



Global generation of test ideals in mixed characteristic and applications

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ABSTRACT

Suppose that X is an integral scheme (quasi-)projective over a complete local ring of mixed characteristic. Using ideas of Takamatsu–Yoshikawa and Bhatt et al., we define a notion of a $+$ -test ideal on X , including for divisors and linear series. We obtain global generation results in this setting that generalize the well-known global generation results obtained via multiplier ideal sheaf techniques in characteristic zero and via test ideals in characteristic $p > 0$. We also obtain applications to the order of vanishing of linear series and to the diminished base locus in mixed characteristic similar to results of Ein–Lazarsfeld–Mustață–Nakamaye–Popa, Nakayama, and Mustață in the equal-characteristic case.

1. Introduction

One of the most useful properties of multiplier ideals $\mathcal{J}(X, \Delta)$ on a projective variety in characteristic zero is Nadel vanishing (a variant of Kawamata–Viehweg vanishing). It states that $H^i(X, \mathcal{O}_X(K_X + M) \otimes \mathcal{J}(X, \Delta)) = 0$ for $i > 0$ assuming that $K_X + M$ is a Cartier divisor such that $M - \Delta$ is ample [Nad89, Laz04]. This in turn implies effective global generation results. For instance, on a nonsingular projective variety X of dimension n , we have that

$$\mathcal{J}(X, \Delta) \otimes \mathcal{O}_X(K_X + M + mL)$$

is globally generated for $m \geq n$ when L is an ample divisor such that $\mathcal{O}_X(L)$ is globally generated; see [Laz04, Proposition 9.4.26].

Building on global generation results for $\mathcal{O}_X(K_X + mL)$ for mildly singular varieties in positive characteristic [Smi97, Kee08, Har01], similar results were also obtained for test ideals $\tau(X, \Delta)$ in characteristic $p > 0$, where instead of using resolution of singularities and Nadel/Kawamata–Viehweg vanishing, one uses Frobenius and Serre/Fujita vanishing. Indeed, we have that if M and L are as above, then

$$\tau(X, \Delta) \otimes \mathcal{O}_X(K_X + M + mL)$$

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not only is globally generated but is globally generated by the subspace of Frobenius stable sections

$$S^0(X, \Delta, \mathcal{O}_X(K_X + M + mL)) \subseteq H^0(X, \mathcal{O}_X(K_X + M + mL));$$

see [Sch14, Mus13]. This global generation result has found several important applications, including towards understanding the nonnef locus (or diminished base locus) in positive characteristic in [Mus13].

In mixed characteristic, that is, for schemes defined over $\mathbb{Z}_p$, we have theories of test/multiplier ideals [MS18] for complete local Noetherian rings. However, their properties are still not well understood; for example, we still do not know whether they behave well under localization. For projective schemes over a mixed-characteristic complete local Noetherian ring, we also have the space of $+$ -stable (in other words, trace invariant) sections

$$\mathbf{B}^0(X, \Delta, \mathcal{O}_X(K_X + M + mL)) \subseteq H^0(X, \mathcal{O}_X(K_X + M + mL)),$$

roughly analogous to S^0 above and which was successfully applied to the study of the 3-fold minimal model program in mixed characteristic [TY21, BMP⁺23]. The usefulness of these special sections $\mathbf{B}^0$ is derived from Bhatt's vanishing theorem [Bha20, BMP⁺23], which can be viewed as a variant of Kawamata–Viehweg vanishing, which holds on X^+ (the absolute integral closure of X). In this paper, we use these global sections $\mathbf{B}^0$ to define a mixed-characteristic test ideal.

Suppose that $X \rightarrow \operatorname{Spec} R$ is a normal integral scheme which we assume is projective over a complete local Noetherian ring $(R, \mathfrak{m})$. For simplicity of statements in the introduction, we assume that K_X is Cartier (this holds, for instance, if X is Gorenstein). We *define* the ideal $\tau_+(\mathcal{O}_X, \Delta) \subseteq \mathcal{O}_X$ so that $\mathbf{B}^0(X, \Delta, \mathcal{O}_X(K_X + M))$ globally generates

$$\tau_+(\mathcal{O}_X, \Delta) \otimes \mathcal{O}_X(K_X + M)$$

for any sufficiently ample divisor M .

We then show that this definition is independent of the choice of the sufficiently ample divisor M in Proposition 4.5. When $K_X + \Delta$ is $\mathbb{Q}$ -Cartier, we prove that it agrees with the test ideal if X is of equal characteristic $p > 0$ in Theorem 4.25. We show that it transforms under finite maps with the same formula as the test ideal and multiplier ideal in Theorem 4.21. We also introduce an adjoint variant $\operatorname{adj}_+^D(\mathcal{O}_X, \Delta)$ and show that it restricts to $\tau_+(\mathcal{O}_{D^N}, \operatorname{Diff}_{D^N}(D + \Delta))$ on the normalization D^N of D , producing an adjunction/inversion of adjunction result in Theorem 4.23.

One of the obstructions to the theory of mixed-characteristic test ideals as introduced in [MS21] was the fact that the test ideals did not behave well under localization. However, we are able to prove that the sheaf $\tau_+(\mathcal{O}_X, \Delta)$ we study is defined affine locally on X , at least when $K_X + \Delta$ is $\mathbb{Q}$ -Cartier. That is, we show the following.

THEOREM A (see Section 5). *Suppose that R is a complete local Noetherian domain and $T = R[x_1, \dots, x_m]/J$ is a normal finite type domain over R . Suppose that $U = \operatorname{Spec} T$ and that Δ_U is a divisor on U so that $K_U + \Delta_U$ is $\mathbb{Q}$ -Cartier. Finally, suppose that X_1, X_2 are two integral normal projective compactifications of U over $\operatorname{Spec} R$ with Δ_1, Δ_2 on X_1, X_2 such that $K_{X_i} + \Delta_i$ is $\mathbb{Q}$ -Cartier and $\Delta_i|_U = \Delta_U$. Then we have that*

$$\tau_+(\mathcal{O}_X, \Delta_1)|_U = \tau_+(\mathcal{O}_X, \Delta_2)|_U.$$

It follows that we can define $\tau_+(U, \Delta_U)$ to be this restriction. Furthermore, the formation of $\tau_+(U, \Delta_U)$ commutes with localization by a single element of T (in other words, restricting

to a smaller affine subset of U). Finally, since any quasi-projective scheme Y over $\operatorname{Spec} R$ is the union of such affine schemes, we have a well-defined notion of $\tau_+(\mathcal{O}_Y, \Delta)$.

We prove the following optimal global generation property, hinted at on the first page.

THEOREM B (Corollary 6.2). *With notation as above, assuming that $K_X + M$ is a Cartier divisor such that $M - \Delta$ is big and semi-ample, $\mathcal{O}_X(L)$ is a globally generated and ample line bundle, and $m \geq n$, where $n = \dim X_{\mathfrak{m}}$ is the dimension of the closed fiber, then*

$$\tau_+(\mathcal{O}_X, \Delta) \otimes \mathcal{O}_X(K_X + M + mL)$$

is globally generated by $\mathbf{B}^0(X, \Delta, \mathcal{O}_X(K_X + M + mL))$.

Note that our method of proof is via a Skoda-type complex; see [Laz04, Section 9.6.C]. A similar method was also used in characteristic $p > 0$ in [ST14]. In characteristic zero, a common proof of the analogous result is obtained via Castelnuovo–Mumford regularity [Laz04], and that approach was also used in characteristic $p > 0$ in [Kee08, Sch14, Mus13, Sat18]. Related results were also obtained in positive and mixed characteristic [Smi97, BMP⁺23] by using the fact that the (completion of the) graded absolute integral closure of the graded section ring is big Cohen–Macaulay.

In fact, inspired by our Skoda-type argument, we also introduce a notion of test ideals of ideal sheaves $\tau(\mathcal{O}_X, \mathfrak{a}^t)$ and show a Skoda-type result for these sheaves, see Theorem 6.6, and deduce a new Briançon–Skoda-type theorem in mixed characteristic.

Using our global generation result, we follow the ideas of [Mus13] in characteristic $p > 0$, which itself uses the ideas of [Nak04, ELM⁺06]. We obtain the following two results analogous to some of the main results of those papers.

THEOREM C (Theorem 7.6). *Let Z be a closed integral subscheme of a regular integral scheme X projective over our complete local Noetherian ring $\operatorname{Spec} R$ and D a big $\mathbb{Q}$ -divisor. Then $\operatorname{ord}_Z(\|D\|)$ is independent of the representative in the numerical class of D .*

THEOREM D (Theorem 7.7). *With notation as above, let Z be a closed irreducible reduced subscheme of a regular scheme X . Assume that $R/\mathfrak{m}$ is infinite. If D is big, then the following are equivalent:*

- (a) *We have that $Z \not\subset \mathbf{B}_-(D)$.*
- (b) *There exists a divisor G such that $Z \not\subset \mathbf{B}_s|mD + G|$ for all $m \geq 1$.*
- (c) *There exists an integer $M > 0$ such that $\operatorname{ord}_Z(|mD|) \leq M$ for all $m \gg 0$.*
- (d) *We have that $\operatorname{ord}_Z(\|D\|) = 0$.*
- (e) *The ideal $\tau_+(\omega_X, \lambda \cdot \|D\|)$ does not vanish along Z for any $\lambda > 0$.*

See also [CDB13, Sat18, Mur21] from some related results on singular varieties in equal characteristic and compare with [BBP13, Conjecture 2.7].

Using the same method as in the proof of the above result, we also observe that if $z \in X$ is a point (possibly nonclosed) so that X_z is regular and $\operatorname{mult}_z(\Delta) < 1$, then $\tau_+(\mathcal{O}_X, \Delta)$ agrees with $\mathcal{O}_X$ in a neighborhood of z ; see Corollary 7.8. Note that similar statements are known in positive characteristic; see [KKP⁺21, Theorem 1.1], where it is also shown that such bounds are essentially optimal; see also [Sat18].

2. Preliminaries

2.1 Local and Matlis duality

We briefly review local and Matlis duality.

Suppose that $(R, \mathfrak{m}, k)$ is a complete local ring and $E = E_R(k)$ is the injective hull of the residue field. Recall that the exact functor $\bullet^\vee = \text{Hom}(\bullet, E)$ is faithful, takes Noetherian modules to Artinian modules (and Artinian modules to Noetherian ones), and for any finitely generated R -module (or complex in $D_{\text{fg}}^b(R)$), we have a natural isomorphism of functors from the identity to $\bullet^{\vee\vee}$.

We fix a normalized dualizing complex in the sense of [Har66]; that is, $\omega_R^\bullet$ has its first nonzero cohomology in degree $-\dim R$. For any finite-type scheme $f: X \rightarrow \text{Spec } R$, we fix $\omega_X^\bullet = f^! \omega_R^\bullet$.

LEMMA 2.1. *Suppose that X is projective over a complete local ring $(R, \mathfrak{m}, k = R/\mathfrak{m})$. Suppose further that $\mathcal{F}$ is a coherent sheaf on X . Let $\bullet^\vee = \text{Hom}_R(\bullet, E)$ denote Matlis duality, where $E = E_k$ is the injective hull of the residue field. Then we have the following natural isomorphism in the derived category:*

$$\mathbf{R}\Gamma(X, \mathbf{R}\mathcal{H}\text{om}_X(\mathcal{F}, \omega_X^\bullet))^\vee \cong \mathbf{R}\Gamma_{\mathfrak{m}}(\mathbf{R}\Gamma(X, \mathcal{F})).$$

In particular, if $\mathcal{F} = \mathcal{L}^{-1}$ is a line bundle and X is Cohen–Macaulay of dimension n (so that $\omega_X^\bullet = \omega_X[n]$), then

$$(H^i(X, \omega_X \otimes \mathcal{L}))^\vee = H_{\mathfrak{m}}^{n-i}(\mathbf{R}\Gamma(X, \mathcal{L}^{-1})).$$

Proof. This is a combination of Grothendieck duality and local duality. Indeed, we have

$$\mathbf{R}\Gamma(X, \mathbf{R}\mathcal{H}\text{om}_X(\mathcal{F}, \omega_X^\bullet))^\vee \cong \mathbf{R}\text{Hom}_R(\mathbf{R}\Gamma(X, \mathcal{F}), \omega_R^\bullet)^\vee \cong \mathbf{R}\Gamma_{\mathfrak{m}}(\mathbf{R}\Gamma(X, \mathcal{F})),$$

where the first isomorphism is Grothendieck duality and the second is local duality. $\square$

In fact, due to a spectral sequence in low-degree terms argument, for $i = 0$ in the above, we do not even need to assume that X is Cohen–Macaulay [BMP⁺23, Section 2.2].

2.2 X^+ and vanishing in mixed characteristic

Suppose that X is an excellent integral scheme, and fix $\overline{K(X)}$ to be the absolute integral closure of the fraction field of X . By X^+ we mean the limit over all finite surjective morphisms $f: Y \rightarrow X$ with Y normal and integral with a fixed embedding $K(Y) \subseteq \overline{K(X)}$ which determines the map f ; note that X^+ is a scheme by [Sta23, Tag 01YV] as all the maps in the system are affine. If $\nu: X^+ \rightarrow X$ is the canonical map, we can describe the structure sheaf of X^+ as

$$\nu_* \mathcal{O}_{X^+} = \text{colim}_{f: Y \rightarrow X} f_* \mathcal{O}_Y.$$

It is important to note that X^+ is typically not Noetherian even if X is. If X is excellent and reduced with irreducible components $X_1, \dots, X_t$, then by X^+ we mean the disjoint union of the X_i^+ .

As mentioned in the introduction, Bhatt’s vanishing theorem can be thought of as a variant of Kawamata–Viehweg vanishing on X^+ . Because of its utility in our treatment of a mixed-characteristic test ideals, we record it here for future reference.

THEOREM 2.2 ([BMP⁺23, Section 3.7]). *Suppose that (T, x) is an excellent local ring of residue characteristic $p > 0$. Let $\pi: X \rightarrow \text{Spec } T$ be a proper map with X integral. Suppose that $L \in \text{Pic}(X)$ is a big and semi-ample line bundle. Then for all $b < 0$ and all $i < \dim(X)$, we have that $H^i(\mathbf{R}\Gamma_x(\mathbf{R}\Gamma(X^+, L^b))) = 0$.*

2.3 Base loci

Let D be a divisor on a normal scheme X which is projective over a local ring $(R, \mathfrak{m})$. We recall that $\text{Bs } |D|$ is used to denote the base locus of the linear system $|D|$. This is the subset of X defined by the intersection of all elements $G \in |D|$ (if $|D| = \emptyset$, then we let $\text{Bs } |D| = X$). The stable base locus of a $\mathbb{Q}$ -divisor D is the subset of X defined by $\mathbf{B}(D) = \bigcap_m \text{Bs } |mD|$, where we consider $m \geq 1$ such that mD has integer coefficients. Note that $\mathbf{B}(D) = \text{Bs } |mD|$ for any $m > 0$ sufficiently divisible (if $\kappa(D) = 0$, then we set $\mathbf{B}(D) = X$). If D is pseudo-effective, then

$$\mathbf{B}_-(D) := \bigcup_A \mathbf{B}(D + A)$$

denotes the diminished base locus, where the union is over ample $\mathbb{Q}$ -divisors A . Note that $\mathbf{B}_-(D)$ is a countable union of closed subsets of X .

3. $+$ -stable global sections

We introduce a slight variant of the global section theory introduced in [TY21] and [BMP⁺23] which focuses more on adjoint line bundles $\omega_X \otimes \mathcal{L}$. We use the following setting.

NOTATION 3.1. Suppose that $(R, \mathfrak{m})$ is a complete Noetherian local ring with residual characteristic $p > 0$, and suppose that $X \rightarrow \text{Spec } R$ is a proper map from a normal integral scheme of dimension d . Fix an algebraic closure $\bar{K}(X)$ of the fraction field of X (that is, a geometric generic point of X), and consider systems of finite maps $f: Y \rightarrow X$ as in Section 2.2. Using the notation of Section 2.2, we fix the canonical map $\nu: X^+ \rightarrow X$.

For each $\mathbb{Q}$ -divisor B on X , we set

$$\nu_* \mathcal{O}_{X^+}(\nu^* B) = \text{colim}_{f: Y \rightarrow X} f_* \mathcal{O}_Y(\lfloor f^* B \rfloor),$$

where $f: Y \rightarrow X$ runs over finite dominant maps, and hence obtain the sheaf $\mathcal{O}_{X^+}(\nu^* B)$ on X^+ . Notice that for large enough $f: Y \rightarrow X$, we have that $f^* B$ has integer coefficients.

DEFINITION 3.2 ($+$ -stable sections). Suppose that $(R, \mathfrak{m})$ is a complete Noetherian local ring with residual characteristic $p > 0$, and suppose that $X \rightarrow \text{Spec } R$ is a proper map from a normal integral scheme of dimension d . For a divisorial sheaf $\mathcal{M} = \mathcal{O}_X(M)$ and $\mathbb{Q}$ -divisor $B \geq 0$, we define

$$\mathbf{B}^0(X, B, \mathcal{O}_X(K_X + M))$$

to be the R -Matlis dual of the image

$$\text{Image}(H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X, \mathcal{O}_X(-M))) \rightarrow H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X^+, \mathcal{O}_{X^+}(-\nu^* M + \nu^* B))))).$$

In particular, it is a submodule of $H^0(X, \mathcal{O}_X(K_X + M))$, the Matlis dual of $H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X, \mathcal{O}_X(-M)))$.

This notation varies slightly from what is presented in [BMP⁺23], where they work with $\mathbf{B}^0(X, B, \mathcal{O}_X(N))$ and hence consider the Matlis dual of the image of

$$H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X, \mathcal{O}_X(-N + K_X))) \rightarrow H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X^+, \mathcal{O}_{X^+}(\nu^*(-N + K_X + B)))).$$

In this paper, we replace N with $K_X + M$, and so we obtain the formula in Definition 3.2.

We have the following immediate properties.

LEMMA 3.3. *With notation as in Definition 3.2:*

- (a) *If $B' \geq B$, then $\mathbf{B}^0(X, B', \mathcal{O}_X(K_X + M)) \subseteq \mathbf{B}^0(X, B, \mathcal{O}_X(K_X + M))$.*

- (b) The space of $+$ -stable global sections $\mathbf{B}^0(X, B, \mathcal{O}_X(K_X + M))$ only depends on the linear equivalence class of M . That is, if $M \sim M'$, an isomorphism $H^0(X, \mathcal{O}_X(M)) \cong H^0(X, \mathcal{O}_X(M'))$ induces

$$\mathbf{B}^0(X, B, \mathcal{O}_X(K_X + M)) \cong \mathbf{B}^0(X, B, \mathcal{O}_X(K_X + M')).$$

- (c) If $F \geq 0$ is a Weil divisor, then there is a commutative diagram

$$\begin{array}{ccc} \mathbf{B}^0(X, B, \mathcal{O}_X(K_X + M - F)) & \xrightarrow{\sim} & \mathbf{B}^0(X, B + F, \mathcal{O}_X(K_X + M)) \\ \downarrow & & \downarrow \\ H^0(X, \mathcal{O}_X(K_X + M - F)) & \hookrightarrow & H^0(X, \mathcal{O}_X(K_X + M)), \end{array}$$

where the map in the top row is an isomorphism.

