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Del Pezzo quintics as equivariant compactifications of vector groups

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Abstract

We study faithful actions with a dense orbit of abelian unipotent groups on quintic del Pezzo varieties over a field of characteristic zero. Such varieties are forms of linear sections of the Grassmannian of planes in a 5-dimensional vector space. We characterize which smooth forms admit these types of actions and show that in case of existence, the action is unique up to equivalence by automorphisms. We also give a similar classification for mildly singular quintic del Pezzo threefolds and surfaces.

1. Introduction

Vector group varieties are defined by analogy to toric varieties as varieties X endowed with an effective action of an abelian unipotent group $\mathbb{U} \cong \mathbb{G}_a^n$ with a Zariski dense open orbit. For varieties defined over a field of characteristic zero, the group $\mathbb{U}$ then embeds equivariantly as the open orbit, making X into a partial equivariant completion of $\mathbb{U}$. The study of such equivariant completions which are Fano varieties was initiated by Hassett and Tschinkel in [HT99] with views towards Manin's conjecture on the asymptotic distribution of rational points of bounded height over number fields (see for example [CT02, CT12]). Over algebraically closed fields of characteristic zero, besides projective spaces of every dimension, many families of Fano varieties and other Mori fibre spaces including for instance smooth projective quadrics, Grassmannians and flag varieties are known to be vector group varieties; see for example [Arz11, AS11, AZ22, Dev15, FH20, FM19, HM20, Nag22, Sha09].

Of particular interest in this context is the question of the classification of possible equivalence classes of structures of vector group variety up to isomorphisms on a given variety. Indeed, it was observed by Hassett and Tschinkel [HT99] that in contrast to toric structures, vector group variety structures on $\mathbb{P}^n_{\mathbb{C}}$, $n \geqslant 2$, are not unique and that for $n \geqslant 6$, there are even infinitely many equivalence classes of such structures. In contrast, it is known that over algebraically closed fields of characteristic zero, Grassmannians other than projective spaces [AS11, Dev15] and smooth quadrics [Sha09] admit a unique equivalence class of structures of vector group variety. For Fano varieties of Picard rank 1 over the field of complex numbers, Fu and Hwang [FH14] gave a uniform

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characterization of uniqueness of vector group variety structures in terms in the smoothness of the variety of minimal rational tangents (VMRT) of a dominating family of minimal rational curves.

In this article, we consider the problem of existence and uniqueness of vector group variety structures on smooth and mildly singular quintic del Pezzo varieties over an arbitrary field k of characteristic zero. By definition, a *smooth quintic del Pezzo variety* is a smooth geometrically connected projective k-variety X whose base extension $X_{\bar{k}} = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$ to an algebraic closure $\bar{k}$ of k has an ample invertible sheaf $\mathcal{L}$ of degree 5 such that $\omega_{X_{\bar{k}}}^{\vee} \cong \mathcal{L}^{\otimes(n-1)}$. If $n = 2$, then $X_{\bar{k}}$ is a del Pezzo surface of degree 5. On the other hand, by [Fuj81], a smooth quintic del Pezzo k-variety X exists if and only if $n \leqslant 6$ and furthermore, for each $n = 3, 4, 5, 6$, $X_{\bar{k}}$ is unique up to isomorphism, isomorphic to any smooth linear section of the Grassmannian $\mathrm{G}(2, 5) \subset \mathbb{P}^9_{\bar{k}}$ of 2-dimensional vector subspaces of $\bar{k}^{\oplus 5}$. For $n \leqslant 3$, the automorphism group of $X_{\bar{k}}$ is finite if $n = 2$ and isomorphic to $\mathrm{PGL}_2(\bar{k})$ if $n = 3$ (see for example [CS16, Proposition 7.1.10]). In particular, it cannot contain any abelian unipotent subgroup defining a vector group variety structure on X. Over the field of complex numbers, existence and uniqueness of vector group variety structures on the remaining varieties, which are all Fano of Picard rank 1, has been settled affirmatively by Fu and Hwang as a consequence of a series of articles [FH18, FH20] devoted to the broader study using VMRT techniques of so-called Euler-symmetric projective varieties.

