)$ be the morphism on presheaves given by precomposition and $F_X^s \colon \mathrm{dAnSt}_k \to \mathcal{X}_X$ the induced map on sheaves. We then obtain the following result.

Lemma 3.7. *The functor*

$$F_X^s \colon \mathrm{dAnSt}_k \longrightarrow \mathcal{X}_X$$

commutes with colimits, and it coincides with the restriction of F_X^p. In particular, one has

$$\mathrm{Map}_{\mathcal{X}_X}\big(\mathbf{1}_X, F_X^s\big(Y^{\mathrm{an}}\big)\big) \simeq \mathrm{Map}_{\mathrm{dAnSt}_k}\big(X, Y^{\mathrm{an}}\big) \,, \tag{3.8}$$

where $\mathbf{1}_X$ denotes the final object of $\mathcal{X}_X$ and $Y \in \mathrm{dSt}_k$.

Proof. The functor F_X^s commutes with colimits thanks to [PY16, Lemma 2.19] because F_X is a cocontinuous morphism of sites. Similarly, it coincides with the restriction of F_X^p in virtue of [PY16, Lemma 2.13] because F_X is continuous. Finally, we observe that the identification (3.8) follows from the fact that F_X^s coincides with the restriction of F_X^p to dAnSt_k. □

Remark 3.9. The functor $F_X \colon \mathrm{dAfd}_X \to \mathrm{dAfd}_k$ does not preserve (finite) products. Therefore, it follows that F_X^s is *not* part of a geometric morphism of ∞-topoi. Nevertheless, it still has a left adjoint $(F_X)_s$ and a right adjoint ${}_sF_X$. However, $(F_X)_s$ does *not* commute with finite limits.

We now turn to describe $\mathrm{Map}_{\mathrm{R}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{\acute{e}t}}(k))}\big(X^{\mathrm{alg}}, Y\big)$ as the global sections of a sheaf on dAfd_X. The main point of doing this is that the new formulation will make sense for an arbitrary $Y \in \mathrm{dSt}_k$. The sheaf of sections of $\mathcal{O}_X^{\mathrm{alg}}$ provides us with a functor

$$G_X \colon \mathrm{dAfd}_X \longrightarrow \mathrm{dAff}_k$$

which informally sends $U \to X$ to

$$G_X(U) \coloneqq \mathrm{Spec}\big(\mathcal{O}_X^{\mathrm{alg}}(U)\big) \,.$$

Notice that the functor G_X does not factor through $\mathrm{dAff}_k^{\mathrm{afp}}$ and moreover is not continuous because it does not commute with fibre products (not even along étale morphisms). However, we can at least prove that it is cocontinuous.

Lemma 3.10. *The functor $G_X \colon (\mathrm{dAfd}_X, \tau_{\mathrm{\acute{e}t}}) \to (\mathrm{dAff}_k, \tau_{\mathrm{\acute{e}t}})$ is a cocontinuous morphism of sites.*

Proof. Consider the functor

$$\overline{\mathcal{O}} \colon \mathrm{CAlg}_k^{\mathrm{op}} \longrightarrow \mathcal{X}_X \,, \quad R \longmapsto \mathrm{Map}_{\mathrm{CAlg}_k}(R, \mathcal{O}(-)) \in \mathcal{X}_X \,.$$

We first show that $\overline{\mathcal{O}}$ takes covers to effective epimorphisms. Write $\mathcal{G}_{\text{ét}}(k)$ for the geometric envelope of $\mathcal{T}_{\text{ét}}(k)$, which can be explicitly described as the full subcategory of $\mathrm{CAlg}_k^{\mathrm{op}}$ spanned by compact objects; see [Lur11a, Definition 4.3.13, Proposition 4.3.15]. Notice that $\overline{\mathcal{O}}$ takes arbitrary colimits in CAlg_k to limits in $\mathcal{X}_X$. In particular, its restriction to $\mathcal{G}_{\text{ét}}(k)$ is left exact. Moreover, its further restriction to $\mathcal{T}_{\text{ét}}(k)$ canonically coincides with $\mathcal{O}$ itself, and it is therefore a $\mathcal{T}_{\text{ét}}(k)$-structure. It follows from the proof of [Lur11a, Proposition 3.4.7] that the restriction of $\overline{\mathcal{O}}$ to $\mathcal{G}_{\text{ét}}(k)$ is a $\mathcal{G}_{\text{ét}}(k)$-structure on $\mathcal{X}_X$. In particular, $\overline{\mathcal{O}}$ takes admissible maps between algebras of finite type (which are exactly $\tau_{\text{ét}}$-coverings) to effective epimorphisms in $\mathcal{X}$.

To check that $\overline{\mathcal{O}}$ takes all $\tau_{\text{ét}}$-coverings to effective epimorphisms, we note that it suffices to check on stalks. Then to check that $\overline{\mathcal{O}}(f)$ is an effective epimorphism, we consider an étale covering $R \to \prod R_i$ and need to produce a lift in the diagram $\mathcal{O}_{X,x} \leftarrow R \to \prod R_i$. Using the local representation of an étale map by a standard étale map, one may see that the étale covering is pro-admissible.

Then by [Lur11a, Proposition 1.3.10], pro-admissible maps form part of a factorization system, see [Lur09, Definition 5.2.8.8], on CAlg_k, considered as $\mathrm{Pro}(\mathcal{G}_{\text{ét}}(k))$. The right set of this factorization system is given by local morphisms, and in particular, $\mathcal{O}_{X,x} \to k$ is right orthogonal to $R \to \prod R_i$, which shows that the desired lift exists [Lur09, Remark 5.2.8.2].

Now let $U \in \mathrm{dAfd}_X$, and fix an étale cover

$$\mathcal{O}(U) \longrightarrow \prod_i R_i \,.$$

The above observation implies that the map

$$\coprod_i \overline{\mathcal{O}}(R_i) \longrightarrow \overline{\mathcal{O}}(\mathcal{O}(U))$$

is an effective epimorphism. In particular, it is an effective epimorphism of sheaves after applying π_0, and thus for every $V \in \mathcal{X}$ and every $f \in \pi_0\overline{\mathcal{O}}(\mathcal{O}(U))(V)$, we can find an effective epimorphism $\coprod V_j \to V$ in $\mathcal{X}_X$ such that for every j there exist some i and some element $f_{ij} \in \pi_0\overline{\mathcal{O}}(R_i)(V_j)$ whose image via

$$\overline{\mathcal{O}}(R_i)(V_j) \longrightarrow \overline{\mathcal{O}}(\mathcal{O}(U))(V_j)$$

coincides with the image of f via the restriction $\overline{\mathcal{O}}(\mathcal{O}(U))(V) \to \overline{\mathcal{O}}(\mathcal{O}(U))(V_j)$. Applying this reasoning to the case $V = U$ and

$$f \coloneqq \mathrm{id}_{\mathcal{O}(U)} \in \overline{\mathcal{O}}(\mathcal{O}(U))(U) = \mathrm{Map}_{\mathrm{CAlg}_k}(\mathcal{O}(U), \mathcal{O}(U)) \,,$$

we deduce the existence of an effective epimorphism $\coprod U_j \to U$ and factorizations $\mathcal{O}(U) \to R_i \to \mathcal{O}(U_j)$. The proof is therefore complete. □

Corollary 3.11. *Let $X \in \mathrm{dAnSt}_k$ be a derived k-analytic stack. Then the functor*

$$G_X^s \colon \mathrm{dSt}_k \longrightarrow \mathcal{X}_X$$

induced by $G_X \colon \mathrm{dAfd}_X \to \mathrm{dAff}_k$ commutes with colimits. In particular, if $U \to Y$ is an effective epimorphism and $U_\bullet$ is its Čech nerve, then the canonical morphism

$$|G_X^s(U_\bullet)| \longrightarrow G_X^s(Y) \tag{3.12}$$

is an equivalence.

Proof. Lemma 3.10 guarantees that the morphism of sites $G_X \colon (\mathrm{dAfd}_X, \tau_{\text{ét}}) \to (\mathrm{dAff}_k, \tau_{\text{ét}})$ is cocontinuous. Therefore, [PY16, Lemma 2.19] shows that it induces a well-defined ∞-functor

$$G^s_X \colon \mathrm{dSt}_k \longrightarrow \mathcal{X}_X$$

which is furthermore left adjoint to ${}_sG_X$. In particular, G^s_X commutes with arbitrary colimits. □

Now let $Y \in \mathrm{dSt}_k^{\mathrm{afp}}$ be a derived stack locally almost of finite presentation. Then Y^{an} is defined as an object in dAnSt_k. In particular, for every $X \in \mathrm{dAn}_k$ both $F^s_X\big(Y^{\mathrm{an}}\big)$ and $G^s_X(Y)$ are defined. The main goal of this section is to prove that they are canonically equivalent whenever Y is furthermore geometric.

Let $U \in \mathrm{dAfd}_X$, and represent it as $U = (\mathcal{U}, \mathcal{O}_U)$. Therefore, $U^{\mathrm{alg}} = \big(\mathcal{U}, \mathcal{O}_U^{\mathrm{alg}}\big)$ and the universal property of the Spec functor of [Lur11a, § 2.2] induces a natural transformation in ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\text{ét}}(k))$

$$\varepsilon_U \colon U^{\mathrm{alg}} \longrightarrow \mathrm{Spec}\big(\mathcal{O}_U^{\mathrm{alg}}(U)\big)\,.$$

For any $Y \in \mathrm{dAff}_k^{\mathrm{afp}}$ this provides us with a natural transformation

$$\alpha_{U,Y} \colon \mathrm{Map}_{\mathrm{dSt}_k}\big(\mathrm{Spec}\big(\mathcal{O}_U^{\mathrm{alg}}(U)\big), Y\big) \longrightarrow \mathrm{Map}_{{}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\text{ét}}(k))}\big(U^{\mathrm{alg}}, Y\big) \simeq \mathrm{Map}_{\mathrm{dAnSt}_k}\big(U, Y^{\mathrm{an}}\big)\,.$$

Notice that

$$\mathrm{Map}_{\mathrm{dSt}_k}\big(\mathrm{Spec}\big(\mathcal{O}_U^{\mathrm{alg}}(U)\big), Y\big) \simeq G^p_X(Y)(U)$$

and

$$\mathrm{Map}_{\mathrm{dAnSt}_k}\big(U, Y^{\mathrm{an}}\big) \simeq F^s_X\big(Y^{\mathrm{an}}\big)(U)\,.$$

As $G^p \circ j_p$ commutes with colimits, the morphisms $\alpha_{U,Y}$ extend to a natural transformation between functors $\mathrm{PSh}\big(\mathrm{dAff}_k^{\mathrm{afp}}\big) \to \mathrm{PSh}\big(\mathrm{dAfd}_X\big)$:

$$\widetilde{\alpha} \colon G^p_X \circ j_p \longrightarrow F^s_X \circ (-)^{\mathrm{an}} \circ j_s \simeq F^s_X \circ (-)^{\mathrm{an,afp}}\,,$$

where j_p and j_s are the functors induced by the morphism of sites (3.2). The equivalence $(-)^{\mathrm{an}} \circ j_s \simeq (-)^{\mathrm{an,afp}}$ is the one provided by Lemma 3.5. As $F^s_X \circ (-)^{\mathrm{an,afp}}$ is a sheaf, we see that this natural transformation induces

$$\alpha \colon G^s_X \circ j_s \longrightarrow F^s_X \circ (-)^{\mathrm{an,afp}}\,.$$

We can now state the main theorem of this section.

THEOREM 3.13. *Let $X \in \mathrm{dAnSt}_k$ be a derived analytic stack. If $Y \in \mathrm{dSt}_k^{\mathrm{afp}}$ is a locally geometric derived stack locally almost of finite presentation, the morphism*

$$\alpha_Y \colon G^s_X(Y) \longrightarrow F^s_X\big(Y^{\mathrm{an}}\big)$$

is an equivalence in $\mathcal{X}_X$.

In particular, α_Y induces an equivalence

$$\mathrm{Map}_{\mathcal{X}_X}\big(\mathbf{1}_X, G^s_X(Y)\big) \simeq \mathrm{Map}_{\mathcal{X}_X}\big(\mathbf{1}_X, F^s_X\big(Y^{\mathrm{an}}\big)\big)\,.$$

In virtue of Lemma 3.7, we can identify the right-hand side with $\mathrm{Map}_{\mathrm{dAnSt}_k}\big(X, Y^{\mathrm{an}}\big)$. The left-hand side instead plays the role of $\mathrm{Map}_{\mathrm{dSt}_k}\big(X^{\mathrm{alg}}, Y\big)$. However, since the functor $G_X \colon \mathrm{dAfd}_X \to \mathrm{dAff}_k$ is not continuous, the functor G^s_X is typically not a right adjoint. This prevents us from rewriting $\mathrm{Map}_{\mathcal{X}_X}\big(\mathbf{1}_X, G^s_X(Y)\big)$ as a mapping space computed in dSt_k. We will nevertheless see that one can effectively use Theorem 3.13 in order to deal with the analytification of higher geometric stacks such as $\mathbf{Perf}_k$.

Proof of Theorem 3.13. We first show the theorem if Y is geometric, proceeding by induction on the geometric level of Y. First suppose that $Y = \operatorname{Spec}(A)$ is affine. For any $U \in \mathrm{dAfd}_X$ we have

$$\begin{aligned} G^p_X(Y)(U) &\simeq \mathrm{Map}_{\mathrm{dSt}_k}\big(\operatorname{Spec}\big(\mathcal{O}^{\mathrm{alg}}_U(U)\big), Y\big) \simeq \mathrm{Map}_{\mathrm{R}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{\acute{e}t}}(k))}\big(U^{\mathrm{alg}}, Y\big) \\ &\simeq \mathrm{Map}_{\mathrm{R}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))}\big(U, Y^{\mathrm{an}}\big) \simeq \mathrm{Map}_{\mathrm{dAnSt}_k}\big(U, Y^{\mathrm{an}}\big) \\ &\simeq F^s_X\big(Y^{\mathrm{an}}\big)(U)\,. \end{aligned}$$

The composition is $\widetilde{\alpha}$. As it is an equivalence and $F^s_X\big(Y^{\mathrm{an}}\big)$ is a sheaf, we conclude that $G^p_X(Y) \simeq G^s_X(Y)$ and that $\alpha\colon G^s_X(Y) \to F^s_X\big(Y^{\mathrm{an}}\big)$ is an equivalence as well.

Now let Y be an n-geometric derived stack locally almost of finite presentation. Choose an n-atlas $u\colon U \to Y$, and let $U_\bullet$ be its Čech nerve. Then Lemma 3.7 and Corollary 3.11 imply that

$$|G^s_X(U_\bullet)| \simeq G^s_X(Y)\,, \quad \big|F^s_X\big(U^{\mathrm{an}}_\bullet\big)\big| \simeq F^s_X\big(Y^{\mathrm{an}}\big)\,.$$

As the natural transformation $\alpha_{U_n}\colon G^s_X\big(U_n\big) \to F^s_X\big(U^{\mathrm{an}}_n\big)$ is an equivalence for every $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ by the induction hypothesis, we conclude that $\alpha_Y\colon G^s_X(Y) \to F^s_X\big(Y^{\mathrm{an}}\big)$ is an equivalence as well.

It remains to extend to the locally geometric case. For this it is sufficient to recall again that G^s_X commutes with colimits by Corollary 3.11 and F^s_X commutes with colimits by Lemma 3.7. □

3.3 Controlling the analytification

In this paper we are mostly concerned with the following type of question. Suppose that the element $X \in \mathrm{dSt}^{\mathrm{afp}}_k$ is a derived geometric stack locally almost of finite presentation. Its analytification X^{an} is obtained via a left Kan extension. This prevents us from providing an easy description of X^{an} in terms of its functor of points. Nevertheless, when X itself parametrizes algebraic families of certain kinds of objects (such as vector bundles, principal G-bundles, perfect complexes, morphisms between algebraic stacks, etc.), there is often an analytic analogue Y parametrizing analytic families of the same type of objects. It is then a natural question to compare the analytification of X with its analytic counterpart Y. Our current goal is to describe a general strategy to prove similar statements (see Proposition 3.14 for a precise statement and a proof). In the rest of the paper, we will repeatedly apply this strategy.

To start we assume given a locally geometric derived stack locally almost of finite presentation $X \in \mathrm{dSt}^{\mathrm{afp}}_k$, an analytic stack $Y \in \mathrm{dAnSt}_k$ and a morphism

$$\varepsilon\colon X^{\mathrm{an}} \longrightarrow Y\,,$$

which we wish to prove is an equivalence. Notice that we do not assume a priori that Y is geometric. It is enough to check that ε induces an equivalence

$$\mathrm{Map}_{\mathrm{dAnSt}_k}\big(U, X^{\mathrm{an}}\big) \simeq \mathrm{Map}_{\mathrm{dAnSt}_k}(U, Y)$$

for all $U \in \mathrm{dAnSt}_k$. Using Lemma 3.7, we see that it is enough to check that ε induces an equivalence

$$F^s_U\big(X^{\mathrm{an}}\big) \longrightarrow F^s_U(Y)\,.$$

Using Theorem 3.13 and the local geometricity of X, we can replace $F^s_U\big(X^{\mathrm{an}}\big)$ with $G^s_U(X)$. In this way, we get rid of the analytification. However, checking in practice that the morphism

$$G^s_U(X) \longrightarrow F^s_U\big(X^{\mathrm{an}}\big) \longrightarrow F^s_U(Y)$$

induced by ε is an equivalence is as difficult as the original problem of proving that ε is an equivalence. The reason is that, once again, $G^s_U(X)$ is not explicitly defined but is rather the result of a sheafification process.

In the non-archimedean setting it happens that in the situations we will consider in the subsequent sections, the map

$$G_U^p(X) \longrightarrow F_U^s(Y)$$

is already an equivalence. This can ultimately be traced back to Tate's acyclicity and Kiehl's theorem (see for instance Lemma 4.6). This implies that $G_U^p(X)$ is a sheaf and therefore that $G_U^p(X) \simeq G_U^s(X)$. In the complex case, this statement is typically false. To remedy this, we are lead to work with compact Stein spaces (see Definition 2.14).

Proposition 3.14. *Let $X \in \mathrm{dSt}_k^{\mathrm{afp}}$ be a locally geometric derived stack locally almost of finite presentation. Let $Y \in \mathrm{dAnSt}_k$ be a derived analytic stack, and let*

$$\varepsilon \colon X^{\mathrm{an}} \longrightarrow Y$$

be a morphism in dAnSt_k. Suppose the following:

1. *If k is a non-archimedean field, then for every $U \in \mathrm{dAfd}_k$ the map ε induces an equivalence*
$$G_U^p(X) \longrightarrow F_U^s(Y) .$$
2. *If $k = \mathbb{C}$, then for every $U \in \mathrm{dStn}_{\mathbb{C}}$ and every compact Stein subset $K \subset U$, the morphism ε induces an equivalence*
$$\operatorname*{colim}_{K \subset V \subset U} G_U^p(X)(V) \simeq \operatorname*{colim}_{K \subset V \subset U} F_U^s(Y)(V) . \tag{3.15}$$

Then $\varepsilon \colon X^{\mathrm{an}} \to Y$ is an equivalence.

Proof. As we already discussed, combining Lemma 3.7 and Theorem 3.13, it is enough to check that ε induces an equivalence

$$G_U^s(X) \longrightarrow F_U^s(Y)$$

for every derived k-affinoid (respectively, Stein) space U. In the non-archimedean situation, the hypothesis guarantees that $G_U^p(X)$ is a sheaf and that it is equivalent to $F_U^s(Y)$. Since $G_U^s(X)$ is the sheafification of $G_U^p(X)$, we conclude that $G_U^s(X) \simeq F_U^s(Y)$ via the morphism induced by ε. Therefore, ε is an equivalence.

In the $\mathbb{C}$-analytic setting, we first use the correspondence provided by Theorem 2.15 to recast $G_U^p(X)$ and $F_U^s(Y)$ as presheaves defined on compact subsets of U. We will abuse notation and write $G_U^p(X)(K)$ for $\mathrm{Lan}_u\big(G_U^p\big)(K)$ if K is a compact Stein space in U. Then the hypothesis guarantees that

$$G_U^p(X)(K) \simeq \operatorname*{colim}_{K \subset V \subset U} G_U^p(X)(V) \simeq \operatorname*{colim}_{K \subset V \subset U} F_U^s(Y)(V) \simeq F_U^s(Y)(K) .$$

Therefore, $G_U^p(X)$ is a sheaf on compact Stein subsets of U which is furthermore equivalent to $F_U^s(Y)$. As compact Stein subsets of U form a basis for U, the conclusion now follows from Lemma 2.16. □

4. Analytic perfect complexes

As usual, we let k be either the field of complex numbers or a non-archimedean field equipped with a non-trivial valuation. In this section we are concerned with the derived analytic stack $\mathbf{AnPerf}_k$ parametrizing families of perfect complexes over derived analytic spaces (see below for its precise definition). Our main goal is to prove that there is a natural equivalence

$$\mathbf{Perf}_k^{\mathrm{an}} \simeq \mathbf{AnPerf}_k .$$

See Proposition 4.9. The proof is based on the general method described in Proposition 3.14. Building on the results obtained in [PY21], it is easy to verify the assumptions of that proposition in the non-archimedean setting. On the other hand, verifying the hypotheses in the $\mathbb{C}$-analytic situation requires a lot of extra work. For this reason, the biggest part of this section is essentially $\mathbb{C}$-analytic in nature, and the main object of study is the category of perfect complexes on a compact Stein, seen as a pro-object in dAn_k.

4.1 The stack of perfect complexes

We start with the basic definitions. Let $\mathcal{X}$ be an ∞-topos, and let $\mathcal{O} \in \mathrm{CAlg}_k(\mathcal{X})$ be a sheaf of connective derived k-algebras. Formally speaking, we set

$$\mathrm{CAlg}_k(\mathcal{X}) \coloneqq \mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k)}(\mathcal{X}) .$$

It can be naturally identified with the ∞-category of sheaves of simplicial commutative k-algebras on $\mathcal{X}$. For every object $U \in \mathcal{X}$ we can form a stably symmetric monoidal ∞-category $\mathcal{O}|_U$-Mod. Its objects are the sheaves of $\mathcal{O}|_U$-modules in the ∞-topos $\mathcal{X}_{/U}$. See [Lur11b, § 2.1].

These categories glue together into a sheaf on $\mathcal{X}$ with values in $\mathrm{Cat}_\infty^{\mathrm{st},\otimes}$, the ∞-category of stably symmetric monoidal ∞-categories. We denote the resulting functor by

$$\mathcal{O}\text{-Mod} \colon \mathcal{X}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty^{\mathrm{st},\otimes} .$$

Notice that when $\mathrm{char}(k) = 0$, the existence of this ∞-functor follows from the technology developed in [Lur17, § 7.3.4], and notably the equivalence

$$\mathcal{O}|_U\text{-Mod} \simeq \mathrm{Sp}(\mathrm{CAlg}(\mathcal{X}_{/U})_{/\mathcal{O}|_U}) ,$$

which reduces the ∞-functoriality of the categories $\mathcal{O}|_U$-Mod to the ∞-functoriality of the comma categories $\mathcal{X}_{/U}$. Here $\mathrm{Sp}(-)$ denotes the ∞-category of spectrum objects; see [Lur17, § 1.4.2]. When $\mathrm{char}(k) > 0$, the same strategy applies, but we have to use instead the identification

$$\mathcal{O}|_U\text{-Mod} \simeq \mathrm{Sp}(\mathrm{Ab}(\mathrm{CAlg}(\mathcal{X}_{/U})_{/\mathcal{O}|_U})) ,$$

proven in [PY20b, Corollary 8.3].

For every $U \in \mathcal{X}$ we let $\mathrm{Perf}^{\mathrm{strict}}(U)$ denote the smallest full stable subcategory of $\mathcal{O}|_U$-Mod closed under retracts and containing $\mathcal{O}|_U$. Restriction along morphisms $V \to U$ in $\mathcal{X}$ preserves strict perfect complexes. So the assignment $U \mapsto \mathrm{Perf}^{\mathrm{strict}}(U)$ can be promoted to a sub-presheaf of $\mathcal{O}$-Mod. We let $\mathrm{Perf}_{\mathcal{X},\mathcal{O}}$ denote its sheafification, computed in the ∞-category $\mathrm{Cat}_\infty^{\mathrm{st},\mathrm{idem}}$ of idempotent complete stable ∞-categories. It is straightforward to observe that the symmetric monoidal structure on $\mathcal{O}|_U$-Mod induces a symmetric monoidal structure on $\mathrm{Perf}(U)$, and so we can actually promote $\mathrm{Perf}_{\mathcal{X},\mathcal{O}}$ to a sheaf with values in idempotent complete stably symmetric monoidal ∞-categories.

When $\mathcal{X} = \mathrm{dSt}_k$ is the ∞-topos of derived stacks and $\mathcal{O}$ is the global section functor, we denote $\mathrm{Perf}_{\mathcal{X},\mathcal{O}}$ simply by Perf_k. According to our general convention, we denote by $\mathbf{Perf}_k$ the associated $\mathcal{S}$-valued stack, determined by the relation

$$\mathbf{Perf}_k(X) \coloneqq \mathrm{Perf}_k(X)^{\simeq} .$$

It coincides with the usual stack of perfect complexes (see [TV07]). Now let $\mathcal{X} = \mathrm{dAnSt}_k$ be the ∞-topos of derived analytic stacks. The functor

$$\mathrm{dAfd}_k^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty^{\mathrm{st},\otimes}$$

sending $U = (\mathcal{U}, \mathcal{O}_U)$ to $\mathcal{O}_U^{\mathrm{alg}}$-Mod extends to a functor $\mathcal{O}\text{-Mod} \colon \mathrm{dAnSt}_k^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{st},\otimes}$. In this case we simply denote by AnMod_k the stack $\mathcal{O}$-Mod and by AnPerf_k the stack $\mathrm{Perf}_{\mathrm{dAnSt}_k,\mathcal{O}}$. When $X \in \mathrm{dAnSt}_k$, we set

$$\mathcal{O}_X\text{-Mod} \coloneqq \mathrm{AnMod}_k(X)\,, \quad \mathrm{Perf}(X) \coloneqq \mathrm{AnPerf}_k(X)\,.$$

When X is a derived Stein (respectively, derived k-affinoid) space, we can also identify X with a $\mathcal{T}_{\mathrm{an}}(k)$-structured topos. In this case, the above notation is compatible with [Lur11b, § 2.1]. Notice that $\mathcal{O}_X$-Mod has a canonical t-structure, where connective objects are defined locally.

Lemma 4.1. *Let $X \in \mathrm{dAnSt}_k$ be a derived analytic stack. Then the stable ∞-category $\mathcal{O}_X$-Mod has a t-structure where an object $\mathcal{F}$ is connective if and only if for every morphism $f\colon U \to X$ with $U \in \mathrm{dAfd}_k$, the pullback $f^*\mathcal{F} \in \mathcal{O}_U^{\mathrm{alg}}$-Mod is connective.*

Proof. This is clear for $X \in \mathrm{dAfd}_k$ using the t-structure on $\mathcal{O}_{\mathcal{X}}^{\mathrm{alg}}$-modules for an ∞-topos $\mathcal{X}$; see [Lur11b, Proposition 2.1.3]. For general X we need to define the t-structure on a limit of categories. We may define connective objects locally and then use the two parts of [Lur17, Proposition 1.4.4.11] to extend to a uniquely defined t-structure. ☐

4.2 Analytification of $\mathbf{Perf}_k$

We define the analytification $\mathbf{Perf}_k^{\mathrm{an}}$ as in Section 3 by first restricting to $\mathrm{dAff}_k^{\mathrm{afp}}$ and then performing left Kan extension along the analytification $(-)^{\mathrm{an}}\colon \mathrm{dAff}_k^{\mathrm{afp}} \to \mathrm{dAfd}_k$. In a similar way, we define the analytification $\mathrm{Perf}_k^{\mathrm{an}}$ of the Cat_∞-valued stack Perf_k.

Remark 4.2. The maximal ∞-groupoid functor $(-)^{\simeq}\colon \mathrm{Cat}_\infty \to \mathcal{S}$ does not commute with colimits in general. Therefore, for $U \in \mathrm{dAfd}_k$ we can no longer identify $\mathbf{Perf}_k^{\mathrm{an}}(U)$ with $\big(\mathrm{Perf}_k^{\mathrm{an}}(U)\big)^{\simeq}$.

Since $\mathrm{Perf}_k^{\mathrm{an}}$ is defined by left Kan extension, in order to give a morphism[7]

$$\varepsilon^*\colon \mathrm{Perf}_k^{\mathrm{an}} \longrightarrow \mathrm{AnPerf}_k\,, \tag{4.3}$$

it is enough to produce a natural transformation

$$\mathrm{Perf}_k \longrightarrow \mathrm{AnPerf}_k \circ (-)^{\mathrm{an}}\,.$$

If $X \in \mathrm{dAff}_k^{\mathrm{afp}}$, then $X^{\mathrm{an}} \in \mathrm{dAfd}_k$, and therefore the underlying $\mathcal{T}_{\mathrm{\acute{e}t}}(k)$-structured topos $\big(X^{\mathrm{an}}\big)^{\mathrm{alg}}$ is well defined. Pulling back along the natural map

$$\theta_X\colon \big(X^{\mathrm{an}}\big)^{\mathrm{alg}} \longrightarrow X$$

provides an analytification functor that respects perfect complexes:

$$(-)_X^{\mathrm{an}}\colon \mathrm{Perf}(X) \longrightarrow \mathrm{Perf}\big(\big(X^{\mathrm{an}}\big)^{\mathrm{alg}}\big)\,.$$

Observe that, according to our definitions, $\mathrm{Perf}\big(\big(X^{\mathrm{an}}\big)^{\mathrm{alg}}\big) \simeq \mathrm{Perf}\big(X^{\mathrm{an}}\big)$. The same construction also provides a morphism $\mathbf{Perf}_k^{\mathrm{an}} \to \mathbf{AnPerf}_k$, which we still denote by ε^*.

Our goal is to prove that $\varepsilon^*\colon \mathbf{Perf}_k^{\mathrm{an}} \to \mathbf{AnPerf}_k$ is an equivalence, and we will do so by verifying the hypotheses of Proposition 3.14. Therefore, fix $X \in \mathrm{dAfd}_k$. The first step is to make the construction of the morphism

$$\varepsilon_X^*\colon G_X^p(\mathbf{Perf}_k) \longrightarrow F_X^s(\mathbf{AnPerf}_k)$$

[7] The notation suggests that this morphism is induced by pullback along a certain map ε. We will make this idea explicit below.

explicit. Let $U \in X_{\text{ét}}$, and define

$$A_U \coloneqq \Gamma\big(U; \mathcal{O}_U^{\mathrm{alg}}\big) .$$

Then

$$G_X^p(\mathbf{Perf}_k)(U) \simeq \mathrm{Perf}(A_U)^{\simeq} , \quad \text{while} \quad F_X^s(\mathbf{AnPerf}_k)(U) \simeq \mathrm{Perf}(U)^{\simeq} .$$

The universal property of Spec proven in [Lur11a, Theorem 2.2.12] provides us with a canonical morphism in ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\text{ét}}(k))$

$$\varepsilon_U \colon (\mathcal{U}, \mathcal{O}_U^{\mathrm{alg}}) \longrightarrow \mathrm{Spec}(A_U) .$$

Pullback along ε_U provides a functor

$$\widetilde{\varepsilon}_U^* \colon \mathcal{O}_{A_U}\text{-Mod} \longrightarrow \mathcal{O}_U\text{-Mod} .$$

This is simply the pullback along ε_U, but we reserve the notation ε_U^* for the restriction

$$\varepsilon_U^* \colon A_U\text{-Mod} \simeq \mathrm{QCoh}(\mathrm{Spec}(A_U)) \xhookrightarrow{i_U} \mathcal{O}_{A_U}\text{-Mod} \xrightarrow{\widetilde{\varepsilon}_U^*} \mathcal{O}_U\text{-Mod} .$$

The functor ε_U^* preserves perfect complexes and therefore further restricts to

$$\varepsilon_U^* \colon \mathrm{Perf}(A_U) \longrightarrow \mathrm{Perf}(U) ,$$

which coincides with the functor induced by $\varepsilon^* \colon \mathrm{Perf}_k^{\mathrm{an}} \to \mathrm{AnPerf}_k$.

