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On the global generation of higher direct images of pluricanonical bundles

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Abstract

Given a fibration f between two projective manifolds X and Y, we discuss the effective generation of the higher direct images $R^i f_*(K_X^m)$, where K_X^m is the mth tensor power of the canonical bundle of X. In particular, we answer two questions proposed by Popa–Schnell (2014).

1. Introduction

Assume that $f\colon X \to Y$ is a fibration, that is, a surjective morphism with connected fibres between two projective manifolds X and Y. Denote by K_X^m the mth tensor power of the canonical bundle K_X of X. The positivity of the associated higher direct image $R^i f_*(K_X^m)$ is of significant importance for understanding the geometry of this fibration. Fruitful results have been obtained on this subject, such as [BP08, Ber09, Hör10, Kaw81, Kaw82, Kol86a, Kol86b, Kol87, Vie82, Vie83].

Popa and Schnell [PS14] proved the following result inspired by the brilliant work of Viehweg [Vie82, Vie83] and Kollár [Kol86a, Kol86b].

Theorem 1.1 (Popa–Schnell). *Let $f\colon X \to Y$ be a fibration between two projective manifolds with $\dim Y = n$, and let A be an ample and globally generated line bundle on Y. If $m \geqslant 1$ is an integer, then the sheaf*

$$f_*\big(K_X^m\big) \otimes A^l$$

is 0-regular, and therefore globally generated, for $l \geqslant m(n+1)$.

Popa and Schnell then proposed the question of whether the similar result holds for higher direct images; see [PS14, Question after Corollary 2.10]. In this paper, we give a positive answer to this question in some sense. Indeed, since by Proposition 2.2 (see Remark 2.3),

$$f_*\big(K_X^m \otimes \mathscr{I}\big(f, \big\|K_X^{m-1}\big\|\big)\big) = f_*\big(K_X^m\big)\,,$$

it is reasonable to involve the asymptotic multiplier ideal when we consider higher direct images. So our main result is as follows; Theorem 1.1 is the case $i = 0$.

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Theorem 1.2. *Let $f\colon X\to Y$ be a fibration between two projective manifolds with $\dim Y=n$, and let A be an ample and globally generated line bundle on Y. If $m\geqslant 1$ is an integer, then the sheaf*

$$R^i f_*\big(K_X^m\otimes \mathscr{I}(f,\|K_X^{m-1}\|)\big)\otimes A^l$$

is 0-regular, and therefore globally generated, for $i\geqslant 0$ and $l\geqslant m(n+1)$.

We use the strategy in [Kol86a, Kol86b] to prove Theorem 1.2. The idea is expanded as follows. First we prove a Kollár-type vanishing theorem.

Theorem 1.3. *Let $f\colon X\to Y$ be a fibration between two projective manifolds with $\dim Y=n$, and let A be an ample and globally generated line bundle on Y. If $m\geqslant 1$ is an integer, then for any $l\geqslant (m-1)n+m$, $i\geqslant 0$ and $q>0$,*

$$H^q\big(Y,R^i f_*\big(K_X^m\otimes \mathscr{I}(f,\|K_X^{m-1}\|)\big)\otimes A^l\big)=0\,.$$

The Kollár-type vanishing theorem has been extensively studied. See, for example, [Eno93, Fuj12, FM21, GM17, Kol86a, Kol86b, Mat14, Mat16, Ohs84, Wu20]. However, we cannot directly apply any result among these papers to obtain Theorem 1.3. The reason is that in general there does not exist a metric φ on K_X such that

$$i\Theta_{K_X,\varphi}\geqslant 0 \quad\text{and}\quad \mathscr{I}(\varphi)=\mathscr{I}(f,\|K_X\|)\,,$$

where $\Theta_{K_X,\varphi}$ is the curvature associated with φ. We will construct a suitable metric on $K_X\otimes f^*A^{n+1}$ to overcome this problem. More details are given in Section 3.1.

Now Theorem 1.2 is a direct consequence of Theorem 1.3 combined with Castelnuovo–Mumford regularity [Mum66] (also see Section 2.4).

After that, we prove a generic vanishing theorem for the higher direct images, which answers another question of Popa and Schnell [PS14] in some sense; see [PS14, Question after Corollary 5.4]. Note that Shibata [Shi16] provided an example to show that the $R^i f_*(K_X^m)$, $m\geqslant 2$, are not necessarily GV-sheaves (see Definition 2.4; cf. [PP11]). Hence it is quite natural to consider instead $R^i f_*\big(K_X^m\otimes \mathscr{I}(f,\|K_X^{m-1}\|)\big)$.

Theorem 1.4. *Let $f\colon X\to Y$ be a morphism from a projective manifold X to an abelian variety Y. Then the sheaf*

$$R^i f_*\big(K_X^m\otimes \mathscr{I}(f,\|K_X^{m-1}\|)\big)$$

is a GV-sheaf for every $i\geqslant 0$ and $m\geqslant 1$.

This result leads in turn to the following vanishing and generation results, which are stronger than those for morphisms to arbitrary varieties.

Corollary 1.5. *If $f\colon X\to Y$ is a morphism from a projective manifold to an abelian variety and A is an ample line bundle on Y, then for every $i\geqslant 0$ and $m\geqslant 1$ one has the following:*

1. *The sheaf $R^i f_*\big(K_X^m\otimes \mathscr{I}(f,\|K_X^{m-1}\|)\big)$ on Y is nef (see Section 2.3).*
2. *We have $H^q\big(Y,R^i f_*\big(K_X^m\otimes \mathscr{I}(f,\|K_X^{m-1}\|)\big)\otimes A\big)=0$ for all $q>0$.*
3. *The sheaf $R^i f_*\big(K_X^m\otimes \mathscr{I}(f,\|K_X^{m-1}\|)\big)\otimes A^2$ is globally generated.*

We conclude with some further discussions. Note that after [PS14], there are several references such as [Den21, DM19, Dut20, Iwa20] which aim to improve Theorem 1.1. It is remarkable that there is no more global generation required for A in order to give the effective lower bound of l

such that $f_*(K_X^m)\otimes A^l$ is (generically) globally generated. Hence it is natural to try to remove the global generation condition in our theorems.

