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Compactifying the rank 2 Hitchin system via spectral data on semistable curves

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Abstract

We study resolutions of the rational map to the moduli space of stable curves that associates with a point in the Hitchin base the spectral curve. In the rank 2 case, the answer is given in terms of the space of quadratic multi-scale differentials introduced by Bainbridge, Chen, Gendron, Grushevsky and Möller. This space defines a compactification (of the projectivization) of the regular locus of the $\mathrm{GL}(2,\mathbb{C})$-Hitchin base and provides a compactification of the Hitchin system by compactified Jacobians of pointed stable curves.

We show how the classical $\mathrm{GL}(2,\mathbb{C})$- and $\mathrm{SL}(2,\mathbb{C})$-spectral correspondence extend to the compactified Hitchin system by a correspondence along an admissible cover between torsion-free rank 1 sheaves and (multi-scale) Higgs pairs of rank 2.

1. Introduction

The complexity of the Hitchin fibration on the moduli space of Higgs bundles stems from the variety of singularities of the spectral curve. Already in the case of $\mathrm{GL}(2,\mathbb{C})$-Higgs bundles, non-reduced curves and all singularities locally given by the equation $\lambda^2 - z^\ell$ have to be considered. In this paper, we investigate the possible modifications of the Hitchin base over which the family of smooth spectral curves on the regular locus can be extended as a family of *semistable curves*. One might ask for a minimal modification or for a modification that allows for a nice modular interpretation and a spectral correspondence. We give an answer to both questions in the rank 2 case.

The construction of the modified Hitchin base is motivated by the compactification of strata of quadratic differentials in [BCG+19a] and works in families as the Riemann surface X varies. In particular, it allows us to study the (modified) Higgs moduli space on any stable degeneration of X. Our definition of multi-scale Higgs pairs works uniformly for smooth and stable X. We compare this viewpoint with existing constructions on degenerating families.

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Singular fibers. We start with a summary of the knowledge about the singular Hitchin fibers. In general, the spectral curve is a complex algebraic curve that can be reducible and non-reduced, with planar singularities. The corresponding Hitchin fibers were put in bijection with torsion-free sheaves on the spectral curve by [BNR89] and [Sch98]. For $G = \mathrm{SL}(2,\mathbb{C})$, their geometry was studied in [GO13] using parabolic modules. From the point of view of algebraically completely integrable systems, the singular fibers were studied in [Hit19], establishing lower-dimensional integrable systems supported on the critical locus. The fibers of these sub-integrable systems are abelian varieties and were augmented in [Hor22] by non-abelian coordinates to a full description of singular Hitchin fibers with integral spectral curve for $G = \mathrm{SL}(2,\mathbb{C})$. The non-abelian parameters can be easily understood as *stratified space*. The global description is more delicate, revealing a list of moduli of non-abelian (Hecke) parameters indexed by the singularities of type A. For A_k with $k = 1, 2$, one obtains a $\mathbb{P}^1$, for $k = 3, 4$ a $\mathbb{P}(1,1,2)$ and in general a complex projective variety of dimension $\lfloor k/2 \rfloor$. Another complication is a relation between the abelian and non-abelian parameters in the case of multi-branched singularities. One motivation for the present work is to reduce the complexity by only allowing nodes for singularities.

The spectral map and its resolution. Let $\mathbb{A}_k^\circ$ be the Hitchin base for $\mathrm{SL}(k,\mathbb{C})$, the sum of the vector spaces of global k-differentials on a fixed complex projective curve X for $k \geqslant 2$. Associating with X and a point in the Hitchin base the spectral curve defines a rational map to the moduli space of stable curves that we call the *spectral map*. Our first result in Section 4 determines the resolution of this rational map in the case $k = 2$, and equivalently for $\mathrm{GL}(2,\mathbb{C})$-Higgs bundles. There are several ways to interpret this problem. A minimal resolution of a rational map is always defined as the closure of the graph, in our special case inside $H^0\big(X, K_X^2\big) \times \overline{\mathcal{M}}_{4g-3}$ since $\mathbb{A}_2^\circ = H^0\big(X, K_X^2\big)$. We characterize the points in this closure in Theorem 4.1. Our characterization relates the position of the marked points on each component of the spectral curve to the existence of a twisted quadratic differential, in spirit close to [BCG+18].

Working with this minimal resolution as a replacement of the moduli space of Higgs bundles has several flaws. One major issue is the absence of well-defined coverings of the limiting spectral curves to (a modification of) the original curve. Hence there is no way to define a spectral correspondence in this setting. From this perspective, it is more natural to instead look at the admissible cover compactification of the spectral map, that is, consider the closure $\widetilde{\mathcal{H}}_g$ of the image of the map $H^0\big(X, K^2\big) \to \mathcal{H}_g$, where $\mathcal{H}_g$ is the admissible cover compactification of simply branched covers of nodal curves of (arithmetic) genus g by nodal curves of genus $\hat{g} = 4g - 3$. Now the pushforward of a coherent sheaf on the covering curve will yield a sheaf on the base curve. However, since $\widetilde{\mathcal{H}}_g$ does not store scaling information for the differentials, we do not quite get Higgs bundles there.

A "compactification" of strata of differentials with prescribed type of zeros is the space of multi-scale differentials that was introduced in [BCG+19a] for abelian differentials and extended to k-differentials in [CMZ19]. (Strictly speaking, only the $\mathbb{C}^*$-quotient by the action of rescaling the differentials is a compact space.) To obtain a space resolving the spectral map, we look at the compactification of the open strata of quadratic differentials of type $\mu = \big(1^{4g-4}\big)$; we denote the moduli space of quadratic multi-scale differentials of type μ by $\overline{\mathcal{Q}}_{g,n}(\mu)$.

A point in $\overline{\mathcal{Q}}_{g,n}(\mu)$ contains the datum of a $(4g-4)$-pointed stable curve $(X, \mathbf{z})$ together with an order on the dual graph that partitions X into the subcurves X_i at various levels; see Section 3 for details. It also contains a collection $\mathbf{q} = (q_i)$ of non-zero quadratic differentials indexed by the levels. These differentials are required to vanish at the marked points and satisfy compatibility conditions at the nodes. Finally, it contains the datum of a double covering $\widehat{X} \to X$ such that the pullback of $\mathbf{q}$ is square of a collection $\boldsymbol{\lambda} = (\lambda_i)$ of abelian differentials. In particular, there is a birational forgetful map

$$\overline{\mathcal{Q}}_{g,n}(\mu) \to \mathcal{H}_g$$

and, by composition, also a birational map to the minimal resolution of the spectral map. Obviously, the space of multi-scale differentials also comes with a natural forgetful map to $\overline{\mathcal{M}}_g$.

We first study the fiber $\mathbb{B}_{X_{\rm st}} \subset \overline{\mathcal{Q}}_{g,n}(\mu)$ over a fixed Riemann surface $X_{\rm st}$ and later return to global aspects. By definition, a quadratic multi-scale differential is contained in $\mathbb{B}_{X_{\rm st}}$ if the stable unpointed curve underlying $(X, \mathbf{z})$ is $X_{\rm st}$. We call $\mathbb{B}_{X_{\rm st}}$ the *modified Hitchin base* since it comes with a natural forgetful birational surjective map $b\colon \mathbb{B}_{X_{\rm st}} \to \mathbb{A}_2^\circ \setminus \{0\}$. In Section 5, we show that this map is a bijection over the union of the regular locus and the locus where merely two points collide. If $g(X_{\rm st}) = 2$, the modified Hitchin base does not depend on $X_{\rm st}$, and we exhibit its complete geometry in Example 5.3. However, if $g(X_{\rm st}) \geqslant 3$, then $\mathbb{B}_{X_{\rm st}}$ does depend on $X_{\rm st}$.

Multi-scale spectral data. We propose to turn the classical spectral correspondence upside down and *define* a modification of the space of Higgs bundles to be the universal compactified Jacobian over the modified Hitchin base. The details are most transparently stated in the case of trace-free $\mathrm{GL}(2,\mathbb{C})$-Higgs bundles. We address the additional twists in the $\mathrm{SL}(2,\mathbb{C})$-case in Section 10. The general $\mathrm{GL}(2,\mathbb{C})$ case simply requires us to record an abelian differential in addition to the data used in the trace-free $\mathrm{GL}(2,\mathbb{C})$ case.

A trace-free $\mathrm{GL}(2,\mathbb{C})$-*multi-scale spectral datum on* $X_{\rm st}$ is a quadratic multi-scale differential $(X, \mathbf{q}) \in \mathbb{B}_{X_{\rm st}}$ together with a torsion-free rank 1 sheaf $\mathcal{F}$ on the associated double cover $\widehat{X}$ of X. The stack $\mathrm{SD}_{X_{\rm st}}$ of multi-scale spectral data on $X_{\rm st}$ is a universally closed smooth Artin stack and comes with natural forgetful maps $h\colon \mathrm{SD}_{X_{\rm st}} \to \mathbb{B}_{X_{\rm st}}$ and $\mathrm{SD}_{X_{\rm st}} \to \overline{\mathcal{M}}_{4g-3}$. The latter map justifies our claim to work in a setting of semistable curves. The next statement summarizes the geometry of the space $\mathrm{SD}_{X_{\rm st}}$.

Proposition 1.1. *There is a rational map $S\colon \mathrm{SD}_{X_{\rm st}} \dashrightarrow \mathrm{Higgs}^\circ_{\mathrm{GL}(2,\mathbb{C})}(X_{\rm st}) \setminus \mathcal{N}$ to the space of trace-free $\mathrm{GL}(2,\mathbb{C})$-Higgs bundles on $X_{\rm st}$ with image in the complement of the nilpotent cone $\mathcal{N}$, such that the diagram*

$$\begin{tikzcd} \mathrm{SD}_{X_{\rm st}} \arrow[r, dashed, "S"] \arrow[d, "h"] & \mathrm{Higgs}^\circ_{\mathrm{GL}(2,\mathbb{C})}(X_{\rm st}) \setminus \mathcal{N} \arrow[d, "\mathrm{Hit}"] \\ \mathbb{B}_{X_{\rm st}} \arrow[r, "b"] & \mathbb{A}_2 \setminus \{0\} \end{tikzcd} \tag{1.1}$$

commutes. The rational map S is defined and is an isomorphism over the locus where the quadratic differential $q \in \mathbb{A}_2$ has simple zeros or at most one double zero.

The definition of $\mathrm{SD}_{X_{\rm st}}$ requires the choice of a numerical polarization on the double cover $\widehat{X}$. In our setup, a natural choice is the pointed canonical polarization. With respect to this polarization, the degree d component of $\mathrm{SD}_{X_{\rm st}}$ is a proper Deligne–Mumford stack if and only if $\gcd(d - 2g + 2, 6g - 6) = 1$.

The $P = W$ conjecture has recently motivated a lot of research on the perverse filtration of the Hitchin map. For example, in [dCHM21], the authors determine support loci with the help of compactified Jacobians on versal deformations of the singular fiber. It would be interesting to see if the (well-understood) map h could lead to additional insights.

The space $\mathrm{SD}_{X_{\mathrm{st}}}$ has two more features that we address in the next paragraphs.

Multi-scale spectral correspondence. While defined in terms of spectral data, that is, by data on the "spectral" curve $\widehat{X}$, the stack $\mathrm{SD}_{X_{\mathrm{st}}}$ has an interpretation as moduli stack of Higgs pairs via a spectral correspondence. To begin with, we specify certain Higgs pairs that appear. As in [BCG+19a] and as in the definition of $\mathrm{SD}_{X_{\mathrm{st}}}$, these Higgs pairs will be defined level by level on some pointed stable curve $(X, \mathbf{z})$ that stabilizes to its top-level curve, which is X_{st}, when the marked points are forgotten. We define a *trace-free multi-scale* $\mathrm{GL}(2,\mathbb{C})$*-Higgs pair* to be a tuple consisting of the following objects. First, it contains a *special torsion-free rank* 2 *sheaf* $\mathcal{E}$ on X; that is, $\mathcal{E}$ is required to be locally free, except for a special local form at all nodes. Second, it contains an equivalence class of a collection of trace-free Higgs fields $\mathbf{\Phi} = (\phi_i)$ on each level of X. These levelwise Higgs fields are meromorphic with poles and zeros of higher order at the preimages of the nodes. We refer the reader to Definition 7.4 for full details. We emphasize that the setup allows X_{st} to vary. The following is the special case of Theorem 7.10, in which X_{st} is fixed but may itself be a stable curve. It should be viewed as a stable curve generalization of the classical Beauville–Narasimhan–Ramanan (BNR) correspondence recalled for comparison in Section 2.

Theorem 1.2. *Given a quadratic multi-scale differential* $(\widehat{X} \to X, \mathbf{z}, \mathbf{q})$, *there is a bijective correspondence between*

(i) *torsion-free rank* 1 *sheaves on* $\widehat{X}$ *and*

(ii) *special pairs of a torsion-free rank* 2 *sheaf* $\mathcal{E}$ *on* X *and a trace-free Higgs field* $\mathbf{\Phi}$ *with determinant* $\mathbf{q}$.

Equivalently, there is bijective correspondence between trace-free $\mathrm{GL}(2,\mathbb{C})$*-multi-scale spectral data and trace-free multi-scale* $\mathrm{GL}(2,\mathbb{C})$*-Higgs pairs.*

Moreover, the correspondence respects natural notions of (semi)stability on spectral data and Higgs pairs.

The precise definition of stability is given in Section 7. In Section 10, we define a substack of $\mathrm{SD}_{X_{\mathrm{st}}}$ of spectral data satisfying a Prym condition. We show that Theorem 1.2 specializes to a correspondence between torsion-free sheaves satisfying the Prym condition and multi-scale $\mathrm{SL}(2,\mathbb{C})$-Higgs pairs generalizing the classical $\mathrm{SL}(2,\mathbb{C})$-spectral correspondence (Theorem 10.1).

In the case of a quadratic differential on X_{st} with one double zero and all other zeros simple, the original spectral curve is nodal and $\widehat{X}$ is the semistable model obtained from the normalization of the spectral curve by putting a rational bridge at the preimages of the node. In this way, one obtains an admissible cover $\widehat{X} \to X$, where X is the smooth curve X_{st} augmented by a rational tail with a node at the double zero. A multi-scale Higgs pair on X restricts to a meromorphic Higgs bundle on the rational tail such that at the preimage of the node, the Higgs field is diagonal with a pole of order 3. Such meromorphic Higgs bundles appeared recently in the work of Tulli [Tul19], who showed that a certain moduli space of Higgs bundles of this kind on a rational curve realizes the Ooguri–Vafa space. The Ooguri–Vafa hyperkähler metric was conjectured in [Nei14] to be part of the local model for the approximate description of the Hitchin hyperkähler metric at a generic point of the discriminant locus.

Comparison to the original Hitchin fiber. In Section 9, we will compare the original Hitchin fibers to the ϕ-compactified Jacobians over the modified Hitchin base with respect to the pointed canonical polarization ϕ. There are similarities in special cases, mostly the cases of quadratic differentials with at most double zeros, but in general the fibers look quite different. For example, the classical Hitchin fibers are stratified by fiber bundles over the Jacobian of the normalized spectral curve. In the present work, the singular spectral curves are replaced by pointed stable curves. Their compactified Jacobians instead are stratified by fiber bundles over the Jacobian of the normalized pointed stable curve with the normalized spectral curve being only one of several connected components. Moreover, the compactified Jacobians of pointed stable curves have irreducible components indicated by the multi-degrees compatible with the stability condition. In contrast, in the classical setting, the $\mathrm{GL}(2,\mathbb{C})^{\circ}$-Hitchin fibers are irreducible whenever there is at least one zero of the quadratic differential of odd order. We will study some special cases in more detail to give references for these differences.

Degeneration of the curve X. Applying degeneration techniques is one of several reasons to consider the moduli space of Higgs bundles on a family of curves $\mathcal{X}$ degenerating to a stable curve. Our construction of the stack of multi-scale spectral data automatically works in families. For comparison, we summarize the known constructions.

A natural approach is to compactify the family of moduli spaces on the generic fiber by the moduli space of Hitchin pairs $(\mathcal{E}, \Phi)$ on the special fiber X made up of a *torsion-free sheaf* $\mathcal{E}$ and a morphism $\mathcal{E} \to \mathcal{E} \otimes \omega_X$. Using Simpson's method for constructing moduli spaces, one can define a moduli space of semistable Hitchin pairs on X. However, this moduli space is missing some desirable properties. For example, there is no well-defined Hitchin map on the moduli space of Hitchin pairs on the stable curve X.

A resolution to these drawbacks is suggested in the work of Balaji, Barik and Nagaraj [BBN16], however only in the case of a family degenerating to a stable curve with a single node. Building on the work of Nagaraj–Seshadri [NS99], the idea is to consider a modification $\mathcal{X}^{\mathrm{mod}}$ of the family of curves obtained by blowing up $\mathcal{X}$ repeatedly at the node. One can show that for any family of torsion-free sheaves, there exists a modification $\mathcal{X}^{\mathrm{mod}}$ such that it corresponds to a family of locally free sheaves on $\mathcal{X}^{\mathrm{mod}}$. Hence a family of Higgs pairs on $\mathcal{X}$ corresponds to a family of Higgs bundles on some modification $\mathcal{X}^{\mathrm{mod}}$. This data of $(\mathcal{X}^{\mathrm{mod}}, E, \Phi)$ is referred to as a Gieseker–Hitchin pair. Using locally free sheaves, the moduli space of Gieseker–Hitchin pairs allows the definition of a Hitchin morphism extending the classical Hitchin map for the generic fiber of $\mathcal{X}$.

Our approach is opposite. Our degeneration of the Hitchin system on a family of curves $\mathcal{X}$ as above features a morphism to the (modified) Hitchin base by definition. Then a spectral correspondence dictates a moduli space of multi-scale $\mathrm{GL}(2,\mathbb{C})$-Higgs pairs on the pointed stable curve X associated with it. When the zeros of the quadratic differential do not collide, we encounter the Hitchin pairs of the naive approach mentioned before with a special local form at all nodes. However, our definition of semistability is different. Taking into account the zeros $\mathbf{z} = (z_1, \ldots, z_{4g-4})$ of the quadratic differential, we consider a polarization associated with the family of stable pointed curves $(\mathcal{X}, \mathbf{z})$. This is natural and necessary in our setting as the families of unpointed curves used to define the modified Hitchin base are not necessarily stable. It would be interesting to see how this compares to the moduli space of Hitchin pairs on stable curves and to the approach of Balaji, Barik and Nagaraj. We hope to come back to this question in future work.

In contrast, we emphasize that limits obtained by rescaling the Higgs fields (for which, for example, the asymptotics of the hyperkähler metric were studied in a series of works Mazzeo, Swoboda, Weiß, Witt and Fredrickson) are not covered by the scope of this paper since the starting point is a compactification of the *projectivization* of a space of quadratic differentials.

2. The $\mathrm{SL}(2,\mathbb{C})$- and the $\mathrm{GL}(2,\mathbb{C})$-Hitchin system

In this section, we recall basic terminology for the Hitchin system in the rank 2 case, that is, for $\mathrm{SL}(2,\mathbb{C})$, for $\mathrm{GL}(2,\mathbb{C})$ and for the trace-free variant of the latter. The results here, the description of the Hitchin fibers and the BNR correspondence (after [BNR89]), are well known and stated for comparison with our results over the modified Hitchin base.

$\mathrm{GL}(2,\mathbb{C})$-*Higgs bundles.* Let X be a Riemann surface of genus $g \geqslant 2$ and K its canonical bundle. A $\mathrm{GL}(2,\mathbb{C})$-*Higgs bundle* is a pair (E,Φ) of a locally free sheaf of rank 2 and a Higgs field $\Phi \in H^0(X,\mathrm{End}(E)\otimes K)$. A $\mathrm{GL}(2,\mathbb{C})$-Higgs bundle is called *stable* (respectively, *semistable*) if for all Φ-invariant subbundles $L \subset E$,

$$\deg(L) < \deg(E)/2 \quad (\text{respectively, } \deg(L) \leqslant \deg(E)/2)\,. \tag{2.1}$$

Let $\mathrm{Higgs}^{\mathrm{GL}_2}(X)$ denote the moduli space of semistable $\mathrm{GL}(2,\mathbb{C})$-Higgs bundles on X. This is an algebraic variety with the moduli of stable Higgs bundles as a smooth subvariety of dimension $8g-6$; see [Nit91]. The $\mathrm{GL}(2,\mathbb{C})$-Hitchin map

$$\mathrm{Hit}\colon \mathrm{Higgs}^{\mathrm{GL}_2}(X) \to \mathbb{A}_2 := H^0(X,K) \oplus H^0\big(X,K^2\big)\,, \quad (E,\Phi) \mapsto (-\operatorname{Tr}(\phi), \det(\Phi)) \tag{2.2}$$

is proper and surjective [Hit87, Nit91] and determines the characteristic polynomial of the Higgs field.

$\mathrm{SL}(2,\mathbb{C})$-*Higgs bundles.* An $\mathrm{SL}(2,\mathbb{C})$-*Higgs bundle* is a pair (E,Φ) as above, but now we require that the determinant of E and the trace of Φ are trivial. The moduli space $\mathrm{Higgs}^{\mathrm{SL}_2}(X)$ of semistable $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles on X is an algebraic variety. It contains the moduli of stable Higgs bundles as a smooth subvariety of dimension $6g-6$; see [Nit91]. The $\mathrm{SL}(2,\mathbb{C})$-Hitchin map

$$\mathrm{Hit}_{\mathrm{SL}_2}\colon \mathrm{Higgs}^{\mathrm{SL}_2}(X) \to \mathbb{A}_2^\circ = H^0\big(X,K^2\big)\,, \quad (E,\Phi) \mapsto \det(\Phi) \tag{2.3}$$

is again proper and surjective.

Variants: Twisted Higgs bundles and/or fixed determinant. We relax the conditions for Higgs bundles in two respects. An M-*twisted rank* 2 *Higgs bundle* is a pair (E,Φ) of a locally free sheaf E of rank 2 and a Higgs field $\Phi \in H^0(X,\mathrm{End}(E)\otimes M)$. Let Λ be a line bundle on X. An M-*twisted Higgs bundle with fixed determinant* Λ is a pair (E,Φ) of a locally free sheaf E of rank 2 such that $\det(E) = \Lambda$ and a Higgs field $\Phi \in H^0(X,\mathrm{End}(E)\otimes M)$ such that $\operatorname{tr}(\Phi) = 0$. That is, as in the $\mathrm{SL}(2,\mathbb{C})$-case, we impose simultaneously the restriction on the determinant of E and on the trace of Φ. The notion of (semi)stability is still defined using (2.1).

Let $\mathrm{Higgs}_\Lambda(X,M)$ denote the moduli space of semistable M-twisted rank 2 Higgs bundle with fixed determinant Λ on X. This is an algebraic variety with the moduli of stable Higgs bundles as a smooth subvariety [Nit91]. When $\deg(M) \geqslant 2g-2$, its dimension is given by $3\deg(M)$. When $\deg(M) > (g+1)/2$, this holds for generic (X,M). Now the Hitchin map reads as

$$\mathrm{Hit}_{\Lambda,M}\colon \mathrm{Higgs}_\Lambda(X,M) \to H^0\big(X,M^2\big)\,, \quad (E,\Phi) \mapsto \det(\Phi)\,.$$

Here $\dim H^0\big(X,M^2\big) = 2\deg M - g + 1$ as long as $\deg M > g-1$. More generally, for generic

(X, M), we have

$$\dim H^0\bigl(X, M^2\bigr) = \begin{cases} 2\deg M - g + 1 & \text{for } \deg M > (g-1)/2\,, \\ 0 & \text{for } \deg M \leqslant (g-1)/2\,. \end{cases}$$

The Hitchin system. The cotangent bundle to the moduli space of stable rank 2 locally free sheaves (respectively, of such sheaves with trivial determinant) is a dense subset of $\mathrm{Higgs}^{\mathrm{GL}_2}(X)$ (respectively, of $\mathrm{Higgs}^{\mathrm{SL}_2}(X)$), and its holomorphic symplectic structure extends to the whole space. The Hitchin map restricted to pairs (q_1, q_2) with discriminant $\Delta = q_2 - \frac{1}{4}q_1^2$ (respectively, $\Delta = q_2$) having only simple zeros is an algebraically completely integrable system, called the $\mathrm{GL}(2,\mathbb{C})$-Hitchin system (respectively, the $\mathrm{SL}(2,\mathbb{C})$-Hitchin system). In the twisted case, if MK^{-1} has a section, then the space $\mathrm{Higgs}_\Lambda(X, M)$ carries the structure of a Poisson manifold [Bot95, Mar94].

The spectral curve. Let $p_K\colon K \to X$ be the total space of the canonical bundle, and let $\eta\colon K \to p_K^*K$ be the tautological section. For $q_i \in H^0\bigl(X, K^i\bigr)$, the *spectral curve* Σ is the zero divisor of the characteristic section

$$\eta^2 + \eta \cdot p_K^* q_1 + p_K^* q_2 \in H^0\bigl(K, p_K^* K^2\bigr)\,. \tag{2.4}$$

The spectral cover $\pi := p_K|_\Sigma\colon \Sigma \to X$ is an analytic covering factoring through the involution $\sigma\colon \Sigma \to \Sigma$ interchanging the sheets. We also abusively write $\eta \in H^0(\Sigma, \pi^*K)$ for the restriction to the spectral cover. We also frequently view the tautological section as a section $\lambda \in H^0(\Sigma, K_\Sigma)$. The set of branch points is $B = \mathrm{div}(\Delta) \subset X$. The section $\eta^\circ := \eta + \frac{1}{2}\pi^* q_1 \in H^0(\Sigma, \pi^*K)$ vanishes on the ramification divisor $\widehat{B} = \mathrm{div}(\eta^\circ)$ and is odd with respect to the involution; that is, $\sigma^*\eta^\circ = -\eta^\circ$.

The spectral curve is smooth if and only if Δ has simple zeros. The corresponding Hitchin fibers are torsors over an abelian variety. The following rank 2 version of the BNR correspondence is the model for our correspondence statement. See [Sch98] for the general version of integral spectral curves, including stability considerations.