Notice that we also have a functor in the opposite direction:

$$\Gamma(U; -) \colon \mathcal{O}_U\text{-Mod} \longrightarrow A_U\text{-Mod} .$$

Observe that ε_U^* and $\Gamma(U;-)$ are not adjoint to each other. However, the inclusion $i_U \colon A_U\text{-Mod} \hookrightarrow \mathcal{O}_{A_U}\text{-Mod}$ admits a left adjoint

$$L_U \colon \mathcal{O}_{A_U}\text{-Mod} \longrightarrow A_U\text{-Mod} ,$$

and, similarly, the functor $\widetilde{\varepsilon}_U^* \colon \mathcal{O}_{A_U}\text{-Mod} \to \mathcal{O}_U\text{-Mod}$ admits a right adjoint

$$\widetilde{\varepsilon}_{U*} \colon \mathcal{O}_U\text{-Mod} \longrightarrow \mathcal{O}_{A_U}\text{-Mod} .$$

Then $\Gamma(U;-)$ is naturally equivalent to the functor $L_U \circ \widetilde{\varepsilon}_{U*}$, and we have two canonical zig-zags of natural transformations

$$\begin{array}{ccccccc} & L_U \circ i_U & & & & \widetilde{\varepsilon}_U^* \circ \widetilde{\varepsilon}_{U*} & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ \Gamma(U;-) \circ \varepsilon_U^* & & \mathrm{Id}_{A_U\text{-Mod}} , & & \varepsilon_U^* \circ \Gamma(U;-) & & \mathrm{Id}_{\mathcal{O}_U\text{-Mod}} . \end{array} \tag{4.4}$$

Notice that $L_U \circ i_U \to \mathrm{Id}_{A_U\text{-Mod}}$ is always an equivalence. In particular, we obtain a well-defined natural transformation

$$\delta \colon \mathrm{Id}_{A_V\text{-Mod}} \longrightarrow \Gamma(U;-) \circ \varepsilon_U^* .$$

We now summarize the most basic properties of the functors we introduced so far.

Proposition 4.5. *Let $U \in \mathrm{dAfd}_k$ be a derived k-affinoid (respectively, Stein) space. Then:*

(1) *The lax symmetric monoidal functor*

$$\Gamma(U;-) \colon \mathcal{O}_U\text{-Mod} \longrightarrow A_U\text{-Mod}$$

is t-exact and symmetric monoidal when restricted to the full subcategory $\mathrm{Coh}(U)$ of unbounded complexes with coherent cohomology.

(2) *The functor*

$$\varepsilon_U^* \colon A_U\text{-Mod} \longrightarrow \mathcal{O}_U\text{-Mod}$$

is t-exact and monoidal, and its restriction to $\mathrm{Coh}(A_U)$ *factors through* $\mathrm{Coh}(U)$*.*

(3) *The restriction of the composition*

$$\Gamma(U;-) \circ \varepsilon_U^* \colon A_U\text{-Mod} \longrightarrow A_U\text{-Mod}$$

to $\mathrm{Coh}(A_U)$ *factors through* $\mathrm{Coh}(A_U)$*.*

(4) *The natural transformation*

$$\delta \colon \mathrm{Id}_{A_U\text{-Mod}} \longrightarrow \Gamma(U;-) \circ \varepsilon_U^*$$

is an equivalence when restricted to $\mathrm{Coh}(A_U)$*.*

(5) *The functor* $\varepsilon_U^* \colon A_U\text{-Mod} \to \mathcal{O}_U\text{-Mod}$ *commutes with filtered colimits, and it is conservative.*

Proof. In the non-archimedean case points (1)–(4) have been proved in [PY21, Theorems 3.1 and 3.4]. In the $\mathbb{C}$-analytic setting point (1) follows from Cartan's theorem B. The t-exactness part of point (2) follows from the flatness result proven in Lemma A.3. At this point, we are left to check that ε_U^* takes $\mathrm{Coh}^\heartsuit(A_U)$ to $\mathrm{Coh}^\heartsuit(U)$. This follows immediately from the fact that A_U is taken to $\mathcal{O}_U$ and the fact that every object in $\mathrm{Coh}^\heartsuit(A_U)$ admits a finite presentation. Point (3) follows immediately from point (4). Point (4) is a consequence of Cartan's Theorem B and point (2).

We now prove point (5). Observe that the inclusion

$$A_U\text{-Mod}^\heartsuit \hookrightarrow \mathcal{O}_{A_U}\text{-Mod}^\heartsuit$$

commutes with filtered colimits. Since $A_U\text{-Mod} \hookrightarrow \mathcal{O}_{A_U}\text{-Mod}$ is t-exact and fully faithful, it follows that it also commutes with filtered colimits. Therefore, ε_U^* has the same property. We now prove conservativity. Since ε_U^* is an exact functor, it is enough to prove that if $\mathcal{F}$ is such that $\varepsilon_U^*(\mathcal{F}) \simeq 0$, then $\mathcal{F} \simeq 0$. Since ε_U^* is t-exact and the t-structures on A_U-Mod and $\mathcal{O}_U$-Mod are complete, we can reduce to the case where $\mathcal{F} \in A_U\text{-Mod}^\heartsuit$. In this case we can write

$$\mathcal{F} \simeq \bigcup_\alpha \mathcal{F}_\alpha\,,$$

where the union ranges over all finitely generated A_U-submodules of $\mathcal{F}$. Since ε_U^* is t-exact, we see that $\varepsilon_U^*(\mathcal{F}_\alpha)$ is a submodule of $\varepsilon_U^*(\mathcal{F})$. This implies that $\varepsilon_U^*(\mathcal{F}_\alpha) = 0$. Since $\mathcal{F}_\alpha \in \mathrm{Coh}^\heartsuit(A_U)$, point (4) implies that $\mathcal{F}_\alpha = 0$ for every α. In particular, $\mathcal{F} = 0$, whence the conclusion. □

In the non-archimedean case, we can strengthen the above result.

Lemma 4.6 (cf. [PY21, Theorem 3.4]). *Let U be a derived k-affinoid space, and let* $A_U \coloneqq \Gamma\big(U; \mathcal{O}_U^{\mathrm{alg}}\big)$*. Then the functors ε_U^* and $\Gamma(U;-)$ realize an equivalence of stable ∞-categories*

$$\mathrm{Coh}(U) \simeq \mathrm{Coh}(A_U)\,,$$

which furthermore restricts to an equivalence

$$\mathrm{Perf}(U) \simeq \mathrm{Perf}(A_U)\,.$$

In particular, there is an equivalence $\mathrm{Perf}(U)^\simeq \simeq \mathrm{Perf}(A_U)^\simeq$*.*

Proof. Theorem 3.4 in [PY21] shows that the global section functor induces a t-exact equivalence of stable ∞-categories

$$\mathrm{Coh}(U) \simeq \mathrm{Coh}(A_U)\,.$$

On the other hand, we know that $\mathrm{Perf}(A_U)$ coincides with the smallest full stable subcategory of $\mathrm{Coh}^-(A_U)$ closed under retracts and containing A_U. As A_U is mapped to $\mathcal{O}_U$ under the above equivalence, we see that $\mathrm{Perf}(A_U)$ is mapped into $\mathrm{Perf}(U)$. It is therefore sufficient to prove that the global section functor takes $\mathrm{Perf}(U)$ to $\mathrm{Perf}(A_U)$. Fix $\mathcal{F} \in \mathrm{Perf}(U)$, and let $\{U_i\}$ be a finite derived affinoid cover of U so that $\mathcal{F}|_{U_i}$ belongs to $\mathrm{Perf}^{\mathrm{strict}}(U_i)$. Let $A_i \coloneqq \Gamma\big(U_i; \mathcal{O}_{U_i}^{\mathrm{alg}}\big)$. In this case, we immediately see that $\Gamma(U_i; \mathcal{F}|_{U_i}) \in \mathrm{Perf}(A_i)$. In particular, $\Gamma(U_i; \mathcal{F}|_{U_i})$ has finite tor-amplitude. As the maps $A \to A_i$ are faithfully flat and

$$\Gamma(U_i; \mathcal{F}|_{U_i}) \simeq \Gamma(X; \mathcal{F}) \otimes_A A_i \,,$$

we conclude that $\Gamma(X;\mathcal{F})$ also has finite tor-amplitude by [Lur18, Proposition 2.8.4.2 (5)]. In particular, it belongs to $\mathrm{Perf}(A)$. □

This verifies the hypotheses of Proposition 3.14 for the map $\varepsilon^* \colon \mathbf{Perf}_k^{\mathrm{an}} \to \mathbf{AnPerf}_k$ in the non-archimedean setting. In the $\mathbb{C}$-analytic case the analogue of Lemma 4.6 is simply false: indeed, the restriction of the functor $\Gamma(U;-)$ to $\mathrm{Perf}(U)$ does not factor through $\mathrm{Perf}(A_U)$. However, Proposition 3.14 asks for a different statement: we have to check that for every compact Stein subset K of X, the map $\varepsilon^* \colon \mathbf{Perf}_k^{\mathrm{an}} \to \mathbf{AnPerf}_k$ induces an equivalence

$$\operatorname*{colim}_{K \subset U \subset X} \mathrm{Perf}(A_U)^{\simeq} \longrightarrow \operatorname*{colim}_{K \subset U \subset X} \mathrm{Perf}(U)^{\simeq} \,, \tag{4.7}$$

where the colimits run through the open Stein neighbourhoods U of K inside X. In order to prove that (4.7) is an equivalence, it is easier (and more natural) to work with the stacks with values in Cat_∞. We will prove below that $\varepsilon^* \colon \mathrm{Perf}_{\mathbb{C}}^{\mathrm{an}} \to \mathrm{AnPerf}_{\mathbb{C}}$ induces an equivalence

$$\operatorname*{colim}_{K \subset U \subset X} \mathrm{Perf}(A_U) \longrightarrow \operatorname*{colim}_{K \subset U \subset X} \mathrm{Perf}(U) \,.$$

However, since the maximal ∞-groupoid functor $(-)^{\simeq}$ does not commute with colimits, it is not straightforward to deduce that (4.7) is an equivalence from the above statement. To circumvent this problem, we prove the following stronger statement.

Theorem 4.8. *Let $X \in \mathrm{dAn}_{\mathbb{C}}$, and let K be a compact Stein subset of X. The functors*

$$\varepsilon_U^* \colon \mathrm{Perf}(A_U) \longrightarrow \mathrm{Perf}(U)$$

induce a morphism

$$\varepsilon_{(K)}^* \colon \text{“}\operatorname*{colim}_{K \subset U \subset X}\text{”}\, \mathrm{Perf}(A_U) \longrightarrow \text{“}\operatorname*{colim}_{K \subset U \subset X}\text{”}\, \mathrm{Perf}(U)$$

in $\mathrm{Ind}\big(\mathrm{Cat}_\infty^{\mathrm{st},\otimes}\big)$, which is furthermore an equivalence.

The proof of this theorem is technical, and it will occupy the rest of this section. Before delving into the details, let us record its main consequence.

Proposition 4.9. *Let k be either the field of complex numbers or a non-archimedean field equipped with a non-trivial valuation. The natural morphism $\varepsilon^* \colon \mathbf{Perf}_k^{\mathrm{an}} \to \mathbf{AnPerf}_k$ is an equivalence of derived analytic stacks. In particular, $\mathbf{AnPerf}_k$ is a locally geometric derived analytic stack.*

Proof. We know that $\mathbf{Perf}_k$ is a locally geometric stack locally almost of finite presentation. It is therefore enough to check that the hypotheses of Proposition 3.14 are satisfied. Fix a derived Stein (respectively, k-affinoid) space U, and let $V \subset U$ be an open Stein subspace (respectively, k-affinoid domain embedding). Unravelling the definitions, we see that

$$G_U^p(\mathbf{Perf}_k)(V) \simeq \mathrm{Perf}(A_V)^{\simeq} \,, \quad F_U^s(\mathbf{AnPerf}_k)(V) \simeq \mathrm{Perf}(V)^{\simeq} \,.$$

In the non-archimedean case the conclusion therefore follows from Lemma 4.6. In the $\mathbb{C}$-analytic case the equivalence of ind-objects provided by Theorem 4.8 induces an equivalence

$$\underset{K \subset V \subset U}{\text{“colim”}} \operatorname{Perf}(A_V)^{\simeq} \longrightarrow \underset{K \subset V \subset U}{\text{“colim”}} \operatorname{Perf}(V)^{\simeq}$$

in $\operatorname{Ind}(\mathcal{S})$. By realizing this equivalence of ind-objects, we see that the hypotheses of Proposition 3.14 are satisfied. The second statement follows at once because the analytification functor preserves locally geometric stacks. □

4.3 Proof of Theorem 4.8

We now turn to the proof of Theorem 4.8. As in Lemma 4.6, we will deduce the statement from the analogous statement concerning unbounded complexes with coherent cohomology. As we already remarked, the difference from the non-archimedean setting is that the composition

$$\operatorname{Coh}(U) \longrightarrow \mathcal{O}_U\text{-Mod} \xrightarrow{\Gamma(U;-)} A_U\text{-Mod}$$

does not factor through $\operatorname{Coh}(A_U)$. We therefore lack a candidate for the inverse of ε_U^*. When working with ind-objects, however, the situation improves thanks to the following couple of lemmas. For the notion of relatively compact, see Definition A.1.

Lemma 4.10. *Let $U \in \mathrm{dStn}_{\mathbb{C}}$, and let $V \Subset U$ be a relatively compact Stein subset. Then the composition*

$$\operatorname{Coh}(U) \xrightarrow{(-)|_V} \operatorname{Coh}(V) \xrightarrow{\Gamma(V;-)} A_V\text{-Mod}$$

factors, as a symmetric monoidal functor, through $\operatorname{Coh}(A_V)$. Moreover, if $\mathcal{F} \in \operatorname{Perf}(U)$, then $\Gamma(V; \mathcal{F}|_V) \in \operatorname{Perf}(A_V)$.

Proof. We know from Proposition 4.5 that $\Gamma(V;-)$ is t-exact and monoidal. Therefore, it is enough to check that when $\mathcal{F} \in \operatorname{Coh}^{\heartsuit}(U)$, we have $\Gamma(V;\mathcal{F}|_V) \in \operatorname{Coh}^{\heartsuit}(A_V)$. This follows at once from Cartan's Theorem B and [PY16, Lemma 8.12].

Now suppose $\mathcal{F} \in \operatorname{Perf}(U)$. We have to prove that $\Gamma(V;\mathcal{F}|_V)$ has finite tor-amplitude. If $\mathcal{F} \in \operatorname{Perf}^{\mathrm{strict}}(U)$, then $\mathcal{F}|_V \in \operatorname{Perf}^{\mathrm{strict}}(V)$ and therefore $\Gamma(V;\mathcal{F}|_V) \in \operatorname{Perf}^{\mathrm{strict}}(A_V)$. In general, we can find a cover $\{U_i\}_{i\in I}$ of U such that $\mathcal{F}|_{U_i}$ belongs to $\operatorname{Perf}^{\mathrm{strict}}(U_i)$. Now choose a cover $\{V_j\}_{j\in J}$ of V satisfying the following properties:

(1) For every $j \in J$ there exists an $i \in I$ such that $V_j \subset U_i$.
(2) For every $j \in J$ the open V_j is Stein and relatively compact inside V.

It follows that each $\Gamma(V_j;\mathcal{F}|_{V_j})$ is perfect over A_{V_j}. Since $V_j \Subset V \Subset U$, we can apply Lemma A.3 to deduce that the family of maps $\{A_V \to A_{V_j}\}$ is faithfully flat. Similarly, Corollary A.4 shows that the natural morphism

$$\Gamma(V;\mathcal{F}|_V) \otimes_{A_V} A_{V_j} \longrightarrow \Gamma(V_j;\mathcal{F}|_{V_j})$$

is an equivalence. At this point, [Lur18, Proposition 2.8.4.2(5)] implies that $\Gamma(V;\mathcal{F}|_V)$ has finite tor-amplitude over A_V, completing the proof. □

This lemma shows that the functor $\Gamma(U;-)\colon \operatorname{Coh}(U) \to A_U\text{-Mod}$ induce a morphism in $\operatorname{Ind}\big(\mathrm{Cat}_\infty^{\mathrm{st},\otimes}\big)$

$$\Gamma_{(K)}\colon \underset{K \subset V \subset U}{\text{“colim”}} \operatorname{Coh}(U) \longrightarrow \underset{K \subset V \subset U}{\text{“colim”}} \operatorname{Coh}(A_U)\,.$$

We now wish to prove that $\Gamma_{(K)}$ and $\varepsilon_{(K)}^*$ form an equivalence of ind-objects. We need the following lemma.

Lemma 4.11. *Let U be a derived Stein space, and let $W \Subset V \Subset U$ be two nested relatively compact Stein subsets. For any $\mathcal{F} \in \mathrm{Coh}(U)$ the $\mathcal{O}_{A_W}$-module $\widetilde{\varepsilon}_{W*}(\mathcal{F}|_W)$ is a coherent sheaf over* $\mathrm{Spec}(A_W)$.

Proof. We already know from Lemma 4.10 that the global sections of $\widetilde{\varepsilon}_{W*}(\mathcal{F}|_W)$ belong to $\mathrm{Coh}(A_W)$. It is therefore enough to prove that all its cohomologies are quasi-coherent. For this, it is enough to check that for every principal open W' of $\mathrm{Spec}(A_W)$, the canonical map

$$\Gamma(W;\mathcal{F}|_W) \otimes_{A_W} A_{W'} \longrightarrow \big(\widetilde{\varepsilon}_{W*}(\mathcal{F}|_W)\big)(W')$$

is an equivalence. We immediately observe that

$$\widetilde{\varepsilon}_{W*}(\mathcal{F}|_W)(W') = \mathcal{F}\big(\varepsilon_W^{-1}(W')\big)\,.$$

It follows from the discussion in [GR84, § 1.4.4] and from the reconstruction theorem proved in [GR79, § IV.7.4] that $\widetilde{W'} := \varepsilon_W^{-1}(W')$ is itself a Stein space. Furthermore, we have $\widetilde{W'} \Subset V$. We can therefore apply Corollary A.4 to the sequence of nested derived Stein subsets $\widetilde{W'} \Subset V \Subset U$ to deduce that

$$\Gamma(W;\mathcal{F}|_W) \otimes_{A_W} A_{\widetilde{W'}} \simeq \Gamma(V;\mathcal{F}|_V) \otimes_{A_V} A_{\widetilde{W'}}\,.$$

The proof is therefore complete. □

Lemma 4.12. *Let U be a derived Stein space, and let $W \Subset V \Subset U$ be two nested relatively compact Stein subsets. Let $\rho_{U,W} \colon \mathcal{O}_U\text{-Mod} \to \mathcal{O}_W\text{-Mod}$ be the restriction functor, and let $j_U \colon \mathrm{Coh}(U) \hookrightarrow \mathcal{O}_U\text{-Mod}$ be the natural inclusion. Then:*

(1) *The natural transformation*

$$\widetilde{\varepsilon}_{W*} \circ \rho_{U,W} \circ j_U \longrightarrow i_W \circ L_W \circ \widetilde{\varepsilon}_{W*} \circ \rho_{U,W} \circ j_U$$

is an equivalence.

(2) *The natural transformation*

$$\vartheta \colon \varepsilon_W^* \circ \Gamma(W;-) \circ \rho_{U,W} \circ j_U \longrightarrow \rho_{U,W} \circ j_U$$

induced by the zig-zag (4.4) and by the previous point is an equivalence.

Proof. Point (1) is a direct consequence of Lemma 4.11 because $\widetilde{\varepsilon}_{W*} \circ \rho_{U,W} \circ j_U$ factors through $\mathrm{Coh}(A_W)$ and the unit of the adjunction $L_W \dashv i_W$ is an equivalence on the objects belonging to A_W-Mod. As for point (2), all the functors appearing are t-exact. It is therefore sufficient to check that θ is an equivalence when evaluated on objects in $\mathrm{Coh}^\heartsuit(U)$. This follows immediately from [PY16, Lemma 8.11]. □

We are now ready to state and prove the key result.

Theorem 4.13. *Let $X \in \mathrm{dAn}_{\mathbb{C}}$, and let K be a compact Stein subset of X. The morphism*

$$\varepsilon^*_{(K)} \colon \underset{K \subset U \subset X}{\text{“colim”}} \mathrm{Coh}(A_U) \longrightarrow \underset{K \subset U \subset X}{\text{“colim”}} \mathrm{Coh}(U) \tag{4.14}$$

is an equivalence in $\mathrm{Ind}\big(\mathrm{Cat}_\infty^{\mathrm{st},\otimes}\big)$, whose inverse is given by $\Gamma_{(K)}$.

Proof. Proposition 4.5 implies that the natural transformation $\delta \colon \mathrm{Id}_{A_U\text{-Mod}} \to \Gamma(U;-) \circ \varepsilon_U^*$ is an equivalence when evaluated on objects in $\mathrm{Coh}(A_U)$, for every open Stein neighbourhood U of K in X. This shows that $\Gamma_{(K)} \circ \varepsilon^*_{(K)}$ is equivalent to the identity of the left-hand side of (4.14).

For the other direction, the natural transformation θ we constructed in Lemma 4.12 provides a path between morphisms in $\mathrm{Ind}\big(\mathrm{Cat}_\infty^{\mathrm{st},\otimes}\big)$

$$\varepsilon^*_{(K)} \circ \Gamma_{(K)} \longrightarrow \mathrm{Id}_{\mathrm{Coh}((K)_U)} ,$$

where $\mathrm{Coh}((K)_U)$ denotes the right-hand side of (4.14). Moreover, Lemma 4.12 shows that this morphism is invertible, thereby completing the proof. □

From here, deducing Theorem 4.8 is straightforward.

Proof of Theorem 4.8. It is enough to check that the functors $\varepsilon^*_{(K)}$ and $\Gamma_{(K)}$ restrict to morphism of ind-objects

$$\varepsilon^*_{(K)} \colon \underset{K\subset U\subset X}{\text{“colim”}} \mathrm{Perf}(A_U) \longrightarrow \underset{K\subset U\subset X}{\text{“colim”}} \mathrm{Perf}(U) , \quad \Gamma_{(K)} \colon \underset{K\subset U\subset X}{\text{“colim”}} \mathrm{Perf}(U) \longrightarrow \underset{K\subset U\subset X}{\text{“colim”}} \mathrm{Perf}(A_U) .$$

In the case of $\varepsilon^*_{(K)}$ this follows directly from the construction, while for $\Gamma_{(K)}$ this has already been in checked in Lemma 4.10. □

By realizing the equivalences of ind-objects obtained in Theorems 4.13 and 4.8, we obtain the following weaker result, which is closer in spirit to [Tay02, Proposition 11.9.2]. Before stating it, let us fix some notation: first of all, we write

$$\varepsilon^*_K \colon \underset{K\subset U\subset X}{\mathrm{colim}} \mathrm{Perf}(A_U) \longrightarrow \underset{K\subset U\subset X}{\mathrm{colim}} \mathrm{Perf}(U)$$

and

$$\Gamma_K \colon \underset{K\subset U\subset X}{\mathrm{colim}} \mathrm{Perf}(U) \longrightarrow \underset{K\subset U\subset X}{\mathrm{colim}} \mathrm{Perf}(A_U)$$

for the realizations of $\varepsilon^*_{(K)}$ and $\Gamma_{(K)}$. We also set

$$A_K \coloneqq \underset{K\subset U\subset X}{\mathrm{colim}} A_U .$$

Then we have the following.

Corollary 4.15. *Let $X \in \mathrm{dAn}_{\mathbb{C}}$, and let K be a compact Stein inside X. Then there is a canonical equivalence*

$$\mathrm{Perf}(A_K) \simeq \underset{K\subset U\subset X}{\mathrm{colim}} \mathrm{Perf}(U) .$$

Proof. By realizing the equivalence of Theorem 4.8, we see that the functors Γ_K and ε^*_K induce an equivalence

$$\underset{K\subset U}{\mathrm{colim}} \mathrm{Perf}(A_U) \simeq \underset{K\subset U\subset X}{\mathrm{colim}} \mathrm{Perf}(U) .$$

On the other hand, it is proven in [Lur18, Corollary 4.5.1.8] that the construction $R \mapsto \mathrm{Perf}(R)$ commutes with filtered colimits. Therefore, we also have an equivalence

$$\mathrm{Perf}(A_K) \simeq \underset{K\subset U\subset X}{\mathrm{colim}} \mathrm{Perf}(A_U) .$$

The conclusion follows. □

Let us give another application that stems from the combination of Corollary 4.15 and Theorem 2.15.

Proposition 4.16. *Let X be a derived $\mathbb{C}$-analytic space. Then the subcategory $\mathrm{Perf}(X) \subset \mathrm{Coh}^-(X)$ coincides exactly with the subcategory of dualizable objects.*

Proof. Denote by AnPerf_X and AnCoh^-_X the restrictions of AnPerf and AnCoh^- to X^{top}. Let $\mathcal{F} \in \mathrm{Coh}^-(X)$ be a dualizable object. Then we observe that for every compact Stein subset $K \subset X$, the object $\Gamma_{(K)}(\mathcal{F})$ is dualizable in $\mathrm{Coh}^-(A_K)$, and hence it belongs to $\mathrm{Perf}(A_K)$. We now use the equivalence

$$\mathrm{Perf}((K)_X) \simeq \mathrm{Perf}(A_K)$$

provided by Corollary 4.15, and we observe that under the equivalence

$$\mathrm{Sh}\big(X^{\mathrm{top}}; \mathrm{Cat}_\infty^{\mathrm{st,idem}}\big) \simeq \mathrm{Sh}_{\mathcal{K}}\big(X^{\mathrm{top}}, \mathrm{Cat}_\infty^{\mathrm{st,idem}}\big)$$

provided by Theorem 2.15, the sheaf AnPerf_X corresponds to the stack sending a compact Stein subset $K \subset X$ to $\mathrm{Perf}((K)_X)$. Therefore, $\mathcal{F}$ defines a global section of AnPerf_X; that is, $\mathcal{F} \in \mathrm{Perf}(X)$.

Conversely, suppose $\mathcal{F} \in \mathrm{Perf}(X)$. Then the functor

$$\mathcal{F} \otimes_{\mathcal{O}_X} - \colon \mathcal{O}_X\text{-Mod} \longrightarrow \mathcal{O}_X\text{-Mod}$$

is left adjoint to

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, -) \colon \mathcal{O}_X\text{-Mod} \longrightarrow \mathcal{O}_X\text{-Mod}\,.$$

In particular, we have unit and counit transformations

$$\mathcal{G} \longrightarrow \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})\,, \quad \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{G}$$

satisfying the triangular identities. Furthermore, as $\mathcal{F}$ is a perfect complex, the canonical morphism

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})$$

is an equivalence. Taking $\mathcal{G} = \mathcal{O}_X$, we obtain the evaluation and coevaluation morphisms for $\mathcal{F}$ and $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. All that is left to check is therefore that $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ is perfect. Combining once again Theorems 4.15 and 2.15, we are reduced to checking that for every compact Stein subset $K \subset U$, we have

$$\Gamma_{(K)}(\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)) \in \mathrm{Perf}(A_K)\,.$$

However, for every open Stein neighbourhood V of K inside X, one has

$$\Gamma(V; \mathcal{H}\mathrm{om}_{\mathcal{O}_V}(\mathcal{F}|_V, \mathcal{O}_V)) \simeq \mathrm{Hom}_{A_V}(\Gamma(V; \mathcal{F}|_V), A_V)\,.$$

Hence

$$\Gamma_{(K)}(\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)) \simeq \mathrm{Hom}_{A_K}(\Gamma_{(K)}(\mathcal{F}), A_K) \in \mathrm{Perf}(A_K)\,.$$

This completes the proof. □

5. GAGA properties

In this section we discuss several variations of the GAGA property on derived stacks locally almost of finite presentation. We verify that they are satisfied in a number of different examples, proving a relative version of Proposition 4.9.

5.1 (Universally) GAGA stacks

We start by generalizing the natural transformation $\varepsilon^* \colon \mathrm{Perf}_k^{\mathrm{an}} \to \mathrm{AnPerf}_k$ introduced in the previous section. Fix $X \in \mathrm{dAff}_k^{\mathrm{afp}}$ and $U \in \mathrm{dAfd}_k$. As usual, we set $A_U \coloneqq \Gamma\big(U; \mathcal{O}_U^{\mathrm{alg}}\big)$. The counit of the analytification adjunction $\varepsilon_X \colon \big(X^{\mathrm{an}}\big)^{\mathrm{alg}} \to X$ induces a well-defined morphism

$$\varepsilon_{X,U} \colon \big(X^{\mathrm{an}} \times U\big)^{\mathrm{alg}} \longrightarrow \big(X^{\mathrm{an}}\big)^{\mathrm{alg}} \times \mathrm{Spec}(A_U) \longrightarrow X \times \mathrm{Spec}(A_U)\,.$$

The pullback functor along $\varepsilon_{X,U}$ induces a well-defined symmetric monoidal functor

$$\varepsilon_{X,U}^* \colon \mathrm{QCoh}(X \times \mathrm{Spec}(A_U)) \longrightarrow \mathcal{O}_{X^{\mathrm{an}} \times U}\text{-Mod}\,,$$

which is natural in both X and U. Moreover, it restricts to a symmetric monoidal functor

$$\varepsilon_{X,U}^* \colon \mathrm{Perf}(X \times \mathrm{Spec}(A_U)) \longrightarrow \mathrm{Perf}\big(X^{\mathrm{an}} \times U\big)\,.$$

The naturality in X allows us to extend this map by colimits. Therefore, we obtain a commutative square

$$\begin{array}{ccc} \mathrm{Perf}(X \times \mathrm{Spec}(A_U)) & \xrightarrow{\varepsilon_{X,U}^*} & \mathrm{Perf}\big(X^{\mathrm{an}} \times U\big) \\ \big\downarrow & & \big\downarrow \\ \mathrm{QCoh}(X \times \mathrm{Spec}(A_U)) & \xrightarrow{\varepsilon_{X,U}^*} & \mathcal{O}_{X^{\mathrm{an}} \times U}\text{-Mod} \end{array}$$

for every $X \in \mathrm{dSt}_k^{\mathrm{afp}}$. When $U = \mathrm{Sp}(k)$, we write ε_X^* instead of $\varepsilon_{X,\mathrm{Sp}(k)}^*$.

Definition 5.1. Let $X \in \mathrm{dSt}_k^{\mathrm{afp}}$ be a derived stack locally almost of finite presentation.

(1) We say that X *satisfies the GAGA property* if the functor $\varepsilon_X^* \colon \mathrm{QCoh}(X) \to \mathcal{O}_{X^{\mathrm{an}}}\text{-Mod}$ is conservative and t-exact and the functor

$$\varepsilon_X^* \colon \mathrm{Perf}(X) \longrightarrow \mathrm{Perf}\big(X^{\mathrm{an}}\big)$$

is an equivalence of ∞-categories.

(2) Let k be a non-archimedean field, and let $U \in \mathrm{dAfd}_k$ be a derived k-affinoid space. We say that X *satisfies the GAGA property relative to* U if the functor

$$\varepsilon_{X,U}^* \colon \mathrm{QCoh}(\mathrm{Spec}(A_U) \times X) \longrightarrow \mathcal{O}_{U \times X^{\mathrm{an}}}\text{-Mod}$$

is conservative and t-exact and the functor

$$\varepsilon_{X,U}^* \colon \mathrm{Perf}(\mathrm{Spec}(A_U) \times X) \longrightarrow \mathrm{Perf}\big(U \times X^{\mathrm{an}}\big)$$

is an equivalence.

(3) Let $U \in \mathrm{dStn}_{\mathbb{C}}$ be a derived Stein space, and let $K \subset U$ be a compact Stein subset. We say that X *satisfies the pro-GAGA property relative to* $(K)_U$ if for every Stein neighbourhood V of K inside U, the map

$$\varepsilon_{X,V}^* \colon \mathrm{QCoh}(\mathrm{Spec}(A_V) \times X) \longrightarrow \mathcal{O}_{V \times X^{\mathrm{an}}}\text{-Mod}$$

is conservative and t-exact and the morphism

$$\varepsilon_{X,(K)}^* \colon \underset{K \subset V \subset U}{\text{“colim”}} \mathrm{Perf}(\mathrm{Spec}(A_V) \times X) \longrightarrow \underset{K \subset V \subset U}{\text{“colim”}} \mathrm{Perf}\big(V \times X^{\mathrm{an}}\big)\,.$$

induced by the functors $\varepsilon_{X,V}^*$ is an equivalence in $\mathrm{Ind}\big(\mathrm{Cat}_\infty^{\mathrm{st},\otimes}\big)$. We say that X *satisfies the GAGA property relative to* U if it satisfies the pro-GAGA property relative to $(K)_U$ for every compact Stein subset $K \subset U$.