However, since Theorem 1.2 depends highly on Castelnuovo–Mumford regularity, it is not easy to make such an extension. Currently, we can only make the following generalisation of Theorem 1.3 and also prove a Kollár-type injectivity theorem. These results seem to be of independent interest.

Theorem 1.6. *Let $f\colon X\to Y$ be a smooth fibration between two projective manifolds with $\dim Y=n$, and let A be an ample line bundle on Y. If $m\geqslant 1$ is an integer, then the following results hold:*

(1) *For any $q>0$, $i\geqslant 0$ and $l>(m-1)(n+2)$,*
$$H^q\big(Y,R^if_*\big(K_X^m\otimes\mathscr{I}\big(f,\|K_X^{m-1}\|\big)\big)\otimes A^l\big)=0\,.$$

(2) *For any integer $j\geqslant 0$ and a (non-zero) section s of f^*A^j, the multiplication map induced by the tensor product with s*
$$\Phi\colon H^q\big(X,K_X^m\otimes f^*A^l\otimes\mathscr{I}\big(f,\|K_X^{m-1}\|\big)\big)\longrightarrow H^q\big(X,K_X^m\otimes f^*A^{l+j}\otimes\mathscr{I}\big(f,\|K_X^{m-1}\|\big)\big)$$
is (well defined and) injective for any $q\geqslant 0$ and $l>(m-1)(n+2)$.

This paper is organised as follows. We first recall some background material in Section 2, including the asymptotic multiplier ideal sheaf and the definition of GV-sheaves in the sense of Pareschi and Popa [PP11]. Then, we prove Theorems 1.2 and 1.3 in Section 3, Theorem 1.4 in Section 4 and Theorem 1.6 in Section 5.

2. Preliminaries

In this section we introduce some basic material. Assume that $f\colon X\to Y$ is a fibration between two projective manifolds and L is a holomorphic line bundle on X. Moreover, L^k refers to the kth tensor power with the convention that $L^0=\mathcal{O}_X$ and $L^k=(L^*)^{-k}$ for $k<0$.

2.1 The asymptotic multiplier ideal sheaf

This part is mostly collected from [Laz04].

First recall the definition of the multiplier ideal sheaf associated with an ideal sheaf $\mathfrak{a}\subset\mathcal{O}_X$ and a positive real number c. Let $\mu\colon\tilde{X}\to X$ be a smooth modification such that $\mu^*\mathfrak{a}=\mathcal{O}_{\tilde{X}}(-E)$, where $E+\mathrm{except}(\mu)$ has simple normal crossing support. Here $\mathrm{except}(\mu)$ is the exceptional divisor of μ. Then the multiplier ideal sheaf is defined as
$$\mathscr{I}(c\cdot\mathfrak{a}):=\mu_*\mathcal{O}_{\tilde{X}}\big(K_{\tilde{X}/X}-\lfloor cE\rfloor\big)\,.$$
Here $\lfloor cE\rfloor$ means the round-down.

Now suppose that L is a line bundle on X whose restriction to a general fibre of f has non-negative Iitaka dimension. For a positive integer k, there is a naturally defined homomorphism
$$\rho_k\colon f^*f_*L^k\longrightarrow L^k\,.$$
The relative base-ideal $\mathfrak{a}_{k,f}$ of $|L^k|$ is then defined as the image of the induced homomorphism
$$f^*f_*L^k\otimes L^{-k}\longrightarrow\mathcal{O}_X\,.$$

Hence for a given positive real number c, we have the multiplier ideal sheaf $\mathscr{I}\big((c/k)\cdot\mathfrak{a}_{k,f}\big)$, which is also denoted by $\mathscr{I}\big(f,(c/k)\big|L^k\big|\big)$. Hence

$$\mathscr{I}\left(f,\frac{c}{k}\big|L^k\big|\right)\subseteq\mathcal{O}_X\,.$$

It is not hard to verify that for every integer $p\geqslant 1$ one has the inclusion

$$\mathscr{I}\left(f,\frac{c}{k}\big|L^k\big|\right)\subseteq\mathscr{I}\left(f,\frac{c}{pk}\big|L^{pk}\big|\right).$$

Therefore, the family of ideals

$$\left\{\mathscr{I}\left(f,\frac{c}{k}\big|L^k\big|\right)\right\}_{(k\geqslant 0)}$$

has a unique maximal element from the ascending chain condition on ideals.

Definition 2.1. The relative asymptotic multiplier ideal sheaf associated with f, c and $|L|$,

$$\mathscr{I}(f,c\|L\|)\,,$$

is defined to be the unique maximal member of the family of ideals $\big\{\mathscr{I}\big(f,(c/k)\big|L^k\big|\big)\big\}_{(k\geqslant 0)}$.