Proof. Claims (a) and (b) follow directly from the definition. For claim (c), notice that because the sheaf $\mathcal{O}_X(-M + F)/\mathcal{O}_X(-M)$ has support of dimension less than $d = \dim X$, we have that

$$H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X, \mathcal{O}_X(-M))) \rightarrow H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X, \mathcal{O}_X(-M + F)))$$

surjects by [Sta23, Tag 0A4R]. Hence they both have the same image in

$$H_{\mathfrak{m}}^d(X^+, \mathcal{O}_{X^+}(-\nu^*(M - F) + \nu^*B)) = H_{\mathfrak{m}}^d(X^+, \mathcal{O}_{X^+}(-\nu^*M + \nu^*(B + F))).$$

This proves claim (c). $\square$

Remark 3.4 (Making $K_X + M$ Cartier). Frequently, it is desirable that the divisor $K_X + M$ considered above be Cartier. If $K_X + M$ is a Weil divisor, we may reduce to the case where $K_X + M$ is Cartier at least when X is *projective* over $\mathrm{Spec} R$. Simply select a Weil divisor $D > 0$ such that $M + K_X + D$ is Cartier, and use Lemma 3.3(c) above to see that $\mathbf{B}^0(X, B + D, \mathcal{O}_X(K_X + M + D)) = \mathbf{B}^0(X, B, \mathcal{O}_X(K_X + M))$.

3.1 The PLT/adjoint variant of $\mathbf{B}^0$

We now move on to the purely log terminal (PLT)/adjoint ideal variant of $\mathbf{B}^0$. Suppose that $D = \sum_{i=1}^t D_i$ is a reduced divisor. We shall also have a $\mathbb{Q}$ -divisor $B \geq 0$, where frequently B and D have no common components. For each prime divisor D_i , with associated prime ideal sheaf $\mathcal{O}_X(-D_i)$, we fix an integral subscheme $D_i^+ \subseteq X^+$ lying over D_i . This gives us a prime ideal sheaf lying over $\mathcal{O}_X(-D_i)$ which we write as $\mathcal{O}_{X^+}(-D_i^+)$. It is also a colimit of compatible ideal sheaves $\mathcal{O}_Y(-D_{i,Y})$ on each finite dominant $f: Y \rightarrow X$ factoring $X^+ \rightarrow X$.

DEFINITION 3.5. With notation as above, we set $\mathbf{B}_D^0(X, D + B, \mathcal{O}_X(K_X + M))$ to be the R -Matlis dual of the image

$$\mathrm{Image} \left(H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X, \mathcal{O}_X(-M))) \rightarrow \bigoplus_{i=1}^t H_{\mathfrak{m}}^d(X^+, \mathcal{O}_{X^+}(-\nu^*M + \nu^*(D + B) - D_i^+)) \right).$$

It is an R -submodule of $H^0(X, \mathcal{O}_X(K_X + M))$.

Remark 3.6 (A variant of the definition when the boundary has no D). Notice that the sheaf $\mathcal{O}_X(-M)/\mathcal{O}_X(-M - D)$ has support of dimension less than d , and so

$$H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X, \mathcal{O}_X(-M - D))) \twoheadrightarrow H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X, \mathcal{O}_X(-M)))$$

is surjective. In particular,

$$\text{Image} \left(H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X, \mathcal{O}_X(-M-D))) \rightarrow \bigoplus_{i=1}^t H_{\mathfrak{m}}^d(X^+, \mathcal{O}_{X^+}(-\nu^*M + \nu^*(D+B) - D_i^+)) \right)$$

coincides with the image in Definition 3.5. Its Matlis dual is then the subset of $H^0(X, \mathcal{O}_X(K_X + M + D))$ given by the image of $\mathbf{B}_D^0(X, D+B, \mathcal{O}_X(K_X + M))$ via the inclusion $H^0(X, \mathcal{O}_X(K_X + M)) \subseteq H^0(X, \mathcal{O}_X(K_X + M + D))$.

We could consider the R -Matlis dual of

$$\text{Image} \left(H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X, \mathcal{O}_X(-M-D))) \rightarrow \bigoplus_{i=1}^t H_{\mathfrak{m}}^d(X^+, \mathcal{O}_{X^+}(-\nu^*M + \nu^*B - D_i^+)) \right). \quad (3.1)$$

That is a subset of $H^0(X, \mathcal{O}_X(K_X + D + M))$. On the other hand, the image of (3.1) is the same as the image of

$$\text{Image} \left(H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X, \mathcal{O}_X(-M-D))) \rightarrow \bigoplus_{i=1}^t H_{\mathfrak{m}}^d(X^+, \mathcal{O}_{X^+}(-\nu^*(M+D) + \nu^*(B+D) - D_i^+)) \right),$$

whose Matlis dual is by definition $\mathbf{B}_D^0(X, D+B, \mathcal{O}_X(K_X + D + M))$.

In the case where D is not prime, we must pay special attention to the direct sums considered above. Note that the use of the direct sum separates the contribution from each D_i (this partially explains why this a PLT variant). Additionally, because the Galois action is independent on each direct summand, we see that $\mathbf{B}_D^0$ is independent of the choice of each D_i^+ ; see [Sta23, Tag 0BRK].

LEMMA 3.7. *With notation as in Definition 3.5:*

(a) *If $B' \geq B$, then*

$$\mathbf{B}_D^0(X, D+B', \mathcal{O}_X(K_X + M)) \subseteq \mathbf{B}_D^0(X, D+B, \mathcal{O}_X(K_X + M)).$$

(b) *The space of $+$ -stable global sections $\mathbf{B}_D^0(X, B+D, \mathcal{O}_X(K_X + M))$ only depends on the linear equivalence class of M . That is, if $M \sim M'$, an isomorphism $H^0(X, \mathcal{O}_X(K_X + M)) \cong H^0(X, \mathcal{O}_X(K_X + M'))$ induces*

$$\mathbf{B}_D^0(X, B+D, \mathcal{O}_X(K_X + M)) \cong \mathbf{B}_D^0(X, B+D, \mathcal{O}_X(K_X + M')).$$

(c) *For any $0 < \epsilon \leq 1$, we have that*

$$\begin{aligned} \mathbf{B}^0(X, D+B, \mathcal{O}_X(K_X + M)) &\subseteq \mathbf{B}_D^0(X, D+B, \mathcal{O}_X(K_X + M)) \\ &\subseteq \mathbf{B}^0(X, (1-\epsilon)D+B, \mathcal{O}_X(K_X + M)). \end{aligned}$$

(d) *If $F \geq 0$ is a Weil divisor, then there is a commutative diagram*

$$\begin{array}{ccc} \mathbf{B}_D^0(X, B+D, \mathcal{O}_X(K_X + M - F)) & \xrightarrow{\sim} & \mathbf{B}_D^0(X, B+D+F, \mathcal{O}_X(K_X + M)) \\ \downarrow & & \downarrow \\ H^0(X, \mathcal{O}_X(K_X + M - F)) & \hookrightarrow & H^0(X, \mathcal{O}_X(K_X + M)) \end{array}$$

inducing an isomorphism of the top row.

Proof. As in Lemma 3.3, we see that claims (a) and (b) follow from the definition.

Claim (c) follows from the inclusions

$$\begin{aligned}\mathcal{O}_{X+}(-\nu^*M + \nu^*((1-\epsilon)D + B)) &\subseteq \mathcal{O}_{X+}(-\nu^*M + \nu^*(D + B) - D_i^+) \\ &\subseteq \mathcal{O}_{X+}(-\nu^*M + \nu^*(D + B)).\end{aligned}$$

Part (d) follows in the same way as Lemma 3.3(c). $\square$

We recall the following dual formulations of $\mathbf{B}^0$ and $\mathbf{B}_D^0$.

LEMMA 3.8 ([BMP⁺23, Sections 4.8, 4.13, 4.24, and 4.25]). *With notation as in Definition 3.2, and with $f: Y \rightarrow X$ as in Notation 3.1, the following are equal:*

- (a) $\mathbf{B}^0(X, B, \mathcal{O}_X(K_X + M))$,
- (b)

$$\bigcap_{f: Y \rightarrow X} \operatorname{im}(H^0(Y, \mathcal{O}_Y(K_Y + \lceil f^*(M - B) \rceil)) \rightarrow H^0(X, \mathcal{O}_X(K_X + M))),$$

- (c) the projection to $H^0(X, \mathcal{O}_X(K_X + M))$ from the inverse limit as below:

$$\operatorname{Image}\left(\lim_{f: Y \rightarrow X} H^0(Y, \mathcal{O}_Y(K_Y + \lceil f^*(M - B) \rceil)) \rightarrow H^0(X, \mathcal{O}_X(K_X + M))\right).$$

Finally, in the case where $M - B$ is $\mathbb{Q}$ -Cartier, we may instead let $f: Y \rightarrow X$ run over alterations from integral normal schemes (with a chosen embedding $K(Y) \subseteq \overline{K(X)}$ as in the finite case).

The analogous result for the adjoint variants $\mathbf{B}_D^0$ also holds, and the following are equal:

- (a) $\mathbf{B}_D^0(X, B + D, \mathcal{O}_X(K_X + M))$,
- (b)

$$\bigcap_{f: Y \rightarrow X} \operatorname{im}\left(H^0\left(Y, \bigoplus_{i=1}^t \mathcal{O}_Y(K_Y + D_{i,Y} + \lceil f^*(M - B - D) \rceil)\right) \rightarrow H^0(X, \mathcal{O}_X(K_X + M))\right),$$

- (c) the projection to $H^0(X, \mathcal{O}_X(K_X + M))$ from the inverse limit as below:

$$\operatorname{Image}\left(\lim_{f: Y \rightarrow X} H^0\left(Y, \bigoplus_{i=1}^t \mathcal{O}_Y(K_Y + D_{i,Y} + \lceil f^*(M - B - D) \rceil)\right) \rightarrow H^0(X, \mathcal{O}_X(K_X + M))\right).$$

PROPOSITION 3.9. *Suppose that $\pi: X' \rightarrow X$ is a proper birational map between normal schemes with X, B, M as above, where B is $\mathbb{Q}$ -Cartier and M is Cartier. Then we have that*

$$\mathbf{B}^0(X, B, \mathcal{O}_X(K_X + M)) \cong \mathbf{B}^0(X', \pi^*B, \mathcal{O}_{X'}(K_{X'} + \pi^*M)).$$

Proof. A similar argument can be found in the proof of [BMP⁺23, Lemma 4.19]. Recall that a proper birational map between normal schemes is an alteration. For any alteration $f: Y \rightarrow X'$, we have the following commutative diagram:

$$\begin{array}{ccc} H^0(Y, \mathcal{O}_Y(K_Y + \lceil f^*(\pi^*(M - B)) \rceil)) & \longrightarrow & H^0(X', \mathcal{O}_{X'}(K_{X'} + \pi^*M)) \\ \downarrow & & \downarrow \\ H^0(Y, \mathcal{O}_Y(K_Y + \lceil (\pi \circ f)^*(M - B) \rceil)) & \longrightarrow & H^0(X, \mathcal{O}_X(K_X + M)). \end{array}$$

Applying the alteration version of Lemma 3.8 to this diagram, which we can do since $M - B$ is $\mathbb{Q}$ -Cartier, yields the desired claim. $\square$

We will need the following result.

PROPOSITION 3.10. *With notation as above, suppose that $H^0(X, \mathcal{O}_X) = R \subseteq S$ is finite étale, where S is also a domain (and hence local and complete). Suppose that M is Weil a divisor on X . Let $X_S = X \times_R S$ denote the base change, and likewise with B_S and M_S . Then,*

$$\mathbf{B}^0(X, B, \mathcal{O}_X(K_X + M)) \otimes_R S = \mathbf{B}^0(X_S, B_S, \mathcal{O}_{X_S}(K_{X_S} + M_S)).$$

Proof. Consider the map

$$H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X, \mathcal{O}_X(-M))) \rightarrow H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X^+, \mathcal{O}_{X^+}(-\nu^*M + \nu^*B))) \quad (3.2)$$

whose image I_1 is dual to $\mathbf{B}^0(X, B, \mathcal{O}_X(K_X + M))$. Tensoring this map with S yields

$$H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X, \mathcal{O}_X(-M)) \otimes_R S) \rightarrow H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X^+, \mathcal{O}_{X^+}(-\nu^*M + \nu^*B)) \otimes_R S).$$

The left side is $H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X_S, \mathcal{O}_{X_S}(-M_S)))$, but the right side is slightly more complicated. The point is, choosing an embedding of $K(X_S) \subseteq \overline{K(X)}$, we see that $X^+ \rightarrow X$ already factors through X_S , and so $(X^+)_S = X^+ \times_{\mathrm{Spec} R} \mathrm{Spec} S$ will break up into multiple components since even $S \otimes_R S$ will not be a domain.

Indeed, let $S' \supseteq S \supseteq R$ denote the integral closure of R in the Galois closure $K(S')$ of $K(S)$ over $K(R)$ (in $K(X^+)$). Then $(X^+)_S = X^+ \times_{\mathrm{Spec} S'} \mathrm{Spec} S' \times_{\mathrm{Spec} R} \mathrm{Spec} S$. Notice that in fact $S' \otimes_R S$ is a direct sum of $[K(S) : K(R)]$ copies of S' . Hence we see that $(X^+)_S$ is simply a product of copies of X^+ .

Furthermore, the map $X_S \rightarrow (X^+)_S$ will be the product of maps corresponding to the different ways to embed $K(X_S) \subseteq K(X^+) = \overline{K(X)}$. Since any such embeddings will differ by a Galois action on $K(X^+)$, we see that each such induced homomorphism $H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X_S, \mathcal{O}_{X_S}(-M_S))) \rightarrow H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X^+, \mathcal{O}_{X^+}(-\nu^*M + \nu^*B)))$ has the same kernel (and hence isomorphic images). Thus we see that the S -Matlis dual of the image of

$$H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X, \mathcal{O}_X(-M))) \otimes_R S \rightarrow H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X^+, \mathcal{O}_{X^+}(-\nu^*M + \nu^*B))) \otimes_R S \quad (3.3)$$

is indeed $\mathbf{B}^0(X_S, B_S, \mathcal{O}_{X_S}(K_{X_S} + M_S))$. The image I_2 of (3.3) is also the tensor product of the image I_1 of (3.2) since $R \rightarrow S$ is flat. In other words $I_2 = I_1 \otimes_R S$.

Let $n = \dim R = \dim S$. Notice that $E_S = \mathrm{Hom}_R(S, E_R)$ since $R \rightarrow S$ is finite, and so

$$\mathrm{Hom}_S(I_2, E_S) \cong \mathrm{Hom}_R(I_2, E_R) = \mathrm{Hom}_R(S \otimes_R I_1, E_R) = S \otimes \mathrm{Hom}_R(I_1, E_R).$$

Here the first isomorphism is true since $R \rightarrow S$ is finite, and the final isomorphism can be found in [Bou98, Section I.2, Exercise 14, p. 45] since S is finite and E_R is injective. $\square$

We also recall [BMP⁺23, Theorem 7.2].

THEOREM 3.11 (Lifting sections). *With notation as in Definition 3.2, suppose that $K_X + M$ is Cartier and that $M - D - B$ is $\mathbb{Q}$ -Cartier, big, and semi-ample. Suppose that $D^N \rightarrow D$ is the normalization and $\kappa: D^N \rightarrow X$ is the canonical map, and write $\kappa^*\mathcal{O}_X(K_X + M) \cong \mathcal{O}_{D^N}(K_{D^N} + M_{D^N})$ for some Weil divisor M_{D^N} . Then the map $H^0(X, \mathcal{O}_X(K_X + M)) \rightarrow H^0(D^N, \mathcal{O}_{D^N}(K_{D^N} + M_{D^N}))$ induces a surjection*

$$\mathbf{B}_D^0(X, D + B, \mathcal{O}_X(K_X + M)) \twoheadrightarrow \mathbf{B}^0(D^N, \mathrm{Diff}_{D^N}(B + D), \mathcal{O}_{D^N}(K_{D^N} + M_{D^N})).$$

Remark 3.12 (Lifting sections without assuming that $K_X + M$ is Cartier). Suppose that $X \rightarrow \mathrm{Spec} R$ is projective and that $K_X + M$ is not Cartier but that $K_X + M + F$ is Cartier for some

effective divisor F with no common components with D . Replacing M with $M + F$ and B with $B + F$, we may apply Theorem 3.11, to obtain a surjection

$$\mathbf{B}_D^0(X, D + B + F, \mathcal{O}_X(K_X + M + F)) \rightarrow \mathbf{B}^0(D^N, \text{Diff}_{D^N}(B + D + F), \kappa^* \mathcal{O}_X(K_X + M + F)).$$

However, by Lemma 3.7(d), we see that the left side is isomorphic to $\mathbf{B}_D^0(X, D + B, \mathcal{O}_X(K_X + M))$. Thus $\mathbf{B}_D^0(X, D + B, \mathcal{O}_X(K_X + M))$ surjects onto a meaningful analog of $\mathbf{B}^0(D^N, \text{Diff}_{D^N}(B + D), \mathcal{O}_{D^N}(K_{D^N} + M_{D^N}))$. Note that the term F contributes to the different (or perhaps more accurately, the fact that $K_X + M$ is not Cartier contributes to the different).

4. A global mixed-characteristic theory of test ideals

Based on the equal-characteristic setting, at least for big enough twists by an ample line bundle, one may expect that the sections $\mathbf{B}^0$ ought to globally generate certain important subsheaves of $\mathcal{O}_X(K_X + M)$. In equal characteristic, under suitable assumptions, analogs of $\mathbf{B}^0$ generate multiplier ideals and test ideals tensored with $\mathcal{O}_X(K_X + M)$ (see [Laz04, Chapter 10] and [Sch14, Mus13]).

In this section, we introduce the sheaves generated by $\mathbf{B}^0$. Note that the test ideals and modules introduced in [MS21] require the ring we are working with to be complete. While we believe that some of these ideas still work for noncomplete bases as in [TY21], as of now we seem to get stronger results for complete bases, and so we restrict to that setting.

Throughout this section, we work in the following setting.

SETTING 4.1. Fix a complete local ring R with residual characteristic $p > 0$, and suppose that X is a normal integral scheme projective over R . We fix a very ample line bundle $\mathcal{L} = \mathcal{O}_X(L)$ on X .

Typically, we also assume that $H^0(X, \mathcal{O}_X) = R$. Indeed, we may reduce to this case by replacing R with $H^0(X, \mathcal{O}_X)$ if necessary. Since $H^0(X, \mathcal{O}_X)$ is a finite extension of R , this does not impact the computation of Matlis duality.