Here, since we work over arbitrary fields k of characteristic zero, possibly non-closed, the actual question comes down to determining and classifying k-forms of vector group variety structures on del Pezzo quintics. Of course, the non-existence of such structures after base extension to an algebraic closure is a clear obstruction for existence of these structures over the given base field. But on the other hand, neither the existence nor its combination with uniqueness up to equivalence after base extension is enough in general to conclude, say by Galois descent arguments, the existence of such structures defined over the base field. For instance, a smooth n-dimensional quadric in $\mathbb{P}^{n+1}_{\mathbb{Q}}$ without $\mathbb{Q}$-rational point does not admit any vector group variety structure defined over $\mathbb{Q}$, even though its base extension to $\bar{\mathbb{Q}}$ admits infinitely many such structures, which, as a consequence of [Sha09], are all equivalent. Our approach is thus by necessity different from that using VMRT techniques in [FH18, FH20, FM19], which, in particular, depend on the Cartan–Fubini analytic extension theorem [HM01] that has no arguably straightforward counterpart over arbitrary fields. Instead, we build on elementary birational geometry of the Grassmannian $\mathrm{G}(2, 5)$, its linear Schubert subvarieties and their associated rational projections, material which, over algebraically closed fields, goes back to a classical article of Todd [Tod30]. Our first main result is a classification of k-forms of vector group variety structures on smooth del Pezzo quintics of dimension $n \geqslant 4$ which can be summarized as follows.

Theorem 1.1. *For a smooth quintic del Pezzo k-variety X_n of dimension $n \in \{4, 5, 6\}$, the following hold:*

(i) *If $n = 6$, then X_6 admits a vector group variety structure if and only if it has a k-rational point. If so, then X_6 is isomorphic to $\mathrm{G}(2, 5) \subset \mathbb{P}^9_k$ and it admits a unique vector group variety structure up to equivalence.*

(ii) *If $n = 4, 5$, then X_n is unique up to isomorphism, isomorphic to any smooth section of $\mathrm{G}(2, 5) \subset \mathbb{P}^9_k$ by a linear subspace of codimension $6 - n$. Furthermore, it admits precisely one vector group variety structure up to equivalence.*

In a second step, we apply the same methods to k-forms of vector group variety structures on mildly singular quintic del Pezzo threefolds and surfaces. The case of quintic del Pezzo surfaces with canonical singularities has already been fully settled by Derenthal and Loughran [DL10], who studied vector group variety structures on del Pezzo surfaces with canonical singularities over arbitrary fields of characteristic zero. We therefore mainly focus on the case of quintic del Pezzo threefolds with terminal singularities. We obtain the following characterization, which says in particular that in contrast to smooth del Pezzo quintics of dimension 4 and 5 and canonical del Pezzo quintic surfaces, which have no non-trivial k-forms for any field k, trinodal del Pezzo quintics threefolds do in general have non-trivial k-forms.

Theorem 1.2. *A quintic del Pezzo threefold X_3 with terminal singularities admits a vector group variety structure if and only if its base extension to $\bar{k}$ has precisely three ordinary double points. In this case, the vector group structure is unique up to isomorphism.*

Furthermore, isomorphism classes of such threefolds are in one-to-one correspondence with $\mathrm{PGL}_2(k)$-orbits of smooth zero-dimensional subschemes of $\mathbb{P}^1_k$ of length 3.

The article is organized as follows. In § 2, we collect standard facts on Grassmannians and basic properties of vector groups and their actions on varieties. Section 3 is devoted to a review of certain classes of linear Schubert subvarieties of the Grassmannian $\mathrm{G}(2,5)$ and its smooth linear sections, and of their associated rational linear projections. These preliminary results are then applied in § 4 and § 5 to derive the proofs of Theorems 1.1 and 1.2, respectively.

2. Preliminaries

We work over a field k of characteristic zero, with a fixed algebraic closure $\bar{k}$ and associated Galois group $\Gamma = \mathrm{Gal}(\bar{k}/k)$. We consider Γ as the profinite group $\varprojlim \mathrm{Gal}(k'/k)$ endowed with the profinite topology, the limit being taken over the directed set of finite Galois extensions $k \subset k'$ of k, each group $\mathrm{Gal}(k'/k)$ being endowed with the discrete topology. A k-variety is an integral k-scheme of finite type.

2.1 Notation and conventions

For vector bundles and projective bundles, we follow [Gro61]. Namely, given a quasi-coherent sheaf $\mathcal{F}$ on a scheme X, we let $\mathrm{Sym}\,\mathcal{F}$ be the symmetric algebra of $\mathcal{F}$, and we denote by $p\colon \mathbb{V}_X(\mathcal{F}) = \mathrm{Spec}_X(\mathrm{Sym}^{\cdot}\,\mathcal{F}) \to X$ the "vector bundle" over X associated with $\mathcal{F}$ and by

$$\pi\colon \mathbb{P}_X(\mathcal{F}) = \mathrm{Proj}_X(\mathrm{Sym}\,\mathcal{F}) \cong (\mathbb{V}_X(\mathcal{F}) \setminus 0_X)/\mathbb{G}_{m,X} \to X\,,$$

where 0_X is the zero section of p, its associated "projective bundle". We denote by $\mathcal{O}_{\mathbb{P}_X(\mathcal{F})}(1)$ the canonical coherent invertible sheaf of $\mathcal{O}_{\mathbb{P}_X(\mathcal{F})}$-modules associated with $\mathrm{Sym}\,\mathcal{F}$ viewed as a graded sheaf of modules over itself with the degree shifted by 1. When $\mathcal{E}$ is a coherent locally free sheaf, $\mathbb{V}_X(\mathcal{E})$ is a usual Zariski locally trivial vector bundle of finite rank and $\mathbb{P}_X(\mathcal{E})$ is its associated projective bundle of lines.