Theorem 2.1 ([Hit87, BNR89]). *Let $(q_1, q_2) \in H^0(X, K) \oplus H^0\bigl(X, K^2\bigr)$ be differentials such that Δ has simple zeros. Then the fiber $\mathrm{Hit}^{-1}(q_1, q_2)$ of the Hitchin map is a torsor over the Jacobian of the spectral cover.*

In the $\mathrm{SL}(2,\mathbb{C})$ case, the fiber $\mathrm{Hit}_{\mathrm{SL}_2}^{-1}(q)$ over a quadratic differential q with simple zeros is a torsor over the Prym variety

$$\ker(\mathsf{Nm}_\pi) = \bigl\{L \in \mathrm{Pic}^0(\Sigma) \mid L \otimes \sigma^* L = \mathcal{O}_\Sigma\bigr\}\,. \tag{2.5}$$

More generally, let $q \in H^0\bigl(X, M^2\bigr)$ be a section with simple zeros. Then the fiber of the Hitchin map in the M-twisted case with determinant Λ is

$$\mathrm{Hit}_{\Lambda,M}^{-1}(q) \cong \bigl\{L \in \mathrm{Pic}^{\deg M + \deg \Lambda}(\Sigma) \mid L \otimes \sigma^* L = \pi^*(M \otimes \Lambda)\bigr\}\,. \tag{2.6}$$

This is a torsor over the Prym variety $\ker(\mathsf{Nm}_\pi)$ having dimension $g - 1 + \deg(M)$.

Proof. If we start with any line bundle L on Σ, then $E = \pi_* L$ together with the π-pushforward

$$\Phi = \pi_*\eta\colon \pi_* L \to \pi_*(L \otimes \pi^* K) = \pi_* L \otimes K$$

of the multiplication map by η gives a Higgs bundle (E, Φ). By the Cayley–Hamilton theorem and the definition of η in (2.4), we find that $(E, \Phi) \in \mathrm{Hit}^{-1}(q_1, q_2)$; see [BNR89, Proposition 3.6].

This Higgs bundle is indeed stable since a destabilizing subbundle would yield a factorization of the characteristic polynomial [Sch98, Lemma 3.2], contradicting the hypothesis that Δ (respectively, q) has simple zeros.

Conversely, let (E, Φ) be a Higgs bundle. The Higgs field, considered as a map $E \otimes K^{-1} \to E$, makes E into a $\pi_*\mathcal{O}_\Sigma$-module since $\pi_*\mathcal{O}_\Sigma = \mathcal{O}_X \oplus K^{-1}$, and this is the same datum as a line bundle on Σ. Clearly, the two constructions are inverse to each other. This completes the first claim.

We need another viewpoint that realizes the converse construction by computing the eigensheaf of the Higgs field pulled back to Σ. In a holomorphic chart (U, z), we write $\eta^\circ = f dz$ with f holomorphic or with a simple zero. Then $E = \pi_* L$ is locally generated (as $\mathcal{O}_X$-module) by 1 and f. The map

$$L|_U \to (\pi^*\pi_* L) \otimes \mathcal{O}\big(\widehat{B}\big)\big|_U, \quad 1 \mapsto 1 \otimes 1 + f \otimes \frac{1}{f}$$

is independent of the chosen local frame and injective and thus defines an embedding. It exhibits

$$L = \ker\big((\Phi - \eta \mathrm{Id}) \otimes \mathrm{Id}_{\mathcal{O}(\widehat{B})}\big) \subset (\pi^*\pi_* L) \otimes \mathcal{O}\big(\widehat{B}\big). \tag{2.7}$$

We now turn to the $\mathrm{SL}(2,\mathbb{C})$ case. To prove the most general claim, we will construct a morphism from the torsor over the Prym variety to the Hitchin fiber. Let $L \in \mathrm{Pic}(\Sigma)$ with $L \otimes \sigma^* L = \pi^*(M \otimes \Lambda)$. As above, let $E := \pi_* L$. The equation (see [Har77, Exercise IV.2.6])

$$\det(E) = \mathsf{Nm}_\pi(L) \otimes \det(\pi_*\mathcal{O}_\Sigma), \tag{2.8}$$

pulls back to

$$\pi^* \det(E) = L \otimes \sigma^* L \otimes \mathcal{O}\big(\widehat{B}\big)^{-1} = \pi^*(M \otimes \Lambda) \otimes \mathcal{O}\big(\widehat{B}\big)^{-1}, \tag{2.9}$$

where $\widehat{B} = \mathrm{div}\,(\eta)$ is the ramification divisor of π. Therefore, in the twisted context now $\mathcal{O}\big(\widehat{B}\big) = \pi^* M$. The pullback $\pi^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(\Sigma)$ is injective as $\pi\colon \Sigma \to X$ is not unbranched. Hence $\det(E) = \Lambda$. By the very definition of $\eta = \eta^\circ$, we have $\mathrm{Tr}(\Phi) = 0$ and $\det(\Phi) = q$. Summing up, (E, Φ) defines a Higgs bundle in $\mathrm{Hit}^{-1}_{\Lambda,M}(q)$.

Starting with (E, Φ), we first compute the two eigensheaves as in (2.7)

$$L = \ker\big((\pi^*\Phi - \eta\,\mathrm{Id}) \otimes \mathrm{Id}_{\mathcal{O}(\widehat{B})}\big), \quad L' = \ker\big((\pi^*\Phi + \eta\,\mathrm{Id}) \otimes \mathrm{Id}_{\mathcal{O}(\widehat{B})}\big).$$

As $\sigma^*\eta = -\eta$, we have $L' = \sigma^* L$. The inclusions into $\pi^* E \otimes \mathcal{O}\big(\widehat{B}\big)$ define an exact sequence

$$0 \to L \oplus \sigma^* L \to \pi^* E \otimes \mathcal{O}\big(\widehat{B}\big) \to \mathcal{O}_{\widehat{B}} \to 0$$

(see [Hor22, Theorem 5.5]). Computing the determinant bundles yields $L \otimes \sigma^* L = \pi^*(\Lambda)\big(\widehat{B}\big) = \pi^*(\Lambda \otimes M)$. This is the Prym condition. □

Reducing to trace-free $\mathrm{GL}(2,\mathbb{C})$. In the following, we will restrict our study to trace-free $\mathrm{GL}(2,\mathbb{C})$-Higgs bundles and $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles. We will refer to trace-free $\mathrm{GL}(2,\mathbb{C})$-Higgs bundles as $\mathrm{GL}(2,\mathbb{C})^\circ$-*Higgs bundles*. The smooth Hitchin fibers depend only on the discriminant $\Delta \in H^0\big(X, K^2\big)$ up to isomorphism. In general, we have an isomorphism

$$\mathrm{Hit}^{-1}(q_1, q_2) \to \mathrm{Hit}^{-1}(0, \Delta), \quad (E, \Phi) \mapsto \big(E, \Phi - \tfrac{1}{2}\,\mathrm{Id}_E \otimes q_1\big).$$

In particular, the rational map $\mathbb{A}_2 \to \overline{\mathcal{M}}_{4g-3}$ to the moduli space of stable curves that associates with a point in the regular locus of $\mathbb{A}_2$ the spectral curve is constant when only the abelian differential in $\mathbb{A}_2$ is varied. This justifies the restriction to $\mathbb{A}_2^\circ$ in terms of the spectral map.

3. The compactification of strata of quadratic differentials

We recall the construction of a smooth compactification of strata of k-differentials from [CMZ18] and [BCG$^+$19a], specialized to the case of quadratic differentials and later to the principal stratum where all zeros are simple. The fiber of this compactification over a smooth curve is a modification on the space of quadratic differentials, which we analyze in Section 5. Finally, we compare this compactification with the incidence variety compactification from [BCG$^+$19b], the naive closure of strata in the Hodge bundle over $\overline{\mathcal{M}}_{g,n}$.

Let $\mu = (m_1, \dots, m_n)$ be a *type* of a quadratic differential, that is, integers m_i with $\sum m_i = 2(2g-2)$. From here on, we will mainly be interested in the case $\mu = \left(1^{4g-4}\right)$; we drop the argument μ in that case. Let $\mathcal{Q}_{g,n}(\mu)$ be the moduli space of quadratic differentials (X, q) with n labeled special points where q has a zero or pole of order m_i for $i = 1, \dots, n$. We state the goal of the construction and explain the missing notation subsequently.

THEOREM 3.1. *There exists a smooth Deligne–Mumford stack $\overline{\mathcal{Q}}_{g,n}(\mu)$, the moduli space of* quadratic multi-scale differentials, *with the following properties:*

(i) *The space $\mathcal{Q}_{g,n}(\mu)$ is dense in $\overline{\mathcal{Q}}_{g,n}(\mu)$.*

(ii) *The boundary $D = \overline{\mathcal{Q}}_{g,n}(\mu) \smallsetminus \mathcal{Q}_{g,n}(\mu)$ is a normal crossing divisor.*

(iii) *The rescaling action of $\mathbb{C}^*$ on $\mathcal{Q}_{g,n}(\mu)$ extends to $\overline{\mathcal{Q}}_{g,n}(\mu)$, and the resulting projectivization $\mathbb{P}\overline{\mathcal{Q}}_{g,n}(\mu)$ is a proper smooth stack.*

(iv) *The space $\overline{\mathcal{Q}}_{g,n}(\mu)$ is immersed in the compactification $\Xi\overline{\mathcal{M}}_{\widehat{g},\{\widehat{n}\}}(\widehat{\mu})$ of a stratum of abelian differentials with partially labeled points.*

Canonical double cover. The notion of *canonical double cover* $\widehat{\pi}\colon \widehat{X} \to X$ associated with a non-zero quadratic differential q on a smooth curve X is ubiquitous to the literature on half-translation surfaces. See for example [BCG$^+$19b, Section 2.1] for various methods of construction. One possible definition of the canonical double cover is the normalization of the spectral curve; that is, $\widehat{X} = \Sigma^\nu$. In particular, the canonical double cover is always smooth. It is irreducible if and only if q is not the square of an abelian differential on X. However, the genus depends on the profile of the zeros on q. To specify the type of the double cover, we let

$$\widehat{\mu} := \big(\underbrace{\widehat{m}_1, \dots, \widehat{m}_1}_{\gcd(2,m_1)}, \underbrace{\widehat{m}_2, \dots, \widehat{m}_2}_{\gcd(2,m_2)}, \dots, \underbrace{\widehat{m}_n, \dots, \widehat{m}_n}_{\gcd(2,m_n)}\big)\,, \tag{3.1}$$

where $\widehat{m}_i := ((2+m_i)/\gcd(2,m_i)) - 1$. We let $\widehat{g} = g(\widehat{X})$ and $\widehat{n} = \sum_i \gcd(2, m_i)$. Suppose that there are s_1 points of odd order and s_2 points of even order, with $s_1 + s_2 = n$. Then by the cyclic covering constructions (see [EV86, Section 3] or [BCG$^+$19b, Section 2.1]), these quantities are related by

$$\widehat{g} = 2g - 1 + \frac{s_1}{2} \quad \text{and} \quad \widehat{n} = s_1 + 2s_2\,.$$

Ordered vs. unordered points. The points of departure, the space of quadratic differentials $\mathcal{Q}_{g,n}(\mu)$ and the compactification $\overline{\mathcal{Q}}_{g,n}(\mu)$, come with n ordered marked points. The same holds for the compactification $\Xi\overline{\mathcal{M}}_{\widehat{g},\widehat{n}}(\widehat{\mu})$ of the moduli space of abelian differentials of type $\widehat{\mu}$. Given μ (or $\widehat{\mu}$), there is a natural subgroup of the symmetric group S_n (or $S_{\widehat{n}}$) that acts by permuting points with the same order. We indicate the corresponding quotients by the action of the symmetric group by brackets $[n]$, for example, $\overline{\mathcal{Q}}_{g,[n]}(\mu)$. In the case of a signature $\widehat{\mu}$ arising from a double covering as in (3.1), there is the subgroup of $S_{\widehat{n}}$ generated by the commuting involutions swapping the pair of points stemming from the same even m_i. The quotient of $\Xi\overline{\mathcal{M}}_{\widehat{g},\widehat{n}}(\widehat{\mu})$ by this involution is the space $\Xi\overline{\mathcal{M}}_{\widehat{g},\{\widehat{n}\}}(\widehat{\mu})$ appearing in item (iv) of Theorem 3.1.

Multi-scale differentials. We now recall the definition of quadratic multi-scale differentials to the extent we need it. The first piece of datum is a pointed stable curve $(\widehat{X},\widehat{\mathbf{z}})$ of genus $\widehat{g}$, where the tuple $\widehat{\mathbf{z}}$ consists of $\widehat{n}$ non-singular points on $\widehat{X}$. Second, there is an involution $\sigma\colon \widehat{X} \to \widehat{X}$ that fixes s_1 points and has s_2 orbits of length 2 on $\widehat{\mathbf{z}}$. We let $\pi\colon \widehat{X} \to X$ be the corresponding quotient map and let $\mathbf{z}$ be the (ordered) n-tuple of images of $\widehat{\mathbf{z}}$, where $n = s_1 + s_2$.

Third, we need a *covering* $\pi\colon \widehat{\Gamma} \to \Gamma$ *of enhanced level graphs*. That is, $\widehat{\Gamma}$ and Γ are the dual graphs of $\widehat{X}$ and X, respectively, both are provided with a level structure, and each edge carries an enhancement that we denote by $\widehat{\kappa}_{\widehat{e}} \in \mathbb{Z}_{\geqslant 0}$ for $\widehat{e} \in \widehat{\Gamma}$ and by $\kappa_e \in \mathbb{Z}_{\geqslant 0}$ for $e \in \Gamma$, subject to the constraints given below.

A level structure is by definition a weak total order on the vertices; any two vertices can be compared, and equality is permitted. The top level is usually normalized to be level zero and depicted on top; see the examples in Figures 1 and 2. The lower levels are then referred to by -1, -2, etc. The covering of graphs, abusively also called π, is level-preserving and the quotient map induced by the involution σ. The level structure on the graph allows us to call edges *horizontal* if they start and end at the same level, and *vertical* otherwise.

The enhancement κ_e of an edge e of Γ is even if and only if e has two preimages in $\widehat{\Gamma}$. In this case, $\widehat{\kappa}_{\widehat{e}_j} = \kappa_e/2$ for either of the two preimages $\widehat{e}_j$ of e. If κ_e is odd, then $\widehat{\kappa}_{\widehat{e}} = \kappa_e$. Moreover, the enhancement κ_e is zero if and only if the edge e is horizontal. These conditions reflects the branching rule of canonical covers; see below. Boundary strata of $\overline{\mathcal{Q}}_{g,[n]}(\mu)$ are labeled by the coverings $\pi\colon \widehat{\Gamma} \to \Gamma$, or usually just by $\widehat{\Gamma}$. In the next item, we clarify which finite number of enhanced level graphs $\widehat{\Gamma}$ actually occur as labels of a boundary stratum.

Fourth, the core datum of a multi-scale differential is a *twisted differential*. This is a collection of differentials $\boldsymbol{\lambda} = (\lambda_v)_{v\in V(\widehat{\Gamma})}$ subject to the following conditions. At the marked legs, the differential has a zero of order $\widehat{m}_i$ (or a pole, depending on the sign of $\widehat{m}_i$). At the upper end of a vertical edge e, there is a zero of order $\widehat{\kappa}_e - 1$. At the lower end of e or at horizontal edges, there is a pole of order $-\widehat{\kappa}_e - 1$, with matching residues along horizontal edges. Finally, the residues at the lower ends of the edges are constrained by a global residue condition (GRC). See [BCG+18, Section 3] for a discussion and examples of those conditions.

Fifth, there is a collection of *prong-matchings* $\boldsymbol{\tau}$, one for each vertical edge, denoted by τ_e. A prong-matching is a bijection of the outgoing horizontal prongs of the differential at the upper end of e with the incoming horizontal prongs of the differential at the lower end of e, reversing the cyclic order on prongs from the planar structure. Prong-matchings determine the branches when smoothing a stable curve with a twisted differential. Prong-matchings play virtually no role in this paper, but recording them is necessary to construct the smooth stack in Theorem 3.1. See [BCG+19a, Section 5] for details of the definition and [CMZ22] for examples of the combinatorics of these objects.

So far the tuple of objects $(\widehat{X},\widehat{\mathbf{z}},\widehat{\Gamma},\boldsymbol{\lambda},\boldsymbol{\tau})$ without the conditions on the existence of involutions or the map π defines an (ordinary) multi-scale differential, that is, points in the space $\Xi\overline{\mathcal{M}}_{\widehat{g},\widehat{n}}(\widehat{\mu})$. The last point, and the reason for the name, is to define the equivalence relation. For this purpose, we group the differentials λ_v for all vertices v on the same level $-i$ to a tuple λ_i. Consequently, we write $\boldsymbol{\lambda} = (\lambda_i)_{i\in L(\widehat{\Gamma})}$, where $L(\widehat{\Gamma}) = L(\Gamma)$ is the set of levels of these two graphs. The group $\mathbb{C}^*$ rescales such a tuple λ_i simultaneously. We apply this for all levels except for the top level (which is rescaled in projectivization, see item (iii) of Theorem (3.1)). Suppose that Γ has L levels below zero. A cover of the resulting torus $(\mathbb{C}^*)^L$ acts by rescaling λ_i and rotating the prong-matching simultaneously. The product is called the level rotation torus in [BCG+19a, Section 6] and defines the equivalence relation we need.

In the last step, we characterize $\overline{\mathcal{Q}}_{g,n}(\mu)$ locally (in the domain) as subspace of $\Xi\overline{\mathcal{M}}_{\widehat{g},\{\widehat{n}\}}(\widehat{\mu})$. For this, in addition to the (ordinary) multi-scale differential, we need the involution σ and the quotient map π, as above. Finally, we require that the collection $\boldsymbol{\lambda}$ is anti-invariant under σ; that is, $\sigma^*\boldsymbol{\lambda} = -\boldsymbol{\lambda}$. This implies that there is a collection $\mathbf{q} = (q_i)_{i\in L(\Gamma)}$ of quadratic multi-scale differentials on the subcurves X_i of level $-i$ with simple zeros at the marked points $\mathbf{z} := \widehat{\pi}(\widehat{\mathbf{z}})$ such that $\lambda_i^2 = \widehat{\pi}^* q_i$. To sum up, on the pointed stable curve $(X, \mathbf{z})$, the data of a quadratic multi-scale differential is the tuple $(\Gamma, \mathbf{q}, \boldsymbol{\tau})$.

Universal family. So far, we have given multi-scale differentials pointwise. The complete definition for families is lengthy as it requires us to encode the passage from the smooth situation (where no prong-matching is present) to boundary points (where prong-matchings are part of the datum). The notion of rescaling ensemble in [BCG+19a] serves this purpose. We only give the consequences here.

By construction, $\overline{\mathcal{Q}}_{g,n}(\mu)$ comes with a universal family $\widehat{f}\colon \widehat{\mathcal{X}} \to \overline{\mathcal{Q}}_{g,n}(\mu)$ (all to be interpreted in the sense of stacks, that is, on an étale chart), a universal family $f\colon \mathcal{X} \to \overline{\mathcal{Q}}_{g,n}(\mu)$ and the universal double covering $\widehat{\pi}\colon \widehat{\mathcal{X}} \to \mathcal{X}$. There are also the universal families of multi-scale differentials that we continue to denote by $\boldsymbol{\lambda}$ and q, as we did for single fibers.

Examples. We give three examples of graphs with two levels and without horizontal edges. These thus correspond to boundary divisors in $\overline{\mathcal{Q}}_{g,n}(\mu)$; that is, we moreover specialize to $\mu = (1^{4g-4})$. In these graphs, a dot is a vertex of genus zero; otherwise, the genus is written inside or next to the vertex. The number in the square is the enhancement. The other numbers written next to half-edges are the orders of zeros of poles that the (abelian or quadratic) multi-scale differential is required to have. In all of the cases, the graphs Γ are given together with the double cover $\widehat{\pi}\colon \widehat{\Gamma} \to \Gamma$.

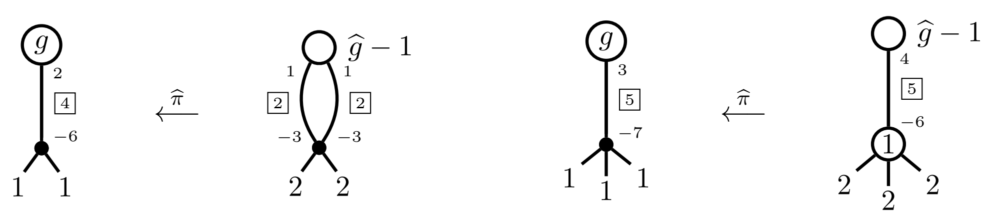

Figure 1. Level graphs of two (left), respectively three (right), zeros coming together and all other zeros simple

The covering viewpoint. Since the total space $\mathcal{X}$ of the universal family over $\overline{\mathcal{Q}}_{g,n}(1^{4g-4})$ is not smooth, the double cover $\widehat{\pi}$ is not given "as usual" in the context of spectral curves by a section of a square of a line bundle. However, for a fixed point of $\overline{\mathcal{Q}}_{g,n}(1^{4g-4})$ corresponding to a boundary stratum labeled by $\widehat{\Gamma}$, or more generally for an equisingular deformation in such a stratum, the cover restricted to each level is given in this usual way. We determine here the line bundle and the section in terms of graph data.

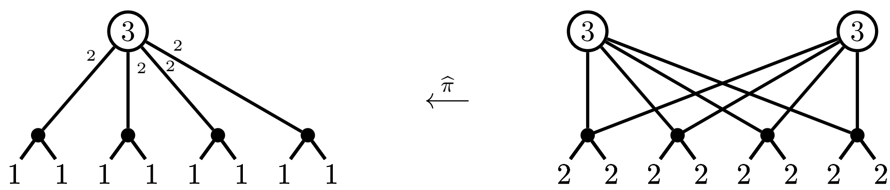

Figure 2. Level graphs of the locus where the quadratic differential in $\mathcal{Q}_{3,4}(1^8)$ becomes a global square with zeros coming together pairwise

Consider the bundle on the level i subcurve given by

$$M_i = \omega_X|_{X_i}(K_i)\,, \quad K_i = \Bigg(-\sum_{e^- \in X_i} \left\lfloor \frac{\kappa_e}{2} \right\rfloor e^- + \sum_{e^+ \in X_i} \left\lceil \frac{\kappa_e}{2} \right\rceil e^+ \Bigg), \tag{3.2}$$

where we sum over all preimages $e^\pm \in X_i$ of the nodes e that connect to upper/lower levels with respect to level i. Then we have $q_i \in H^0\big(X_i, M_i^{\otimes 2}\big)$ with only simple zeros, namely at the marked points in X_i and at the nodes where κ_e is odd, by the definition of compatibility with the level structure. Consequently, the covering

$$\widehat{\pi}_i\colon \widehat{X}_i \to X_i \tag{3.3}$$

is the spectral curve for the pair (M_i, q_i).

The incidence variety compactification (*IVC*). Instead of the space of multi-scale differentials $\overline{\mathcal{Q}}_{g,n}(\mu)$, one can consider the compactification $\overline{\mathcal{Q}}_{g,n}^{\mathrm{IVC}}(\mu)$ of the stratum $\mathcal{Q}_{g,n}(\mu)$ inside the total space of the second Hodge bundle $\Omega^2\overline{\mathcal{M}}_{g,n}$, the space of pointed stable quadratic differentials. There is a forgetful map

$$f_{\mathrm{IVC}}\colon \overline{\mathcal{Q}}_{g,n}(\mu) \to \overline{\mathcal{Q}}_{g,n}^{\mathrm{IVC}}(\mu)$$

that takes a multi-scale differential $\big(\pi\colon \widehat{X} \to X, \widehat{\mathbf{z}}, \widehat{\Gamma}, \boldsymbol{\lambda}, \boldsymbol{\tau}\big)$ and

(i) forgets the prong-matching $\boldsymbol{\tau}$;

(ii) forgets the covering surface $\widehat{X}$, its marked points $\widehat{\mathbf{z}}$, the abelian differentials $\boldsymbol{\lambda}$ and the graph covering, just retaining $(X, \mathbf{z}, \Gamma, \mathbf{q})$;

(iii) forgets the enhancements on Γ but keeps the level structure;

(iv) allows rescaling the lower-level components of $\mathbf{q}$ individually on each irreducible component of the stable curve.

Said differently, points in $\overline{\mathcal{Q}}_{g,n}^{\mathrm{IVC}}(\mu)$ are given by a pointed stable curve $(X, \mathbf{z}, \Gamma, \mathbf{q})$ with a non-enhanced level graph and a twisted quadratic differential compatible with Γ. The latter is defined by the existence of a double covering such that the pullback is an (abelian) differential in the above sense. Obviously, the map f_{IVC} is equivariant under the rescaling actions and defines a map on the projectivizations.

Passing to $\overline{\mathcal{Q}}_{g,n}^{\mathrm{IVC}}(\mu)$, we lose smoothness (singularities are, for example, not $\mathbb{Q}$-factorial), the normal crossing boundary divisor, the interpretation as a functor of multi-scale differentials, but we retain an object whose points are more easily described.

4. Resolutions of the spectral map

In this section, we outline the definitions of the spectral map and its variants for the $\mathrm{GL}(k,\mathbb{C})$-Hitchin base. For the case $k=2$, we give a characterization of the image and the points in the minimal resolution.

Let X_{st} be a smooth curve and $\mathbb{A}_k=\bigoplus_{i=1}^k H^0\big(X_{\mathrm{st}},K_{X_{\mathrm{st}}}^{\otimes i}\big)$ be the $\mathrm{GL}(k,\mathbb{C})$-Hitchin base. For a collection $\mathbf{q}=(q_1,\dots,q_k)$ of differentials, that is, for a point in $\mathbb{A}_k$, there is the corresponding spectral curve $\Sigma_{\mathbf{q}}$. We let $\Delta_{\mathbf{q}}$ be the discriminant of $\mathbf{q}$ and $\Delta_{\mathbb{A}}=\{\Delta_{\mathbf{q}}=0\}\subset\mathbb{A}_k$ the discriminant locus.