We aim to introduce a notion of test ideals on X .

4.1 Test ideals via global sections

We begin with the following lemma.

LEMMA 4.2. *With notation as above, let M and N be divisors on X . Then, the image of the canonical multiplication map*

$$\mathbf{B}^0(X, B, \mathcal{O}_X(K_X + M)) \otimes_R H^0(X, \mathcal{O}_X(N)) \rightarrow H^0(X, \mathcal{O}_X(K_X + M + N))$$

is contained in $\mathbf{B}^0(X, B, \mathcal{O}_X(K_X + M + N))$. Similarly, for $\mathbf{B}_D^0$, the image of

$$\mathbf{B}_D^0(X, D + B, \mathcal{O}_X(K_X + M)) \otimes_R H^0(X, \mathcal{O}_X(N)) \rightarrow H^0(X, \mathcal{O}_X(K_X + M + N))$$

is contained in $\mathbf{B}_D^0(X, D + B, \mathcal{O}_X(K_X + M + N))$.

Proof. Fix a choice of R -module generators of $y_1, \dots, y_m \in H^0(X, \mathcal{O}_X(N))$. We will pull these back to global sections $\pi^* y_i$ of $\mathcal{O}_Y(\pi^* N)$ for each finite cover $\pi: Y \rightarrow X$ from a normal integral scheme. Now, suppose that $z \in \mathbf{B}^0(X, B, \mathcal{O}_X(K_X + M))$; then for each finite morphism $\pi: Y \rightarrow X$ as before with $\pi^* B$ a Weil divisor, there exists a $z_Y \in H^0(Y, \mathcal{O}_Y(K_Y - \pi^* B + \pi^* M))$ mapping to z via the corresponding trace map. Hence $z_Y \otimes \pi^* y_i \in H^0(Y, \mathcal{O}_Y(K_Y - \pi^* B + \pi^* M + \pi^* N))$ maps to $z \otimes y_i$. The argument for $\mathbf{B}_D^0$ is the same. This proves the lemma. $\square$

DEFINITION 4.3 (Test modules). Suppose that X and $\mathcal{L}$ are as in Setting 4.1. For each i , we define the subsheaf $\mathcal{N}_i \subseteq \omega_X \otimes \mathcal{L}^i$ generated by $\mathbf{B}^0(X, B, \omega_X \otimes \mathcal{L}^i) \subseteq H^0(X, \omega_X \otimes \mathcal{L}^i)$ and define $\mathcal{J}_i := \mathcal{N}_i \otimes \mathcal{L}^{-i}$, a subsheaf of ω_X . Because $\mathcal{L}$ is globally generated, Lemma 4.2 implies that $\mathcal{J}_i \subseteq \mathcal{J}_{i+1} \subseteq \omega_X$, and hence $\mathcal{J}_i = \mathcal{J}_{i+1}$ for large enough i . One may then define the $+$ -test submodule $\tau_+^{\mathcal{L}}(\omega_X, B)$ to be $\mathcal{J}_i$ for $i \gg 0$. If $B = 0$, we simply write $\tau_+^{\mathcal{L}}(\omega_X)$.

DEFINITION 4.4 (Adjoint modules). Likewise, with notation as in Definition 3.5, the *adjoint $+$ -test module* $\text{adj}_+^{D, \mathcal{L}}(\omega_X(D), B)$ is defined as follows. For each $i > 0$, let $\mathcal{N}_i \subseteq \omega_X(D) \otimes \mathcal{L}^i = \mathcal{O}_X(K_X + D) \otimes \mathcal{L}^i$ be the submodule generated by $\mathbf{B}_D^0(X, B + D, \mathcal{O}_X(K_X + D) \otimes \mathcal{L}^i)$. Setting $\mathcal{J}_i = \mathcal{N}_i \otimes \mathcal{L}^{-i} \subseteq \omega_X(D)$, we see that $\mathcal{J}_i \subseteq \mathcal{J}_{i+1}$ by Lemma 4.2. We define $\text{adj}_+^{D, \mathcal{L}}(\omega_X(D), B) := \mathcal{J}_i$ for $i \gg 0$.

4.2 Independence of the choice of ample $\mathcal{L}$

Our next goal is to show that $\tau_+^{\mathcal{L}}(\omega_X, B)$ is independent of the choice of the ample line bundle $\mathcal{L}$.

PROPOSITION 4.5. *Given any two very ample line bundles $\mathcal{L}$ and $\mathcal{L}'$, we have that*

$$\tau_+^{\mathcal{L}}(\omega_X, B) = \tau_+^{\mathcal{L}'}(\omega_X, B).$$

Moreover, for any sufficiently ample line bundle $\mathcal{A}$, the sheaf $\tau_+(\omega_X, B) \otimes \mathcal{A}$ is generated by $\mathbf{B}^0(X, B, \omega_X \otimes \mathcal{A})$. The analogous statements also hold for $\text{adj}_+^{D, \mathcal{L}}(\omega_X(D), B)$.

Proof. It is clear from the definition that we have equality when $\mathcal{L}' = \mathcal{L}^k$ for any $k > 0$ since the $\mathcal{N}_i$ are ascending. Hence, taking high enough powers of $\mathcal{L}'$ again if necessary, we may assume that $\mathcal{L}' = \mathcal{L} \otimes \mathcal{G}$, where $\mathcal{G}$ is a globally generated (even very ample) line bundle.

Lemma 4.2 implies that $\tau_+^{\mathcal{L}}(\omega_X, B) \subseteq \tau_+^{\mathcal{L}'}(\omega_X, B)$. Reversing the roles of $\mathcal{L}$ and $\mathcal{L}'$ proves the reverse inclusion, and hence we have shown that

$$\tau_+^{\mathcal{L}}(\omega_X, B) = \tau_+^{\mathcal{L}'}(\omega_X, B).$$

The proof for $\text{adj}_+^D(\omega_X(D), B)$ is the same.

Now suppose that $\mathcal{A}$ is sufficiently ample and in particular $\mathcal{A} \otimes \mathcal{L}^{-i}$ is very ample for some $i \gg 0$. Let $M \otimes \mathcal{A} \subseteq \omega_X \otimes \mathcal{A}$ be the module generated by $\mathbf{B}^0(X, B, \omega_X \otimes \mathcal{A})$. Arguing as above, we see that $\mathbf{B}^0(X, B, \omega_X \otimes \mathcal{L}^i) \otimes H^0(X, \mathcal{A} \otimes \mathcal{L}^{-i})$ is contained in $\mathbf{B}^0(X, B, \omega_X \otimes \mathcal{A})$ and hence $\tau_+^{\mathcal{L}}(\omega_X, B) \subseteq M$. On the other hand, since $\mathbf{B}^0(X, B, \omega_X \otimes \mathcal{A}) \otimes H^0(X, \mathcal{A}^{j-1})$ is contained in $\mathbf{B}^0(X, B, \omega_X \otimes \mathcal{A}^j)$ for any $j > 0$, it follows that $M \subseteq \tau_+^{\mathcal{A}}(\omega_X, B)$. Since we have already seen that $\tau_+^{\mathcal{A}}(\omega_X, B) = \tau_+^{\mathcal{L}}(\omega_X, B)$, the claim follows. The proof for $\text{adj}_+^D(\omega_X(D), B)$ is analogous. $\square$

NOTATION 4.6. In view of Proposition 4.5, we omit the $\mathcal{L}$ from the notation of $\tau_+^{\mathcal{L}}(\omega_X, B)$ (respectively, $\text{adj}_+^{D, \mathcal{L}}(\omega_X(D), B)$) and simply write $\tau_+(\omega_X, B)$ (respectively, $\text{adj}_+^D(\omega_X(D), B)$).

We make the following observation.

PROPOSITION 4.7. *For $i \gg 0$, we have that*

$$H^0(X, \tau_+(\omega_X, B) \otimes \mathcal{L}^i) = \mathbf{B}^0(X, B, \omega_X \otimes \mathcal{L}^i)$$

and similarly that

$$H^0(X, \text{adj}_+^D(\omega_X(D), B) \otimes \mathcal{L}^i) = \mathbf{B}_D^0(X, D + B, \omega_X(D) \otimes \mathcal{L}^i).$$

Proof. By Definition 4.3, we have that $\tau_+(\omega_X, B) \otimes \mathcal{L}^i = \mathcal{J}_i \otimes \mathcal{L}^i = \mathcal{N}_i$ is generated by $\mathbf{B}^0(X, B, \omega_X \otimes \mathcal{L}^i)$ for $i \gg 0$. We must then prove that the induced inclusion $\mathbf{B}^0(X, B, \omega_X \otimes \mathcal{L}^i) \subseteq H^0(X, \tau_+(\omega_X, B) \otimes \mathcal{L}^i)$ is an equality for $i \gg 0$.

Without loss of generality, we may assume that $H^0(X, \mathcal{O}_X) = R$. Let $S = \bigoplus_{i \geq 0} H^0(X, \mathcal{L}^i)$ be the graded ring whose degree zero part is the complete local ring $(R, \mathfrak{m})$. Fix $N \gg 0$. We consider the graded S -module $J := \bigoplus_{i \geq N} \mathbf{B}^0(X, B, \omega_X \otimes \mathcal{L}^i)$. Lemma 4.2 proves that J is a graded S -submodule of the graded canonical module ω_S (which in large degrees is $H^0(X, \omega_X \otimes \mathcal{L}^i)$, see [BMP⁺23, Section 5]). The sheaf $\mathcal{J}$ associated with the graded module J is a coherent subsheaf of ω_X on X . Furthermore, we have that $H^0(X, \mathcal{J} \otimes \mathcal{L}^i)$ is the i th graded piece $[J]_i$ of J in sufficiently high degree i by [Gro61, Theorem 2.3.1] (a similar statement is found in [Har77, Exercise 2.5.9(b)] under unnecessary geometric hypotheses). Furthermore, $H^0(X, \mathcal{J} \otimes \mathcal{L}^i)$ globally generates $\mathcal{J} \otimes \mathcal{L}^i$ for $i \gg 0$. Therefore, $\mathcal{J} \otimes \mathcal{L}^i = \tau_+(\omega_X, B) \otimes \mathcal{L}^i$ since the right side is generated by $\mathbf{B}^0(X, B, \omega_X \otimes \mathcal{L}^i)$. This proves the desired result. The adjoint statement in the corollary follows from the same reasoning with minor modifications. $\square$

LEMMA 4.8 (Basic facts about τ_+ and adj_+). *With notation as above:*

- (a) If $B' \geq B$, then $\tau_+(\omega_X, B') \subseteq \tau_+(\omega_X, B)$ and likewise $\text{adj}_+^D(\omega_X(D), B') \subseteq \text{adj}_+^D(\omega_X(D), B)$.
- (b) If $F \geq 0$ is a Cartier divisor, then $\tau_+(\omega_X, B + F) = \tau_+(\omega_X, B) \otimes \mathcal{O}_X(-F)$ and $\text{adj}_+^D(\omega_X(D), B + F) = \text{adj}_+^D(\omega_X(D), B) \otimes \mathcal{O}_X(-F)$.
- (c) For any $0 < \epsilon \leq 1$, we have that

$$\tau_+(\omega_X, D + B) \subseteq \text{adj}_+^D(\omega_X(D), D + B) \subseteq \tau_+(\omega_X, (1 - \epsilon)D + B).$$

Proof. The first part follows directly from the corresponding inclusions of $\mathbf{B}^0$ and $\mathbf{B}_D^0$ in Lemmas 3.3 and 3.7.

For part (b), fix M sufficiently ample so that $M - F$ is also sufficiently ample. Notice that $\mathbf{B}^0(X, B + F, \mathcal{O}_X(K_X + M)) \subseteq H^0(X, \mathcal{O}_X(K_X + M))$ globally generates $\tau_+(\omega_X, B + F) \otimes \mathcal{O}_X(M)$. However, using Lemma 3.3(c), we see that $\mathbf{B}^0(X, B + F, \mathcal{O}_X(K_X + M)) \cong \mathbf{B}^0(X, B, \mathcal{O}_X(K_X + M - F)) \subseteq H^0(X, \mathcal{O}_X(K_X + M - F))$ also globally generates $\tau_+(\omega_X, B) \otimes \mathcal{O}_X(M - F)$. Hence $\tau_+(\omega_X, B) \otimes \mathcal{O}_X(M - F) = \tau_+(\omega_X, B + F) \otimes \mathcal{O}_X(M)$, which proves part (b) for τ_+ . The version for adj_+^D is the same except that we use Lemma 3.7(d).

We now check part (c). First observe that the R -module $\mathbf{B}_D^0(X, B + 2D, \mathcal{O}_X(K + D + M))$ globally generates $\text{adj}_+^D(\omega_X(D), D + B) \otimes \mathcal{O}_X(M)$ for M sufficiently ample.

Then by Lemma 3.7(c) and (d), we have that

$$\begin{aligned} \mathbf{B}^0(X, D + B, \mathcal{O}_X(K_X + M)) &= \mathbf{B}^0(X, 2D + B, \mathcal{O}_X(K_X + D + M)) \\ &\subseteq \mathbf{B}_D^0(X, 2D + B, \mathcal{O}_X(K_X + D + M)) \\ &\subseteq \mathbf{B}^0(X, D + (1 - \epsilon)D + B, \mathcal{O}_X(K_X + D + M)) \\ &= \mathbf{B}^0(X, (1 - \epsilon)D + B, \mathcal{O}_X(K_X + M)). \end{aligned}$$

The first term generates $\tau_+(\omega_X, D + B)$, the last generates $\tau_+(\omega_X, (1 - \epsilon)D + B)$, and the third line generates $\text{adj}_+^D(\omega_X(D), D + B)$. The result follows. $\square$

Before we move on to comparing our test submodule with the multiplier submodule, we show that $\tau_+(\omega_X, B)$ is nonzero.

LEMMA 4.9. *With notation as above, $\tau_+(\omega_X, B)$ and $\text{adj}_+^D(\omega_X(D), B)$ are nonzero.*

Proof. We may assume that X has mixed characteristic because if X can be defined in characteristic $p > 0$, the result is well known.

For the first statement, it suffices to show that $\mathbf{B}^0(X, B, \omega_X \otimes \mathcal{L}^i) \neq 0$ for $i \gg 0$. Choose $D \geq B$ such that $D \sim mL$ for some $m > 0$. It suffices to show that $0 \neq \mathbf{B}^0(X, D, \omega_X \otimes \mathcal{O}_X(iL))$

by Lemma 3.3(a). Now, by Lemma 3.3(b) and (c), we have that

$$\mathbf{B}^0(X, 0, \mathcal{O}_X(K_X + (i - m)L)) \cong \mathbf{B}^0(X, 0, \mathcal{O}_X(K_X + iL - D)) \cong \mathbf{B}^0(X, D, \mathcal{O}_X(K_X + iL)),$$

and so we must show that $\mathbf{B}^0(X, 0, \mathcal{O}_X(K_X + (i - m)L)) \neq 0$. Thus, since $i \gg 0$, we must show that $\mathbf{B}^0(X, 0, \mathcal{O}_X(K_X + iL)) \neq 0$.

If $S = \bigoplus_{i \geq 0} H^0(X, \mathcal{L}^i)$ is the section ring with respect to $\mathcal{L}$, using [BMP⁺23, Proposition 5.5], we see that

$$\mathbf{B}^0(X, 0, \mathcal{O}_X(K_X + iL)) = [\tau_{S^+, \text{gr}}(\omega_S)]_i,$$

where $\tau_{S^+, \text{gr}}(\omega_S)$ is defined as in [BMP⁺23, Section 5]. This is equivalent to showing that the map $H_{\mathfrak{m}_S}^{d+1}(S) \rightarrow H_{\mathfrak{m}_S}^{d+1}(S^{+, \text{gr}})$ is nonzero. It is thus sufficient to show that $\tau_{\widehat{S^+, \text{gr}}}(\omega_{\widehat{S}}) \neq 0$ since this is nonzero if and only if the map on local cohomology above is nonzero (this object is defined in [MS21, Section 5]). Because we can take a Noether normalization $A \subseteq \widehat{R}$ via the Cohen structure theorem, we have that $\tau_{\widehat{S^+, \text{gr}}}(\omega_{\widehat{S}}) \neq 0$ by [MS21, Theorem 5.13].

For the statement regarding $\text{adj}_+^D(\omega_X(D), B)$, notice that

$$\begin{aligned} 0 &\neq \mathbf{B}^0(X, B + D, \omega_X \otimes \mathcal{L}^i) \\ &\subseteq \mathbf{B}_D^0(X, B + D, \omega_X \otimes \mathcal{L}^i) \\ &= \mathbf{B}_D^0(X, B + 2D, \omega_X(D) \otimes \mathcal{L}^i) \\ &\subseteq \mathbf{B}_D^0(X, B + D, \omega_X(D) \otimes \mathcal{L}^i) \end{aligned}$$

by Lemma 3.7. The latter generates $\text{adj}_+^D(\omega_X(D), B)$. $\square$

Remark 4.10. Notice that in the case where $B = 0$ and X is nonsingular, we are not (yet) asserting that $\tau_+(\omega_X) = \omega_X$. This is true and follows from Proposition 4.24, or from [BMP⁺23, Theorem 5.8], but it is not obvious based on our definitions.

4.3 Blowups and alterations

Our next goal is to discuss the behavior of τ_+ under proper birational maps, or even alterations, and to relate τ_+ with the multiplier ideal if a resolution of singularities exists.

PROPOSITION 4.11. *With notation as above, suppose additionally that B is $\mathbb{Q}$ -Cartier. For any alteration $\pi: Y \rightarrow X$, we have that*

$$\tau_+(\omega_X, B) \subseteq \text{Tr } \pi_* \mathcal{O}_Y(K_Y - \lfloor \pi^* B \rfloor),$$

where $\text{Tr}: \pi_* \omega_Y \rightarrow \omega_X$ is the Grothendieck trace.

Proof. By Proposition 4.7, it suffices to show that

$$\mathbf{B}^0(X, B, \omega_X \otimes \mathcal{L}^i) \subseteq H^0(X, (\text{Tr } \pi_* \mathcal{O}_Y(K_Y - \lfloor \pi^* B \rfloor)) \otimes \mathcal{L}^i)$$

for $i \gg 0$ and $\mathcal{L}$ ample. But for any i ,

$$\begin{aligned} \mathbf{B}^0(X, B, \omega_X \otimes \mathcal{L}^i) &\subseteq \text{Image}(H^0(X, (\pi_* \mathcal{O}_Y(K_Y - \lfloor \pi^* B \rfloor)) \otimes \mathcal{L}^i) \xrightarrow{\text{Tr}} H^0(X, \omega_X \otimes \mathcal{L}^i)) \\ &\subseteq H^0(X, (\text{Tr } \pi_* \mathcal{O}_Y(K_Y - \lfloor \pi^* B \rfloor)) \otimes \mathcal{L}^i), \end{aligned}$$

which completes the proof. $\square$

Localizing to characteristic zero, we obtain the following.

