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Multiplicative bounds for measures of irrationality on complete intersections

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Abstract

We show that measures of irrationality on very general codimension two complete intersections and very general complete intersection surfaces are multiplicative in the degrees of the defining equations. This confirms some cases of a conjecture of Bastianelli, De Poi, Ein, Lazarsfeld, and Ullery. Our methods involve studying the numerical invariants of curves on complete intersections.

1. Introduction

In recent years, there has been growing interest in studying measures of irrationality for projective varieties. As a higher-dimensional generalization of gonality, these birational invariants quantify in various ways how far a given variety is from being rational. We will focus primarily on two of these measures, the *degree of irrationality* and the *covering gonality*, which are defined for an irreducible projective variety X as follows:

$$\operatorname{irr}(X) = \min\left\{\delta > 0 \mid \exists \text{ degree } \delta \text{ rational covering } X \dashrightarrow \mathbb{P}^{\dim X}\right\},$$
$$\operatorname{cov.gon}(X) = \min\left\{c > 0 \,\middle|\, \begin{array}{c}\text{Given a general point } p \in X,\ \exists \text{ irreducible}\\ \text{curve } C \subseteq X \text{ through } p \text{ with gonality } c\end{array}\right\}.$$

There has been a significant amount of activity and progress towards studying the case of hypersurfaces in projective space. Specifically, let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a smooth hypersurface of degree d and dimension $n \geqslant 2$. If X is very general of degree $d \geqslant 2n+1$, then Bastianelli, De Poi, Ein, Lazarsfeld, and Ullery [BDE+17] have shown that $\operatorname{irr}(X) = d-1$. In the same paper, the authors proved that if X is arbitrary and $d \geqslant n+2$, then $\operatorname{cov.gon}(X) \geqslant d-n$. Smith [Smi22] later extended these covering gonality results to positive characteristic. The central theme of [BDE+17] is that the positivity properties of the canonical bundle yield lower bounds for measures of irrationality. For very general hypersurfaces of degree $d \geqslant 2n+2$, Bastianelli, Ciliberto, Flamini, and Supino [BCFS19] computed more precisely $\operatorname{cov.gon}(X) \approx d - 2\sqrt{n}$.

A logical next step is to investigate the behavior of these invariants for complete intersection varieties in projective space. Let $X \subset \mathbb{P}^{n+e}_{\mathbb{C}}$ be a complete intersection variety of type $(d_1, \ldots, d_e)$. By projecting from points on X, as with hypersurfaces, there is always a naive upper bound of $d_1 \cdots d_e - e$. For complete intersections over $\mathbb{C}$, the same techniques in [BDE+17] yield lower

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bounds for the covering gonality which are additive in the degrees d_i. However, it has been conjectured [BDE+17, Problem 4.1] that measures of irrationality on complete intersections should be *multiplicative* in the degrees d_i. As evidence, Lazarsfeld [Laz97, Exercise 4.12] produced a multiplicative bound for the gonality of a smooth complete intersection curve $C \subset \mathbb{P}^{e+1}_{\mathbb{C}}$ of type $(d_1, \ldots, d_e)$ with $2 \leqslant d_1 \leqslant \cdots \leqslant d_e$:

$$\operatorname{gon}(C) \geqslant (d_1 - 1) d_2 \cdots d_e\,.$$

Further refinements due to Hotchkiss, Lau, and Ullery [HLU20] showed that when $4 \leqslant d_1 < d_2 \leqslant \cdots \leqslant d_e$ holds, the gonality of the curve C is realized by projection from a suitable linear subspace. In higher dimensions, Stapleton used results about Seshadri constants on hypersurfaces which were due to Ito [Ito14] to give superadditive bounds for the covering gonality of very general codimension two complete intersections (see [Sta17, Theorem 5.4]). Afterwards, Stapleton and Ullery [SU20] computed the degree of irrationality for codimension two complete intersections of type $(2, d)$ and $(3, d)$.

Our first result shows that the covering gonality of very general codimension two complete intersections is multiplicative.

Theorem A. *Let $X \subset \mathbb{P}^{n+2}_{\mathbb{C}}$ be a very general smooth complete intersection of type (a, b) and dimension $n \geqslant 2$. If $a, b \geqslant 3n$, then*

$$\operatorname{cov.gon}(X) \geqslant \frac{2}{3(n+1)^2} \cdot ab\,.$$

Since $\operatorname{irr}(X) \geqslant \operatorname{cov.gon}(X)$, we obtain the same lower bound for the degree of irrationality. Next, we focus on the case of complete intersection surfaces in arbitrary codimension and give multiplicative bounds for their covering gonality.

Theorem B. *Let $X \subset \mathbb{P}^{e+2}_{\mathbb{C}}$ be a very general smooth complete intersection surface of type $(d_1, \ldots, d_e)$ and codimension $e \geqslant 2$. There exist positive constants $A = A(e)$ and $B = B(e)$ such that if $d_i \geqslant A$ for all $1 \leqslant i \leqslant e$, then*

$$\operatorname{cov.gon}(X) \geqslant B \cdot d_1 \cdots d_e\,.$$

See the end of § 2, Proposition 3.9, and Remark 5.4 for explicit constants.

We will now outline the main ideas behind Theorem A. By an argument of Stapleton [Sta17, § 5.2], Theorem A reduces to showing that certain families of line bundles are nef.

Theorem 1.1. *Let $Y \subset \mathbb{P}^{n+2}_{\mathbb{C}}$ denote a very general hypersurface of degree a, and let $X \in |\mathcal{O}_Y(b)|$ be a smooth divisor on Y. Suppose $b \geqslant a \geqslant 3n$, and fix an integer*

$$r \leqslant \frac{2}{3(n+1)^2} ab\,.$$

Consider any set of r distinct points $p_1, \ldots, p_r \in X$, let $\mu\colon \widetilde{Y} \to Y$ be the blow-up of Y at these points with exceptional divisor E_i over p_i, and set $H = \mu^* \mathcal{O}_Y(1)$. Then the following divisor on $\widetilde{Y}$ is nef:

$$L := bH - \sum_{i=1}^{r} (n+1) E_i\,.$$

The statement in Theorem 1.1 can be thought of as a multi-point Seshadri constant on Y (compare with [BDH+09, Definition 1.9]), where the points are constrained to lie on the subvariety X. Similar statements about multi-point Seshadri constants in special configurations have only been studied in the case of $\mathbb{P}^2$; see [Pok19].

Since we are considering arbitrary configurations of distinct points on X, we cannot apply the degeneration techniques of [Bir99] which were used in [Ito14]. Instead, we argue by contradiction. The failure of the line bundle L to be nef means that there exists a curve $\widetilde{C} \subset \widetilde{Y}$ which intersects negatively against L. This translates into saying that the image curve $C := \mu(\widetilde{C})$ passes through the points p_i with large multiplicities m_i. We then relate this to the geometry of curves on very general complete intersections by writing down a simple inequality of the form

$$(\text{i}) \leqslant f(m_i) \leqslant (\text{ii}) + (\text{iii})\,.$$

Roughly speaking, the term $f(m_i)$ is a function involving the multiplicities m_i, and the three terms above correspond to (i) the denefing condition which generates $\widetilde{C}$, (ii) the total discrepancy between the arithmetic genus and the geometric genus of C, and (iii) a correction term which can be taken care of using induction. In particular, term (ii) will require lower bounds on the geometric genus of C, which follow from work of Ein [Ein88] and Voisin [Voi96, Voi98]. The end goal is to use this inequality to derive a contradiction.

Theorem B reduces to a similar statement about nef line bundles on the blow-up of a complete intersection threefold Y having special degrees (see Theorem 2.6). However, this will require more tools; in particular, we will give a more precise estimate for the arithmetic genus of C in terms of the multiplicities m_i and construct lower bounds for the degree of C, which involve certain degeneration arguments of Kollár [BCC92]. We will also need an additional degeneration argument to show that Theorem 2.6 implies Theorem B.

In § 2, we will present a key reduction step (Proposition 2.3), which first appeared in the thesis of Stapleton [Sta17, § 5.2]. In § 3, we collect several tools that will be used to control the numerical invariants of curves on complete intersections. In § 4, we prove Theorem 1.1. The proof of Theorem 2.6 takes up most of § 5. Throughout the paper, we work over $\mathbb{C}$.

2. Reduction step

In this section, we will show how Theorems A and B follow from the nefness of certain families of line bundles. First, we introduce an important definition (note the minor discrepancy in numbering compared to [Sta17, Definition 5.13]).

Definition 2.1. A line bundle L on a variety X *separates r points on an open set* if there exists a Zariski-open $U \subset X$ such that for every finite set of r points $Z := \{p_1, \ldots, p_r\} \subset U$, the restriction map $H^0(X, L) \to H^0(X, L|_Z)$ is surjective.

Our starting point is the following result, which shows that positivity properties of the canonical bundle lead to lower bounds on the covering gonality (see [BDE+17, Theorem 1.10] and [Sta17, Remark 5.14]).