For a generic choice of $\mathbf{q}$, the $k(k-1)$-differential $\Delta_{\mathbf{q}}$ has simple zeros, thus in fact $n=k(k-1)(2g-2)$ of them. In this case, the spectral curve $\Sigma_{\mathbf{q}}$ is a smooth curve of genus $\widehat{g}$. Associating with $(X_{\mathrm{st}},\mathbf{q})$ the spectral curve $\Sigma_{\mathbf{q}}$ thus defines a rational map

$$\psi_{X_{\mathrm{st}},k}\colon \mathbb{A}_k \dashrightarrow \overline{\mathcal{M}}_{\widehat{g}}$$

that we call the *spectral map*. Like every rational map, this map can be resolved by taking the closure $\mathcal{R}_{g,k}^{X_{\mathrm{st}}}$ of the graph of $\psi_{X_{\mathrm{st}},k}$ inside $\mathbb{A}_k\times\overline{\mathcal{M}}_{\widehat{g}}$. We denote the resolution by

$$\widetilde{\psi}_{X_{\mathrm{st}},k}\colon \mathcal{R}_{g,k}^{X_{\mathrm{st}}}\to\overline{\mathcal{M}}_{\widehat{g}}\,.$$

The general goal is a characterization of the points in $\mathcal{R}_{g,k}^{X_{\mathrm{st}}}$ or at least of the stable curves that arise as their $\widetilde{\psi}_{X_{\mathrm{st}},k}$-images.

All these definitions have analogs in the trace-free case, where the abelian differentials are zero. That is, on the $\mathrm{SL}(k,\mathbb{C})$- (or trace-free) Hitchin base $\mathbb{A}_k^{\circ}=\bigoplus_{i=2}^k H^0\big(X_{\mathrm{st}},K_{X_{\mathrm{st}}}^{\otimes i}\big)$, there is the rational map $\psi^{\circ}_{X_{\mathrm{st}},k}\colon\mathbb{A}_k^{\circ}\dashrightarrow\overline{\mathcal{M}}_{\widehat{g}}$, whose resolution we denote by $\mathcal{R}_{g,k}^{X_{\mathrm{st}},\circ}$.

We give an explicit description of the points in resolutions $\mathcal{R}_{g,k}^{X_{\mathrm{st}}}$ and $\mathcal{R}_{g,k}^{X_{\mathrm{st}},\circ}$ for the case $k=2$. The answer is very close to the compactification of [BCG+18] and [BCG+19b] and requires mainly to keep track of rational tails correctly.

Theorem 4.1. *A stable curve $\widetilde{X}$ of genus $\widehat{g}=4g-3$ is in the image of the $\mathrm{SL}(2,\mathbb{C})$-spectral map $\widetilde{\psi}^{\circ}_{X_{\mathrm{st}}}$ if and only if*

(i) *there exists a level structure on the dual graph Γ of $\widetilde{X}$,*

(ii) *there is an involution σ on $\widetilde{X}$ such that the quotient of the top-level curve by σ is isomorphic to X_{st},*

(iii) *each σ-fixed self-node of $\widetilde{X}$ can be replaced by a rational bridge so that the resulting semistable curve $\widehat{X}$ has a σ-anti-invariant twisted differential of type $\mu=2^{4g-4}$ compatible with one (equivalently, any) level structure on the dual graph of $\widehat{X}$ extending the one on Γ.*

In particular, the top level of Γ has a unique vertex or two vertices exchanged by σ.

The $\mathrm{GL}(2,\mathbb{C})$-spectral map $\widetilde{\psi}_{X_{\mathrm{st}}}$ has the same image.

We use two variants of the spectral map for the proof. First, the spectral maps $\psi_{X_{\mathrm{st}},k}$ can be assembled to a rational map ψ_k from the fiber product of bundles $\Omega^i\mathcal{M}_g$ of i-differentials $(1\leqslant i\leqslant k)$ over $\mathcal{M}_g$. Second, the target point of the map $\psi_{X_{\mathrm{st}},k}$ only depends on the class of the point $\mathbb{A}_k$ up to the action of $\mathbb{C}^*$ rescaling the ith component with exponent i. We denote this map by

$$\psi_k^{\mathbb{P}}\colon \big(\times_{\mathcal{M}_g}\Omega^i\mathcal{M}_g\big)/\mathbb{C}^*\dashrightarrow\overline{\mathcal{M}}_{\widehat{g}}$$

and its resolution by $\mathcal{R}^{\mathbb{P}}_{g,k}$. All these objects obviously have their trace-free variants, decorated by a superscript $\circ$.

Interpolation by Hurwitz spaces. Let $k = 2$ from now on. By definition, the space $\mathcal{R}^{X}_{g,2}$ is closely related to the incidence variety compactification, or rather the quotient

$$\overline{\mathcal{Q}}^{\mathrm{IVC}}_{g} := \overline{\mathcal{Q}}^{\mathrm{IVC}}_{g,4g-4}(1^{4g-4})/S_{4g-4}$$

by the permutation group of the marked points. The following space of admissible covers interpolates between the two, dominating both birationally but with neither map being the identity.

We take $\overline{\mathcal{H}}_g$ to be the admissible cover compactification of the Hurwitz space of degree 2 covers $\pi\colon \widehat{X} \to X$ with simple branching over $4g-4$ (unmarked) points with $g(X) = g$ (and thus $g(\widehat{X}) = \widehat{g}$). The Hurwitz space comes with source and target maps

$$\overline{\mathcal{H}}_g \to \overline{\mathcal{M}}_{\widehat{g}}\,, \quad \overline{\mathcal{H}}_g \to \overline{\mathcal{M}}_g\,.$$

We consider the subspace

$$\widetilde{\mathcal{H}}^{\circ}_g \subset \overline{\mathcal{H}}_g$$

defined as the closure of the locus where π is a covering between smooth curves and the branch divisor in X is the zero locus of a quadratic differential. By definition, there are rational maps

$$\widetilde{\mathcal{H}}^{\circ}_g \dashrightarrow \Omega^2\mathcal{M}_g/\mathbb{C}^*\,, \quad \widetilde{\mathcal{H}}^{\circ}_g \dashrightarrow \Omega^2\overline{\mathcal{M}}_{g,4g-4}/\mathbb{C}^*\,. \tag{4.1}$$

These define forgetful maps

$$f_1\colon \widetilde{\mathcal{H}}^{\circ}_g \to \mathcal{R}^{\mathbb{P},\circ}_{g,2}\,, \quad f_2\colon \widetilde{\mathcal{H}}^{\circ}_g \to \mathbb{P}\overline{\mathcal{Q}}^{\mathrm{IVC}}_{g}\,.$$

Here the map f_1 is induced by taking the product of the first map in (4.1) and the source map of the Hurwitz space. Recall that we can think of $\mathcal{R}^{\mathbb{P},\circ}_{g,2}$ as the closure of the graph of $\psi^{\mathbb{P}}_{k}$. The map f_2 is induced by the second map in (4.1). They are well defined since both the domain and the range are defined as closures of loci on which the map is obviously well defined. Theorem 4.1 will follow from the following statements, where the first is a direct consequence of the definitions.

Lemma 4.2. *The map f_2 is quasi-finite, surjective and generically bijective. The cardinality of the f_2-fiber over $(X, \mathbf{z}, \Gamma, \mathbf{q})$ is precisely the number of double coverings $\pi\colon \widehat{\Gamma} \to \Gamma$ of enhanced level graphs with given target graph.*

The passage between the two-level graphs in the f_2-fibers is called “criss-cross” in [BCG+19b, Example 4.3].

Lemma 4.3. *Consider a point in $\mathcal{R}^{\mathbb{P},\circ}_{g,2}$ represented by a triple $(\widetilde{X}, X_{\mathrm{st}}, q_2)$. There exists a unique semistable model $\widehat{X} \to \widetilde{X}$, which has an admissible cover $\pi\colon \widehat{X} \to X$, such that the stabilization of X is X_{st}.*

Proof. Using admissible reduction applied to the germ of a family with special fiber $(\widetilde{X}, X_{\mathrm{st}}, q_2)$, we obtain a semistable model $\widehat{X} \to \widetilde{X}$ and an admissible cover $\pi\colon \widehat{X} \to X$. By the process of admissible reduction, X is given by X_{st} augmented by rational tails. On all irreducible components of $\widehat{X}$ of genus greater than 1, the (“hyperelliptic”) involution σ is uniquely determined. Moreover, there is at least one irreducible component (“top level”) where q_2 is non-zero by the definition of where we analyze f_1. On this component, the differential q_2 determines the double cover and the involution σ. The argument of uniqueness for elliptic tails of $\widehat{X}$ is now similar, using the point of attachment (or the exchanged pair of points of attachment) to a higher-genus curve or “higher-level curve.” □

Lemma 4.4. *The map f_1 is surjective, and injective over $\{q_2 \neq 0\}$ if the model $\widehat{X}$ defined by Lemma 4.3 has no rational tails. However, the fibers of f_1 are positive-dimensional in general even over $\{q_2 \neq 0\}$.*

Proof. The surjectivity follows from the properness of the space of admissible covers.

By Lemma 4.3, the non-uniqueness stems precisely from a chain of rational tails that is contracted under the stabilization $\widehat{X} \to \widetilde{X}$. For example, take X to be a genus 2 curve $X_0 = X_{\mathrm{st}}$, attach a rational tail T_1 with two branch points and to that another tail T_2 with two branch points. Then the double cover $\widehat{X} \to X$ is unramified over the nodes, and the stable model $\widetilde{X}$ of $\widehat{X}$ has two irreducible components: an unramified double cover of X_0 augmented by a singular elliptic curve meeting the top-level component in two nodes. (One might refer to this singular elliptic curve as fish bridge.) However, the admissible cover records the cross-ratio of the four special points on T_1, while the image in $\mathcal{R}^{\mathbb{P}}_{2,2}$ loses that information. □

Proof of Theorem 4.1. This is a restatement of the surjectivity of the map f_1 together with the "almost bijectivity" of f_2 and the main theorem of [BCG+19b, Theorem 1.5] that characterizes boundary points in $\overline{\mathcal{Q}}_2^{\mathrm{IVC}}$, hence in $\mathcal{R}_{g,2}^{X_{\mathrm{st}},\circ}$.

The statement about the $\mathrm{GL}(2,\mathbb{C})$-spectral map $\widetilde{\psi}_{X,2}$ follows from the observation that for a dense open subset of $\mathbb{A}_2$, the spectral double cover is given by a quadratic differential, the discriminant $\Delta_{\mathbf{q}}$. Since on this locus, the spectral maps have the same image, the same holds for the closure. □

5. The modified Hitchin base

The target of the usual $\mathrm{SL}(2,\mathbb{C})$-Hitchin map given in (2.3) is the space $\mathbb{A}_2^\circ = H^0\big(X_{\mathrm{st}}, K^2\big)$ of quadratic differentials on a fixed smooth curve X_{st}. We refer to this vector space as the *Hitchin base*. Both in the compactification and the resolution of the spectral map, the fibers of the forgetful map $\psi\colon \overline{\mathcal{Q}}_{g,[4g-4]}\big(1^{4g-4}\big) \to \overline{\mathcal{M}}_g$ play an important role. We call this fiber $\mathbb{B}_{X_{\mathrm{st}}} = \psi^{-1}([X_{\mathrm{st}}])$ the *modified Hitchin base* and describe the geometry of this DM-stack in more detail.

We describe the enhanced level graphs Γ that appear in $\mathbb{B}_{X_{\mathrm{st}}}$ if X_{st} is a smooth curve:

(i) The dual graph Γ is a tree.
(ii) The top level has a unique vertex corresponding to the irreducible curve X_{st} of genus g.
(iii) There are no horizontal edges in Γ (and thus in $\widehat{\Gamma}$).

Proposition 5.1. *There is a ("forgetful") morphism $b\colon \mathbb{B}_{X_{\mathrm{st}}} \to H^0\big(X_{\mathrm{st}}, K^2\big)$. This map is birational, and it is an isomorphism over the locus where there is at most one non-simple zero that is of order 2.*

Proof. The morphism b is defined by assigning to a quadratic multi-scale differential in $\mathbb{B}_{X_{\mathrm{st}}}$ the quadratic differential on top level $X_0 = X_{\mathrm{st}}$. If the zeros are pairwise disjoint, then the construction of $\overline{\mathcal{Q}}_{g,[4g-4]}\big(1^{4g-4}\big)$ does not modify the differential.

Whenever there is only one double zero, the construction of $\overline{\mathcal{Q}}_{g,[4g-4]}\big(1^{4g-4}\big)$ replaces the smooth curve with double zeros by a rational tail with two zeros as in Figure 2, left. Since these rational tails have three (unordered) marked points and thus have no moduli, the map b is still an isomorphism over this locus. □

Remark 5.2. Since the level graphs Γ parameterizing boundary strata of the modified Hitchin base arise from the collision of points (only), they are always of compact type; that is, $h^1(\Gamma) = 0$.

Example 5.3. The case $g = 2$ is particularly simple since the (second-order) Weierstrass sequences are the same for all curves, and yet this case shows that b is not birational beyond the locus claimed in Proposition 5.1.

Consider $\mathbb{P}^2 = \mathbb{P}\big(H^0\big(X_{\mathrm{st}}, K^{\otimes 2}\big)\big)$ for any X_{st} with $g(X_{\mathrm{st}}) = 2$. A basis of one-forms is $\{\omega_1 = dx/y, \omega_2 = xdx/y\}$, and a basis of two-forms is $\{\omega_1^2, \omega_1\omega_2, \omega_2^2\}$. In particular, all quadratic differentials are invariant under the hyperelliptic involution. This implies that the locus of differentials of type $(2,1,1)$ consists of six lines (called W_i in Figure 3 that shows only $i = 1, 2$), one for each of the six Weierstrass points. Note that the Weierstrass points of the first and the second order agree. The locus where the zeros are of type $(2,2)$ are the pairwise intersection points together with the reducible locus (the $g = 2$ version of Figure 2). This reducible locus is a curve $V \cong \mathbb{P}^1$, embedded via the Veronese embedding into the $\mathbb{P}^2$. The double zeros along the reducible locus are never Weierstrass points, except when they collide to form a fourfold zero. This gives a special point on each of the Weierstrass lines. There are five more special points on each of them, the intersections with the other Weierstrass lines. This gives $6 \cdot 5/2 + 6$ special points in total.

We claim that $b\colon \mathbb{P}\mathbb{B}_{X_{\mathrm{st}}} \to \mathbb{P}^2$ is the blowup of these 15 points $W_i \cap W_j$ and a $\mathbb{P}^1$ with an orbifold point of order 2 blown into the intersection points $W_i \cap V$ for each $i = 1, \ldots, 6$. For the latter, note that this intersection is tangent of order 2. A first blowup makes the intersection transversal, with exceptional divisor E_i. A second blowup creates the curve F_i as exceptional divisor, which is geometrically explained below. After this blowup, E_i becomes a (-2)-curve whose contraction leads to an ordinary double point, equivalently the orbifold point of order 2.

We now justify the decoration in Figure 2. Each double zero produces an $\mathcal{M}_{0,3}$-rational tail. Two double zeros thus give a "cherry"-type level graph. We now have to distinguish cases. Suppose that the double zeros are at two Weierstrass points. This implies that the quadratic differential is the product of two abelian double-zero differentials, hence not a square. The relative scale of the one-forms on the two rational tails is the extra parameter that induces a $\mathbb{P}^1$ instead of a point in the original Hitchin base. This gives the curves D_{ij} in Figure 3 for $1 \leqslant i < j \leqslant 6$. These D_{ij} intersect W_i and W_j at points where the cherry is slanted left or right.

On the other hand, if the double zeros are not Weierstrass points and hence hyperelliptic conjugates, then we are on the Veronese curve, the reducible locus. There, the two differentials on the lower level have to be of the same scale in order to allow a deformation to proper quadratic differentials. This can be seen on the double cover (the $g = 2$ version of Figure 2, right), where the global residue condition forces all residues to be the same. Also, a global dimension count shows that scaling differentials on the lower level independently would give a boundary stratum of the same dimension as the stratum $\mathcal{Q}_{g,[4g-4]}\big(1^{4g-4}\big)$.

A similar constraint applies to a fourfold zero. There, the four marked points on the lower level of a rational tail cannot move in an unconstrained way. (Check on the double cover: The top level is the square of an abelian differential; hence its double cover splits into two components. The double cover of the bottom level is an elliptic curve whose j-invariant is determined by the position of the four simple zeros of the quadratic differential. This is unconstrained since the j-invariant is determined by the absolute periods, which are in the (-1)-eigenspace. However, the point of attachment has to be chosen so that the residue of the differential of type $(2,2,2,2,-4,-4)$ on the elliptic curve is zero by the global residue condition.) Consequently, they can move in a codimension 1 subvariety F_i of this $\mathcal{M}_{0,5}$, where again $i = 1, \ldots, 6$ indexes the Weierstrass points.

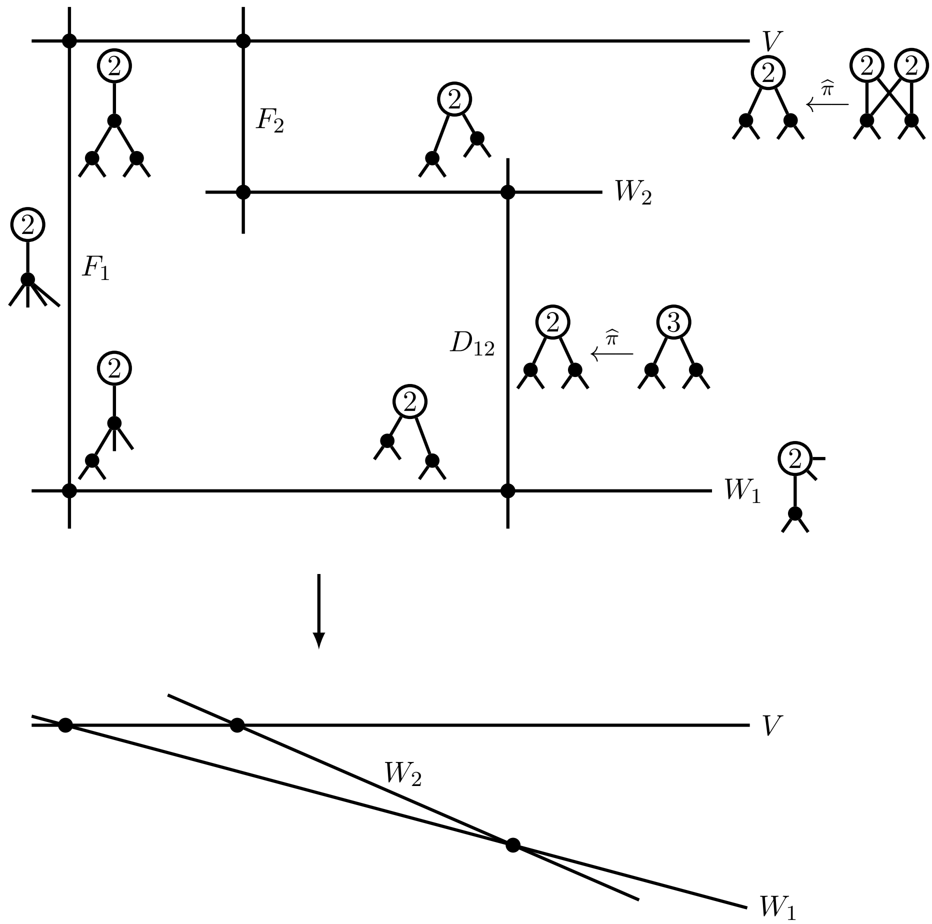

Figure 3. The modified Hitchin base for $g = 2$. The map b to $\mathbb{A}_2^\circ$ contracts the divisors F_i and D_{ij}.

This subvariety intersects the Weierstrass line W_i in the three-level graph, with type $(-8, 2, 1, 1)$ on the middle level and type $(-6, 1, 1)$ on the bottom level. It intersects the Veronese curve (the reducible locus) in the three-level graph given by a cherry with a long stalk; see Figure 3.

To explain the singularity of F_i, we compute that with labeled points and the pole set as $x_5 = \infty$, the equation of the locus of differentials

$$q = (z - x_1)(z - x_2)(z - x_3)(z - x_4)dz^2$$

with vanishing 2-residue at x_5 is precisely the union of the three affine-invariant loci

$$x_i + x_j = x_k + x_\ell$$

for all $\{i, j, k, l\} = \{1, 2, 3, 4\}$. The action of the symmetric group permutes these three loci, and the stabilizer of, say, $x_1 + x_2 = x_3 + x_4$ is generated by the involutions (12) and (34). This results in a quotient singularity at the origin.

Remark 5.4. The dependence of $\mathbb{B}_{X_{\mathrm{st}}}$ on X_{st} for $g(X_{\mathrm{st}}) = 3$ and similarly for higher genus can easily be seen as follows. The modified Hitchin base contains a "boundary" divisor for each two-level graph. Among those graphs is the "compact-type" graph Γ_8 with two vertices and one level, and with all marked points on the bottom level. In this case, the top-level twisted differential belongs to the stratum $\mu = (8)$. This stratum has projectivized dimension 4. The forgetful map of this stratum to $\mathcal{M}_3$ is not dominant, and for X_{st} outside the range of the forgetful map, the modified Hitchin base does not have a divisor corresponding to Γ_8.

6. Universal compactified Jacobians

In this section, we recall the notions leading to a compactification of the universal Jacobian over the moduli space of pointed stable curves. Theorem 6.1 is essentially contained in the literature; see the references in the proof. Fix $g \geqslant 2$ and $n > 0$, and let $d \in \mathbb{Z}$ denote the degree of the line bundles.

First, we let the *unrigidified universal compactified Jacobian* $\widetilde{\mathcal{J}}^d_{g,n}$ be the algebraic stack parameterizing stable curves $\mathcal{X} \to B$ of genus g with n marked points together with a family $\mathcal{F}$ of torsion-free rank 1 sheaves on $\mathcal{X}$ of relative degree d. The restriction of $\widetilde{\mathcal{J}}^d_{g,n}$ to $\mathcal{M}_{g,n}$ is the universal degree d Picard variety. The multiplicative group $\mathbb{G}_m$ acts by rescaling $\mathcal{F}$ in every fiber. We denote by $\widetilde{\mathcal{J}}^d_{g,n} /\!\!/ \mathbb{G}_m$ the rigidification with respect to this group action, the (*rigidified*) *universal compactified Jacobian*. Finally, we let $\overline{\mathcal{J}}^{d,P}_{g,n} \subset \widetilde{\mathcal{J}}^d_{g,n} /\!\!/ \mathbb{G}_m$ be the substack of P-semistable sheaves with respect to a polarization P (see below). It comes with a forgetful morphism $\overline{\mathcal{J}}^{d,P}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. The pointed canonical polarization P_{can} defined in (6.1) is a universal polarization on $\overline{\mathcal{M}}_{g,n}$ invariant under permutations of the numbering of the marked points.

Theorem 6.1. *The universal P-compactified Jacobian $\overline{\mathcal{J}}^{d,P}_{g,n}$ is a universally closed smooth (Artin) stack. If the polarization P is non-degenerate, it is a proper Deligne–Mumford stack.*

The pointed canonical polarization P_{can} of degree d is non-degenerate if and only if $\gcd(d - g + 1, 2g - 2 + n) = 1$.

Nodes of a curve X where $\mathcal{F}$ is not locally free will be called *Neveu–Schwarz nodes* for brevity.[1] The stabilizers of points in $\overline{\mathcal{J}}^d_{g,n}$ may be infinite, in fact precisely if the set of Neveu–Schwarz nodes separates the stable curve. This is possible for degenerate polarizations only. Consequently, $\overline{\mathcal{J}}^d_{g,n}$ is not a Deligne–Mumford stack and not separated (and thus not proper) near these points.

Torsion-free rank 1 *sheaves.* Let X be a nodal curve throughout this section. Recall that a coherent sheaf $\mathcal{F}$ on X is of *rank* 1 if it is of rank 1 at every generic point of X. It is *pure* if the support of every non-zero subsheaf is equal to the support of $\mathcal{F}$. It is *torsion-free* if it is pure and the support is equal to the whole curve X. The *degree* of a torsion-free sheaf $\mathcal{F}$ of rank 1 on a curve X is defined by $\deg(\mathcal{F}) = \chi(\mathcal{F}) - \chi(X)$.

A torsion-free rank 1 sheaf $\mathcal{F}$ on a smooth curve is simply a line bundle. On a nodal curve X, such a sheaf is a line bundle away from the nodes. At a node P, the stalk is isomorphic either to $\mathcal{O}_{X,P}$ or to the maximal ideal $\mathfrak{m}_P$.

If $X = \bigcup_i X_i$ is reducible with say s components, we associate with a torsion-free rank 1 sheaf $\mathcal{F}$ its multi-degree, that is,

$$\deg(\mathcal{F}) = (\deg \mathcal{F}_{X_1}, \ldots, \deg \mathcal{F}_{X_s}),$$

[1] This is inspired by physics terminology; see for example [JKV01]. The nodes where the bundle is locally free are called Ramond nodes.

where $\mathcal{F}_{X_j}$ is the maximal torsion-free quotient of $\mathcal{F}|_{X_j}$. If $f\colon \mathcal{X} \to B$ is a family of curves, then a family of torsion-free rank 1 sheaves $\mathcal{F}$ is a B-flat sheaf $\mathcal{F}$ on $\mathcal{X}$ whose fibers are of rank 1 and torsion-free.