Next, we explain the analytic counterpart of the relative multiple ideal sheaf. By definition,

$$\mathscr{I}(f,c\|L\|)=\mathscr{I}\left(f,\frac{c}{k}\big|L^k\big|\right)=\mathscr{I}\left(\frac{c}{k}\cdot\mathfrak{a}_{k,f}\right)$$

for some k. In this case, we will say that k computes $\mathscr{I}(f,c\|L\|)$. Let U be a local coordinate ball of Y. By definition, we can pick $\{u_1,\dots,u_m\}$ in $\Gamma\big(f^{-1}(U),L^k\big)$ which generate $\mathfrak{a}_{k,f}$ on $f^{-1}(U)$. Let $\varphi_U=\log\big(|u_1|^2+\cdots+|u_m|^2\big)$, which is a singular metric on $L^k|_{f^{-1}(U)}$. We verify that

$$\mathscr{I}\left(f,\frac{c}{k}\big|L^k\big|\right)=\mathscr{I}\left(\frac{c}{k}\varphi_U\right)\quad\text{on } f^{-1}(U)\,.$$

Indeed, let $\mu\colon\tilde{X}\to X$ be a smooth modification of $\mathfrak{a}_{k,f}$. Then $\mu^*\mathfrak{a}_{k,f}=\mathcal{O}_{\tilde{X}}(-E)$, with E such that $E+\mathrm{except}(\mu)$ has simple normal crossing support. Now it is computed in [Dem12] that

$$\mathscr{I}\left(\frac{c}{k}\varphi_U\right)=\mu_*\mathcal{O}_{\tilde{X}}\left(K_{\tilde{X}/X}-\left\lfloor\frac{c}{k}E\right\rfloor\right)\ \text{on } f^{-1}(U)\,,$$

which coincides with the definition of $\mathscr{I}\big(f,(c/k)\big|L^k\big|\big)$. Furthermore, if $v_1,\dots,v_m$ are alternative generators and $\psi_U=\log\big(|v_1|^2+\cdots+|v_m|^2\big)$, obviously we have $\mathscr{I}\big((c/k)\varphi_U\big)=\mathscr{I}\big((c/k)\psi_U\big)$. Hence all the $\mathscr{I}\big((c/k)\varphi_U\big)$ patch together to give a globally defined multiplier ideal sheaf $\mathscr{I}\big((c/k)\varphi\big)$ such that

$$\mathscr{I}\left(\frac{c}{k}\varphi\right)=\mathscr{I}\left(f,\frac{c}{k}\big|L^k\big|\right)=\mathscr{I}(f,c\|L\|)\,.$$

Note that $\big\{f^{-1}(U),(1/k)\varphi_U\big\}$ does not give a globally defined metric on L in general. The notation $(1/k)\varphi$ is interpreted as the collection of functions $\big\{f^{-1}(U),(1/k)\varphi_U\big\}$ by abusing the notation; it is called the collection of (local) singular metrics on L associated with $\mathscr{I}(f,c\|L\|)$. Certainly, it depends on the choice of k and is not unique.

The following elementary property is collected from [Laz04].

Proposition 2.2 (cf. [Laz04, Proposition 11.2.15]). *Let $f\colon X\to Y$ be a fibration between two projective manifolds and L be a line bundle on X. Given a positive integer k, let*

$$\mathfrak{a}_{k,f}=\mathfrak{a}\big(f,\big|L^k\big|\big)$$

be the relative base-ideal of $|L^k|$ *relative to* f. *Then the canonical map* $\rho_k\colon f^*f_*L^k \to L^k$ *factors through the inclusion* $L^k \otimes \mathscr{I}(f, \|L^k\|)$; *that is,* $\mathfrak{a}_{k,f} \subseteq \mathscr{I}(f, \|L^k\|)$. *Equivalently, the natural map*

$$f_*\big(L^k \otimes \mathscr{I}(f, \|L^k\|)\big) \longrightarrow f_*(L^k)$$

is an isomorphism.

Remark 2.3. For any $0 \leqslant m \leqslant k$ it follows from Proposition 2.2 that

$$f_*\big(L^k \otimes \mathscr{I}(f, \|L^m\|)\big) = f_*(L^k)\,,$$

where we are using the evident fact that $\mathscr{I}(f, \|L^k\|) \subseteq \mathscr{I}(f, \|L^m\|)$.

2.2 GV-sheaves in the sense of Pareschi and Popa

2.2.1 *Definition.* We now concentrate on the case of more specific morphisms $f\colon X \to Y$, where X is a projective manifold and Y is an abelian variety. We denote by P the normalised Poincaré bundle on the product $Y \times \mathrm{Pic}^0(Y)$ and by P_α its restriction to the slice $Y \times \{\alpha\}$; this is of course just a different name for the point $\alpha \in \mathrm{Pic}^0(Y)$.

Definition 2.4 (cf. [PP11]). A coherent sheaf $\mathcal{F}$ on X is said to be a *GV*-sheaf (where *GV* stands for "generic vanishing") if

$$\operatorname{codim}_{\mathrm{Pic}^0(Y)}\big\{\alpha \in \mathrm{Pic}^0(Y) \mid H^q(X, \mathcal{F} \otimes f^*P_\alpha) \neq 0\big\} \geqslant q$$

for every $q \geqslant 0$.

2.2.2 *A brief review of former results.* If f is generically finite, then K_X is a *GV*-sheaf by a special case of the generic vanishing theorem of Green and Lazarsfeld [GL87]. This result was generalised by Hacon [Hac04], to the effect that for an arbitrary f the higher direct images $R^if_*(K_X)$ are *GV*-sheaves on Y for all $i \geqslant 0$. On the other hand, there exist simple examples showing that even when f is generically finite, the powers K_X^m with $m \geqslant 2$ are not necessarily *GV*-sheaves; see [PP11, Example 5.6]. Therefore, it is quite surprising that Popa and Schnell [PS14] showed that the $f_*(K_X^m)$ are *GV*-sheaves on Y for all $m \geqslant 1$. They then asked whether the higher direct images are also *GV*-sheaves. We positively answer this question in some sense in Theorem 1.4; that is, we prove that the $R^if_*(K_X^m \otimes \mathscr{I}(f, \|K_X^{m-1}\|))$ are *GV*-sheaves.