Proposition 2.2. *Let X be a smooth projective variety, and suppose that there exists an integer r such that the canonical bundle K_X separates r points on an open set. Then*

$$\operatorname{cov.gon}(X) \geqslant r + 1\,.$$

We will apply this proposition to prove the following.

Proposition 2.3 (See [Sta17, § 5.2]). *Consider the varieties $X \subset Y \subset \mathbb{P}^{n+e}$, where Y is a complete intersection of dimension $n+1$ and type $(a_1, \ldots, a_{e-1})$, and $X \in |\mathcal{O}_Y(a_e)|$ is a complete intersection of dimension n and type $(a_1, \ldots, a_e)$. Suppose that there exists a positive integer r*

such that on the blow-up $\mu\colon \widetilde{Y} \to Y$ *along any distinct points* $p_1, \ldots, p_r \in X$ *with exceptional divisors* $E_1, \ldots, E_r$*, the line bundle*

$$L := \mu^* \mathcal{O}_Y(X) - \sum_{i=1}^{r} (n+1) E_i$$

is nef and big. Then $\operatorname{cov.gon}(X) \geqslant r+1$.

Proof. By Kawamata–Viehweg vanishing for big and nef line bundles,

$$\begin{aligned} H^1\left(Y, (K_Y + \mathcal{O}_Y(X)) \otimes \mathcal{I}_{\{p_1,\ldots,p_r\}}\right) &= H^1\left(\widetilde{Y}, \mu^*(K_Y + \mathcal{O}_Y(X)) - \sum_{i=1}^{r} E_i\right) \\ &= H^1\big(\widetilde{Y}, K_{\widetilde{Y}} + L\big) = 0\,. \end{aligned}$$

Here we use the fact that $K_{\widetilde{Y}} \cong \mu^* K_Y + nE$. The vanishing above gives a surjection

$$H^0(Y, K_Y + \mathcal{O}_Y(X)) \twoheadrightarrow H^0\left(Y, (K_Y + \mathcal{O}_Y(X)) \otimes \mathcal{O}_{\{p_1,\ldots,p_r\}}\right).$$

In other words, sections of the adjoint bundle $K_Y + \mathcal{O}_Y(X)$ separate any finite set of r distinct points in X. By the adjunction formula, $K_X \cong (K_Y + \mathcal{O}_Y(X))\big|_X$, and hence sections of K_X separate any finite set of r distinct points in X. Proposition 2.2 implies that $\operatorname{cov.gon}(X) \geqslant r+1$. □

Once we prove the nefness of L, it will follow that L is big by checking the numerical condition $L^{\dim \widetilde{Y}} > 0$ (see [Laz04, Theorem 2.2.16]), and we will conclude that $\operatorname{cov.gon}(X) \geqslant r+1$.

Set-up 2.4 (Codimension two complete intersections)*.* Let $n \geqslant 2$ be an integer, and let $X \subset Y \subset \mathbb{P}^{n+2}$, where $Y \in |\mathcal{O}_{\mathbb{P}^{n+2}}(a)|$ is a very general hypersurface and $X \in |\mathcal{O}_Y(b)|$ is a smooth divisor of dimension n such that $b \geqslant a \geqslant 3n$.

Granting Theorem 1.1 for now, we will first show how it implies Theorem A.

Proof of Theorem A. In the setting of Set-up 2.4, fix $n \geqslant 2$, choose $b, a \geqslant 3n$, and set $r = \lfloor 2ab/(3(n+1)^2) \rfloor$. Theorem 1.1 shows that for any tuple of r distinct points $p_1, \ldots, p_r \in X$, the divisor $L := bH - \sum_{i=1}^{r}(n+1)E_i$ on the blow-up $\mu\colon \widetilde{Y} \to Y$ is nef, where $H = \mu^*\mathcal{O}_Y(1)$. It is straightforward to check that $(L^{n+1}) > 0$ on $\widetilde{Y}$:

$$\left(bH - \sum_{i=1}^{r}(n+1)E_i\right)^{n+1} = ab^{n+1} - r \cdot (n+1)^{n+1} \geqslant ab^{n+1} - \frac{2}{3}(n+1)^{n-1}ab > 0$$

holds as long as $b \geqslant a \geqslant n+1$, so L is also big. By Proposition 2.3,

$$\operatorname{cov.gon}(X) \geqslant r+1 \geqslant \frac{2}{3(n+1)^2} \cdot ab\,.$$ □

Set-up 2.5 (Complete intersection surfaces)*.* Set $n = 2$, and let $e \geqslant 2$ be an integer. Consider e integers $a_e \geqslant a_{e-1} \geqslant \cdots \geqslant a_1$ given by

$$a_i := (e+1)! \cdot q_i \quad \text{for } i = 1, \ldots, e-1\,,$$

where the q_i are positive integers which are pairwise coprime and $q_1, \ldots, q_{e-1} > 2^{e+1}$. Let $X \subset Y \subset \mathbb{P}^{e+2}$, where Y is a smooth complete intersection threefold of type $(a_1, \ldots, a_{e-1})$ and X is a very general smooth complete intersection surface of type $(a_1, \ldots, a_{e-1}, a_e)$.

We will prove the following.

Theorem 2.6. *Recall Set-up 2.5, and fix a positive integer*

$$r \leqslant \frac{2}{3^4(3e+2)((e+1)!)^e} \cdot a_1 \cdots a_e\,.$$

Consider any set of r distinct points $p_1, \ldots, p_r \in Y$, let $\mu\colon \widetilde{Y} \to Y$ be the blow-up of Y at these points with exceptional divisor E_i over p_i, and set $H = \mu^*\mathcal{O}_Y(1)$. Then the following divisor on $\widetilde{Y}$ is nef:

$$a_e H - \sum_{i=1}^{r} 3E_i\,.$$

Granting Theorem 2.6 for now, we will use it to prove Theorem B.

Proof of Theorem B. In the setting of Set-up 2.5, fix $e \geqslant 2$, and set

$$r = \left\lfloor \frac{2}{3^4(3e+2)((e+1)!)^e} \cdot a_1 \cdots a_e \right\rfloor .$$

Theorem 2.6 shows that for any tuple of r distinct points $p_1, \ldots, p_r \in X$, the divisor $L := a_e H - \sum_{i=1}^{r} 3E_i$ on the blow-up $\mu\colon \widetilde{Y} \to Y$ is nef, where $H = \mu^*\mathcal{O}_Y(1)$. It is straightforward to check that $(L^3) > 0$ on $\widetilde{Y}$, so L is also big. By Proposition 2.3,

$$\operatorname{cov.gon}(X) \geqslant r + 1 \geqslant \frac{2}{3^4(3e+2)((e+1)!)^e} \cdot a_1 \cdots a_e\,.$$

So far, this only gives lower bounds on the covering gonality of complete intersection surfaces of special degrees. In contrast with the proof of Theorem A, here we will need an extra degeneration step. Consider a very general complete intersection surface $X' \subset \mathbb{P}^{e+2}$ of type $(d_1, \ldots, d_e)$, where $d_1 \leqslant \cdots \leqslant d_e$. According to Remark 5.4, if d_1 is sufficiently large, then we may choose $a_1, \ldots, a_e$ such that

$$(a_1, \ldots, a_{e-1}, a_e) := ((e+1)!q_1, \ldots, (e+1)!q_{e-1}, d_e)\,,$$

where the integers q_i are distinct primes and $d_i/2 < a_i \leqslant d_i$ for $1 \leqslant i \leqslant e-1$. Let us degenerate X' to a union of varieties, with one component consisting of a very general complete intersection $X \subset \mathbb{P}^{e+2}$ of type $(a_1, \ldots, a_e)$. The family can be chosen so that the total space is irreducible. More concretely, one may take a pencil given by

$$\{F_{a_e} = 0\} \cap \bigcap_{i=1}^{e-1} \{tF_{d_i} + G_{a_i} \cdot H_{d_i - a_i} = 0\} \subset \mathbb{P}^{e+2} \times \mathbb{A}^1_t\,,$$

where the F, G, and H are sufficiently general hypersurfaces whose degrees are equal to their subscripts. In this situation, X' is the general member of the family, and X is a component of the central fiber $t = 0$ cut out by the equations $\{F_{a_e} = G_{a_1} = \cdots = G_{a_{e-1}} = 0\} \subset \mathbb{P}^{e+2}$.

We claim that in this situation,

$$\operatorname{cov.gon}(X') \geqslant \operatorname{cov.gon}(X)\,;$$

this relies upon a strengthened version of [GK19, Proposition 2.2] involving families where the central fiber is possibly reducible, as follows.