Polarizations. A *numerical d-polarization* on the stable curve $X = \bigcup_{i=1}^s X_i$ is a tuple $\phi = (\phi_1, \dots, \phi_s)$ of rational numbers with total degree $|\phi| = \sum \phi_i = d$. We can alternatively view it as a function on the vertices of the dual graph Γ of X. A *universal numerical d-polarization* ϕ for $\overline{\mathcal{M}}_{g,n}$ is a collection of numerical d-polarizations ϕ_Γ for all dual graphs of curves in $\overline{\mathcal{M}}_{g,n}$ subject to the compatibility condition that for a contraction $c\colon \Gamma_1 \to \Gamma_2$ of dual graphs,

$$\phi_{\Gamma_2}(v) = \sum_{w \in c^{-1}(v)} \phi_{\Gamma_1}(w)\,.$$

We often drop the index Γ if it is clear from the context. In order to specify a universal polarization, it is often convenient to use a flat vector bundle P of some rank r and degree $r(d-g+1) > 0$ on the universal family $\mathcal{X}$. With such a vector bundle P, we associate the tuple

$$\phi = \phi(P) = \left(\frac{\deg P|_{X_1}}{r} + \frac{\deg(\omega_{X_1})}{2}, \dots, \frac{\deg P|_{X_s}}{r} + \frac{\deg(\omega_{X_s})}{2} \right).$$

In fact, every numerical polarization can be given by a vector bundle; see [KP19, Remark 4.6]. Of special interest further on is the following polarization. Let

$$\omega_{\mathcal{X}}(\mathbf{z}) := \omega_{\mathcal{X}} \left(\sum_{i=1}^n z_i \right)$$

be the canonical bundle twisted by the universal sections z_i. Then

$$P_{\mathrm{can}} = \omega_{\mathcal{X}}(\mathbf{z})^{\otimes d-g+1} \oplus \mathcal{O}_{\mathcal{X}}^{2g-3+n} \tag{6.1}$$

defines a universal d-polarization, the *pointed canonical polarization* $\phi_{\mathrm{can}} = \phi(P_{\mathrm{can}})$. In fact, there is a class of polarizations (discussed in, for example, [Mel15, Section 4.4.2]) obtained by varying the coefficients of z_i in the twisted canonical bundle, and we focus on the case of coefficients $+1$.

Stability. The torsion-free rank 1 sheaf $\mathcal{F}$ is called *ϕ-semistable* if for every non-empty proper subcurve $Y \subset X$, the "basic inequality"

$$\deg(\mathcal{F}_Y) \geqslant \sum_{X_i \subset Y} \phi_i - \frac{|Y \cap Y^c|}{2} \tag{6.2}$$

holds, where Y^c is the complement of Y in X. The sheaf $\mathcal{F}$ is called *ϕ-stable* if the above inequality is strict for all Y.

The polarization ϕ is called *non-degenerate* if the right-hand side of (6.2) does not assume integral values for any dual graph Γ and for any proper subcurve Y of a curve X with dual graph Γ. In particular, there are no strictly semistable sheaves with respect to a non-degenerate polarization.

It will be convenient to give an equivalent formulation in terms of Euler characteristics of the restriction of $\mathcal{F}$. Namely, since

$$\chi(\mathcal{F}_Y) = \deg \mathcal{F}_Y + 1 - g_Y = \deg \mathcal{F}_Y - \frac{1}{2}\big(\deg \omega_X|_Y - |Y \cap Y^c|\big)\,,$$

we conclude that (6.2) is equivalent to

$$\chi(\mathcal{F}_Y) \geqslant \frac{\deg P_Y}{\operatorname{rk} P}\,. \tag{6.3}$$

Balanced line bundles on quasi-stable curves. An alternative and essentially equivalent viewpoint uses balanced line bundles on quasi-stable curves. Here a semistable pointed curve $(X, \mathbf{z})$ is called *quasi-stable* if all ("exceptional") destabilizing components are rational curves without marked points and if these are disjoint and not contained in rational tails. A line bundle $\mathcal{L}$ is called *semibalanced* if (in the unpointed case) the *basic inequality*

$$\left|\deg(\mathcal{L}|_Z) - \sum_{X_i \subset Z} \phi_i^{\mathrm{can}}\right| \leqslant \frac{|Y \cap Y^c|}{2} \tag{6.4}$$

holds. It is called *balanced* if moreover $\deg(L|_E) = 1$ for every exceptional component of X. (See [Mel11] for the generalization to the pointed case, where rational tails have to be taken into account correctly so as to obtain a notion compatible with the morphism induced by forgetting a marked point.) The equivalence of the two notions is clarified by [EP16, Propositions 5.4 and 6.2], see also [KP19, Remark 5.14]: The pushforward of a balanced line bundle under the stabilization map is a torsion-free rank 1 sheaf, and the balancing condition (6.4) translates into (6.2). (On the level of coarse moduli spaces, this is the main content of Pandharipande's compactification [Pan96].)

The set of all universal numerical d-polarizations fits into a space $V^d_{g,n}$ analyzed in detail in [KP19]. Each $\phi \in V^d_{g,n}$ defines a substack $\overline{\mathcal{J}}^{d,\phi}_{g,n} \subset \widetilde{\mathcal{J}}^d_{g,n} /\!\!/ \mathbb{G}_m$, and [KP19] studies the dependence of the geometry of the substack on ϕ.

The versal deformations of sheaves on nodal curves. To justify the smoothness of $\overline{\mathcal{J}}^d_{g,n}$ and to prepare for the next section, we recall the local deformation theory of sheaves on nodes from [CKV15]. Let $\widehat{R} = \mathbb{C}[[x,y,t]]/(xy-t)$ be the complete local algebra of a node and $\mathcal{X} = \operatorname{Spec} \widehat{R}$ the family of curves degenerating to a node. The miniversal deformation ring of a locally free sheaf on a node is simply the base ring $\mathbb{C}[[t]]$ that parameterizes the degeneration to the node. The miniversal deformation ring of the torsion-free rank 1 sheaf $\mathfrak{m} \subset \widehat{R}_0 = \mathbb{C}[[x,y]]/(xy)$ is

$$S = \mathbb{C}[[a,b,t]]/(ab-t)\,; \tag{6.5}$$

see [CKV15, Lemma 3.13]. In fact, the miniversal deformation of $\mathfrak{m}$ on the pullback family $\mathcal{X}_S = \mathcal{X} \times_{\operatorname{Spec} \mathbb{C}[[t]]} \operatorname{Spec} S$ is

$$\mathcal{I} = \langle x-a, y-b \rangle\,. \tag{6.6}$$

It is also convenient to present $\mathcal{I}$ as $\mathbb{C}[[x,y,a,b]]/(xy-ab)$-module as

$$\mathcal{I} = \langle s_1, s_2 \mid xs_1 = -as_2,\ ys_2 = -bs_1 \rangle\,. \tag{6.7}$$

Proof of Theorem 6.1. The construction of the stack as geometric invariant theory (GIT) quotient is given in [Mel09], building on the earlier work of Caporaso [Cap94]. The isomorphism to the functor described here is given in [EP16, Theorem 6.3]. The source cited above in fact works without marked points, but [KP19, Remark 5.14] explains how to fix this.

The miniversal deformations ring of any torsion-free rank 1 sheaf at any node is smooth, combining (6.5) and the obvious locally free case. The full deformation space is a product of the deformations at the node and a power series ring since the forgetful map from deformations of the pair (X, F) to the product of local deformations at the node is formally smooth; see [CKV15, Section 6].

The stack $\widetilde{\mathcal{J}}^d_{g,n} /\!\!/ \mathbb{G}_m$ satisfies the valuative criterion for properness by [Est01, Theorem 32]. Since $\overline{\mathcal{J}}^d_{g,n}$ is a closed substack and moreover quasi-compact, it is universally closed.

The non-degeneracy of the canonical polarization under the gcd hypothesis follows from [Cap08]. In this case, the complement of the set of Neveu–Schwarz nodes is connected (otherwise, applying the definition of semistability to the two components violates non-degeneracy), and the automorphism group of the rigidified functor is trivial [CKV15, Theorem A]. Hence the stack is Deligne–Mumford in this case.

For the second statement, we only need to check the pointed canonical variant of the criterion for the polarization to be non-degenerate. Suppose that the polarization has this property. Then for any subcurve Y, say with $|Y \cap Y^c| = k$, we must have

$$\begin{aligned} \mathbb{Z} \not\ni \frac{\deg(\omega_Y)}{2} + \frac{\deg P|_Y}{2g-2+n} - \frac{k}{2} \\ = (g_Y - 1) + \frac{(d-g+1)(2g_Y - 2 + k + n_Y)}{2g-2+n}\,, \end{aligned} \tag{6.8}$$

so that, in particular, the second summand is not integral. We may prove this for any "banana curve," that is, with two irreducible components connected by $k = 2$ nodes. We may take any $g_Y \leqslant g-2$ with $n_Y = n$ or $g_Y = g-1$ with $n_Y = n-1$ to get a pointed stable curve. This implies that $2g_Y - 2 + k + n_Y$ attains all integers less than or equal to $2g-2+n-1$. Hence if $\gcd(d-g+1, 2g-2+n) = \ell > 1$, we may arrange for $2g_Y - 2 + k + n_Y = (2g-2+n)/\ell$ and get a contradiction.

Conversely, if $\gcd(d-g+1, 2g-2+n) = 1$, then (6.8) follows since

$$2g_Y - 2 + k + n_Y < 2g - 2 + n$$

for any subcurve of a pointed stable curve. $\square$

7. The spectral correspondence

In this section, we define an extension of the notion of spectral data on smooth Hitchin fibers to spectral data on semistable curves. Our notion of spectral data builds on the universal compactified Jacobian over the modified Hitchin base constructed in the previous sections. When specifying Higgs-related data, we write "$\mathrm{GL}(2,\mathbb{C})^\circ$" as shorthand for "trace-free $\mathrm{GL}(2,\mathbb{C})$."

We start by defining multi-scale spectral data pointwise and in families. Then we analyze the pushforward of multi-scale spectral data. This motivates the definition of multi-scale Higgs pairs and allows us to formulate spectral correspondences: a pointwise statement in Theorem 7.10 and a version for families in Theorem 7.12. These theorems will be proven in the following section.

7.1 Multi-scale spectral data

Let $(X, \mathbf{z})$ be a pointed stable curve.

Definition 7.1. A $\mathrm{GL}(2,\mathbb{C})^\circ$*-multi-scale spectral datum* $\big(\widehat{\Gamma}, \mathbf{q}, \boldsymbol{\tau}, \mathcal{F}\big)$ of degree $\widehat{d}$ on $(X, \mathbf{z})$ is a quadratic multi-scale differential $\big(\widehat{\Gamma}, \mathbf{q}, \boldsymbol{\tau}\big)$ together with a torsion-free rank 1 sheaf $\mathcal{F}$ of degree $\widehat{d}$ on $\widehat{X}$.

Let $\widehat{P}$ be a polarization on $\widehat{X}$. A $\mathrm{GL}(2,\mathbb{C})^\circ$-multi-scale spectral datum is called $\widehat{P}$*-semistable* if $\mathcal{F}$ is $\widehat{P}$-semistable.

Semistable multi-scale spectral data are points of the fiber product

$$\mathrm{SD}_g = \overline{\mathcal{Q}}_{g,[4g-4]} \times_{\overline{\mathcal{M}}_{\widehat{g},[4g-4]}} \overline{\mathcal{J}}^{\widehat{d},\widehat{P}}_{\widehat{g},4g-4} .$$

We comment on changing the polarization and on changing the Lie group below. As a consequence of Theorems 3.1 and 6.1, we record the following.

Proposition 7.2. *Let $\widehat{P}$ be a universal polarization on $\overline{\mathcal{M}}_{\widehat{g},\{4g-4\}}$. The space of multi-scale spectral data SD_g is a smooth Artin stack that admits an action of $\mathbb{C}^*$. The quotient $\mathbb{P}\,\mathrm{SD}_g = \mathrm{SD}_g/\mathbb{C}^*$ is universally closed. It is a proper Deligne–Mumford stack if $\widehat{P}$ is non-degenerate.*

From the definition of SD_g, it is clear how to define a family of multi-scale spectral data. We record it in order to fix notation.

Definition 7.3. A *germ of a family of multi-scale* $\mathrm{GL}(2,\mathbb{C})^\circ$*-spectral data* over a family $\mathcal{X} \to S$ is a tuple $\big(\widehat{\Gamma}, \mathbf{q}, \boldsymbol{\tau}, \mathcal{F}\big)$ of a family of multi-scale quadratic differentials $\big(\widehat{\Gamma}, \mathbf{q}, \boldsymbol{\tau}\big)$ on $\mathcal{X}$ and flat family of torsion-free sheaves $\mathcal{F}$ on $\widehat{\mathcal{X}}$.

One obtains the notion of a family of multi-scale $\mathrm{GL}(2,\mathbb{C})^\circ$-spectral data by patching together these germs. We refer to [BCG+19a, Section 7] for further details on the "sheafification process."

7.2 The local form of the pushforward

We analyze the local form of the pushforward of a family of multi-scale spectral data $\big(\widehat{\Gamma}, \mathbf{q}, \boldsymbol{\tau}, \mathcal{F}\big)$ at the nodes. We restrict the double covering $\widehat{\pi}\colon \widehat{\mathcal{X}} \to \mathcal{X}$ to a neighborhood $\mathcal{X}$ of a deformation of a node e and the (σ-invariant) preimage. We may pretend that this is a node joining levels 0 and -1 and label all objects accordingly. There are four cases to consider depending on whether κ_e is odd or even and whether $\mathcal{F}$ is locally free in the special fiber or not. When κ_e is odd, the node has a single preimage $\widehat{e}$ fixed by the involution σ. When κ_e is even, the node has two preimages $\widehat{e}_1$, $\widehat{e}_2$ interchanged by σ.

We will assume that the multi-scale abelian differential $\boldsymbol{\lambda}$ on $\widehat{\mathcal{X}}$ is given in the normal form of [BCG+19a, Theorem 4.3]

$$\lambda_0 = \big(x^{\kappa_{\widehat{e}}} + r\big)\frac{dx}{x}\,, \quad \lambda_{-1} = -\Big(y^{-\kappa_{\widehat{e}}} + \frac{r}{t^{\alpha\kappa_{\widehat{e}}}}\Big)\,\frac{dy}{y}$$

over a deformation of the node of the form $\{xy = t^\alpha\}$. Here the residue $r \in \mathbb{C}[[t]]$ can be assumed to be divisible by $t^{\alpha\kappa_e}$. The abelian differential on the canonical double cover $\widehat{X}$ is σ-anti-symmetric. Hence the residue is zero at nodes $\widehat{e}$ fixed by σ (see also [BCG+19b, Theorem 3.1]).

7.2.1 *κ_e odd and $\mathcal{F}$ is locally free.* The covering is given by the inclusion of rings

$$\begin{aligned} \widehat{\pi}^\sharp\colon\ & R := \mathbb{C}[[u,v,t]]/\big(uv - t^{2\alpha}\big) \ \to\ \hat{R} = \mathbb{C}[[x,y,t]]/(xy - t^\alpha) \\ & u \mapsto x^2\,, \quad v \mapsto y^2\,, \quad t \mapsto t \end{aligned} \tag{7.1}$$

for some α determined by the family of curves and $\mathcal{F} \cong \widehat{R}$, so that

$$\mathcal{E} := \widehat{\pi}_* \widehat{R} = R \oplus \langle x, y\rangle_R\,. \tag{LF/fix}$$

The divisor X_1 is given by $u = 0$ in these coordinates, so u is invertible on $\mathcal{X} \setminus X_1$, and so $\mathcal{E}|_{\mathcal{X}\setminus X_1} = 1 + \langle x\rangle_R$ (since $y = x \cdot t^a/u$). Similarly, $\mathcal{E}|_{\mathcal{X}\setminus X_0} = 1 + \langle y\rangle_R$. Multiplication with λ_i on $\mathcal{F}$

induces Higgs fields

$$\Phi_0\colon \mathcal{E}|_{\mathcal{X}\setminus X_1} \to \mathcal{E}\otimes\omega_{\mathcal{X}}|_{\mathcal{X}\setminus X_1}\,, \quad \Phi_1\colon \mathcal{E}|_{\mathcal{X}\setminus X_0} \to \mathcal{E}\otimes\omega_{\mathcal{X}}|_{\mathcal{X}\setminus X_0}$$

given in this basis by

$$\begin{aligned} \Phi_0 = \widehat{\pi}_*\lambda_0|_{\mathcal{X}\setminus X_1}\colon &\quad 1\mapsto u^{(\kappa_{\widehat{e}}-1)/2}x\otimes\frac{du}{u}\,, \quad x\mapsto u^{(\kappa_{\widehat{e}}+1)/2}\otimes\frac{du}{u}\,, \\ \Phi_1 = \widehat{\pi}_*\lambda_1|_{\mathcal{X}\setminus X_0}\colon &\quad 1\mapsto v^{-(\kappa_{\widehat{e}}+1)/2}y\otimes\frac{dv}{v}\,, \quad y\mapsto v^{-(\kappa_{\widehat{e}}-1)/2}\otimes\frac{dv}{v}\,. \end{aligned} \tag{7.2}$$

7.2.2 κ_e *odd and* $\mathcal{F}$ *is Neveu–Schwarz.* Using the same covering (7.1), the sheaf $\mathcal{F}$ is now locally the pullback of the miniversal deformation $\mathcal{I}$ from (6.6) with t replaced by t^α. Consequently,

$$\mathcal{E} := \widehat{\pi}_*\mathcal{I} = \langle x-a, y-b, x(x-a), y(y-b)\rangle_R\,. \tag{NS/fix}$$

For $u\neq 0$ (respectively, $v\neq 0$), this generating set simplifies to

$$\mathcal{E}|_{\mathcal{X}\setminus X_1} = R(x-a)\oplus Rx(x-a) \quad (\text{respectively, } \mathcal{E}|_{\mathcal{X}\setminus X_0} = R(y-b)\oplus Ry(y-b))\,.$$

The two Higgs fields are given by

$$\begin{aligned} \Phi_0\colon &\quad (x-a)\mapsto u^{(\kappa_{\widehat{e}}-1)/2}x(x-a)\otimes\frac{du}{u}\,, \quad x(x-a)\mapsto u^{(\kappa_{\widehat{e}}+1)/2}(x-a)\otimes\frac{du}{u}\,, \\ \Phi_1\colon &\quad (y-b)\mapsto v^{-(\kappa_{\widehat{e}}+1)/2}y(y-b)\otimes\frac{dv}{v}\,, \quad y(y-b)\mapsto v^{-(\kappa_{\widehat{e}}-1)/2}(y-b)\otimes\frac{dv}{v}\,. \end{aligned}$$

7.2.3 κ_e *even and* $\mathcal{F}$ *is locally free.* The covering is given by the inclusion

$$\begin{aligned} \widehat{\pi}^\sharp = \left(\widehat{\pi}_1^\sharp,\widehat{\pi}_2^\sharp\right)\colon\ R := \frac{\mathbb{C}[[u,v,t]]}{uv-t^\alpha} &\ \to\ \widehat{R} = \frac{\mathbb{C}[[x_1,y_1,t]]}{x_1y_1-t^\alpha}\oplus\frac{\mathbb{C}[[x_2,y_2,t]]}{x_2y_2-t^\alpha} =:\ \widehat{R}_1\oplus\widehat{R}_2 \\ u\mapsto (x_1,x_2)\,, \quad v\mapsto (y_1,y_2)\,, &\qquad t\mapsto t \end{aligned} \tag{7.3}$$

of rings for some α. Note the index $j=1,2$ of $\widehat{R}_j$ and $i=0,-1$ of the levels of the special fiber X_i. Let $\mathcal{F}$ be free module of rank 1 over $\widehat{R}$ and $\mathcal{E}=\widehat{\pi}_*\mathcal{F}$. Then

$$\mathcal{E} = \widehat{\pi}_*\mathcal{F} = \widehat{\pi}_{1*}\mathcal{F}|_{\widehat{U}_1}\oplus\widehat{\pi}_{2*}\mathcal{F}|_{\widehat{U}_2}\,, \tag{LF/swap}$$

where $\widehat{U}_j = \operatorname{Spec}(\widehat{R}_j)$ with $j=1,2$. The multi-scale abelian differential $\boldsymbol{\lambda}$ is σ-antisymmetric and hence given by levelwise abelian differentials

$$\begin{aligned} \lambda_0 &= \left(\left(x_1^{\kappa_e/2}+r\right)\frac{dx_1}{x_1}, -\left(x_2^{\kappa_e/2}+r\right)\frac{dx_2}{x_2}\right), \\ \lambda_{-1} &= \left(-\left(y_1^{-\kappa_e/2}+\frac{r}{t^{\alpha\kappa_{\widehat{e}}}}\right)\frac{dy_1}{y_1}, \left(y_2^{-\kappa_e/2}+\frac{r}{t^{\alpha\kappa_{\widehat{e}}}}\right)\frac{dy_2}{y_2}\right). \end{aligned}$$

There exist unique abelian differentials η_i on $\mathcal{X}\setminus X_{-1-i}$ such that

$$\widehat{\pi}^*\eta_0 = \left(x_1^{\kappa_e/2}+r\right)\frac{dx_1}{x_1} \quad\text{and}\quad \widehat{\pi}^*\eta_{-1} = -\left(y_1^{-\kappa_e/2}+\frac{r}{t^{\alpha\kappa_{\widehat{e}}}}\right)\frac{dy_1}{y_1}.$$

The Higgs fields are diagonal with respect to the given splitting of $\mathcal{E}$. They are given by

$$\Phi_i = \widehat{\pi}_*\lambda_i|_{\mathcal{X}\setminus X_{-1-i}} = \begin{pmatrix}\eta_i & 0\\ 0 & -\eta_i\end{pmatrix} \tag{7.4}$$

for $i=0,-1$ with respect to such a frame.

7.2.4 κ_e *even and* $\mathcal{F}$ *is Neveu–Schwarz.* There are two cases to consider depending on whether $\mathcal{F}$ is Neveu–Schwarz at one or at both nodes interchanged by σ. However, they work quite similarly. First consider the case where $\mathcal{F}$ is Neveu–Schwarz at the node $\widehat{e}_1$ and locally free at $\widehat{e}_2$. To take into account the equisingular deformation of $\mathcal{F}_{\widehat{e}_1} \cong \mathfrak{m}_{\widehat{e}_1}$, we have to work over the miniversal deformation ring S recorded in (6.5). To compute the fiber product with the covering (7.3), let $R = \mathbb{C}[u,v,a,b,t]/(uv - t^\alpha, ab - t^\alpha)$ and

$$\widehat{R}_j = \mathbb{C}[x_j, y_j, a, b, t]/(x_j y_j - t^\alpha, ab - t^\alpha)$$

for $j = 1, 2$. Then the covering over S is given by

$$\widehat{\pi}^\sharp\colon R \to \widehat{R}_1 \oplus \widehat{R}_2\,, \quad u \mapsto (x_1, x_2), v \mapsto (y_1, y_2)\,.$$

Let $\mathcal{I}_1 = \langle x_1 - a, y_1 - b\rangle \subset \widehat{R}_1$ be the miniversal deformation of $\mathfrak{m}_{\widehat{e}_1}$. Then the family of torsion-free sheaves $\mathcal{F}$ is given by

$$\mathcal{F} = \mathcal{I}_1 \oplus \widehat{R}_2 = \langle x_1 - a, y_1 - b\rangle \oplus \widehat{R}_2\,. \tag{7.5}$$

Consequently,

$$\mathcal{E} := \widehat{\pi}_*\mathcal{F} = \langle u - a, v - b\rangle \oplus R \subset R^2 \tag{NS/swap}$$

is a deformation of $\mathfrak{m}_e \oplus R_e$. The local form of the Higgs field is given by the same matrix as in (7.4).

In the case of $\mathcal{F}$ being Neveu–Schwarz at both nodes $\widehat{e}_1$, $\widehat{e}_2$, we have to consider the fiber product of the covering (7.3) with the direct sum of two miniversal deformation rings $S_1 \oplus S_2$. The remainder of the construction can be easily adapted. In this case, the family of torsion-free sheaves $\mathcal{E} = \widehat{\pi}_*\mathcal{F}$ is a deformation of $\mathfrak{m}_e \oplus \mathfrak{m}_e$ at the node.

7.2.5 *Horizontal nodes.* At horizontal nodes, the canonical covering $\widehat{\pi}\colon \widehat{X} \to X$ is unbranched, and the local models look the same as in the case $\kappa_e \equiv 0 \bmod 2$. Instead of considering two irreducible components on two different levels $i = 0, -1$, we have two irreducible components on one level i, which we index by a, b so that on the a-component, we have x-coordinates and on the b-component, we have y-coordinates. The only difference with vertical nodes is that the componentwise abelian differentials glue to a section of $\omega_{\widehat{X}}$, respectively $\omega_{\widehat{X}}$. In local coordinates as in (7.3), they are given by

$$\lambda_a = \left(r\frac{dx_1}{x_1}, -r\frac{dx_2}{x_2}\right), \quad \lambda_b = \left(-r\frac{dy_1}{y_1}, r\frac{dy_2}{y_2}\right),$$

with residue $r \in \mathbb{C}$. This yields a gluing of the componentwise Higgs fields

$$\Phi_a = \begin{pmatrix} r\dfrac{du}{u} & 0 \\ 0 & -r\dfrac{du}{u} \end{pmatrix}, \quad \Phi_b = \begin{pmatrix} -r\dfrac{dv}{v} & 0 \\ 0 & r\dfrac{dv}{v} \end{pmatrix}$$

to a Higgs field $\Phi\colon \mathcal{E} \to \mathcal{E} \otimes \omega$.

7.3 Multi-scale $\mathrm{GL}(2,\mathbb{C})^\circ$-Higgs pairs

The previous computations motivate the following definition.

Definition 7.4. A *multi-scale* $\mathrm{GL}(2,\mathbb{C})^\circ$*-Higgs pair* of degree d on a $(4g-4)$-pointed stable curve $(X, \mathbf{z})$ is a tuple $(\widehat{\Gamma}, \mathcal{E}, \boldsymbol{\Phi}, \boldsymbol{\tau})$ consisting of a level graph structure on a double covering $\widehat{\pi}\colon \widehat{\Gamma} \to \Gamma$ of the dual graph of X, a torsion-free rank 2 sheaf $\mathcal{E}$ on X, Higgs fields $\boldsymbol{\Phi} = (\Phi_0, \ldots, \Phi_L)$ if Γ

has L levels below zero and prong-matchings $\boldsymbol{\tau}$ for the collection $\mathbf{q} = (q_i)$ of differentials with simple zeros at the marked points, subject to the following conditions:

(i) If we denote by $\mathcal{E}_i$ the restriction $\mathcal{E}|_{X_i}$ mod torsion, then Φ induces
$$\Phi_i \colon \mathcal{E}_i \to \mathcal{E}_i \otimes M_i \,, \tag{7.6}$$
where the M_i are the twists of the canonical bundle defined in (3.2), such that $q_i = \det(\Phi_i)$.