Note that the $R^if_*(K_X^m)$ are not necessarily *GV*-sheaves. In fact, in [Shi16] T. Shibata constructed such a counterexample; see Example 4.5 there. After that, T. Shibata proved that the $R^if_*(K_X^m)$ are *GV*-sheaves under the assumptions that $\dim X = 2$ and $\kappa(X) \geqslant 0$; see [Shi16, Proposition 4.8]. We will give a brief illustration (see Remark 4.1) that our Theorem 1.4 actually implies this result. Therefore, it is quite natural to consider the twist by the asymptotic multiplier ideal.

2.3 Nef coherent sheaves

With any coherent sheaf $\mathcal{F}$ on a projective manifold Y, one associates the scheme [Har77]

$$\mathbb{P}(\mathcal{F}) := \mathrm{Proj}\big(\oplus_{m\geqslant 0}\mathrm{Sym}^m\mathcal{F}\big)$$

and an invertible sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ on $\mathbb{P}(\mathcal{F})$. Then we have the following definition.

Definition 2.5. A coherent sheaf $\mathcal{F}$ is said to be nef if $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ is.

2.4 Castelnuovo–Mumford regularity

Finally, we review the definition of and a basic result concerning Castelnuovo–Mumford regularity [Mum66].

Definition 2.6. Let X be a projective manifold and L an ample and globally generated line bundle on X. Given an integer m, a coherent sheaf F on X is m-regular with respect to L if for all $i \geqslant 1$, we have $H^i\big(X, F \otimes L^{m-i}\big) = 0$.

Theorem 2.7 (cf. [Mum66]). *Let X be a projective manifold and L an ample and globally generated line bundle on X. If F is a coherent sheaf on X that is m-regular with respect to L, then the sheaf $F \otimes L^m$ is globally generated.*

3. Main theorem

3.1 The vanishing theorem

We first prove Theorem 1.3. We will apply the following Kollár-type vanishing theorem.

Theorem 3.1 (cf. [FM21, Theorem D]). *Let $f\colon X \to Y$ be a surjective morphism from a compact Kähler manifold X onto a projective variety Y. Let F be a holomorphic line bundle on X with a (singular) metric φ_F such that $i\Theta_{F,\varphi_F} \geqslant 0$. Let N be a holomorphic line bundle on X. Assume that there exist two positive integers a and b and an ample line bundle A on Y such that $N^a = f^*A^b$. Then*

$$H^q(Y, R^i f_*(K_X \otimes F \otimes \mathscr{I}(\varphi_F) \otimes N)) = 0$$

for every $q > 0$ and $i \geqslant 0$.

We cannot pick K_X^{m-1} as F in Theorem 3.1 since in general we cannot obtain a metric ψ on K_X such that

$$i\Theta_{K_X,\psi} \geqslant 0 \quad \text{and} \quad \mathscr{I}((m-1)\psi) = \mathscr{I}\big(f, \big\|K_X^{m-1}\big\|\big)\,.$$

Instead we consider $F = K_X^{m-1} \otimes f^*A^{(m-1)(n+1)}$, where $n = \dim Y$.

Proof of Theorem 1.3. Let $\{U_\alpha\}$ be a finite local coordinate chart of Y. Let p be a divisible and large enough integer which computes $\mathscr{I}\big(f, \big\|K_X^{m-1}\big\|\big)$, and let $\mathfrak{a}_{p,f}$ be the relative base-ideal of $\big|K_X^p\big|$. Then

$$\mathscr{I}\big(p^{-1}(m-1)\cdot \mathfrak{a}_{p,f}\big) = \mathscr{I}\big(f, \big\|K_X^{m-1}\big\|\big)\,.$$

Pick $\{u_{i,\alpha}\} \subseteq \Gamma\big(f^{-1}(U_\alpha), K_X^p\big)$ to be the local generators of $\mathfrak{a}_{p,f}$.

Now $f_*\big(K_X^p\big) \otimes A^{p(n+1)}$ is globally generated by Theorem 1.1. Using this fact, we will show that there exist sections

$$v_{ij,\alpha} \subseteq H^0\big(X, K_X^p \otimes f^*A^{p(n+1)}\big)$$

that together define an ideal sheaf $\mathfrak{a}$ with

$$\mathscr{I}\big(p^{-1}(m-1)\cdot \mathfrak{a}\big) = \mathscr{I}\big(p^{-1}(m-1)\mathfrak{a}_{p,f}\big)\,.$$

In fact, if we denote, by abusing the notation, by

$$H^0\big(Y, f_*\big(K_X^p\big) \otimes A^{p(n+1)}\big) \quad \text{and} \quad H^0\big(X, K_X^p \otimes f^*A^{p(n+1)}\big)$$

the trivial vector bundles on Y and X, respectively, then the morphisms

$$H^0\big(Y, f_*(K_X^p) \otimes A^{p(n+1)}\big) \otimes \mathcal{O}_Y \longrightarrow f_*\big(K_X^p\big) \otimes A^{p(n+1)}\,,$$
$$H^0\big(X, K_X^p \otimes f^*A^{p(n+1)}\big) \otimes \mathcal{O}_X \longrightarrow f^*f_*\big(K_X^p\big) \otimes f^*A^{p(n+1)}$$

are both surjective by definition. Therefore,

$$H^0\big(X, K_X^p \otimes f^*A^{p(n+1)}\big) \otimes K_X^{-p} \longrightarrow f^*f_*\big(K_X^p\big) \otimes K_X^{-p} \otimes f^*A^{p(n+1)}$$

is also surjective. In particular, we have the following surjection:

$$H^0\big(X, K_X^p \otimes f^*A^{p(n+1)}\big) \otimes K_X^{-p} \longrightarrow \mathfrak{a}_{p,f} \otimes f^*A^{p(n+1)}\,, \tag{3.1}$$

where $\mathfrak{a}_{p,f} \otimes f^*A^{p(n+1)}$ by definition is the image of the natural morphism

$$f^*f_*\big(K_X^p\big) \otimes K_X^{-p} \otimes f^*A^{p(n+1)} \longrightarrow f^*A^{p(n+1)}\,.$$