COROLLARY 4.12. *Fix notation as in Proposition 4.11, including that B is $\mathbb{Q}$ -Cartier. Let $R' = R[1/p]$, so for instance if R is a mixed-characteristic discrete valuation ring, then R' is a field of characteristic zero. Then,*

$$\tau_+(\omega_X, B)|_{X_{R'}} \subseteq \mathcal{J}(\omega_{X_{R'}}, B|_{X_{R'}}) = \mathcal{J}(X_{R'}, (-K_X + B)|_{X_{R'}}),$$

where $X_{R'}$ is the base change.

In the above, $\mathcal{J}(\omega_{X_{R'}}, B|_{X_{R'}})$ is defined to be

$$\pi_* \mathcal{O}_{Y_{R'}}(\lceil K_{Y_{R'}} - \pi^* B|_{X_{R'}} \rceil),$$

where $\pi: Y_{R'} \rightarrow X_{R'}$ is any log resolution of singularities. It coincides with the multiplier (fractional-)ideal sheaf $\mathcal{J}(X_{R'}, (-K_X + B)|_{X_{R'}})$ by definition.

Proof. Fix a proper birational map $\pi: Y \rightarrow X$ from a normal Y such that $Y_{R'} \rightarrow X_{R'}$ is a log resolution of singularities. Arguing as above and using that Tr is simply an inclusion since π is birational, we have

$$\tau_+(\omega_X, B)|_{X_{R'}} \subseteq \text{Tr} \pi_* \mathcal{O}_Y(K_Y - \lfloor \pi^* B \rfloor)|_{X_{R'}} = (\pi|_{Y_{R'}})_* \mathcal{O}_{Y_{R'}}(K_{Y_{R'}} - \lfloor \pi^* B|_{X_{R'}} \rfloor).$$

The right side is $\mathcal{J}(\omega_{X_{R'}}, B|_{X_{R'}})$ by definition. $\square$

One expects that the containment in the statement of Corollary 4.12 is an equality. In fact, we have learned that forthcoming work by the authors of [BMP⁺23] will address related questions.

Proposition 4.11 implies the following; also compare with [Mus13, Proposition 3.3].

LEMMA 4.13. *For every irreducible closed subset $Z \subseteq X$ such that the generic point η_Z is contained in the regular locus of X and any $\lambda > 0$, we have*

$$\text{ord}_Z(\tau_+(\omega_X, \lambda \Delta)) > \lambda \cdot \text{ord}_Z(\Delta) - \text{codim}(Z, X).$$

Proof. Since, as observed above, $\tau_+(\omega_X, \lambda \cdot \Delta) \subseteq \text{Tr} \pi_* \mathcal{O}_Y(K_Y - \lfloor \lambda \cdot \pi^* \Delta \rfloor)$, we may consider the blowup $\pi: Y \rightarrow X$ of X along Z and the exceptional divisor E . We may restrict to an open subset of X containing η_Z so that we may assume that Z is regular. If $c = \text{codim}(Z, X)$, then $K_{Y/X} = (c-1)E$ and $\lfloor \lambda \cdot \pi^* \Delta \rfloor \geq \lfloor \lambda \cdot \text{ord}_Z(\Delta) \rfloor E$. But then, using the fact that on this open subset, we can assume that $K_X = 0$, we have that

$$\text{Tr} \pi_* \mathcal{O}_Y(K_Y - \lfloor \lambda \cdot \pi^* \Delta \rfloor) \subseteq \text{Tr} \pi_* \mathcal{O}_Y(((c-1) - \lfloor \lambda \cdot \text{ord}_Z(\Delta) \rfloor)E) = \mathcal{J}_Z^{(p)},$$

where $p = \lfloor \lambda \cdot \text{ord}_Z(\Delta) \rfloor - (c-1) > \lambda \cdot \text{ord}_Z(\Delta) - c$, and the claim follows. $\square$

4.4 $\dashv$ -test ideals

In this subsection, we define $\dashv$ -test ideals $\tau_+(\mathcal{O}_X, \Delta) \subseteq \mathcal{O}_X$ instead of submodules of ω_X . Indeed, by Lemma 4.8(b), for a Cartier divisor $H \geq 0$,

$$\tau_+(\omega_X, B + H) = \tau_+(\omega_X, B) \otimes \mathcal{O}_X(-H).$$

Inspired by this, we make the following definition.

DEFINITION 4.14 ($\tau_+(\omega_X)$ version 2). Suppose that B is an arbitrary $\mathbb{Q}$ -divisor (not necessarily effective). We define the $\dashv$ -test submodule $\tau_+(\omega_X, B)$ to be $\tau_+(\omega_X, B + H) \otimes \mathcal{O}_X(H)$, where H is a Cartier divisor such that $H + B \geq 0$.

Likewise, we define $\text{adj}_+^D(\omega_X(D), B) = \text{adj}_+^D(\omega_X(D), B + H) \otimes \mathcal{O}_X(H)$. It is straightforward to verify that these definitions are independent of the choice of H .

Notice that if Δ is noneffective, we can have that $\tau_+(\omega_X, \Delta) \not\subseteq \omega_X$. We use this to obtain the following notion of test *ideal*.

DEFINITION 4.15. With X as above, fix a $\mathbb{Q}$ -divisor $\Delta \geq 0$. Fix a canonical divisor K_X , and write $\omega_X = \mathcal{O}_X(K_X)$. We then define the test ideal to be

$$\tau_+(\mathcal{O}_X, \Delta) := \tau_+(\omega_X, K_X + \Delta).$$

If $\Delta = 0$, then we write $\tau_+(\mathcal{O}_X)$ for $\tau_+(\mathcal{O}_X, 0)$. Likewise, if D is reduced, we write

$$\text{adj}_+^D(\mathcal{O}_X, \Delta) = \text{adj}_+^D(\omega_X(D), K_X + D + \Delta).$$

Note that we do not necessarily assume that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier, although that hypothesis will be convenient in many cases.

Remark 4.16. This notation conflicts slightly with the notation of [MST⁺22]. There, in the case where $X = \text{Spec } R$, the notation $\text{adj}_{R \rightarrow (R/I_D)^+}^D(R, D + \Delta)$ was used for the object we are calling $\text{adj}_+^D(\mathcal{O}_X, \Delta)$.

LEMMA 4.17. With X and Δ as above, suppose that $\mathcal{M} = \mathcal{O}_X(M)$ is a sufficiently ample line bundle. Then $\mathbf{B}^0(X, \Delta, \mathcal{M}) \subseteq H^0(X, \mathcal{M})$ generates $\tau_+(\mathcal{O}_X, \Delta) \otimes \mathcal{M}$.

Likewise, if D is a reduced divisor, then $\mathbf{B}_D^0(X, D + \Delta, \mathcal{M})$ generates $\text{adj}_+^D(\mathcal{O}_X, \Delta) \otimes \mathcal{M}$.

Proof. For the first statement, fix a Cartier divisor $H \geq 0$ such that $K_X + H \geq 0$, and choose L sufficiently ample (so that $L - H$ is also sufficiently ample). We know that

$$\tau_+(\mathcal{O}_X, \Delta) \otimes \mathcal{O}_X(L - H) = \tau_+(\omega_X, K_X + \Delta) \otimes \mathcal{O}_X(L - H) = \tau_+(\omega_X, K_X + \Delta + H) \otimes \mathcal{O}_X(L)$$

is globally generated by $\mathbf{B}^0(X, K_X + \Delta + H, \mathcal{O}_X(K_X + L))$.

Choose $M = L - H$; then by Lemma 3.3(c),

$$\mathbf{B}^0(X, K_X + \Delta + H, \mathcal{O}_X(K_X + L)) = \mathbf{B}^0(X, \Delta, \mathcal{O}_X(K_X + L - K_X - H)) = \mathbf{B}^0(X, \Delta, \mathcal{O}_X(M)).$$

The first result follows.

The second statement is similar. We have that

$$\text{adj}_+^D(\mathcal{O}_X, \Delta) \otimes \mathcal{O}_X(L - H) = \text{adj}_+^D(\omega_X(D), K_X + D + \Delta + H) \otimes \mathcal{O}_X(L)$$

is globally generated by $\mathbf{B}_D^0(X, K_X + 2D + \Delta + H, \mathcal{O}_X(K_X + D + L))$. But setting $M = L - H$, we have that

$$\mathbf{B}_D^0(X, K_X + 2D + \Delta + H, \mathcal{O}_X(K_X + D + L)) = \mathbf{B}_D^0(X, D + \Delta, \mathcal{O}_X(M)),$$

which completes the proof. $\square$

LEMMA 4.18. With notation as in Definition 4.15 and assuming that $\Delta \geq 0$, we have that $\tau_+(\mathcal{O}_X, \Delta)$ is an ideal sheaf. Likewise, $\text{adj}_+^D(\mathcal{O}_X, \Delta) \subseteq \mathcal{O}_X$.

Proof. We have that $\mathbf{B}^0(X, \Delta, \mathcal{M}) \subseteq H^0(X, \mathcal{M})$ globally generates $\tau_+(\mathcal{O}_X, \Delta) \otimes \mathcal{M} \subseteq \mathcal{M}$ by Lemma 4.17. In particular, $\tau_+(\mathcal{O}_X, \Delta) \subseteq \mathcal{O}_X$. The statement for $\text{adj}_+^D(\mathcal{O}_X, \Delta)$ is similar. $\square$

Remark 4.19 (The quasi-Gorenstein case). If X is quasi-Gorenstein (meaning that K_X is Cartier), then it follows from our definition that $\tau_+(\mathcal{O}_X, \Delta) = \tau_+(\omega_X, K_X + \Delta) = \tau_+(\omega_X, \Delta) \otimes \mathcal{O}_X(-K_X)$. Hence

$$\tau_+(\mathcal{O}_X, \Delta) \otimes \omega_X = \tau_+(\omega_X, \Delta).$$

We have the following containments.

LEMMA 4.20. *With notation as above, if $\Delta \geq 0$ is a $\mathbb{Q}$ -divisor and D is a reduced divisor, then*

$$\tau_+(\mathcal{O}_X, D + \Delta) \subseteq \text{adj}_+^D(\mathcal{O}_X, \Delta) \subseteq \tau_+(\mathcal{O}_X, (1 - \epsilon)D + \Delta)$$

for all $0 < \epsilon \leq 1$.

Proof. By Lemma 4.8(c), replacing B with $K_X + B$, we have that

$$\tau_+(\omega_X, K_X + D + B) \subseteq \text{adj}_+^D(\omega_X(D), K_X + D + B) \subseteq \tau_+(\omega_X, K_X + (1 - \epsilon)D + B).$$

By definition, this is

$$\tau_+(\mathcal{O}_X, D + B) \subseteq \text{adj}_+^D(\mathcal{O}_X, B) \subseteq \tau_+(\mathcal{O}_X, (1 - \epsilon)D + B). \quad \square$$

We point out the following transformation rules for finite maps.

THEOREM 4.21 (Transformation rules for finite maps). *With notation as above, suppose that $f: Y \rightarrow X$ is a finite dominant map between normal integral schemes. Then for any $B \geq 0$, we have that*

$$\text{Tr}(f_*\tau_+(\omega_Y, f^*B)) = \tau_+(\omega_X, B).$$

Furthermore, if $\Delta \geq 0$ and f is generically separable with ramification divisor Ram , then we have that

$$\text{Tr}(f_*\tau_+(\mathcal{O}_Y, f^*\Delta - \text{Ram})) = \tau_+(\mathcal{O}_X, \Delta).$$

Proof. By the inverse limit description of $\mathbf{B}^0$ found in Lemma 3.8, we know that

$$\mathbf{B}^0(X, B, \omega_X \otimes \mathcal{L}) = \text{Tr}(\mathbf{B}^0(Y, f^*B, \omega_Y \otimes f^*\mathcal{L})).$$

Choosing $\mathcal{L}$ sufficiently ample so that $f^*\mathcal{L}$ is also sufficiently ample, we have, using Proposition 4.7, that

$$H^0(X, \tau_+(\omega_X, B) \otimes \mathcal{L}) = \text{Tr}(H^0(Y, \tau_+(\omega_Y, f^*B) \otimes f^*\mathcal{L})).$$

Now, replacing $\mathcal{L}$ with a power if necessary, we have that $(f_*\tau_+(\omega_Y, f^*B)) \otimes \mathcal{L}$ is also globally generated as a $\mathcal{O}_X$ -module. Its image under the composition

$$f_*(\tau_+(\omega_Y, f^*B) \otimes f^*\mathcal{L}) \hookrightarrow f_*(\omega_Y \otimes f^*\mathcal{L}) \xrightarrow{\text{Tr}} \omega_X \otimes \mathcal{L}$$

is thus globally generated. Furthermore, if $J \otimes \mathcal{L} \subseteq \omega_X \otimes \mathcal{L}$ is that image, we have a surjection of global sections (again replacing $\mathcal{L}$ with a power if necessary)

$$H^0(X, f_*(\tau_+(\omega_Y, f^*B) \otimes f^*\mathcal{L})) \rightarrow H^0(X, J \otimes \mathcal{L}).$$

But the image of those global sections is $\mathbf{B}^0(X, B, \omega_X \otimes \mathcal{L}) = H^0(X, \tau_+(\omega_X, B) \otimes \mathcal{L})$. Thus $J \otimes \mathcal{L} = \tau_+(\omega_X, B) \otimes \mathcal{L}$, and the first statement is proven.

The second statement follows from the first via the techniques we have already used (or by mimicking the argument above), once one notes that $K_Y = f^*K_X + \text{Ram}$. $\square$

Étale extensions of the base also behave well.

PROPOSITION 4.22. *With notation as above, suppose that $H^0(X, \mathcal{O}_X) = R \subseteq S$ is a finite étale extension. Then*

$$\tau_+(\omega_X, B) \otimes_R S = \tau_+(\omega_{X_S}, B_S),$$

where X_S, B_S denote the base change.

Proof. This is a direct consequence of Proposition 3.10. $\square$

THEOREM 4.23 (Adjunction and inversion thereof). *With X as above, suppose that $K_X + D + \Delta$ is $\mathbb{Q}$ -Cartier, where D is reduced, $\Delta \geq 0$, and D and Δ have no common components. Then*

$$\mathrm{adj}_+^D(\mathcal{O}_X, \Delta) \cdot \mathcal{O}_{D^N} = \tau_+(\mathcal{O}_{D^N}, \mathrm{Diff}_{D^N}(D + \Delta)).$$

Proof. Choose $L = K_X + M$ with $\mathcal{L} = \mathcal{O}_X(L)$ sufficiently ample on X whose restriction is also sufficiently ample on D^N . We have a surjection from Theorem 3.11:

$$\mathbf{B}_D^0(X, D + \Delta, \mathcal{L}) \rightarrow \mathbf{B}^0(D^N, \mathrm{Diff}_{D^N}(D + \Delta), \mathcal{L}|_{D^N}).$$

By Lemma 4.17, the left side generates $\mathrm{adj}_+^D(\mathcal{O}_X, \Delta) \otimes \mathcal{L}$ as a $\mathcal{O}_X$ -module, and the right side generates $\tau_+(\mathcal{O}_{D^N}, \mathrm{Diff}_{D^N}(D + \Delta)) \otimes \mathcal{L}|_{D^N}$ as a $\mathcal{O}_{D^N}$ -module. In particular, the image of $\mathrm{adj}_+^D(\mathcal{O}_X, \Delta)$ under the map $\mathcal{O}_X \rightarrow \mathcal{O}_{D^N}$ generates the ideal $\tau_+(\mathcal{O}_{D^N}, \mathrm{Diff}_{D^N}(D + \Delta))$. This is what we wanted to show. $\square$

4.5 Test ideals on the nonsingular locus

PROPOSITION 4.24. *With (X, B) as above, let $U \subseteq X$ denote the locus where X is regular, B is simple normal crossings, and $\lfloor B \rfloor = 0$. Then*

$$\tau_+(\omega_X, B)|_U = \omega_U \quad \text{and} \quad \tau_+(\mathcal{O}_X, B)|_U = \mathcal{O}_U.$$

Proof. Choose a (not necessarily closed) point $x \in U$. We proceed by induction on the codimension/height of x as a point of X . If x is the generic point of X , there is nothing to prove since these sheaves are nonzero by Lemma 4.9.

Since x is nonsingular, we may pick a K_X that does not contain x in its support. Choosing a Weil divisor $E \geq 0$ also not containing x in its support so that $K_X + E \geq 0$ is Cartier, we have that

$$\tau_+(\omega_X, B) \supseteq \tau_+(\omega_X, B + K_X + E) = \tau_+(\mathcal{O}_X, B + E) \subseteq \tau_+(\mathcal{O}_X, B).$$

It then suffices to show that $\tau_+(\mathcal{O}_X, B + E)$ agrees with $\mathcal{O}_X$ at x to show both statements. We then replace B with $B + E$ and forget about E (note that $\lfloor B \rfloor$ may no longer equal zero, but it is zero in a neighborhood of x).

Choose a prime divisor D passing through x that is nonsingular at x so that $B \vee D$ is simple normal crossings at x . There are two ways to do this: If $x \notin \mathrm{Supp} B$, then choose D arbitrarily but nonsingular at x . Otherwise, choose D to be one of the components of $\mathrm{Supp} B$ at x . Either way, write $B \vee D = B' + D$, where B' does not have D in its support. Finally, choose a $\mathbb{Q}$ -divisor $B'' \geq 0$ not containing x in its support so that $K_X + D + B' + B''$ is $\mathbb{Q}$ -Cartier. Since

$$(1 - \epsilon)D + B' + B'' \geq B$$

for $\epsilon \gg 0$, we have that

$$\mathrm{adj}_+^D(\mathcal{O}_X, B' + B'') \subseteq \tau_+(\mathcal{O}_X, (1 - \epsilon)D + B' + B'') \subseteq \tau_+(\mathcal{O}_X, B),$$

where the first containment is Lemma 4.20. Thus it suffices to show that $\mathrm{adj}_+^D(\mathcal{O}_X, B' + B'')_x = \mathcal{O}_{X,x}$.