(4) We say that X *satisfies the universal GAGA property* if it satisfies the GAGA property relative to U for every $U \in \mathrm{dAfd}_k$.

Fix $X \in \mathrm{dSt}_k^{\mathrm{afp}}$. For every $S \in \mathrm{dAff}_k^{\mathrm{afp}}$ the analytification functor

$$\mathrm{Perf}(X \times S) \longrightarrow \mathrm{Perf}\big(X^{\mathrm{an}} \times S^{\mathrm{an}}\big)$$

induces a natural transformation of Cat_∞-valued stacks

$$\mathbf{Map}(X, \mathrm{Perf}_k) \longrightarrow \mathbf{AnMap}\big(X^{\mathrm{an}}, \mathrm{AnPerf}_k\big) \circ (-)^{\mathrm{an}},$$

which restricts to a natural transformation

$$\mathbf{Map}(X, \mathbf{Perf}_k) \longrightarrow \mathbf{AnMap}\big(X^{\mathrm{an}}, \mathbf{AnPerf}_k\big) \circ (-)^{\mathrm{an}}.$$

By adjunction, this morphism determines a map

$$\mu_X \colon \mathbf{Map}(X, \mathbf{Perf}_k)^{\mathrm{an}} \longrightarrow \mathbf{AnMap}\big(X^{\mathrm{an}}, \mathbf{AnPerf}_k\big).$$

The universal GAGA property enables us to check that μ_X is an equivalence.

Proposition 5.2. *Let $X \in \mathrm{dSt}_k^{\mathrm{afp}}$ be a derived stack locally almost of finite presentation. Suppose that*

(1) *the mapping stack $\mathbf{Map}(X, \mathbf{Perf}_k)$ is locally geometric and locally almost of finite presentation;*
(2) *the stack X satisfies the universal GAGA property.*

Then the canonical morphism

$$\mu_X \colon \mathbf{Map}(X, \mathbf{Perf}_k)^{\mathrm{an}} \longrightarrow \mathbf{AnMap}(X, \mathbf{AnPerf}_k)$$

is an equivalence.

Proof. We apply Proposition 3.14. Notice that for $U \in \mathrm{dAfd}_k$ and $V \in U_{\mathrm{\acute{e}t}}$, we have

$$G_U^p(\mathbf{Map}(X, \mathbf{Perf}_k))(V) \simeq \mathrm{Perf}(X \times \mathrm{Spec}(A_V))^{\simeq},$$
$$F_U^s(\mathbf{AnMap}(X, \mathrm{AnPerf}_k))(V) \simeq \mathrm{Perf}\big(X^{\mathrm{an}} \times V\big)^{\simeq}.$$

In the non-archimedean case the conclusion therefore follows directly from the assumption on X. In the $\mathbb{C}$-analytic case we have to check that for every compact Stein subset $K \subset U$, the natural map

$$\operatorname*{colim}_{K \subset V \subset U} \mathrm{Perf}(X \times \mathrm{Spec}(A_V))^{\simeq} \longrightarrow \operatorname*{colim}_{K \subset V \subset U} \mathrm{Perf}\big(X^{\mathrm{an}} \times V\big)^{\simeq}$$

is an equivalence. Since X satisfies the universal GAGA property, the natural map

$$\text{“}\operatorname*{colim}_{K \subset V \subset U}\text{”}\, \mathrm{Perf}(X \times \mathrm{Spec}(A_V)) \longrightarrow \text{“}\operatorname*{colim}_{K \subset V \subset U}\text{”}\, \mathrm{Perf}\big(X^{\mathrm{an}} \times V\big)$$

is an equivalence in $\mathrm{Ind}\big(\mathrm{Cat}_\infty^{\mathrm{st},\otimes}\big)$. The conclusion therefore follows by applying the maximal ∞-groupoid functor $(-)^{\simeq}$ and then realizing the equivalence in $\mathrm{Ind}(\mathcal{S})$. □

The following example is of course expected.

Example 5.3. A proper derived geometric stack locally almost of finite presentation over $\mathbb{C}$ satisfies the GAGA property. Indeed, it follows from [Por19, Theorem 7.2] that the analytification functor induces an equivalence $\mathrm{Perf}(X) \simeq \mathrm{Perf}\big(X^{\mathrm{an}}\big)$. The argument given in loc. cit. is an extension to the derived setting of the analogous statement for underived stacks, which has been proven in [PY16] in both the $\mathbb{C}$-analytic and the non-archimedean setting. The same extension argument works in the non-archimedean case, which shows that X satisfies the GAGA property in this situation too. Furthermore, the map $\mathrm{QCoh}(\mathrm{Spec}(A_U) \times X) \longrightarrow \mathcal{O}_{U \times X^{\mathrm{an}}}\text{-Mod}$ is conservative and t-exact: this easily follows by choosing a smooth hypercover of X by derived affine schemes and observing that t-exactness and conservativeness can be checked locally with respect to this hypercover. In the affine case, the result follows from the flatness of the natural map $\left(X^{\mathrm{an}}\right)^{\mathrm{alg}} \to X$; see [Por19, Corollary 5.15] in the $\mathbb{C}$-analytic case and [PY20b, Proposition 4.17] in the k-analytic case.

This example covers a great variety of situations. Indeed, the following are special cases of proper derived geometric stacks locally almost of finite presentation over k:

(1) proper schemes and algebraic spaces,

(2) proper Deligne–Mumford stacks,
(For instance, if X is a smooth and proper algebraic variety over k and $\overline{\mathcal{M}}_{g,n}(X)$ denotes the moduli stack of stable curves of genus g with n marked points, then $\overline{\mathcal{M}}_{g,n}(X)$ is a proper Deligne–Mumford stack. The same holds true for its natural derived enhancement.)

(3) higher classifying stacks $\mathrm{K}(G,n)$, where G is a compact (and abelian if $n \geqslant 2$) algebraic group scheme.
(For the case of $\mathrm{B}G$ with G reductive, see Section 5.2.5.)

We would like to prove that a proper derived geometric stack locally almost of finite presentation over k also satisfies the universal GAGA property. Notice that when $X = \mathrm{Spec}(k)$, saying that X satisfies the GAGA property relative to U is equivalent to Lemma 4.6 (in the k-analytic case) and to Theorem 4.8 (in the $\mathbb{C}$-analytic case). Extending these results to a more general X requires some effort. We start by dealing with the non-archimedean case, where the argument is technically easier. However, we first state explicitly a lemma implicitly used in [Por19].

Lemma 5.4. *Let X be a derived geometric stack locally almost of finite presentation over k. Let $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}^{\mathrm{b}}(X)$. Then the canonical map*

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_X}\left(\mathcal{F}, \mathcal{G}\right)^{\mathrm{an}} \longrightarrow \mathcal{H}\mathrm{om}_{\mathcal{O}_{X^{\mathrm{an}}}}\left(\mathcal{F}^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}}\right)$$

is an equivalence.

Proof. This question is local on X, and we can therefore suppose that X is affine. Notice that the map

$$\gamma_{\mathcal{F},\mathcal{G}} \colon \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})^{\mathrm{an}} \longrightarrow \mathcal{H}\mathrm{om}_{\mathcal{O}_{X^{\mathrm{an}}}}\left(\mathcal{F}^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}}\right)$$

is defined for all $\mathcal{F}, \mathcal{G} \in \mathrm{QCoh}(X)$. It is tautologically an equivalence when $\mathcal{F} = \mathcal{O}_X$. From here, it follows that it is an equivalence whenever $\mathcal{F}$ is perfect. When $\mathcal{F} \in \mathrm{Coh}^{\mathrm{b}}(X)$ is arbitrary, we use [Lur17, Proposition 7.2.4.11(5)] to write some shift of $\mathcal{F}$ as a geometric realization $\mathcal{F} \simeq |\mathcal{P}_\bullet|$ of a simplicial diagram $\mathcal{P}_\bullet$ such that each $\mathcal{P}_n$ is perfect. By shifting $\mathcal{G}$ if necessary, we may assume that $\mathcal{F}$ itself is $|\mathcal{P}_\bullet|$. We then have

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \simeq \lim_{[n] \in \mathbf{\Delta}} \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{P}_n, \mathcal{G})\,.$$

On the other hand, since $(-)^{\mathrm{an}}$ commutes with arbitrary colimits, we have a canonical equivalence $\mathcal{F}^{\mathrm{an}} \simeq |\mathcal{P}_\bullet^{\mathrm{an}}|$. As a consequence, we obtain the equivalence

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_{X^{\mathrm{an}}}}\left(\mathcal{F}^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}}\right) \simeq \lim_{[n] \in \mathbf{\Delta}} \mathcal{H}\mathrm{om}_{\mathcal{O}_{X^{\mathrm{an}}}}\left(\mathcal{P}_n^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}}\right).$$

It is therefore enough to prove that the natural map

$$\Big(\lim_{[n] \in \mathbf{\Delta}} \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{P}_n, \mathcal{G})\Big)^{\mathrm{an}} \longrightarrow \lim_{[n] \in \mathbf{\Delta}} \mathcal{H}\mathrm{om}_{\mathcal{O}_X}\left(\mathcal{P}_n, \mathcal{G}\right)^{\mathrm{an}}$$

is an equivalence. In order to check this, it is enough to check that for every integer $m \in \mathbb{Z}$ the above map induces an isomorphism

$$\pi_m\Big(\lim_{[n]\in\mathbf{\Delta}} \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{P}_n, \mathcal{G})\Big)^{\mathrm{an}} \longrightarrow \pi_m \lim_{[n]\in\mathbf{\Delta}} \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{P}_n, \mathcal{G})^{\mathrm{an}} .$$

Since the analytification is t-exact, we are therefore reduced to checking that the map

$$\Big(\pi_m\Big(\lim_{[n]\in\mathbf{\Delta}} \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{P}_n, \mathcal{G})\Big)\Big)^{\mathrm{an}} \longrightarrow \pi_m\Big(\lim_{[n]\in\mathbf{\Delta}} \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{P}_n, \mathcal{G})^{\mathrm{an}}\Big)$$

is an isomorphism. Now observe that there exists an $m' \gg 0$ such that

$$\pi_m\Big(\lim_{[n]\in\mathbf{\Delta}} \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{P}_n, \mathcal{G})\Big) \simeq \pi_m\Big(\lim_{[n]\in\mathbf{\Delta}_{\leqslant m'}} \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{P}_n, \mathcal{G})\Big)$$

and similarly

$$\pi_m\Big(\lim_{[n]\in\mathbf{\Delta}} \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{P}_n, \mathcal{G})^{\mathrm{an}}\Big) \simeq \pi_m\Big(\lim_{[n]\in\mathbf{\Delta}_{\leqslant m'}} \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{P}_n, \mathcal{G})^{\mathrm{an}}\Big) .$$

Since $\mathbf{\Delta}_{\leqslant m}$ is a finite category and $(-)^{\mathrm{an}}$ is an exact functor between stable ∞-categories, we deduce that the canonical map

$$\Big(\lim_{[n]\in\mathbf{\Delta}_{\leqslant m'}} \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{P}_n, \mathcal{G})\Big)^{\mathrm{an}} \longrightarrow \lim_{[n]\in\mathbf{\Delta}_{\leqslant m'}} \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{P}_n, \mathcal{G})^{\mathrm{an}}$$

is an equivalence. The conclusion follows. □

Theorem 5.5. *Let k be either the field of complex numbers or a non-archimedean field equipped with a non-trivial valuation. Let X be a proper derived geometric stack locally almost of finite presentation over k. Then X satisfies the universal GAGA property.*

In particular, if X is as above, Proposition 5.2 gives us the following result.

Corollary 5.6. *Let k be either the field of complex numbers or a non-archimedean field equipped with a non-trivial valuation. Let X be a derived geometric stack locally almost of finite presentation over k. Assume that*

(1) *the stack X is proper;*
(2) *the stack $\mathbf{Map}(X, \mathbf{Perf}_k)$ is locally geometric and locally almost of finite presentation.*

Then the canonical map

$$\mu_X \colon \mathbf{Map}(X, \mathbf{Perf}_k)^{\mathrm{an}} \longrightarrow \mathbf{AnMap}\big(X^{\mathrm{an}}, \mathbf{AnPerf}_k\big)$$

is an equivalence.

Remark 5.7. In Corollary 5.6, the need for the geometricity and locally almost of finite presentation assumption as well on $\mathbf{Map}(X, \mathbf{Perf}_k)$ ultimately comes from Theorem 3.13. The main theorem of [TV07] shows that this assumption is satisfied when X is a smooth and proper scheme over k. More generally, this problem can be seen as a particular instance of the geometricity of $\mathbf{Map}(X, Y)$ for $X, Y \in \mathrm{dSt}_k$. In [Lur12, Proposition 3.3.8] it is shown that $\mathbf{Map}(X, Y)$ is geometric and locally almost of finite presentation whenever X is a proper and flat derived algebraic space and Y is a derived Deligne–Mumford stack locally almost of finite presentation.

These results can be improved: we expect $\mathbf{Map}(X, \mathbf{Perf}_k)$ to be locally geometric whenever X is proper and of finite tor-amplitude. The main tool to prove this theorem is the version of

Artin–Lurie's representability theorem for derived Artin stacks that will appear in [Lur18, Chapter 27]. As usual, the critical assumptions to be verified are the integrability of $\mathbf{Map}(X, \mathbf{Perf}_k)$ and the existence of its cotangent complex. The latter can easily be established by combining properness and finite tor-amplitude following the method of [Lur11d, Proposition 3.3.23]. Notice that we do not need the functor f_+ to be defined on the whole $\mathrm{QCoh}(X \times \mathbf{Map}(X, \mathbf{Perf}_k))$ but that it is enough to have it defined on $\mathrm{Perf}(X \times \mathbf{Map}(X, \mathbf{Perf}_k))$; see [PY21, Lemma 8.4] for a similar situation where f_+ can only be defined for perfect complexes. On the other hand, the integrability of $\mathbf{Map}(X, \mathbf{Perf}_k)$ can be reduced to the statement of the formal GAGA equivalence for the stack $X \times \mathrm{Spec}(R) \to \mathrm{Spec}(R)$, for R a local complete noetherian ring. This result can be obtained by extension from the analogous statements for proper schemes over $\mathrm{Spec}(R)$ in the same way as in [PY16].

Finally, let us remark that, in the absence of the strong version of Artin–Lurie's representability theorem, one can combine the main theorem of [Ols06] with the weak version of Lurie's representability theorem [Lur18, Theorem 18.1.0.2] to deduce the representability of $\mathbf{Map}(X, \mathbf{Perf}_k)$ from the representability of its truncation. In order for this method to go through, one needs to assume X to be proper and flat over k.

We now turn to the proof of the theorem.

Proof of Theorem 5.5. Let us first deal with the non-archimedean setting. In this case we have to check that the analytification functor

$$(-)^{\mathrm{an}} \colon \mathrm{Perf}(\mathrm{Spec}(A_V) \times X) \longrightarrow \mathrm{Perf}\big(V \times X^{\mathrm{an}}\big) \tag{5.8}$$

is an equivalence. We will in fact prove that the analytification functor

$$(-)^{\mathrm{an}} \colon \mathrm{Coh}(X \times \mathrm{Spec}(A_V)) \longrightarrow \mathrm{Coh}\big(X^{\mathrm{an}} \times V\big) \tag{5.9}$$

is an equivalence. Notice that the flatness of derived analytification proved in [PY20b, Proposition 4.17] together with [Lur18, Proposition 2.8.4.2(5)] implies that an $\mathcal{F} \in \mathrm{Coh}(X \times \mathrm{Spec}(A_V))$ is a perfect complex if and only if $\mathcal{F}^{\mathrm{an}}$ belongs to $\mathrm{Perf}\big(X^{\mathrm{an}} \times V\big)$. From this remark and the claimed equivalence, it follows immediately that (5.8) is an equivalence as well.

We start by proving that the functor (5.9) is fully faithful. Recall that pushing forward along the natural closed immersions

$$\mathrm{t}_0(X \times \mathrm{Spec}(A_V)) \hookrightarrow X \times \mathrm{Spec}(A_V)\,, \quad \mathrm{t}_0\big(X^{\mathrm{an}} \times V\big) \hookrightarrow X^{\mathrm{an}} \times V$$

induces equivalences of abelian categories

$$\mathrm{Coh}^{\heartsuit}(X \times \mathrm{Spec}(A_V)) \simeq \mathrm{Coh}^{\heartsuit}(\mathrm{t}_0(X \times \mathrm{Spec}(A_V)))\,, \quad \mathrm{Coh}^{\heartsuit}(X^{\mathrm{an}} \times V) \simeq \mathrm{Coh}^{\heartsuit}(\mathrm{t}_0(X^{\mathrm{an}} \times V))\,.$$

Notice that $\mathrm{t}_0(X \times \mathrm{Spec}(A_V)) \simeq \mathrm{t}_0(X) \times \mathrm{Spec}(\pi_0(A_V))$. Furthermore, $\pi_0(A_V) \simeq \Gamma\big(\mathrm{t}_0(V); \mathcal{O}^{\mathrm{alg}}_{\mathrm{t}_0(V)}\big)$. Applying [PY16, Theorem 7.1], we see that the diagram

$$\begin{array}{ccc} \mathrm{Coh}^{\heartsuit}(\mathrm{t}_0(X) \times \mathrm{Spec}(\pi_0(A_V))) & \xrightarrow{(-)^{\mathrm{an}}} & \mathrm{Coh}^{\heartsuit}\big(\mathrm{t}_0\big(X^{\mathrm{an}}\big) \times \mathrm{t}_0(V)\big) \\ \big\downarrow p_* & & \big\downarrow p_*^{\mathrm{an}} \\ \mathrm{Coh}(\pi_0(A_V)) & \longrightarrow & \mathrm{Coh}(\mathrm{t}_0(V)) \end{array}$$

commutes. Since the diagram

$$\begin{array}{ccc} \mathrm{Coh}(\pi_0(A_V)) & \longrightarrow & \mathrm{Coh}(\mathrm{t}_0(V)) \\ \downarrow & & \downarrow \\ \mathrm{Coh}(A_V) & \longrightarrow & \mathrm{Coh}(V) \end{array}$$

commutes as well, we may form a cube with five commuting faces and deduce that its sixth face, the diagram

$$\begin{array}{ccc} \mathrm{Coh}^{\heartsuit}(X \times \mathrm{Spec}(A_V)) & \longrightarrow & \mathrm{Coh}^{\heartsuit}\big(X^{\mathrm{an}} \times V\big) \\ \big\downarrow q_* & & \big\downarrow q_*^{\mathrm{an}} \\ \mathrm{Coh}(A_V) & \longrightarrow & \mathrm{Coh}(V)\,, \end{array}$$

also commutes. From here, proceeding by induction on the cohomological amplitude as in the proof of [Por19, Theorem 7.1], we deduce that the diagram (written in *homological* convention)

$$\begin{array}{ccc} \mathrm{Coh}^{-}(X \times \mathrm{Spec}(A_V)) & \longrightarrow & \mathrm{Coh}^{-}(X^{\mathrm{an}} \times V) \\ \big\downarrow q_* & & \big\downarrow q_*^{\mathrm{an}} \\ \mathrm{Coh}(A_V) & \longrightarrow & \mathrm{Coh}(V) \end{array}$$

commutes as well. Now let $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}^{\mathrm{b}}(X \times \mathrm{Spec}(A_V))$. Applying Lemma 5.4, we see that

$$\mathcal{H}\mathrm{om}_{X \times \mathrm{Spec}(A_V)}(\mathcal{F}, \mathcal{G})^{\mathrm{an}} \simeq \mathcal{H}\mathrm{om}_{X^{\mathrm{an}} \times V}\big(\mathcal{F}^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}}\big)\,. \tag{5.10}$$

Notice furthermore that the functor $\mathrm{Coh}(A_V) \to \mathrm{Coh}(V)$ coincides with the equivalence provided by [PY21, Theorem 3.4]. Combining the equivalence $\mathrm{Coh}(A_V) \simeq \mathrm{Coh}(V)$ with (5.10) and with the commutativity of the above diagram, we deduce that the analytification functor restricts to a fully faithful functor

$$\mathrm{Coh}^{\mathrm{b}}(X \times \mathrm{Spec}(A_V)) \longrightarrow \mathrm{Coh}\big(X^{\mathrm{an}} \times V\big)\,.$$

From here, a second induction on the cohomological amplitude such as the one that can be found in [Por19, Theorem 7.2] proves that the functor (5.9) is also fully faithful.

For essential surjectivity we first use [PY16, Theorem 7.4] to deduce that the analytification induces an equivalence

$$\mathrm{Coh}^{\heartsuit}(X \times \mathrm{Spec}(A_V)) \simeq \mathrm{Coh}^{\heartsuit}\big(X^{\mathrm{an}} \times V\big)\,.$$

Next we bootstrap on this using the full faithfulness of (5.9) to deduce that the analytification functor on unbounded coherent sheaves is also essentially surjective. The conclusion follows.

We now turn to the $\mathbb{C}$-analytic situation. In this case we have to prove that for every compact Stein subset $K \subset U$, the morphism

$$\underset{K \subset V \subset U}{\text{“colim”}}\, \mathrm{Perf}(X \times \mathrm{Spec}(A_V)) \longrightarrow \underset{K \subset V \subset U}{\text{“colim”}}\, \mathrm{Perf}\big(X^{\mathrm{an}} \times V\big)$$

is an equivalence in $\mathrm{Ind}\big(\mathrm{Cat}_\infty^{\mathrm{st},\otimes}\big)$. Just as in the non-archimedean setting, we prove below that actually the map

$$\underset{K \subset V \subset U}{\text{“colim”}}\, \mathrm{Coh}(X \times \mathrm{Spec}(A_V)) \longrightarrow \underset{K \subset V \subset U}{\text{“colim”}}\, \mathrm{Coh}\big(X^{\mathrm{an}} \times V\big) \tag{5.11}$$

is an equivalence, where the two colimits range over the open Stein neighbourhoods of K inside U. Notice that when $X = \mathrm{Spec}(k)$, this is exactly the result proven in Theorem 4.13.

In order to prove that the functor (5.11) is an equivalence, we will need the following two claims, which will be proved below:

(1) For every open Stein neighbourhood V of K in U, the analytification map
$$\mathrm{Coh}(X \times \mathrm{Spec}(A_V)) \longrightarrow \mathrm{Coh}\big(X^{\mathrm{an}} \times V\big)$$
is fully faithful (see Proposition 5.13 below).

(2) Let $W \Subset V \Subset U$ be two relatively compact Stein neighbourhoods of K inside U. Then the map
$$\mathrm{Coh}\big(X^{\mathrm{an}} \times V\big) \longrightarrow \mathrm{Coh}\big(X^{\mathrm{an}} \times W\big)$$
factors through $\mathrm{Coh}(X \times \mathrm{Spec}(A_W)) \to \mathrm{Coh}\big(X^{\mathrm{an}} \times W\big)$ (see Proposition 5.15 below).

We can therefore promote the functors of assertion (2) to a morphism
$$\underset{K \subset V \subset U}{\text{“colim”}} \mathrm{Coh}\big(X^{\mathrm{an}} \times V\big) \longrightarrow \underset{K \subset V \subset U}{\text{“colim”}} \mathrm{Coh}(X \times \mathrm{Spec}(A_V))\,.$$
It is easily checked that this forms an equivalence together with the functor (5.11). □

To prove the claims, we need the following preliminary result.

Lemma 5.12. *Let X be a proper derived geometric $\mathbb{C}$-stack. Let $U \in \mathrm{dStn}_{\mathbb{C}}$ be a derived Stein space. Write $A_U \coloneqq \Gamma(U; \mathcal{O}_U)$, and let $p_U \colon X \times \mathrm{Spec}(A_U) \to \mathrm{Spec}(A_U)$ and $p_U^{\mathrm{an}} \colon X^{\mathrm{an}} \times U \to U$ be the two canonical projections. Then the diagram (written in* homological *convention)*
$$\begin{array}{ccc} \mathrm{Coh}^-(X \times \mathrm{Spec}(A_U)) & \xrightarrow{\varepsilon^*_{X,U}} & \mathrm{Coh}^-\big(X^{\mathrm{an}} \times U\big) \\ \downarrow{\scriptstyle p_{U*}} & & \downarrow{\scriptstyle p^{\mathrm{an}}_{U*}} \\ \mathrm{Coh}^-(\mathrm{Spec}(A_U)) & \xrightarrow{\varepsilon^*_U} & \mathrm{Coh}^-(U) \end{array}$$
canonically commutes. Here ε^*_U denotes the functor introduced in Section 4.2.

Proof. Proceeding by induction on the cohomological amplitude as in the proof of [Por19, Theorem 7.1], we see that it is enough to prove that the diagram
$$\begin{array}{ccc} \mathrm{Coh}^{\heartsuit}(X \times \mathrm{Spec}(A_U)) & \xrightarrow{\varepsilon^*_{X,U}} & \mathrm{Coh}^{\heartsuit}\big(X^{\mathrm{an}} \times U\big) \\ \downarrow{\scriptstyle p_{U*}} & & \downarrow{\scriptstyle p^{\mathrm{an}}_{U*}} \\ \mathrm{Coh}^-(\mathrm{Spec}(A_U)) & \xrightarrow{\varepsilon^*_U} & \mathrm{Coh}^-(U) \end{array}$$
commutes.

We first deal with the case where X is a proper derived $\mathbb{C}$-*scheme*. Fix an object $\mathcal{F} \in \mathrm{Coh}^{\heartsuit}(X \times \mathrm{Spec}(A_U))$. In this case, the Čech complex computing both $p_{U*}(\mathcal{F})$ and $p^{\mathrm{an}}_{U*}(\mathcal{F}^{\mathrm{an}})$ is cohomologically bounded. As Proposition 4.5 shows that ε^*_U is t-exact, we deduce that $\varepsilon^*_U(p_{U*}(\mathcal{F}))$ is also cohomologically bounded. We are therefore left to check that the canonical map
$$\gamma_{\mathcal{F}} \colon \varepsilon^*_U(p_{U*}(\mathcal{F})) \longrightarrow p^{\mathrm{an}}_{U*}\big(\mathcal{F}^{\mathrm{an}}\big)$$
between objects in $\mathrm{Coh}^{\mathrm{b}}(U)$ is an equivalence. Let $\mathcal{G} \coloneqq \mathrm{fib}(\gamma_{\mathcal{F}})$. Equivalently, we have to prove that $\mathcal{G} \simeq 0$. Since $\mathcal{G}$ is cohomologically bounded, the cohomological Nakayama's lemma implies that it is enough to check that for every closed point $x \colon \mathrm{Sp}(\mathbb{C}) \to U$, one has $x^*\mathcal{G} \simeq 0$. On the

other hand, the derived base change and its analytic counterpart[8] imply that the two diagrams

$$\begin{array}{ccc} \mathrm{Coh}^+(X \times \mathrm{Spec}(A_U)) & \xrightarrow{(\mathrm{id}_X \times x)^*} & \mathrm{Coh}^+(X) \\ \big\downarrow p_{U*} & & \big\downarrow p_* \\ \mathrm{Coh}^+(A_U) & \xrightarrow{\quad x^* \quad} & \mathrm{Coh}^+(\mathbb{C}) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Coh}^+\big(X^{\mathrm{an}} \times U\big) & \xrightarrow{(\mathrm{id}_{X^{\mathrm{an}}} \times x)^*} & \mathrm{Coh}^+\big(X^{\mathrm{an}}\big) \\ \big\downarrow p^{\mathrm{an}}_{U*} & & \big\downarrow p^{\mathrm{an}}_* \\ \mathrm{Coh}^+(U) & \xrightarrow{\quad x^* \quad} & \mathrm{Coh}^+(\mathbb{C}) \end{array}$$

are commutative. In this way, we can reduce to the case where $U = \mathrm{Sp}(\mathbb{C})$, and in this case the statement follows from the equivalences

$$\mathrm{Coh}^\heartsuit(X \times \mathrm{Spec}(A_U)) \simeq \mathrm{Coh}^\heartsuit(\mathrm{t}_0(X) \times \mathrm{Spec}(\pi_0(A_U)))\,,$$
$$\mathrm{Coh}^\heartsuit\big(X^{\mathrm{an}} \times U\big) \simeq \mathrm{Coh}^\heartsuit(\mathrm{t}_0(X)^{\mathrm{an}} \times \mathrm{t}_0(U))$$

and [PY16, Theorem 7.1] (in fact, the classical GAGA theorem that can be found in [Gro63, Exposé XII, Théorème 4.4] is enough for this step).

At this point, we proceed by induction on the geometric level of X. We notice that the same proof as in [PY16, Theorem 7.1] applies. The reader should be wary that in this case too, the noetherian induction has to be performed on X (and not on $X \times \mathrm{Spec}(A_U)$). The reader should also be aware that in loc. cit. the cohomological convention was used, while in this paper we are following the homological one. □

Proposition 5.13. *Let X be a proper derived geometric $\mathbb{C}$-stack. Let $U \in \mathrm{dStn}_{\mathbb{C}}$ be a derived Stein space. Then the analytification functor*

$$\mathrm{Coh}(X \times \mathrm{Spec}(A_U)) \longrightarrow \mathrm{Coh}\big(X^{\mathrm{an}} \times U\big)$$

is fully faithful.

Proof. Fix $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(X \times \mathrm{Spec}(A_U))$. We have to prove that the natural morphism

$$\psi_{\mathcal{F},\mathcal{G}} \colon \mathrm{Map}^{\mathrm{st}}_{X \times \mathrm{Spec}(A_U)}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Map}^{\mathrm{st}}_{X^{\mathrm{an}} \times U}\big(\mathcal{F}^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}}\big)$$

is an equivalence. Let $p \colon X \times \mathrm{Spec}(A_U) \to \mathrm{Spec}(\mathbb{C})$ and $p^{\mathrm{an}} \colon X^{\mathrm{an}} \times U \to \mathrm{Sp}(\mathbb{C})$ be the two canonical maps to the point. It follows from the definitions that we have natural equivalences

$$\mathrm{Map}^{\mathrm{st}}_{X \times \mathrm{Spec}(A_U)}(\mathcal{F}, \mathcal{G}) \simeq \tau_{\geqslant 0} p_* \mathcal{H}\mathrm{om}_{X \times \mathrm{Spec}(A_U)}(\mathcal{F}, \mathcal{G})\,,$$
$$\mathrm{Map}^{\mathrm{st}}_{X^{\mathrm{an}} \times U}\big(\mathcal{F}^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}}\big) \simeq \tau_{\geqslant 0} p^{\mathrm{an}}_* \mathcal{H}\mathrm{om}_{X^{\mathrm{an}} \times U}\big(\mathcal{F}^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}}\big)\,.$$

If $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}^{\mathrm{b}}(X \times \mathrm{Spec}(A_U))$, then the same argument as given in Lemma 5.4 shows that the canonical map

$$\zeta_{\mathcal{F},\mathcal{G}} \colon \mathcal{H}\mathrm{om}_{X \times \mathrm{Spec}(A_U)}(\mathcal{F}, \mathcal{G})^{\mathrm{an}} \longrightarrow \mathcal{H}\mathrm{om}_{X^{\mathrm{an}} \times U}\big(\mathcal{F}^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}}\big)$$

is an equivalence. Moreover, in this case $\mathcal{H}\mathrm{om}_{X \times \mathrm{Spec}(A_U)}(\mathcal{F}, \mathcal{G})$ belongs to $\mathrm{Coh}^-(X \times \mathrm{Spec}(A_U))$, and therefore we can use Lemma 5.12 to deduce that the canonical map

$$\varepsilon_U^*\big(p_{U*}\big(\mathcal{H}\mathrm{om}_{X \times \mathrm{Spec}(A_U)}(\mathcal{F}, \mathcal{G})\big)\big) \longrightarrow p^{\mathrm{an}}_{U*}\big(\mathcal{H}\mathrm{om}_{X \times \mathrm{Spec}(A_U)}(\mathcal{F}, \mathcal{G})^{\mathrm{an}}\big)$$

is an equivalence. Composing with the equivalence $\zeta_{\mathcal{F},\mathcal{G}}$, applying the global section functor $\Gamma(U; -)$ and using Proposition 4.5(4), we deduce that the canonical map

$$p_* \mathcal{H}\mathrm{om}_{X \times \mathrm{Spec}(A_U)}(\mathcal{F}, \mathcal{G}) \longrightarrow p^{\mathrm{an}}_* \mathcal{H}\mathrm{om}_{X^{\mathrm{an}} \times U}\big(\mathcal{F}^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}}\big)$$

[8] Since $x \colon \mathrm{Sp}(\mathbb{C}) \hookrightarrow U$ is a closed immersion, the analytic base change follows from the unramifiedness of $\mathcal{T}_{\mathrm{an}}(\mathbb{C})$. This can be proved as in [PY21, Lemma 6.4]; the key ingredients in the derived setting are [Lur11c, Propositions 11.12(3) and 12.10].

is an equivalence. Therefore, the conclusion follows in the case where $\mathcal{F}$ and $\mathcal{G}$ are (locally) cohomologically bounded. At this point, the argument given in [Por19, proof of Theorem 7.2] shows that the map $\psi_{\mathcal{F},\mathcal{G}}$ is an equivalence whenever $\mathcal{F}, \mathcal{G}$ belong to $\mathrm{Coh}(X)$. □

For later use we record the following useful consequence.