Let $\{s_j\}$ be a set of global sections that generates $f^*A^{p(n+1)}$. Due to (3.1), all of the sections $u_{i,\alpha} \otimes s_j$ extend over X as global sections $v_{ij,\alpha}$ of

$$H^0\big(X, K_X^p \otimes f^*A^{p(n+1)}\big)\,.$$

Since $\{s_j\}$ generates $f^*A^{p(n+1)}$, as i, j vary, the sections $v_{ij,\alpha}$ generate $\mathfrak{a}_{p,f}$ on $f^{-1}(U_\alpha)$.

Now the sections $v_{ij,\alpha}$, as i, j, α vary, together define an ideal sheaf $\mathfrak{a}$. Next we prove that

$$\mathscr{I}\big(p^{-1}(m-1)\cdot\mathfrak{a}\big) = \mathscr{I}\big(f, \big\|K_X^{m-1}\big\|\big)\,.$$

Let $\mathfrak{a}_\alpha$ be the restriction of $\mathfrak{a}_{p,f}$ on $f^{-1}(U_\alpha)$. Then by the choice of $\{u_{i,\alpha}\}$, for every α we claim that

$$\mathscr{I}\big(p^{-1}(m-1)\cdot\mathfrak{a}\big) = \mathscr{I}\big(p^{-1}(m-1)\cdot\mathfrak{a}_\alpha\big) \text{ on } f^{-1}(U_\alpha). \tag{3.2}$$

In fact, $\mathscr{I}\big(p^{-1}(m-1)\cdot\mathfrak{a}_\alpha\big)$ is just the restriction of $\mathscr{I}\big(f, p^{-1}(m-1)|K_X^p|\big)$ on $f^{-1}(U_\alpha)$. However,

$$\mathscr{I}\big(f, p^{-1}(m-1)\big|K_X^p\big|\big) = \mathscr{I}\big(f, \big\|K_X^{m-1}\big\|\big)$$

is now the unique maximal element in

$$\left\{\mathscr{I}\left(f, \frac{1}{k}\big|K_X^{k(m-1)}\big|\right)\right\}_{(k\geqslant 0)}\,.$$

So $\mathscr{I}\big(f, p^{-1}(m-1)\big|K_X^p\big|\big)$, as well as $\mathscr{I}\big(p^{-1}(m-1)\cdot\mathfrak{a}_\alpha\big)$, should be stable. In other words, if u is a section of $\Gamma\big(f^{-1}(U_\alpha), K_X^p\big)$, we must have

$$\mathscr{I}\big(p^{-1}(m-1)\cdot\mathfrak{a}_\alpha\big) = \mathscr{I}\big(f, p^{-1}(m-1)\big|K_X^p\big|\big)\big|_{f^{-1}(U_\alpha)} = \mathscr{I}\big(p^{-1}(m-1)\cdot(\mathfrak{a}_\alpha \cup (u))\big)\,.$$

Here $\mathfrak{a}_\alpha \cup (u)$ refers to the ideal sheaf generated by both of $\mathfrak{a}_\alpha$ and u.

On the other hand, by construction the sections $v_{ij,\alpha}$, as i, j vary, also generate $\mathfrak{a}_\alpha$ on $f^{-1}(U_\alpha)$. By stability, the sections $v_{ij,\alpha}$, as i, j, α vary, will lead to the same multiplier ideal sheaf here. This implies (3.2). Then

$$\mathscr{I}\big(p^{-1}(m-1)\cdot\mathfrak{a}\big) = \mathscr{I}\big(p^{-1}(m-1)\cdot\mathfrak{a}_{p,f}\big) = \mathscr{I}\big(f, \big\|K_X^{m-1}\big\|\big)$$

on $f^{-1}(U_\alpha)$, hence everywhere. Let χ be the (singular) metric on $K_X^p \otimes f^*A^{p(n+1)}$ defined by the sections $v_{ij,\alpha}$ as i, j, α vary. Based on the discussions above, we conclude that χ has positive associated curvature current and

$$\mathscr{I}\big(p^{-1}(m-1)\chi\big) = \mathscr{I}\big(f, \big\|K_X^{m-1}\big\|\big)\,.$$

Now let

$$(F, \varphi_F) = \big(K_X^{m-1} \otimes f^*A^{(m-1)(n+1)}, p^{-1}(m-1)\chi\big) \quad \text{and} \quad N = f^*A^{l-(m-1)(n+1)}\,.$$

Since $l \geqslant (m-1)n+m$ by hypothesis, $l-(m-1)(n+1) \geqslant 1$ and N is a positive multiple of f^*A. Applying Theorem 3.1 (with the same notation as there), we then obtain the desired vanishing result from the fact that

$$R^if_*(K_X \otimes F \otimes \mathscr{I}(\varphi_F) \otimes N) = R^if_*\big(K_X^m \otimes \mathscr{I}\big(f, \|K_X^{m-1}\|\big)\big) \otimes A^l\,. \qquad \square$$

3.2 Global generation

Using Theorem 1.3, we can prove the global generation of higher direct images, namely Theorem 1.2, based on Castelnuovo–Mumford regularity.