Proposition 2.7. *Let $f\colon \mathcal{X} \to T$ be a flat family of projective varieties over a pointed irreducible one-dimensional base $(T, 0)$. Assume that the total space $\mathcal{X}$ is irreducible, and suppose that for* *all $0 \neq t \in T$, the fiber $\mathcal{X}_t$ is irreducible with covering gonality at most d. Then every component $\mathcal{X}' \subset \mathcal{X}_0^{\mathrm{red}}$ of the reduced special fiber has covering gonality at most d.*

Proof. Adapting the proof of [GK19, Proposition 2.2], the key point is that irreducibility of the total space of the family means that if the base of the covering family coming from the compactified Kontsevich moduli space of stable maps is irreducible (this may be assumed), then the covering family covers $\mathcal{X}$ and hence *automatically* covers every component of the central fiber [Kol96]. By dimensionality reasons, there must be a component of the central fiber of the covering family which is a divisor on $\mathcal{X}$ and dominates $\mathcal{X}'$. The rest of argument follows through. □

By Proposition 2.7, we have

$$\operatorname{cov.gon}(X') \geqslant \operatorname{cov.gon}(X) \geqslant \frac{2}{3^4(3e+2)((e+1)!)^e} \cdot \frac{d_1}{2} \cdots \frac{d_{e-1}}{2} \cdot d_e \,,$$

which simplifies to give the desired bound with a constant of

$$B(e) = \frac{2}{3^4(3e+2)((e+1)!)^e 2^{e-1}} \,.$$

This completes the proof of Theorem B. □

Remark 2.8. Note that Theorems 1.1 and 2.6 are false without the very general assumption. One may take a complete intersection Y which contains a line and then choose points which all lie on the line. In fact, this gives a glimpse of why the arguments in § 3 are crucial.

3. Numerical invariants of curves on complete intersections

In this section, we will collect some results about the geometry of curves in complete intersections, which will be used in the proofs of Theorems 1.1 and 2.6. We begin by giving lower bounds for the geometric genus of curves on generic complete intersections, which arise from calculations of Ein [Ein88] and Voisin [Voi96, Voi98] (for comparison, see [BDE+17, proof of Proposition 3.8]).

Proposition 3.1. *Let $X \subset \mathbb{P}^{n+e}$ be a very general complete intersection of dimension $n \geqslant 2$ and type $(d_1, \ldots, d_e)$. For any integral curve $C \subset X$, we have*

$$p_g(C) \geqslant 1 + \frac{1}{2}\left(\sum_{i=1}^{e} d_i - 2n - e\right) \cdot \deg_{\mathbb{P}^{n+e}}(C)\,.$$

Proof. Consider the spaces $V^{d_i} := H^0(\mathbb{P}^{n+e}, \mathcal{O}_{\mathbb{P}^{n+e}}(d_i))$ for $d_i \geqslant 2$, and let $V = \prod_i V^{d_i}$. Consider the universal complete intersection $\mathcal{X} \subseteq V \times \mathbb{P}^{n+e}$ of type $(d_1, \ldots, d_e)$ with the two projections $\mathrm{pr}_1 \colon \mathcal{X} \to V$ and $\mathrm{pr}_2 \colon \mathcal{X} \to \mathbb{P}^{n+e}$. Let $v = \dim V$, and suppose that a very general complete intersection of type $(d_1, \ldots, d_e)$ in $\mathbb{P}^{n+e}$ contains an irreducible curve of geometric genus g. By standard arguments, there is a diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{X} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ T & \xrightarrow{\rho} & V\,, \end{array}$$

where $\pi\colon \mathcal{C} \to T$ is a family of curves of geometric genus g whose general member $\mathcal{C}_t = \pi^{-1}(t)$ is smooth, ρ is étale, and $f_t\colon \mathcal{C}_t \to X_{\rho(t)}$ is birational onto its image. In this setting, Ein and Voisin show that if $t \in T$ is a general point, then

$$\Omega_{\mathcal{C}}^{v+1} \otimes \left((\mathrm{pr}_2 \circ f)^* \mathcal{O}_{\mathbb{P}^{n+e}}(2(n+e) - \sum_{i=1}^{e} d_i - e) \right)\Big|_{\mathcal{C}_t}$$

is generically generated by its global sections (cf. [Voi96, proof of Theorem 1.4]). This implies that the canonical bundle of the general curve $\mathcal{C}_t$ is of the form

$$K_{\mathcal{C}_t} \cong \left(\sum_{i=1}^{e} d_i - 2n - e \right) H_{\mathcal{C}_t} + (\text{effective})\,,$$

where $H_{\mathcal{C}_t}$ is the pull-back of the hyperplane bundle from $\mathbb{P}^{n+e}$. Comparing degrees on both sides, we arrive at the desired result. □

We will also need the following result.

Lemma 3.2. *Let $C \subset \mathbb{P}^N$ (for $N \geqslant 3$) be a reduced and irreducible curve of degree k with a finite collection of points p_i $(i = 1, \dots, \ell)$ which have multiplicity m_i. After a generic projection of $\varphi\colon C \to C' \subset \mathbb{P}^2$, the multiplicities of the image points $\varphi(p_i)$ in C' remain the same. This leads to the estimate*

$$\sum_{i=1}^{\ell} \frac{m_i(m_i-1)}{2} \leqslant p_a(C') - p_g(C') = \frac{(k-1)(k-2)}{2} - p_g(C)\,.$$

Remark 3.3. Given a smooth variety X and a curve $C \subset X$, the multiplicity of C at a point p is equal to the intersection of the strict transform $\widetilde{C}$ against the exceptional divisor E_p of the blow-up $\mu\colon \widetilde{X} \to X$ at p (see [Ful98, § 4.3, p. 79]).

In order to prove Theorem 2.6, we will need a finer analysis of the contribution of the multiplicity of a singular point to the arithmetic genus of a curve. This is captured in the following result.

Proposition 3.4. *Let V be a smooth variety of dimension n. Let $C \subset V$ be an irreducible and reduced curve with a singular point p of multiplicity $m := \mathrm{mult}_p C$. Then the discrepancy between the arithmetic genus and the geometric genus of C is bounded from below by*

$$p_a(C) - p_g(C) \geqslant \frac{(n-1)((n-1)!)^{1/(n-1)}}{n} m^{n/(n-1)} - (n-1)m\,.$$

Proof. Consider the blow-up $\widetilde{V} := \mathrm{Bl}_p V \xrightarrow{\pi} V$, with exceptional divisor E. Let $\widetilde{C} \subset \widetilde{V}$ be the strict transform of C. We may push forward the ideal sheaf sequence for $\widetilde{C} \subset \widetilde{V}$ along π to obtain

$$0 \to \mathcal{I}_C \to \mathcal{O}_V \to \pi_* \mathcal{O}_{\widetilde{C}} \to \tau := R^1 \pi_* \mathcal{I}_{\widetilde{C}} \to 0\,.$$

To compute the length of the sheaf τ, we will use the theorem on formal functions. Write $\mathcal{E}_\ell$ for the ℓth infinitesimal neighborhood of E. Then

$$\left(R^1 \pi_* \mathcal{I}_{\widetilde{C}}\right)^\wedge = \varprojlim_{\ell} H^1\left(\mathcal{I}_{\widetilde{C}} \otimes \mathcal{O}_{\mathcal{E}_\ell}\right),$$

so

$$\mathrm{length}(\tau) = \lim_{\ell \to \infty} h^1\left(\mathcal{I}_{\widetilde{C}} \otimes \mathcal{O}_{\mathcal{E}_\ell}\right).$$

The spaces on the right can be studied using the sequences

$$0 \to \mathcal{I}_{\widetilde{C}}\big|_E(\ell-1) \to \mathcal{I}_{\widetilde{C}} \otimes \mathcal{O}_{\mathcal{E}_\ell} \to \mathcal{I}_{\widetilde{C}} \otimes \mathcal{O}_{\mathcal{E}_{\ell-1}} \to 0\,.$$

Define

$$\begin{aligned} k_\ell &= \dim\ker\big(H^0\big(\mathcal{I}_{\widetilde{C}} \otimes \mathcal{O}_{\mathcal{E}_{\ell-1}}\big) \to H^1\big(\mathcal{I}_{\widetilde{C}}\big|_E(\ell-1)\big)\big)\,,\\ c_\ell &= \dim\operatorname{coker}\big(H^0\big(\mathcal{I}_{\widetilde{C}} \otimes \mathcal{O}_{\mathcal{E}_{\ell-1}}\big) \to H^1\big(\mathcal{I}_{\widetilde{C}}\big|_E(\ell-1)\big)\big)\,. \end{aligned}$$

Since all higher cohomology H^i terms are zero for $i \geqslant 2$, it follows that

$$\begin{aligned} h^0\big(\mathcal{I}_{\widetilde{C}} \otimes \mathcal{O}_{\mathcal{E}_\ell}\big) &= h^0\big(\mathcal{I}_{\widetilde{C}}\big|_E(\ell-1)\big) + k_\ell\,,\\ h^1\big(\mathcal{I}_{\widetilde{C}} \otimes \mathcal{O}_{\mathcal{E}_\ell}\big) &= c_\ell + h^1\big(\mathcal{I}_{\widetilde{C}} \otimes \mathcal{O}_{\mathcal{E}_{\ell-1}}\big)\,. \end{aligned} \tag{3.1}$$

In addition,

$$c_\ell - k_\ell = h^1\big(\mathcal{I}_{\widetilde{C}}\big|_E(\ell-1)\big) - h^0\big(\mathcal{I}_{\widetilde{C}} \otimes \mathcal{O}_{\mathcal{E}_{\ell-1}}\big)\,. \tag{3.2}$$

We make the following assertion.