Corollary 5.14. *Let X be a proper derived geometric $\mathbb{C}$-stack, and let $j \colon X_{\mathrm{red}} \to X$ be the canonical map. Let $U \in \mathrm{dSt}_{\mathbb{C}}$ be a derived affinoid. Then for $\mathcal{F} \in \mathrm{Coh}\big(U \times X^{\mathrm{an}}\big)$ the following conditions are equivalent:*

(1) *The coherent sheaf $\mathcal{F}$ is algebraizable; that is, it belongs to the essential image of the functor $\mathrm{Coh}(\mathrm{Spec}(A_U) \times X) \to \mathrm{Coh}\big(U \times X^{\mathrm{an}}\big)$.*

(2) *The discrete sheaf $\mathrm{H}^i(\mathcal{F})$ is algebraizable for every $i \in \mathbb{Z}$.*

If furthermore $\mathcal{F} \in \mathrm{Coh}^+\big(U \times X^{\mathrm{an}}\big)$, then the above conditions are equivalent to the following:

(3) *The pullback $(\mathrm{id}_U \times j^{\mathrm{an}})^* \mathcal{F} \in \mathrm{Coh}\big(U \times X^{\mathrm{an}}_{\mathrm{red}}\big)$ is algebraizable.*

Proof. Since the analytification functor $(-)^{\mathrm{an}} \colon \mathrm{Coh}(\mathrm{Spec}(A_U) \times X) \to \mathrm{Coh}\big(U \times X^{\mathrm{an}}\big)$ is t-exact, it commutes with both the limit and the colimit of the Postnikov tower. This shows immediately that $\mathcal{F}$ is algebraizable if and only if for every $n, m \in \mathbb{Z}$ the sheaf $\tau_{\leqslant n} \tau_{\geqslant m}(\mathcal{F})$ is algebraizable. Moreover, Proposition 5.13 shows that this functor is also fully faithful. A simple induction on the number of non-vanishing cohomology groups therefore implies the equivalence between conditions (1) and (2).

Now assume that $\mathcal{F} \in \mathrm{Coh}^+(U \times X^{\mathrm{an}})$. Then the implication (1) ⇒ (3) is clear. Let us prove that condition (3) implies condition (2). Since $\mathcal{F}$ is eventually connective, we can choose the minimum integer i such that $\mathrm{H}^i(\mathcal{F})$ is non-zero. Using the fibre sequence

$$\tau_{\leqslant i+1} \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathrm{H}^i(\mathcal{F}) \, ,$$

we see that it is enough to prove that $\mathrm{H}^i(\mathcal{F})$ is algebraizable. We can furthermore replace both U and X with their truncations and therefore assume that they are underived. Let $\mathcal{J}$ be the nilradical ideal sheaf of X, let $\mathcal{J}^{\mathrm{an}}$ be its analytification, and let $\mathcal{J}^{\mathrm{an}}_U$ be pullback of $\mathcal{J}^{\mathrm{an}}$ along $U \times X^{\mathrm{an}} \to X^{\mathrm{an}}$. Since X is proper, there exists an integer n such that $\mathcal{J}^n = 0$. We now observe that

$$\mathrm{H}^i(\mathcal{F}) / \mathcal{J}^{\mathrm{an}}_U \mathrm{H}^i(\mathcal{F}) \simeq \mathrm{L}^0 \big(\mathrm{id}_U \times j^{\mathrm{an}}\big)^* \mathrm{H}^i(\mathcal{F}) \simeq \mathrm{H}^i\big(\big(\mathrm{id}_U \times j^{\mathrm{an}}\big)^* \mathcal{F}\big) \, .$$

This implies that $\mathrm{H}^i(\mathcal{F}) / \mathcal{J}^{\mathrm{an}}_U$ is algebraizable. Proceeding by induction on m as in [PY16, Theorem 5.13], we see that $\mathrm{H}^i(\mathcal{F}) / \big(\mathcal{J}^{\mathrm{an}}_U\big)^m \mathrm{H}^i(\mathcal{F})$ is algebraizable for every $m \geqslant 1$. Taking $m = n$ we conclude that $\mathrm{H}^i(\mathcal{F})$ is algebraizable, thus completing the proof. □

At this point the only missing piece needed for the proof of Theorem 5.5 is the following.

Proposition 5.15. *Let X be a proper derived geometric $\mathbb{C}$-stack. Let $U \in \mathrm{dStn}_{\mathbb{C}}$ be a derived Stein space, and let $W \Subset V \Subset U$ be a nested sequence of relatively compact open Stein subsets of U. Then there exists a functor $\mathrm{Coh}(X^{\mathrm{an}} \times U) \to \mathrm{Coh}(X \times \mathrm{Spec}(A_W))$ making the diagram*

$$\begin{array}{ccc} & & \mathrm{Coh}(X \times \mathrm{Spec}(A_W)) \\ & \nearrow & \big\downarrow (-)^{\mathrm{an}} \\ \mathrm{Coh}\big(X^{\mathrm{an}} \times U\big) & \longrightarrow & \mathrm{Coh}\big(X^{\mathrm{an}} \times W\big) \end{array}$$

commutative.

Proof. Using Corollary 5.14, we see that it is enough to prove the same statement at the level of hearts. Using Proposition 5.13, we see that the relative analytification functor

$$(-)^{\mathrm{an}} \colon \mathrm{Coh}^{\heartsuit}(X \times \mathrm{Spec}(A_W)) \longrightarrow \mathrm{Coh}^{\heartsuit}\big(X^{\mathrm{an}} \times W\big) \tag{5.16}$$

is fully faithful. Therefore, it is enough to prove that the restriction functor

$$\mathrm{Coh}^{\heartsuit}\big(X^{\mathrm{an}} \times U\big) \longrightarrow \mathrm{Coh}^{\heartsuit}\big(X^{\mathrm{an}} \times W\big)$$

factors through the essential image of $(-)^{\mathrm{an}}$.

We first deal with the case where X is a scheme. Notice that, using the equivalences

$$\mathrm{Coh}^{\heartsuit}(X \times \mathrm{Spec}(A_U)) \simeq \mathrm{Coh}^{\heartsuit}(\mathrm{t}_0(X) \times \mathrm{Spec}(\pi_0(A_U)))\,,$$
$$\mathrm{Coh}^{\heartsuit}\big(X^{\mathrm{an}} \times U\big) \simeq \mathrm{Coh}^{\heartsuit}\big(\mathrm{t}_0(X)^{\mathrm{an}} \times \mathrm{t}_0(U)\big)\,,$$

we can assume that both X and U (and hence V and W) are underived. Under this hypothesis, we proceed by noetherian induction on the dimension of X. Using Chow's lemma as in [Gro63, Exposé XII, Théorème 4.4], we are readily reduced to the case of projective space, $X = \mathbb{P}^n_{\mathbb{C}}$. Write $\mathbf{P}^n_{\mathbb{C}} \coloneqq \big(\mathbb{P}^n_{\mathbb{C}}\big)^{\mathrm{an}}$. Let

$$p_U \colon \mathbb{P}^n_{\mathbb{C}} \times \mathrm{Spec}(A_U) \longrightarrow \mathrm{Spec}(A_U)\,, \quad p_U^{\mathrm{an}} \colon \mathbf{P}^n_{\mathbb{C}} \times U \longrightarrow U\,,$$
$$q_U \colon \mathbb{P}^n_{\mathbb{C}} \times \mathrm{Spec}(A_U) \longrightarrow \mathbb{P}^n_{\mathbb{C}}\,, \qquad q_U^{\mathrm{an}} \colon \mathbf{P}^n_{\mathbb{C}} \times U \longrightarrow \mathbf{P}^n_{\mathbb{C}}$$

be the natural projections. For $m \in \mathbb{Z}$ we write

$$\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}} \times \mathrm{Spec}(A_U)}(m) \coloneqq q_U^* \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(m)\,, \quad \mathcal{O}_{\mathbf{P}^n_{\mathbb{C}} \times U}(m) \coloneqq q_U^{\mathrm{an}*} \mathcal{O}_{\mathbf{P}^n_{\mathbb{C}}}(m)\,.$$

Given $\mathcal{F} \in \mathrm{Coh}^{\heartsuit}\big(\mathbf{P}^n_{\mathbb{C}} \times U\big)$, we consider the canonical map

$$\phi_m \colon \mathcal{G} \coloneqq \mathrm{L}^0 p_U^{\mathrm{an}*} \mathrm{R}^0 p_{U*}^{\mathrm{an}}(\mathcal{F}(-m)) \otimes \mathcal{O}_{\mathbf{P}^n_{\mathbb{C}} \times U}(m) \longrightarrow \mathcal{F}\,.$$

For every point $x \in \overline{V}$ there is an integer $m_x \in \mathbb{Z}$ such that for $m \geqslant m_x$ the pullback of this morphism along $\mathrm{id}_{\mathbf{P}^n_{\mathbb{C}}} \times x \colon \mathbf{P}^n_{\mathbb{C}} \to \mathbf{P}^n_{\mathbb{C}} \times U$ becomes surjective. As both $\mathcal{G}$ and $\mathcal{F}$ are coherent, Nakayama's lemma implies that there exists a neighbourhood U_x of x such that for $m \geqslant m_u$ the map ϕ_m becomes surjective when restricted to U_x. As $\overline{V}$ is compact, we can therefore find an open Stein subset U' of U containing $\overline{V}$ and an integer m such that the restriction of ϕ_m to U' is surjective. In particular, the restriction of ϕ_m to V is surjective. Repeating the same reasoning for the kernel of ϕ on U', we find a second open Stein subset U'' of U containing $\overline{V}$ such that $\mathcal{F}|_{U''}$ admits a presentation of the form

$$\mathcal{H} \xrightarrow{f} \mathcal{G} \longrightarrow \mathcal{F}|_{U''} \longrightarrow 0\,,$$

where $\mathcal{H}$ and $\mathcal{G}$ can be written as

$$\mathcal{H} \simeq \mathrm{L}^0 p_{U''}^{\mathrm{an}*}(\mathcal{H}_0) \otimes \mathcal{O}_{\mathbf{P}^n_{\mathbb{C}} \times U''}(m_2)\,, \quad \mathcal{G} \simeq \mathrm{L}^0 p_{U''}^{\mathrm{an}*}(\mathcal{G}_0) \otimes \mathcal{O}_{\mathbf{P}^n_{\mathbb{C}} \times U''}(m_1)$$

for $\mathcal{H}_0, \mathcal{G}_0 \in \mathrm{Coh}^{\heartsuit}(U'')$ and $m_1, m_2 \gg 0$. In particular, the same remains true after restricting to V. Using Theorem 4.13, we see that $\mathcal{H}_0|_W, \mathcal{G}_0|_W$ come from objects $\mathcal{H}_0^{\mathrm{alg}}$ and $\mathcal{G}_0^{\mathrm{alg}}$ in $\mathrm{Coh}^{\heartsuit}(A_W)$ via the functor ε_W^*. Since we already argued that the functor (5.16) is fully faithful, we can find a map

$$f^{\mathrm{alg}} \colon p_W^*\big(\mathcal{H}_0^{\mathrm{alg}}\big) \otimes \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}} \times \mathrm{Spec}(A_W)}(m_2) \longrightarrow p_W^*\big(\mathcal{G}_0^{\mathrm{alg}}\big) \otimes \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}} \times \mathrm{Spec}(A_W)}(m_1)$$

whose analytification coincides with the map $f \colon \mathcal{H} \to \mathcal{G}$. Set $\mathcal{F}^{\mathrm{alg}} \coloneqq \mathrm{coker}\big(f^{\mathrm{alg}}\big)$. Then we have $\big(\mathcal{F}^{\mathrm{alg}}\big)^{\mathrm{an}} \simeq \mathcal{F}$; that is, $\mathcal{F}$ belongs to the essential image of the analytification functor. At this point the extension to a generic proper geometric stack X goes as in [PY16, Theorem 7.4]. □

5.2 More examples

Building on Theorem 5.5, we can prove that a number of different stacks satisfy the universal GAGA property. We start by discussing the case of formal completions. Next we study three stacks coming from Hodge theory, the de Rham, Betti and Dolbeault stacks. Although we briefly recall their definitions below, we refer the reader to [PTVV13, § 2.1] for a more thorough discussion. Finally, we consider the case of $\mathrm{B}G$ where G is a complex reductive algebraic group.

5.2.1 *Formal completions.* Let X be a derived geometric stack locally almost of finite presentation over k, and let $Y \hookrightarrow X$ be a closed immersion. We also suppose that Y is locally almost of finite presentation.

Definition 5.17. Let $\mathrm{Nil}_{/X}(Y)$ be the full subcategory of $(\mathrm{dSt}_k)_{Y_{\mathrm{red}}//X}$ spanned by morphisms $Y_{\mathrm{red}} \to Z \to X$, where Z is a derived geometric stack locally almost of finite presentation and the map $Y_{\mathrm{red}} \to Z$ induces an equivalence $Y_{\mathrm{red}} \simeq Z_{\mathrm{red}}$.

Definition 5.18. Let X, Y be derived geometric stacks locally almost of finite presentation over k, and let $i \colon Y \hookrightarrow X$ be a closed immersion. We define the *formal completion of X along Y* as the colimit

$$X^\wedge_Y \coloneqq \operatorname*{colim}_{Z \in \mathrm{Nil}_{/X}(Y)} Z \,.$$

The same definitions can be applied in the analytic setting. We have the following global analogue of [Por19, Proposition 8.2].

Lemma 5.19. *The analytification functor induces an equivalence of ∞-categories*

$$(-)^{\mathrm{an}} \colon \mathrm{Nil}_{/X}(Y) \simeq \mathrm{Nil}_{/X^{\mathrm{an}}}\big(Y^{\mathrm{an}}\big) \,.$$

Proof. We first remark that for every derived geometric stack Z locally almost of finite presentation over k, the canonical map

$$(Z_{\mathrm{red}})^{\mathrm{an}} \longrightarrow \big(Z^{\mathrm{an}}\big)_{\mathrm{red}}$$

is an equivalence. This implies that the analytification functor induces a well-defined map

$$(-)^{\mathrm{an}} \colon \mathrm{Nil}_{/X}(Y) \longrightarrow \mathrm{Nil}_{/X^{\mathrm{an}}}\big(Y^{\mathrm{an}}\big) \,.$$

Let $U_\bullet$ be an affine hypercover of X, and let $Y_\bullet \coloneqq Y \times_X U_\bullet$. Then for any $[n] \in \boldsymbol{\Delta}$ the map $Y_n \to U_n$ is a closed immersion, and in particular Y_n is an affine derived scheme almost of finite presentation. We have canonical equivalences

$$\mathrm{Nil}_{/X}(Y) \simeq \lim_{\boldsymbol{\Delta}} \mathrm{Nil}_{/U_\bullet}(Y_\bullet) \,, \quad \mathrm{Nil}_{/X^{\mathrm{an}}}\big(Y^{\mathrm{an}}\big) \simeq \lim_{\boldsymbol{\Delta}} \mathrm{Nil}_{/U_\bullet^{\mathrm{an}}}\big(Y_\bullet^{\mathrm{an}}\big) \,.$$

We are therefore reduced to the case where X itself is affine.

We first prove that it is fully faithful. Let $Z, T \in \mathrm{Nil}_{/Y}(X)$. Since X is affine, we have

$$\mathrm{Map}_{Y^{\mathrm{an}}_{\mathrm{red}}//X^{\mathrm{an}}}\big(Z^{\mathrm{an}}, T^{\mathrm{an}}\big) \simeq \mathrm{Map}_{(Y^{\mathrm{an}}_{\mathrm{red}})^{\mathrm{alg}}//X}\big(\big(Z^{\mathrm{an}}\big)^{\mathrm{alg}}, T\big) \,.$$

Now fix $T \in \mathrm{Nil}_{/Y}(X)$, and consider the full subcategory $\mathcal{C}$ of $\mathrm{Nil}_{/X}(Y)$ spanned by those Z for which the canonical map

$$\mathrm{Map}_{Y_{\mathrm{red}}//X}(Z, T) \longrightarrow \mathrm{Map}_{(Y_{\mathrm{red}})^{\mathrm{alg}}//X}\big(\big(Z^{\mathrm{an}}\big)^{\mathrm{alg}}, T\big)$$

is an equivalence. We observe that

(1) the object Y_{red} belongs to $\mathcal{C}$;

(2) the category $\mathcal{C}$ is closed under colimits in $\mathrm{Nil}_{/X}(Y)$.

Proceeding by induction, we are therefore left to check that if $Z \in \mathcal{C}$ and $M \in \mathrm{Coh}^{\geqslant 1}(Z)$, then $Z[M] \in \mathcal{C}$. This follows at once from [PY20b, Theorem 5.21]. This shows that the functor is fully faithful. For essential surjectivity recall from [Por19, Proposition 8.1] that a derived geometric analytic stack Z is algebraizable if and only if $\mathrm{t}_0(Z)$ is algebraizable. Since $\mathrm{t}_0(Z)$ is algebraizable if and only if $\mathrm{t}_0(Z)_{\mathrm{red}} = Z_{\mathrm{red}}$ is algebraizable, we conclude that the above functor is essentially surjective. □

Corollary 5.20. *Let X, Y be derived geometric stacks locally almost of finite presentation over k. The canonical morphism*

$$\big(X^\wedge_Y\big)^{\mathrm{an}} \longrightarrow \big(X^{\mathrm{an}}_{Y^{\mathrm{an}}}\big)^\wedge$$

is an equivalence.

Proof. This follows from the fact that $(-)^{\mathrm{an}}$ commutes with colimits by construction and from Lemma 5.19. □

Proposition 5.21. *Let X, Y be derived geometric stacks locally almost of finite presentation over k. Let $U \in \mathrm{dAfd}_k$ be a derived k-affinoid (respectively, Stein) space. The canonical map*

$$(-)^{\mathrm{an}} \colon \mathrm{QCoh}\big(\mathrm{Spec}(A_U) \times X^\wedge_Y\big) \longrightarrow \mathcal{O}_{U\times (X^\wedge_Y)^{\mathrm{an}}}\text{-Mod} \tag{5.22}$$

is conservative and t-exact.

Proof. For every $Z \in \mathrm{Nil}_{/X}(Y)$ we let

$$j_Z \colon Z \longrightarrow X^\wedge_Y\,, \quad j_{Z^{\mathrm{an}}} \colon Z^{\mathrm{an}} \longrightarrow (X_Y)^\wedge$$

be the two canonical maps. The morphisms

$$j_Z^* \colon \mathrm{QCoh}\big(\mathrm{Spec}(A_U) \times X^\wedge_Y\big) \longrightarrow \mathrm{QCoh}(\mathrm{Spec}(A_U) \times Z)\,,$$
$$j_{Z^{\mathrm{an}}}^* \colon \mathcal{O}_{U\times (X^\wedge_Y)^{\mathrm{an}}}\text{-Mod} \longrightarrow \mathcal{O}_{U\times Z^{\mathrm{an}}}\text{-Mod}$$

are jointly conservative as Z varies in $\mathrm{Nil}_{/X}(Y)$. Since Z is a derived geometric stack, Example 5.3 implies that the map

$$(-)^{\mathrm{an}} \colon \mathrm{QCoh}(\mathrm{Spec}(A_U) \times Z) \longrightarrow \mathcal{O}_{U\times Z^{\mathrm{an}}}\text{-Mod}$$

is conservative. As

$$\mathrm{QCoh}\big(\mathrm{Spec}(A_U) \times X^\wedge_Y\big) \simeq \lim_{Z\in \mathrm{Nil}_{/X}(Y)} \mathrm{QCoh}(\mathrm{Spec}(A_U) \times Z)\,,$$
$$\mathcal{O}_{U\times (X^\wedge_Y)^{\mathrm{an}}}\text{-Mod} \simeq \lim_{Z\in \mathrm{Nil}_{/X}(Y)} \mathcal{O}_{U\times Z^{\mathrm{an}}}\text{-Mod}\,,$$

it follows that (5.22) is conservative.

It is also clear that the functor (5.22) is left t-exact. Now let $\mathcal{F} \in \mathrm{QCoh}(\mathrm{Spec}(A_U) \times X^\wedge_Y)^{\leqslant 0}$. We have to prove that for every $\mathcal{G} \in \mathcal{O}_{U\times (X^\wedge_Y)^{\mathrm{an}}}\text{-Mod}^{\geqslant 1}$ we have

$$\mathrm{Map}_{\mathcal{O}_{U\times (X^\wedge_Y)^{\mathrm{an}}}\text{-Mod}}\big(\mathcal{G}, \mathcal{F}^{\mathrm{an}}\big) \simeq 0\,.$$

By definition, we have

$$\mathrm{Map}_{\mathcal{O}_{U\times (X^\wedge_Y)^{\mathrm{an}}}\text{-Mod}}\big(\mathcal{G}, \mathcal{F}^{\mathrm{an}}\big) \simeq \lim_{Z\in \mathrm{Nil}_{/X}(Y)} \mathrm{Map}_{\mathcal{O}_{U\times Z^{\mathrm{an}}}}\big(j^*_{Z^{\mathrm{an}}}\mathcal{G}, j^*_{Z^{\mathrm{an}}}\mathcal{F}^{\mathrm{an}}\big)\,.$$

However, $j^*_{Z^{\mathrm{an}}}\mathcal{F}^{\mathrm{an}} \simeq \big(j^*_Z\mathcal{F}\big)^{\mathrm{an}}$ and the analytification functor

$$(-)^{\mathrm{an}} \colon \mathrm{QCoh}(\mathrm{Spec}(A_U) \times Z) \longrightarrow \mathcal{O}_{U\times Z^{\mathrm{an}}}\text{-Mod}$$

is t-exact. Since in this case $j^*_{Z^{\mathrm{an}}}\mathcal{G} \in \mathcal{O}_{U\times Z^{\mathrm{an}}}\text{-Mod}^{\geqslant 1}$ by the definition of the t-structure on $\mathcal{O}_{U\times (X^\wedge_Y)^{\mathrm{an}}}$-Mod, the conclusion follows. □

Proposition 5.23. *Let X, Y be derived geometric stacks locally almost of finite presentation over k. Suppose furthermore that Y is proper. Then the formal completion $X^\wedge_Y$ satisfies the universal GAGA property.*

Proof. Since the analytification is defined as a left Kan extension, we have a natural equivalence

$$\big(X^\wedge_Y\big)^{\mathrm{an}} \simeq \big(X^{\mathrm{an}}\big)^\wedge_{Y^{\mathrm{an}}}\,.$$

We have

$$\mathrm{Perf}\big(X^\wedge_Y \times \mathrm{Spec}(A_U)\big) \simeq \lim_{Z\to X} \mathrm{Perf}\big(Z \times \mathrm{Spec}(A_U)\big)$$

and similarly

$$\mathrm{Perf}\big(\big(X^{\mathrm{an}}\big)^\wedge_{Y^{\mathrm{an}}} \times U\big) \simeq \lim_{Z\to X} \mathrm{Perf}\big(Z^{\mathrm{an}} \times U\big)\,.$$

Notice that each Z is still a proper derived geometric stack locally almost of finite presentation. In the non-archimedean setting, Theorem 5.5 shows that the analytification functor

$$\mathrm{Perf}(Z \times \mathrm{Spec}(A_U)) \longrightarrow \mathrm{Perf}\big(Z^{\mathrm{an}} \times U\big)$$

is an equivalence. In the $\mathbb{C}$-analytic case as always Theorem 5.5 shows that for every compact Stein subset $K \subset U$ and every open Stein neighbourhood V of K, the functor

$$\mathrm{Perf}(Z \times \mathrm{Spec}(A_V)) \longrightarrow \mathrm{Perf}\big(Z^{\mathrm{an}} \times V\big)$$

is fully faithful. In particular, we obtain that

$$\mathrm{Perf}\big(X^\wedge_Y \times \mathrm{Spec}(A_U)\big) \longrightarrow \mathrm{Perf}\big(\big(X^{\mathrm{an}}\big)^\wedge_{Y^{\mathrm{an}}} \times U\big)$$

is fully faithful as well. In order to prove that

$$\underset{K\subset V\subset U}{\text{“colim”}}\, \mathrm{Perf}\big(X^\wedge_Y \times \mathrm{Spec}(A_V)\big) \longrightarrow \underset{K\subset V\subset U}{\text{“colim”}}\, \mathrm{Perf}\big(\big(X^{\mathrm{an}}\big)^\wedge_{Y^{\mathrm{an}}} \times V\big)$$

is an equivalence in $\mathrm{Ind}\big(\mathrm{Cat}^{\mathrm{st},\otimes}_\infty\big)$, it is therefore enough to prove that given a relatively compact Stein open neighbourhood $W \Subset V$ of K in V, the restriction functor

$$\mathrm{Perf}\big(\big(X^{\mathrm{an}}\big)^\wedge_{Y^{\mathrm{an}}} \times V\big) \longrightarrow \mathrm{Perf}\big(\big(X^{\mathrm{an}}\big)^\wedge_{Y^{\mathrm{an}}} \times W\big)$$

factors through $\mathrm{Perf}(X^\wedge_Y \times \mathrm{Spec}(A_W))$. This fact follows directly from Proposition 5.15. Notice that the choice of W does not depend on $Z \in \mathrm{Nil}_{/X}(Y)$. □

5.2.2 *De Rham stacks.* The de Rham stack can be defined in both the algebraic and analytic setting as follows. Let

$$j \colon \mathrm{Aff}^{\mathrm{red}}_k \longrightarrow \mathrm{dAff}_k$$

be the natural inclusion. This is a continuous and cocontinuous morphism of sites with respect to the étale topology on both sides. In particular, the functor

$$j^s \colon \mathrm{dSt}_k \longrightarrow \mathrm{St}\big(\mathrm{Aff}^{\mathrm{red}}_k, \tau_{\text{ét}}\big)$$

admits both a left adjoint j_s and a right adjoint ${}_sj$. We set

$$(-)_{\mathrm{dR}} \coloneqq {}_sj \circ j^s\,, \quad (-)_{\mathrm{red}} \coloneqq j_s \circ j^s\,.$$

It can be shown that when $X = \mathrm{Spec}(A)$, we have $X_{\mathrm{red}} \simeq \mathrm{Spec}(\pi_0(A)_{\mathrm{red}})$, while it is always true that

$$X_{\mathrm{dR}}(\mathrm{Spec}(A)) \simeq X(\mathrm{Spec}(\pi_0(A))_{\mathrm{red}}) .$$

The same definitions can be carried over in the analytic setting, using $\mathrm{Afd}_k^{\mathrm{red}}$ instead of $\mathrm{Aff}_k^{\mathrm{red}}$. See [Por17b, § 3] for the $\mathbb{C}$-analytic case.

LEMMA 5.24. *Let $X \in \mathrm{dSt}_k^{\mathrm{afp}}$ be a derived stack locally almost of finite presentation. Then there is a canonical map*

$$\left(X_{\mathrm{dR}}\right)^{\mathrm{an}} \longrightarrow \left(X^{\mathrm{an}}\right)_{\mathrm{dR}}$$

which is furthermore an equivalence when X is a smooth geometric stack.

Proof. We observe that the analytification functor takes $\mathrm{Aff}_k^{\mathrm{red,afp}}$ to $\mathrm{Afd}_k^{\mathrm{red}}$. In particular, the natural transformation $(-)^{\mathrm{an}} \circ (-)_{\mathrm{dR}} \to (-)_{\mathrm{dR}} \circ (-)^{\mathrm{an}}$ is simply a Beck–Chevalley transformation. When X is smooth and geometric, we observe that the maps

$$X \longrightarrow X_{\mathrm{dR}} , \quad X^{\mathrm{an}} \longrightarrow (X^{\mathrm{an}})_{\mathrm{dR}}$$

are effective epimorphisms. Their Čech nerves can be identified with the simplicial objects $\left(X^\bullet\right)^\wedge_X$ and $\left(\left(X^{\mathrm{an}}\right)^\bullet\right)^\wedge_{X^{\mathrm{an}}}$ given by the formal completions of X^n and $\left(X^{\mathrm{an}}\right)^n$ along the small diagonals. The conclusion now follows from Corollary 5.20. □

PROPOSITION 5.25. *Let X be a smooth geometric stack locally almost of finite presentation over k. Then for any $U \in \mathrm{dAfd}_k$ the canonical map*

$$(-)^{\mathrm{an}} \colon \mathrm{QCoh}(\mathrm{Spec}(A_U) \times X_{\mathrm{dR}}) \longrightarrow \mathcal{O}_{U \times X_{\mathrm{dR}}^{\mathrm{an}}}\text{-Mod}$$

is conservative and t-exact.

Proof. Write $X^\bullet / X_{\mathrm{dR}}$ and $(X^{\mathrm{an}})^\bullet / X_{\mathrm{dR}}^{\mathrm{an}}$ to denote the Čech nerves of the maps $X \to X_{\mathrm{dR}}$ and $X^{\mathrm{an}} \to X_{\mathrm{dR}}^{\mathrm{an}}$. Since X is smooth, we have

$$X_{\mathrm{dR}} \simeq \left| X^\bullet / X_{\mathrm{dR}} \right| , \quad X_{\mathrm{dR}}^{\mathrm{an}} \simeq \left| \left(X^{\mathrm{an}}\right)^\bullet / X_{\mathrm{dR}}^{\mathrm{an}} \right| .$$

In turn, this provides canonical equivalences

$$\mathrm{QCoh}(\mathrm{Spec}(A_U) \times X_{\mathrm{dR}}) \simeq \lim_{\mathbf{\Delta}} \mathrm{QCoh}(\mathrm{Spec}(A_U) \times (X^\bullet / X_{\mathrm{dR}})) ,$$

$$\mathcal{O}_{U \times X_{\mathrm{dR}}^{\mathrm{an}}}\text{-Mod} \simeq \lim_{\mathbf{\Delta}} \mathcal{O}_{U \times ((X^{\mathrm{an}})^\bullet / X_{\mathrm{dR}}^{\mathrm{an}})}\text{-Mod} .$$

The argument given in [Por17b, Proposition 5.1] shows that we can endow $\mathrm{QCoh}(\mathrm{Spec}(A_U) \times X_{\mathrm{dR}})$ and $\mathcal{O}_{U \times X_{\mathrm{dR}}^{\mathrm{an}}}$-Mod with t-structures characterized by the fact that the forgetful functors

$$\mathrm{QCoh}(\mathrm{Spec}(A_U) \times X_{\mathrm{dR}}) \longrightarrow \mathrm{QCoh}(\mathrm{Spec}(A_U) \times X) , \quad \mathcal{O}_{U \times X_{\mathrm{dR}}^{\mathrm{an}}}\text{-Mod} \longrightarrow \mathcal{O}_{U \times X}\text{-Mod}$$

are t-exact. Moreover, the analytification functor is obtained by passing to the limit from the analytification functors

$$(-)^{\mathrm{an}} \colon \mathrm{QCoh}(\mathrm{Spec}(A_U) \times (X^\bullet / X_{\mathrm{dR}})) \longrightarrow \mathcal{O}_{U \times ((X^{\mathrm{an}})^\bullet / X_{\mathrm{dR}}^{\mathrm{an}})}\text{-Mod} .$$

Since we can identify X^n / X_{dR} and $\left(X^{\mathrm{an}}\right)^n / X_{\mathrm{dR}}^{\mathrm{an}}$ with the formal completion of the small diagonal in X^n and in $\left(X^{\mathrm{an}}\right)^n$, respectively, the conclusion now follows from Proposition 5.21. □

PROPOSITION 5.26. *Let X be a smooth and proper geometric stack locally almost of finite presentation over k. Then X_{dR} satisfies the universal GAGA property.*

Proof. Let us first assume that k is a non-archimedean field. Then thanks to Proposition 5.25, we only have to check that the canonical map

$$\mathrm{Perf}(\mathrm{Spec}(A_U) \times X_{\mathrm{dR}}) \longrightarrow \mathrm{Perf}\big(U \times X^{\mathrm{an}}_{\mathrm{dR}}\big)$$

is an equivalence for every $U \in \mathrm{dAfd}_k$. Let $X^\bullet/X_{\mathrm{dR}}$ and $(X^{\mathrm{an}})^\bullet/X^{\mathrm{an}}_{\mathrm{dR}}$ be the Čech nerves of the maps $X \to X_{\mathrm{dR}}$ and $X^{\mathrm{an}} \to X^{\mathrm{an}}_{\mathrm{dR}}$, respectively. Then

$$\mathrm{Perf}(\mathrm{Spec}(A_U) \times X_{\mathrm{dR}}) \simeq \lim_{\boldsymbol{\Delta}} \mathrm{Perf}(\mathrm{Spec}(A_U) \times (X^\bullet/X_{\mathrm{dR}}))$$

and

$$\mathrm{Perf}\big(U \times X^{\mathrm{an}}_{\mathrm{dR}}\big) \simeq \lim_{\boldsymbol{\Delta}} \mathrm{Perf}\big(U \times \big((X^{\mathrm{an}})^\bullet/X^{\mathrm{an}}_{\mathrm{dR}}\big)\big)\,.$$

The conclusion now follows directly from Proposition 5.23.