Now, by Theorem 4.23,

$$\mathrm{adj}_+^D(\mathcal{O}_X, B' + B'') \cdot \mathcal{O}_{D^N} = \tau_+(\mathcal{O}_{D^N}, \mathrm{Diff}_{D^N}(D + B' + B'')),$$

so it suffices to show the theorem for the right side since D is normal at x . Since D is regular at x and $\mathrm{Diff}_{D^N}(D + B' + B'')$ is simple normal crossings at x , we are done by induction. $\square$

4.6 A brief comparison with characteristic $p > 0$

Suppose briefly that R is an F -finite field¹ of characteristic $p > 0$ and $X \rightarrow \operatorname{Spec} R$ is projective, where X is normal and integral. Everything we have defined so far still makes sense. Indeed, in the case where B is $\mathbb{Q}$ -Cartier and M is Cartier, we have that $\mathbf{B}^0(X, \Delta, \mathcal{O}_X(K_X + M))$ agrees with $T^0(X, M - \Delta)$ as defined in [BST15, Section 6]. In that case, we find that

$$\mathbf{B}^0(X, B, \mathcal{O}_X(K_X + M)) = \operatorname{Image}(H^0(Y, \mathcal{O}_Y(K_Y - f^*B + f^*M)) \rightarrow H^0(X, \mathcal{O}_X(K_X + M)))$$

for a sufficiently large finite map or alteration $f: Y \rightarrow X$. This stabilization does not occur in mixed characteristic; see [BMP⁺23, Example 4.14]. On the other hand, for such a sufficiently large finite map, we also have that

$$\tau(\omega_X, B) = \operatorname{Image}(f_*\mathcal{O}_Y(K_Y - f^*B) \rightarrow \omega_X),$$

where $\tau(\omega_X, B)$ is the test submodule; see for instance [BST15, Definition 2.33 and the proof of Theorem 4.6].

THEOREM 4.25. *With notation as above, if B is $\mathbb{Q}$ -Cartier, we have that*

$$\tau_+(\omega_X, B) = \tau(\omega_X, B).$$

Hence, if $\Delta \geq 0$ is a $\mathbb{Q}$ -divisor such that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier, then we have that

$$\tau_+(\mathcal{O}_X, \Delta) = \tau(\mathcal{O}_X, \Delta).$$

Proof. Choose $g: X' \rightarrow X$ with g^*B Cartier. Since $\operatorname{Tr}(g_*\tau_+(\omega_{X'}, g^*B)) = \tau_+(\omega_X, B)$ by Theorem 4.21 and $\operatorname{Tr}(g_*\tau(\omega_{X'}, g^*B)) = \tau(\omega_X, B)$ by [BST15, Proposition 4.4], we may replace X with X' and so assume that B is Cartier. But now

$$\tau_+(\omega_X, B) = \tau_+(\omega_X) \otimes \mathcal{O}_X(-B) \quad \text{and} \quad \tau(\omega_X, B) = \tau(\omega_X) \otimes \mathcal{O}_X(-B),$$

and so we may assume that $B = 0$.

Let $F: X \rightarrow X$ be the absolute Frobenius. Then $\operatorname{Tr}_F(\tau_+(\omega_X)) = \tau_+(\omega_X)$. But $\tau(\omega_X)$ is the smallest nonzero module with this property, so we have that $\tau(\omega_X) \subseteq \tau_+(\omega_X)$.

For the reverse inclusion, notice that for a sufficiently large $f: Y \rightarrow X$ with

$$\tau(\omega_X) = \operatorname{Image}(f_*\omega_Y \rightarrow \omega_X),$$

we have that

$$\mathbf{B}^0(X, B, \mathcal{O}_X(K_X + M)) = \operatorname{Image}(H^0(Y, \mathcal{O}_Y(K_Y + f^*M)) \rightarrow H^0(X, \mathcal{O}_X(K_X + M)))$$

is contained in $H^0(X, \tau(\omega_X) \otimes M)$. Hence, if we choose M ample enough so that $\mathbf{B}^0(X, 0, \mathcal{O}_X(K_X + M))$ globally generates $\tau_+(\omega_X) \otimes \mathcal{O}_X(M)$, then we obtain that $\tau_+(\omega_X) \subseteq \tau(\omega_X)$. This finishes the proof of the first statement.

The second statement follows from the first since we have that $\tau_+(\mathcal{O}_X, \Delta) = \tau_+(\omega_X, K_X + \Delta)$ and $\tau(\mathcal{O}_X, \Delta) = \tau(\omega_X, K_X + \Delta)$. $\square$

5. Passing to affine charts and localization

Our goal in this section is to show that τ_+ is actually well defined affine locally. First we point out a transformation rule for birational maps. The same containment also holds in characteristic $p > 0$.

¹We expect that everything works for more general complete local Noetherian bases, but we restrict to this case here to keep the arguments short.

PROPOSITION 5.1. *Suppose that X is normal and integral, projective over a Noetherian complete local ring $(R, \mathfrak{m})$ of mixed characteristic. Suppose additionally that B is $\mathbb{Q}$ -Cartier on X and that $\pi: Y \rightarrow X$ is any projective birational map. Then*

$$\tau_+(\omega_X, B) \subseteq \pi_*\tau_+(\omega_Y, \pi^*B).$$

Proof. Fix $\mathcal{L} = \mathcal{O}_X(L)$ sufficiently ample on X . Then we know that

$$\mathbf{B}^0(X, B, \mathcal{O}_X(K_X + L)) = \mathbf{B}^0(Y, \pi^*B, \mathcal{O}_Y(K_Y + \pi^*L))$$

by Proposition 3.9. Now, $\mathbf{B}^0(Y, \pi^*B, \mathcal{O}_Y(K_Y + \pi^*L))$ generates a $\mathcal{O}_Y$ -subsheaf $J \otimes \pi^*\mathcal{L} \subseteq \mathcal{O}_Y(K_Y + \pi^*L)$. Note that $\pi_*J \supseteq \tau_+(\omega_X, B)$ since the global generators of the $\mathcal{O}_Y$ -sheaf $J \otimes \pi^*\mathcal{L}$ also generate the $\mathcal{O}_X$ -sheaf $\tau_+(\omega_X, B) \otimes \mathcal{L}$.

If we twist by a sufficiently ample $\mathcal{M} = \mathcal{O}_Y(M)$ on Y , then since we have the multiplication map of Lemma 4.2,

$$\mathbf{B}^0(Y, \pi^*B, \mathcal{O}_Y(K_Y + \pi^*L)) \otimes H^0(Y, \mathcal{O}_Y(M)) \rightarrow \mathbf{B}^0(Y, \pi^*B, \mathcal{O}_Y(K_Y + \pi^*L + M)),$$

we see that $J \otimes \mathcal{O}_Y(\pi^*L + M) \subseteq \tau_+(\omega_Y, \pi^*B) \otimes \mathcal{O}_Y(\pi^*L + M)$ and so $J \subseteq \tau_+(\omega_Y, \pi^*B)$. Hence we have that $\tau_+(\omega_X, B) \subseteq \pi_*J \subseteq \pi_*\tau_+(\omega_Y, \pi^*B)$, as desired. $\square$

Remark 5.2. Note that we cannot expect equality here. Indeed, [CMP⁺21, Example 4.14] provides a counterexample. Explicitly, the singularity $R = W(k)[[x, y]]/(p^2 + y^3 + z^5)$, where k is an algebraically closed field of characteristic $p > 0$, is a rational Gorenstein singularity (see [Lip69, Proposition 8.1]), and so if one resolves the singularity $\pi: Y \rightarrow X = \text{Spec } R$, we see that $\tau_+(\omega_Y) = \omega_Y$ and, since R has rational singularities, $\pi_*\omega_Y = \omega_X$. On the other hand, by [CMP⁺21, Example 4.14], we see that R is not a splinter, and it follows that $\tau_+(\omega_R) \subsetneq \omega_R$.

THEOREM 5.3. *Suppose that (X, B) is a pair as above, where B is $\mathbb{Q}$ -Cartier and $U \subseteq X$ is an open set. Now suppose that $g: X' \rightarrow X$ is a projective birational map that is an isomorphism over U with $U' = g^{-1}(U)$. Then we have that*

$$\tau_+(\omega_{X'}, g^*B)|_{U'} = \tau_+(\omega_X, B)|_U.$$

Proof. Fix a sufficiently ample line bundle $\mathcal{A} = \mathcal{O}_X(A)$ on X . Notice that if $\{U_i\}$ is an open cover of U , and we prove the result for each U_i , then we have also proven it for U . Indeed, given two subsheaves of ω_U , if they agree on an open cover, then they are equal. Hence, when working at a particular point, we may replace U with a smaller open neighborhood of that point whenever it is helpful. In particular, we may assume that U is affine and that U is the complement of some ample divisor $H \sim A$ (possibly replacing A by a multiple).

CLAIM 5.4. *Shrinking U if necessary and making $\mathcal{A}$ more ample if necessary, we may assume that there exists an effective Cartier divisor E on X' such that $E \cap U' = \emptyset$ and $\mathcal{A}' = g^*\mathcal{A}(-E) = \mathcal{O}_{X'}(g^*A - E)$ is ample.*

Proof of Claim 5.4. We know that g is the blowup of some ideal sheaf $\mathcal{J}$; see [Har77, Theorem II.7.17]. Fix a point $x \in U$. If $\mathcal{J}$ does not vanish at x , we are done since we can write $\mathcal{J} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-E)$ (replacing $\mathcal{A}$ with a power if necessary). If it does vanish at x , we must replace $\mathcal{J}$ with an ideal sheaf, with the same blowup, that does not vanish at x .

We know that $\mathcal{J}$ is locally principal at x since its blowup is an isomorphism at x . We may assume that $\mathcal{J} \otimes \mathcal{A}$ is globally generated (replacing $\mathcal{A}$ with a power). Choose an element $s \in H^0(X, \mathcal{J} \otimes \mathcal{A}) \subseteq H^0(X, \mathcal{A})$ that generates $\mathcal{J} \otimes \mathcal{A}$ at x . Then if $D \sim A$ is the Cartier divisor corresponding to $s \in H^0(X, \mathcal{A})$, the fractional ideal $\mathcal{J} \otimes \mathcal{O}_X(D)$ agrees with $\mathcal{O}_X$ in

a neighborhood of x . Shrinking U if necessary (which implicitly increases $\mathcal{A}$ since U is the complement of some divisor H linearly equivalent to A), we can assume that $(\mathcal{J} \otimes \mathcal{O}_X(D))|_U = \mathcal{O}_U$. It follows that $\mathcal{J} \otimes \mathcal{O}_X(D - mH)$ is contained in $\mathcal{O}_X$ for $m \gg 0$, it agrees with $\mathcal{O}_X$ at x , and it has the same blowup as $\mathcal{J}$. This proves the claim. $\square$

There is an effective Cartier divisor F on X such that $F \cap U = \emptyset$ and $g^*F \geq E$. Now take $F = mH$ for some $m \gg 0$.

Since $\tau_+(\omega_X, B + kF) = \tau_+(\omega_X, B) \otimes \mathcal{O}_X(-kF)$ by Lemma 4.8, then $\tau_+(\omega_X, B + kF)|_U \cong \tau_+(\omega_X, B)|_U$. Similarly, $\tau_+(\omega_{X'}, g^*(B + kF)) = \tau_+(\omega_{X'}, g^*B) \otimes \mathcal{O}_{X'}(-kg^*F)$ so that $\tau_+(\omega_{X'}, g^*(B + kF))|_{U'} \cong \tau_+(\omega_{X'}, g^*B)|_{U'}$. By Proposition 3.9 (see also [BMP⁺23, Lemma 4.19]), we have that

$$\mathbf{B}^0(X, B, \omega_X \otimes \mathcal{A}^i) = \mathbf{B}^0(X', g^*B, \omega_{X'} \otimes g^*\mathcal{A}^i) \supset \mathbf{B}^0(X', g^*B, \omega_{X'} \otimes \mathcal{A}'^i) \quad (5.1)$$

with $\mathcal{A}'$ as defined in Claim 5.4. The last inclusion is induced by the natural map $\mathcal{A}'^i \cong \mathcal{O}_{X'}(i(g^*A - E)) \subseteq \mathcal{O}_{X'}(ig^*A)$. Therefore, taking $i \gg 0$ we obtain that

$$\tau_+(\omega_{X'}, g^*B)|_{U'} \subseteq \tau_+(\omega_X, B)|_U.$$

The reverse inclusion follows from Proposition 5.1. $\square$

LEMMA 5.5. *Suppose that X is a normal projective integral scheme over a Noetherian ring R and $U \subseteq X$ is an affine open set, the complement of an ample effective Cartier divisor H . Suppose that $B_U \geq 0$ is a $\mathbb{Q}$ -trivial divisor. Then there exists a $\mathbb{Q}$ -Cartier divisor B on X such that $B|_U = B_U$.*

Remark. That B_U is a $\mathbb{Q}$ -trivial divisor means that $nB_U \sim 0$ for some $n > 0$. This is not a particularly restrictive condition on an affine open set since one may always restrict to an affine open cover trivializing any $\mathbb{Q}$ -Cartier divisor.

Proof. Write $D_U = nB_U = \text{div}(f)$ for some $f \in \Gamma(U, \mathcal{O}_U)$. Then viewing $f \in K(X)$, it defines a rational divisor D' on X with $D'|_U = nB_U$. However, D' is possibly not effective but $D' + mH$ is effective for some $m > 0$, and so we set $B = \frac{1}{n}(D' + mH)$. $\square$

DEFINITION 5.6. Suppose that $(R, \mathfrak{m})$ is a complete local ring with positive-characteristic residue field. Suppose further that T is a finite-type R -algebra. Let $U = \text{Spec } T$, and suppose that $B_U \geq 0$ is a $\mathbb{Q}$ -trivial divisor on U . Take a projective scheme X over $\text{Spec } R$, where $\text{Spec } T = U = X \setminus V(H)$ is an affine open subset, the complement of a very ample divisor. Suppose that $B \geq 0$ is a $\mathbb{Q}$ -Cartier-divisor on X such that $B|_U = B_U$.

Then define

$$\tau_+(\omega_U, B_U) = \tau_+(\omega_X, B)|_U.$$

We show that this definition is independent of the choices of X and B .

PROPOSITION 5.7. *The sheaf $\tau_+(\omega_U, B_U)$ is independent of the compactifications X and B .*

Proof. First suppose that B_1, B_2 are two effective $\mathbb{Q}$ -Cartier divisors on X that restrict to B_U on U . Let $\pi: Y \rightarrow X$ be a finite cover from a normal integral scheme such that both π^*B_1 and π^*B_2 are Cartier. Let $V = \pi^{-1}(U)$. Since $\text{Tr}: \pi_*\omega_V \rightarrow \omega_X$ satisfies $\text{Tr}(\tau_+(\omega_Y, \pi^*B_i)) = \tau_+(\omega_X, B_i)$ for $i = 1, 2$ by Theorem 4.21, it suffices to prove the result on $V \subseteq Y$. In particular, replacing X with Y , we may assume that B_U, B_1, B_2 are all Cartier. Furthermore, we may assume that $B_2 \geq B_1$ with $B_2 = B_1 + G$, where $G|_U = 0$. Then we have that

$$\tau_+(\omega_X, B_2)|_U = (\tau_+(\omega_X, B_1) \otimes \mathcal{O}_X(-G))|_U = \tau_+(\omega_X, B_1)|_U,$$

and we have thus shown that our definition is independent of the choice of B .

Now suppose that X_1, X_2 are two compactifications of U . Passing to finite covers as above, we may assume that B_U, B_1, B_2 are all Cartier divisors. Since $\tau_+(\omega_{X_i}, B_i) = \tau_+(\omega_{X_i}) \otimes \mathcal{O}_X(-B_i)$, it suffices to show that $\tau_+(\omega_{X_1})|_U = \tau_+(\omega_{X_2})|_U$. We then have that $X_1 \dashrightarrow X_2$ is a rational map between projective schemes over a Noetherian local ring R . We resolve the indeterminacies of this rational map. That is, we have

$$\begin{array}{ccc} & X_3 & \\ \swarrow & & \searrow \\ X_1 & \xrightarrow[\mu]{} & X_2, \end{array}$$

where both diagonal arrows are blowups and are isomorphisms over U . But now we are done by Theorem 5.3. $\square$

COROLLARY 5.8. *Suppose that X is a quasi-projective normal integral scheme over a complete local ring $(R, \mathfrak{m})$. Further suppose that $B \geq 0$ on X is a $\mathbb{Q}$ -Cartier divisor. Then there exists a well-defined $\tau_+(\omega_X, B)$ which restricts to $\tau_+(\omega_{X'}, B|_{X'})$ for any open $X' \subseteq X$. If additionally X is projective, note that this agrees with τ_+ as defined in Notation 4.6. In particular, if $X = \operatorname{Spec} T$ is affine, then we can define $\tau_+(\omega_X, B)$ commuting with localization at a single element.*

Proof. Since $\tau_+(\omega_X, B)$ is a subsheaf of ω_X , it suffices to check this on a sufficiently fine affine cover. But now the result follows from Proposition 5.7. $\square$

LEMMA 5.9. *Suppose that X is quasi-projective over a complete local ring $(R, \mathfrak{m})$ with residual characteristic $p > 0$ and $B \geq 0$. Suppose further that $G \geq 0$ is a Cartier divisor. Then*

$$\tau_+(\omega_X, B + G) = \tau_+(\omega_X, B) \otimes \mathcal{O}_X(-G).$$

Proof. This may be checked locally, and so we may assume that G is a principal divisor ($G \sim 0$). Thus we may compactify to $\overline{X}, \overline{B}$, and $\overline{G}$, where $\overline{B}$ is $\mathbb{Q}$ -Cartier and $\overline{G}$ is Cartier. Now the result follows from Lemma 4.8(b). $\square$

DEFINITION 5.10. Suppose that X is quasi-projective over a complete local ring $(R, \mathfrak{m})$. Suppose that $\Delta \geq 0$ is a $\mathbb{Q}$ -divisor such that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier. Then we define

$$\tau_+(\mathcal{O}_X, \Delta) = \tau_+(\omega_X, K_X + \Delta + G) \otimes \mathcal{O}_X(G),$$

where G is a Cartier divisor such that $K_X + \Delta + G \geq 0$. It follows from Lemma 5.9 that this is independent of the choice of G in the definition.

Remark 5.11. Suppose that T is a normal domain essentially of finite type over a complete local ring $(R, \mathfrak{m})$ with residue field of characteristic $p > 0$. Then for any $\mathbb{Q}$ -Cartier divisor $B \geq 0$ on $\operatorname{Spec} T$, we can define $\tau_+(\omega_T, B)$, whose formation commutes with localization. The point is that we can unlocalize to a finite-type R -algebra domain, where B is still $\mathbb{Q}$ -Cartier. Likewise, we can define $\tau_+(T, \Delta)$, where $K_T + \Delta$ is $\mathbb{Q}$ -Cartier.

Remark 5.12. Suppose for instance that $X = \operatorname{Spec} R$. Then $\tau_+(\mathcal{O}_X, B)$ is nothing other than the test ideal $\tau_{R^+}(R, B)$ as defined in [MS21]. We then obtain a theory of localization, but it is nothing more than localizing this ideal $\tau_{R^+}(R, B)$.

6. Effective global generation of τ_+ and a Skoda-type theorem

Fix a normal integral scheme X , projective over a complete local Noetherian domain $(R, \mathfrak{m})$. Suppose that $H^0(X, \mathcal{O}_X) = R$. Suppose that $X_{\mathfrak{m}}$ is the special fiber, and set $n_0 = \dim X_{\mathfrak{m}}$.