Proof of Theorem 1.2. Since $l \geqslant m(n+1)$, we have $l-q \geqslant (m-1)n+m$ when $q \leqslant n$. It then follows from Theorem 1.3 that for every $q \geqslant 1$

$$H^q\big(Y, R^if_*\big(K_X^m \otimes \mathscr{I}\big(f, \|K_X^{m-1}\|\big)\big) \otimes A^{l-q}\big) = 0\,.$$

Hence the sheaf $R^if_*\big(K_X^m \otimes \mathscr{I}\big(f, \|K_X^{m-1}\|\big)\big) \otimes A^l$ is 0-regular with respect to A. So it is globally generated, by Theorem 2.7. $\square$

4. Generic vanishing

We first prove that the higher direct images are GV-sheaves.

Proof of Theorem 1.4. In view of [PP11, PS14], it is enough to show that for every finite étale morphism $\beta\colon Z \to Y$ of abelian varieties and an ample and globally generated line bundle H on Z, we have

$$H^q\big(Z, H^l \otimes \beta^*R^if_*\big(K_X^m \otimes \mathscr{I}\big(f, \|K_X^{m-1}\|\big)\big)\big) = 0 \tag{4.1}$$

for every $q > 0$ with l large enough. Here *large enough* means that there exists a bound d depending only on n and m such that vanishing (4.1) holds for any Z and H as long as $l \geqslant d$. In particular, we cannot apply the Serre asymptotic vanishing theorem [Har77] here.

Now we prove this vanishing result. Let $W := Z \times_Y X$ be the fibre product [Har77]. Then we have the following commutative diagram:

$$\begin{array}{ccc} W & \xrightarrow{\alpha} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[\beta]{} & Y\,. \end{array}$$

By construction, α is also finite and étale. Hence we have $\alpha^*K_X = K_W$ and

$$\alpha^*\mathscr{I}\big(f, \|K_X^{m-1}\|\big) = \mathscr{I}\big(g, \|K_W^{m-1}\|\big)$$

by the behaviour of asymptotic multiplier ideals under étale covers (cf. [Laz04, Theorem 11.2.16]).

By the flat base change theorem [Har77],

$$\beta^*R^if_*\big(K_X^m \otimes \mathscr{I}\big(f, \|K_X^{m-1}\|\big)\big) \simeq R^ig_*\big(K_W^m \otimes \mathscr{I}\big(f, \|K_W^{m-1}\|\big)\big)\,.$$

Hence we obtain the vanishing (4.1) by Theorem 1.3. $\square$

Remark 4.1. A variant of Theorem 1.4 can actually recover the following proposition, which is originally obtained in [Shi16].

Proposition 4.2 (cf. [Shi16, Proposition 4.8]). *Let $f\colon X \to Y$ be a morphism from a smooth projective surface X to an abelian variety Y. Assume $\kappa(X) \geqslant 0$. Then the $R^i f_*(K_X^m)$ are GV-sheaves for all $i \geqslant 0$ and $m \geqslant 1$.*

Re-proof via Theorem 1.3. The proof is similar to the one in [Shi16], hence we only sketch it.

We may assume without loss of generality that $\dim f(X) \geqslant 1$. Then it is enough to show that $R^1 f_*(K_X^m)$ is a GV-sheaf. Since $\kappa(X) \geqslant 0$, we can take a series of contractions of (-1)-curves $\varepsilon\colon X \to X'$ such that $K_{X'}$ is semi-ample, and obtain a natural morphism $f'\colon X' \to Y$ such that $f' \circ \varepsilon = f$. Combined with the Leray spectral sequence [Har77], it is left to prove that

$$H^q\big(Y, f'_* R^1\varepsilon_*\big(K_X^m\big) \otimes H^l\big) = 0\,, \tag{4.2}$$

$$H^q\big(Y, R^1 f'_*\big(K_{X'}^m\big) \otimes H^l\big) = 0 \tag{4.3}$$

for every $q > 0$ and any ample and globally generated line bundle H on Y with l *large enough*. The vanishing (4.2) is quite obvious for dimension reasons, while (4.3) is a direct consequence of Theorem 1.3 since the semi-ampleness implies that $\mathscr{I}\big(f', \big\|K_{X'}^{m-1}\big\|\big) = \mathcal{O}_{X'}$. □

Once we know generic vanishing, the situation is in fact much better than what we obtained for morphisms to arbitrary varieties.

Proof of Corollary 1.5. In view of [PS14, Corollary 5.4], everything is immediately verified once we know that $R^i f_*\big(K_X^m \otimes \mathscr{I}\big(f, \big\|K_X^{m-1}\big\|\big)\big)$ is a GV-sheaf. Hence we omit the proof here. □

5. Further discussions

In this section, we prove Theorem 1.6 based on results in [Den21, DM19, Dut20, Iwa20]. More precisely, we will apply the following theorem.

Theorem 5.1 (cf. [Iwa20, Theorem 1.4]). *Let $f\colon X \to Y$ be a fibration between two projective manifolds with $\dim Y = n$, and let A be an ample line bundle on Y. For any integer $m \geqslant 1$ and $l \geqslant \frac{1}{2}n(n-1) + m(n+1)$, the sheaf*

$$f_*\big(K_X^m\big) \otimes A^l$$

is generated by the global sections at a regular value y of f.

The proof of Theorem 1.6 then involves the same argument as Theorem 1.3 with slight adjustment.

Proof of Theorem 1.6. (1) Let $\{U_\alpha\}$ be a finite local coordinate chart of Y. Let p be a divisible and large enough integer which computes $\mathscr{I}\big(f, \big\|K_X^{m-1}\big\|\big)$. Moreover, $f^*A^{p(n+2)}$ is globally generated and

$$p \geqslant \tfrac{1}{2}n(n-1)\,. \tag{5.1}$$

Let $\mathfrak{a}_{p,f}$ be the relative base-ideal of $\big|K_X^p\big|$, and let $\{u_{i,\alpha}\} \subseteq \Gamma\big(f^{-1}(U_\alpha), K_X^p\big)$ be local generators.