We now turn to the $\mathbb{C}$-analytic case. Therefore, fix a compact Stein subset $K \subset U$. Since we can identify both X^n/X_{dR} and $(X^{\mathrm{an}})^n/X^{\mathrm{an}}_{\mathrm{dR}}$ with the formal completion of the small diagonal in X^n and in $(X^{\mathrm{an}})^n$, we can use Proposition 5.23 to deduce that for every open Stein neighbourhood V of K in U, the map

$$\mathrm{Perf}\big(\mathrm{Spec}(A_V) \times \big(X^n/X_{\mathrm{dR}}\big)\big) \longrightarrow \mathrm{Perf}\big(V \times \big((X^{\mathrm{an}})^n/X^{\mathrm{an}}_{\mathrm{dR}}\big)\big)$$

is fully faithful. Therefore, to prove that the map

$$\underset{K \subset V \subset U}{\text{“colim”}} \mathrm{Perf}\big(\mathrm{Spec}(A_V) \times X_{\mathrm{dR}}\big) \longrightarrow \underset{K \subset V \subset U}{\text{“colim”}} \mathrm{Perf}\big(V \times X^{\mathrm{an}}_{\mathrm{dR}}\big)$$

is an equivalence in $\mathrm{Ind}\big(\mathrm{Cat}^{\mathrm{st},\otimes}_\infty\big)$, it is enough to check that if $W \Subset V$ is a relatively compact open Stein neighbourhood of K in V, then the restriction functor

$$\mathrm{Perf}\big(V \times X^{\mathrm{an}}_{\mathrm{dR}}\big) \longrightarrow \mathrm{Perf}\big(W \times X^{\mathrm{an}}_{\mathrm{dR}}\big)$$

factors through $\mathrm{Perf}(\mathrm{Spec}(A_W) \times X^{\mathrm{an}}_{\mathrm{dR}})$. This follows immediately from Proposition 5.15. □

5.2.3 *Betti stacks.* The canonical functor $\pi \colon \mathrm{dAff}^{\mathrm{afp}}_k \to \{*\}$ induces an adjunction

$$\pi_s \colon \mathrm{dSt}^{\mathrm{afp}}_k \rightleftarrows \mathcal{S} : \pi^s\,.$$

Given a space $K \in \mathcal{S}$, we set

$$K_{\mathrm{B}} \coloneqq \pi^s(K)\,.$$

We refer to K_{B} as the *Betti stack associated with B*. In other words, K_{B} is the sheafification of the constant presheaf with values K. We similarly define $K^{\mathrm{an}}_{\mathrm{B}}$ as the constant analytic stack associated with K.

Lemma 5.27. *There is a canonical equivalence* $(K_{\mathrm{B}})^{\mathrm{an}} \simeq K^{\mathrm{an}}_{\mathrm{B}}$.

Proof. Let us denote by φ the derived analytification functor

$$\varphi \coloneqq (-)^{\mathrm{an}} \colon \mathrm{dAff}^{\mathrm{afp}}_k \longrightarrow \mathrm{dAn}_k\,.$$

Then $\varphi^s(K^{\mathrm{an}}_{\mathrm{B}}) \simeq \pi^s(K) \simeq K_{\mathrm{B}}$. Consequently,

$$\mathrm{Map}_{\mathrm{dAnSt}_k}\big((K_{\mathrm{B}})^{\mathrm{an}}, K^{\mathrm{an}}_{\mathrm{B}}\big) \simeq \mathrm{Map}_{\mathrm{dSt}^{\mathrm{afp}}_k}\big(K_{\mathrm{B}}, \varphi^s(K^{\mathrm{an}}_{\mathrm{B}})\big)\,.$$

Therefore, the identity of K_{B} corresponds to a canonical morphism $(K_{\mathrm{B}})^{\mathrm{an}} \to K^{\mathrm{an}}_{\mathrm{B}}$. We now observe that this morphism is an equivalence when $K \simeq *$, and moreover both $(-)^{\mathrm{an}}$ and the formation of $K^{\mathrm{an}}_{\mathrm{B}}$ commute with arbitrary colimits. The conclusion therefore follows. □

Proposition 5.28. *Let $K \in \mathcal{S}$ be a space. Then for any $U \in \mathrm{dAfd}_k$ the Betti stack K_{B} satisfies the universal GAGA property.*

Proof. We first observe that

$$\mathrm{QCoh}(K_{\mathrm{B}} \times \mathrm{Spec}(A_U)) \simeq \mathrm{Fun}(K, A_U\text{-Mod})\,, \quad \mathcal{O}_{K_{\mathrm{B}}^{\mathrm{an}} \times U}\text{-Mod} \simeq \mathrm{Fun}(K, \mathcal{O}_U\text{-Mod})\,. \tag{5.29}$$

Moreover, the analytification functor is simply obtained by composition with the analytification functor

$$\varepsilon_U^* \colon A_U\text{-Mod} \longrightarrow \mathcal{O}_U\text{-Mod}\,.$$

As Proposition 4.5 guarantees that ε_U^* is t-exact and conservative, we deduce that the same goes for the functor (5.29). Next, in the non-archimedean case, the equivalence $\mathrm{Perf}(A_U) \simeq \mathrm{Perf}(U)$ immediately implies that the analytification

$$\mathrm{Perf}(K_{\mathrm{B}} \times \mathrm{Spec}(A_U)) \longrightarrow \mathrm{Perf}\big(K_{\mathrm{B}}^{\mathrm{an}} \times U\big)$$

is an equivalence. In the $\mathbb{C}$-analytic case, fix a compact Stein subset $K \subset U$. Then Proposition 5.13 implies that for each open Stein neighbourhood V of K inside U, the canonical map

$$\mathrm{Perf}(K_{\mathrm{B}} \times \mathrm{Spec}(A_V)) \longrightarrow \mathrm{Perf}\big(K_{\mathrm{B}}^{\mathrm{an}} \times V\big)$$

is fully faithful, while Lemma 4.10 implies that if $W \Subset V$ is a relatively compact open Stein neighbourhood of K inside V, then the restriction

$$\mathrm{Perf}\big(K_{\mathrm{B}}^{\mathrm{an}} \times V\big) \longrightarrow \mathrm{Perf}\big(K_{\mathrm{B}}^{\mathrm{an}} \times W\big)$$

factors through $\mathrm{Perf}(K_{\mathrm{B}} \times \mathrm{Spec}(A_W))$. This implies that the canonical map

$$\underset{K \subset V \subset U}{\text{“colim”}} \mathrm{Perf}(K_{\mathrm{B}} \times \mathrm{Spec}(A_V)) \longrightarrow \underset{K \subset V \subset U}{\text{“colim”}} \mathrm{Perf}\big(K_{\mathrm{B}}^{\mathrm{an}} \times V\big)$$

is an equivalence. □

5.2.4 *Dolbeault stacks.* The Dolbeault stack of a derived formal stack X appears in Simpson's non-abelian Hodge theory in dealing with Higgs bundles. It is defined as follows: let

$$\mathrm{T}X \coloneqq \mathrm{Spec}_X(\mathrm{Sym}_{\mathcal{O}_X}(\mathbb{L}_X))$$

be the derived tangent bundle to X. Let $\widehat{\mathrm{T}X}$ be the formal completion of $\mathrm{T}X$ along the zero section. Using [Lur17, Proposition 4.2.2.9], we can convert the natural commutative group structure of $\mathrm{T}X$ relative to X (seen as an associative one) into a simplicial diagram $\mathrm{T}^\bullet X \colon \mathbf{\Delta}^{\mathrm{op}} \to (\mathrm{dSt}_k)_{/X}$. Unwinding the definitions, we see that $\mathrm{T}^\bullet X$ can be identified with the n-fold product $\mathrm{T}X \times_X \cdots \times_X \mathrm{T}X$. The zero section $X \to \mathrm{T}X$ allows us to promote $\mathrm{T}^\bullet X$ to a simplicial diagram

$$\mathrm{T}^\bullet X \colon \mathbf{\Delta}^{\mathrm{op}} \longrightarrow (\mathrm{dSt}_k)_{X//X}\,.$$

Formal completion along the natural maps $X \to \mathrm{T}^n X$ provides us with a new simplicial object

$$\widehat{\mathrm{T}^\bullet X} \colon \mathbf{\Delta}^{\mathrm{op}} \longrightarrow (\mathrm{dSt}_k)_{/X}\,.$$

Definition 5.30. The *Dolbeault stack of X* is the geometric realization

$$X_{\mathrm{Dol}} \coloneqq \big|\widehat{\mathrm{T}^\bullet X}\big| \in (\mathrm{dSt}_k)_{/X}\,.$$

The Dolbeault stack can be defined directly at the analytical level by the exact same procedure. We have the following.

Lemma 5.31. *Let X be a derived geometric k-stack locally almost of finite presentation. Then there is a natural equivalence $(X_{\mathrm{Dol}})^{\mathrm{an}} \simeq (X^{\mathrm{an}})_{\mathrm{Dol}}$.*

Proof. Combining the universal property of the analytification and of the cotangent complex, we find a canonical comparison map

$$(\mathbb{L}_X)^{\mathrm{an}} \longrightarrow \mathbb{L}^{\mathrm{an}}_{X^{\mathrm{an}}} .$$

We claim that this map is an equivalence. Since both sides satisfy smooth descent in X, it suffices to check this when X is a derived affine almost of finite presentation. In this case the statement directly follows from [PY20b, Theorem 5.21].

From here, the conclusion follows directly from Corollary 5.20 and from the fact that the analytification functor $(-)^{\mathrm{an}} \colon \mathrm{dSt}_k^{\mathrm{afp}} \to \mathrm{dAnSt}_k$ commutes with finite limits and arbitrary colimits. □

Proposition 5.32. *Let X be a proper derived geometric k-stack. For any $U \in \mathrm{dAfd}_k$ the Dolbeault stack X_{Dol} satisfies the universal GAGA property.*

Proof. Since the face maps in the simplicial diagram $\widehat{\mathrm{T}^\bullet X}$ are flat, we deduce directly from Proposition 5.21 that the canonical map

$$\mathrm{QCoh}(X_{\mathrm{Dol}} \times \mathrm{Spec}(A_U)) \longrightarrow \mathcal{O}_{X^{\mathrm{an}}_{\mathrm{Dol}} \times U}\text{-Mod}$$

is conservative and t-exact.

In the non-archimedean case, Proposition 5.23 implies that the analytification functor induces an equivalence

$$\mathrm{Perf}\big(\widehat{\mathrm{T}^n X} \times \mathrm{Spec}(A_U)\big) \simeq \mathrm{Perf}\big(\widehat{\mathrm{T}^n X^{\mathrm{an}}} \times U\big)$$

for every $U \in \mathrm{dAfd}_k$ and every $n \geqslant 0$. Therefore, we deduce that the canonical map

$$\mathrm{Perf}\big(X_{\mathrm{Dol}} \times \mathrm{Spec}(A_U)\big) \longrightarrow \mathrm{Perf}\big(X^{\mathrm{an}}_{\mathrm{Dol}} \times U\big)$$

is an equivalence as well. In the $\mathbb{C}$-analytic case, we deduce from Proposition 5.23 that each map

$$\mathrm{Perf}\big(\widehat{\mathrm{T}^n X} \times \mathrm{Spec}(A_U)\big) \longrightarrow \mathrm{Perf}\big(\widehat{\mathrm{T}^n X^{\mathrm{an}}} \times U\big)$$

is fully faithful, and therefore that for every $U \in \mathrm{dSt}_{\mathbb{C}}$ the functor

$$\mathrm{Perf}\big(X_{\mathrm{Dol}} \times \mathrm{Spec}(A_U)\big) \longrightarrow \mathrm{Perf}\big(X^{\mathrm{an}}_{\mathrm{Dol}} \times U\big)$$

is fully faithful. Now let $K \subset U$ be a compact Stein subset. It is enough to prove that if $W \Subset V$ are two open Stein neighbourhoods of K inside U, with W relatively compact inside V, then the canonical map

$$\mathrm{Perf}\big(X^{\mathrm{an}}_{\mathrm{Dol}} \times V\big) \longrightarrow \mathrm{Perf}\big(X^{\mathrm{an}}_{\mathrm{Dol}} \times W\big)$$

factors through $\mathrm{Perf}\big(X_{\mathrm{Dol}} \times \mathrm{Spec}(A_W)\big)$. This follows once again from Proposition 5.15. □

5.2.5 *Classifying stack of a complex reductive group.* In the previous sections we discussed several examples of derived stacks satisfying the universal GAGA property. All the examples we considered so far are consequences of the analysis carried out in order to prove Theorem 5.5. We now consider a different kind of example.

Let G be a connected reductive group over $\mathbb{C}$. Then $\mathrm{B}G$ is a smooth geometric stack, but it is not proper in the sense of [PY16, Definition 4.8]. We nevertheless can prove the following result.

Proposition 5.33. *If G is a connected reductive group over $\mathbb{C}$, then $\mathrm{B}G$ satisfies the GAGA property.*

Proof. Let us start by remarking that since the analytification functor commutes with colimits, we have a canonical equivalence $(\mathrm{B}G)^{\mathrm{an}} \simeq \mathrm{B}(G^{\mathrm{an}})$. We will therefore use the notation $\mathrm{B}G^{\mathrm{an}}$ since no confusion can arise. Next, we observe that the argument given in Example 5.3 shows that the canonical map

$$\mathrm{QCoh}(\mathrm{B}G) \longrightarrow \mathcal{O}_{\mathrm{B}G^{\mathrm{an}}}\text{-Mod}$$

is t-exact and conservative. All we are left to check is therefore that the canonical functor

$$\mathrm{Perf}(\mathrm{B}G) \longrightarrow \mathrm{Perf}\big(\mathrm{B}G^{\mathrm{an}}\big)$$

is an equivalence. We will prove more generally that the morphism

$$\mathrm{Coh}(\mathrm{B}G) \longrightarrow \mathrm{Coh}\big(\mathrm{B}G^{\mathrm{an}}\big)$$

is an equivalence of stable ∞-categories. Since both sides are equipped with complete t-structures and the functor between them is t-exact, we are reduced to proving the following two statements:

(1) The analytification functor

$$\mathrm{Coh}^{\heartsuit}(\mathrm{B}G) \longrightarrow \mathrm{Coh}\big(\mathrm{B}G^{\mathrm{an}}\big)$$

is fully faithful.

(2) The functor

$$\mathrm{Coh}^{\heartsuit}(\mathrm{B}G) \longrightarrow \mathrm{Coh}^{\heartsuit}\big(\mathrm{B}G^{\mathrm{an}}\big)$$

is essentially surjective.

Notice that the second statement is entirely classical and is, in fact, well known. In the case where G is semi-simple, it is proven for instance in [Tay02, Corollary 15.8.7]. For tori it is well known. Finally, a general reductive group G admits a finite cover by a product of a torus and a semi-simple group, and from here it is straightforward to obtain the statement for G.

It is therefore enough to prove the first statement. Since we work over $\mathbb{C}$, it follows from [HR15] that $\mathrm{QCoh}(\mathrm{B}G)$ is compactly generated and in particular that

$$\mathrm{QCoh}(\mathrm{B}G) \simeq \mathrm{D}\big(\mathrm{QCoh}^{\heartsuit}(\mathrm{B}G)\big)\,.$$

Since G is reductive, for $M, N \in \mathrm{QCoh}^{\heartsuit}(\mathrm{B}G)$ the mapping space $\mathrm{Map}_{\mathrm{QCoh}(\mathrm{B}G)}(M,N)$ is discrete and coincides with the hom set $\mathrm{Hom}_G(M,N)$. We let M^{an} and N^{an} denote the analytic representations of G^{an} associated with M and N. Since M and N are coherent, we have an equivalence

$$\mathrm{Hom}_G(M,N) \simeq \mathrm{Hom}_{G^{\mathrm{an}}}\big(M^{\mathrm{an}}, N^{\mathrm{an}}\big) \simeq \pi_0 \,\mathrm{Map}_{\mathrm{QCoh}(\mathrm{B}G^{\mathrm{an}})}\big(M^{\mathrm{an}}, N^{\mathrm{an}}\big)\,,$$

which readily follows from [Tay02, Corollary 15.8.7]. In order to complete the proof, we have to check that $\pi_i \,\mathrm{Map}_{\mathrm{QCoh}(\mathrm{B}G^{\mathrm{an}})}\big(M^{\mathrm{an}}, N^{\mathrm{an}}\big) \simeq 0$ for $i \neq 0$. We denote by $\mathcal{H}\big(G^{\mathrm{an}}\big)$ the category of holomorphic representations of G^{an} on topological vector spaces with continuous G^{an}-invariant maps between them. We now invoke the results of [HM66]. Since G is reductive, it has a maximal compact subgroup, which is ample in the sense of [HM66]. Moreover, any finite-dimensional representation of G^{an} is complete and locally convex. Therefore, [HM66, Proposition 2.3] implies that N^{an} is holomorphically injective, and therefore that

$$\mathrm{Hom}_{G^{\mathrm{an}}}\big(M^{\mathrm{an}}, N^{\mathrm{an}}\big) \simeq \mathrm{Map}_{\mathrm{D}(\mathcal{H}(G^{\mathrm{an}}))}\big(M^{\mathrm{an}}, N^{\mathrm{an}}\big)\,.$$

We are now reduced to proving that there is an equivalence

$$\mathrm{Map}_{\mathrm{D}(\mathcal{H}(G^{\mathrm{an}}))}\big(M^{\mathrm{an}}, N^{\mathrm{an}}\big) \simeq \mathrm{Map}_{\mathcal{O}_{\mathrm{B}G^{\mathrm{an}}}\text{-Mod}}\big(M^{\mathrm{an}}, N^{\mathrm{an}}\big)\,.$$

We will use the completed tensor product $\hat{\otimes}$ of locally convex spaces. As we are working with global sections over Stein spaces, the locally convex spaces are nuclear, see [Dem12, Proposition IX.5.18], and there is no distinction between projective and injective tensor product. In particular, $\hat{\otimes}$ preserves subspaces; see [Dem12, Proposition IX.5.6].

Let $(G^{\mathrm{an}})^\bullet$ be the Čech nerve of the map $p\colon \mathrm{Sp}(\mathbb{C}) \to \mathrm{B}G^{\mathrm{an}}$. We can compute the Exts in $\mathcal{O}_{\mathrm{B}G^{\mathrm{an}}}$-Mod by means of the equivalence

$$\mathcal{O}_{\mathrm{B}G^{\mathrm{an}}}\text{-Mod} \simeq \lim_{[n]\in\mathbf{\Delta}} \mathcal{O}_{(G^{\mathrm{an}})^{\times n}}\text{-Mod}\,.$$

Write $q_n\colon \left(G^{\mathrm{an}}\right)^{\times n} \to \mathrm{Sp}(\mathbb{C})$ and $p_n\colon \left(G^{\mathrm{an}}\right)^{\times n} \to \mathrm{B}G$ for the standard projection maps. We have canonical identifications $p_n \simeq p \circ q_n$. The cosimplicial object computing $\mathrm{Map}_{\mathrm{B}G^{\mathrm{an}}}(M^{\mathrm{an}}, N^{\mathrm{an}})$ has

$$\mathrm{Map}_{\mathrm{Coh}^\heartsuit((G^{\mathrm{an}})^{\times n})}\left(p_n^* M^{\mathrm{an}}, p_n^* N^{\mathrm{an}}\right) \simeq \mathrm{Map}_{\mathbb{C}}\left(p^* M^{\mathrm{an}}, q_{n*} p_n^* N^{\mathrm{an}}\right)$$

in degree n. As $\left(G^{\mathrm{an}}\right)^{\times n}$ is Stein and p^*M^{an} and p^*N^{an} are globally finitely generated in the sense of [PY16, Lemma 8.11], we have an equivalence

$$\mathrm{Hom}_{\mathrm{Coh}((G^{\mathrm{an}})^{\times n})}\left(p_n^* M^{\mathrm{an}}, p_n^* N^{\mathrm{an}}\right) \simeq \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}((G^{\mathrm{an}})^{\times n})}\left(q_{n*} p_n^* M^{\mathrm{an}}, q_{n*} p_n^* N^{\mathrm{an}}\right)\,.$$

Here the superscript cts denotes the subset of continuous maps for the unique complete topology on the global sections of a coherent sheaf over a Stein space. We further have the following equivalence:

$$\mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}((G^{\mathrm{an}})^{\times n})}\left(q_{n*} p_n^* M^{\mathrm{an}}, q_{n*} p_n^* N^{\mathrm{an}}\right) \simeq \mathrm{Hom}^{\mathrm{cts}}_{\mathbb{C}}\left(p^* M^{\mathrm{an}}, p^* N^{\mathrm{an}} \,\widehat{\otimes}\, \mathcal{O}((G^{\mathrm{an}})^{\times n})\right)\,.$$

We claim that there is a further isomorphism

$$\mathrm{Hom}^{\mathrm{cts}}_{\mathbb{C}}\left(p^* M^{\mathrm{an}}, p^* N^{\mathrm{an}} \,\widehat{\otimes}\, \mathcal{O}\left(\left(G^{\mathrm{an}}\right)^{\times n}\right)\right) \simeq \mathrm{Hom}_{\mathcal{H}(G^{\mathrm{an}})}\left(M^{\mathrm{an}}, N^{\mathrm{an}} \,\widehat{\otimes}\, \mathcal{O}\left(\left(G^{\mathrm{an}}\right)^{\times n+1}\right)\right)\,.$$

This would follow if we could show that the forgetful functor from $\mathcal{H}\left(G^{\mathrm{an}}\right)$ to topological vector spaces has a right adjoint given by $-\hat{\otimes}\mathcal{O}\left(G^{\mathrm{an}}\right)$, equivalently that pushforward from the point to $\mathrm{B}G^{\mathrm{an}}$ is given by $-\hat{\otimes}\mathcal{O}\left(G^{\mathrm{an}}\right)$. Unfortunately, this situation does not seem to be treated in the literature and goes beyond the scope of this article. We will prove a weaker statement that is sufficient for our purposes, using the fact that M is finite-dimensional.

Firstly, observe that $\mathrm{Hom}_{\mathcal{H}(G^{\mathrm{an}})}\left(M, \mathcal{O}\left(G^{\mathrm{an}}\right)\right) \cong \mathrm{Hom}(M, \mathbb{C})$. There is clearly an embedding of the right-hand side into the left-hand side as $\mathcal{O}\left(G^{\mathrm{an}}\right)$ contains the regular functions on G. But this embedding is surjective as any G-equivariant map from M to $\mathcal{O}\left(G^{\mathrm{an}}\right)$ is determined by the image of a basis of M evaluated at the identity; that is, the space of such maps has dimension $\dim M$.

Now we consider $\mathrm{Hom}_{\mathcal{H}(G^{\mathrm{an}})}\left(M, V\hat{\otimes}\mathcal{O}\left(G^{\mathrm{an}}\right)\right)$, where we write V for the trivial G-representation $N\hat{\otimes}\mathcal{O}\left(G^{\mathrm{an}}\right)^{\times n}$. Any function from M factors through some $V_i\hat{\otimes}\mathcal{O}\left(G^{\mathrm{an}}\right)$, where V_i is a finite-dimensional subspace of V. Thus we need to compute $\mathrm{Hom}_{\mathcal{H}(G^{\mathrm{an}})}(M, \operatorname{colim} V_i\hat{\otimes}\mathcal{O}(G^{\mathrm{an}}))$, where we take the colimit over all finite-dimensional subspaces. Now Hom out of M commutes with filtered colimits over admissible maps, and we are reduced to showing that for V' finite-dimensional $\mathrm{Hom}_{\mathcal{H}(G^{\mathrm{an}})}\left(M, V' \otimes \mathcal{O}\left(G^{\mathrm{an}}\right)\right) \cong \mathrm{Hom}(M, V')$, which readily follows from the case where V' is 1-dimensional. Then $\mathrm{Hom}_{\mathcal{H}(G^{\mathrm{an}})}\left(M, \operatorname{colim} V_i \otimes \mathcal{O}\left(G^{\mathrm{an}}\right)\right) \cong \mathrm{Hom}^{\mathrm{cts}}\left(M, \operatorname{colim} V_i\right) \cong \mathrm{Hom}^{\mathrm{cts}}(M, V)$. This proves the claim.

Now we have a complex of holomorphic representations of G^{an} which computes cohomology in $\mathcal{H}\left(G^{\mathrm{an}}\right)$. The complex $\left(\mathcal{O}\left(G^{\mathrm{an}}\right)^{\hat{\otimes}\bullet+1} \otimes N\right)$ is quasi-isomorphic to N by the usual bar complex arguments, and by [HM66, Proposition 2.1], it is levelwise injective, thus injective as it is bounded below. We note that the G^{an}-action on $\mathcal{O}\left(G^{\mathrm{an}}\right)^{\hat{\otimes} n+1}$ is on the last factor since

$(q_n)_* p_n^* \mathcal{N} \cong \mathcal{O}(G^{\mathrm{an}})^{\otimes n} \otimes \mathcal{N}$ has the trivial G^{an}-action as $p^* \mathcal{N}$ forgets the G^{an}-action. This gives $\mathrm{Hom}_{\mathcal{H}(G^{\mathrm{an}})}\big(M, N \hat{\otimes} \mathcal{O}(G^{\mathrm{an}})^{\times \bullet + 1}\big) \simeq \mathrm{Hom}_{\mathcal{H}(G^{\mathrm{an}})}(M, N)$

This shows that the Čech complex computation recovers the (trivial) holomorphic group cohomology of G^{an}. □

Remark 5.34. In the above example we used in an essential way that G is a reductive group over the field of complex numbers. We do not know what happens if G is a reductive group over a non-archimedean field, but it would be interesting to know if the same property holds.

6. Tannaka duality

In this section we prove the main theorem of this paper. Our goal is to find sufficient conditions on a geometric derived stack Y and a (not necessarily geometric) derived stack X so that the canonical morphism

$$\mathbf{Map}_{\mathrm{dSt}_k}(X, Y)^{\mathrm{an}} \longrightarrow \mathbf{Map}_{\mathrm{dAnSt}_k}(X^{\mathrm{an}}, Y^{\mathrm{an}})$$

is an equivalence. When both X and Y are proper underived schemes, Serre's GAGA theorem and a simple graph argument imply that

$$\mathrm{Hom}_{\mathrm{Sch}_k}(X, Y) \simeq \mathrm{Hom}_{\mathrm{An}_k}(X^{\mathrm{an}}, Y^{\mathrm{an}}) \, .$$

The relative version of the GAGA theorem implies that the same holds true for the hom schemes. Unfortunately, this argument breaks down when Y is no longer proper, or when Y is taken to be a stack (in both cases, it is the graph argument which fails). The idea to fix this problem was first introduced by Lurie [Lur04]. He allows X to be a proper Deligne–Mumford stack over $\mathbb{C}$ and Y to be a geometric stack satisfying several conditions making Tannakian reconstruction for Y possible. Lurie contents himself with proving that under these assumptions the canonical map

$$\mathrm{Map}_{\mathrm{St}_k}(X, Y) \longrightarrow \mathrm{Map}_{\mathrm{AnSt}_k}(X^{\mathrm{an}}, Y^{\mathrm{an}})$$

is an equivalence. Our goal is to generalize this result in several directions: firstly, we aim to prove a relative version of the result; that is, we consider mapping stacks rather than just mapping spaces. Secondly, we want to allow X to be a more general object than a Deligne–Mumford stack. For instance, in Section 7 we will be interested in the situation where X is S_{dR} for S a smooth and proper k-scheme. Finally, we want to allow X, Y and the mapping stacks to be derived.

The general strategy for the proof of our main theorem is the same as the one employed in [Lur04]. However, the technical tools used in [Lur04]d needed to be generalized and sharpened in order to apply to the situations we are concerned with. These improved tools also form the other main theorems of this paper. Notably, we are referring to Theorems 3.13, 4.15 and 5.5.

Throughout this section all ∞-categories are assumed to be k-linear (in the sense of [GH15]), and all functors are assumed to be k-linear. If $\mathcal{C}$ and $\mathcal{D}$ are two k-linear ∞-categories, we let $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ denote the ∞-category of *k-linear* and exact functors.

6.1 Analytification and Tannaka duality

We start this section by briefly reviewing the notion and the machinery of Tannaka duality. The main references for the algebraic theory are [Lur11b] and [Lur18, § III.9]. Our goal is to study how the Tannaka property interacts with the analytification functor. Recall that we have an

∞-functor

$$\mathrm{Perf}_k \colon \mathrm{dAff}_k^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty^{\mathrm{st},\otimes}$$

with values in stably symmetric monoidal ∞-categories that sends $\mathrm{Spec}(A)$ to the ∞-category $\mathrm{Perf}(A)$, equipped with its canonical symmetric monoidal structure. We also have at our disposal an ∞-functor

$$\mathrm{QCoh}_k \colon \mathrm{dAff}_k^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty^{\mathrm{st},\otimes}$$

sending $\mathrm{Spec}(A)$ to the ∞-category $\mathrm{QCoh}(A) \simeq A\text{-Mod}$ equipped with its natural symmetric monoidal structure. Given stably symmetric monoidal ∞-categories $\mathcal{C}$ and $\mathcal{D}$, we denote by $\mathrm{Fun}_{\mathrm{ex}}^{\otimes}(\mathcal{C}, \mathcal{D})$ the ∞-category of symmetric monoidal exact functors from $\mathcal{C}$ to $\mathcal{D}$. Recall that both Perf_k and QCoh_k satisfy étale descent. In particular, they extend to functors

$$\mathrm{Perf}_k, \mathrm{QCoh}_k \colon \mathrm{dSt}_k^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty^{\mathrm{st},\otimes} ,$$

and in particular for every pair of derived stacks locally almost of finite presentation, X and Y, we obtain morphisms

$$P \colon \mathrm{Map}_{\mathrm{dSt}_k}(X, Y) \longrightarrow \mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(X))$$

and

$$\widehat{P} \colon \mathrm{Map}_{\mathrm{dSt}_k}(X, Y) \longrightarrow \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(Y), \mathrm{QCoh}(X)) .$$

Definition 6.1. We say that a derived stack $Y \colon \mathrm{dAff}_k^{\mathrm{op}} \to \mathcal{S}$ is *weakly Tannakian* (or that Y *satisfies the weak Tannaka property*) if it satisfies the following condition:

(1) For any derived stack $X \colon \mathrm{dAff}_k^{\mathrm{op}} \to \mathcal{S}$, the ∞-functor

$$\widehat{P} \colon \mathrm{Map}_{\mathrm{dSt}_k}(X, Y) \longrightarrow \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(Y), \mathrm{QCoh}(X))$$

is fully faithful.