THEOREM 6.1 (Effective global generation). *With notation as above, let $\mathcal{A} = \mathcal{O}_X(A)$ be a globally generated ample line bundle, $\Gamma \geq 0$ be a $\mathbb{Q}$ -divisor, and L be a divisor such that $L - \Gamma$ is big and semi-ample. Then*

$$\tau_+(\omega_X, \Gamma) \otimes \mathcal{O}_X(nA + L)$$

is globally generated by $\mathbf{B}^0(X, \Gamma, \omega_X \otimes \mathcal{O}_X(nA + L))$ for all $n \geq n_0 = \dim X_{\mathfrak{m}}$.

Our proof is inspired by similar arguments in characteristic $p > 0$ found in [ST14], which themselves are based upon arguments for multiplier ideals found in [Laz04]; see also [EL99, Section 1].

Proof. Set $R = H^0(X, \mathcal{O}_X)$. If the residue field $k = R/\mathfrak{m}$ is infinite, we may fix global generators $y_1, \dots, y_{n+1} \in H^0(X, \mathcal{A})$, where $n \geq n_0$ and n_0 is the dimension of the fiber. The point is that sections in $H^0(X, \mathcal{A})$ globally generate $\mathcal{A}$ if and only if their images globally generate $\mathcal{A}|_{X_{\mathfrak{m}}}$, which is projective of dimension n over an infinite field.

However, if the residue field k is finite, then even though $\mathcal{A}$ is globally generated, it is not obvious that it can be globally generated by n_0 elements. But, if k is finite, then for any $n \geq n_0$, we can find $n + 1$ generating sections $y_1, \dots, y_{n+1}$ in $H^0(X, \mathcal{A}) \otimes_R S = H^0(X_S, \mathcal{A}_S)$ for some finite étale extension $R \subseteq S$ (inducing a finite separable extension of k). Since global generation of a sheaf is unaffected by faithfully flat base change, by using Propositions 3.10 and 4.22, we may replace R with S , X with $X_S = X \times_{\text{Spec } R} \text{Spec } S$, etc., and so either way we have our desired global generators y_i .

Consider the Skoda/Koszul complex determined by the y_i :

$$0 \rightarrow \mathcal{F}_{n+1} \rightarrow \mathcal{F}_n \rightarrow \dots \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_0 \rightarrow 0,$$

where $\mathcal{F}_i = \mathcal{O}_X(-iA)^{\oplus \binom{n+1}{i}}$. Since A is a Cartier divisor, this complex is exact; see [BH93, Proposition 1.6.5(c)]. Let X^+ denote the absolute integral closure of X as in Section 2.2, and use $\rho: X^+ \rightarrow X$ to denote the canonical map. We pull back our Skoda/Koszul sequence to X^+ . Twisting by the line bundle $\mathcal{O}_{X^+}(\rho^*((n+1)A + L - \Gamma))$ and applying the functor $\mathcal{H}om_{\mathcal{O}_{X^+}}(\bullet, \mathcal{O}_{X^+})$, we obtain

$$0 \leftarrow \mathcal{G}_{n+1} \leftarrow \mathcal{G}_n \leftarrow \dots \leftarrow \mathcal{G}_1 \leftarrow \mathcal{G}_0 \leftarrow 0,$$

where $\mathcal{G}_i = \mathcal{O}_{X^+}(\rho^*(-(n+1-i)A - L + \Gamma))^{\oplus \binom{n+1}{i}}$. This is still exact since our original complex can be viewed as a locally free resolution of a locally free sheaf. We can also do the same thing on X , twisting by $(n+1)A + L$, dualizing, and setting $\mathcal{K}_i = \mathcal{O}_X(-(n+1-i)A - L)$. Notice that since $\Gamma \geq 0$, we have canonical maps $\mathcal{K}_i \rightarrow \rho_* \mathcal{G}_i$ for each i .

Finally, if $d = \dim X$, taking top local cohomology yields a diagram

$$\begin{array}{ccccccc} 0 & \longleftarrow & H_{\mathfrak{m}}^d(\mathcal{G}_{n+1}) & \longleftarrow & \dots & \longleftarrow & H_{\mathfrak{m}}^d(\mathcal{G}_2) & \longleftarrow & H_{\mathfrak{m}}^d(\mathcal{G}_1) & \xleftarrow{\alpha} & H_{\mathfrak{m}}^d(\mathcal{G}_0) & \longleftarrow & 0 \\ & & \uparrow & & & & \uparrow & & \uparrow_{j_1} & & \uparrow_{j_0} & & \\ & & H_{\mathfrak{m}}^d(\mathcal{K}_{n+1}) & \longleftarrow & \dots & \longleftarrow & H_{\mathfrak{m}}^d(\mathcal{K}_2) & \longleftarrow & H_{\mathfrak{m}}^d(\mathcal{K}_1) & \longleftarrow & H_{\mathfrak{m}}^d(\mathcal{K}_0) & & \end{array}$$

Here we are abusing and condensing notation and writing $H_{\mathfrak{m}}^d(\mathcal{G}_i)$ instead of $H_{\mathfrak{m}}^d(R\Gamma(X, \rho_* \mathcal{G}_i))$. The lower local cohomologies $H_{\mathfrak{m}}^i(\mathcal{G}_j)$ for $i < d$ are all zero by Bhatt's vanishing theorem (Theorem 2.2), see [Bha20, BMP⁺23], so the top row is in fact exact, and in particular the map

labeled α is injective. The bottom row is not necessarily exact though. We now provide a less condensed view of the square involving α , j_0 , and j_1 :

$$\begin{array}{ccc} \bigoplus_{n+1} H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X^+, \mathcal{O}_{X^+}(\rho^*(-nA - L + \Gamma)))) & \xleftarrow{\alpha} & H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X^+, \mathcal{O}_{X^+}(\rho^*(-(n+1)A - L + \Gamma)))) \\ \uparrow j_1 & & \uparrow j_0 \\ \bigoplus_{n+1} H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X, \mathcal{O}_X(-nA - L))) & \xleftarrow{\quad} & H_{\mathfrak{m}}^d(\mathbf{R}\Gamma(X, \mathcal{O}_X(-(n+1)A - L))). \end{array}$$

Applying Matlis duality $\bullet^\vee = \text{Hom}(\bullet, E)$, where $E = E_{R/\mathfrak{m}}$ is the injective hull of the residue field of R , we see that

$$\bigoplus_{n+1} \mathbf{B}^0(X, \Gamma, \mathcal{O}_X(K_X + nA + L)) = \text{Image}(j_1)^\vee \rightarrow \text{Image}(j_0)^\vee = \mathbf{B}^0(X, \Gamma, \mathcal{O}_X(K_X + (n+1)A + L))$$

surjects. In other words, the multiplication map

$$\mathbf{B}^0(X, \Gamma, \mathcal{O}_X(K_X + nA + L)) \otimes H^0(X, \mathcal{O}_X(A)) \rightarrow \mathbf{B}^0(X, \Gamma, \mathcal{O}_X(K_X + (n+1)A + L))$$

surjects. In fact, running through the same argument, replacing $n+1$ with $n+t$ and composing, we see that

$$\mathbf{B}^0(X, \Gamma, \mathcal{O}_X(K_X + nA + L)) \otimes H^0(X, \mathcal{O}_X(tA)) \rightarrow \mathbf{B}^0(X, \Gamma, \mathcal{O}_X(K_X + (n+t)A + L))$$

also surjects. Now $\mathbf{B}^0(X, \Gamma, \mathcal{O}_X(K_X + (n+t)A + L))$ globally generates $\tau_+(\omega_X, \Gamma) \otimes \mathcal{O}_X((n+t)A + L)$ for $t \gg 0$ by the second part of Proposition 4.5. This completes the proof. $\square$

We use this to deduce the following statement, which more closely matches the existing literature in characteristic $p > 0$.

COROLLARY 6.2. *With notation as above, suppose that $\Delta \geq 0$ and that M is a Cartier divisor such that $M - K_X - \Delta$ is big and semi-ample. Additionally, suppose that A is globally generated and ample. Then if n_0 is the dimension of the fiber $X_{\mathfrak{m}}$, we have for any $n \geq n_0$ that*

$$\tau_+(\mathcal{O}_X, \Delta) \otimes \mathcal{O}_X(nA + M)$$

is globally generated by $\mathbf{B}^0(X, \Delta, \mathcal{O}_X(nA + M))$.

Proof. Choose a Cartier divisor $D \geq 0$ such that $K_X + D \geq 0$. Then we have that

$$\tau_+(\omega_X, \Delta + K_X + D) = \tau_+(\mathcal{O}_X, \Delta + D) = \tau_+(\mathcal{O}_X, \Delta) \otimes \mathcal{O}_X(-D).$$

Set $L = M + D$, and we observe that $L - \Delta - K_X - D = M + D - \Delta - K_X - D = M - K_X - \Delta$ is big and semi-ample. In particular, we have that

$$\tau_+(\omega_X, \Delta + K_X + D) \otimes \mathcal{O}_X(nA + L) = \tau_+(\mathcal{O}_X, \Delta) \otimes \mathcal{O}_X(nA + M)$$

is globally generated by

$$\mathbf{B}^0(X, \Delta + K_X + D, \omega_X \otimes \mathcal{O}_X(nA + L)) = \mathbf{B}^0(X, \Delta, \mathcal{O}_X(nA + M)),$$

which is what we wanted to show. $\square$

6.1 Skoda-type result

In the previous sections, we considered $+$ -test ideals of divisor pairs (X, Δ) . In characteristic zero and $p > 0$, it is also natural to consider pairs $(X, \mathfrak{a}^t)$, where $\mathfrak{a} \subseteq \mathcal{O}_X$ is an ideal sheaf and $t > 0$ is a real or rational number. One sometimes even considers triples $(X, \Delta, \mathfrak{a}^t)$. While we

believe that the work we have done in previous sections generalizes to this setting of triples, we do not pursue this. Instead, we prove a Skoda-type result in the case where $\Delta = 0$ for the object we define as $\tau_+(\omega_X, \mathfrak{a}^t)$.

Let X be a normal integral scheme projective over a complete Noetherian local ring $(R, \mathfrak{m})$. Suppose that $\mathfrak{a} \subseteq \mathcal{O}_X$ is an ideal. Let $\mu: X' \rightarrow X$ be the normalization of the blowup of $\mathfrak{a}$ so that $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$, where $F \geq 0$ is a divisor. In particular, $\bar{\mathfrak{a}} = \mu_* \mathcal{O}_{X'}(-F)$ is the integral closure of $\mathfrak{a}$ (see [Laz04, Section 9.6.A]). For any rational number $t > 0$, we define the *$\mathfrak{a}$ -test submodule associated with $\mathfrak{a}^t$* , denoted by $\tau_+(\omega_X, \mathfrak{a}^t)$, to be the subsheaf of ω_X such that

$$\tau_+(\omega_X, \mathfrak{a}^t) \otimes \mathcal{L} \subseteq \omega_X \otimes \mathcal{L}$$

is globally generated by

$$\begin{aligned} \mathbf{B}^0(X', tF, \mathcal{O}_{X'}(K_{X'} + L')) &= \mathbf{B}^0(X', \{tF\}, \mathcal{O}_{X'}(K_{X'} - \lfloor tF \rfloor + L')) \\ &\subseteq H^0(X', \mathcal{O}_{X'}(K_{X'} + L')) \\ &\subseteq H^0(X, \mathcal{O}_X(K_X + L)), \end{aligned}$$

where $\mathcal{L} = \mathcal{O}_X(L)$ is sufficiently ample, $\mathcal{L}' = \mu^* \mathcal{L}$, and $L' = \mu^* L$. Arguing as in previous sections, it is straightforward to see that $\tau_+(\omega_X, \mathfrak{a}^t)$ is independent of the choice of the sufficiently ample line bundle $\mathcal{L}$.

Next, suppose that $\mathcal{A} = \mathcal{O}_X(A)$ is a sufficiently ample line bundle on X so that $\mathcal{A}$ and $\mathfrak{a} \otimes \mathcal{A}$ are globally generated. Then if $\mathcal{A}' = \pi^* \mathcal{A}$, we have that $\mathcal{A}'(-F) = \mathcal{O}_{X'}(A' - F)$ is globally generated. In some cases, we replace $\mathcal{L}$ with $\mathcal{L} \otimes \mathcal{A}^{\lfloor t \rfloor}$. This is harmless since we have simply made our line bundle even more ample.

Instead of setting X' to be the normalized blowup, by Proposition 3.9, we can set $X' \rightarrow X$ to be any projective birational map from a normal scheme X' dominating the normalized blowup since $\mathbf{B}^0$ is unaffected. Note that $\mathbf{B}^0(X', tF, \mathcal{O}_{X'}(K_{X'} + L')) \subseteq H^0(X', \mathcal{O}_{X'}(\lceil K_{X'} - tF + L' \rceil))$, and so we have that

$$\tau_+(\omega_X, \mathfrak{a}^t) \otimes \mathcal{L} \subseteq \mu_* \mathcal{O}_{X'}(\lceil K_{X'} - tF + L' \rceil)$$

since the right side is globally generated by a subspace of $H^0(X, \mathcal{O}_X(K_X + L))$ for L sufficiently ample. Hence, if for instance X' is a log resolution of $(X, \mathfrak{a})$, we have an inclusion to the corresponding multiplier submodule $\tau_+(\omega_X, \mathfrak{a}^t) \subseteq \mathcal{J}(\omega_X, \mathfrak{a}^t)$.

If X is $\mathbb{Q}$ -Gorenstein, one can likewise define the *$\mathfrak{a}$ -test ideal associated with $\mathfrak{a}^t$* , denoted by $\tau_+(\mathcal{O}_X, \mathfrak{a}^t)$, to be the ideal sheaf $\mathcal{J}$, so that $\mathcal{J} \otimes \mathcal{L}$ is globally generated by

$$\mathbf{B}^0(X', tF, \mathcal{O}_{X'}(K_{X'} - \mu^* K_X + L')) \subseteq H^0(X, \mathcal{O}_X(L))$$

for $\mathcal{L} = \mathcal{O}_X(L)$ sufficiently ample. If K_X is Cartier, we see directly that $\tau_+(\mathcal{O}_X, \mathfrak{a}^t) \otimes \omega_X = \tau_+(\omega_X, \mathfrak{a}^t)$. Arguing as above, we again obtain results of the form

$$\tau_+(\mathcal{O}_X, \mathfrak{a}^t) \subseteq \mathcal{J}(\mathcal{O}_X, \mathfrak{a}^t).$$

Remark 6.3 (Even more general definitions). In fact, given a $\mathbb{Q}$ -divisor $\Delta \geq 0$ such that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier, and given ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_m$ and rational numbers $t_1, \dots, t_m \geq 0$, we can define the *$\mathfrak{a}$ -test ideal associated with $(X, \Delta, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_m^{t_m})$* , denoted by

$$\tau_+(\mathcal{O}_X, \Delta, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_m^{t_m}),$$

to be the ideal sheaf $\mathcal{J}$ such that $\mathcal{J} \otimes \mathcal{L}$ is generated by

$$\mathbf{B}^0(X', t_1 F_1 + \cdots + t_m F_m, \mathcal{O}_{X'}(K_{X'} - \mu^*(K_X + \Delta) + L')) \subseteq H^0(X, \mathcal{L})$$

for $\mathcal{L}$ sufficiently ample, where $\mu: X' \rightarrow X$ is a projective birational map from a normal integral scheme X that factors through the blowups of the ideal sheaves $\mathfrak{a}_i$ and where $\mathfrak{a}_i \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F_i)$. We will not work in this generality, although we believe that the many results of this paper generalize to this setting.

Remark 6.4. This is not the only natural way to define $\tau_+(\omega_X, \mathfrak{a}^t)$. For instance, in [MS18] and [ST21], when $X = \operatorname{Spec} R$, the ideal $\tau_+(\mathfrak{a}^t)$ was essentially defined as the sum of the $\tau_+((t/m)\operatorname{div}(f))$ for $f \in \mathfrak{a}^m$. This is also roughly equivalent to how the generalized test ideals $\tau(\mathfrak{a}^t)$ of [HY03] are defined in characteristic $p > 0$. In fact, we use a variant of this approach below in Section 7. Our approach in this section instead is inspired by a characterization of test ideals from [ST14] which itself is inspired by the classical definition of multiplier ideals. We briefly compare these two constructions in our global setting.

Again set $\mathcal{A}$ such that $\mathfrak{a} \otimes \mathcal{A}$ is globally generated. Since $\mu^*((1/m)\Gamma) \geq F$ for any $\Gamma \in |\mathcal{A}^m \otimes \mathfrak{a}^m| \subseteq |\mathcal{A}^m|$, by Lemma 3.3 and Proposition 3.9, we see that $\mathbf{B}^0(X', tF, \mathcal{O}_{X'}(K_{X'} + L'))$ contains

$$\mathbf{B}^0\left(X', t \cdot \mu^*\left(\frac{1}{m}\Gamma\right), \mathcal{O}_{X'}(K_{X'} + L')\right) \cong \mathbf{B}^0\left(X, \frac{t}{m}\Gamma, \mathcal{O}_X(K_X + L)\right).$$

Putting all this together, one sees that we have a containment

$$\sum_{\Gamma \in |\mathcal{A}^m \otimes \mathfrak{a}^m|} \tau_+\left(\omega_X, \frac{t}{m} \cdot \Gamma\right) \subseteq \tau_+(\omega_X, \mathfrak{a}^t).$$

We do not know if we always have equality in the inclusion above even in the local case where $X = \operatorname{Spec} R$. It would also be natural to expect that we have equality without the sum for a single sufficiently general Γ .

One advantage of our new definition is that it is clear that $\tau_+(\omega_X, \mathfrak{a}^t) = \tau_+(\omega_X, \bar{\mathfrak{a}}^t) = \tau_+(\omega_X, \overline{\mathfrak{a}^t})$, where $\bar{\bullet}$ denotes the integral closure. This is not clear for the sum-style definition.

We also have the following.