(ii) The traces of the Higgs fields vanish; that is, $\operatorname{Tr}(\Phi_i) = 0$.

(iii) The collection $\mathbf{q} = (q_i)$ together with $\boldsymbol{\tau}$ is a multi-scale differential of type $\mu = \left(1^{4g-4}\right)$ compatible with the level graph Γ.

(iv) For each node e, there exists an analytic neighborhood U such that $(\mathcal{E}, \boldsymbol{\Phi})|_U$ is given by the specialization to $t = 0$ of one of the local forms of Section 7.2.

Two Higgs pairs are called *equivalent* if they are in the same orbit of the level rotation torus, acting by rescaling the Higgs fields Φ_i for $i < 0$ and simultaneously on $\boldsymbol{\tau}$.

Note that the condition on q_i to form a multi-scale differential implies, in particular, that all the Higgs fields Φ_i are non-zero.

Remark 7.5. At a node e with κ_e even, including the case of horizontal nodes, condition (iv) is equivalent to the existence of a splitting of $\mathcal{E}$ into rank 1 subsheaves in an analytic neighborhood of e such that the Higgs fields $\Phi_{\ell(e^+)}$ and $\Phi_{\ell(e^-)}$ are diagonal with respect to the splitting. Here, $\ell(e^\pm)$ denote the respective levels of the preimages of the node e. However, for $\kappa_e \equiv 1 \bmod 2$ we are missing an abstract reformulation of condition (iv).

Definition 7.6. We define stability on multi-scale Higgs pairs as follows:

(i) A *polarization* P of a multi-scale $\mathrm{GL}(2,\mathbb{C})$-Higgs pair of degree d is a locally free sheaf P on X such that $\deg(P) = \operatorname{rk}(P)(d - 2g + 2)$.

(ii) Let P be a polarization of multi-scale $\mathrm{GL}(2,\mathbb{C})$-Higgs pairs of degree d on X. Then a multi-scale $\mathrm{GL}(2,\mathbb{C})$-Higgs pair is called *semistable* if for every subsheaf $\mathcal{G} \hookrightarrow \mathcal{E}$ that is $\boldsymbol{\Phi}$-stable (that is, such that there is a factorization $\Phi_i \colon \mathcal{G}|_{X_i} \to \mathcal{G}|_{X_i} \otimes M_i$), we have
$$\chi(\mathcal{G}) \leqslant \frac{\sum_{v=1}^{s} \operatorname{rk}(\mathcal{G}|_{X_v}) \deg(P|_{X_v})}{2 \operatorname{rk} P} \,. \tag{7.7}$$
The sum runs over all irreducible components $X_v \subset X$ for $v = 1, \ldots, s$.

The same convention, an index $v = 1, \ldots, s$ referring to vertices $v \in \Gamma$ and thus to irreducible components of X, is used frequently from here on.

Example 7.7. Let $\mathcal{L}$ be a polarizing line bundle on X. Then
$$P = \mathcal{L}^{\otimes d-2g+2} \oplus \mathcal{O}_X^{\deg \mathcal{L} - 1}$$
defines a polarization of multi-scale $\mathrm{GL}(2,\mathbb{C})$-Higgs pairs of degree d on X. We recover the Simpson's p-stability condition for Higgs pair with respect to $\mathcal{L}$
$$\frac{\chi(\mathcal{G})}{\sum_{v=1}^{s} \operatorname{rk}(\mathcal{G}|_{X_v}) \deg(\mathcal{L}|_{X_v})} \leqslant \frac{\chi(\mathcal{E})}{2 \deg(\mathcal{L})}$$
(cf. [BBN16, Definition 2.2]). A universal choice for $\mathcal{L}$ over $\overline{\mathcal{Q}}_{g,4g-4}\left(1^{4g-4}\right)$ is the pullback of the pointed canonical bundle $\omega_X(\mathbf{z})$ along $\overline{\mathcal{Q}}_{g,4g-4}\left(1^{4g-4}\right) \to \overline{\mathcal{M}}_{g,4g-4}$. It is invariant under permuting the marked points and hence descends to the quotient stack by the action of the symmetric group.

Lemma 7.8. *If P is polarization of multi-scale* $\mathrm{GL}(2,\mathbb{C})$*-Higgs pairs of degree d, then $\widehat{P} = \widehat{\pi}^* P \oplus \mathcal{O}_{\widehat{X}}^{\mathrm{rk}(P)}$ is polarization of degree $\widehat{d} = d + 2g - 2$ in the sense of Section 6.*

We refer to $\widehat{P}$ as the *induced polarization* on $\widehat{X}$.

Proof. We have $\mathrm{rk}\big(\widehat{P}\big) = 2\,\mathrm{rk}(P)$ and

$$\deg\big(\widehat{P}\big) = 2\,\mathrm{rk}(P)(d - 2g - 2) = \mathrm{rk}\big(\widehat{P}\big)(\widehat{d} - 4g + 4)\,.$$

Hence $\widehat{P}$ satisfies the condition for a polarization of torsion-free rank 1 sheaves of degree $\widehat{d} = d + 2g - 2$. □

Remark 7.9. It is not clear how to define a pushforward of polarizations in terms of the bundles $\widehat{P}$ and P. However, the stability condition associated with a polarization $\widehat{P}$ of degree $\widehat{d}$ on $\widehat{X}$ is uniquely defined by the *slopes* (cf. (6.3))

$$\hat{s}_v := \frac{\deg \widehat{P}|_{\widehat{X}_v}}{\mathrm{rk}\,\widehat{P}}\,.$$

We can define a pushforward of a stability condition in terms of these slopes by

$$s_v^\flat = \frac{1}{\sharp\widehat{\pi}^{-1}(v)} \sum_{\hat{v}\in\widehat{\pi}^{-1}(v)} \hat{s}_{\hat{v}}$$

for all irreducible components X_v. This is compatible with the notion of pullback of a polarization in Lemma 7.8.

Now, the pointwise version of the correspondence is the following generalization of Theorem 1.2.

Theorem 7.10. *Let X be a pointed stable curve. Let P be a polarization of multi-scale* $\mathrm{GL}(2,\mathbb{C})$*-Higgs pairs of degree d on X and $\widehat{P}$ the induced polarization on $\widehat{X}$. Then there is a bijection between $\widehat{P}$-(semi)stable* $\mathrm{GL}(2,\mathbb{C})^\circ$*-multi-scale spectral data on $\widehat{X}$ and equivalence classes of P-(semi)stable multi-scale* $\mathrm{GL}(2,\mathbb{C})^\circ$*-Higgs pairs on X.*

Our BNR correspondence is in fact functorial, that is, works in families. To state this, we globalize Definition 7.4. Since multi-scale differentials are defined by gluing (equivalence classes of) germs, we will state the correspondence at the level of germs. For a germ of a family of stable curves $f\colon \mathcal{X} \to S$, we define $\mathcal{X}_{i^\complement} = \mathcal{X} \setminus \bigcup_{j\neq i} X_j$, the complement of the curves in the special fiber that are not at level i.

There is one major difference in the form of the Higgs field between Definitions 7.4 and 7.11 already present in the construction of multi-scale differentials: In the pointwise situation, and more generally for equisingular deformations, the nodes are given by sections of the family. They can be used for twisting line bundles and hence for the definition of the M_i in (7.6). For families, say with smooth generic fiber, the nodes are of higher codimension. There, alternatively, we define the Higgs field in (7.8) away from the nodes. Extension across codimension 2 allows us to recover the local structure near the node, and so the two definitions are compatible under pullback.

We emphasize that the subsequent definition treats only the case of families with smooth generic fiber. We leave it to the reader to state the obvious generalization to the general case where some nodes remain equisingular while others are smoothened.

Definition 7.11. A *germ of a family of multi-scale* $\mathrm{GL}(2,\mathbb{C})^\circ$*-Higgs pairs* on a germ $f\colon \mathcal{X} \to S$ of a family of $(4g-4)$-pointed stable curves with smooth generic fiber and special fiber $(X, \mathbf{z})$ is a tuple $\big(\widehat{\Gamma}, \mathcal{E}, \mathbf{\Phi}, \boldsymbol{\tau}\big)$ consisting of a level graph structure on a double covering $\widehat{\pi}\colon \widehat{\Gamma} \to \Gamma$ of the dual graph of X, a flat family of torsion-free sheaves $\mathcal{E}$ of rank 2 on $\mathcal{X}$, Higgs fields $\mathbf{\Phi} = (\Phi_0, \dots, \Phi_L)$ if Γ has L levels below zero, a collection $\mathbf{q} = (q_i)$ of differentials $q_i = \det(\Phi_i)$ with simple zeros at the marked points and prong-matchings $\boldsymbol{\tau}$ for $\mathbf{q}$ at the persistent nodes, defined with the following conditions:

(i) There are $\mathcal{O}_{\mathcal{X}_{i}^{\complement}}$-module homomorphisms

$$\Phi_i\colon \mathcal{E}|_{\mathcal{X}_{i}^{\complement}} \to \mathcal{E}|_{\mathcal{X}_{i}^{\complement}} \otimes \omega_{\mathcal{X}/S}|_{\mathcal{X}_{i}^{\complement}} \tag{7.8}$$

such that $q_i = \det(\Phi_i)$

(ii) The traces of the Higgs fields vanish; that is, $\mathrm{Tr}(\Phi_i) = 0$.

(iii) The collection $\mathbf{q} = (q_i)$ together with $\boldsymbol{\tau}$ is a multi-scale differential of type $\mu = \big(1^{4g-4}\big)$ compatible with the level graph Γ.

(iv) At every node e, the pair $(\mathcal{E}, \mathbf{\Phi})$ is given by a base change of the local forms described in Section 7.2.

Two germs of families of Higgs pairs are called *equivalent* if they are in the same orbit of a section over S of the level rotation torus, acting by rescaling the Higgs fields Φ_i for $i < 0$ and simultaneously on $\boldsymbol{\tau}$.

A germ of a family of Higgs pairs is called *semistable* if fiberwise the stability condition (7.7) holds.

We have prepared the statement of the correspondence in families.

Theorem 7.12. *Let $f\colon \mathcal{X} \to S$ be a germ of a family of pointed stable curves with smooth generic fiber, and let P be a polarization on $\mathcal{X}$. Then there is a bijection between germs of $\widehat{P}$-(semi)stable* $\mathrm{GL}(2,\mathbb{C})^\circ$*-multi-scale spectral data on $\mathcal{X}$ and equivalence classes of germs of P-(semi)stable multi-scale* $\mathrm{GL}(2,\mathbb{C})^\circ$*-Higgs pairs on $\mathcal{X}$.*

7.4 The pushforward correspondence

One direction of Theorem 7.10 without addressing stability yet is the following proposition.

Proposition 7.13. *Let $\big(\widehat{\pi}\colon \widehat{X} \to X, \mathbf{q}, \boldsymbol{\tau}, \mathcal{F}\big)$ be a* $\mathrm{GL}(2,\mathbb{C})^\circ$-multi-scale spectral datum *on X or on a germ of a family $f\colon \mathcal{X} \to S$. Then $\mathcal{E} = \widehat{\pi}_*\mathcal{F}$, together with the Higgs fields $\Phi_i = \widehat{\pi}_*(\,\cdot\,\lambda_i)$, defines a* multi-scale $\mathrm{GL}(2,\mathbb{C})^\circ$-Higgs pair $\big(\widehat{\Gamma}, \mathcal{E}, \mathbf{\Phi}, \boldsymbol{\tau}\big)$ *on X or on $\mathcal{X}$, respectively.*

Proof. Since $\widehat{\pi}|_{X_i}$ is a double covering given by (M_i, q_i) as defined in (3.3), the classical BNR correspondence restricted to the X_i implies the claim on the trace and the determinant of the Φ_i. In the case of a smooth generic fiber, it is obvious that the Higgs field is a map as required by (7.8). In the equisingular case (7.6), the condition on the range (being $\mathcal{E} \otimes M_i$) is a consequence of the local descriptions of the Φ_i in Section 7.2, which in turn follows from the orders of zeros and poles of λ_i imposed by being compatible with $\widehat{\Gamma}$.

The level rotation torus acts simultaneously on the abelian differentials λ_i and the prong-matching. This induces an action on $\Phi_i = \widehat{\pi}_*(\,\cdot\,\lambda_i)$ and thus on its determinant. This is obviously compatible with the action on $q_i = \lambda_i^2$ and shows that pushforward is well defined on equivalence classes.

The claim about stability will be proven separately in Proposition 8.3. $\square$

8. Proof of the spectral correspondence

We first define an inverse map to the pushforward correspondence considered above. In the case of a family of curves $\mathcal{X} \to S$ with generically smooth fiber over a smooth base scheme S, this is an easy task: We can apply the classical case as in the proof of Theorem 2.1 over $\widehat{\mathcal{X}} \setminus \widehat{N}$, where $\widehat{N}$ is the nodal locus in the fibers of the family, and then push forward along the inclusion $j : \widehat{\mathcal{X}} \setminus \widehat{N} \to \widehat{\mathcal{X}}$. A flat family of torsion-free sheaves is reflexive (see [NS99, Section 4, p. 191]) and hence determined by its values on a codimension 2 subset [Har80, Proposition 1.4].

However, this argument does not work in the equisingular case as here the nodes have codimension 1. Here we need to give an explicit construction. To emphasize the functoriality of the construction, we instead give an explicit construction using the special frames constructed in Section 7.2 in both cases.

The second task in this section is to verify stability conditions in this correspondence.

8.1 The pullback correspondence over a generically smooth family of curves

Let $f\colon \mathcal{X} \to S$ be a germ of a family of $(4g-4)$-pointed stable curves. Let $\big(\widehat{\Gamma}, \mathcal{E}, \mathbf{\Phi}, \boldsymbol{\tau}\big)$ be a germ of a family of multi-scale $\mathrm{GL}(2,\mathbb{C})^{\circ}$-Higgs pairs on $\mathcal{X}$. First we want to recover a locally free sheaf $\mathcal{F}_i$ on $\mathcal{X}_{i\complement}$. To do so, we apply the classical pullback correspondence from Theorem 2.1 to $(\mathcal{E}, \mathbf{\Phi})$. For each level i, let

$$\widehat{\mathcal{B}}_i = \frac{1}{2}\mathrm{div}\,(\lambda_i) \in \mathsf{Div}^{+}\big(\widehat{\mathcal{X}}_{i\complement}\big)$$

be the zero divisor of the family of abelian differentials on $\mathcal{X}_{i\complement}$. We define

$$\mathcal{F}_i := \ker\big(\widehat{\pi}_i^{*}\Phi_i - \lambda_i\,\mathrm{Id}_{\widehat{\pi}^{*}\mathcal{E}}\big) \otimes \mathcal{O}\big(\widehat{\mathcal{B}}_i\big)\,,$$

which is a locally free sheaf on $\widehat{\mathcal{X}}_{i\complement}$. These locally free sheaves naturally glue on the smooth fibers of $\widehat{\mathcal{X}} \to S$, where the differentials λ_i differ by an invertible function on S. This defines a locally free sheaf $\mathcal{F}'$ on $\widehat{\mathcal{X}} \setminus \widehat{N}$. We will now describe how to extend this locally free sheaf over the nodes $\widehat{N}$ with respect to local frames. This will define the torsion-free rank 1 sheaf $\mathcal{F}$ on $\widehat{\mathcal{X}}$.

In the local considerations, we can restrict to two levels 0, -1 meeting in one node e in the special fiber X. (The case of a horizontal node is dealt with in the same way, necessarily falling into the "swapped node" subcases. There, in the following local computations, the indices $i = 0$ and $i = 1$ that usually refer to the level should be used to enumerate the two branches of the node.) We have to consider several special cases in parallel to Section 7.2. As we mentioned above, whenever the family of $\widehat{\mathcal{X}}$ is normal, the sheaf constructed below is equal to the reflexive sheaf $j_*\mathcal{F}'$. In the following, for all modules over a ring $\widehat{R}$ that are defined as a tensor product, we act by $\widehat{R}$ on the left factor.

8.1.1 *Fixed node and $\mathcal{E}$ has a free summand locally near the node.* This is to say that we start with $\mathcal{E} = R \oplus \langle x, y\rangle_R$ as in (LF/fix) and the Higgs fields as in (7.2). In this case,

$$\ker(\widehat{\pi}^{*}\Phi_0 - \lambda_0\,\mathrm{Id}) = \left\langle 1 \otimes 1 + \frac{1}{x} \otimes x \right\rangle \subset \mathcal{O}\big(\widehat{X}_1\big) \otimes \widehat{\pi}^{*}\mathcal{E}|_{\widehat{\mathcal{X}}\setminus\widehat{X}_1} = \widehat{R} \otimes \widehat{R}|_{\widehat{\mathcal{X}}\setminus\widehat{X}_1}\,,$$

and similarly

$$\ker(\widehat{\pi}^{*}\Phi_1 - \lambda_1\,\mathrm{Id}) = \left\langle 1 \otimes 1 + \frac{1}{y} \otimes y \right\rangle \subset \mathcal{O}\big(\widehat{X}_0\big) \otimes \widehat{\pi}^{*}\mathcal{E}|_{\widehat{\mathcal{X}}\setminus\widehat{X}_0} = \widehat{R} \otimes \widehat{R}|_{\widehat{\mathcal{X}}\setminus\widehat{X}_0}\,.$$

These two sheaves glue to a locally free sheaf $\mathcal{F}'$ on the complement $\mathcal{X} \setminus \{\widehat{e}\}$ of the node. More precisely, the generating sections glue on the set $\{t \neq 0\}$ by

$$1 \otimes 1 + \frac{t^\alpha y^2}{t^\alpha y^2} \frac{1}{x} \otimes x = 1 \otimes 1 + \frac{1}{y} \otimes y\,.$$

Hence $\mathcal{F}'$ is free of rank 1 over $\mathcal{X} \setminus \{\widehat{e}\}$, and consequently $\mathcal{F}'$ extends over the node as a free rank 1 module $\widehat{R}$ over $\mathcal{X}$. This defines $\mathcal{F}$ at $\widehat{e}$.

8.1.2 *Fixed node and $\mathcal{E}$ has no free summand.* We now start with $\mathcal{E}$ as in (NS/fix). Now

$$\ker(\widehat{\pi}^*\Phi_0 - \lambda_0 \operatorname{Id}) = \left\langle 1 \otimes (x-a) + \frac{1}{x} \otimes x(x-a) \right\rangle \subset \mathcal{O}\big(\widehat{X}_1\big) \otimes \widehat{\pi}^*\mathcal{E}|_{\widehat{\mathcal{X}}\setminus \widehat{X}_1} = \widehat{R} \otimes \widehat{R}|_{\widehat{\mathcal{X}}\setminus \widehat{X}_1}\,,$$

and similarly

$$\ker(\widehat{\pi}^*\Phi_1 - \lambda_1 \operatorname{Id}) = \left\langle 1 \otimes (y-b) + \frac{1}{y} \otimes y(y-b) \right\rangle \subset \mathcal{O}\big(\widehat{X}_0\big) \otimes \widehat{\pi}^*\mathcal{E}|_{\widehat{\mathcal{X}}\setminus \widehat{X}_0} = \widehat{R} \otimes \widehat{R}|_{\widehat{\mathcal{X}}\setminus \widehat{X}_0}\,.$$

The two generators e_0, e_1 satisfy the relations $ye_0 = -ae_1$ and $xe_1 = -be_0$ for $t \neq 0$, where they are both defined. In particular, they glue to a locally free sheaf $\mathcal{F}'$ on $\mathcal{X} \setminus \{\widehat{e}\}$. Notice that there is an isomorphism

$$\mathcal{I}|_{\widehat{\mathcal{X}}\setminus\{\widehat{e}\}} \to \mathcal{F}'\,, \quad s_1 \mapsto e_0, s_2 \mapsto e_1\,.$$

Hence we can extend the module $\mathcal{F}'$ over $\widehat{e}$ by $\mathcal{I}$. This defines $\mathcal{F}$ at $\widehat{e}$.

8.1.3 *Swapped node and $\mathcal{E}$ is locally free.* With $\mathcal{E}$ written in a frame as in (LF/swap) and the Higgs field from (7.4), the kernel $\ker(\widehat{\pi}^*\Phi_0 - \lambda_0 \operatorname{Id})$ is generated (on $\mathcal{X} \setminus X_1$) by the line bundle generated by the first coordinate on U_1 and by the second coordinate on U_2. The same holds for the kernel $\ker(\widehat{\pi}^*\Phi_1 - \lambda_1 \operatorname{Id})$. Consequently, these coordinate line bundles extend $\mathcal{F}'$ over both nodes $\widehat{e}_1$, $\widehat{e}_2$ to a line bundle $\mathcal{F}$ on $\mathcal{X}$.

8.1.4 *Swapped node and $\mathcal{E}$ is not locally free.* We again only treat the case where $\mathcal{E}_e \cong \mathfrak{m}_e \oplus R$. The case where $\mathcal{E}_e \cong \mathfrak{m}_e \oplus \mathfrak{m}_e$ works along the same lines. With $\mathcal{E}$ written in a frame as in (NS/swap) and the Higgs field still as in (7.4), we define $\mathcal{F}$ on the complement of X_1 and X_0 with the help of the eigenspaces

$$\ker(\widehat{\pi}^*\Phi_{i=0} - \lambda_{i=0} \operatorname{Id}_{\widehat{\pi}^*\mathcal{E}}|_{x_j\neq 0}) \quad \text{and} \quad \ker(\widehat{\pi}^*\Phi_{i=1} - \lambda_{i=1} \operatorname{Id}_{\widehat{\pi}^*\mathcal{E}}|_{y_j\neq 0})\,, \text{ respectively}\,.$$

Both these kernel are locally free, generated by $s_1 = (x_1 - a, 1)$ and $s_2 = (y_1 - b, 1)$, respectively. These glue over $t \neq 0$ by

$$s_1 = \left(-\frac{a}{y_1}, 1\right) s_2 \;\Leftrightarrow\; s_2 = \left(-\frac{b}{x_1}, 1\right) s_1$$

to a locally free sheaf $\mathcal{F}'$ defined away from the nodes $\widehat{e}_1$, $\widehat{e}_2$. The restriction to the first coordinate U_1 can be extended by $\mathcal{I}_1$ over the node. The restriction to the second coordinate U_2 extends by the free module R_2. This defines $\mathcal{F}$.

8.2 The pullback correspondence in the equisingular case

For notational simplicity, we consider the pullback correspondence for multi-scale $\mathrm{GL}(2,\mathbb{C})^\circ$-Higgs pairs $(\mathcal{E}, \mathbf{\Phi})$ on a single stable curve X. In this case, the levelwise abelian differentials λ_i can be interpreted as sections of $\widehat{\pi}_i^* M_i$ having only simple zeros. Denote their divisors by

$\widehat{B}_i = \operatorname{div}(\lambda_i)$. On each level, we define the locally free sheaf $\mathcal{F}_i = \ker(\widehat{\pi}_i\Phi_i - \lambda_i \operatorname{Id}_{\widehat{\pi}_i^*\mathcal{E}}) \otimes \mathcal{O}(\widehat{B}_i)$. We have to glue these locally free sheaves levelwise to define a torsion-free sheaf $\mathcal{F}$ on $\widehat{X}$. This gluing will again be defined with respect to the special frames constructed in Section 7.2. It will become apparent that the construction given here agrees with the construction of the previous section restricted to the special fiber.

8.2.1 *Swapped nodes.* The argument in the *swapped node cases* carries over to equisingular families without any change. Let us give some details in the *swapped node, Neveu–Schwarz* case. To define $\mathcal{F}$, we choose a frame of $\mathcal{E}$ diagonalizing Φ. Then

$$\widehat{\pi}^*(\mathcal{E},\Phi)|_{U_1} = \left(\mathcal{I}_1 \oplus \sigma^*\widehat{R}_2, \begin{pmatrix} \lambda|_{U_1} & 0 \\ 0 & -\lambda|_{U_1} \end{pmatrix}\right).$$

The generator s_1 of $\mathcal{I}_1$ generates the eigenspace $\ker(\widehat{\pi}^*\Phi_{i=0} - \lambda_{i=0} \operatorname{Id}_{\widehat{\pi}^*\mathcal{E}}|_{y_j=0})|_{U_1}$ freely. Similarly, the generator s_2 of $\mathcal{I}_1$ freely generates the eigenspace $\ker(\widehat{\pi}^*\Phi_{i=1} - \lambda_{i=1} \operatorname{Id}_{\widehat{\pi}^*\mathcal{E}}|_{x_j=0})|_{U_1}$. These two generators glue back inside $\widehat{\pi}^*\mathcal{E}$ to $\mathcal{I}_1$. In the same way, one recovers $\widehat{R}_2$ on U_2. This defines $\mathcal{F}$ at $\widehat{e}_1$, $\widehat{e}_2$.

8.2.2 *Fixed node, $\mathcal{E}$ has a free summand.* We are in the situation of (7.1) specialized to $t = 0$. The levelwise locally free rank 1 sheaves are given by

$$\mathcal{F}_0 = \ker(\widehat{\pi}^*\Phi_0 - \lambda_0 \operatorname{Id}_{\widehat{\pi}^*\mathcal{E}}) \otimes \mathcal{O}\big(\widehat{B}_0\big), \quad \mathcal{F}_1 = \ker(\widehat{\pi}^*\Phi_1 - \lambda_1 \operatorname{Id}_{\widehat{\pi}^*E}) \otimes \mathcal{O}\big(\widehat{B}_1\big).$$

In local coordinates, the eigensheaves are generated by

$$s_0 = \frac{1}{x} \otimes x + 1 \otimes 1 \in \Gamma(X_0, \widehat{\pi}^*\mathcal{E}|_{X_0} \otimes \mathcal{O}(\widehat{e}_0)),$$
$$s_1 = \frac{1}{y} \otimes y + 1 \otimes 1 \in \Gamma(X_1, \widehat{\pi}^*\mathcal{E}|_{X_1} \otimes \mathcal{O}(\widehat{e}_1)).$$

We define a local generator for $\mathcal{F}$ by gluing these two generators.