We say that a derived stack $Y \colon \mathrm{dAff}_k^{\mathrm{op}} \to \mathcal{S}$ is *Tannakian* (or that Y *satisfies the Tannaka property*) if it is weakly Tannakian and it satisfies the following supplementary condition:

(2) The essential image of the ∞-functor $\widehat{P}$ consists of exact symmetric monoidal functors $\mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ that commute with colimits and preserve both connective objects and flat objects.

Notice that a symmetric monoidal functor $F \colon \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ preserves dualizable objects. As $\mathrm{Perf}(Y)$ and $\mathrm{Perf}(X)$ coincide with the full subcategories of $\mathrm{QCoh}(Y)$ and $\mathrm{QCoh}(X)$ spanned exactly by dualizable objects, we conclude that each such functor gives rise to a functor $F \colon \mathrm{Perf}(Y) \to \mathrm{Perf}(X)$, which is again symmetric monoidal. This argument also shows that the inclusion $\mathrm{Perf}(X) \hookrightarrow \mathrm{QCoh}(X)$ induces an equivalence

$$\mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(X)) \xrightarrow{\sim} \mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{QCoh}(X)) .$$

Lemma 6.2. *Suppose that* $\mathrm{QCoh}(Y)$ *is compactly generated by perfect complexes; that is, we have* $\mathrm{QCoh}(Y) \simeq \mathrm{Ind}(\mathrm{Perf}(Y))$. *Then* Y *is weakly Tannakian if and only if the functor* $P \colon \mathrm{Map}_{\mathrm{dSt}_k}(X, Y) \to \mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(X))$ *is fully faithful.*

Proof. Under the assumption $\mathrm{QCoh}(Y) \simeq \mathrm{Ind}(\mathrm{Perf}(Y))$, we have

$$\mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(X)) \simeq \mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{QCoh}(X)) \simeq \mathrm{Fun}_L^{\otimes}(\mathrm{QCoh}(Y), \mathrm{QCoh}(X)) ,$$

where the subscript L denotes the full subcategory spanned by those functors that commute with arbitrary colimits. The conclusion follows. $\square$

Example 6.3. Theorem 9.3.0.3 of [Lur18] guarantees that if k is a field of characteristic zero and X is a geometric stack with affine diagonal, then it is Tannakian. Notice that the notion of geometric stack in loc. cit. is different from the one used in this paper. See Remark 2.12.

Remark 6.4. In this section we do *not* need to assume k to be of characteristic zero. Nevertheless, since all known criteria to guarantee that a derived stack X is Tannakian in the sense of Definition 6.1, it is likely that in order to apply our main theorem, Theorem 6.14, one needs to assume characteristic zero. As our main applications take place in the $\mathbb{C}$-analytic setting, this is not an issue for us.

Definition 6.1 is difficult to export to the analytic setting. The main obstruction is that there is no truly satisfactory notion of quasi-coherent sheaf in the analytic setting. (After the initial version of this paper was completed, a major breakthrough was obtained in this direction by Clausen and Scholze using the ∞-category of quasi-coherent liquid sheaves $C(X, X)$; see [CS22, Lecture VI].)

However, perfect complexes make perfect sense, and in particular for every pair of derived analytic stacks $X, Y \colon \mathrm{dAfd}_k^{\mathrm{op}} \to \mathcal{S}$, we have a natural map

$$P \colon \mathrm{Map}_{\mathrm{dAnSt}_k}(X, Y) \longrightarrow \mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(X)) .$$

In virtue of Lemma 6.2, we might be tempted to use the map P to define at least the weak Tannakian property in the analytic setting. However, this would still not be a satisfactory notion: indeed, the condition $\mathrm{QCoh}(Y) \simeq \mathrm{Ind}(\mathrm{Perf}(Y))$ is a rather strong one, which in particular implies that $\mathrm{Perf}(Y)$ is a saturated stable ∞-category in the sense of [TV08a]. Theorem 1.1 in [TV08a] implies that an analytic space X is algebraizable if and only if $\mathrm{Perf}(X)$ is saturated. In the case where Y actually comes from analytification, we can prove the following result.

THEOREM 6.5. *Let $Y \in \mathrm{dSt}_k^{\mathrm{afp}}$ be a derived stack locally almost of finite presentation. Suppose that*

(1) *the stable ∞-category $\mathrm{QCoh}(Y)$ satisfies $\mathrm{QCoh}(Y) \simeq \mathrm{Ind}(\mathrm{Perf}(Y))$;*
(2) *Y is Tannakian.*

Then for every $X \in \mathrm{dAnSt}_k$ the composition

$$\mathrm{Map}_{\mathrm{dAnSt}_k}(X, Y^{\mathrm{an}}) \xrightarrow{P} \mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y^{\mathrm{an}}), \mathrm{Perf}(X)) \longrightarrow \mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(X)) \qquad (6.6)$$

is fully faithful. Here the second map is the one induced by the analytification functor $\mathrm{Perf}(Y) \to \mathrm{Perf}(Y^{\mathrm{an}})$.

Proof. We adapt the strategy of Proposition 3.14 in the current setting. Let us denote the composite functor (6.6) by

$$\tau_{X,Y} \colon \mathrm{Map}_{\mathrm{dAnSt}_k}(X, Y^{\mathrm{an}}) \longrightarrow \mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(X)) .$$

This map is functorial in both X and Y. Notice that the left- and right-hand sides commute separately with colimits in $X \in \mathrm{dAnSt}_k$. We can therefore reduce to the case where X is a derived k-affinoid (respectively, Stein) space to begin with. Introduce the presheaf

$$T_Y \colon \mathrm{dAfd}_X^{\mathrm{op}} \longrightarrow \mathrm{Cat}_{\infty}$$

sending a map $U \to X$ to the ∞-category $\mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(U))$. Notice that T_Y is in fact a sheaf, and the maps $\tau_{U,Y}$ assemble into a natural transformation

$$\tau_Y \colon F_X^s(Y^{\mathrm{an}}) \longrightarrow T_Y .$$

Let $\overline{\tau}_Y$ denote the composition

$$\overline{\tau}_Y \colon G^p_X(Y) \longrightarrow G^s_X(Y) \xrightarrow{\alpha_Y} F^s_X\big(Y^{\mathrm{an}}\big) \xrightarrow{\tau_Y} T_Y ,$$

where α_Y is the map from Theorem 3.13. Now recall that the ∞-topos underlying X has enough points. Since full faithfulness can be tested on stalks and since $G^s_X(Y)$ is the sheafification of $G^p_X(Y)$ (hence they have the same stalks), we are reduced to proving that $\overline{\tau}_Y$ is fully faithful.

We start dealing with the k-analytic case. Fix $U \in \mathrm{dAfd}_X$, and let $A_U \coloneqq \Gamma(U; \mathcal{O}^{\mathrm{alg}}_U)$. Then, unwinding the definitions, we have

$$G^p_X(Y)(U) \simeq \mathrm{Map}_{\mathrm{dSt}_k}(\mathrm{Spec}(A_U), Y) .$$

Since Y is Tannakian and $\mathrm{QCoh}(Y) \simeq \mathrm{Ind}(\mathrm{Perf}(Y))$, the canonical map

$$P \colon \mathrm{Map}_{\mathrm{dSt}_k}(\mathrm{Spec}(A_U), Y) \longrightarrow \mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(A_U))$$

is fully faithful. On the other hand,

$$T_Y(U) \simeq \mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(U)) .$$

Using Lemma 4.6, we see that the global section functor $\Gamma(U; -)$ induces an equivalence

$$\mathrm{Perf}(U) \xrightarrow{\sim} \mathrm{Perf}(A_U) ,$$

whence the full faithfulness of $\overline{\tau}_Y$.

We now turn to the $\mathbb{C}$-analytic case. Using the correspondence provided by Theorem 2.15, we extend both $G^p_X(Y)$ and T_Y to compact subsets of X. We let once again $\overline{\tau}_Y$ denote the induced natural transformation between them. Using Lemma 2.16 we see that it is enough to prove that for every compact Stein subset $K \subset X$ of X, the natural map

$$\overline{\tau}_{Y,(K)_X} \colon G^p_X(Y)((K)_X) \longrightarrow T_Y((K)_X)$$

is fully faithful. Unravelling the definitions, we see that we have to prove that the functor

$$\operatorname*{colim}_{K \subset U \subset X} \mathrm{Map}_{\mathrm{dSt}_k}(\mathrm{Spec}(A_U), Y) \longrightarrow \operatorname*{colim}_{K \subset U \subset X} \mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(U)) \tag{6.7}$$

is fully faithful. Here the colimit is taken over all Stein open neighbourhoods U of K inside X. We start by dealing with the left-hand side. Since Y is Tannakian and $\mathrm{QCoh}(Y) \simeq \mathrm{Ind}(\mathrm{Perf}(Y))$, we have fully faithful embeddings

$$\mathrm{Map}_{\mathrm{dSt}_k}(\mathrm{Spec}(A_U), Y) \hookrightarrow \mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(A_U)) .$$

Let $\mathrm{Fun}^{\otimes}_{\mathrm{ind}}$ denote the mapping space in the ∞-category $\mathrm{Ind}(\mathrm{Cat}^{\mathrm{st},\otimes}_{\infty})$. Then we have a tautological equivalence

$$\operatorname*{colim}_{K \subset U \subset X} \mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(A_U)) \simeq \mathrm{Fun}^{\otimes}_{\mathrm{ind}}\big(\mathrm{Perf}(Y), \operatorname*{``colim"}_{K \subset U \subset X} \mathrm{Perf}(A_U)\big) .$$

On the other hand, we also have

$$\operatorname*{colim}_{K \subset U \subset X} \mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(U)) \simeq \mathrm{Fun}^{\otimes}_{\mathrm{ind}}\big(\mathrm{Perf}(Y), \operatorname*{``colim"}_{K \subset U \subset X} \mathrm{Perf}(U)\big) .$$

Notice that the functor $\Gamma_{(K)}$ of Theorem 4.13 induces an equivalence in $\mathrm{Ind}\big(\mathrm{Cat}^{\mathrm{st},\otimes}_{\infty}\big)$

$$\Gamma_{(K)} \colon \operatorname*{``colim"}_{K \subset U \subset X} \mathrm{Perf}(U) \xrightarrow{\sim} \operatorname*{``colim"}_{K \subset U \subset X} \mathrm{Perf}(A_U) ,$$

making the diagram

$$\begin{array}{ccc} \operatorname{colim}_{K\subset U\subset X} \operatorname{Map}_{\mathrm{dSt}_{\mathbb{C}}}(\operatorname{Spec}(A_U), Y) & \longrightarrow & \operatorname{colim}_{K\subset U\subset X} \operatorname{Map}_{\mathrm{dAnSt}_{\mathbb{C}}}(U, Y^{\mathrm{an}}) \\ \downarrow & & \downarrow \\ \operatorname{Fun}^{\otimes}_{\mathrm{ind}}\big(\operatorname{Perf}(Y), \text{“colim”}_{K\subset U\subset X} \operatorname{Perf}(A_U)\big) & \xrightarrow{\Gamma_{(K)}} & \operatorname{Fun}^{\otimes}_{\mathrm{ind}}\big(\operatorname{Perf}(Y), \text{“colim”}_{K\subset U\subset X} \operatorname{Perf}(U)\big) \end{array}$$

commutative. Notice that the composition of the right vertical functor with the top horizontal one coincides with (6.7). As the left vertical arrow is fully faithful and the bottom horizontal one is an equivalence, we can therefore conclude that (6.7) is fully faithful, thus completing the proof. □

Corollary 6.8. *Let $Y \in \mathrm{dSt}_k^{\mathrm{afp}}$ be a derived stack locally almost of finite presentation. Suppose that*

(1) *the stable ∞-category $\operatorname{QCoh}(Y)$ is compactly generated by perfect complexes; that is, $\operatorname{QCoh}(Y) \simeq \operatorname{Ind}(\operatorname{Perf}(Y))$;*
(2) *Y satisfies the GAGA property (cf. Definition 5.1(1));*
(3) *Y is Tannakian.*

Then for every derived analytic stack $X\colon \mathrm{dAfd}_k^{\mathrm{op}} \to \mathcal{S}$, the functor

$$P\colon \operatorname{Map}_{\mathrm{dAnSt}_k}(X, Y^{\mathrm{an}}) \longrightarrow \operatorname{Fun}^{\otimes}\big(\operatorname{Perf}(Y^{\mathrm{an}}), \operatorname{Perf}(X)\big)$$

is fully faithful.

Proof. Since Y satisfies the GAGA property, we see that the analytification functor $\operatorname{Perf}(Y) \to \operatorname{Perf}(Y^{\mathrm{an}})$ is an equivalence of ∞-categories. In particular, we are reduced to checking that the composition

$$\operatorname{Map}_{\mathrm{dAnSt}_k}(X, Y^{\mathrm{an}}) \xrightarrow{P} \operatorname{Fun}^{\otimes}\big(\operatorname{Perf}(Y^{\mathrm{an}}), \operatorname{Perf}(X)\big) \longrightarrow \operatorname{Fun}^{\otimes}(\operatorname{Perf}(Y), \operatorname{Perf}(X))$$

is fully faithful. We have shown in Theorem 6.5 that this is true even without the assumption that Y satisfies the GAGA property. □

In the proof of Theorem 6.14, we will need some control over the essential image of the functor

$$\operatorname{Map}_{\mathrm{dAnSt}_k}(X, Y^{\mathrm{an}}) \longrightarrow \operatorname{Fun}^{\otimes}(\operatorname{Perf}(Y), \operatorname{Perf}(X))$$

that we just proved is fully faithful. Already in the algebraic case, it is unreasonable to expect a characterization of the essential image (unless more restrictive hypotheses are formulated on Y; see [Bha16, Theorem 2.1]). Under the assumption $\operatorname{QCoh}(Y) \simeq \operatorname{Ind}(\operatorname{Perf}(Y))$, we have an equivalence

$$\operatorname{Fun}^{\otimes}_{L}(\operatorname{Perf}(Y), \mathcal{O}_X\text{-Mod}) \simeq \operatorname{Fun}^{\otimes}_{L}(\operatorname{QCoh}(Y), \mathcal{O}_X\text{-Mod})\,.$$

It is then much more reasonable to expect to be able to characterize the essential image of the fully faithful functor

$$\operatorname{Map}_{\mathrm{dAnSt}_k}(X, Y^{\mathrm{an}}) \longrightarrow \operatorname{Fun}^{\otimes}_{L}(\operatorname{QCoh}(Y), \mathcal{O}_X\text{-Mod})\,,$$

As usual, there is a difference between the k-analytic case and the $\mathbb{C}$-analytic one: in the former, we do obtain a characterization of the above functor whenever $X \in \mathrm{dAfd}_k$; in the latter, however, we are forced to replace X with a compact Stein subspace.

We start with the following definition.

Definition 6.9. Let $X \in \mathrm{dAfd}_k$ be a derived k-affinoid (respectively, Stein) space. We say that an object $\mathcal{F} \in \mathcal{O}_X\text{-Mod}$ is *flat* if for every $\mathcal{G} \in \mathrm{Coh}^\heartsuit(X)$ the tensor product $\mathcal{F} \otimes \mathcal{G}$ belongs to $\mathcal{O}_X\text{-Mod}^\heartsuit$.

Lemma 6.10. *Let $Y \in \mathrm{dSt}_k^{\mathrm{afp}}$ be a derived stack locally almost of finite presentation. Suppose that Y satisfies the same assumptions as in Theorem 6.5 and that it is moreover geometric. Then for every derived k-affinoid (respectively, Stein) space $X \in \mathrm{dAfd}_k$, the essential image of the functor*

$$\mathrm{Map}_{\mathrm{dAnSt}_k}\big(X, Y^{\mathrm{an}}\big) \longrightarrow \mathrm{Fun}_L^{\otimes}(\mathrm{QCoh}(Y), \mathcal{O}_X\text{-Mod})$$

factors through the full subcategory spanned by those functors $F \in \mathrm{Fun}_L^{\otimes}(\mathrm{QCoh}(Y), \mathcal{O}_X\text{-Mod})$ that preserve flat objects and connective objects and take the full subcategory $\mathrm{Perf}(Y)$ of $\mathrm{QCoh}(Y)$ to $\mathrm{Perf}(X)$.

Proof. Let $f\colon X \to Y^{\mathrm{an}}$ be a given morphism. Then its image F in $\mathrm{Fun}_L^{\otimes}(\mathrm{QCoh}(Y), \mathcal{O}_X\text{-Mod})$ is obtained by extending by colimits along $\mathrm{Perf}(Y) \hookrightarrow \mathrm{QCoh}(Y)$ the composition

$$\mathrm{Perf}(Y) \longrightarrow \mathrm{Perf}\big(Y^{\mathrm{an}}\big) \xrightarrow{f^*} \mathrm{Perf}(X) \longrightarrow \mathcal{O}_X\text{-Mod}\,.$$

Since $\mathrm{Perf}(Y) \hookrightarrow \mathrm{QCoh}(Y)$ is fully faithful, we see that F takes $\mathrm{Perf}(Y)$ to $\mathrm{Perf}(X)$ by construction.

For the other statements we proceed by induction on the geometric level of Y. First suppose that $Y = \mathrm{Spec}(A)$ is affine. Let $\mathcal{F} \in \mathrm{QCoh}(Y)$ be a flat object. In this case Lazard's theorem [Lur17, Theorem 7.2.2.15(1)] implies that $\mathcal{F}$ can be written as filtered colimit of finitely generated free A-modules. In particular, $f^*(\mathcal{F})$ can also be written as filtered colimit of free $\mathcal{O}_X$-modules, and hence it is flat. Similarly, [Lur17, Proposition 1.4.4.11] implies that $\mathrm{QCoh}(Y)^{\geqslant 0}$ coincides with the smallest full subcategory of $\mathrm{QCoh}(Y)$ closed under colimits and extensions and containing A. Since the functor $f^*\colon \mathrm{QCoh}(Y) \to \mathcal{O}_X\text{-Mod}$ is exact and commutes with filtered colimits, it commutes with arbitrary colimits. The conclusion now follows from the fact that connective objects in $\mathcal{O}_X\text{-Mod}$ are stable under colimits.

Now assume that the statements have been proven for n-geometric derived stacks, and let Y be an $(n+1)$-geometric derived stack. Let $u\colon U \to Y$ be a smooth atlas, and let $U_\bullet$ be its Čech nerve. Then

$$\big|U_\bullet^{\mathrm{an}}\big| \simeq Y^{\mathrm{an}}\,.$$

Given $f\colon X \to Y^{\mathrm{an}}$, we therefore see that, up to a cover $V \to X$, we can suppose that f factors through U^{an}:

$$\begin{array}{ccccc} & & & \overset{g}{\dashrightarrow} & U^{\mathrm{an}} \\ & & & & \big\downarrow{\scriptstyle u^{\mathrm{an}}} \\ V & \longrightarrow & X & \xrightarrow{f} & Y^{\mathrm{an}}\,. \end{array}$$

Since we can check that f^* commutes with flat objects and connective objects locally on X, we can assume from the very beginning that f factors as $f \simeq u^{\mathrm{an}} \circ g$. The inductive hypothesis guarantees that g^* commutes with flat objects and connective objects. The same is true for $(u^{\mathrm{an}})^*$ because u is a smooth atlas and the analytification functor $\mathrm{QCoh}(U) \to \mathcal{O}_{U^{\mathrm{an}}}\text{-Mod}$ commutes with flat objects and connective objects. Therefore, the conclusion follows. □

Our goal is to prove that the converse to Lemma 6.10 holds. We start by dealing with the non-archimedean case, where the converse holds literally.

Proposition 6.11. *Let k be a non-archimedean field, and let $Y \in \mathrm{dSt}_k^{\mathrm{afp}}$ be a derived k-stack locally almost of finite presentation. Suppose that Y satisfies the same assumptions as in Theorem 6.5. Then for every derived k-affinoid space $X \in \mathrm{dAfd}_k$, the essential image of the functor*

$$\mathrm{Map}_{\mathrm{dAnSt}_k}\left(X, Y^{\mathrm{an}}\right) \longrightarrow \mathrm{Fun}_L^{\otimes}(\mathrm{QCoh}(Y), \mathcal{O}_X\text{-Mod})$$

contains all those functors that preserve flat objects and connective objects and take the full subcategory $\mathrm{Perf}(Y)$ of $\mathrm{QCoh}(Y)$ to $\mathrm{Perf}(X)$.

Proof. Let $A := \Gamma\left(X; \mathcal{O}_X^{\mathrm{alg}}\right)$. Lemma 4.6 provides us with a canonical equivalence $\mathrm{Perf}(X) \simeq \mathrm{Perf}(A)$. Let $F \in \mathrm{Fun}_L^{\otimes}(\mathrm{QCoh}(Y), \mathcal{O}_X\text{-Mod})$ be a functor satisfying the conditions in the statement of the proposition. By assumption, F restricts to a symmetric monoidal functor $\overline{F}\colon \mathrm{Perf}(Y) \to \mathrm{Perf}(X)$. Using the above equivalence, we can redefine $\overline{F}$ as a functor $\overline{F}\colon \mathrm{Perf}(Y) \to \mathrm{Perf}(A)$. Consider the extension $\widetilde{F}\colon \mathrm{QCoh}(Y) \to A\text{-Mod}$. The functor $\varepsilon_X^*\colon A\text{-Mod} \to \mathcal{O}_X\text{-Mod}$ which we defined in Section 4.2 commutes with filtered colimits. As a consequence, we can identify the composition

$$\mathrm{QCoh}(Y) \xrightarrow{\widetilde{F}} A\text{-Mod} \xrightarrow{\varepsilon_X^*} \mathcal{O}_X\text{-Mod}$$

with the original functor F. Since ε_X^* is conservative, strong monoidal, t-exact and preserves flat and coherent objects, we see that $\widetilde{F}$ preserves flat objects and connective objects. Therefore, we can use the Tannakian property of Y to see that $\widetilde{F}$ comes from a map $\widetilde{f}\colon \mathrm{Spec}(A) \to Y$. Let $f\colon X \to Y^{\mathrm{an}}$ be the image of $\widetilde{f}$ via the canonical map

$$\mathrm{Map}_{\mathrm{dSt}_k}(\mathrm{Spec}(A), Y) = G_X^p(Y)(X) \longrightarrow F_X^s\left(Y^{\mathrm{an}}\right) = \mathrm{Map}_{\mathrm{dAnSt}_k}\left(X, Y^{\mathrm{an}}\right).$$

Since the diagram

$$\begin{array}{ccc}
\mathrm{Map}_{\mathrm{dSt}_k}(\mathrm{Spec}(A), Y) & \longrightarrow & \mathrm{Map}_{\mathrm{dAnSt}_k}\left(X, Y^{\mathrm{an}}\right) \\
\downarrow & & \downarrow \\
\mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(A)) & \longrightarrow & \mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(X)) \\
\downarrow & & \downarrow \\
\mathrm{Fun}_L^{\otimes}(\mathrm{QCoh}(Y), A\text{-Mod}) & \longrightarrow & \mathrm{Fun}_L^{\otimes}(\mathrm{QCoh}(Y), \mathcal{O}_X\text{-Mod})
\end{array}$$

commutes, the conclusion follows. □

Proposition 6.12. *Let $Y \in \mathrm{dSt}_{\mathbb{C}}^{\mathrm{afp}}$ be a derived stack locally almost of finite presentation satisfying the same assumptions as in Theorem 6.5. Let $X \in \mathrm{dAfd}_{\mathbb{C}}$ be a derived Stein space, and let $V \Subset U \Subset X$ be a nested sequence of relatively compact Stein subspaces. Let $F \in \mathrm{Fun}_L^{\otimes}(\mathrm{QCoh}(Y), \mathcal{O}_X\text{-Mod})$ be a functor preserving perfect complexes, flat objects and connective objects. Then there exists a map $f\colon V \to Y^{\mathrm{an}}$ so that the diagram*

$$\begin{array}{ccc}
\mathrm{QCoh}(Y) & \xrightarrow{F} & \mathcal{O}_X\text{-Mod} \\
\downarrow & & \downarrow \\
\mathcal{O}_{Y^{\mathrm{an}}}\text{-Mod} & \xrightarrow{f^*} & \mathcal{O}_V\text{-Mod}
\end{array}$$

commutes.

Proof. Since $F\colon \mathrm{QCoh}(Y) \to \mathcal{O}_X\text{-Mod}$ preserves perfect complexes, it restricts to a functor $\mathrm{Perf}(Y) \to \mathrm{Perf}(X)$. Furthermore, since $\mathrm{QCoh}(Y) \simeq \mathrm{Ind}(\mathrm{Perf}(Y))$ by assumption, we see that the extension of

$$\mathrm{Perf}(Y) \longrightarrow \mathrm{Perf}(X) \longrightarrow \mathcal{O}_X\text{-Mod}$$

along $\mathrm{Perf}(Y) \hookrightarrow \mathrm{QCoh}(Y)$ coincides with F. Let $A_X \coloneqq \Gamma(X; \mathcal{O}_X^{\mathrm{alg}})$, $A_U \coloneqq \Gamma(U; \mathcal{O}_U^{\mathrm{alg}})$ and $A_V \coloneqq \Gamma(V; \mathcal{O}_V^{\mathrm{alg}})$. Using Lemma 4.10, we obtain a well-defined functor

$$\overline{F} \colon \mathrm{Perf}(Y) \longrightarrow \mathrm{Perf}(X) \longrightarrow \mathrm{Perf}(A_U)\,.$$

Consider the functor $\widetilde{F} \colon \mathrm{QCoh}(Y) \to A_U\text{-Mod}$ obtained by extending by filtered colimits the composition

$$\mathrm{Perf}(Y) \xrightarrow{\overline{F}} \mathrm{Perf}(A_U) \hookrightarrow A_U\text{-Mod}$$

along the inclusion $\mathrm{Perf}(Y) \hookrightarrow \mathrm{QCoh}(Y)$. Lemma 4.12 implies that the composition

$$\mathrm{Perf}(X) \xrightarrow{\Gamma(U;-)} \mathrm{Perf}(A_U) \longrightarrow \mathrm{Perf}(A_V) \xrightarrow{\varepsilon_V^*} \mathrm{Perf}(V)$$

coincides with the restriction functor $\mathrm{Perf}(X) \to \mathrm{Perf}(V)$. It follows that the outer diagram in

$$\begin{array}{ccc} \mathrm{Perf}(Y) & \longrightarrow & \mathrm{Perf}(X) \\ \downarrow & & \downarrow \\ \mathrm{Perf}(A_U) & \xrightarrow{\varepsilon_U^*} & \mathrm{Perf}(U) \\ \downarrow & & \downarrow \\ \mathrm{Perf}(A_V) & \xrightarrow{\varepsilon_V^*} & \mathrm{Perf}(V) \end{array}$$

commutes. In turn, this implies that the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(Y) & \xrightarrow{F} & \mathcal{O}_X\text{-Mod} \\ \downarrow{\scriptstyle \widetilde{F}} & & \downarrow \\ A_V\text{-Mod} & \xrightarrow{\varepsilon_V^*} & \mathcal{O}_V\text{-Mod} \end{array} \tag{6.13}$$

commutes. Now recall from Proposition 4.5 that the functor ε_V^* commutes with filtered colimits, and it is conservative, t-exact and preserves flat, coherent and connective objects. Therefore, we deduce that the symmetric monoidal functor $\widetilde{F} \colon \mathrm{QCoh}(Y) \to A_V\text{-Mod}$ commutes with colimits and preserves flat objects and connective objects. In particular, [Lur18, Theorem 9.3.0.3] implies the existence of a map $g \colon \mathrm{Spec}(A_V) \to Y$ so that $\widetilde{F} \simeq g^*$. The map g defines an element in $G_X^p(Y)(V)$. Let f be the image of g via the canonical map

$$G_X^p(Y)(V) \longrightarrow G_X^s(Y)(V) \simeq F_X^s(Y^{\mathrm{an}})(V) \simeq \mathrm{Map}_{\mathrm{dAnSt}_k}(V, Y^{\mathrm{an}})\,.$$

Then unravelling the definitions, we see that the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(Y) & \xrightarrow{\widetilde{F}} & A_V\text{-Mod} \\ \downarrow & & \downarrow{\scriptstyle \widetilde{\varepsilon_V^*}} \\ \mathcal{O}_{Y^{\mathrm{an}}}\text{-Mod} & \xrightarrow{f^*} & \mathcal{O}_V\text{-Mod} \end{array}$$

commutes. Combining this with the commutativity of the diagram (6.13), we see that the conclusion follows. □

6.2 Mapping stacks with Tannakian target

We now turn to the main result of this paper.

Theorem 6.14. *Let $X, Y \in \mathrm{dSt}_k^{\mathrm{afp}}$ be a derived stacks locally almost of finite presentation. Assume that*

(1) *the stack Y is a geometric, Tannakian stack such that $\mathrm{QCoh}(Y) \simeq \mathrm{Ind}(\mathrm{Perf}(Y))$;*
(2) *the mapping stack $\mathbf{Map}(X,Y)$ is locally geometric and locally almost of finite presentation.*

If X satisfies the GAGA property, then the canonical map

$$\mathrm{Map}(X,Y) \longrightarrow \mathrm{Map}\big(X^{\mathrm{an}}, Y^{\mathrm{an}}\big)$$

is an equivalence. Furthermore, if X satisfies the universal GAGA property, then the canonical map

$$\mathbf{Map}(X,Y)^{\mathrm{an}} \longrightarrow \mathbf{AnMap}\big(X^{\mathrm{an}}, Y^{\mathrm{an}}\big)$$

is an equivalence.

Proof. We first observe that when $U = \mathrm{Sp}(k)$, we have

$$\mathbf{Map}(X,Y)^{\mathrm{an}}(U) \simeq \mathrm{Map}(X,Y)\,,$$

so the first statement follows from the proof of the second.[9]

Now let $U \in \mathrm{dAfd}_k$ be a derived k-affinoid (respectively, Stein) space. Assume that X satisfies the GAGA property relative to U. We will prove that the natural morphism

$$\mathbf{Map}(X,Y)^{\mathrm{an}}(U) \longrightarrow \mathbf{AnMap}\big(X^{\mathrm{an}}, Y^{\mathrm{an}}\big)(U)$$

is an equivalence. We work on $\mathcal{X}_U$. Using Lemma 3.7, we are reduced to checking that the natural morphism

$$F_U^s\big(\mathbf{Map}(X,Y)^{\mathrm{an}}\big) \longrightarrow F_U^s\big(\mathbf{AnMap}\big(X^{\mathrm{an}}, Y^{\mathrm{an}}\big)\big)$$

is an equivalence. Using Theorem 3.13 and the fact that $\mathbf{Map}(X,Y)$ is geometric, we obtain a natural equivalence

$$G_U^s(\mathbf{Map}(X,Y)) \longrightarrow F_U^s\big(\mathbf{Map}(X,Y)^{\mathrm{an}}\big)\,.$$

We consider as in Proposition 3.14 the induced map

$$G_U^p(\mathbf{Map}(X,Y)) \longrightarrow F_U^s\big(\mathbf{AnMap}\big(X^{\mathrm{an}}, Y^{\mathrm{an}}\big)\big)\,. \tag{6.15}$$

We first deal with the non-archimedean case. In this case we claim that the map (6.15) is an equivalence. Fix an étale map $V \to U$ from a derived k-affinoid space V. Let

$$A_V \coloneqq \Gamma\big(V; \mathcal{O}_V^{\mathrm{alg}}\big)\,.$$

Then

$$G_U^p(\mathbf{Map}(X,Y))(V) \simeq \mathrm{Map}_{\mathrm{dSt}_k}(\mathrm{Spec}(A_V), \mathbf{Map}(X,Y)) \simeq \mathrm{Map}_{\mathrm{dSt}_k}(\mathrm{Spec}(A_V) \times X, Y)\,.$$

Since Y is Tannakian, we have a fully faithful embedding

$$Q\colon\ \mathrm{Map}_{\mathrm{dSt}_k}(\mathrm{Spec}(A_V) \times X, Y) \hookrightarrow \mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(\mathrm{Spec}(A_V) \times X)\,.$$

On the other hand, Theorem 6.5 provides us with a fully faithful embedding

$$\begin{aligned} Q^{\mathrm{an}}\colon\ & F_U^s\big(\mathbf{AnMap}\big(X^{\mathrm{an}}, Y^{\mathrm{an}}\big)\big)(V) \\ & \simeq \mathrm{Map}_{\mathrm{dAnSt}_k}\big(V \times X^{\mathrm{an}}, Y^{\mathrm{an}}\big) \hookrightarrow \mathrm{Fun}^{\otimes}\big(\mathrm{Perf}(Y), \mathrm{Perf}\big(V \times X^{\mathrm{an}}\big)\big)\,. \end{aligned}$$

[9] Note that since $\mathrm{Sp}(\mathbb{C})$ is a compact Stein space (and may be considered as a constant pro-object), this is indeed a special case of our result. Of course a direct proof in this case would be significantly easier than in the relative case that is our main interest.