We write inequality (5.1) as

$$p(n+2) \geqslant \tfrac{1}{2}n(n-1) + p(n+1)\,.$$

So by Theorem 5.1 the sheaf $f_*(K_X^p)\otimes A^{p(n+2)}$ is globally generated on Y. Now let $\{s_j\}$ be a set of global sections that generates $f^*A^{p(n+2)}$. Then as is shown in the proof of Theorem 1.3, all of the sections $u_{i,\alpha}\otimes s_j$ extend over X as sections $v_{ij,\alpha}$ in $H^0\big(X, K_X^p\otimes f^*A^{p(n+2)}\big)$. The sections $v_{ij,\alpha}$, as i, j, α vary, together define a (singular) metric χ on $K_X^p\otimes f^*A^{p(n+2)}$ with positive curvature current. Then by the same argument as Theorem 1.3, we obtain

$$\mathscr{I}\big(p^{-1}(m-1)\chi\big)=\mathscr{I}\big(f,\|K_X^{m-1}\|\big).$$

Let

$$(F,\varphi_F)=\big(K_X^{m-1}\otimes f^*A^{(m-1)(n+2)},p^{-1}(m-1)\chi\big),$$

and let $N=f^*A^{l-(m-1)(n+2)}$. Since $l>(m-1)(n+2)$ by hypothesis, $l-(m-1)(n+2)\geqslant 1$. Now we have

$$R^if_*(K_X\otimes F\otimes\mathscr{I}(\varphi_F)\otimes N)=R^if_*\big(K_X^m\otimes\mathscr{I}\big(f,\|K_X^{m-1}\|\big)\big)\otimes A^l.$$

Then by Theorem 3.1 (with the same notation as there), we obtain the desired vanishing result.

(2) The strategy is to apply the following injectivity theorem in [Mat14].

Theorem 5.2 (cf. [Mat14, Theorem 1.5]). *Let (F,φ_F) and (M,φ_M) be line bundles with (singular) metrics on a compact Kähler manifold X. Assume the following conditions:*

(a) *There exists a subvariety Z on X such that φ_F and φ_M are smooth on $X\setminus Z$.*
(b) *We have $i\Theta_{F,\varphi_F}\geqslant\gamma$ and $i\Theta_{M,\varphi_M}\geqslant\gamma$ on X for some smooth $(1,1)$-form γ on X.*
(c) *We have $i\Theta_{F,\varphi_F}\geqslant\varepsilon i\Theta_{M,\varphi_M}$ for some positive number $\varepsilon>0$.*

Then for a (non-zero) section s of M with $\sup_X|s|_{\varphi_M}<\infty$, the multiplication map induced by the tensor product with s,

$$\Phi_s\colon H^q(X,K_X\otimes F\otimes\mathscr{I}(\varphi_F))\longrightarrow H^q(X,K_X\otimes F\otimes M\otimes\mathscr{I}(\varphi_F+\varphi_M)),$$

is (well defined and) injective for any q.

From part (1) we know that $\big(K_X^{m-1}\otimes f^*A^{(m-1)(n+2)},p^{-1}(m-1)\chi\big)$ is a Hermitian line bundle with positive curvature current and satisfies

$$\mathscr{I}\big(p^{-1}(m-1)\chi\big)=\mathscr{I}\big(f,\|K_X^{m-1}\|\big).$$

In particular, χ is smooth outside a subvariety. Let ψ be a smooth metric on f^*A with positive curvature. Let

$$(F,\varphi_F)=\big(K_X^{m-1}\otimes f^*A^l,p^{-1}(m-1)\chi+(l-(m-1)(n+2))\psi\big)$$

and $(M,\varphi_M)=\big(f^*A^j,j\psi\big)$. Since $l>(m-1)(n+2)$ by hypothesis, $l-(m-1)(n+2)\geqslant 1$. Therefore, let $\varepsilon=1/j$; we have $i\Theta_{F,\varphi_F}\geqslant\varepsilon i\Theta_{M,\varphi_M}\geqslant 0$. Moreover, since ψ is smooth,

$$\mathscr{I}(\varphi_F+\varphi_M)=\mathscr{I}\big(p^{-1}(m-1)\chi\big)=\mathscr{I}\big(f,\|K_X^{m-1}\|\big).$$

So the proof is finished by directly applying Theorem 5.2. □

Remark 5.3. Note that the ingredient here is the fact that $f_*(K_X^p)\otimes A^{p(n+2)}$ is globally generated on the whole Y. This is deduced from the main results in [Den21, DM19, Dut20, Iwa20] due to the assumption that f is smooth.

We hope to prove Theorem 1.6 without requiring that f is a smooth fibration in the future, which would make Theorem 1.6 more widely applicable.

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References

Ber09 B. Berndtsson, *Curvature of vector bundles associated to holomorphic fibrations*, Ann. of Math. (2) **169** (2009), no. 2, 531–560; doi:10.4007/annals.2009.169.531.

BP08 B. Berndtsson and M. Păun, *Bergman kernels and the pseudoeffectivity of relative canonical bundles*, Duke Math. J. **145** (2008), no. 2, 341–378; doi:10.1215/00127094-2008-054.

Dem12 J.-P. Demailly, *Analytic methods in algebraic geometry*, Surv. Mod. Math., vol. 1 (International Press, Somerville, MA; Higher Education Press, Beijing, 2012).

Den21 Y. Deng, *Applications of the Ohsawa–Takegoshi Extension Theorem to Direct Image Problems*, Int. Math. Res. Not. IMRN **2021** (2021), no. 23, 17611–17633; doi:10.1093/imrn/rnaa018.