LEMMA 6.5. *With notation as above, for any rational $t > 0$, we have a natural inclusion*

$$\mathfrak{a}^{[t]} \cdot \tau_+(\omega_X) \subseteq \tau_+(\omega_X, \mathfrak{a}^t).$$

Furthermore, if X is $\mathbb{Q}$ -Gorenstein, then

$$\mathfrak{a}^{[t]} \cdot \tau_+(\mathcal{O}_X) \subseteq \tau_+(\mathcal{O}_X, \mathfrak{a}^t).$$

Proof. Since $\tau_+(\omega_X, \mathfrak{a}^{[t]}) \subseteq \tau_+(\omega_X, \mathfrak{a}^t)$, we may replace t with $[t]$ and so assume that t is an integer. Fix $\mathcal{A}$ ample such that $\mathfrak{a} \otimes \mathcal{A}$ is globally generated, which implies that $\mathfrak{a}^t \otimes \mathcal{A}^t$ is also globally generated, and hence there is a surjection

$$H^0(X, \mathfrak{a}^t \otimes \mathcal{A}^t) \otimes \mathcal{A}^{-t} \rightarrow \mathfrak{a}^t \subseteq \mathcal{O}_X.$$

Thus $\mathfrak{a}^t = \sum_{\Gamma \in |\mathcal{A}^t \otimes \mathfrak{a}^t|} \mathcal{O}_X(-\Gamma)$. Arguing as in the remark above, we have that

$$\begin{aligned} \mathfrak{a}^t \cdot \tau_+(\omega_X) &= \tau_+(\omega_X) \cdot \left(\sum_{\Gamma \in |\mathcal{A}^t \otimes \mathfrak{a}^t|} \mathcal{O}_X(-\Gamma) \right) = \sum_{\Gamma \in |\mathcal{A}^t \otimes \mathfrak{a}^t|} \tau_+(\omega_X) \cdot \mathcal{O}_X(-\Gamma) \\ &= \sum_{\Gamma \in |\mathcal{A}^t \otimes \mathfrak{a}^t|} \tau_+(\omega_X, \Gamma) \subseteq \tau_+(\omega_X, \mathfrak{a}^t). \end{aligned}$$

The statement about $\mathcal{O}_X$ follows similarly. □

THEOREM 6.6. *Suppose that X is normal, integral, and projective over a complete local Noetherian domain $(R, \mathfrak{m})$, and suppose that $0 \neq \mathfrak{a}$ is an ideal sheaf on X . Let $\mu: X' \rightarrow X$ denote the normalized blowup of $\mathfrak{a}$, and let $n = \dim X'_\mathfrak{m}$. Then for any rational number $t \geq n + 1$, we have that $\tau_+(\omega_X, \mathfrak{a}^t) = \mathfrak{a} \cdot \tau_+(\omega_X, \mathfrak{a}^{t-1})$.*

Furthermore, if X is $\mathbb{Q}$ -Gorenstein, we have that $\tau_+(\mathcal{O}_X, \mathfrak{a}^t) = \mathfrak{a} \cdot \tau_+(\mathcal{O}_X, \mathfrak{a}^{t-1})$.

In the case where $X = \operatorname{Spec} R$, then $\dim X'_\mathfrak{m} \leq \dim X - 1$. Hence we recover the usual Skoda statement in the local case. As is typical, by iterating this result, if t is an integer, then we have that

$$\mathfrak{a}^{t-n} \cdot \tau_+(\omega_X, \mathfrak{a}^n) = \tau_+(\omega_X, \mathfrak{a}^t)$$

and in the $\mathbb{Q}$ -Gorenstein case that

$$\mathfrak{a}^{t-n} \cdot \tau_+(\mathcal{O}_X, \mathfrak{a}^n) = \tau_+(\mathcal{O}_X, \mathfrak{a}^t).$$

Proof. We first handle the case of $\tau_+(\omega_X, \dots)$ and then explain how to modify the proof for the $+$ -test ideal case. Write $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$ for some effective divisor F . We will assume that $\mathcal{A}$ and $\mathcal{A} \otimes \mathfrak{a}$ are globally generated and $\mathcal{L}$ is a sufficiently ample line bundle. The proof now mimics the proof of Theorem 6.1, but we use X'^+ instead of X^+ .

Note that $\mu^* H^0(X, \mathcal{A} \otimes \mathfrak{a}) \cong H^0(X', \mathcal{A}'(-F))$ generates $\mathcal{A}'(-F)$ and we may pick sections $s_0, \dots, s_n$ generating $\mathcal{A}'(-F)$ (this is because $\dim X'_\mathfrak{m} = n$). Notice that if the residue field is not infinite, we may need to take an étale base change of R as in the proof of Theorem 6.1, but this is not an obstruction due to Proposition 3.10. We then have an exact Skoda/Koszul complex on X'

$$0 \rightarrow \mathcal{F}_{n+1} \rightarrow \mathcal{F}_n \rightarrow \dots \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_0 \rightarrow 0,$$

where $\mathcal{F}_i = \wedge^i V \otimes \mathcal{O}_{X'}(-i(A' - F))$, V is the free $R = H^0(X, \mathcal{O}_X)$ -module on the s_i so that $\wedge^i V = R^{\oplus \binom{n+1}{i}}$, and $A' = \mu^* A$. Let L be a sufficiently ample line bundle on X , and set $L' = \mu^* L$. Let $\rho': X'^+ \rightarrow X'$ be the canonical map, and pull back this complex to X'^+ . It stays exact since we are pulling back a resolution of a locally free sheaf $\mathcal{F}_0$. We now twist by $\rho'^*((n+1)A' + L' - tF) = \rho'^*((n+1)(A' - F) + L' - (t-n-1)F)$, which is a line bundle on X' and not just a $\mathbb{Q}$ -line bundle. Furthermore, there exists a finite morphism $g': Y' \rightarrow X'$ where $g'^*(L' - (t-i)F)$ is in fact a big and semi-ample Cartier divisor for $i = 0, \dots, n+1$. Next, apply the functor $\mathcal{H}\operatorname{om}_{\mathcal{O}_{X'^+}}(\bullet, \mathcal{O}_{X'^+})$ to obtain an exact complex

$$0 \leftarrow \mathcal{G}_{n+1} \leftarrow \mathcal{G}_n \leftarrow \dots \leftarrow \mathcal{G}_1 \leftarrow \mathcal{G}_0 \leftarrow 0,$$

where

$$\begin{aligned} \mathcal{G}_i &= \mathcal{O}_{X'^+}(\rho'^*(-(n+1-i)(A' - F) - L' + (t-n-1)F)) \otimes \wedge^i V \\ &= \mathcal{O}_{X'^+}(\rho'^*(-(n+1-i)A' - L' + (t-i)F)) \otimes \wedge^i V. \end{aligned}$$

Taking local cohomology, we obtain the commutative diagram

$$\begin{array}{ccc} V \otimes H_\mathfrak{m}^d(\mathbf{R}\Gamma(X'^+, \mathcal{O}_{X'^+}(\rho'^*(-nA' - L' + (t-1)F)))) & \xhookrightarrow{\alpha} & H_\mathfrak{m}^d(\mathbf{R}\Gamma(X'^+, \mathcal{O}_{X'^+}(\rho'^*(-(n+1)A' - L' + tF)))) \\ \uparrow j_1 & & \uparrow j_0 \\ V \otimes H_\mathfrak{m}^d(\mathbf{R}\Gamma(X', \mathcal{O}_{X'}(-nA' - L'))) & \longleftarrow & H_\mathfrak{m}^d(\mathbf{R}\Gamma(X', \mathcal{O}_{X'}(-(n+1)A' - L'))) \end{array}$$

Here α is injective by Bhatt vanishing (Theorem 2.2) since A' and $g'^*(L' - (t-i)F)$ are big and semi-ample (note that L was arbitrarily ample).

Applying Matlis duality $\bullet^\vee = \text{Hom}(\bullet, E)$, where $E = E_{R/\mathfrak{m}}$ is the injective hull of the residue field of R , we see that

$$\begin{aligned} V \otimes \mathbf{B}^0(X', (t-1)F, \mathcal{O}_{X'}(K_{X'} + nA' + L')) &= \text{Image}(j_1)^\vee \\ &\rightarrow \text{Image}(j_0)^\vee = \mathbf{B}^0(X', tF, \mathcal{O}_{X'}(K_{X'} + (n+1)A' + L')) \end{aligned}$$

surjects. Pushing forward via $\mu: X' \rightarrow X$ (and remembering that the sections s_i are elements of $H^0(X, \mathcal{A} \otimes \mathfrak{a}) \subseteq H^0(X, \mathcal{A})$), we have a surjection

$$\tau_+(\omega_X, \mathfrak{a}^{t-1}) \otimes \mathcal{A}^n \otimes \mathcal{L} \otimes H^0(X, \mathcal{A} \otimes \mathfrak{a}) \rightarrow \tau_+(\omega_X, \mathfrak{a}^t) \otimes \mathcal{A}^{n+1} \otimes \mathcal{L}.$$

It then follows that $\mathfrak{a} \cdot \tau_+(\omega_X, \mathfrak{a}^{t-1}) = \tau_+(\omega_X, \mathfrak{a}^t)$. This completes the proof of the first case.

The second case follows similarly, but replace the sufficiently ample L by $L - K_X$ (and hence L' by $L' - \mu^*K_X$). In particular, our Y' should also be such that the pullback of μ^*K_X is Cartier. Additionally, because μ^*K_X is only a $\mathbb{Q}$ -divisor, $\lfloor - \rfloor$ must be inserted in various places such as the displayed commutative diagram. The rest of the proof then follows analogously. $\square$

6.2 An application to the Briançon–Skoda theorem

As suggested by S. Takagi, we also show how the above can be used to show the following version of the Briançon–Skoda theorem for $\mathbb{Q}$ -Gorenstein splinters of mixed characteristic. Since splinters are pseudo-rational by [Bha20], the Briançon–Skoda theorem for splinters follows from the work of Lipman–Tessier [LT81, Theorem 2.1 and Corollary 2.2]. However, instead of simply looking at the analytic spread of the ideal, the Lipman–Tessier bound also considers the codimension of the associated primes of powers of I , which do not appear in our corollary. Related versions of this result also follow from [HM21, Theorem 2.7]; cf. [Hei97] and [MST⁺22, Section 4].

COROLLARY 6.7. *Suppose that $(R, \mathfrak{m})$ is a complete normal $\mathbb{Q}$ -Gorenstein local domain of residual characteristic $p > 0$ and $X = \text{Spec } R$. Suppose that $I \subseteq R$ is an ideal of analytic spread $n + 1$. Then for any integer $j > 0$,*

$$\overline{I^{j+n}} \cdot \tau_+(\mathcal{O}_X) \subseteq I^j,$$

where $\overline{}$ denotes integral closure. Furthermore, if $(R, \mathfrak{m})$ is not necessarily complete but is instead an excellent local splinter (meaning that it is a summand of every module finite ring extension, or equivalently that X is $\dagger$ -regular/ T -regular), then we have that

$$\overline{I^{j+n}} \subseteq I^j.$$

Remark. The analytic spread m of I is the dimension of $R[It]/\mathfrak{m}R[It]$, where $R[It]$ is the Rees algebra of I . Hence $m - 1 = n = \dim X'_\mathfrak{m}$.

The proof we use is essentially the same as the one contained in [Laz04, Section 9.6].

Proof. For the second statement, we may also assume that R is complete (the splinter property is preserved by [DT23]). We will now use the theory we have developed thus far, applied in the case where $X = \text{Spec } R$. We have that $\overline{I^m} \cdot \tau_+(\mathcal{O}_X) \subseteq \tau_+(\mathcal{O}_X, \overline{I^m}) = \tau_+(\mathcal{O}_X, I^m)$ for all m by Lemma 6.5. Therefore,

$$\overline{I^{j+n}} \cdot \tau_+(\mathcal{O}_X) \subseteq \tau_+(\mathcal{O}_X, I^{j+n}) = I^j \cdot \tau_+(\mathcal{O}_X, I^n) \subseteq I^j.$$

When R is a splinter, we have that $\tau_+(\mathcal{O}_X) = \mathcal{O}_X$ (since every finite ring extension of R splits), and so the proof is complete. $\square$

If X is not $\mathbb{Q}$ -Gorenstein but $\Delta \geq 0$ is such that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier, we believe a slight generalization of the same proof also shows that $\overline{I^{j+n}} \cdot \tau_+(\mathcal{O}_X, \Delta) \subseteq I^j$.

7. $+$ -test modules of linear series

In this section, we will generalize results of [Laz04, Mus13, Sat18] on the global generation of test ideals (or multiplier ideals) of linear series from the equal-characteristic case to the mixed-characteristic case. We then apply this to study the order of vanishing of linear series and to the diminished base locus as in [Nak04, ELM⁺06, Mus13]; note that some related results were also obtained in the singular case in [CDB13, Sat18, Mur21]. Many of our arguments were inspired by those in [Mus02]. We begin by defining the corresponding test ideals for linear series.

SETTING 7.1. Throughout this section, suppose that X is a Noetherian normal integral scheme, projective over a complete local Noetherian ring $(R, \mathfrak{m})$ of mixed characteristic $(0, p)$. We *also* assume that X is a regular scheme (in other words, it is nonsingular) and $\mathcal{L} = \mathcal{O}_X(L)$ is a line bundle.

DEFINITION 7.2. Let $W \subseteq H^0(X, \mathcal{L})$ be a subspace and $\lambda > 0$ a positive rational number. Then the $+$ -test module corresponding to the linear series $|W|$ is

$$\tau_+(\omega_X, \lambda \cdot |W|) := \sum_{D \in |W|} \tau_+(\omega_X, \lambda \cdot D).$$

Since X is Noetherian, we may pick finitely many $D_1, \dots, D_r \in |W|$ such that $\tau_+(\omega_X, \lambda \cdot |W|) = \sum_{i=1}^r \tau_+(\omega_X, \lambda \cdot D_i)$.

We also define $\tau_+(\mathcal{O}_X, \lambda \cdot |W|) = \tau_+(\omega_X, \lambda \cdot |W|) \otimes \omega_X^{-1}$.

Remark 7.3. Note that we could have also defined $\tau_+(\omega_X, \lambda \cdot |W|)$ as in the previous section as $\tau_+(\omega_X, \mathfrak{b}_{|W|}^\lambda)$. This is what is done in the classical situation of the multiplier ideal and also what was done in positive characteristic in [Mus13, Sat18] (although in those cases, the sum formulation of the multiplier ideal as above is known to agree with $\tau_+(\omega_X, \mathfrak{b}_{|W|}^\lambda)$). While we expect similar statements for that latter definition, we do not pursue that approach here.

If D is a $\mathbb{Q}$ -divisor with $\kappa(D) \geq 0$, then mD is a divisor such that $|mD| \neq \emptyset$ for all sufficiently divisible $m > 0$. We let

$$\tau_+(\omega_X, \lambda \cdot ||D||) = \bigcup_{m>0} \tau_+\left(\omega_X, \frac{\lambda}{m} \cdot |mD|\right),$$

where we ignore the terms such that $|mD| = \emptyset$ (or, equivalently, we let $\tau_+(\omega_X, \frac{\lambda}{m} \cdot |mD|) = 0$ if $|mD| = \emptyset$). Recall that if mD is not necessarily an integral divisor, then $|mD| = \lfloor |mD| \rfloor + \{mD\}$. We also let

$$\tau_+(\mathcal{O}_X, \lambda \cdot ||D||) = \bigcup_{m>0} \tau_+\left(\mathcal{O}_X, \frac{\lambda}{m} \cdot |mD|\right) = \tau_+(\omega_X, \lambda \cdot ||D||) \otimes \omega_X^{-1}.$$

LEMMA 7.4. *We have that $\tau_+(\omega_X, \lambda \cdot ||D||) = \tau_+(\omega_X, \frac{\lambda}{m} \cdot |mD|)$ for any sufficiently divisible $m > 0$.*

Proof. Note that if $G \in |mD| \neq \emptyset$, then for any $r > 0$, we have that $rG \in |rmD|$ and therefore

$$\begin{aligned} \tau_+\left(\omega_X, \frac{\lambda}{m} \cdot |mD|\right) &= \sum_{G \in |mD|} \tau_+\left(\omega_X, \frac{\lambda}{m} G\right) = \sum_{G \in |mD|} \tau_+\left(\omega_X, \frac{\lambda}{mr} \cdot rG\right) \\ &\subseteq \tau_+\left(\omega_X, \frac{\lambda}{mr} \cdot |mrD|\right). \end{aligned}$$

Because X is Noetherian, the set $\{\tau_+(\omega_X, (\lambda/m) \cdot |mD|) \text{ s.t. } |mD| \neq \emptyset\}$ has a maximal element, and the claim follows. $\square$

LEMMA 7.5. *With the above notation (and continuing to assume that X is nonsingular):*

- (a) *If $|D| \neq \emptyset$, then $\tau_+(\omega_X, \lambda \cdot |D|) \subseteq \tau_+(\omega_X, \lambda \cdot ||D||)$.*
- (b) *If D has $\mathbb{Z}$ -coefficients and $|D|$ is nonempty, then $\tau_+(\omega_X, |D|) = \mathfrak{b}_{|D|} \otimes \omega_X$ or, in other words, $\tau_+(\mathcal{O}_X, |D|) = \mathfrak{b}_{|D|}$.*
- (c) *If $\lambda < \mu$, then $\tau_+(\omega_X, \mu \cdot ||D||) \subseteq \tau_+(\omega_X, \lambda \cdot ||D||)$.*
- (d) *Let $k > 0$ be an integer; then $\tau_+(\omega_X, \lambda \cdot ||D||) = \tau_+(\omega_X, (\lambda/k) \cdot ||kD||)$.*
- (e) *If $L - \lambda D$ is big and semi-ample and A is ample and globally generated, then*

$$\tau_+(\omega_X, \lambda \cdot ||D||) \otimes \mathcal{O}_X(nA + L) = \tau_+(\mathcal{O}_X, \lambda \cdot ||D||) \otimes \mathcal{O}_X(L + K_X + nA)$$

is generated by a sublinear series of $H^0(X, \mathcal{O}_X(K_X + nA + L))$, where $n = \dim X_{\mathfrak{m}}$.

- (f) *If $D - G$ is semi-ample, then $\tau_+(\omega_X, \lambda \cdot ||G||) \subseteq \tau_+(\omega_X, \lambda \cdot ||D||)$.*

Proof. Claim (a) follows from the definition.

Claim (b) follows because X is nonsingular, and so by Lemma 3.7,

$$\tau_+(\omega_X, |D|) = \sum_{G \in |D|} \tau_+(\omega_X, G) = \sum \omega_X(-G) = \left(\sum \mathcal{O}_X(-G) \right) \otimes \omega_X = \mathfrak{b}_{|D|} \otimes \omega_X.$$

Claim (c) follows because for any sufficiently divisible $m > 0$, we have that

$$\tau_+(\omega_X, \lambda \cdot ||D||) = \tau_+\left(\omega_X, \frac{\lambda}{m} \cdot |mD|\right) \supset \tau_+\left(\omega_X, \frac{\mu}{m} \cdot |mD|\right) = \tau_+(\omega_X, \mu \cdot ||D||).$$

Claim (d) follows because for any sufficiently divisible $m > 0$, we have that

$$\begin{aligned} \tau_+(\omega_X, \lambda \cdot ||D||) &= \tau_+\left(\omega_X, \frac{\lambda}{mk} \cdot |mkD|\right) = \tau_+\left(\omega_X, \frac{\lambda/k}{m} \cdot |m(kD)|\right) \\ &= \tau_+\left(\omega_X, \frac{\lambda}{k} \cdot ||kD||\right). \end{aligned}$$

To see claim (e), for some $G \in |mL|$ note that by Theorem 6.1, $\tau_+(\omega_X, (\lambda/m) \cdot G) \otimes \mathcal{O}_X(nA + L)$ is generated by $\mathbf{B}^0(X, (\lambda/m) \cdot G, \omega_X \otimes \mathcal{O}_X(nA + L))$. Thus $\tau_+(\omega_X, \lambda \cdot ||D||) \otimes \mathcal{O}_X(nA + L)$ is generated by

$$\sum_{i=1}^r \mathbf{B}^0\left(X, \frac{\lambda}{m} \cdot G_i, \omega_X \otimes \mathcal{O}_X(nA + L)\right) \subseteq H^0(X, \mathcal{O}_X(K_X + nA + L))$$

for appropriately chosen $G_i \in |mL|$.