Claim 3.5. *With the set-up above:*

(i) *The dimensions $h^1\big(\mathcal{I}_{\widetilde{C}} \otimes \mathcal{O}_{\mathcal{E}_{\ell-1}}\big)$ are nondecreasing in ℓ and stabilize for $\ell \gg 0$.*

(ii) *For any $\ell \geqslant 1$,*

$$h^1\big(\mathcal{I}_{\widetilde{C}} \otimes \mathcal{O}_{\mathcal{E}_\ell}\big) = -\sum_{i=0}^{\ell-2} \chi\big(\mathcal{I}_{\widetilde{C}}\big|_E(i)\big) + h^1\big(\mathcal{I}_{\widetilde{C}}\big|_E(\ell-1)\big) + k_\ell\,.$$

Granting Claim 3.5 for now, we will use it to complete the proof of Proposition 3.4. Observe that

$$\chi\big(\mathcal{I}_{\widetilde{C}}\big|_E(i)\big) = \chi(\mathcal{O}_E(i)) - \chi\big(\mathcal{O}_{\widetilde{C}\cap E}(i)\big) = h^0(\mathcal{O}_E(i)) - \operatorname{mult}_p C = \binom{i+n-1}{n-1} - m\,.$$

Negate both sides to get

$$-\chi\big(\mathcal{I}_{\widetilde{C}}\big|_E(i)\big) = m - \binom{i+n-1}{n-1}. \tag{3.3}$$

Set ℓ_0 to be the largest integer such that

$$m - \binom{(\ell_0-2)+n-1}{n-1} \geqslant 0\,.$$

We may assume $m \geqslant 1$, so that $\ell_0 \geqslant 2$. This has two consequences:

$$m - \binom{\ell_0-2+n}{n-1} < 0 \implies \frac{(\ell_0-2+n)^{n-1}}{(n-1)!} > m \implies \ell_0 > ((n-1)!\cdot m)^{1/(n-1)} - n + 2. \tag{3.4}$$

$$-\binom{\ell_0-2+n}{n} = -\frac{(\ell_0-2+n)}{n}\cdot\binom{(\ell_0-2)+n-1}{n-1} \geqslant -\frac{(\ell_0-2+n)}{n}\cdot m\,. \tag{3.5}$$

Now we can apply the formula in Claim 3.5 with $\ell = \ell_0$ to get

$$\begin{aligned} \operatorname{length}(\tau) &\geqslant -\sum_{i=0}^{\ell_0-2} \chi\big(\mathcal{I}_{\widetilde{C}}\big|_E(i)\big)\\ &= \sum_{i=0}^{\ell_0-2}\left[m - \binom{i+n-1}{n-1}\right] \quad \text{by (3.3)} \end{aligned}$$

$$
\begin{aligned}
&= m(\ell_0 - 1) - \binom{\ell_0 - 2 + n}{n} \\
&\geqslant m \cdot \left(\ell_0 - 1 - \frac{(\ell_0 - 2 + n)}{n}\right) \quad \text{by (3.5)} \\
&> m \cdot \left(\frac{n-1}{n} \cdot \left[((n-1)! \cdot m)^{1/(n-1)} - n + 2\right] - 2 + \frac{2}{n}\right) \quad \text{by (3.4)} \\
&= \frac{(n-1)((n-1)!)^{1/(n-1)}}{n} m^{n/(n-1)} - (n-1)m.
\end{aligned}
$$

As for Claim 3.5, part (i) follows from the equation $h^1\big(\mathcal{I}_{\widetilde{C}} \otimes \mathcal{O}_{\mathcal{E}_\ell}\big) = c_\ell + h^1\big(\mathcal{I}_{\widetilde{C}} \otimes \mathcal{O}_{\mathcal{E}_{\ell-1}}\big)$ and the fact that $H^1\big(\mathcal{I}_{\widetilde{C}}\big|_E(\ell - 1)\big) = 0$ for $\ell \gg 0$. For part (ii), we will argue by induction. The base case $\ell = 1$ follows from relations (3.1) and (3.2). Now assume that the equation holds for some positive integer $\ell = j - 1$. By these same relations,

$$
\begin{aligned}
h^1\big(\mathcal{I}_{\widetilde{C}} \otimes \mathcal{O}_{\mathcal{E}_j}\big) &= h^1\big(\mathcal{I}_{\widetilde{C}} \otimes \mathcal{O}_{\mathcal{E}_{j-1}}\big) + c_j \\
&= h^1\big(\mathcal{I}_{\widetilde{C}} \otimes \mathcal{O}_{\mathcal{E}_{j-1}}\big) + h^1\big(\mathcal{I}_{\widetilde{C}}\big|_E(j-1)\big) - h^0\big(\mathcal{I}_{\widetilde{C}} \otimes \mathcal{O}_{\mathcal{E}_{j-1}}\big) + k_j \\
&= h^1\big(\mathcal{I}_{\widetilde{C}} \otimes \mathcal{O}_{\mathcal{E}_{j-1}}\big) + h^1\big(\mathcal{I}_{\widetilde{C}}\big|_E(j-1)\big) - \big(h^0\big(\mathcal{I}_{\widetilde{C}}\big|_E(j-2)\big) + k_{j-1}\big) + k_j\,.
\end{aligned}
$$

We may rewrite the inductive hypothesis as

$$
h^1\big(\mathcal{I}_{\widetilde{C}} \otimes \mathcal{O}_{\mathcal{E}_{j-1}}\big) - h^0\big(\mathcal{I}_{\widetilde{C}}\big|_E(j-2)\big) = -\sum_{i=0}^{j-2} \chi\big(\mathcal{I}_{\widetilde{C}}\big|_E(i)\big) + k_{j-1}\,.
$$

Therefore,

$$
h^1\big(\mathcal{I}_{\widetilde{C}} \otimes \mathcal{O}_{\mathcal{E}_j}\big) = -\sum_{i=0}^{j-2} \chi\big(\mathcal{I}_{\widetilde{C}}\big|_E(i)\big) + h^1\big(\mathcal{I}_{\widetilde{C}}\big|_E(j-1)\big) + k_j\,,
$$

which completes the induction. □

For dimension counts, the following expression will be useful.

Lemma 3.6. *Let $Y \subset \mathbb{P}^{n+f}$ be a complete intersection of dimension $n \geqslant 2$ and type $(a_1, \dots, a_f)$. Then*

$$
h^0(Y, \mathcal{O}(\ell)) = \sum_{j_1=0}^{a_1-1} \cdots \sum_{j_f=0}^{a_f-1} h^0\big(\mathbb{P}^n, \mathcal{O}(\ell - j_1 - \cdots - j_f)\big)\,.
$$

Proof. Consider the Koszul resolution for a complete intersection $Y \subset \mathbb{P}^{n+f}$ of dimension $n \geqslant 2$ and type $(a_1, \dots, a_f)$. Using the vanishing of higher cohomology of line bundles on $\mathbb{P}^{n+f}$, we have

$$
\begin{aligned}
h^0(Y, \mathcal{O}(\ell)) = \Bigg[h^0(\mathbb{P}^{n+f}, \mathcal{O}(\ell)) &- \sum_{j=1}^{f} h^0\big(\mathbb{P}^{n+f}, \mathcal{O}(\ell - a_j)\big) \\
&+ \sum_{1 \leqslant u < v \leqslant n+1} h^0\big(\mathbb{P}^{n+f}, \mathcal{O}(\ell - a_u - a_v)\big) - \cdots \Bigg].
\end{aligned}
$$

We may rearrange the terms in pairs as follows:

$$
\begin{aligned}
&[h^0\big(\mathbb{P}^{n+f}, \mathcal{O}(\ell)\big) - h^0\big(\mathbb{P}^{n+f}, \mathcal{O}(\ell - a_1)\big)] \\
&\qquad - \sum_{j=2}^{f} \big[h^0\big(\mathbb{P}^{n+f}, \mathcal{O}(\ell - a_j)\big) - h^0\big(\mathbb{P}^{n+f}, \mathcal{O}(\ell - a_j - a_1)\big)\big] + \cdots
\end{aligned}
$$

$$= \left[\binom{\ell + n + f}{n + f} - \binom{\ell - a_1 + n + f}{n + f} \right] - \sum_{j=2}^{f} \left[\binom{\ell - a_j + n + f}{n + f} - \binom{\ell - a_j - a_1 + n + f}{n + f} \right] + \cdots ,$$

where the convention we adopt is that the binomial coefficients are zero if the upper index is smaller than the lower index. The expression in each pair of brackets can be replaced by a sum of binomial coefficients. For instance,

$$\binom{\ell + n + f}{n + f} - \binom{\ell - a_1 + n + f}{n + f} = \sum_{i=0}^{a_1 - 1} \binom{\ell - i + n + f - 1}{n + f - 1}.$$