There is a finite number of choices of coordinates x, y, u, v such that the covering has the above form. They differ by multiplication with constants in $\mathbb{C}^\times$. This does not affect the generators s_0, s_1. However, we work with respect to a special frame of $(\mathcal{E}, \mathbf{\Phi})$ here, and this choice does affect the definition of the generator of $\mathcal{F}$. The following lemma proves that the choices of special frames are in one-to-one correspondence with choices of frames of the locally free sheaf $\mathcal{F}$. In particular, $\mathcal{F}$ is well defined.

Lemma 8.1. *Fix local coordinates x, y, u, v such that the covering is given by (7.1) with $t = 0$. The choices of special frames in (LF/fix) such that the Higgs field is given by (7.2) are in one-to-one correspondence with the choices of frames $\mathcal{F} \cong \hat{R}$ under the spectral correspondence.*

Proof. It is easy to see using the Higgs field that a special frame as in (LF/fix) is uniquely determined by choosing a generator of the locally free part s_1. Let $\widehat{1}$ denote the background generator for $\mathcal{F}$. Choosing the generator $\phi \cdot \widehat{1}$ with $\phi \in \widehat{R}^\times$ results in the new generator $t_1 s_1$ with $t_1 = \phi_1 + \phi_2 x + \phi_3 y$ with ϕ_1 the even part of ϕ and $\phi_2 x$, $\phi_3 y$ the odd parts in the x- and y-coordinates, respectively. For the converse, choose a generator of the locally free part $t_1 s_1$ with $t_1 = e + fx + gy$ with respect to the background frame for $e \in R^\times$, $f \in \mathbb{C}[[u]]^\times$ and $g \in \mathbb{C}[[v]]^\times$. It is easy to compute that the new frame of $\mathcal{F}$ under the spectral correspondence will be $(\pi^\sharp(e) + \pi^\sharp(f)x + \pi^\sharp(g)y) \cdot \widehat{1}$. Every element of $\widehat{R}^\times$ can be described in this way. $\square$

8.2.3 *Fixed node and $\mathcal{E}$ has no free summand.* We are in the situation of (NS/fix) specialized to $t=0$. We think of the pullback $\widehat{\pi}^*\mathcal{E}$ as the left $\widehat{R}$-module $\widehat{R}\otimes_R \widehat{R}$. In this representation, the pullback of the Higgs field Φ_i acts by multiplying with λ_i from the right. To recover $\mathcal{I}$, we first compute the eigensheaf of the Higgs fields. As in the previous case, this will not quite define $\mathcal{I}$ due to the twisting by the pushforward at a branch point (cf. the proof of Theorem 2.1). Let

$$e_1 := x\otimes(x-a)+1\otimes x(x-a)\,,\quad e_2 := b(x\otimes 1+1\otimes x)\,,$$
$$e_3 := y\otimes(y-b)+1\otimes y(y-b)\,,\quad e_4 := a(y\otimes 1+1\otimes y)\,.$$

It is easy to check that

$$e_1,e_2\in\ker(\widehat{\pi}^*\Phi_0-\lambda_0\,\mathrm{Id}_{\widehat{\pi}^*\mathcal{E}_0})\quad\text{and}\quad e_3,e_4\in\ker(\widehat{\pi}^*\Phi_1-\lambda_1\,\mathrm{Id}_{\widehat{\pi}^*\mathcal{E}_1})\,.$$

These sections generate the restricted eigensheaves. However, this does not immediately give us generators for the restrictions of $\mathcal{I}$. More precisely, for $b\neq 0$, $a=0$, the section e_2 generates $\mathcal{I}_0(-\widehat{e}_0)$, and e_3 generates $\mathcal{I}_1(-\widehat{e}_1)$. For $a\neq 0$, $b=0$, the section e_1 generates $\mathcal{I}_0(-\widehat{e}_0)$, and e_4 generates $\mathcal{I}_1(-\widehat{e}_1)$. Finally, for $a=b=0$, the section e_1 generates $\mathcal{I}_0(-\widehat{e}_0)$, and e_3 generates $\mathcal{I}_1(-\widehat{e}_1)$. Again, we have to glue sections with simple poles at the preimages of the node to obtain global generators s_1, s_2 of $\mathcal{I}$:

(i) To obtain s_1, we glue $(1/x)e_1$ on $\widehat{X}_0$ to $-(1/y)e_4=-a\big((1/y)\otimes y+1\otimes 1\big)$ on $\widehat{X}_1$.

(ii) To obtain s_2, we glue $(1/y)e_3$ on $\widehat{X}_1$ to $-(1/x)e_2=-b\big((1/x)\otimes x+1\otimes 1\big)$ on $\widehat{X}_0$.

These are the restrictions of the generators used in Section 8.1.2 to the special fiber. By definition, the resulting generators satisfy the relations

$$ys_1=-as_2\,,\quad xs_2=-bs_1\,.$$

We define $\mathcal{F}$ by gluing $\mathcal{F}_0$ to $\mathcal{F}_1$ by the $\widehat{R}$-module $\mathcal{I}$ generated by these sections. Again, this construction depends on the choice of a special frame of $(\mathcal{E},\boldsymbol{\Phi})$. The choice of such a frame is equivalent to a choice of generators for $\mathcal{I}$.

Lemma 8.2. *The choices of special frames described by (NS/fix) such that the Higgs fields have the prescribed form are in one-to-one correspondence with the choices of generating sets $s_1,s_2\in\mathcal{I}$ such that $ys_1=-as_2$, $xs_2=-bs_1$.*

Proof. First one shows that all generators of $\mathcal{I}$ satisfying the relations are given by ϕs_1, ϕs_2 with $\phi\in\widehat{R}^\times$. We can decompose $\phi=\phi_1+\phi_2x+\phi_3y$ with $\phi_i\in R$. Then the corresponding generators of $\pi_*\mathcal{I}$ are

$$t_1'=\phi_1(x-a)+\phi_2x(x-a)-\phi_3ay\,,\quad t_3'=xt_1'\,,$$
$$t_2'=\phi_1(y-b)-\phi_2bx+\phi_3y(y-b)\,,\quad t_4'=yt_2'\,.$$

Let $t_1=x-a$, $t_2=y-b$, $t_3=x(x-a)$, $t_4=y(y-b)$ be the standard generators of the special frame in (NS/fix). It is easy to see that any generators t_1', t_2', t_3', t_4' of $\mathcal{E}$ such that the Higgs field has the desired form are uniquely determined by the choice of t_1', t_2'. Now an easy but tedious computation using the relations of $\mathcal{E}$ shows that all generating sets are of the form described above. □

8.3 The proofs of Theorems 7.12 and 7.10

Proof of Theorem 7.12. Let $f\colon\mathcal{X}\to S$ be a germ of a family of $(4g-4)$-pointed stable curves. Let $\big(\widehat{\pi}\colon\widehat{\mathcal{X}}\to\mathcal{X},\mathbf{q},\boldsymbol{\tau},\mathcal{F}\big)$ be a germ of families of $\mathrm{GL}(2,\mathbb{C})^\circ$-spectral data. Let $(\mathcal{E},\boldsymbol{\Phi})$ be the associated germ of families of multi-scale $\mathrm{GL}(2,\mathbb{C})^\circ$-Higgs pairs defined by pushforward. For each level, we recover the restrictions of $\mathcal{F}$ to $\mathcal{X}_{i\complement}$ by $\ker(\widehat{\pi}_i^*\Phi_i - \lambda_i \operatorname{Id}_{\widehat{\pi}^*\mathcal{E}}) \otimes \mathcal{O}(\widehat{\mathcal{B}}_i)$. For each fiber of f, this is the classical spectral correspondence revisited in Theorem 2.1. These eigensheaves glue over $t \neq 0$ to a locally free sheaf $\mathcal{F}'$ on $\widehat{\mathcal{X}} \setminus \widehat{N}$, where $\widehat{N}$ is the set of nodes of the singular fibers of $\widehat{\mathcal{X}} \to S$. Then by the local computations above, $\mathcal{F} \cong j_*\mathcal{F}'$. More precisely, the choice of a local frame of $\mathcal{F}$ at the nodes of $\widehat{X}$ determines special frames of $(\mathcal{E}, \mathbf{\Phi})$ at the nodes on X, which in turn determine special frames of $j_*\mathcal{F}'$ by the local computation above. It is with respect to these special frames of $\mathcal{F}$ and $j_*\mathcal{F}'$ that the isomorphism extends over the nodes of $\widehat{X}$.

For the converse, start with a germ of families of multi-scale $\mathrm{GL}(2,\mathbb{C})^\circ$-Higgs pairs $(\widehat{\Gamma}, \mathcal{E}, \mathbf{\Phi}, \boldsymbol{\tau})$ on $\mathcal{X}$. Then the pullback construction yields a torsion-free sheaf $\mathcal{F} = j_*\mathcal{F}'$ such that by construction, there is an isomorphism of $\widehat{\pi}_*(\mathcal{F}, \boldsymbol{\lambda}) \cong (\mathcal{E}, \mathbf{\Phi})$ on $\mathcal{X} \setminus N$, where N is the set of nodes of the singular fibers of $\mathcal{X} \to S$. Again, this follows from the classical spectral correspondence applied to the fibers of f. The choice of a special frame at each node $e \in \mathcal{X}$ determines a frame of $\mathcal{F}$ at the preimages of the node. This in turn determines a special frame of $\widehat{\pi}_*(\mathcal{F}, \boldsymbol{\lambda})$. By the local spectral correspondences, the isomorphism on $\mathcal{X} \setminus N$ extends over the nodes with respect to the special frames to an isomorphism $(\mathcal{E}, \mathbf{\Phi}) \cong \widehat{\pi}_*(\mathcal{F}, \boldsymbol{\lambda})$. That the spectral correspondence preserves the notions of stability will be proven separately in Proposition 8.3. ☐

Proof of Theorem 7.10. Let $(\widehat{\pi}\colon \widehat{X} \to X, \mathbf{q}, \boldsymbol{\tau}, \mathcal{F})$ be a multi-scale $\mathrm{GL}(2,\mathbb{C})^\circ$-spectral datum. Let $(\mathcal{E}, \mathbf{\Phi})$ be the associated multi-scale $\mathrm{GL}(2,\mathbb{C})^\circ$-Higgs pairs defined by pushforward. By Theorem 2.1 applied to the case of M_i-twisted Higgs pairs on X_i, we recover the restrictions $\mathcal{F}_i$ as

$$\ker(\Phi_i - \lambda_i \operatorname{Id}_{\pi_i^* E}) \otimes \mathcal{O}(\widehat{B}_i)\,.$$

The gluing of the restrictions was described locally at all nodes. This determines a torsion-free sheaf $\mathcal{F}'$ on $\widehat{X}$. We claim that $\mathcal{F} \cong \mathcal{F}'$. This can again be checked with respect to frames. Choices of frames of $\mathcal{F}$ at $\widehat{N}$ determine special frames of $(\mathcal{E}, \mathbf{\Phi})$ at N. These in turn induce frames of $\mathcal{F}'$ at $\widehat{N}$ by the local construction. With respect to such a framing, the componentwise isomorphism $\mathcal{F}_i' \cong \mathcal{F}_i$ extends over the nodes.

For the converse, consider a multi-scale $\mathrm{GL}(2,\mathbb{C})^\circ$-Higgs pair $(\mathcal{E}, \mathbf{\Phi})$ on X. The levelwise eigensheaves glue to a torsion-free sheaf $\mathcal{F}$ as described above. The pushforward $\widehat{\pi}_*(\mathcal{F}, \lambda)$ is a multi-scale $\mathrm{GL}(2,\mathbb{C})^\circ$-Higgs pair such that by the classical spectral correspondence, the restrictions to the levels are isomorphic to the restriction of $(\mathcal{E}, \mathbf{\Phi})$. Choices of special frames of $\mathcal{E}$ at the nodes correspond to choices of frames of $\mathcal{F}$ at their preimages. Hence the local considerations above show that the levelwise isomorphisms extend with respect to special frames; that is, $\widehat{\pi}_*(\mathcal{F}, \lambda) \cong (\mathcal{E}, \mathbf{\Phi})$. That the spectral correspondence preserves the notions of stability will be proven separately in Proposition 8.3. ☐

Proposition 8.3. *Let P be a polarization of multi-scale $\mathrm{GL}(2,\mathbb{C})$-Higgs pairs of degree d on X. A multi-scale $\mathrm{GL}(2,\mathbb{C})^\circ$-Higgs pair $(\mathcal{E}, \mathbf{\Phi})$ is P-(semi)stable if and only if the associated torsion-free sheaf $\mathcal{F}$ is $\widehat{P}$-(semi)stable.*

Proof. As preparation, we show how to translate the stability condition (7.7) into a stability condition on torsion-free quotients of $\mathcal{E}$. Let $\mathcal{E}'$ be a $\mathbf{\Phi}$-invariant subsheaf of $\mathcal{E}$. Then $\mathcal{E}'$ defines three (closed) subcurves $Y_0, Y_1, Y_2 \subset Y$ such that $\mathcal{E}'|_{Y_i}$ has generic rank i. On all irreducible components X_v where q_v is not a square of an abelian differential, there are no Φ_{X_v}-invariant rank 1 subsheaves of $\mathcal{E}|_{X_v}$. Hence for all $X_v \subset Y_1$, the meromorphic quadratic differential q_v is the square of an abelian differential. Consider the exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{Q} \to 0\,.$$

Then $\mathcal{E}'$ satisfies (7.7) if and only if $\mathcal{Q}$ satisfies the inequality

$$\chi(\mathcal{Q}) \geqslant \chi(\mathcal{E}) - \frac{\sum_{v=1}^{s} \operatorname{rk} \mathcal{G}_{X_v} \deg P_{X_v}}{2 \operatorname{rk} P} = \frac{\deg P_{Y_1} + 2 \deg P_{Y_0}}{2 \operatorname{rk}(P)} . \tag{8.1}$$

A priori the quotient $\mathcal{Q}$ has torsion. We did not put any saturation condition on the subsheaf $\mathcal{E}' \subset \mathcal{E}$; hence the support of the torsion submodule $\mathsf{Tor}(\mathcal{Q})$ can be any finite collection of points in X. Recall that for a coherent sheaf $\mathcal{S}$ on X and a proper subcurve $U \subset X$, we denote by $\mathcal{S}_U$ the torsion-free pullback to the subcurve $U \subset X$. By the definition of the subcurves Y_j, we have $\mathcal{Q}_{Y_0 \cup Y_1} = \mathcal{Q}/\mathsf{Tor}(\mathcal{Q})$. The multi-scale Higgs fields $\mathbf{\Phi}$ induces levelwise Higgs fields on $\mathcal{Q}$ as $\mathcal{E}'$ was $\mathbf{\Phi}$-invariant. These Higgs fields preserve the torsion submodule. Hence the kernel of $\mathcal{E} \to \mathcal{Q}_{Y_0 \cup Y_1}$ is again preserved by $\mathbf{\Phi}$. This kernel has Euler characteristic at least $\chi(\mathcal{E}')$. Therefore, it is enough to check the inequality (8.1) for torsion-free quotients $\mathcal{Q}$ with Higgs fields that descend levelwise.

So let $\mathcal{Q}$ be a torsion-free quotient of $\mathcal{E}$ such that the multi-scale Higgs field $\mathbf{\Phi}$ descends. We claim that under these conditions, there exists a proper subcurve $\widehat{W} \subset \widehat{X}$ such that $\widehat{\pi}_* \mathcal{F}_{\widehat{W}} = \mathcal{Q}$. To prove this claim, first notice that pushforward along $\widehat{\pi}$ commutes with torsion-free pullback to subcurves and hence $\mathcal{E}_{Y_0 \cup Y_1} = (\widehat{\pi}_* \mathcal{F})_{Y_0 \cup Y_1} = \widehat{\pi}_*(\mathcal{F}_{\widehat{\pi}^{-1}(Y_0 \cup Y_1)})$. If $Y_1 = \emptyset$, we have $\mathcal{Q} = \mathcal{E}_{Y_0}$, and this proves the claim. If $Y_1 \neq \emptyset$, there is an exact sequence

$$0 \to \mathcal{L} \to \mathcal{E}_{Y_0 \cup Y_1} = \widehat{\pi}_*(\mathcal{F}_{\pi^{-1}(Y_0 \cup Y_1)}) \to \mathcal{Q} \to 0 ,$$

where $\mathcal{L}$ is a torsion-free rank 1 sheaf supported on Y_1. However, by the initial remark, for all $X_v \subset Y_1$, there exist abelian differentials a_v and line bundles $\mathcal{L}_1$, $\mathcal{L}_2$ such that $q_v = -a_v^2$ and $(\mathcal{E}, \mathbf{\Phi})_{X_v} = (\mathcal{L}_1 \oplus \mathcal{L}_2, \operatorname{diag}(a_v, -a_v))$. Let $\widehat{\pi}^{-1}(X_v) = \widehat{X}_{v_1} \cup \widehat{X}_{v_2}$. By definition, $\mathcal{L}_j = \widehat{\pi}_* \mathcal{F}|_{\widehat{X}_{v_j}}$ for $j = 1, 2$. Being Φ-invariant, the kernel $\mathcal{L}$ restricted to X_v must agree with one of the $\mathcal{L}_j$. Hence $\mathcal{Q} = \widehat{\pi}_* \mathcal{F}_{\widehat{W}}$ for a proper subcurve $\widehat{W} \subset \widehat{\pi}^{-1}(Y_0 \cup Y_1) \subset \widehat{X}$ defined by choosing one of the preimages of $\pi^{-1} X_v$ for all $X_v \in Y_1$. This proves the claim in general.

As the Euler characteristic is invariant under pushforward along finite maps, the above inequality (8.1) is equivalent to

$$\chi(\mathcal{F}|_{\widehat{W}}) \geqslant \frac{\deg P_{Y_1} + 2 \deg P_{Y_0}}{2 \operatorname{rk} P} = \frac{\deg \widehat{P}_{\widehat{W}}}{\operatorname{rk} \widehat{P}} .$$

That is the stability condition (6.3) with respect to $\widehat{P}$. □

8.4 Tschirnhausen modules and multi-scale Higgs pairs

Fix a $\mathrm{GL}(2, \mathbb{C})^{\circ}$-multi-scale spectral datum $(\widehat{\pi} \colon \widehat{X} \to X, \mathbf{q}, \boldsymbol{\tau}, \mathcal{F})$. The pushforward of the structure sheaf of the admissible cover $\widehat{\pi} \colon \widehat{X} \to X$ defines an $\mathcal{O}_X$-algebra $\mathcal{A} := \widehat{\pi}_* \mathcal{O}_{\widehat{X}}$. One each irreducible component X_v of X, the covering $\widehat{\pi}$ is defined by an equation $\lambda^2 - q_v$; hence it is affine. By [Har77, Exercise II.5.17], the pushforward induces an equivalence of categories between the category of quasi-coherent sheaves on $\widehat{X}$ and the category of quasi-coherent $\mathcal{A}$-modules on X. In particular, $\mathcal{E} = \widehat{\pi}_* \mathcal{F}$ has an $\mathcal{A}$-module structure. In the following, we want to compare this $\mathcal{A}$-module structure to a multi-scale Higgs pair $(\mathcal{E}, \mathbf{\Phi})$ compatible with $\mathbf{q}$ (cf. Definition 7.4).

The $\mathcal{A}$-module structure is decoded in the action of the Tschirnhausen module on $\mathcal{E}$. The Tschirnhausen module τ is defined through

$$\mathcal{A} = \widehat{\pi}_* \mathcal{O}_{\widehat{X}} = \mathcal{O}_X \oplus \tau .$$

An $\mathcal{A}$-module structure on $\mathcal{E}$ induces a morphism of $\mathcal{O}_X$-modules

$$\varphi \colon \tau \otimes \mathcal{E} \to \mathcal{E}$$

satisfying the relations of the sheaf of $\mathcal{O}_X$-algebras $\mathcal{A}$. On each level X_i, the covering is defined by the projective class of differentials $[q_i] \in \mathbb{P}H^0(X_i, M_i^2)$ with M_i defined in (3.2). This identifies the restriction $\tau|_{X_i} = M_i^{-1}$ (see for example [Hor22, Corollary 4.11]). Choosing a representative q_i is equivalent to choosing an embedding $X_i \subset \mathsf{Tot}(M_i)$. By the classical BNR correspondence, such a choice induces a Higgs field

$$\Phi_i \colon \mathcal{E}_i \to \mathcal{E}_i \otimes M_i$$

such that $\Phi_i^2 + \mathrm{Id}_{\mathcal{E}} \otimes q_i = 0$. We will show the converse: A choice of levelwise Higgs fields compatible with the quadratic multi-scale differential $\mathbf{q}$ induces an $\mathcal{A}$-module structure on $\mathcal{E}$. A local description of τ is apparent from the construction of the pushforward in Section 7.2. Thereby τ is locally free at each node where κ is even, and isomorphic to the maximal ideal $\mathfrak{m}$ at each node where κ is odd.

Lemma 8.4. *Let X' be the partial normalization of X at all nodes e with κ_e an odd number. Let $(\widehat{\Gamma}, \mathcal{E}, \boldsymbol{\Phi}, \boldsymbol{\tau})$ be a multi-scale $\mathrm{GL}(2,\mathbb{C})^\circ$-Higgs pair with respect to $\mathbf{q}$. Define $\varphi_i \colon M_i^{-1} \otimes \mathcal{E}_i \to \mathcal{E}_i$ to be the morphism of $\mathcal{O}_{X_i}$-sheaves obtained from Φ_i by tensoring with M_i^{-1}. A morphism of $\mathcal{O}_{X'}$-modules $\varphi_{X'} \colon \tau|_{X'} \otimes \mathcal{E}|_{X'} \to \mathcal{E}|_{X'}$ such that the restriction to X_i is given by φ_i defines a unique morphism of $\mathcal{O}_X$-sheaves*

$$\varphi \colon \tau \otimes \mathcal{E} \to \mathcal{E}\,.$$

Proof. Let $\iota \colon X' \to X$ be the inclusion. By the local description of τ given above, we have $\iota_* \tau|_{X'} = \tau$. The pushforward of the restricted morphism $\tau|_{X'} \otimes \mathcal{E}|_{X'} \to \mathcal{E}|_{X'}$ defines a morphism

$$f \colon \tau \otimes \iota_*\mathcal{E}|_{X'} \to \iota_*\mathcal{E}|_{X'}\,.$$

At a node e with κ_e odd, $\mathcal{E}$ is locally given by $R \oplus \mathfrak{m}$ or $\mathfrak{m} \oplus \mathfrak{m}$ (see (LF/fix), (NS/fix)). Hence there is an exact sequence

$$0 \to \mathcal{E} \to \iota_*\mathcal{E}|_{X'} \to \mathcal{S} \to 0\,,$$

where $\mathcal{S}$ is a coherent sheaf supported at the nodes e with κ_e odd with stalks of length at most 1. As $\widehat{\pi}$ is affine, $\mathcal{A}$ is flat and so is τ. Hence we obtain a diagram

$$\begin{tikzcd}
0 \arrow[r] & \tau \otimes \mathcal{E} \arrow[r] \arrow[d, dashed] & \tau \otimes \iota_*\mathcal{E}|_{X'} \arrow[r] \arrow[d, "f"] & \tau \otimes \mathcal{S} \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \mathcal{E} \arrow[r] & \iota_*\mathcal{E}|_{X'} \arrow[r] & \mathcal{S} \arrow[r] & 0\,.
\end{tikzcd}$$

By the local description of the Higgs field (7.2), the map f descends to the quotient by zero and hence defines a map on the kernels. This is the desired morphism $\varphi \colon \tau \otimes \mathcal{E} \to \mathcal{E}$. □

Proposition 8.5. *A multi-scale $\mathrm{GL}(2,\mathbb{C})^\circ$-Higgs pair $(\widehat{\Gamma}, \mathcal{E}, \boldsymbol{\Phi}, \boldsymbol{\tau})$ with respect to $\mathbf{q}$ induces the structure of an $\mathcal{A}$-module on $\mathcal{E}$.*

Proof. Let $j \colon X' \to X$ be the partial normalization at the nodes e with odd κ_e. Recall that $\tau_{X'}$ is an invertible sheaf. The levelwise Higgs fields $\boldsymbol{\Phi}$ can be glued to a morphism of $\mathcal{O}_{X'}$-modules $\mathcal{E}|_{X'} \to \mathcal{E}|_{X'} \otimes \tau|_{X'}^{-1}$. To see this, recall that the levelwise Higgs fields are sections $\Phi_i \in H^0(X_i, \mathrm{End}(\mathcal{E}_i) \otimes M_i)$. Hence there is only one choice of gluing for each level passage. However, $\tau|_{X'}^{-1}$ is locally free at each node of X'. Let U be a neighborhood of a node e with κ_e even such that $\tau^{-1}|_U \cong \mathcal{O}_U$. Then we can define a $\tau^{-1}|_U$-action by letting $1 \in \mathcal{O}_U$ act by $\Phi_{\ell^+(e)}$ on the higher level and by $\Phi_{\ell^-(e)}$ on the lower. This defines a unique morphism $\mathcal{E}|_{X'} \to \mathcal{E}|_{X'} \otimes \tau|_{X'}^{-1}$. By Lemma 8.4, it can be extended to a morphism of $\mathcal{O}_X$-sheaves $\varphi \colon \tau \otimes \mathcal{E} \to \mathcal{E}$. We are left with showing that this morphism induces a $\widehat{\pi}_*\mathcal{O}_{\widehat{X}}$- module structure on $\mathcal{E}$.