Dut20 Y. Dutta, *On the Effective Freeness of the Direct Images of Pluricanonical Bundles*, Ann. Inst. Fourier (Grenoble) **70** (2020), no. 4, 1545–1561; doi:10.5802/aif.3351.

DM19 Y. Dutta and T. Murayama, *Effective generation and twisted weak positivity of direct images*, Algebra Number Theory **13** (2019), no. 2, 425–454; doi:10.2140/ant.2019.13.425.

Eno93 I. Enoki, *Kawamata-Viehweg vanishing theorem for compact Kähler manifolds*, Einstein metrics and Yang–Mills connections (Sanda, 1990), Lecture Notes in Pure and Appl. Math., vol. 145 (Dekker, New York, 1993), 59–68.

Fuj12 O. Fujino, *A transcendental approach to Kollár's injectivity theorem*, Osaka J. Math. **49** (2012), no. 3, 833–852.

FM21 O. Fujino and S. Matsumura, *Injectivity theorem for pseudo-effective line bundles and its applications*, Trans. Amer. Math. Soc. Ser. B **8** (2021), 849–884; doi:10.1090/btran/86.

GM17 Y. Gongyo and S. Matsumura, *Versions of injectivity and extension theorems*, Ann. Sci. Éc. Norm. Supér. (4) **50** (2017), no. 2, 479–502; doi:10.24033/asens.2325.

GL87 M. Green and R. Lazarsfeld, *Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville*, Invent. Math. **90** (1987), no. 2, 389–407; doi:10.1007/BF01388711.

Hac04 C. D. Hacon, *A derived category approach to generic vanishing*, J. reine angew. Math. **575** (2004), 173–187; doi:10.1515/crll.2004.078.

Har77 R. Hartshorne, *Algebraic geometry*, Grad. Texts in Math., vol. 52 (Springer-Verlag, New York-Heidelberg, 1977); doi:10.1007/978-1-4757-3849-0.

Hör10 A. Höring, *Positivity of direct image sheaves—a geometric point of view*, Enseign. Math. (2) **56** (2010), no. 1-2, 87–142; doi:10.4171/LEM/56-1-4.

Iwa20 M. Iwai, *On the global generation of direct images of pluri-adjoint line bundles*, Math. Z. **294** (2020), no. 1-2, 201–208; doi:10.1007/s00209-019-02266-0.

Kaw81 Y. Kawamata, *Characterization of abelian varieties*, Compos. Math. **43** (1981), no. 2, 253–276.

Kaw82 ———, *A generalization of Kodaira–Ramanujam's vanishing theorem*, Math. Ann. **261** (1982), no. 1, 43–46; doi:10.1007/BF01456407.

Kol86a J. Kollár, *Higher direct images of dualizing sheaves. I*, Ann. of Math. (2) **123** (1986), no. 1, 11–42; doi:10.2307/1971351.

Kol86b ———, *Higher direct images of dualizing sheaves. II*, Ann. of Math. (2) **124** (1986), no. 1, 171–202; doi:10.2307/1971390.

Kol87 ———, *Subadditivity of the Kodaira dimension: fibers of general type*, Algebraic Geometry (Sendai, 1985), Adv. Stud. Pure Math., vol. 10 (North-Holland, Amsterdam, 1987), 361–398; doi:10.2969/aspm/01010361.

Laz04 R. Lazarsfeld, *Positivity in algebraic geometry. II*, Ergeb. Math. Grenzgeb. (3), vol. 49 (Springer-Verlag, Berlin, 2004); doi:10.1007/978-3-642-18808-4.

Mat14 S. Matsumura, *A Nadel vanishing theorem via injectivity theorems*, Math. Ann. **359** (2014), no. 3-4, 785–802; doi:10.1007/s00208-014-1018-6.

Mat16 ———, *A vanishing theorem of Kollár–Ohsawa type*, Math. Ann. **366** (2016), no. 3-4, 1451–1465; doi:10.1007/s00208-016-1371-8.

Mum66 D. Mumford, *Lectures on Curves on an Algebraic Surface*, Ann. of Math. Stud., vol. 59 (Princeton Univ. Press, Princeton, NJ, 1966); doi:10.1515/9781400882069.

Ohs84 T. Ohsawa, *Vanishing theorems on complete Kähler manifolds*, Publ. Res. Inst. Math. Sci. **20** (1984), no. 1, 21–38; doi:10.2977/prims/1195181825.

PP11 G. Pareschi and M. Popa, *GV-sheaves, Fourier–Mukai transform, and generic vanishing*, Amer. J. Math. **133** (2011), no. 1, 235–271; doi:10.1353/ajm.2011.0000.

PS14 M. Popa and C. Schnell, *On direct images of pluricanonical bundles*, Algebra Number Theory **8** (2014), no. 9, 2273–2295; doi:10.2140/ant.2014.8.2273.

Shi16 T. Shibata, *On generic vanishing for pluricanonical bundles*, Michigan Math. J. **65** (2016), no. 4, 873–888; doi:10.1307/mmj/1480734025.

Vie82 E. Viehweg, *Die Additivität der Kodaira Dimension für projektive Faserräume über Varietäten des allgemeinen Typs*, J. reine angew. Math. **330** (1982), 132–142; doi:10.1515/crll.1982.330.132.

Vie83 ———, *Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces*, Algebraic Varieties and Analytic Varieties (Tokyo, 1981), Adv. Stud. Pure Math., vol. 1 (North-Holland, Amsterdam, 1983), 329–353; doi:10.2969/aspm/00110329.

Wu20 J. Wu, *A Kollár-type vanishing theorem*, Math. Z. **295** (2020), no. 1-2, 331–340; doi:10.1007/s00209-019-02406-6.

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