Similarly, the next term can be rewritten as

$$\binom{\ell - a_j + n + f}{n + f} - \binom{\ell - a_j - a_1 + n + f}{n + f} = \sum_{i=0}^{a_1 - 1} \binom{\ell - i - a_j + n + f - 1}{n + f - 1},$$

and so on. Adding up the contribution of each term, it follows that

$$h^0(Y, \mathcal{O}(\ell)) = \sum_{i=0}^{a_1 - 1} h^0(W, \mathcal{O}(\ell - i)),$$

where $W \subset \mathbb{P}^{n+f-1}$ is a complete intersection of dimension n and type $(a_2, \ldots, a_f)$ (compare with the Koszul resolution for W). By repeating this process, we obtain

$$h^0(Y, \mathcal{O}(\ell)) = \sum_{j_1=0}^{a_1 - 1} \cdots \sum_{j_f=0}^{a_f - 1} h^0\big(\mathbb{P}^n, \mathcal{O}(\ell - j_1 - \cdots - j_f)\big). \qquad \square$$

Another ingredient that goes into the proof of Theorem 2.6 is an estimate of the minimal degree of curves contained in a generic complete intersection of special type (see Proposition 3.9). The following results are a straightforward generalization of ideas mostly due to Kollár [BCC92]. One may also compare with [Pau22, § 2].

Lemma 3.7 (cf. [Pau22, Lemma 8]). *Let d, k be integers. Assume that there is a smooth projective variety Y of dimension n with a line bundle L such that $L^n = d$ and $k \mid B \cdot L$ holds for every curve $B \subset Y$. If $X \subset \mathbb{P}^{n+1}$ is a very general hypersurface of degree d and $C \subset X$ is any curve, then $k \big| n! \cdot \deg C$.*

Example 3.8 (Van Geemen). Let (A, L) be a very general abelian variety of dimension $m \geqslant 3$ with a polarization of type $(1, \ldots, 1, \delta)$. Then $\mathrm{NS}(A) = \mathbb{Z}[L]$ and $L^m = m! \cdot \delta$. In $H^1(A, \mathbb{Z})$, choose a basis $\{dx_i\}$ such that

$$c_1(L) = dx_1 \wedge dx_{m+1} + \cdots + dx_{m-1} \wedge dx_{2m-1} + \delta \cdot dx_m \wedge dx_{2m}.$$

Write $\omega_i := dx_i \wedge dx_{m+i}$ for $i = 1, \ldots, m$. Then

$$\frac{c_1(L)^{\wedge(m-1)}}{(m-1)!} = \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_{m-1} + \delta \cdot \sum_{i=1}^{m-1} \omega_1 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega_m$$

represents an indivisible integral class in $H^{2m-2}(A, \mathbb{Z})$. By the hard Lefschetz theorem, the cohomology class of every curve in A is a rational multiple of $c_1(L)^{\wedge(m-1)}$ and hence an integral multiple of $c_1(L)^{\wedge(m-1)}/(m-1)!$, so $m\delta \mid C \cdot L$ for every curve $C \subset A$.

Proposition 3.9. *Fix integers $q_1, \dots, q_f > 2^{n+f-1}$ such that $(q_i, q_j) = 1$ for all $1 \leqslant i < j \leqslant f$. Let $X \subset \mathbb{P}^{n+f}$ be a very general complete intersection of dimension n and type*

$$((n+f-1)! \cdot q_1, \dots, (n+f-1)! \cdot q_f)\,.$$

If $C \subset X$ is a curve, then $q_1 \cdots q_f \mid (n+f-2)! \cdot \deg_{\mathbb{P}^{n+f}} C$.

Proof. Fix i, and let $q_i > 2^{n+f-1}$ be an integer as above. If (A, L) is a very general abelian variety of dimension $n+f-1$ with a polarization of type $(1, \dots, 1, q_i)$, then Debarre–Hulek–Spandaw [DHS94] have shown that L is very ample. Example 3.8 demonstrates that $(n+f-1) \cdot q_i \mid C \cdot L$ for every curve $C \subset A$. By applying Lemma 3.7 to the pair (A, L) and using the fact that $L^{n+f-1} = (n+f-1)! \cdot q_i$, we see that if $X_{(n+f-1)! \cdot q_i} \subset \mathbb{P}^{n+f}$ is a generic hypersurface of degree $(n+f-1)! \cdot q_i$ and $C \subset X_{(n+f-1)! \cdot q_i}$ is any curve, then

$$(n+f-1)q_i \mid (n+f-1)! \cdot \deg C \implies q_i \mid (n+f-2)! \cdot \deg C\,.$$

Now we can vary this argument for all $i = 1, \dots, f$; in other words, we will apply it to the f hypersurfaces that intersect to give X. From the hypothesis that $(q_i, q_j) = 1$ for all $1 \leqslant i < j \leqslant f$, it follows that for any curve $C \subset X$, we have $q_1 \cdots q_f \mid (n+f-2)! \cdot \deg C$. □

Proposition 3.9 suggests that there should be uniform lower bounds for the degrees of curves in very general complete intersections of sufficiently large degrees. In fact, we conjecture that the following should hold.

Conjecture 3.10. *Given a very general complete intersection variety $X \subset \mathbb{P}^{n+f}$ of dimension n and type $(d_1, \dots, d_f)$ where the degrees d_i are sufficiently large, any curve in X has degree at least $d_1 \cdots d_f$.*

If the conjecture is true, the proof of Theorem B can be streamlined by eliminating the need for Proposition 2.7 and Remark 5.4.

4. Proof of Theorem 1.1

Recalling Set-up 2.4, we will induct on the number of points r. For the base case, the statement is true for $r \leqslant 2$ as soon as $b \geqslant 2n+2$ since $\mathcal{O}_Y(1)$ is very ample. Here we use the fact that $b \geqslant 3n$ and $n \geqslant 2$. By induction, we may assume that the theorem holds for $r = s$, where

$$2 \leqslant s \leqslant \frac{2}{3(n+1)^2} ab - 1\,.$$

We want to prove that the theorem holds for $r = s+1$.

Suppose for the sake of contradiction that the theorem fails when $r = s+1$. Then there exist $p_1, \dots, p_{s+1} \in X$ such that the corresponding divisor

$$L := bH - \sum_{i=1}^{s+1} (n+1) E_i$$

on the blow-up $\widetilde{Y} \xrightarrow{\mu} Y$ is not nef. Here, $H = \mu^* \mathcal{O}_Y(1)$, and E_i is the exceptional divisor over p_i.

By definition, this means that there is an integral curve $\widetilde{C} \subset \widetilde{Y}$ such that

$$L \cdot \widetilde{C} < 0\,.$$

We claim that $\widetilde{C}$ cannot be contained in some exceptional divisor E_j; otherwise, $H \cdot \widetilde{C} = 0$, and the negativity lemma would give $E_j \cdot \widetilde{C} \leqslant 0$, which together contradict the fact that $L \cdot \widetilde{C} < 0$. By the Lefschetz hyperplane theorem and Poincaré duality, $\widetilde{C}$ is numerically a $\mathbb{Q}$-linear combination of terms involving H^n and E_i^n (note that the mixed terms involving $H \cdot E_i$ must vanish because we have blown up a collection of points). One can check that $H^{n+1} = a$ and $(-E_i)^{n+1} = -1$. Furthermore, the intersection numbers $\widetilde{C} \cdot H \geqslant 1$ and $\widetilde{C} \cdot E_i \geqslant 0$ must be integers. It follows that the numerical class of $\widetilde{C}$ is given by

$$\widetilde{C} \equiv_{\text{num}} \frac{k}{a} H^n + (-1)^n \sum_{i=1}^{s+1} m_i E_i^n \,, \tag{4.1}$$

where $k \geqslant 1$ is the degree of the image curve $C := \mu(\widetilde{C}) \subset \mathbb{P}^{n+2}$ and the $m_i \geqslant 0$ are the multiplicities of C at p_i. Note that

$$L \cdot \widetilde{C} < 0 \implies \sum_{i=1}^{s+1} m_i > \frac{1}{n+1} bk \,.$$

Fixing $\sum m_i$, the quantity $\sum m_i^2$ is minimized when all of the m_i are the same. Therefore,

$$\sum_{i=1}^{s+1} m_i > \frac{1}{n+1} bk =: \gamma \implies \sum_{i=1}^{s+1} m_i^2 > \left(\frac{\gamma}{s+1}\right)^2 \cdot (s+1) = \frac{b^2 k^2}{(n+1)^2 (s+1)} \,. \tag{4.2}$$

This step also follows from applying Jensen's inequality with equal weights using the convex function $f(x) = x^2$.