We can realize $\mathcal{A} = \operatorname{Spec} \mathsf{Sym}(\tau)/\mathcal{I}$, where $\mathcal{I}$ is an $\mathsf{Sym}(\tau)$-ideal sheaf. We need to show that the morphism $\tau \otimes \mathcal{E} \to \mathcal{E}$ satisfies the relation of $\mathcal{I}$. Restricted to levels, this means that the Higgs fields satisfy $\Phi_i^2 + \mathrm{Id}_{\mathcal{E}} \otimes q_i = 0$, which is satisfied by construction. At a node e with κ even, the Tschirnhausen module is the module of odd local functions under $\sigma\colon \widehat{X} \to \widehat{X}$. On the other hand, $\lambda_{\ell^+(e)}$, $\lambda_{\ell^-(e)}$ are odd with respect to σ and non-vanishing in a neighborhood of the node as sections of $\widehat{\pi}^* M_{\ell^+(e)}$, $\widehat{\pi}^* M_{\ell^-(e)}$. Hence $\Phi_{\ell^+(e)}$, $\Phi_{\ell^-(e)}$, which correspond to multiplication by $\lambda_{\ell^+(e)}$, $\lambda_{\ell^-(e)}$ by definition, induce an $\mathcal{A}$-module structure on $\mathcal{E}$ in a neighborhood of the node e. At a node e with κ_e odd, the module $\mathcal{A}$ is given by $\mathcal{A} = 1 \oplus \langle x, y\rangle$ with relations $x^2 = u$, $y^2 = v$, $xy = 0$. The first two conditions are satisfied as a consequence of $\Phi_i^2 + \mathrm{Id}_{\mathcal{E}} \otimes q_i = 0$. The third condition is satisfied by extending the levelwise Higgs fields by zero to the other levels. In summary, we have shown that the morphism φ' defines an $\mathcal{A}$-module structure on $\mathcal{E}$. □

Alternative proof of Theorem 7.10. Let $\big(\widehat{\pi}\colon \widehat{X} \to X, \mathbf{q}, \boldsymbol{\tau}, \mathcal{F}\big)$ be a $\mathrm{GL}(2,\mathbb{C})^{\circ}$-spectral datum. Then $\widehat{\pi}_* \mathcal{F}$ has an $\widehat{\pi}_* \mathcal{O}_{\widehat{X}}$-module structure. Hence it corresponds to a unique quasi-coherent sheaf on $\widehat{X}$. This recovers $\mathcal{F}$.

For the converse, a multi-scale Higgs pair $(\mathcal{E}, \boldsymbol{\Phi})$ induces a $\widehat{\pi}_* \mathcal{O}_{\widehat{X}}$-module structure on $\mathcal{E}$ by Proposition 8.5. Hence it determines a unique quasi-coherent sheaf $\mathcal{F}$ such that $\mathcal{E} = \widehat{\pi}_* \mathcal{F}$. Using the special frames of $\mathcal{E}$ given in Section 7.2 at each node, we conclude that $\mathcal{F}$ is torsion-free of rank 1. □

9. Comparison to the original Hitchin fibers

In this section, for a smooth curve X_{st}, we compare the fibers of the compactified Jacobian, hence the fibers of the map $h\colon \mathrm{SD}_{X_{\mathrm{st}}} \to \mathbb{B}_{X_{\mathrm{st}}}$ over the modified Hitchin base, with the fibers of the Hitchin map Hit; that is, we compare the vertical arrows in Proposition 1.1. To do so, we work on the level of moduli spaces instead of stacks. For concreteness, in this section, we work with the weighted canonical polarization $\widehat{P} = \widehat{P}_{\mathrm{can}}$. For other polarizations, the differences of the fibers are similar. This polarization is indeed induced by the polarization of the pointed stable curves by the pointed canonical bundle. In particular, the coarse moduli spaces for the $\widehat{P}$-compactified Jacobian functor can be obtained by Simpson's construction of moduli spaces of semistable sheaves. The points of this moduli space correspond to Jordan–Hölder equivalence classes of semistable torsion-free sheaves on $\widehat{X}$.

The fibers of both maps, the Hitchin fibration and our variant using the universal Jacobian, are stratified into semi-abelian varieties, but the combinatorics and the semi-abelian varieties in the strata are quite different.

The singular fibers of the Hitchin map. In the case where q is not a global square, the geometry of the singular Hitchin fibers $\mathrm{Hit}^{-1}(X_{\mathrm{st}}, q)$ was analyzed in [Hor22] using Hecke modifications. There is a stratification

$$\mathrm{Hit}^{-1}(X_{\mathrm{st}}, q) = \bigcup_D \mathcal{S}_D\,, \tag{9.1}$$

where D runs over effective divisors on X_{st} such that $\operatorname{div}(q) - 2D$ is also effective. There are exact sequences

$$0 \to (\mathbb{C}^*)^{r_1} \times \mathbb{C}^{r_2} \to \mathcal{S}_D \to \mathrm{Pic}^d(\Sigma^n) \to 0\,, \tag{9.2}$$

where Σ^n denotes the normalization of the spectral curve and r_1, r_2, d can be read off from q and D; see the examples below. Splitting $\mathbb{C} = \mathbb{C}^* \cup \{0\}$ gives the stratification into semi-abelian varieties.

The fibers of the universal Jacobian. We next describe the fibers of the modified Hitchin map h over a multi-scale differential. It only depends on the underlying semistable curve $\widehat{X}$. By definition, it is the fiber of the universal compactified Jacobian $\overline{\mathcal{J}}^d_{g,n}$ over $\widehat{X}$. This fiber admits a stratification

$$h^{-1}\big(\widehat{X}, \mathbf{z}, \mathbf{q}\big) = \bigcup_{\substack{(\mathbf{d},N) \\ \phi^{\mathrm{can}}\text{-semistable}}} \mathrm{Pic}^{\mathbf{d}}\big(\widehat{X}_N\big) \tag{9.3}$$

by semi-abelian varieties, where N is a subset of the nodes of $\widehat{X}$ and $\widehat{X}_N$ is the partial normalization there. The semi-abelian variety of line bundles with multi-degree $\mathbf{d}$ is denoted by $\mathrm{Pic}^{\mathbf{d}}$. Here the inclusion $\mathrm{Pic}^{\mathbf{d}}\big(\widehat{X}_N\big) \to h^{-1}\big(\widehat{X}, \mathbf{z}, \mathbf{q}\big)$ is defined by pushforward along the partial normalization $\widehat{X}_N \to \widehat{X}$. The notion of semistability is designed so that the right-hand side is a finite union. This stratification is essentially in the paper of Caporaso [Cap94] or Oda–Seshadri [OS79]; see [CC19] for in-depth examples and [MMUV22] for a modern overview.

Recall from the construction of the modified Hitchin base that if the image of $\big(\widehat{X}, \mathbf{z}, \mathbf{q}\big)$ under the forgetful map b is q, then the top-level curve $\widehat{X}_0$ is the normalization of the spectral curve associated with X_{st} and the top-level differential $q_0 = q$.

Proposition 9.1. *Suppose that q is not a global square. Each of the strata of $\mathrm{Hit}^{-1}(X_{\mathrm{st}}, q)$ and of $h^{-1}\big(\widehat{X}, \mathbf{z}, \mathbf{q}\big)$ has a map to $\mathrm{Jac}^d\big(\widehat{X}_0\big)$. If q has an even-order zero, these maps do not glue to a global map to $\mathrm{Jac}^d\big(\widehat{X}_0\big)$, neither for the original Hitchin fiber, nor for $h^{-1}\big(\widehat{X}, \mathbf{z}, \mathbf{q}\big)$.*

Proof. The existence of the map on each stratum is obvious from the definition in (9.3) for the universal Jacobian and from the description of the spectral data after (9.1) for the Hitchin fiber. The non-existence of a map to $\mathrm{Jac}^d\big(\widehat{X}_0\big)$ on the universal compactified Jacobian is clear from the non-uniqueness of stable multi-degrees in (9.3). For the Hitchin fiber, the non-existence of this map is stated in [Hor22, Example 8.3]; see also [GO13, Section 5]. (With an analogous argument, one shows that the map to $\mathrm{Jac}^d\big(\widehat{X}_0\big)$ does not extend to the irreducible components of the universal compactified Jacobian.) □

We now compare the two fibers in typical loci; see Figure 1.

9.1 Quadratic differentials with one zero of order $m = 2k$: Banana curves

The special case $k = 1$ is illustrated in Figure 1, left. In general, there are $2k$ simple zeros of $\mathbf{q}$ on the bottom level X_{-1} of the pointed stable curve X, and the genera of the covering curves are $\widehat{g}_0 = g\big(\widehat{X}_0\big) = 4g - 3 - k$ and $\widehat{g}_1 = g\big(\widehat{X}_{-1}\big) = k - 1$.

Proposition 9.2. *For $\gcd(d + 2g - 2, 6g - 6) = 1$, the fibers $\mathrm{Hit}^{-1}(X_{\mathrm{st}}, q)$ and $h^{-1}\big(\widehat{X}, \mathbf{z}, \mathbf{q}\big)$ have different numbers of irreducible components for each $k \geqslant 1$. For $d = 2g - 2 \bmod 6g - 6$, they have the same number of irreducible components for all $k \geqslant 1$.*

For the generic point in the singular locus of the Hitchin base, in the case of one double zero (that is, $k = 1$), the strata of (9.3) and of (9.1) of the same dimension are isomorphic.

For $k \geqslant 2$, not even the top-dimensional strata are isomorphic.

In fact, given a singularity of the spectral curve, one defines an associated moduli space $\mathrm{Heck}(q)$ of allowable Hecke modifications, and one obtains a map from a fiber bundle of spectral data $\mathcal{S}$

$$\mathrm{Heck}(q) \to \mathcal{S} \to \mathrm{Jac}\big(\widehat{X}_0\big)$$

to the $\mathrm{Hit}^{-1}(X_{\mathrm{st}}, q)$ by applying Hecke modifications. We have $\mathrm{Heck}(q) \cong \mathbb{P}^1$ for $k = 1$, while $\mathrm{Heck}(q) \cong \mathbb{P}(1,1,2)$ is a weighted projective space for $k = 2$. The map to the Hitchin fiber is birational but no isomorphism. For example, for $k = 1$, the $\mathbb{P}^1$-bundle is twisted by gluing the zero section over $L \in \mathrm{Jac}^d\big(\widehat{X}_0\big)$ to the ∞-section over $L(p_+ - q_+)$, where p_+, q_+ are the preimages of the singularity in $\widehat{X}_0 = \Sigma^n$.

As preparation for the proof, we list the possible multi-degrees $\mathbf{d}$ appearing in (9.3); see [Cap94, Example 7.3]. If the polarization is non-degenerate (see Theorem 6.1 for when this holds for the weighted canonical polarization), the possible multi-degrees of the line bundles in $h^{-1}\big(\widehat{X}, \mathbf{z}, \mathbf{q}\big)$ are a shift of $(1,0)$, $(0,1)$, otherwise a shift of $(2,0)$, $(1,1)$, $(0,2)$. In the first case, both multi-degrees are stable and correspond to irreducible components of $h^{-1}\big(\widehat{X}, \mathbf{z}, \mathbf{q}\big)$. In the second, only the middle multi-degree is stable and corresponds to an irreducible component. The other two are strictly semistable. By the identification of numerical stability conditions with the GIT-stability conditions [Ale04] (see also [CKV15, Fact 2.8]), these multi-degrees correspond to strata in the boundary of this irreducible component. So in this case, $h^{-1}\big(\widehat{X}, \mathbf{z}, \mathbf{q}\big)$ is irreducible. Concretely, for the canonical pointed numerical polarization, the torsion-free pullbacks to the components must satisfy the conditions

$$\deg\big(\mathcal{F}_{\widehat{X}_0}\big) \geqslant \widehat{d} + \frac{k-2}{3} + \widehat{d}\frac{1-2k}{6g-6}, \quad \deg\big(\mathcal{F}_{\widehat{X}_{-1}}\big) \geqslant \widehat{d}\frac{2k-1}{6g-6} - \frac{k+4}{3}.$$

Proof. The irreducibility of the Hitchin fibers with irreducible spectral curve and at least one zero of odd order is proven in [Hor22, Corollary 8.5]. In fact, the stratum $\mathcal{S}_0$ is dense. The count above shows that the fibers in the compactified Jacobian are reducible for $\gcd\big(\widehat{d}, 6g-6\big) = 1$ and always irreducible for $\widehat{d} = 4g-4 \bmod 6g-6$ for all $k \geqslant 1$. This proves the first claim by noting that $\widehat{d} = d + 2g - 2$.

In the Hit-fibers, the rank of the abelian part is the genus of the normalization of the spectral curve, which is the same as the top level curve $\widehat{X}_0$. In the h-fibers, the rank of the abelian part is $g\big(\widehat{X}_0\big) + g\big(\widehat{X}_1\big)$. Since for $k \geqslant 2$, the bottom-level curve has positive genus, this proves the last statement. For $k = 1$, the closed strata are in both cases the Jacobians of $\widehat{X}_0$, and the open strata are $\mathbb{C}^*$-extension of this Jacobian. This follows from (9.3) and since $r_1 = 1$ and $r_2 = 0$ in (9.2) in this case by [Hor22, Theorem 6.2]. □

9.2 Quadratic differentials with one zero of order $m = 2k+1$: Compact type

The special case $k = 1$ is illustrated in Figure 1, right. In general, there are $2k+1$ simple zeros of $\mathbf{q}$ on the bottom level X_{-1} of the pointed stable curve X. The genera of the covering curves are $\widehat{g}_0 = g\big(\widehat{X}_0\big) = 4g - 3 - k$ and $\widehat{g}_1 = g\big(\widehat{X}_{-1}\big) = k$.

Proposition 9.3. *The fibers $\mathrm{Hit}^{-1}(X_{\mathrm{st}}, q)$ and $h^{-1}\big(\widehat{X}, \mathbf{z}, \mathbf{q}\big)$ are irreducible.*

Neither the fibers nor their strata are isomorphic for any $k \geqslant 1$.

In this case, the pointed canonical stability conditions are

$$\deg\big(\mathcal{F}_{\widehat{X}_0}\big) \geqslant \widehat{d} - \widehat{d}\frac{k}{3g-3} + \frac{k}{3}, \quad \deg\big(\mathcal{F}_{\widehat{X}_{-1}}\big) \geqslant \widehat{d}\frac{k}{3g-3} - \frac{k}{3} - 1.$$

Proof. The irreducibility of $\mathrm{Hit}^{-1}(X_{\mathrm{st}}, q)$ is shown in [Hor22, Corollary 7.10]; see also [GO13]. The irreducibility of $h^{-1}(\widehat{X}, \mathbf{z}, \mathbf{q})$ is discussed in [Cap94, Example 7.1]. There are cases where there are two strictly semistable multi-degrees (d_1+1, d_2) and (d_1, d_2+1). However, up to Jordan–Hölder equivalence, they can be represented by the pushforward of a locally free sheaf of multi-degree (d_1, d_2) along the normalization map of $\widehat{X}$ (or equivalently by the locally free sheaf on a semistable model with one rational bridge). Hence the irreducibility still holds.

The strata of the Hitchin fiber in (9.1) are semi-abelian varieties with abelian part $\mathrm{Jac}(\widehat{X}_0)$. The stratification (9.3) is reduced to a single stratum isomorphic to $\mathrm{Jac}(\widehat{X}_0) \times \mathrm{Jac}(\widehat{X}_1)$. Since $g(\widehat{X}_{-1}) = k > 0$, the remaining claims follow. $\square$

In fact, in this case, the fibers of $\mathrm{Hit}^{-1}(X_{\mathrm{st}}, q)$ over $\mathrm{Jac}^d(\widehat{X}_0)$ are isomorphic to $\mathrm{Heck}(q) \cong \mathbb{P}^1$ for $k = 1$, while $\mathrm{Heck}(q) \cong \mathbb{P}(1,1,2)$ is a weighted projective space for $k = 2$.

9.3 The square of an abelian differential with simple zeros

In this case, the spectral curve has two irreducible components interchanged by σ. We compare the fibers in the generic stratum where α with $\alpha^2 = q$ has only simple zeros. The level graph and that of the covering are given in Figure 2. To discuss a concrete example, we consider the case $g = 3$ given there. The double cover $\widehat{X}$ has two irreducible components on level 0 of genus 3, denoted by $\widehat{X}_{01}$, $\widehat{X}_{02}$, and four components of genus 0 on the lower level, $\widehat{X}_{11}, \ldots, \widehat{X}_{14}$. We have

$$\deg\big(\omega_{\widehat{X}}(\widehat{\mathbf{z}})|_{\widehat{X}_{01}}\big) = \deg\big(\omega_{\widehat{X}}(\widehat{\mathbf{z}})|_{\widehat{X}_{02}}\big) = 8\,,$$
$$\deg\big(\omega_{\widehat{X}}(\widehat{\mathbf{z}})|_{\widehat{X}_{1j}}\big) = 2 \quad \text{for all } i = 1, \ldots, 4\,.$$

Computing the stability conditions with respect to the $\widehat{P}$-polarization of degree $\widehat{d} = 2g-2$, we obtain: Let $Y \subset X$ be a connected subsurface. If Y contains one component on the top level and $0 \leqslant i \leqslant 4$ components on the bottom level, then $\deg(\mathcal{F}_Y) \geqslant (2-i)/3$. If Y contains the top level and $1 \leqslant i \leqslant 3$ components on the bottom level, then $\deg(\mathcal{F}_Y) \geqslant 2(i-1)/3$. If $Y = X_{1j}$, then $\deg(\mathcal{F}_Y) \geqslant -4/3$. We list all possible multi-degrees up to permutation on the first two or last four entries.

$$(2,2,0,0,0,0)\,,\ (3,1,0,0,0,0)\,,\ (2,3,-1,0,0,0,0)\,,\ (3,3,-1,-1,0,0)\,,$$
$$(3,4,-1,-1,-1,0)\,,\ (4,4,-1,-1,-1,-1)\,,\ (2,4,-1,-1,0,0)\,.$$

All but the last multi-degree are stable and correspond to irreducible components of the $\widehat{P}$-compactified Jacobian. The last multi-degree is strictly semistable. Let $Y \subset \widehat{X}$ be a proper subcurve containing one irreducible component on the top level and two irreducible components of the corresponding bottom level such that the multi-degree of the restriction to Y is $(2,-1,-1)$. Then $\deg(\mathcal{F}_Y) = 0$, which yields equality in the stability condition to Y. As above, this multi-degree corresponds to a stratum in the boundary of the compactified Jacobian instead of an irreducible component.

In this case, the classical Hitchin fiber is also reducible. Let us briefly explain how to see that. Let $(E, \Phi) \in \mathrm{Hit}^{-1}(X_{\mathrm{st}}, \alpha^2)$. Define the eigen-line bundles

$$L_1 = \ker(\Phi - i\alpha\,\mathrm{Id}_E)\,, \quad L_2 = \ker(\Phi + i\alpha\,\mathrm{Id}_E)\,.$$

The semistability of (E, Φ) is equivalent to $\deg L_i \leqslant 0$. By [GO13], there is an open subset of the Hitchin fiber where the Higgs field is non-vanishing; that is, for all $x \in X$, we have $\Phi(x) \neq 0$.

Lemma 9.4. *There is a morphism of coherent sheaves*

$$E^\vee \to L_1^\vee \oplus L_2^\vee$$

that is an isomorphism away from $Z(\alpha)$ such that the dualized Higgs field $\Phi^\vee$ is the pullback of the diagonal Higgs field $\mathsf{diag}(i\alpha, -i\alpha)$ on $L_1^\vee \oplus L_2^\vee$ along this map. Furthermore, for Φ non-vanishing, we have $L_1 \otimes L_2 = \omega_X^{-1}$.

Proof. We have an exact sequence of coherent sheaves

$$0 \to L_1 \oplus L_2 \xrightarrow{\iota_1+\iota_2} E \to \mathcal{T} \to 0\,,$$

where ι_1, ι_2 are inclusions of subbundles and $\mathcal{T}$ is a torsion sheaf supported at $Z(\alpha)$. The Higgs field Φ on E induces the diagonal Higgs field on $L_1 \oplus L_2$. The dual of the first map is the desired morphism. Generically, the line subbundles $L_1, L_2 \subset E$ agree to order 1 on $Z(\alpha)$. In this case, $\mathcal{T}$ has a stalk of length 1 at $p \in Z(\alpha)$. Since α is an abelian differential, this yields $\det(\mathcal{T}) = \omega_X$ and hence $L_1 \otimes L_2 = \omega_X^{-1}$. This genericity condition is equivalent to the Higgs field being non-vanishing (cf. [Hor22, Theorem 5.5]). $\square$

The morphism of coherent sheaves described in Lemma 9.4 is referred to as a Hecke transformation (cf. [Hor22, Section 4]). By the last formula of Lemma 9.4 and since $g(X) = 3$, the stability of (E, Φ) in the open stratum is equivalent to

$$-4 < \deg(L_1) < 0\,.$$

Parameterizing the Hecke transformation, we obtain a description of the open stratum as a $(\mathbb{C}^\times)^3$-fiber bundle over

$$\mathrm{Pic}^{-3}(X) \cup \mathrm{Pic}^{-2}(X) \cup \mathrm{Pic}^{-1}(X)\,.$$

The closures of the three connected components of the open stratum are the irreducible components of $\mathrm{Hit}^{-1}(\alpha^2)$. These correspond to multi-degrees $(1, 3, 0, 0, 0)$, $(2, 2, 0, 0, 0, 0)$, $(3, 1, 0, 0, 0, 0)$. The other irreducible components of the $\widehat{P}$-compactified Jacobian do not correspond to the original Hitchin fiber.

10. The fixed-determinant case

The goal of this section is to deduce from the multi-scale spectral correspondence that we proved in Theorem 7.12 a multi-scale version for $\mathrm{SL}(2,\mathbb{C})$-Higgs pairs or more generally multi-scale $\mathrm{GL}(2,\mathbb{C})^\circ$-Higgs pairs with fixed determinant $\mathcal{L}$. This correspondence boils down for smooth curves and quadratic differentials with simple zeros to the classical case recalled in Theorem 2.1. For this purpose, we have to define compactified Prym varieties and determinants for torsion-free sheaves that are locally free except for the special form given in Section 7.2 at the nodes.

There are difficulties to find a good notion of determinant for vector bundles on nodal curves; see [NS97, Section 8], [Sun02] for solutions in some cases. Similarly, for covers of curves with singular target, one has to be careful with the definition of the norm map; see for example [Car22] for various attempts. For this reason, we restrict to the situation of a family of quadratic multi-scale differentials $\left(\widehat{\pi}\colon \widehat{\mathcal{X}} \to \mathcal{X}, \mathbf{q}, \boldsymbol{\tau}\right)$ on a fixed Riemann surface X_{st} and over a special basis. We will develop in this section the notions for the following spectral correspondence.

Theorem 10.1. *Let S be a reduced scheme, and let $\left(\widehat{\pi}\colon \widehat{\mathcal{X}} \to \mathcal{X}, \mathbf{q}, \boldsymbol{\tau}\right)$ be a family of quadratic multi-scale differentials over S with a fixed underlying Riemann surface X_{st}. Let $\mathcal{L}$ be a line* *bundle on X_{st}. The spectral correspondence of Theorem 7.12 restricts to a correspondence between $\widehat{P}$-semistable torsion-free sheaves $\mathcal{F}$ on $\widehat{\mathcal{X}}$ satisfying the $\mathcal{L}$-twisted Prym condition and P-semistable $\mathrm{GL}(2,\mathbb{C})^{\circ}$-Higgs pairs with fixed determinant $\mathcal{L}$.*

The condition on the underlying Riemann surface can be stated by the requiring that the moduli map sends S to the modified Hitchin base $\mathbb{B}_{X_{\mathrm{st}}}$ for a fixed Riemann surface X_{st} of genus g rather than allowing X_0 to vary. We say that such families are "in $\mathbb{B}_{X_{\mathrm{st}}}$" for brevity. Concretely, the fibers of $\mathcal{X}$ are given by X_{st} augmented by rational tails.

Both the Prym condition and the determinant condition depend on a line bundle $\mathcal{L}$ on X_{st}. Since the fibers of $\mathcal{X} \to S$ are curves of compact type, we may extend $\mathcal{L}$ uniquely by the trivial bundle on the rational tails to a line bundle on X that we keep calling $\mathcal{L}$.

In order to generalize the Prym condition from (2.6) to families of stable curves $\widehat{\mathcal{X}}$, we have to allow a twist by components of the special fiber. If the base S is a point, components of the special fiber are no longer divisors, and hence we have to rephrase twisting by its effect on the normalization. The definition requires some preparation. For an admissible double cover $\widehat{\pi}\colon \widehat{X} \to X$, we denote by $\nu\colon X^{\nu} \to X$ and $\widehat{\nu}\colon \widehat{X}^{\nu} \to \widehat{X}$ the normalizations and by $\widehat{\pi}^{\nu}\colon \widehat{X}^{\nu} \to X^{\nu}$ the induced covering of smooth curves. We denote by ν^{T} (and $\widehat{\nu}^{T}$) the torsion-free pullback; that is, $\nu^{T}(\mathcal{G}) = \nu^{*}\mathcal{G}/\mathsf{Tor}(\nu^{*}\mathcal{G})$ for a coherent sheaf $\mathcal{G}$ on X. In the following, we identify the edges $E(\Gamma)$ of the dual graph Γ with the corresponding nodes of X. For each node e of X (respectively, $\widehat{e}$ of $\widehat{X}$), we label its preimages by $\nu^{-1}(e) = \{e_1, e_2\}$ (respectively, $\widehat{\nu}^{-1}(\widehat{e}) = \{\widehat{e}_1, \widehat{e}_2\}$). Since Γ is a tree, we can write the effect of twisting, a priori a sum over vertices of Γ, equivalently as a sum over its edges; see the second sum in the following definition.