Now, since X satisfies the universal GAGA property, the functor

$$\varepsilon^*_{X,V} \colon \mathrm{Perf}(\mathrm{Spec}(A_V) \times X) \simeq \mathrm{Perf}\big(V \times X^{\mathrm{an}}\big)$$

is an equivalence. We therefore obtain the following diagram:

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{dSt}_k}(\mathrm{Spec}(A_V) \times X, Y) & \longrightarrow & \mathrm{Map}_{\mathrm{dAnSt}_k}\big(V \times X^{\mathrm{an}}, Y^{\mathrm{an}}\big) \\ \big\downarrow Q & & \big\downarrow Q^{\mathrm{an}} \\ \mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(\mathrm{Spec}(A_V) \times X)) & \xrightarrow[\varepsilon^*_{X,V}]{\sim} & \mathrm{Fun}^{\otimes}\big(\mathrm{Perf}(Y), \mathrm{Perf}\big(V \times X^{\mathrm{an}}\big)\big) . \end{array}$$

This immediately implies that the top horizontal map is fully faithful. We are therefore left to check that it is essentially surjective. To do so, we fix a morphism

$$f \colon V \times X^{\mathrm{an}} \longrightarrow Y^{\mathrm{an}} .$$

Consider the extended diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{dSt}_k}(\mathrm{Spec}(A_V) \times X, Y) & \longrightarrow & \mathrm{Map}_{\mathrm{dAnSt}_k}\big(V \times X^{\mathrm{an}}, Y^{\mathrm{an}}\big) \\ \big\downarrow Q & & \big\downarrow Q^{\mathrm{an}} \\ \mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(\mathrm{Spec}(A_V) \times X)) & \xrightarrow[\varepsilon^*_{X,V}]{\sim} & \mathrm{Fun}^{\otimes}\big(\mathrm{Perf}(Y), \mathrm{Perf}\big(V \times X^{\mathrm{an}}\big)\big) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Fun}^{\otimes}_L(\mathrm{QCoh}(Y), \mathrm{QCoh}(\mathrm{Spec}(A_V) \times X)) & \xrightarrow[\varepsilon^*_{X,V}]{} & \mathrm{Fun}^{\otimes}_L(\mathrm{QCoh}(Y), \mathcal{O}_{V \times X^{\mathrm{an}}}\text{-Mod}) . \end{array}$$

Let

$$F \colon \mathrm{Perf}(Y) \longrightarrow \mathrm{Perf}(\mathrm{Spec}(A_V) \times X)$$

be the functor corresponding to $f^* \colon \mathrm{Perf}(Y) \to \mathrm{Perf}\big(V \times X^{\mathrm{an}}\big)$ via the equivalence $\varepsilon^*_{X,V}$, and let

$$\widetilde{F} \colon \mathrm{QCoh}(Y) \longrightarrow \mathrm{QCoh}(\mathrm{Spec}(A_V) \times X)$$

be the functor obtained by extension along $\mathrm{Perf}(Y) \hookrightarrow \mathrm{QCoh}(Y) \simeq \mathrm{Ind}(\mathrm{Perf}(Y))$. Since Y is Tannakian, it is enough to check that $\widetilde{F}$ commutes with flat objects and connective objects. We observe that $\varepsilon^*_{X,V} \circ \widetilde{F} \simeq \widetilde{f}^*$, where

$$\widetilde{f}^* \colon \mathrm{QCoh}(Y) \longrightarrow \mathcal{O}_{V \times X^{\mathrm{an}}}\text{-Mod}$$

is the functor obtained by left Kan extension of f^* along $\mathrm{Perf}(Y) \hookrightarrow \mathrm{Ind}(\mathrm{Perf}(Y)) \simeq \mathrm{QCoh}(Y)$. As such, Lemma 6.10 implies that it preserves flat objects and connective objects. We now observe that since X satisfies GAGA relative to V, the functor

$$\varepsilon^*_{X,V} \colon \mathrm{QCoh}(\mathrm{Spec}(A_V) \times X) \longrightarrow \mathcal{O}_{V \times X^{\mathrm{an}}}\text{-Mod}$$

is conservative and t-exact. As flatness and connectivity of an object $\mathcal{F} \in \mathrm{QCoh}(\mathrm{Spec}(A_V) \times X)$ can be checked locally with respect to X, we conclude that $\mathcal{F}$ is flat (respectively, connective) if and only if $\varepsilon^*_{X,V}(\mathcal{F})$ is flat (respectively, connective). Since we know that $\widetilde{f}^*$ preserves flat objects and connective objects, it follows that the same is true for $\widetilde{F}$. Since Y is Tannakian, we see that F comes from a morphism $\overline{f} \colon \mathrm{Spec}(A_V) \times X \to Y$. Moreover, the commutativity of the diagram implies that $\overline{f}^{\mathrm{an}} \simeq f$. This completes the proof in the non-archimedean case.

We now deal with the $\mathbb{C}$-analytic case. We adapt the idea of Proposition 3.14 to this context. As usual, the overall strategy stays the same, but instead of proving that the map (6.15) is an equivalence on the spot, we prove that it becomes an equivalence after passing (via Theorem 2.15) to compact Stein subsets of U. We will conclude by Lemma 2.16. Therefore, let $K \subset U$ be a compact Stein in U, and let $(K)_U$ be the associated pro-object in $\mathrm{dAnSt}_{\mathbb{C}}$. If $V \subset U$ is an open Stein subspace, we let $A_V \coloneqq \Gamma\big(V; \mathcal{O}_V^{\mathrm{alg}}\big)$. Then, unravelling the definitions, we have

$$G_U^p(\mathbf{Map}(X,Y))((K)_U) \simeq \operatorname*{colim}_{K \subset V \subset U} \mathrm{Map}_{\mathrm{dSt}_{\mathbb{C}}}(\mathrm{Spec}(A_V) \times X, Y)\,,$$

where the colimit ranges over all the open Stein neighbourhoods of K inside U. Since Y is Tannakian, we have a fully faithful embedding

$$Q_V \colon \mathrm{Map}_{\mathrm{dSt}_{\mathbb{C}}}(\mathrm{Spec}(A_V) \times X, Y) \hookrightarrow \mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(\mathrm{Spec}(A_V) \times X))\,.$$

Since fully faithful functors are stable under filtered colimits, we obtain a fully faithful inclusion

$$Q_{(K)} \colon \operatorname*{colim}_{K \subset V \subset U} \mathrm{Map}_{\mathrm{dSt}_{\mathbb{C}}}(\mathrm{Spec}(A_V) \times X, Y) \hookrightarrow \operatorname*{colim}_{K \subset V \subset U} \mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(\mathrm{Spec}(A_V) \times X)\,.$$

We work as in Theorem 6.5 and introduce $\mathrm{Fun}^{\otimes}_{\mathrm{ind}}$, the mapping space in $\mathrm{Ind}\big(\mathrm{Cat}_\infty^{\mathrm{st},\otimes}\big)$. Then we have a tautological equivalence

$$\operatorname*{colim}_{K \subset V \subset U} \mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(\mathrm{Spec}(A_V) \times X)) \simeq \mathrm{Fun}^{\otimes}_{\mathrm{ind}}\Big(\mathrm{Perf}(Y), \text{“}\operatorname*{colim}_{K \subset V \subset U}\text{”}\, \mathrm{Perf}(\mathrm{Spec}(A_V) \times X)\Big)\,.$$

On the other hand, Theorem 6.5 provides us with fully faithful embeddings

$$Q_V^{\mathrm{an}} \colon \mathrm{Map}_{\mathrm{dAnSt}_{\mathbb{C}}}\big(V \times X^{\mathrm{an}}, Y^{\mathrm{an}}\big) \hookrightarrow \mathrm{Fun}^{\otimes}\big(\mathrm{Perf}(Y), \mathrm{Perf}\big(V \times X^{\mathrm{an}}\big)\big)\,.$$

These embeddings assemble into a fully faithful functor

$$Q_{(K)}^{\mathrm{an}} \colon \operatorname*{colim}_{K \subset V \subset U} \mathrm{Map}_{\mathrm{dAnSt}_{\mathbb{C}}}\big(V \times X^{\mathrm{an}}, Y^{\mathrm{an}}\big) \hookrightarrow \operatorname*{colim}_{K \subset V \subset U} \mathrm{Fun}^{\otimes}\big(\mathrm{Perf}(Y), \mathrm{Perf}\big(V \times X^{\mathrm{an}}\big)\big)\,.$$

Once again, we can formally rewrite

$$\operatorname*{colim}_{K \subset V \subset U} \mathrm{Fun}^{\otimes}\big(\mathrm{Perf}(Y), \mathrm{Perf}\big(V \times X^{\mathrm{an}}\big)\big) \simeq \mathrm{Fun}^{\otimes}_{\mathrm{ind}}\Big(\mathrm{Perf}(Y), \text{“}\operatorname*{colim}_{K \subset V \subset U}\text{”}\, \mathrm{Perf}\big(V \times X^{\mathrm{an}}\big)\Big)\,.$$

Now consider the following commutative diagram:

$$\begin{array}{ccc}
\operatorname*{colim}_{K \subset V \subset U} \mathrm{Map}_{\mathrm{dSt}_{\mathbb{C}}}(\mathrm{Spec}(A_V) \times X, Y) & \longrightarrow & \operatorname*{colim}_{K \subset V \subset U} \mathrm{Map}_{\mathrm{dAnSt}_{\mathbb{C}}}\big(V \times X^{\mathrm{an}}, Y^{\mathrm{an}}\big) \\
\big\downarrow Q_{(K)} & & \big\downarrow Q_{(K)}^{\mathrm{an}} \\
\mathrm{Fun}^{\otimes}_{\mathrm{ind}}\Big(\mathrm{Perf}(Y), \text{“}\operatorname*{colim}_{K \subset V \subset U}\text{”}\, \mathrm{Perf}(\mathrm{Spec}(A_V) \times X)\Big) & \longrightarrow & \mathrm{Fun}^{\otimes}_{\mathrm{ind}}\Big(\mathrm{Perf}(Y), \text{“}\operatorname*{colim}_{K \subset V \subset U}\text{”}\, \mathrm{Perf}\big(V \times X^{\mathrm{an}}\big)\Big)\,.
\end{array}$$

Recall that X satisfies the GAGA property relative to U (see Definition 5.1(3)). In other words, the canonical morphism

$$\text{“}\operatorname*{colim}_{K \subset V \subset U}\text{”}\, \mathrm{Perf}(\mathrm{Spec}(A_V) \times X) \longrightarrow \text{“}\operatorname*{colim}_{K \subset V \subset U}\text{”}\, \mathrm{Perf}\big(V \times X^{\mathrm{an}}\big)$$

is an equivalence in $\mathrm{Ind}\big(\mathrm{Cat}_\infty^{\mathrm{st},\otimes}\big)$. This implies that the bottom horizontal arrow in the above diagram is an equivalence. As $Q_{(K)}$ and $Q_{(K)}^{\mathrm{an}}$ are fully faithful embeddings, we deduce that the top horizontal morphism is fully faithful as well. We are therefore left to check that it is essentially surjective as well. Let

$$[f] \in \operatorname*{colim}_{K \subset V \subset U} \mathrm{Map}_{\mathrm{dAnSt}_{\mathbb{C}}}\big(V \times X^{\mathrm{an}}, Y^{\mathrm{an}}\big)$$

be any element, and let $f\colon V \times X^{\mathrm{an}} \to Y^{\mathrm{an}}$ be a representative for $[f]$. By construction, we have

$$\begin{aligned} Q^{\mathrm{an}}_{(K)}([f]) &\simeq [f^*] \in \mathrm{Fun}^{\otimes}_{\mathrm{ind}}\big(\mathrm{Perf}(Y), \underset{K \subset V \subset U}{\text{“colim”}} \mathrm{Perf}\big(V \times X^{\mathrm{an}}\big)\big) \\ &\simeq \underset{K \subset V \subset U}{\mathrm{colim}} \mathrm{Fun}^{\otimes}\big(\mathrm{Perf}(Y), \mathrm{Perf}\big(V \times X^{\mathrm{an}}\big)\big) \,. \end{aligned}$$

Via the equivalence

$$\begin{aligned} \varepsilon^*_{X,(K)}\colon \ & \mathrm{Fun}^{\otimes}_{\mathrm{ind}}\big(\mathrm{Perf}(Y), \underset{K \subset V \subset U}{\text{“colim”}} \mathrm{Perf}(\mathrm{Spec}(A_V) \times X)\big) \\ & \longrightarrow \mathrm{Fun}^{\otimes}_{\mathrm{ind}}\big(\mathrm{Perf}(Y), \underset{K \subset V \subset U}{\text{“colim”}} \mathrm{Perf}\big(V \times X^{\mathrm{an}}\big)\big) \,, \end{aligned}$$

we can select an open Stein neighbourhood W of K and a symmetric monoidal functor

$$F_W \colon \mathrm{Perf}(Y) \longrightarrow \mathrm{Perf}(\mathrm{Spec}(A_W) \times X)$$

such that $[\varepsilon^*_{X,W} \circ F_W] \simeq [f^*]$. Without loss of generality, we can suppose that $W \subset V$. Let $f_W \colon W \times X^{\mathrm{an}} \to Y^{\mathrm{an}}$ be the restriction of f along $W \times X^{\mathrm{an}} \to V \times X^{\mathrm{an}}$. Then since $[\varepsilon^*_{X,W} \circ F_W] \simeq [f^*_W]$, we see that up to shrinking W again, we can suppose that there is a natural equivalence

$$\varepsilon^*_{X,W} \circ F_W \simeq f^*_W$$

of functors $\mathrm{Perf}(Y) \to \mathrm{Perf}(W \times X^{\mathrm{an}})$. Let

$$\widetilde{F}_W \colon \mathrm{QCoh}(Y) \longrightarrow \mathrm{QCoh}(\mathrm{Spec}(A_V) \times X)$$

be the extension of F_W along $\mathrm{Perf}(Y) \hookrightarrow \mathrm{QCoh}(Y) \simeq \mathrm{Ind}(\mathrm{Perf}(Y))$. We claim that $\widetilde{F}_W$ commutes with flat objects and connective objects. As Y is Tannakian, this will suffice to complete the proof. To prove the claim, consider the commutative diagram

$$\begin{array}{ccc} & \mathrm{QCoh}(Y) & \\ {}^{\widetilde{F}_W}\swarrow & & \searrow^{f^*_W} \\ \mathrm{QCoh}(\mathrm{Spec}(A_W) \times X) & \xrightarrow{\ \varepsilon^*_{X,W}\ } & \mathcal{O}_{V \times X^{\mathrm{an}}}\text{-Mod} \,. \end{array}$$

Since flatness and connectivity of an object in $\mathrm{QCoh}(\mathrm{Spec}(A_W) \times X)$ can be tested locally on X, we can reduce to the case where X is affine. In this case, the conclusion follows from the fact that $\varepsilon^*_{X,W}$ is t-exact and conservative and from Lemma 6.10. □

COROLLARY 6.16. *Let $X, Y \in \mathrm{dSt}^{\mathrm{afp}}_k$ be derived stacks locally almost of finite presentation over k. Then the natural map*

$$\mathbf{Map}(X,Y)^{\mathrm{an}} \longrightarrow \mathbf{AnMap}\big(X^{\mathrm{an}}, Y^{\mathrm{an}}\big)$$

is an equivalence whenever Y is geometric and Tannakian and $\mathrm{QCoh}(Y) \simeq \mathrm{Ind}(\mathrm{Perf}(Y))$ and whenever X belongs to one of the following cases:

(1) *X is a derived proper geometric stack locally almost of finite presentation over k such that $\mathbf{Map}(X,Y)$ is a geometric stack again locally almost of finite presentation.*
(2) *X is of the form Z_{dR} or Z_{Dol} for some smooth and proper scheme Z.*
(3) *X is of the form K_{B} for some finite homotopy type K.*

Proof. We have to check that X satisfies the universal GAGA property and that $\mathbf{Map}(X,Y)$ is geometric and locally almost of finite presentation. For point (1) being geometric and locally almost of finite presentation is part of the assumption, and the universal GAGA property is exactly the content of Theorem 5.5. In point (2), the geometricity and local finite presentation can be deduced from Lurie's representability theorem. See also the examples following [PTVV13, Theorem 2.15] and [Sim09]. The universal GAGA property for these cases has been verified in Propositions 5.26 and 5.32. Finally, for point (3) it is enough to observe that geometric stacks locally almost of finite presentation are closed under finite limits, while the universal GAGA property has been proven in Proposition 5.28. □

Remark 6.17. We discussed the question of the geometricity of $\mathbf{Map}(X, Y)$ in Remark 5.7. As for the assumptions on Y, let us remark that they are satisfied in the following two important cases:

(1) Y is a quasi-compact quasi-separated Deligne–Mumford stack.

(2) Y is the classifying stack of an affine group scheme of finite type in characteristic zero.

Both examples are Tannakian by [Lur11b, Theorem 3.4.2]. Compact generation is proved in [HR17, Theorem A] (see also Example 9.4) for the first case and in the corollary to [HR17, Theorem B] for the second.

7. Applications

In this section we develop some applications of the main results of this paper.

7.1 Further consequences of Theorem 3.13

Theorem 3.13 is one of the key technical results of this paper. It is certainly the main tool we have to deal with analytification of geometric stacks that are not Deligne–Mumford. We use it here to deduce some other general properties of the analytification functor. The following is a generalization of Lemma 4.6 and Theorem 4.8.

Proposition 7.1. *Let $\mathcal{C} \in \mathrm{Cat}_\infty$, and let $\mathrm{Perf}_k^{\mathcal{C}}$ be the derived stack sending A to the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathrm{Perf}_k(A))$. Similarly, let $\mathrm{AnPerf}_k^{\mathcal{C}}$ be the derived analytic Cat_∞-valued stack sending U to $\mathrm{Fun}(\mathcal{C}, \mathrm{AnPerf}_k(U))$. Then:*

(1) *If k is non-archimedean, then for every derived k-affinoid space $U \in \mathrm{dAfd}_k$, there is a canonical equivalence*
$$\mathrm{Perf}_k^{\mathcal{C}}(A_U) \simeq \mathrm{AnPerf}_k^{\mathcal{C}}(U)\,.$$

(2) *If $k = \mathbb{C}$, then for every derived Stein space $U \in \mathrm{dStn}_{\mathbb{C}}$ and every compact Stein subset K in U, there is a canonical equivalence*
$$\underset{K \subset U \subset X}{\text{“colim”}}\, \mathrm{Perf}_{\mathbb{C}}^{\mathcal{C}}(A_U) \simeq \underset{K \subset U \subset X}{\text{“colim”}}\, \mathrm{AnPerf}_{\mathbb{C}}^{\mathcal{C}}(U)$$
in $\mathrm{Ind}\big(\mathrm{Cat}_\infty^{\mathrm{st},\otimes}\big)$.

Proof. Constructing a natural transformation[10]
$$\eta^{\mathcal{C}} \colon \big(\mathrm{Perf}_k^{\mathcal{C}}\big)^{\mathrm{an}} \longrightarrow \mathrm{AnPerf}_k^{\mathcal{C}}$$
is equivalent to constructing a natural transformation
$$\mathrm{Perf}_k^{\mathcal{C}} \longrightarrow \mathrm{AnPerf}_k^{\mathcal{C}} \circ (-)^{\mathrm{an}}\,.$$
The latter is simply induced by composition with the analytification functor
$$\varepsilon_X^* \colon \mathrm{Fun}(\mathcal{C}, \mathrm{Perf}(X)) \longrightarrow \mathrm{Fun}\big(\mathcal{C}, \mathrm{Perf}\big(X^{\mathrm{an}}\big)\big)\,.$$

[10] See Remark 3.6 for the meaning of analytification for Cat_∞-valued stacks.

At this point, in the non-archimedean setting the conclusion follows immediately from Lemma 4.6. In the $\mathbb{C}$-analytic case consider the functor

$$\mathrm{Fun}(\mathcal{C}, -)\colon \mathrm{Cat}_\infty^{\mathrm{st},\otimes} \longrightarrow \mathrm{Cat}_\infty^{\mathrm{st},\otimes}.$$

Applying the ind-construction, we obtain a functor

$$\mathrm{Ind}(\mathrm{Fun}(\mathcal{C}, -))\colon \mathrm{Ind}\big(\mathrm{Cat}_\infty^{\mathrm{st},\otimes}\big) \longrightarrow \mathrm{Ind}\big(\mathrm{Cat}_\infty^{\mathrm{st},\otimes}\big),$$

which takes an ind-object "colim"$_{i\in I} \mathcal{D}_i$ to "colim"$_{i\in I} \mathrm{Fun}(\mathcal{C}, \mathcal{D}_i)$. Evaluating this functor on the equivalence obtained in Theorem 4.8, we therefore get the equivalence

$$\underset{K\subset U\subset X}{\text{“colim”}} \mathrm{Fun}(\mathcal{C}, \mathrm{Perf}(A_U)) \simeq \underset{K\subset U\subset X}{\text{“colim”}} \mathrm{Fun}(\mathcal{C}, \mathrm{Perf}(U))$$

we were looking for. □

Similarly to what we did in Section 4, we can now obtain the following analogue of Proposition 4.9.

Corollary 7.2. *Let $\mathcal{C} \in \mathrm{Cat}_\infty$ be a compact object, and let $\mathbf{Perf}_k^{\mathcal{C}}$ (respectively, $\mathbf{AnPerf}_k^{\mathcal{C}}$) be the derived stack (respectively, derived analytic stack) associated with $\mathrm{Perf}_k^{\mathcal{C}}$ and $\mathrm{AnPerf}_k^{\mathcal{C}}$. Then the canonical morphism*

$$\big(\mathbf{Perf}_k^{\mathcal{C}}\big)^{\mathrm{an}} \longrightarrow \mathbf{AnPerf}_k^{\mathcal{C}}$$

is an equivalence.

Notice that $\mathbf{Perf}_k^{\mathcal{C}}(X) \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{Perf}(X))^{\simeq}$, which is different from $\mathrm{Fun}(\mathcal{C}, \mathbf{Perf}(X))$.

Proof. In Proposition 7.1 we constructed a morphism $\eta^{\mathcal{C}}\colon \big(\mathrm{Perf}_k^{\mathcal{C}}\big)^{\mathrm{an}} \to \mathrm{AnPerf}_k^{\mathcal{C}}$ which induces a canonical morphism $\big(\mathbf{Perf}_k^{\mathcal{C}}\big)^{\mathrm{an}} \to \mathbf{AnPerf}_k^{\mathcal{C}}$. In order to check that the latter is an equivalence, we verify that the hypotheses of Proposition 3.14 are satisfied. First of all, since $\mathcal{C}$ is compact, $\mathbf{Perf}_k^{\mathcal{C}}$ is still a locally geometric stack locally almost of finite presentation (as follows from [TV07, Theorem 0.2]). Unwinding the definitions, we see that we have to check that the map

$$\mathrm{Fun}(\mathcal{C}, \mathrm{Perf}(A_U))^{\simeq} \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathrm{Perf}(U))^{\simeq}$$

is an equivalence for every $U \in \mathrm{dAfd}_k$ when k is non-archimedean and that

$$\underset{K\subset V\subset U}{\mathrm{colim}} \mathrm{Fun}(\mathcal{C}, \mathrm{Perf}(A_V))^{\simeq} \longrightarrow \underset{K\subset V\subset U}{\mathrm{colim}} \mathrm{Fun}(\mathcal{C}, \mathrm{Perf}(V))^{\simeq}$$

is an equivalence for every $U \in \mathrm{dStn}_{\mathbb{C}}$ and every compact Stein subset K of U. Both statements follow at once from Proposition 7.1. □

The following result answers a question raised by G. Ginot.

Proposition 7.3. *Let I be a finite ∞-category, and let $F\colon I \to \mathrm{dSt}_k^{\mathrm{afp}}$ be a diagram. Suppose that for every $i \in I$ the object $X_i := F(i)$ is a geometric stack. Then*

$$\big(\lim_{i\in I} F(i)\big)^{\mathrm{an}} \longrightarrow \lim_{i\in I} F(i)^{\mathrm{an}}$$

is an equivalence.

Proof. Let $X := \lim_{i\in I} X_i \in \mathrm{dSt}_k$ be the limit. Since I is finite, we see that X is a geometric stack.

It is now enough to check that for every derived k-affinoid (respectively, Stein) space $U \in \mathrm{dAfd}_k$, the canonical map $\mathrm{Map}_{\mathrm{dAnSt}_k}\big(U, X^{\mathrm{an}}\big) \to \lim_{i\in I} \mathrm{Map}_{\mathrm{dAnSt}_k}\big(U, X_i^{\mathrm{an}}\big)$ is an equivalence.

We work in the ∞-topos $\mathcal{X}_U$. Since $\mathrm{Map}_{\mathcal{X}_U}(\mathbf{1}_U, -)$ commutes with limits, Lemma 3.7 implies that it is enough to prove that the canonical map $F_U^s(X^{\mathrm{an}}) \to \lim_{i\in I} F_U^s(X_i^{\mathrm{an}})$ is an equivalence. Since X and all of the X_i are geometric stacks, we can use Theorem 3.13 to obtain (functorial) equivalences

$$G_U^s(X) \simeq F_U^s\big(X^{\mathrm{an}}\big)\,, \quad G_U^s(X_i) \simeq F_U^s\big(X^{\mathrm{an}}\big)\,.$$

Since the sheafification commutes with finite colimits, we see that the canonical map $G_U^s(X) \to \lim_{i\in I} G_U^s(X_i)$ is induced by sheafification from the map $G_U^p(X) \to \lim_{i\in I} G_U^p(X_i)$. It is then sufficient to prove that this second map is an equivalence. For every étale map from a derived k-affinoid (respectively, Stein) space $V \to U$, we can rewrite

$$G_U^p(X)(V) \simeq \mathrm{Map}_{\mathrm{dSt}_k}(\mathrm{Spec}(A_V), X)\,, \quad G_U^p(X_i)(V) \simeq \mathrm{Map}_{\mathrm{dSt}_k}(\mathrm{Spec}(A_V), X_i)\,,$$

where we set $A_V := \Gamma\big(V; \mathcal{O}_V^{\mathrm{alg}}\big)$ as usual. The conclusion now follows from the fact that $X \simeq \lim_{i\in I} X_i$ in dSt_k. □

7.2 The derived period domain

As a further consequence of Corollary 7.2, we can clarify a construction that arises in the study of the derived period map introduced in [DH19]. This is a derived enhancement of Griffith's classical period map, that associates with a smooth projective family of derived stacks a map from the base to a derived period domain. Let us start by briefly recalling the definition of the derived period domain.

Assume that we are given a perfect complex $V \in \mathrm{Perf}_{\mathbb{C}}$, concentrated in degrees 0 to $2n$, equipped with a $2n$-shifted bilinear form $q\colon V\otimes V \to \mathbb{C}[2n]$ that is non-degenerate on cohomology. The typical example is the cohomology complex of a smooth projective variety. With the pair (V, q) we can associate a derived geometric stack $\mathbf{D}_n(V, q)$, which classifies families of decreasing filtrations $\{F^*\}$ of V of length $n+1$ satisfying the following two additional properties:

(1) The filtration $\{F^*\}$ descends to a filtration on cohomology groups of V.

(2) The filtration $\{F^*\}$ satisfies the Hodge–Riemann orthogonality relation with respect to Q; that is, Q vanishes on $F^i \otimes F^{n+1-i}$.

For more precise definitions and statements, we refer to [DH19, Theorem 3.4]. The coarse moduli space of the underived truncation of $\mathbf{D}_n(V, q)$ recovers the closure of the classical period domain.

Next we consider the following conditions:

(3) The filtration $\{F^*\}$ induces Hodge structures on cohomology groups; that is, we have $F^pH^n \oplus \overline{F^{n+1-p}H^n} \cong H^n(V)$.

(4) The shifted bilinear form q satisfies the Hodge–Riemann non-degeneracy conditions on the cohomology groups of V; that is, $i^{p-q}q(\varphi, \overline{\varphi}) > 0$ for $\varphi \in F^pH^{p+q}(V) \cap \overline{F^qH^{p+q}(V)}$.

Conditions (2)–(4) state that the filtration $\{F^*\}$ induces polarized Hodge structures on the cohomology groups of V.

Imposing conditions (3) and (4) singles out an open analytic substack U of the analytification $\mathbf{D}_n(V, q)^{\mathrm{an}}$, which is the derived period domain. In [DH19] it is shown that U is a geometric analytic stack and that it is the target of the derived period map. Using Corollary 7.2, we can give a modular description for U.

Corollary 7.4. *Let (V, q) be a $2n$-shifted quadratic perfect complex, and assume that Q is* *non-degenerate on the cohomology of V. The functor*

$$\mathbf{P}_n(V,q)\colon \mathrm{dStn}^{\mathrm{op}} \longrightarrow \mathcal{S}$$

which sends $S \in \mathrm{dStn}$ to the space of filtrations $\{F^*\}$ on $\pi_S^* V \in \mathrm{Perf}(S)$ satisfying conditions (1) through (4) above is representable by a geometric derived analytic stack. Here $\pi_S\colon S \to \mathrm{Sp}(\mathbb{C})$ denotes the canonical map. Furthermore, $\mathbf{P}_n(V,q)$ coincides with the derived period domain U considered in [DH19].

Proof. It suffices to show that $\mathbf{D}_n(V,q)^{\mathrm{an}}$ is equivalent to the derived analytic moduli stack classifying filtrations satisfying Hodge–Riemann orthogonality, which we will denote by $\mathbf{AnD}_n(V,q)$. Then U and $\mathbf{P}_n(V,q)$ are defined by the same open conditions (3) and (4).

To prove this, we observe that following the proof of [DH19, Theorem 3.4], the stack $\mathbf{D}_n(V,q)$ is constructed in a categorical way from the stack of perfect complexes $\mathbf{Perf}_{\mathbb{C}}$. The stack of $(n+1)$-term filtrations $\mathbf{Filt}_n$ is defined as the stack of n composable morphisms in $\mathbf{Perf}_{\mathbb{C}}$; that is, $\mathbf{Filt}_n \simeq \mathbf{Perf}_{\mathbb{C}}^{\Delta^n}$. The derived flag variety $\mathbf{Flag}_n(V)$ is defined as the fibre at $V \in \mathbf{Perf}_{\mathbb{C}}$ of the natural map $\mathbf{Filt}_n \to \mathbf{Perf}_{\mathbb{C}}$ forgetting the filtration. The forgetful map also induces a forgetful map of morphism stacks $\mathbf{Filt}_1^{\Delta^1} \to \mathbf{Perf}_{\mathbb{C}}^{\Delta^1}$, and we define the stack $\mathbf{Flag}_1(V,q)$ as the fibre of $\mathbf{Filt}_1^{\Delta^1} \to \mathbf{Perf}_{\mathbb{C}}^{\Delta^1}$ over $q\colon \mathrm{Sym}^2 V \to \mathbb{C}[2n]$. Then we define $\mathbf{D}'_n(V,q)$ as the pullback

$$\begin{array}{ccc}
\mathbf{D}'_n(V,q) & \longrightarrow & \mathbf{Flag}_1(V,q) \\
\downarrow & & \downarrow{\scriptstyle e} \\
\mathbf{Flag}_n(V) & \xrightarrow{\ \sigma\ } & \mathbf{Flag}_1\big(\mathrm{Sym}^2 V\big) \times \mathbf{Flag}_1(\mathbb{C}[2n]) .
\end{array}$$

Here e is just given by the evaluation maps at the source and target. The map σ sends a filtration $F^n \to \cdots \to F^0$ to $S \times (0 \to \mathcal{O}[2n])$, where S is the 2-term filtration on $\mathrm{Sym}^2 F^0$ given by the image of $\oplus_i (F^i \otimes F^{n+1-i})$. Thus the points of $\mathbf{D}'_n(V,q)$ are those flags in V satisfying the orthogonality condition (2). The derived stack $\mathbf{D}_n(V,q)$ is defined as the open substack of $\mathbf{D}'_n(V,q)$ satisfying condition (1).