On the other hand, our induction hypothesis implies that the divisor $L_I := bH - \sum_{i \in I} (n+1) E_i$ is nef for any subset $I \subset \{1, 2, \ldots, s+1\}$ with $\# I = s$. Averaging over all I shows that

$$L_{s+1} := \frac{1}{s} \sum_I L_I = \frac{s+1}{s} bH - \sum_{i=1}^{s+1} (n+1) E_i$$

is nef. This implies that $L_{s+1} \cdot \widetilde{C} \geqslant 0$, and hence

$$\sum_{i=1}^{s+1} m_i \leqslant \frac{(s+1)}{s(n+1)} bk \,. \tag{4.3}$$

By Lemma 3.2 and Proposition 3.1 applied to $C \subset Y \subset \mathbb{P}^{n+2}$, we have

$$\begin{aligned} \sum_{i=1}^{r} \frac{m_i (m_i - 1)}{2} &\leqslant \frac{1}{2}(k-1)(k-2) - p_g(C) \\ &\leqslant \frac{1}{2}(k-1)(k-2) - \frac{1}{2}(a - 2n - 3)k - 1 \\ &= \frac{1}{2}(k^2 + (2n - a)k) \,. \end{aligned} \tag{4.4}$$

Next, we can combine this with the inequalities in (4.2) and (4.3):

$$\frac{k^2 b^2}{(n+1)^2 (s+1)} < \sum_{i=1}^{r} m_i^2 = \sum_{i=1}^{r} m_i (m_i - 1) + \sum_{i=1}^{r} m_i \leqslant k^2 + (2n - a)k + \frac{(s+1)}{s(n+1)} bk \,.$$

After simplifying, we get

$$\left[\frac{b^2}{(n+1)^2 \cdot (s+1)} - 1 \right] \cdot k < 2n - a + \frac{(s+1)}{s(n+1)} b \,. \tag{4.5}$$

Note that $b \geqslant a$, so the term on the left-hand side of (4.5) is positive. Recall from our induction set-up that

$$3 \leqslant s+1 \leqslant \frac{2}{3(n+1)^2}ab \quad \text{and} \quad n \geqslant 2 \implies \frac{(s+1)}{s(n+1)} \leqslant \frac{1}{2}\,.$$

Then (4.5) yields

$$k < \frac{(2n-a+b/2)\cdot 2a}{3b-2a} = \frac{1}{6}\left(2a + \frac{(12n-4a)\cdot 2a}{3b-2a}\right) \leqslant \frac{1}{3}a \quad \text{since } a \geqslant 3n\,. \tag{4.6}$$

Since all of the m_i are nonnegative integers (in fact, the induction hypothesis tells us that $m_i \geqslant 1$ for all i), the inequality in (4.4) also gives

$$\frac{1}{2}(k-1)(k-2) - \frac{1}{2}k(a-2n-3) - 1 \geqslant \sum_{i=1}^{r} \frac{m_i(m_i-1)}{2} \geqslant 0 \implies k \geqslant a-2n\,,$$

which contradicts (4.6) as soon as $a \geqslant 3n$. This completes the proof of Theorem 1.1.

5. Proof of Theorem 2.6

Recalling Set-up 2.5, we will induct on the number of points r. Let $\alpha := a_1 \cdots a_{e-1}$ be the degree of the threefold $Y \subset \mathbb{P}^{e+2}$. For the base case $r \leqslant 2$, the statement is trivial as soon as $d \geqslant 2$ since $\mathcal{O}_Y(1)$ is very ample on Y. By induction, we may assume that the theorem holds for $r = s$, where

$$2 \leqslant s \leqslant \frac{2}{3^4(3e+2)((e+1)!)^e} \cdot \alpha a_e - 1\,. \tag{5.1}$$

We want to prove that the theorem holds for $r = s+1$.

Suppose for the sake of contradiction that the theorem fails when $r = s+1$. Then there exist $p_1, \ldots, p_{s+1} \in X$ such that the divisor

$$L := a_e H - \sum_{i=1}^{s+1} 3E_i$$

on the blow-up $\mu\colon \widetilde{Y} \to Y$ is not nef, where $H = \mu^*\mathcal{O}_Y(1)$.

By definition, this means that there is an integral curve $\widetilde{C} \subset \widetilde{Y}$ such that

$$L \cdot \widetilde{C} < 0\,.$$

As in the proof of Theorem 1.1, note that $\widetilde{C}$ cannot be contained in some exceptional divisor E_j. By the Lefschetz hyperplane theorem and Poincaré duality, $\widetilde{C}$ is numerically a $\mathbb{Q}$-linear combination of terms involving H^2 and E_i^2 (note that the mixed terms involving $H \cdot E_i$ must vanish because we have blown up a collection of points). One can check that $H^3 = \alpha$ and $(-E_i)^3 = -1$. Furthermore, the intersection numbers $\widetilde{C} \cdot H \geqslant 1$ and $\widetilde{C} \cdot E_i \geqslant 0$ must be integers. It follows that the numerical class of $\widetilde{C}$ is given by

$$\widetilde{C} \equiv_{\text{num}} \frac{k}{\alpha} H^2 + \sum_{i=1}^{s+1} m_i E_i^2 \tag{5.2}$$

for some integers $k \geqslant 1$ and $m_i \geqslant 0$.

Remark 5.1. Note that k is the degree of the image curve $C := \mu(\widetilde{C}) \subset Y \subset \mathbb{P}^{e+2}$ and the integers m_i are the multiplicities of C at the p_i. By the induction hypothesis, we know that $m_i \geqslant 1$ for all i.

Using the description (5.2) for $\widetilde{C}$, the condition $L \cdot \widetilde{C} < 0$ reduces to

$$\sum_{i=1}^{s+1} m_i > \frac{1}{3} a_e k\,. \tag{5.3}$$

For a fixed $\sum_{i=1}^{s+1} m_i$, the quantity $\sum m_i^{3/2}$ is minimized when all of the m_i are the same, so setting $N := a_e k/3$ for the right-hand side yields

$$\sum_{i=1}^{s+1} m_i^{3/2} > \left(\frac{N}{s+1}\right)^{3/2} \cdot (s+1) = \frac{1}{3^{3/2} \cdot (s+1)^{1/2}} k^{3/2} a_e^{3/2}\,. \tag{5.4}$$

The rest of the proof will be devoted to bounding the expression $\sum_{i=1}^{s+1} m_i^{3/2}$ from above. We will first use a dimension count to approximate C as a complete intersection curve. The key point is that if we can find an effective divisor $V \in |\mathcal{O}_Y(\ell)|$ of degree $\ell \leqslant \frac{1}{3} a_e$ passing through all of the points p_i, then the denefing condition $L \cdot \widetilde{C} < 0$ implies that $\widetilde{V} \cdot \widetilde{C} < 0$, where $\widetilde{V}$ is the strict transform of V. It must then follow that $\widetilde{C} \subset \widetilde{V}$. This is the basic idea behind the following.

Lemma 5.2. *Consider Set-up 2.5. Then there are surfaces $V_j \in |\mathcal{O}_Y(b_j)|$ with $b_j \leqslant \frac{1}{3} a_e$ for $j = 1, 2$ such that their intersection $V_1 \cap V_2$ is a curve* (*not necessarily irreducible*) *which contains C as an irreducible component.*

Proof. Recall that

$$s + 1 \leqslant \frac{2}{3^4 (3e+2)((e+1)!)^e} \cdot \alpha a_e\,.$$

Choose b_1 to be the minimal degree such that there exists a hypersurface $V_1 \in |\mathcal{O}_Y(b_1)|$ passing through all of the points p_i ($1 \leqslant i \leqslant s+1$). We claim that $b_1 \leqslant \lfloor a_e/3 \rfloor$. For $\ell = \lfloor a_e/3 \rfloor$, we can apply Lemma 3.6 with $f = e-1$, $n = 3$, and $Y \subset \mathbb{P}^{e+2}$ a threefold of type $(a_1, \ldots, a_{e-1})$ to see that

$$\begin{aligned}
h^0(Y, \mathcal{O}(\ell)) &= \sum_{j_1=0}^{a_1-1} \cdots \sum_{j_{e-1}=0}^{a_{e-1}-1} h^0\big(\mathbb{P}^3, \mathcal{O}(\ell - j_1 - \cdots - j_{e-1})\big) \\
&\geqslant \sum_{j_1=0}^{\lfloor a_1/(3e) \rfloor} \cdots \sum_{j_{e-1}=0}^{\lfloor a_{e-1}/(3e) \rfloor} h^0\big(\mathbb{P}^3, \mathcal{O}(\ell - j_1 - \cdots - j_{e-1})\big) \quad \text{(by truncating the sum)} \\
&> \sum_{j_1=0}^{\lfloor a_1/(3e) \rfloor} \cdots \sum_{j_{e-1}=0}^{\lfloor a_{e-1}/(3e) \rfloor} \frac{1}{3!} \left(\ell - \left\lfloor \frac{a_1}{3e} \right\rfloor - \cdots - \left\lfloor \frac{a_{e-1}}{3e} \right\rfloor + 1\right)^3 \\
&\geqslant \frac{1}{6} \cdot \frac{a_1}{3e} \cdots \frac{a_{e-1}}{3e} \cdot \left(\left\lfloor \frac{a_e}{3} \right\rfloor - (e-1) \cdot \left\lfloor \frac{a_e}{3e} \right\rfloor + 1\right)^3 \quad (\text{setting } \ell = \lfloor a_e/3 \rfloor) \\
&\geqslant \frac{1}{6 \cdot (3e)^{e+2}} \alpha a_e^3 \ > \ s+1 \quad (\text{for } e \geqslant 2).
\end{aligned}$$