We will now prove claim (f). For $m > 0$ sufficiently divisible, we have that

$$\tau_+(\omega_X, \lambda \cdot ||G||) = \tau_+\left(\omega_X, \frac{\lambda}{m} \cdot |mG|\right), \quad \tau_+\left(\omega_X, \frac{\lambda}{m} \cdot |mD|\right) = \tau_+(\omega_X, \lambda \cdot ||D||).$$

Since $D - G$ is semi-ample, we may assume that $|m(D - G)|$ is base-point-free. For any $F \in |m(D - G)|$, we have an inclusion $W := |mG| + F \subseteq |mD|$. Let $c = \lceil \lambda/m \rceil$, and pick divisors $D_1, \dots, D_r \in |mD|$ and $G_1, \dots, G_s \in |mG|$ such that

$$\tau_+(\omega_X, \lambda \cdot ||D||) = \sum_{i=1}^r \tau_+\left(\omega_X, \frac{\lambda}{m} \cdot D_i\right), \quad \tau_+(\omega_X, \lambda \cdot ||G||) = \sum_{i=1}^s \tau_+\left(\omega_X, \frac{\lambda}{m} \cdot G_i\right).$$

Possibly adding some $G_i + F$ to the set $\{D_1, \dots, D_r\}$, we may also assume that $\{G_1 + F, \dots, G_s + F\} \subseteq \{D_1, \dots, D_r\}$. Thus

$$\begin{aligned} \tau_+(\omega_X, \lambda \cdot \|G\|) \otimes \mathcal{O}_X(-cF) &= \sum_{i=1}^s \tau_+\left(\omega_X, \frac{\lambda}{m} \cdot G_i\right) \otimes \mathcal{O}_X(-cF) = \sum_{i=1}^s \tau_+\left(\omega_X, \frac{\lambda}{m} \cdot G_i + c \cdot F\right) \\ &\subseteq \sum_{i=1}^s \tau_+\left(\omega_X, \frac{\lambda}{m} \cdot (G_i + F)\right) \subseteq \sum_{i=1}^r \tau_+\left(\omega_X, \frac{\lambda}{m} \cdot D_i\right) = \tau_+(\omega_X, \lambda \cdot \|D\|). \end{aligned}$$

Since $|m(D - G)|$ is base-point-free, it follows that $\sum_{F \in |m(D-G)|} \mathcal{O}_X(-F) = \mathcal{O}_X$ and hence

$$\tau_+(\omega_X, \lambda \cdot \|G\|) = \sum_{F \in |m(D-G)|} \tau_+(\omega_X, \lambda \cdot \|G\|) \otimes \mathcal{O}_X(-F) \subseteq \tau_+(\omega_X, \lambda \cdot \|D\|). \quad \square$$

THEOREM 7.6. *Let Z be a closed integral subscheme of a regular scheme X over $\operatorname{Spec} R$ and D a big $\mathbb{Q}$ -divisor. Then $\operatorname{ord}_Z(\|D\|)$ is independent of the representative in the numerical class of D .*

Proof. Recall that

$$\operatorname{ord}_Z(\|D\|) = \lim \frac{1}{m} \operatorname{ord}_Z(|mD|) = \inf \left\{ \frac{1}{m} \operatorname{ord}_Z(|mD|) \right\}.$$

Let $E \equiv D$ be numerically equivalent big $\mathbb{Q}$ -divisors. We must show that $\operatorname{ord}_Z(\|D\|) = \operatorname{ord}_Z(\|E\|)$. Replacing D and E with appropriate multiples, we may assume that D and E have integer coefficients. Pick $l > 0$ such that $lD - (K_X + (n+1)A) \sim G > 0$, where A is ample and globally generated and $n = \dim X_{\mathfrak{m}}$. Then $mE - G \equiv (m-l)D + K_X + (n+1)A$. In particular,

$$mE - G - K_X - (m-l)D - nA \equiv A$$

is ample (and so big and semi-ample).

Since, by Lemma 7.5(e), the sheaf $\tau_+(\mathcal{O}_X, (m-l) \cdot \|D\|) \otimes \mathcal{O}_X(mE - G)$ is globally generated, it follows that

$$\tau_+(\mathcal{O}_X, (m-l) \cdot \|D\|) \subseteq \mathfrak{b}_{|mE-G|},$$

where $\mathfrak{b}_{|mE-G|}$ is the base ideal of the linear series $|mE - G|$. Thus

$$\mathcal{O}_X(-G) \cdot \tau_+(\mathcal{O}_X, (m-l) \cdot \|D\|) \subseteq \mathcal{O}_X(-G) \cdot \mathfrak{b}_{|mE-G|} \subseteq \mathfrak{b}_{|mE|},$$

where the last inequality is induced by the multiplication map $|G| \times |mE - G| \rightarrow |mE|$.

It follows by Lemma 7.5(a)–(d), that for $m > l$, we have

$$\mathcal{O}_X(-G) \cdot \mathfrak{b}_{|mD|} \subseteq \mathcal{O}_X(-G) \cdot \tau_+(\mathcal{O}_X, \|mD\|) \subseteq \mathcal{O}_X(-G) \cdot \tau_+(\mathcal{O}_X, \|(m-l)D\|) \subseteq \mathfrak{b}_{|mE|}.$$

But then

$$\operatorname{ord}_Z(G) + \operatorname{ord}_Z(|mD|) \geq \operatorname{ord}_Z(|mE|),$$

and passing to the limit, we obtain that $\operatorname{ord}_Z(\|D\|) \geq \operatorname{ord}_Z(\|E\|)$. The reverse inequality follows by symmetry. This concludes the proof. $\square$

THEOREM 7.7. *Suppose that $R/\mathfrak{m}$ is infinite. Let Z be a closed irreducible reduced subscheme with associated generic point η_Z . If D is big, then the following are equivalent:*

- (a) We have that $Z \not\subseteq \mathbf{B}_-(D)$.
- (b) There exists a divisor G such that $Z \not\subseteq \mathbf{B}_s |mD + G|$ for all $m \geq 1$.
- (c) There exists an integer $M > 0$ such that $\operatorname{ord}_Z(|mD|) \leq M$ for all $m \gg 0$.

(d) We have that $\text{ord}_Z(|D|) = 0$.

(e) The sheaf $\tau_+(\mathcal{O}_X, \lambda \cdot |D|)$ does not vanish along Z for any $\lambda > 0$.

Proof. (b) $\Rightarrow$ (c). Since D is big, we may pick an integer $m_0 > 0$ and a divisor $T \in |m_0 D - G|$. For any $m > 0$, we may pick a divisor $S_m \in |mD + G|$ whose support does not contain Z . Then $\text{ord}_Z(|(m + m_0)D|) \leq \text{ord}_Z(T + S_m) = M$, where $M = \text{ord}_Z(T)$.

(c) $\Rightarrow$ (d). This is immediate since $\text{ord}_Z(|D|) = \lim(\text{ord}_Z(|mD|)/m) \leq \lim M/m = 0$.

(d) $\Rightarrow$ (e). For inductive purposes, we weaken the hypothesis that X is regular. Instead, we merely assume that X is regular in a neighborhood of η_Z . We may assume that $\lambda = 1$. Since X is regular in a neighborhood of η_Z , there is an isomorphism $\omega_X \cong \mathcal{O}_X$ in such a neighborhood. Fix $m > 0$ such that $\text{ord}_Z(|mD|) < m$. The assumption that $\text{ord}_Z(|mD|) < m$ implies that there exists a $G \in |mD|$ with $\text{ord}_Z(G/m) < 1$. We now proceed by induction on the codimension of Z in X . If Z is a divisor in X , then in a neighborhood of η_Z , the pair $(X, G/m)$ has normal crossings with coefficients less than 1, and so in this neighborhood, we have that

$$\mathcal{O}_X \subseteq \tau_+(\mathcal{O}_X, |G/m|) \subseteq \tau_+(\mathcal{O}_X, |D|),$$

and the claim follows. We will now assume that $\dim X - \dim Z \geq 2$, and we will proceed by induction, restricting to hypersurfaces. Picking H to be sufficiently ample, we can then choose a general element of $H^0(X, I_Z \otimes \mathcal{O}_X(H))$ which yields a divisor $Y \sim H$ such that Y is regular in a neighborhood of η_Z and such that $\text{ord}_Z(G|_Y) = \text{ord}_Z(G) < m$. It is in this last step that we use that the residue field of R is infinite.

Assuming for the moment that Y is reduced, there is an inclusion

$$\text{adj}_+^Y(\mathcal{O}_X, Y + G/m) \subseteq \tau_+(\mathcal{O}_X, (1 - \epsilon)Y + G/m) \subseteq \tau_+(\mathcal{O}_X, G/m), \quad (7.1)$$

and we have the equality

$$\text{adj}_+^Y(\mathcal{O}_X, Y + G/m) \cdot \mathcal{O}_{Y^N} = \tau_+(\mathcal{O}_{Y^N}, \text{Diff}_{Y^N}(Y + G/m)). \quad (7.2)$$

Equations (7.1) and (7.2) combine to tell us that

$$\tau_+(\mathcal{O}_X, G/m) \cdot \mathcal{O}_{Y^N} \supseteq \tau_+(\mathcal{O}_{Y^N}, \text{Diff}_{Y^N}). \quad (7.3)$$

Even if Y is not reduced globally, in a neighborhood of η_Z , it is regular and hence reduced and normal. We then observe that by Proposition 5.7 and Definition 5.10, the inclusion (7.3) still holds in such a neighborhood. Again, in a neighborhood of η_Z , by induction (since the codimension of Z in Y is less than the codimension of Z in X), we have that $\tau_+(\mathcal{O}_{Y^N}, \text{Diff}_{Y^N}(Y + G/m)) = \mathcal{O}_{Y^N} = \mathcal{O}_Y$, and so by (7.3), we see that $\tau_+(\mathcal{O}_X, G/m) = \mathcal{O}_X$ in that neighborhood, and the result follows. Note that for induction purposes, at each induction step if necessary, we may globally replace Y with the normalization of Y_{red} since all computations may be done in a neighborhood of η_Z .

(e) $\Rightarrow$ (b). Pick $G = K_X + (d + 1)H$, where H is very ample and $d = \dim X_{\mathfrak{m}}$. Then $\tau_+(\mathcal{O}_X, |mD|) \otimes \mathcal{O}_X(mD + G)$ is globally generated, and so Z is not contained in the base locus of $|mD + G|$.

(b) $\Rightarrow$ (a). Fix an ample $\mathbb{Q}$ -divisor A . Pick $m > 0$ such that $mA - G$ is ample. Then $\mathbf{B}(D + A) = \mathbf{B}(m(D + A)) \subseteq \mathbf{B}(mD + G)$. Thus $Z \not\subseteq \mathbf{B}(D + A)$, and the claim follows as $\mathbf{B}_-(D) = \bigcup \mathbf{B}(D + A)$.

(a) $\Rightarrow$ (d). By assumption, $Z \not\subseteq \mathbf{B}(D + (1/i)A)$ for any $i > 0$, where A is a fixed very ample divisor. Since D is big, there is an integer $j > 0$ such that $jD - A \sim G \geq 0$. But then $|(k(i+j))D| \supset kG + |ki(D + (1/i)A)|$, so that $\text{ord}_Z(|(k(i+j))D|) \leq k \cdot \text{ord}_Z(G)$ for any sufficiently

divisible $k > 0$. Then

$$\mathrm{ord}_Z(\|D\|) \leq \frac{\mathrm{ord}_Z(|(k(i+j))D|)}{k(i+j)} \leq \frac{k \cdot \mathrm{ord}_Z(G)}{k(i+j)} = \frac{\mathrm{ord}_Z(G)}{i+j}.$$

Note that as j is fixed, the right side tends to zero as i goes to infinity, and we therefore have that $\mathrm{ord}_Z(\|D\|) = 0$. $\square$

In our proof of (d) $\Rightarrow$ (e) above, we showed the following. Compare with [KKP⁺21, Theorem 1.1] and also [Sat18].

COROLLARY 7.8. *Keep the assumptions of Theorem 7.7. If $G \geq 0$ is a $\mathbb{Q}$ -divisor such that $\mathrm{ord}_Z(G) < 1$, then $\tau_+(\mathcal{O}_X, G) = \mathcal{O}_X$ in a neighborhood of the generic point of Z .*

Proof. The proof follows immediately along the lines of the proof of (d) $\Rightarrow$ (e) in Theorem 7.7. $\square$

8. Further open questions

We mention several of the most important open questions associated with the theory developed in this paper.

8.1 Localization and completion

Suppose that X is projective over a complete local domain $(R, \mathfrak{m})$. For any point on $x \in X$, we can define $\tau_+(\mathcal{O}_{X,x}, B)$ as the stalk of $\tau_+(\mathcal{O}_X, B)$ at x . We state what we hope happens when we then complete at x .

CONJECTURE 8.1. *Suppose that X is a normal integral scheme of finite type over a complete local ring $(R, \mathfrak{m})$ such that $R/\mathfrak{m}$ has positive characteristic. Further suppose that $B \geq 0$ is a $\mathbb{Q}$ -divisor on X such that $K_X + B$ is $\mathbb{Q}$ -Cartier. Fix a point $x \in X$ with residual characteristic $p > 0$, and let $S = \mathcal{O}_{X,x}$ denote the stalk at x . Then we have that*

$$\tau_+(\mathcal{O}_X, B)_x \cdot \widehat{S} = \tau_+(\widehat{S}, \widehat{B}) = \tau_{\widehat{R}^+}(\widehat{S}, \widehat{B})$$

and that

$$\tau_+(\mathcal{O}_X, B)_x = \tau_+(\widehat{S}, \widehat{B}) \cap S.$$

8.2 Perturbed test modules

CONJECTURE 8.2. *Suppose that X is a normal integral scheme projective over a complete local ring $(R, \mathfrak{m})$ with residual characteristic $p > 0$. Then for B a $\mathbb{Q}$ -divisor on X ,*

$$\tau_+(\omega_X, B) = \tau_+(\omega_X, B + \epsilon D)$$

for $D > 0$ a Cartier divisor on X and $1 \gg \epsilon > 0$.

Even in the case where $X = \mathrm{Spec} R$, this is not clear. In that case, if instead of using X^+ , one uses a sufficiently large perfectoid big Cohen–Macaulay S -algebra, the conjecture is true; see [MS21]. One way to handle this in the case where $X \neq \mathrm{Spec} R$ would be to try to compute a variant of τ from a graded perfectoid big Cohen–Macaulay S -algebra, where $S = \bigoplus H^0(X, \mathcal{L})$ is a section ring of X with respect to an ample line bundle $\mathcal{L}$.

Alternately, we can bake a perturbation term into the definition as in [MS18]. Indeed, suppose that we have notation as in the conjecture above with $B \geq 0$. For a fixed $D > 0$ and $\epsilon_1 > \epsilon_2$, we see that $\tau_+(\omega_X, B + \epsilon_1 D) \subseteq \tau_+(\omega_X, B + \epsilon_2 D) \subseteq \omega_X$. Thus at least the object $\tau_+(\omega_X, B + \epsilon D)$ is well defined for sufficiently small ϵ by the Noetherian property of ω_X .

8.3 Subadditivity

CONJECTURE 8.3. *Suppose that X is regular. Then*

$$\tau_+(\mathcal{O}_X, \Delta_1 + \Delta_2) \subseteq \tau_+(\mathcal{O}_X, \Delta_1) \cdot \tau_+(\mathcal{O}_X, \Delta_2).$$

We expect this based on the fact that the same formula holds for multiplier ideals in characteristic zero and test ideals in characteristic $p > 0$. In the local case, when $X = \operatorname{Spec} R$, it is proven in [MS18, Section 5] for a variant of perturbed test ideals, using André’s A_∞ instead of R^+ , see [And18, Bha18], but the method works for us when $X = \operatorname{Spec} R$. Indeed, a generalization of this approach to certain general mixed $\mathbf{+}$ -test ideals has since been accomplished by Murayama in [Mur21]. All of these results used a strategy similar to, and inspired by, that of Takagi in characteristic $p > 0$; see [Tak06] (see also [BMS08, Proposition 2.11(d)] for an equivalent-up-to-duality proof in characteristic $p > 0$, or [HY03] for a characteristic $p > 0$ proof corresponding more closely to the classical subadditivity proof for multiplier ideals [DEL00]).

8.4 Discreteness and rationality of jumping numbers

Based on the characteristic zero and $p > 0$ contexts, we would expect the following to hold.

CONJECTURE 8.4. *Suppose that X is $\mathbb{Q}$ -Gorenstein and $\Delta \geq 0$ is a $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor. The set of real numbers $t > 0$ such that $\tau_+(\mathcal{O}_X, t\Delta) \neq \tau_+(\mathcal{O}_X, (t - \epsilon)\Delta)$ for all $1 \gg \epsilon > 0$ is a set of rational numbers without accumulation points in $\mathbb{R}$.*

In characteristic zero, the corresponding statement about the multiplier ideal follows immediately from the fact that multiplier ideals can be computed on a single resolution. In characteristic $p > 0$, however, the corresponding statement requires a more involved proof.

8.5 Alternate descriptions

Suppose that R is a complete local domain with positive-characteristic residue field and S is a normal finite-type R -algebra and $\Delta \geq 0$ a $\mathbb{Q}$ -divisor on $\operatorname{Spec} S$ such that $K_S + \Delta$ is $\mathbb{Q}$ -Cartier. By projectively compactifying $\operatorname{Spec} S$ over $\operatorname{Spec} R$, we can define a test ideal on $\operatorname{Spec} S$ and hence an ideal $\tau_+(S, \Delta) \subseteq S$ of S . Note it could happen that S is purely of characteristic zero.

Question 8.5. Is there a description of $\tau_+(S, \Delta)$ that does not rely on a compactification of $\operatorname{Spec} S$ over $\operatorname{Spec} R$? For instance, does $\tau_+(S, \Delta)$ agree with the $T^0(\operatorname{Spec} S, \Delta, \mathcal{O}_{\operatorname{Spec} S})$ as defined in [TY21]? Is there a description of $\tau_+(S, \Delta)$ via some closure operation as in tight closure?

A positive answer to Conjecture 8.1 might also shed light on this question.

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