We can now perform the exact same constructions starting from $\mathbf{AnPerf}_{\mathbb{C}}$ instead of $\mathbf{Perf}_{\mathbb{C}}$ to obtain $\mathbf{AnD}_n(V,q)$. We first consider $\mathbf{AnFilt}_n \coloneqq \mathbf{AnPerf}_{\mathbb{C}}^{\Delta^n}$. Corollary 7.2 supplies a natural equivalence $\mathbf{AnFilt}_n \simeq \mathbf{Filt}_n^{\mathrm{an}}$. Similarly $\mathbf{AnFlag}_n(V)$, defined as the fibre of $\mathbf{AnPerf}_{\mathbb{C}}^{\Delta_n} \to \mathbf{AnPerf}_{\mathbb{C}}$ over V, is the analytification of $\mathbf{Flag}_n(V)$ since analytification commutes with finite limits in virtue of Proposition 7.3. The same holds for $\mathbf{AnFlag}_1(q)$, and putting all of this together with condition (1), we see that $\mathbf{AnD}_n(V,q)$ is a derived analytic moduli stack which is equivalent to the analytification of $\mathbf{D}_n(V,q)$. □

7.3 The derived Riemann–Hilbert correspondence

One of the main applications of the techniques of this paper is to obtain an extended version of the derived Riemann–Hilbert correspondence first proven in [Por17b]. There, the second-named author, following a suggestion of C. Simpson, introduced for every $\mathbb{C}$-analytic space X a morphism

$$\eta_{\mathrm{RH}}\colon X_{\mathrm{dR}} \longrightarrow X_{\mathrm{B}}$$

called the Riemann–Hilbert transformation. He then showed that if X is smooth, the canonical morphism

$$\eta_{\mathrm{RH}}^*\colon \mathbf{AnMap}(X_{\mathrm{B}}, \mathbf{AnPerf}_{\mathbb{C}}) \longrightarrow \mathbf{AnMap}(X_{\mathrm{dR}}, \mathbf{AnPerf}_{\mathbb{C}})$$

is an equivalence (see [Por17b, Theorem 6.11]).

Corollary 7.5. *Let X be a smooth proper scheme over $\mathbb{C}$. Then η_{RH} induces an equivalence*

$$\mathbf{Map}(X_{\mathrm{B}}, \mathbf{Perf}_{\mathbb{C}})^{\mathrm{an}} \simeq \mathbf{Map}(X_{\mathrm{dR}}, \mathbf{Perf}_{\mathbb{C}})^{\mathrm{an}} .$$

Proof. It follows from Sections 5.2.2 and 5.2.3 that X_{dR} and X_{B} satisfy the universal GAGA property. In other words, we proved that the canonical maps

$$\mathbf{Map}(X_{\mathrm{B}}, \mathbf{Perf}_{\mathbb{C}})^{\mathrm{an}} \longrightarrow \mathbf{AnMap}((X_{\mathrm{B}})^{\mathrm{an}}, \mathbf{AnPerf}_{\mathbb{C}})$$

and

$$\mathbf{Map}(X_{\mathrm{dR}}, \mathbf{Perf}_{\mathbb{C}})^{\mathrm{an}} \longrightarrow \mathbf{AnMap}((X_{\mathrm{dR}})^{\mathrm{an}}, \mathbf{AnPerf}_{\mathbb{C}})$$

are equivalences. Furthermore, we saw in Section 5.2.2 that there is a canonical morphism

$$(X_{\mathrm{dR}})^{\mathrm{an}} \longrightarrow (X^{\mathrm{an}})_{\mathrm{dR}}$$

which is furthermore an equivalence because X is smooth. Since there is an obvious equivalence $(X_{\mathrm{B}})^{\mathrm{an}} \simeq (X^{\mathrm{an}})_{\mathrm{B}}$, we can use the Riemann–Hilbert transformation for X^{an} to obtain the equivalence we are looking for. □

As our last application, we notice that Proposition 6.12 and Theorem 6.14 together imply that the Riemann–Hilbert correspondence with coefficients in an algebraic stack satisfying the Tannakian property is still an equivalence.

Corollary 7.6. *Let $Y \in \mathrm{dSt}_{\mathbb{C}}^{\mathrm{afp}}$ be a derived stack locally almost of finite presentations and satisfying the assumptions of Theorem 6.14. Then for every smooth analytic space X, the Riemann–Hilbert transformation $\eta_{\mathrm{RH}} \colon X_{\mathrm{dR}} \to X_{\mathrm{B}}$ induces an equivalence*

$$\eta_{\mathrm{RH}}^* \colon \mathbf{AnMap}(X_{\mathrm{B}}, Y^{\mathrm{an}}) \longrightarrow \mathbf{AnMap}(X_{\mathrm{dR}}, Y^{\mathrm{an}}) .$$

Proof. Fix a derived Stein space $S \in \mathrm{dStn}_{\mathbb{C}}$. We have to prove that $\eta_{\mathrm{RH}} \colon X_{\mathrm{dR}} \to X_{\mathrm{B}}$ induces an equivalence

$$\eta_{\mathrm{RH}}^* \colon \mathrm{Map}_{\mathrm{dAnSt}_{\mathbb{C}}}(S \times X_{\mathrm{B}}, Y^{\mathrm{an}}) \longrightarrow \mathrm{Map}_{\mathrm{dAnSt}_{\mathbb{C}}}(S \times X_{\mathrm{dR}}, Y^{\mathrm{an}}) .$$

Consider the commutative diagram

$$\begin{array}{ccc}
\mathrm{Map}_{\mathrm{dAnSt}_{\mathbb{C}}}(S \times X_{\mathrm{B}}, Y^{\mathrm{an}}) & \longrightarrow & \mathrm{Map}_{\mathrm{dAnSt}_{\mathbb{C}}}(S \times X_{\mathrm{dR}}, Y^{\mathrm{an}}) \\
\downarrow & & \downarrow \\
\mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(S \times X_{\mathrm{B}})) & \longrightarrow & \mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(S \times X_{\mathrm{dR}})) ,
\end{array}$$

where the vertical morphisms are the ones induced by Theorem 6.5. This proposition shows furthermore that they are fully faithful. The bottom horizontal morphism is an equivalence by virtue of [Por17b, Theorem 6.11]. It follows that the top horizontal functor is fully faithful, too.

We are left to check that it is essentially surjective. Fix a morphism $f \colon S \times X_{\mathrm{dR}} \to Y^{\mathrm{an}}$, and let

$$F \colon \mathrm{Perf}(Y) \longrightarrow \mathrm{Perf}(S \times X_{\mathrm{dR}})$$

be the induced symmetric monoidal functor. Let

$$G \colon \mathrm{Perf}(Y) \longrightarrow \mathrm{Perf}(S \times X_{\mathrm{B}})$$

be the symmetric monoidal functor induced by the equivalence $\eta_{\mathrm{RH}}^* \colon \mathrm{Perf}(S \times X_{\mathrm{B}}) \xrightarrow{\sim} \mathrm{Perf}(S \times X_{\mathrm{dR}})$. We would like to invoke Proposition 6.12, but for this we first have to replace X_{B} with a colimit of derived Stein spaces.

Applying the argument of [PY16, Lemma 5.14 and Remark 5.15], we produce three open hypercovers $W_\bullet$, $V_\bullet$ and $U_\bullet$ of X satisfying the following conditions:

(1) For every integer m the spaces U_m, V_m and W_m are disjoint unions of *contractible* open Stein subspaces of X.
(2) For every integer m we have $W_m \Subset V_m \Subset U_m$.

Observe that for every integer m we have canonical equivalences

$$(U_m)_{\mathrm{B}} \simeq \coprod_{I_U} \mathrm{Sp}(\mathbb{C}) , \quad (V_m)_{\mathrm{B}} \simeq \coprod_{I_V} \mathrm{Sp}(\mathbb{C}) , (W_m)_{\mathrm{B}} \simeq \coprod_{I_W} \mathrm{Sp}(\mathbb{C})$$

and that $|(W_\bullet)_{\mathrm{B}}| \simeq |(V_\bullet)_{\mathrm{B}}| \simeq |(U_\bullet)_{\mathrm{B}}| \simeq X_{\mathrm{B}}$. Therefore, we can represent G as an element in the limit

$$\lim_{m \in \mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\otimes}(\mathrm{Perf}(Y), \mathrm{Perf}(S \times (U_m)_{\mathrm{B}})) .$$

For every integer m denote by G_m the induced symmetric monoidal functor

$$G_m \colon \mathrm{Perf}(Y) \longrightarrow \mathrm{Perf}(S \times (U_m)_{\mathrm{B}}) \simeq \prod_{I_U} \mathrm{Perf}(S) .$$

Let

$$\widetilde{G}_m \colon \mathrm{QCoh}(Y) \longrightarrow \mathcal{O}_{S \times (U_m)_{\mathrm{B}}}\text{-Mod} \simeq \prod_{I_U} \mathcal{O}_S\text{-Mod}$$

be the symmetric monoidal functor obtained by left Kan extension along $\mathrm{Perf}(Y) \hookrightarrow \mathrm{Ind}(\mathrm{Perf}(Y)) \simeq \mathrm{QCoh}(Y)$. We claim that each $\widetilde{G}_m$ commutes with perfect complexes, flat objects and connective objects. Assuming this claim, Proposition 6.12 shows that the composition

$$\mathrm{QCoh}(Y) \xrightarrow{G_m} \mathcal{O}_{S \times (U_m)_{\mathrm{B}}}\text{-Mod} \longrightarrow \mathcal{O}_{S \times (W_m)_{\mathrm{B}}}\text{-Mod}$$

can be represented by a morphism $g_m \colon S \times (W_m)_{\mathrm{B}} \to Y^{\mathrm{an}}$. The full faithfulness provided by Theorem 6.5 shows that the morphisms g_m can be glued back to a morphism $g \colon S \times X_{\mathrm{B}} \to Y^{\mathrm{an}}$. Finally, the construction shows that $g \circ \eta_{\mathrm{RH}} \simeq f$.

We are therefore left to prove the above claim. Reasoning as in [Por17b, Proposition 5.1], we see that pulling back along the canonical morphism $X \to X_{\mathrm{dR}}$ produces a conservative and t-exact functor $\mathcal{O}_{S \times X_{\mathrm{dR}}}\text{-Mod} \to \mathcal{O}_{S \times X}\text{-Mod}$. Let

$$\widetilde{F} \colon \mathrm{QCoh}(Y) \longrightarrow \mathcal{O}_{S \times X_{\mathrm{dR}}}\text{-Mod}$$

be the left Kan extension of F along $\mathrm{Perf}(Y) \hookrightarrow \mathrm{Ind}(\mathrm{Perf}(Y)) \simeq \mathrm{QCoh}(Y)$, and define $\widetilde{G}$ similarly. Then

$$\widetilde{G} \simeq \eta_{\mathrm{RH}}^* \widetilde{(F)} .$$

Invoking [Por17b, Corollary 5.3], it suffices to prove that $\widetilde{F}$ commutes with perfect complexes, flat objects and connective objects.

First observe that there is a canonical equivalence $|(U_\bullet)_{\mathrm{dR}}| \simeq X_{\mathrm{dR}}$. Next consider the following Čech nerve:

$$U_{\bullet,\star} \coloneqq \check{\mathcal{C}}(U_\bullet \longrightarrow (U_\bullet)_{\mathrm{dR}}) .$$

We can identify $U_{\bullet,\star}$ with a bisimplicial object in $\mathrm{dAnSt}_{\mathbb{C}}$. Moreover, [Por17b, Lemma 4.1] provides canonical identifications

$$U_{m,n} \simeq \operatorname*{colim}_{i \in \mathbb{N}} \Delta^{n,(i)}_{U_m} ,$$

where $\Delta^n_{U_m}$ denotes the (small) diagonal of U_m in $(U_m)^{\times n}$ and $\Delta^{n,(i)}_{U_m}$ denotes the ith infinitesimal neighbourhood of $\Delta^n_{U_m}$ inside $(U_m)^{\times n}$. With similar notation, we obtain the following descriptions: Let $J \coloneqq \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \times \mathbb{N}$. Since $\mathrm{dAnSt}_{\mathbb{C}}$ is an ∞-topos, colimits are universal in $\mathrm{dAnSt}_{\mathbb{C}}$, and in particular we obtain

$$\operatorname*{colim}_{([m],[n],i)\in J} S \times \Delta^{n,(i)}_{U_m} \simeq S \times X_{\mathrm{dR}} \, .$$

It follows that we can represent the functor $F\colon \mathrm{Perf}(Y) \to \mathrm{Perf}(S \times X_{\mathrm{dR}})$ as an element in the limit

$$\lim_{([m],[n],i)\in J} \mathrm{Fun}^{\otimes}\big(\mathrm{Perf}(Y), \mathrm{Perf}\big(S \times \Delta^{n,(i)}_{U_m}\big)\big) \, .$$

Let

$$F^i_{m,n}\colon \mathrm{Perf}(Y) \longrightarrow \mathrm{Perf}\big(S \times \Delta^{n,(i)}_{U_m}\big)$$

denote the projection of F on $\mathrm{Perf}\big(S \times \Delta^{n,(i)}_{U_m}\big)$, and let

$$\widetilde{F}^i_{m,n}\colon \mathrm{QCoh}(Y) \longrightarrow \mathcal{O}_{\Delta^{n,(i)}_{U_m}}\text{-}\mathrm{Mod}$$

be the left Kan extension of $F^i_{m,n}$ along $\mathrm{Perf}(Y) \hookrightarrow \mathrm{Ind}(\mathrm{Perf}(Y)) \simeq \mathrm{QCoh}(Y)$. Notice that each $S \times \Delta^{n,(i)}_{U_m}$ is a derived Stein space. Therefore, Lemma 6.10 implies that $\widetilde{F}^i_{m,n}$ preserves perfect complexes, flat objects and connective objects. From here, we deduce that the same is true for $\widetilde{F}$. The proof is therefore complete. □

Appendix. Some lemmas on derived Stein spaces

In this appendix we prove some basic facts on Stein spaces that do not fit in the main body of the text. We mainly focus on the $\mathbb{C}$-analytic setting as the non-archimedean setting has already been addressed in [PY21].

Given a derived $\mathbb{C}$-analytic space $X \in \mathrm{dAn}_{\mathbb{C}}$, we denote by $X_{\text{ét}}$ its small étale site. This is the ∞-site spanned by étale maps $Y \to X$, where Y is a derived Stein space. The truncation functor

$$\mathrm{t}_0\colon X_{\text{ét}} \longrightarrow (\mathrm{t}_0(X))_{\text{ét}}$$

is an equivalence of ∞-categories. This follows directly from [Por19, Lemma 3.4]. In virtue of this fact, we give the following definition.

Definition A.1. Let $f\colon U \to V$ be an open immersion of derived Stein spaces. We say that U *is relatively compact in* V *(via* f*)* if the closure of $\mathrm{t}_0(U)$ inside $\mathrm{t}_0(V)$ is compact. In this case we write $U \Subset V$.

Lemma A.2. *Let U be a derived Stein space, and let $A_U \coloneqq \Gamma\big(U; \mathcal{O}^{\mathrm{alg}}_U\big)$. Then there is a canonical equivalence*

$$\pi_i(A_U) \simeq \Gamma\big(U; \pi_i\big(\mathcal{O}^{\mathrm{alg}}_U\big)\big) \, .$$

Proof. First of all, we observe that $\pi_i\big(\mathcal{O}^{\mathrm{alg}}_U\big)$ is by assumption a coherent sheaf on the underived Stein space $\mathrm{t}_0(U)$. Therefore, Cartan's Theorem B implies that $\Gamma\big(U; \pi_i\big(\mathcal{O}^{\mathrm{alg}}_U\big)\big)$ is concentrated in cohomological degree zero. In turn, this implies that the spectral sequence computing $\Gamma\big(U; \mathcal{O}^{\mathrm{alg}}_U\big)$ degenerates at page E_2, yielding the desired equivalence. □

Lemma A.3. *Let $W \Subset V \Subset U$ be a nested sequence of relatively compact derived Stein spaces. Set*

$$A_U \coloneqq \Gamma\big(U; \mathcal{O}_U^{\mathrm{alg}}\big)\,, \quad A_V \coloneqq \Gamma\big(V; \mathcal{O}_V^{\mathrm{alg}}\big)\,, \quad A_W \coloneqq \Gamma\big(W; \mathcal{O}_W^{\mathrm{alg}}\big)\,.$$

Then the natural map $A_V \to A_W$ is flat.

Proof. Since $W \Subset V$, we can use [PY16, Lemma 8.13] to see that $\pi_0(A_V) \to \pi_0(A_W)$ is flat. All we are left to check is therefore that the canonical map

$$\pi_i(A_V) \otimes_{\pi_0(A_V)} \pi_0(A_W) \longrightarrow \pi_i(A_W)$$

is an isomorphism. Lemma A.2 provides us with natural equivalences

$$\pi_i(A_V) \simeq \Gamma\big(V; \pi_i\big(\mathcal{O}_V^{\mathrm{alg}}\big)\big)\,, \quad \pi_i(A_W) \simeq \Gamma\big(W; \pi_i\big(\mathcal{O}_W^{\mathrm{alg}}\big)\big)\,.$$

We now observe that $\pi_i\big(\mathcal{O}_V^{\mathrm{alg}}\big)$ is a coherent sheaf on the underived Stein space $\mathrm{t}_0(V)$ and furthermore $\pi_i\big(\mathcal{O}_V^{\mathrm{alg}}\big)|_W = \pi_i\big(\mathcal{O}_W^{\mathrm{alg}}\big)$. Therefore, [Dou73, Proposition 2] implies that

$$\Gamma\big(W; \pi_i\big(\mathcal{O}_W^{\mathrm{alg}}\big)\big) \simeq \pi_i\big(\mathcal{O}_V^{\mathrm{alg}}\big)(W) \simeq \Gamma\big(V; \mathcal{O}_V^{\mathrm{alg}}\big) \widehat{\otimes}_{\pi_0(A_V)} \pi_0(A_W)\,.$$

We now observe that $\pi_i\big(\mathcal{O}_V^{\mathrm{alg}}\big)$ is the restriction to $\mathrm{t}_0(V)$ of the coherent sheaf $\pi_0\big(\mathcal{O}_U^{\mathrm{alg}}\big)$ on $\mathrm{t}_0(U)$. Since $V \Subset U$, Lemmas 8.11 and 8.12 of [PY16] imply that $\Gamma\big(V; \mathcal{O}_V^{\mathrm{alg}}\big)$ is finitely generated over $\pi_0(A_V)$. Therefore, the canonical map

$$\Gamma\big(V; \mathcal{O}_V^{\mathrm{alg}}\big) \otimes_{\pi_0(A_V)} \pi_0(A_W) \longrightarrow \Gamma\big(V; \pi_i\big(\mathcal{O}_V^{\mathrm{alg}}\big)\big) \widehat{\otimes}_{\pi_0(A_V)} \pi_0(A_W)$$

is an equivalence. The conclusion follows. □

The same technique used to prove Lemma A.3 also allows us to prove the following more general result.

Corollary A.4. *Let $W \Subset V \Subset U$ be a nested sequence of relatively compact derived Stein spaces. Let A_U, A_V and A_W be defined as in Lemma A.3. Then for any $\mathcal{F} \in \mathrm{Coh}(U)$ the natural map*

$$\gamma_{\mathcal{F}} \colon \Gamma(V; \mathcal{F}|_V) \otimes_{A_V} A_W \longrightarrow \Gamma(W; \mathcal{F}|_W)$$

is an equivalence.

Proof. It is enough to check that for every integer $i \in \mathbb{Z}$ the map $\gamma_{\mathcal{F}}$ induces an isomorphism

$$\pi_i(\Gamma(V; \mathcal{F}|_V) \otimes_{A_V} A_W) \longrightarrow \pi_i(\Gamma(W; \mathcal{F}|_W))\,.$$

Thanks to Lemma A.3, we know that the map $A_V \to A_W$ is flat. As a consequence, the Tor spectral sequence of [Lur17, Proposition 7.2.1.19] degenerates at the page E_2, yielding an equivalence

$$\pi_i\left(\Gamma(V; \mathcal{F}|_V) \otimes_{A_V} A_W\right) \simeq \pi_i(\Gamma(V; \mathcal{F}|_V)) \otimes_{A_V} A_W\,.$$

On the other hand, Cartan's Theorem B supplies a natural equivalence

$$\pi_i(\Gamma(V; \mathcal{F}|_V)) \simeq \Gamma(V; \pi_i(\mathcal{F})|_V)\,.$$

Since $\pi_i(\mathcal{F})$ is a coherent sheaf on the underived Stein space $\mathrm{t}_0(U)$, we can use [Dou73, Proposition 2] to obtain an equivalence

$$\Gamma(V; \pi_i(\mathcal{F})|_V) \widehat{\otimes}_{\pi_0(A_V)} \pi_0(A_W) \simeq \Gamma(W; \pi_i(\mathcal{F})|_W)\,.$$

We now apply [PY16, Lemmas 8.11 and 8.12] to the sheaf $\pi(\mathcal{F})$ to deduce that $\Gamma(V;\pi_i(\mathcal{F})|_V)$ is finitely generated over $\pi_0(A_V)$. In particular, the natural map

$$\Gamma(V;\pi_i(\mathcal{F})|_V)\otimes_{\pi_0(A_V)}\pi_0(A_W)\longrightarrow\Gamma(V;\pi_i(\mathcal{F})|_V)\widehat{\otimes}_{\pi_0(A_V)}\pi_0(A_W)$$

is an isomorphism. The conclusion follows. □

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References

Ber90 V. G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Math. Surveys Monogr., vol. 33 (Amer. Math. Soc., Providence, RI, 1990); doi:10.1090/surv/033.

Ber94 ———, *Vanishing cycles for formal schemes*, Invent. Math. **115** (1994), no. 3, 539–571; doi:10.1007/BF01231772.

Bha16 B. Bhatt, *Algebraization and Tannaka duality*, Camb. J. Math. **4** (2016), no. 4, 403–461; doi:10.4310/CJM.2016.v4.n4.a1.

CS22 D. Clausen and P. Scholze, *Condensed Mathematics and Complex Geometry*, 2022, available at https://people.mpim-bonn.mpg.de/scholze/Complex.pdf.

Del70 P. Deligne, *Équations différentielles à points singuliers réguliers*, Lecture Notes in Math., vol. 163 (Springer-Verlag, Berlin-New York, 1970); doi:10.1007/BFb0061194.

Dem12 J.-P. Demailly, *Complex analytic and differential geometry*, 2012, available at https://www-fourier.ujf-grenoble.fr/~Demailly/manuscripts/agbook.pdf.

DH19 C. Di Natale and J. V. S. Holstein, *The global derived period map*, Adv. Math. **353** (2019), 224–280; doi:10.1016/j.aim.2019.06.022.

DPS22 D.-E. Diaconescu, M. Porta, and F. Sala, *Cohomological Hall algebras and their representations via torsion pairs*, 2022, arXiv:2207.08926.

DPS23 ———, *McKay correspondence, cohomological Hall algebras and categorification I*, Represent. Theory **27** (2023), 933–972; doi:10.1090/ert/649.

Dou73 A. Douady, *Le théorème des images directes de Grauert [d'après Kiehl-Verdier]*, in *Séminaire Bourbaki, 24ème année (1971/1972)*, Exp. No. 404, Lecture Notes in Math., vol. 317 (Springer-Verag, Berlin-New York, 1973), 73–87.

GR03 O. Gabber and L. Ramero, *Almost ring theory*, Lecture Notes in Math., vol. 1800 (Springer-Verlag, Berlin, 2003); doi:10.1007/b10047.

GH15 D. Gepner and R. Haugseng, *Enriched ∞-categories via non-symmetric ∞-operads*, Adv. Math. **279** (2015), 575–716; doi:10.1016/j.aim.2015.02.007.

GR79 H. Grauert and R. Remmert, *Theory of Stein spaces* (translated from the German by A. Huckleberry), Grundlehren math. Wiss., vol. 236 (Springer-Verlag, Berlin-New York, 1979); doi:10.1007/978-1-4757-4357-9.

GR84 ———, *Coherent analytic sheaves*, Grundlehren math. Wiss., vol. 265 (Springer-Verlag, Berlin, 1984); doi:10.1007/978-3-642-69582-7.

Gro63 A. Grothendieck, *Revêtements étales et groupe fondamental. Fasc. II: Exposés 6, 8 à 11* (Inst. Hautes Études Sci., Paris, 1963).

HR15 J. Hall and D. Rydh, *Algebraic groups and compact generation of their derived categories of representations*, Indiana Univ. Math. J. **64** (2015), no. 6, 1903–1923; doi:10.1512/iumj.2015.64.5719.

HR17 ———, *Perfect complexes on algebraic stacks*, Compos. Math. **153** (2017), no. 11, 2318–2367; doi:10.1112/S0010437X17007394.

HM66 G. Hochschild and G. D. Mostow, *Holomorphic cohomology of complex analytic linear groups*, Nagoya Math. J. **27** (1966), no. 2, 531–542; doi:10.1017/S0027763000026362.

KY23 S. Keel and T. Y. Yu, *The Frobenius structure theorem for affine log Calabi-Yau varieties containing a torus*, Ann. of Math. (2) **198** (2023), no. 2, 419–536; doi:10.4007/annals.2023.198.2.1.

KS06 M. Kontsevich and Y. Soibelman, *Affine structures and non-Archimedean analytic spaces*, in *The unity of mathematics*, Progr. Math., vol. 244 (Birkhäuser Boston, Boston, MA, 2006), 321–385; doi:10.1007/0-8176-4467-9_9.

Lur04 J. Lurie, *Tannaka duality for geometric stacks*, 2004, arXiv:math/0412266.

Lur09 ———, *Higher topos theory*, Ann. of Math. Stud., vol. 170 (Princeton Univ. Press, Princeton, NJ, 2009); doi:10.1515/9781400830558.

Lur11a ———, *DAG V: Structured spaces*, 2011, available at https://www.math.ias.edu/~lurie/papers/DAG-V.pdf.

Lur11b ———, *DAG VIII: Quasi-coherent sheaves and Tannaka duality theorems*, 2011, available at https://www.math.ias.edu/~lurie/papers/DAG-VIII.pdf.

Lur11c ———, *DAG IX: Closed immersions*, 2011, available at https://www.math.ias.edu/~lurie/papers/DAG-IX.pdf.

Lur11d ———, *DAG XII: Proper morphisms, completions and the Grothendieck existence theorem*, 2011, available at https://www.math.ias.edu/~lurie/papers/DAG-XII.pdf.

Lur12 ———, *DAG XIV: Representability theorems*, 2012, available at https://www.math.ias.edu/~lurie/papers/DAG-XIV.pdf.

Lur17 ———, *Higher algebra*, 2017, available at https://www.math.ias.edu/~lurie/papers/HA.pdf.

Lur18 ———, *Spectral Algebraic Geometry*, 2018, available at https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf.

MR18 E. Mann and M. Robalo, *Brane actions, categorifications of Gromov–Witten theory and quantum K-theory*, Geom. Topol. **22** (2018), no. 3, 1759–1836; doi:10.2140/gt.2018.22.1759.

Neg19 A. Neguţ, *Shuffle algebras associated to surfaces*, Selecta Math. (N.S.) **25** (2019), no. 3, article no. 36; doi:10.1007/s00029-019-0481-z.

NXY19 J. Nicaise, C. Xu and T. Y. Yu, *The non-archimedean SYZ fibration*, Compos. Math. **155** (2019), no. 5, 953–972; doi:10.1112/s0010437x19007152.

Ols06 M. C. Olsson, Hom*-stacks and restriction of scalars*, Duke Math. J. **134** (2006), no. 1, 139–164; doi:10.1215/S0012-7094-06-13414-2.

PT21 T. Pantev and B. Toën, *Poisson geometry of the moduli of local systems on smooth varieties*, Publ. Res. Inst. Math. Sci. **57** (2021), no. 3–4, 959–991; doi:10.4171/prims/57-3-8.

PT22 ———, *Moduli of flat connections on smooth varieties*, Algebr. Geom. **9** (2022), no. 3, 266–310; doi:10.14231/ag-2022-009.

PTVV13 T. Pantev, B. Toën, M. Vaquié and G. Vezzosi, *Shifted symplectic structures*, Publ. Math. Inst. Hautes Études Sci. **117** (2013), 271–328; doi:10.1007/s10240-013-0054-1.

Por17a M. Porta, *Comparison results for derived Deligne–Mumford stacks*, Pacific J. Math. **287** (2017), no. 1, 177–197; doi:10.2140/pjm.2017.287.177.

Por17b ———, *The derived Riemann–Hilbert correspondence*, 2017, arXiv:1703.03907.

Por19 ______, *GAGA theorems in derived complex geometry*, J. Algebraic Geom. **28** (2019), no. 3, 519–565; doi:10.1090/jag/716.

PS23 M. Porta and F. Sala, *Two-dimensional categorified Hall algebras*, J. Eur. Math. Soc. (JEMS) **25** (2023), no. 3, 1113–1205; doi:10.4171/jems/1303.

PT24 M. Porta and J. B. Teyssier, *Homotopy theory of Stokes structures and derived moduli*, 2024, arXiv:2401.12335.

PY16 M. Porta and T. Y. Yu, *Higher analytic stacks and GAGA theorems*, Adv. Math. **302** (2016), 351–409; doi:10.1016/j.aim.2016.07.017.

PY18 ______, *Derived non-archimedean analytic spaces*, Selecta Math. (N.S.) **24** (2018), no. 2, 609–665; doi:10.1007/s00029-017-0310-1.

PY20a ______, *Non-archimedean quantum K-invariants*, 2020, arXiv:2001.05515.

PY20b ______, *Representability theorem in derived analytic geometry*, J. Eur. Math. Soc. (JEMS) **22** (2020), no. 12, 3867–3951; doi:10.4171/jems/998.

PY21 ______, *Derived Hom spaces in rigid analytic geometry*, Publ. Res. Inst. Math. Sci. **57** (2021), no. 3–4, 921–958; doi:10.4171/prims/57-3-7.

STV15 T. Schürg, B. Toën and G. Vezzosi, *Derived algebraic geometry, determinants of perfect complexes, and applications to obstruction theories for maps and complexes*, J. reine angew. Math. **702** (2015), 1–40; doi:10.1515/crelle-2013-0037.

Sim96 C. Simpson, *Algebraic (geometric) n-stacks*, 1996, arXiv:alg-geom/9609014.

Sim09 ______, *Geometricity of the Hodge filtration on the ∞-stack of perfect complexes over X_{DR}*, Mosc. Math. J. **9** (2009), no. 3, 665–721; doi:10.17323/1609-4514-2009-9-3-665-721.

Tay02 J. L. Taylor, *Several complex variables with connections to algebraic geometry and Lie groups*, Grad. Stud. Math., vol. 46 (Amer. Math. Soc., Providence, RI, 2002); doi:10.1090/gsm/046.

TV07 B. Toën and M. Vaquié, *Moduli of objects in dg-categories*, Ann. Sci. Éc. Norm. Supér. (4) **40** (2007), no. 3, 387–444; doi:10.1016/j.ansens.2007.05.001.

TV08a ______, *Algébrisation des variétés analytiques complexes et catégories dérivées*, Math. Ann. **342** (2008), no. 4, 789–831; doi:10.1007/s00208-008-0257-9.

TV08b B. Toën and G. Vezzosi, *Homotopical algebraic geometry. II. Geometric stacks and applications*, Mem. Amer. Math. Soc. **193** (2008), no. 902; doi:10.1090/memo/0902.

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