Hence, there exists a hypersurface $V_1 \in |\mathcal{O}_Y(\lfloor a_e/3 \rfloor)|$ of degree $b_1 \leqslant \lfloor a_e/3 \rfloor$ which passes through all of the p_i. We claim that $C \subset V_1$. To see this, observe that the class of the strict transform $\widetilde{V_1} \subset \widetilde{Y}$ is given by

$$\widetilde{V_1} \equiv_{\text{lin}} b_1 H - \sum_{i=1}^{s+1} c_i E_i$$

for some integers $c_i \geqslant 1$ with $1 \leqslant i \leqslant s+1$. By comparing with (5.3), we see that the condition $b_1 \leqslant a_e/3$ implies that $\widetilde{V_1} \cdot \widetilde{C} < 0$ and hence $\widetilde{C} \subset \widetilde{V_1}$. Therefore, $C \subset V_1$. Since C is irreducible and passes through all of the p_i by Remark 5.1, we may assume that V_1 is irreducible.

By a similar argument, choose b_2 to be the smallest positive integer such that there exists a section $V_2 \in |\mathcal{O}_{V_1}(b_2)|$ containing p_i ($1 \leqslant i \leqslant s+1$). Since V_1 is a complete intersection, a similar dimension count using Lemma 3.6 for $V_1 \subset \mathbb{P}^{e+2}$ leads to

$$\begin{aligned} h^0(V_1, \mathcal{O}(\ell)) &= \sum_{j_0=0}^{b_1-1} \sum_{j_1=0}^{a_1-1} \cdots \sum_{j_{e-1}=0}^{a_{e-1}-1} h^0\big(\mathbb{P}^2, \mathcal{O}(\ell - j_0 - j_1 - \cdots - j_{e-1})\big) \\ &\geqslant \sum_{j_1=0}^{\lfloor a_1/(3e) \rfloor} \cdots \sum_{j_{e-1}=0}^{\lfloor a_{e-1}/(3e) \rfloor} h^0\big(\mathbb{P}^2, \mathcal{O}(\ell - j_1 - \cdots - j_{e-1})\big) \quad \begin{array}{l}\text{(taking the first term}\\ j_0 = 0 \text{ and truncating)}\end{array} \\ &> \sum_{j_1=0}^{\lfloor a_1/(3e) \rfloor} \cdots \sum_{j_{e-1}=0}^{\lfloor a_{e-1}/(3e) \rfloor} \frac{1}{2}\left(\ell - \left\lfloor \frac{a_1}{3e} \right\rfloor - \cdots - \left\lfloor \frac{a_{e-1}}{3e} \right\rfloor + 1\right)^2 \\ &\geqslant \frac{1}{2} \cdot \frac{a_1}{3e} \cdots \frac{a_{e-1}}{3e} \cdot \left(\left\lfloor \frac{a_e}{3} \right\rfloor - (e-1) \cdot \left\lfloor \frac{a_e}{3e} \right\rfloor + 1\right)^2 \quad (\text{since } \ell = \lfloor a_e/3 \rfloor) \\ &\geqslant \frac{1}{2 \cdot (3e)^{e+1}} \alpha a_e^2 \ > \ s+1\,. \end{aligned}$$

Thus, there exists a hypersurface $V_2 \in |\mathcal{O}_{V_1}(b_2)|$ of degree $b_2 \leqslant \lfloor a_e/3 \rfloor$ passing through all of the p_i. The restriction map $H^0(Y, \mathcal{O}(b_2)) \to H^0(V_1, \mathcal{O}(b_2))$ is surjective (see [Bea96, Lemma VIII.9]), so without loss of generality, we may take V_2 to be a surface in $|\mathcal{O}_Y(b_2)|$ such that the intersection $V_1 \cap V_2$ is a curve (not necessarily irreducible). By the same reasoning as above, the condition $b_2 \leqslant a_e/3$ implies that V_2 contains C. This proves that $C \subset V_1 \cap V_2$. $\square$

We can now give a bound for the arithmetic genus of $C \subset Y$.

Lemma 5.3. *Consider Set-up 2.5. Then the arithmetic genus of C is bounded from above by*

$$p_a(C) \leqslant (a_1 + \cdots + a_{e-1} + 2a_e/3) \cdot \deg C\,.$$

Proof. The approach used here follows from ideas of Castelnuovo [Cas93] about bounding the genus of space curves as well as subsequent work of Harris [Har80]. In summary, let α_ℓ be the dimension of the image of the restriction map

$$H^0(Y, \mathcal{O}(\ell)) \xrightarrow{\rho_\ell} H^0(C, \mathcal{O}(\ell))\,.$$

We would like to bound the differences $\alpha_\ell - \alpha_{\ell-1}$ from below in order to get a lower bound for $h^0(C, \mathcal{O}(\ell))$. For $\ell \gg 0$, we can then apply the Riemann–Roch theorem to obtain a bound on $p_a(C)$.

Let $d = \deg C$, let H be a generic hyperplane section in $\mathbb{P}^{e+2}$, and consider the intersections

$$\Gamma := H \cap C\,, \quad Y_2 := Y \cap H\,.$$

There is another restriction map

$$H^0(Y, \mathcal{I}_\Gamma(\ell)) \xrightarrow{\sigma_\ell} H^0(C, \mathcal{I}_\Gamma(\ell)\big|_C) \cong H^0(C, \mathcal{O}(\ell-1))\,.$$

Since $H^0(Y, \mathcal{O}(\ell-1))$ injects into $H^0(Y, \mathcal{I}_\Gamma(\ell))$ via the multiplication map by the defining equation of H and the kernels of ρ_ℓ and σ_ℓ are isomorphic, that is, equal to $H^0(Y, \mathcal{I}_C(\ell))$, we see that

$$\beta_\ell := h^0(Y, \mathcal{O}(\ell)) - h^0(Y, \mathcal{I}_\Gamma(\ell)) \leqslant \alpha_\ell - \alpha_{\ell-1}\,.$$

The natural restriction maps $H^0(Y,\mathcal{O}(\ell)) \to H^0(Y_2,\mathcal{O}(\ell))$ and $H^0(Y,\mathcal{I}_\Gamma(\ell)) \to H^0(Y_2,\mathcal{I}_\Gamma(\ell))$ are both surjective, so

$$\beta_\ell = h^0(Y_2,\mathcal{O}(\ell)) - h^0(Y_2,\mathcal{I}_\Gamma(\ell)) \,.$$

In order to understand the β_ℓ, set $\gamma_0 := \beta_0$, and define the differences

$$\gamma_\ell := \beta_\ell - \beta_{\ell-1} \,.$$

Furthermore, take a generic hyperplane section $Y_1 := H' \cap Y_2$ (with our convention $\dim Y_j = j$). Assuming that H' is disjoint from Γ, we can write down a long sequence on cohomology:

$$0 \to H^0(Y_2,\mathcal{I}_\Gamma(\ell-1)) \to H^0(Y_2,\mathcal{I}_\Gamma(\ell)) \xrightarrow{(*)} H^0(Y_1,\mathcal{O}(\ell)) \to H^1(Y_2,\mathcal{I}_\Gamma(\ell-1)) \,. \tag{5.5}$$

If we let e_ℓ be the dimension of the image of $(*)$, then

$$e_\ell = h^0(Y_2,\mathcal{I}_\Gamma(\ell)) - h^0(Y_2,\mathcal{I}_\Gamma(\ell-1)) = h^0(Y_2,\mathcal{O}(\ell)) - \beta_\ell - h^0(Y_2,\mathcal{O}(\ell-1)) + \beta_{\ell-1} \,,$$

which can be written as

$$\gamma_\ell = h^0(Y_1,\mathcal{O}(\ell)) - e_\ell \,. \tag{5.6}$$

By Serre vanishing, $\gamma_\ell = 0$ for $\ell \gg 0$. For our purposes, we will need an effective bound for when the numbers γ_ℓ become zero. This is where Lemma 5.2 will be utilized. Recall that $C \subset V_1 \cap V_2$, where the $V_j \in |\mathcal{O}_Y(b_j)|$ are surfaces in Y and $V_1 \cap V_2$ is a (possibly reducible) curve. So

$$\Gamma \subset H \cap V_1 \cap V_2 \,,$$

where the right-hand side is also finite. From the Koszul resolution for $H \cap V_1 \cap V_2 \subset Y_2$, Serre duality, and [Bea96, Lemma VIII.9], it follows that

$$H^1(Y_2,\mathcal{I}_{H\cap V_1\cap V_2}(\ell)) = 0 \quad \text{for } \ell > a_1 + \cdots + a_{e-1} - e - 2 + b_1 + b_2 \,.$$