Given a quasi-coherent sheaf $\mathcal{F}$ on a scheme X and an integer $d \geqslant 0$, we denote by $\rho\colon \mathbb{G}_X(\mathcal{F}, d) \to X$ the Grassmann bundle whose T-points, where $f\colon T \to X$ is any X-scheme, are equivalence classes of coherent locally free quotients $\mathcal{E}$ of $f^*\mathcal{F}$ of constant rank d, two such quotients being called equivalent if the corresponding surjections $q\colon f^*\mathcal{F} \to \mathcal{E}$ and $q'\colon f^*\mathcal{F} \to \mathcal{E}'$ have the same kernel; see for example [Kle69]. We denote by $\rho^*\mathcal{F} \to \mathcal{Q}$ the universal coherent locally free quotient of constant rank d and refer to the kernel $\mathcal{S}$ of this surjection as the universal subsheaf of $\rho^*\mathcal{F}$ so that we have the universal exact sequence $0 \to \mathcal{S} \to \rho^*\mathcal{F} \to \mathcal{Q} \to 0$. For a coherent locally free sheaf $\mathcal{F}$ of constant rank $n \geqslant d+1$, there is a canonical isomorphism

$\mathbb{G}_X(\mathcal{F},d) \to \mathbb{G}_X\big(\mathcal{F}^\vee, n-d\big)$ given on T-points by mapping a quotient $q\colon f^*\mathcal{F} \to \mathcal{E}$ with kernel $\mathcal{K}$ to the quotient $f^*\mathcal{F}^\vee \to \mathcal{K}^\vee$. For $d=1$, we have $\mathbb{G}_X(\mathcal{F},d) \cong \mathbb{P}_X(\mathcal{F})$, and under this isomorphism, the universal exact sequence coincides with the relative Euler exact sequence of $\mathbb{P}_X(\mathcal{F})$

$$0 \to \Omega^1_{\mathbb{P}_X(\mathcal{F})}(1) := \Omega^1_{\mathbb{P}_X(\mathcal{F})} \otimes \mathcal{O}_{\mathbb{P}_X(\mathcal{F})}(1) \to \pi^*\mathcal{F} \to \mathcal{O}_{\mathbb{P}_X(\mathcal{F})}(1) \to 0\,.$$

2.2 Grassmannians

We summarize basic properties of Grassmannian varieties; see for example [LB15] and [Wey03, Chapters 3 and 4] for the details. For a k-vector space V of dimension $n \geqslant 2$ and an integer $1 \leqslant d \leqslant n-1$, the Grassmann bundle $\rho\colon \mathbb{G}_k\big(V^\vee, d\big) \to \operatorname{Spec}(k)$ is the $d(n-d)$-dimensional Grassmannian whose geometric points correspond to equivalence classes of d-dimensional quotients of $V^\vee_{\bar{k}}$, equivalently to d-dimensional $\bar{k}$-vector subspaces E of $V_{\bar{k}}$.

2.2.1 *Tautological sheaves.* We put $V^\vee_{\mathbb{G}_k(V^\vee,d)} = V^\vee \otimes_k \mathcal{O}_{\mathbb{G}_k(V^\vee,d)}$, and we write

$$0 \to \mathcal{S} = \mathcal{S}_{\mathbb{G}_k(V^\vee,d)} \to V^\vee_{\mathbb{G}_k(V^\vee,d)} \to \mathcal{Q} = \mathcal{Q}_{\mathbb{G}_k(V^\vee,d)} \to 0$$