Definition 10.2. Let $\left(\widehat{\pi}\colon \widehat{X} \to X, \mathbf{q}, \boldsymbol{\tau}\right)$ be in $\mathbb{B}_{X_{\mathrm{st}}}$. Let $\mathcal{F} \in \overline{\mathcal{J}}^{\widehat{d},\widehat{P}}_{\widehat{g},\widehat{n}}\left(\widehat{X}\right)$ be a torsion-free rank 1 sheaf. Let $\widehat{N}$ be the set of nodes where $\mathcal{F}$ is not locally free. Then $\mathcal{F}$ satisfies the *$\mathcal{L}$-twisted Prym condition* if for a choice (or equivalently for all choices) $i(e) \in \{1,2\}$ of the preimage of each node $\widehat{e} \in \widehat{N}$ in the normalization, there exists an $m_e \in \mathbb{Z}$ such that

$$\widehat{\nu}^{T}\left(\mathcal{F} \otimes \sigma^{*}\mathcal{F}\left(-\widehat{B}\right)\right) \cong (\widehat{\pi} \circ \widehat{\nu})^{*}\mathcal{L}\left(-\sum_{\widehat{e} \in \widehat{N}} \widehat{e}_{i(e)} + \sigma(\widehat{e}_{i(e)}) + \sum_{e \in E(\Gamma)} m_e \sum_{\widehat{e} \in \pi^{-1}(e)} \widehat{e}_1 - \widehat{e}_2\right). \tag{10.1}$$

The Prym condition defines a subfunctor of the compactified Jacobian for families $\left(\widehat{\pi}\colon \widehat{\mathcal{X}} \to \mathcal{X}, \mathbf{q}, \boldsymbol{\tau}\right) \to S$ with fixed underlying X_{st} and arbitrary base S. We define the *compactified Prym functor* as the functor that associates with a scheme $T \to S$ the set

$$\overline{\mathcal{P}}_{\mathcal{L}}(\widehat{\pi})(T) = \left\{ \mathcal{F} \in \overline{\mathcal{J}}^{\widehat{d},\widehat{P}}_{\widehat{g},\widehat{n}}\left(\widehat{\mathcal{X}} \times_S T\right) \,\middle|\, \begin{array}{l} \mathcal{F} \text{ satisfies the } \mathcal{L}\text{-twisted Prym condition} \\ \text{(10.1) for all geometric points in } T \end{array} \right\}.$$

First we justify the claim that this condition is the specialization of the Prym condition in Theorem 2.1 under degeneration of the quadratic differential, that is, that the various Prym conditions are consistent in families.

Proposition 10.3. *Let $\left(\widehat{\pi}\colon \widehat{\mathcal{X}} \to \mathcal{X}, \mathbf{q}, \boldsymbol{\tau}\right)$ be a germ of families of quadratic multi-scale differentials in $\mathbb{B}_{X_{\mathrm{st}}}$ over a discrete valuation ring (DVR) S with generically smooth fiber. Let η denote the generic point of S. A torsion-free sheaf $\mathcal{F} \in \overline{\mathcal{J}}^{\widehat{d},\widehat{P}}_{\widehat{g},\widehat{n}}\left(\widehat{\mathcal{X}}\right)(S)$ satisfies the $\mathcal{L}$-twisted Prym condition if and only if $\mathcal{F} \otimes \sigma^{*}\mathcal{F}\left(-\widehat{B}\right) \cong \widehat{\pi}^{*}\mathcal{L}$ on $\widehat{\mathcal{X}}_{\eta}$. In particular, $\widehat{d} = \deg \mathcal{L} + 2g - 2$.*

The proposition relies on the following extension result. A crucial ingredient is that the dual graph Γ of the special fiber of $\mathcal{X}$ is a tree.

LEMMA 10.4. *Let* $(\widehat{\pi}\colon \widehat{\mathcal{X}} \to \mathcal{X}, \mathbf{q}, \boldsymbol{\tau})$ *be a germ of families of quadratic multi-scale differentials in* $\mathbb{B}_{X_{\mathrm{st}}}$ *over a DVR* S *with generically smooth fiber* $\mathcal{X}_\eta$ *and special fiber* X*.*

(i) *Let* $\mathcal{L}$ *be a family of line bundles on* $\mathcal{X}_\eta$*. Then there exists an extension of* $\mathcal{L}$ *to* $\mathcal{X}$ *as a locally free rank 1 sheaf, and all possible such extensions differ by twisting with components of the special fiber* X*.*

(ii) *Let* $\mathcal{L}$ *be a family of line bundles on* $\widehat{\mathcal{X}}_\eta$*. Then there exists an extension of* $\mathcal{L}\otimes\sigma^*\mathcal{L}$ *as a locally free rank 1 sheaf, and all possible such extensions differ by twisting with components of the special fiber* $\widehat{X} = \widehat{\mathcal{X}}_s$*.*

Proof. (i) Choose a polarization ϕ on $\mathcal{X}$. Then there exists a semistable extension $\mathcal{F}$ of $\mathcal{L}$ to $\mathcal{X}$. Assume that $\mathcal{F}$ is not locally free. Let $e \in X$ be a node where this happens. Then we can obtain this torsion-free rank 1 sheaf from a locally free rank 1 sheaf $\mathcal{G}$ on a quasi-stable model of X, where the node e is replaced by a rational bridge such that $\mathcal{G}$ has degree 1 on this rational bridge. By assumption, the node e separates $X = Y_1 \cup Y_2$ into disjoint subcurves. Let $d_i = \deg(\mathcal{F}_{Y_i})$. Now it easy to see that $\mathcal{G}(-Y_1)$ has degree $d_1 + 1$ on Y_1, d_2 on Y_2 and degree 0 on the rational bridge. Hence $\mathcal{G}(-Y_1)$ defines an extension of $\mathcal{L}$ that is locally free at the node e. We also did not create any new nodes where $\mathcal{G}(-Y_1)$ is not locally free. Repeating this procedure at every other node where $\mathcal{G}$ is not locally free yields a locally free extension. On the other hand, if we have two extensions of $\mathcal{F}_1$, $\mathcal{F}_2$ of $\mathcal{L}$ as locally free sheaves, then $\mathcal{F}_1 \otimes \mathcal{F}_2^{-1}$ is a locally free sheaf that is trivial when restricted to $\mathcal{X}_\eta$. Hence it is isomorphic to $\mathcal{O}(\sum n_i X_i)$, where the sum is over all irreducible components $X_i \subset X$.

(ii) The proof works similarly although $\widehat{\Gamma}$ is not a tree, thanks to the σ-invariance of the line bundle to be extended: Take an extension $\mathcal{F}$ of $\mathcal{L}$ to $\mathcal{X}$ as a torsion-free rank 1 sheaf. Let $\widehat{e} \in \widehat{X}$ be a node such that $\mathcal{F}$ is not locally free. Choose a quasi-stable model of $\widehat{X}$ so that $\mathcal{F}$ and $\sigma^*\mathcal{F}$ are represented by locally free sheaves $\mathcal{G}$ and $\sigma^*\mathcal{G}$. If $\widehat{e}$ is fixed by the involution σ, then $\mathcal{G}\otimes\sigma^*\mathcal{G}$ has degree 2 on the rational bridge. Hence it is equivalent to a torsion-free sheaf with degree 0 on the rational bridge by twisting with the rational component. If $\widehat{e}$ is not fixed by the involution, then $\sigma^*\mathcal{F}$ is not locally free at $\sigma\widehat{e}$. In particular, $\mathcal{G}\otimes\sigma^*G$ is a locally free sheaf that has the same degree on the two rational bridges corresponding to $\widehat{e}$ and $\sigma\widehat{e}$. The pair of nodes $\widehat{e}$, $\sigma\widehat{e}$ cuts $\widehat{X}$ into two σ-invariant subcurves $\widehat{Y}_1$ and $\widehat{Y}_2$ (the preimages of the components Y_1, Y_2 as defined above using the node $\widehat{\pi}(\widehat{e}) = \widehat{\pi}(\sigma\widehat{e})$). The sheaf $\mathcal{G}\otimes\sigma^*G$ is equivalent to a sheaf with degree 0 on both rational bridges by twisting with one of the components $\widehat{Y}_1$ or $\widehat{Y}_2$. Hence $\mathcal{L}\otimes\sigma^*L$ has an extension that is locally free at $\widehat{e}$ and $\sigma\widehat{e}$. We did not change $\mathcal{F}\otimes\sigma^*\mathcal{F}$ at any other node. Hence, by induction, we obtain a locally free sheaf on $\mathcal{X}$ extending $\mathcal{L}\otimes\sigma^*\mathcal{L}$. The uniqueness up to twisting by components works as before. □

Proof of Proposition 10.3. By Lemma 10.4, we can find a locally free extension $\mathcal{G}$ of $\mathcal{F}\otimes\sigma^*\mathcal{F}$. By the proof of the lemma, the restriction of $\mathcal{G}$ to the components differs from the restriction of $\mathcal{F}\otimes\sigma^*\mathcal{F}$ by twisting with $\widehat{e}_{i(e)} + \sigma^*\widehat{e}_{i(e)}$ for certain preimages determined by $i(e) \in \{1,2\}$ for all $\widehat{e} \in \widehat{N}$. On the other hand, $\mathcal{L}$ is a locally free extension. Hence, again by Lemma 10.4, there exist $m_i \in \mathbb{Z}$ such that

$$\mathcal{G}(-\widehat{B}) = \widehat{\pi}^*\mathcal{L}\Bigg(\sum_{\widehat{X}_i\subset\widehat{X}\ \text{irred.}} m_i\widehat{X}_i\Bigg).$$

Restricted to the special fiber of $\widehat{\mathcal{X}} \to S$, twisting by the irreducible component $\widehat{X}_i$ yields a twist of the torsion-free pullback to the normalization $\widehat{X}^\nu$ by $\sum_{\widehat{e}\in\widehat{X}_i}(\widehat{e}_1 - \widehat{e}_2)$. As $\widehat{\Gamma}$ is obtained from the tree Γ by doubling some edges, this is equivalent to twisting by $\sum_{e\in\Gamma} m_e \sum_{\widehat{e}\in\pi^{-1}(e)}(\widehat{e}_1 - \widehat{e}_2)$.

All together, we obtain formula (10.1). For different choices of the preimages $\widehat{e}_{i(e)} \in \widehat{X}^\nu$ in (10.1), we can compensate by varying the m_e. □

Proposition 10.5. *Let $\big(\widehat{\pi}\colon \widehat{\mathcal{X}} \to \mathcal{X}, \mathbf{q}, \boldsymbol{\tau}\big) \to S$ be a family of quadratic multi-scale differentials in $\mathbb{B}_{X_{\mathrm{st}}}$. Let $\widehat{d} = \deg \mathcal{L} - 2g + 2$. Then $\overline{\mathcal{P}}_{\mathcal{L}}(\widehat{\pi})$ is universally closed in $\overline{\mathcal{J}}^{\widehat{d}, \widehat{P}}_{\widehat{g}, \widehat{n}}$. It is a proper Deligne–Mumford stack if $\widehat{P}$ is non-degenerate.*

Proof. For a family of quadratic differentials on X_{st} with simple zeros and thus a family of smooth curves $\widehat{X} \to S$ the result follows from the properness of the Prym variety. Otherwise, let $\mathcal{F} \in \overline{\mathcal{J}}^{\widehat{d}}_{\widehat{g},\widehat{n}}\big(\widehat{\mathcal{X}}\big)[T]$ be a family of torsion-free sheaves over a DVR T that satisfies the $\mathcal{L}$-twisted Prym condition on the generic point η_T of T. We may assume that $\mathcal{X} \times_S T$ is singular over the special point $t \in T$. Let $\widehat{N}_{\mathrm{per}} = \big\{n_i \colon T \to \widehat{\mathcal{X}} \times_S T\big\}$ be the set of persistent nodes over T. We consider the partial normalization $f \colon \widehat{\mathcal{Y}} \to \widehat{\mathcal{X}}$ at all nodes in $\widehat{N}_{\mathrm{per}}$. Then the family $\widehat{\mathcal{Y}} \to T$ has generically smooth fiber, and the Prym condition (10.1) is a condition on $f^T\mathcal{F}$ over the generic point. When the family of smooth curves $\widehat{\mathcal{Y}} \to T$ develops new nodes over the special point of $t \in T$, the fiber $\mathcal{F}_t$ still satisfies the $\mathcal{L}$-twisted Prym condition by Proposition 10.3.

We are left with considering the case where $\mathcal{F}$ is locally free at a persistent node $n_j(\eta_T)$ but Neveu–Schwarz at $n_j(t)$. Let $\widehat{\mathcal{Y}}_1$ and $\widehat{\mathcal{Y}}_2$ be the connected components of $\widehat{\mathcal{Y}}$ meeting in the node n_j. If a locally free sheaf becomes Neveu–Schwarz at $n_j(s)$, then the degree $\deg\big(f^T\mathcal{F}_{\mathcal{Y}_i}\big)$ drops by 1 for one of the components $\mathcal{Y}_i$ and stays constant on the other component. (See the alternative viewpoint using quasi-stable curves in Section 6 for an explanation.) Let $p_{n_j(s)}$ be the preimage of the node in $\mathcal{Y}_i$. Then the degree drop is compensated in (10.1) by twisting with $p_{n_j(s)} + \sigma p_{n_j(s)}$. This shows that by choosing the correct preimage of the node $n_j(s)$, we can ensure that the formula (10.1) is constant under such a degeneration. In particular, $\mathcal{F}_s$ still satisfies the $\mathcal{L}$-twisted Prym condition. □

For a multi-scale $\mathrm{GL}(2,\mathbb{C})^\circ$-Higgs pair $(\mathcal{E}, \boldsymbol{\Phi})$ on $\big(\widehat{\pi}\colon \widehat{X} \to X, \mathbf{q}, \boldsymbol{\tau}\big)$ in $\mathbb{B}_{X_0}$, we partition the set N of nodes where $\mathcal{E}$ is not locally free into subsets $N = N_{o,f} \cup N_{o,n} \cup N_{e,n} \cup N_{e,nn}$ according to ramification and the local structure as follows:

(i) Let $N_{o,f}$ be the set of nodes where κ is odd and $\mathcal{E}$ is of the form (LF/fix).
(ii) Let $N_{o,n}$ be the set of nodes where κ is odd and $\mathcal{E}$ is of the form (NS/fix).
(iii) Let $N_{e,n}$ be the set of nodes where κ is even and $\mathcal{E}$ is of the form (7.5).
(iv) Let $N_{e,nn}$ be the set of nodes where κ is even and $\mathcal{E}$ is the pushforward of torsion-free sheaf that is Neveu–Schwarz at both nodes in $\widehat{\pi}^{-1}(e)$.

Definition 10.6. Let $\big(\widehat{\pi}\colon \widehat{X} \to X, \mathbf{q}, \boldsymbol{\tau}\big)$ be a quadratic multi-scale differential in $\mathbb{B}_{X_{\mathrm{st}}}$. Let $(\mathcal{E}, \boldsymbol{\Phi})$ be a multi-scale $\mathrm{GL}(2,\mathbb{C})^\circ$-Higgs pair on X. Then $(\mathcal{E}, \boldsymbol{\Phi})$ has *fixed determinant* $\mathcal{L}$ if and only if for a choice (or equivalently all choices) $i(e) \in \{1,2\}$ of the preimage of a node $e \in N$, there exists an $m_e \in \mathbb{Z}$ such that

$$\det\big(\nu^T(\mathcal{E})\big)\Bigg(\sum_{e \in N_{o,f} \cup N_{e,n}} e_{i(e)} + \sum_{e \in N_{o,n} \cup N_{e,nn}} 2e_{i(e)}\Bigg) = \nu^*\mathcal{L}\Bigg(\sum_{e \in E} m_e(e_1 - e_2)\Bigg). \tag{10.2}$$

We call $(\mathcal{E}, \boldsymbol{\Phi})$ a *multi-scale* $\mathrm{SL}(2,\mathbb{C})$*-Higgs pair* if it has fixed determinant equal to $\mathcal{O}_{X_0}$.

Again, we want to show that this condition is a specialization of the fixed-determinant condition for smooth curves.

Proposition 10.7. *Let* $(\mathcal{E}, \mathbf{\Phi})$ *be a multi-scale* $\mathrm{GL}(2,\mathbb{C})^\circ$*-Higgs pair on a germ of families of quadratic multi-scale differentials* $\big(\widehat{\pi}\colon \widehat{\mathcal{X}} \to \mathcal{X}, \mathbf{q}, \boldsymbol{\tau}\big) \to S$ *in* $\mathbb{B}_{X_{\mathrm{st}}}$ *over a DVR* S *with generically smooth fiber. If* $(\mathcal{E}, \mathbf{\Phi})$ *has fixed determinant* $\mathcal{L}$ *on the generic fiber* $\mathcal{X}_\eta$*, then it satisfies* (10.2) *on the special fiber. In particular,* $\deg \mathcal{E} = \deg \mathcal{L}$*.*

Proof. By Lemma 10.4, we can find a locally free extension $\mathcal{G}$ of $\det(\mathcal{E})$ from the generic fiber $\mathcal{X}_\eta$ to $\mathcal{X}$. Since $\mathcal{L}$ is another locally free extension, there exist $m_i \in \mathbb{Z}$ such that

$$\mathcal{G} = \mathcal{L}\bigg(\sum_{X_i \subset X \text{ irred.}} m_i X_i\bigg).$$

By restriction to the special fiber, we obtain

$$\nu^*\mathcal{G} = \nu^*\mathcal{L}\bigg(\sum_{e \in E(\Gamma)} m_e(e_1 - e_2)\bigg).$$

To relate $\nu^*\mathcal{G}$ to $\det(\nu^*\mathcal{E})$, we compare both of them to $\bigwedge^2 \mathcal{E}$. This sheaf is not torsion-free in general. We recover $\mathcal{G}$ by first taking the reflexive hull of $\bigwedge^2 \mathcal{E}|_{\mathcal{X}\setminus E(\Gamma)}$ and then in the "second step" applying Lemma 10.4 to the resulting torsion-free rank 1 sheaf. For a node e, let X_{e_1} and X_{e_2} be the connected components of the normalization of X at e so that $e_1 \in X_{e_1}$ and $e_2 \in X_{e_2}$. For every node e where the reflexive hull is Neveu–Schwarz, the "second step" involves the choice of one of the connected components X_{e_i} by which we twist to obtain a locally free extension. For the comparison with $\det(\nu^T\mathcal{E})$, it is necessary to remember these choices.

(i) If $e \in N_{o,f}$, then locally in a neighborhood U_e of e, $\mathcal{E}$ is isomorphic to $R \oplus \langle x, y\rangle$. Then $\bigwedge^2 \mathcal{E} \cong \langle 1 \otimes x, 1 \otimes y\rangle$ is Neveu–Schwarz at e. Let $\nu^{-1}U = U_{e_1} \sqcup U_{e_2}$ be the connected components of the preimage in the normalization. Assume that we have twisted by X_{e_1} in the "second step." Using the isomorphism $\langle x, y\rangle \cong \langle u, t\rangle$ defined by multiplying by x, we see that $\mathcal{G}_{U_{e_1}} = \det(\mathcal{E}_{U_{e_1}})(e_1)$ and $\mathcal{G}_{U_{e_2}} = \det(\mathcal{E}_{U_{e_2}})$. Alternatively, in case we twisted with X_{e_2} in the "second step," we use the isomorphism $\langle x, y\rangle \cong \langle t, v\rangle$ and recover $\mathcal{G}_{U_{e_1}} = \det(\mathcal{E}_{U_{e_1}})$ and $\mathcal{G}_{U_{e_2}} = \det(\mathcal{E}_{U_{e_2}})(e_2)$.

(ii) Let $e \in N_{o,n}$, and assume that the degree of $\mathcal{E}$ has dropped at X_{e_2}. Then locally at e, the rank 2 torsion-free sheaf $\mathcal{E}$ is isomorphic to $\langle 1_u, v\rangle \oplus \langle x, y\rangle \cong \langle 1_u, v\rangle \oplus \langle u, t\rangle$ (or $\langle 1_u, v\rangle \oplus \langle t, v\rangle$). Here 1_u, 1_v denote the units in the coordinate rings $\mathcal{O}_{U_{e_1}} \cong \mathbb{C}[[u]]$ and $\mathcal{O}_{U_{e_1}} \cong \mathbb{C}[[v]]$, respectively. When choosing to twist by X_{e_1} (respectively, by X_{e_2}) in the "second step," we obtain $\mathcal{G}_{U_{e_1}} = \det(\mathcal{E}_{U_{e_1}})(e_1)$ and $\mathcal{G}_{U_{e_2}} = \det(\mathcal{E}_{U_{e_2}})(e_2)$ (respectively, we obtain $\mathcal{G}_{X_{e_1}} = \det(\mathcal{E}_{X_{e_1}})$ and $\mathcal{G}_{X_{e_2}} = \det(\mathcal{E}_{X_{e_2}})(2e_2)$). However, up to twisting by $e_2 - e_1$, there is no difference between these two choices.

(iii) For the cases where $e \in N_{e,n}$ and $e \in N_{e,nn}$, the argument works exactly as in the proof of Lemma 10.4. Here $\mathcal{E}$ can be identified with $\mathcal{F} \oplus \sigma^*\mathcal{F}$ locally at e.

Summing up, we obtain the formula in (10.2). □

Proof of Theorem 10.1. We formulated the Prym condition and the fixed-determinant condition fiberwise. Hence it is enough to relate the two notions on a single quadratic multi-scale differential $\big(\widehat{\pi}\colon \widehat{X} \to X, \mathbf{q}, \boldsymbol{\tau}\big)$ in $\mathbb{B}_{X_{\mathrm{st}}}$. For a torsion-free sheaf $\mathcal{F}$ on $\widehat{X}$, pushforward and pullback to the normalization commute; that is, $\nu^T\widehat{\pi}_*\mathcal{F} = \widehat{\pi}^\nu_*\widehat{\nu}^T\mathcal{F}$. The result essentially follows from formula (2.9) since the map $\widehat{\pi}^{\nu *}$ is still injective because when restricted to a connected component of $\widehat{X}^\nu$, this map is not unbranched. To give the details, it suffices to check the correspondence for $\mathcal{F}$ a locally free sheaf on $\widehat{X}$ because both formulas (10.1) and (10.2) continue to hold under degeneration of

$\mathcal{F}$ to a torsion-free sheaf. Notice that for $\mathcal{F}$ a locally free sheaf, $N_{o,n} = N_{e,n} = N_{e,nn} = \varnothing$. We convert the fixed-determinant condition into the Prym condition as follows:

$$\begin{aligned}
&\det\big(\nu^T(\mathcal{E})\big)\Bigg(\sum_{e\in N_{o,f}} e_{i(e)}\Bigg) = \nu^*\mathcal{L}\Bigg(\sum_{e\in E(\Gamma)} m_e(e_1-e_2)\Bigg)\\
\Leftrightarrow\ &\big(\widehat{\pi}^{\nu*}\det\big(\nu^T\widehat{\pi}_*\mathcal{F}\big)\big)\Bigg(\sum_{e\in N_{o,f}} 2\widehat{e}_{i(e)}\Bigg) = (\widehat{\pi}^{\nu*}\nu^*\mathcal{L})\Bigg(\sum_{e\in E(\Gamma)} m_e \sum_{\widehat{e}\in\pi^{-1}(e)}(\widehat{e}_1-\widehat{e}_2)\Bigg)\\
\Leftrightarrow\ &(\nu^T\mathcal{F}\otimes\sigma^*\nu^T\mathcal{F})\Bigg(-\nu^*\widehat{B}+\sum_{e\in N_{o,f}} 2\widehat{e}_{i(e)}-\widehat{e}_1-\widehat{e}_2\Bigg) = (\widehat{\nu}^*\widehat{\pi}^*\mathcal{L})\Bigg(\sum_{e\in E(\Gamma)} m_e \sum_{\widehat{e}\in\pi^{-1}(e)}(\widehat{e}_1-\widehat{e}_2)\Bigg).
\end{aligned}$$

Note that for $e \in N_{o,f}$, the preimages in the normalization $\widehat{e}_i$ are branch points of $\widehat{\pi}^{\nu}$. This explains the extra twist on the left-hand side. Finally, $2\widehat{e}_{i(e)} - \widehat{e}_1 - \widehat{e}_2$ is of the form $\pm(\widehat{e}_1 - \widehat{e}_2)$. Bringing these divisors to the right, we see that there exists an $m'_e \in \mathbb{Z}$ such that condition (10.1) is satisfied. This proves the theorem. □

Summary of notation

X_{st}, $(X,\mathbf{z})$	$X_{\mathrm{st}} \in \overline{\mathcal{M}}_g$ and $(X,\mathbf{z}) \in \overline{\mathcal{M}}_{g,n}$ base-stable (pointed) curve
$\pi\colon \Sigma \to X$	spectral cover of $(X,q) \in \mathcal{Q}_g^+(\mu)$
$\widehat{\pi}\colon \widehat{X} \to X$	canonical double cover, with marked points $\widehat{\mathbf{z}}$
$\widehat{g}$	$\widehat{g} = 4g-3$, genus of $\widehat{X}$
e, $\widehat{e}$	nodes of pointed stable curve X and $\widehat{X}$, respectively
N, $\widehat{N}$	sets of nodes of X and $\widehat{X}$, respectively
B, $\widehat{B}$	branch divisor and ramification divisor of $\widehat{\pi}$, respectively
$\sigma\colon \widehat{X} \to \widehat{X}$	involution on $\widehat{X}$ interchanging the sheets
$\nu\colon X^\nu \to X$	normalization of X
$\widehat{\nu}\colon \widehat{X}^\nu \to \widehat{X}$	normalization of $\widehat{X}$
$\omega(\mathbf{z})$, $\widehat{\omega}(\widehat{\mathbf{z}})$	twisted dualizing sheaves $\omega_X(\sum_{i=1}^n z_i)$ and $\omega_{\widehat{X}}(\sum_{i=1}^n \widehat{z}_i)$, respectively
(X,q), $(X,q,\mathbf{z})$	twisted quadratic differential in $\overline{\mathcal{Q}}_g^+(\mu)$ and in $\overline{\mathcal{Q}}_{g,n}^+(\mu)$, respectively
X_i, $\widehat{X}_i$	i-levels of X and $\widehat{X}$, respectively
$\mathcal{X}_{i^{\complement}}$	$\mathcal{X} \setminus \bigcup_{j\neq i} X_j$
$\boldsymbol{\tau}$	prong-matching
L, $\mathcal{L}$	locally free sheaf of rank 1
$\mathcal{F}$	coherent sheaf, often on $\widehat{X}$
$\mathcal{E}$	vector bundle or special rank 2 bundle, often on X
M	locally free sheaf of rank 1, twist of Higgs field
Λ	locally free sheaf of rank 1, fixed determinant

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