The exact sequence $0 \to \mathcal{I}_{H\cap V_1\cap V_2} \to \mathcal{I}_\Gamma \to \mathcal{O}_{H\cap V_1\cap V_2\setminus\Gamma} \to 0$ on Y_2 implies that

$$H^1(Y_2,\mathcal{I}_\Gamma(\ell)) = 0 \quad \text{if } \ell > a_1 + \cdots + a_{e-1} - e - 2 + b_1 + b_2 \,.$$

Since $b_1, b_2 \leqslant a_e/3$, by (5.5) and (5.6), we see that $\gamma_\ell = 0$ for

$$\ell > a_1 + \cdots + a_{e-1} + 2a_e/3 - e - 1 \,.$$

Note that $\beta_i = \sum_{v=0}^{i} \gamma_v$, so

$$\sum_{i=0}^{\ell} \beta_i = \sum_{i=0}^{\ell}\sum_{v=0}^{i} \gamma_v = \sum_{i=0}^{\ell} (\ell - i + 1)\gamma_i \,.$$

From the ideal sheaf sequence for $\Gamma \subset Y$ and the fact that $H^1(Y,\mathcal{I}_\Gamma(\ell)) = 0$ for $\ell \gg 0$, it follows that $\sum_{i=0}^{\ell} \gamma_i = \beta_\ell = d$ for $\ell \gg 0$. We may substitute this into the expression above to get

$$\sum_{i=0}^{\ell} \beta_i = \ell d - \sum_{i=0}^{\ell} (i-1)\gamma_i \,, \quad \ell \gg 0 \,.$$

On the other hand, for $\ell \gg 0$, we have $\alpha_\ell \geqslant \sum_{i=0}^{\ell} \beta_i$. From the definition of α_ℓ and the Riemann–Roch theorem, it follows that for $\ell \gg 0$,

$$p_a(C) = d\ell - \alpha_\ell + 1 \leqslant \sum_{i=0}^{\ell} (i-1)\gamma_i + 1 \,.$$

So we would like to produce a function $\gamma_i^{\max}$ which maximizes the sum above, subject to the constraint $\sum_{i=1}^{\ell}\gamma_i = d$ for $\ell \gg 0$.

Using the fact that $\gamma_\ell = 0$ for $\ell > a_1 + \cdots + a_{e-1} + 2a_e/3 - e - 1$, we can write down a simple example of a function $\gamma_i^{\max}$ that produces a rough upper bound for the arithmetic genus:

$$\gamma_\ell^{\max} = \begin{cases} d & \text{if } \ell = a_1 + \cdots + a_{e-1} + \lfloor 2a_e/3 \rfloor - e - 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

We then obtain

$$\begin{aligned} p_a(C) &\leqslant 1 + \sum_{i=0}^{\ell}(i-1)\gamma_i^{\max} \\ &= 1 + (a_1 + \cdots + a_{e-1} + \lfloor 2a_e/3 \rfloor - e - 2)d \\ &\leqslant (a_1 + \cdots + a_{e-1} + 2a_e/3)\deg C\,. \end{aligned}$$ □

Now we will use the induction hypothesis, which says that the divisor

$$L_I := a_e H - \sum_{i\in I} 3E_i$$

is nef for any subset $I \subset \{1, 2, \ldots, s+1\}$ with $\#I = s$. Averaging over all I shows that

$$L_{s+1} := \frac{1}{s}\sum_I L_I = \frac{s+1}{s}a_e H - \sum_{i=1}^{s+1} 3E_i$$

is nef. This implies that $L_{s+1}\cdot \widetilde{C} \geqslant 0$, and hence

$$\sum_{i=1}^{s+1} m_i \leqslant \frac{s+1}{3s}a_e k\,. \tag{5.7}$$

We can now give an upper bound for

$$\frac{2^{3/2}}{3}\sum_{i=1}^{s+1} m_i^{3/2}$$

by integrating the various estimates that have been established so far. Applying Proposition 3.4 to $p_i \in C \subset Y$ $(1 \leqslant i \leqslant s+1)$ with $n = 3$ gives

$$p_a(C) - p_g(C) \geqslant \sum_{i=1}^{s+1}\left(\frac{2^{3/2}}{3}\cdot m_i^{3/2} - 2m_i\right).$$

This may be rewritten as

$$\sum_{i=1}^{s+1}\frac{2^{3/2}}{3}\cdot m_i^{3/2} \leqslant p_a(C) - p_g(C) + 2\sum_{i=1}^{s+1} m_i \leqslant p_a(C) + a_e k \leqslant \left(a_1 + \cdots + a_{e-1} + \frac{5}{3}a_e\right)k\,,$$

where the second step follows from $s \geqslant 2$ and (5.7) and the third step relies on Remark 5.1 and Lemma 5.3. Next, combine this with the lower bound coming from (5.4):

$$\frac{2^{3/2}}{3^{5/2}\cdot(s+1)^{1/2}}k^{3/2}a_e^{3/2} < \sum_{i=1}^{s+1}\frac{2^{3/2}}{3}\cdot m_i^{3/2} \leqslant \left(a_1 + \cdots + a_{e-1} + \frac{5}{3}a_e\right)k\,.$$

We can square both sides, solve for k, and simplify using the inequalities $a_1 \leqslant \cdots \leqslant a_e$:

$$k < \frac{3^5}{2^3} \cdot \frac{(a_1 + \cdots + a_{e-1} + 5a_e/3)^2}{a_e^3}(s+1) \leqslant \frac{3^5}{2^3} \cdot \frac{2^2[(e+2/3)a_e]^2}{a_e^3}(s+1) = \frac{3^3(3e+2)^2}{2a_e}(s+1)$$
$$\leqslant \frac{(3e+2)}{3 \cdot ((e+1)!)^e} \cdot \alpha < \frac{1}{e!((e+1)!)^{e-1}}\alpha\,,$$

where the last line follows from (5.1).

On the other hand, recall from Set-up 2.5 that the integers a_i are of the form $a_i = (e+1)! \cdot q_i$ for some integers q_i which are pairwise coprime ($i = 1, \ldots, e-1$). By applying Proposition 3.9 to $C \subset Y$ with $n = 3$ and $f = e-1$, it follows that

$$k \geqslant \frac{1}{e!} \cdot q_1 \cdots q_{e-1} = \frac{1}{e!((e+1)!)^{e-1}}\alpha\,,$$

which gives a contradiction. This completes the proof of Theorem 2.6.

Remark 5.4. Lastly, we would like to explain how to choose the numbers a_i with the desired properties in the proof of Theorem B. One method is to invoke a theorem of Sondow [Son09], which can be viewed as an extension of Bertrand's postulate. Let $\pi(x)$ denote the number of primes not exceeding x, and recall that for $n \geqslant 1$, the nth *Ramanujan prime* is the smallest positive integer R_n such that if $x \geqslant R_n$, then $\pi(x) - \pi(\frac{1}{2}x) \geqslant n$.

By [Son09, Theorem 2], the nth Ramanujan prime satisfies the inequalities

$$2n \log 2n < R_n < 4n \log 4n\,, \quad n \geqslant 1\,.$$

For our purposes, the upper bound will suffice. Set $n = e-1$, and let $d_1 \leqslant \cdots \leqslant d_{e-1}$ be integers such that

$$2^{e+2} < \frac{d_1}{(e+1)!}\,. \tag{5.8}$$

For $e \geqslant 2$, this yields $d_i \geqslant d_1 > (e+1)! \cdot 2^{e+2} \geqslant (e+1)! \cdot 4(e-1)\log 4(e-1)$ for all $i = 1, \ldots, e-1$, so Sondow's theorem tells us that there are always $e-1$ primes between $d_i/(2 \cdot (e+1)!)$ and $d_i/(e+1)!$ for $i = 1, \ldots, e-1$. Working in reverse order $i = e-1, e-2, \ldots, 1$, it follows that we can always choose *distinct* primes q_i such that

$$\frac{1}{2} \cdot \frac{d_i}{(e+1)!} < q_i \leqslant \frac{d_i}{(e+1)!}\,.$$

Note that the left inequality together with (5.8) imply that $q_i > 2^{e+1}$, which is what we need for Set-up 2.5 and Proposition 3.9. Finally, we may define $a_i := (e+1)! \cdot q_i$.

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Note added in proof (July 2024). Conjecture 3.10 has now been proved in [CCZ24].

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