for the universal sequence of coherent locally free sheaves on $\mathbb{G}_k\big(V^\vee,d\big)$. The sheaf of Kähler differentials $\Omega^1_{\mathbb{G}_k(V^\vee,d)/k}$ is canonically isomorphic to $\mathcal{H}om(\mathcal{Q},\mathcal{S}) \cong \mathcal{S} \otimes \mathcal{Q}^\vee$, and its determinant $\omega_{\mathbb{G}_k(V^\vee,d)}$ is canonically isomorphic to $(\det \mathcal{S})^{\otimes n-d} \otimes \big(\det \mathcal{Q}^\vee\big)^{\otimes d} \cong \big(\det \mathcal{Q}^\vee\big)^{\otimes n}$. The k-vector spaces $H^0\big(\mathbb{G}_k\big(V^\vee,d\big),\mathcal{Q}\big)$ and $H^0\big(\mathbb{G}_k\big(V^\vee,d\big),\mathcal{S}^\vee\big)$ are canonically isomorphic to $V^\vee$ and V, respectively. We have $H^1\big(\mathbb{G}_k(V^\vee,d),\Omega^1_{\mathbb{G}_k(V^\vee,d)/k}\big) \cong k$, and all other cohomology spaces of $\mathcal{Q}$, $\mathcal{Q}^\vee$, $\mathcal{S}$, $\mathcal{S}^\vee$ and $\Omega^1_{\mathbb{G}_k(V^\vee,d)/k}$ are zero.

2.2.2 *Plücker embedding and automorphisms.* We denote by $j_P\colon \mathbb{G}_k\big(V^\vee,d\big) \to \mathbb{P}_k\big(\Lambda^d V^\vee\big)$ the *Plücker embedding*, that is, the closed immersion determined by the surjection $\Lambda^d V^\vee_{\mathbb{G}_k(V^\vee,d)} \to \det \mathcal{Q}$ induced by the universal quotient homomorphism. Letting $\operatorname{Aut}_k\big(\mathbb{P}_k\big(\Lambda^d V^\vee\big),\mathbb{G}_k\big(V^\vee,d\big)\big)$ be the stabilizer of $j_P\big(\mathbb{G}_k\big(V^\vee,d\big)\big)$ in $\operatorname{Aut}_k\big(\mathbb{P}_k\big(\Lambda^d V^\vee\big)\big)$, it follows from [Cho49] that the composition of the homomorphism of k-group schemes

$$\operatorname{PGL}_k\big(V^\vee\big) = \operatorname{Aut}_k\big(\mathbb{P}_k\big(V^\vee\big)\big) \to \operatorname{Aut}_k\big(\mathbb{P}_k\big(\Lambda^d V^\vee\big),\mathbb{G}_k\big(V^\vee,d\big)\big)\,, \quad \varphi \mapsto \Lambda^d\varphi \tag{2.1}$$

with the restriction homomorphism $\operatorname{Aut}_k\big(\mathbb{P}_k\big(\Lambda^d V^\vee\big),\mathbb{G}_k\big(V^\vee,d\big)\big) \to \operatorname{Aut}_k\big(\mathbb{G}_k\big(V^\vee,d\big)\big)$ is a closed immersion, which is an isomorphism when $n \neq 2d$ (otherwise, its image is a k-subgroup scheme of index 2).

2.3 Vector groups and vector group varieties

2.3.1 *Vector groups.* A *vector k-group* is an abelian unipotent algebraic k-group scheme. By [DG70, § IV.2.4], there is an equivalence between the category of finite-dimensional k-vector spaces and the category of vector k-groups. This equivalence is given by the map associating with a finite-dimensional k-vector space V the k-group scheme $\big(\mathbb{V}_k\big(V^\vee\big),+\big)$, where the comorphism of the k-group scheme structure $+$ is induced by the diagonal homomorphism $V^\vee \to V^\vee \oplus V^\vee$, and with a k-linear homomorphism $f\colon W \to V$ the k-group homomorphism $\big(\mathbb{V}_k\big(W^\vee\big),+\big) \to \big(\mathbb{V}_k\big(V^\vee\big),+\big)$ induced by ${}^tf\colon V^\vee \to W^\vee$. The choice of a k-basis of V determines an isomorphism of k-group schemes $\big(\mathbb{V}_k\big(V^\vee\big),+\big) \cong \mathbb{G}^n_{a,k}$. We will repeatedly use the following simple results.

Lemma 2.1. *With the notation above, the following hold:*

(i) *Every k-subgroup and quotient k-group of a vector k-group is a vector k-group.*

(ii) *Every extension* $0 \to \mathbb{U}' \to \mathbb{U} \to \mathbb{U}'' \to 0$ *of vector* k*-groups has a splitting* $h\colon \mathbb{U} \xrightarrow{\sim} \mathbb{U}' \times \mathbb{U}''$.

(iii) *The subspace* $M^{\mathbb{U}}$ *of* $\mathbb{U}$*-invariants of a rational* $\mathbb{U}$*-module* M *of finite positive dimension is non-zero.*

Recall [MFK94, § I.3] (see also [HL10, § 4.2]) that a *quasi-coherent* G*-sheaf of* $\mathcal{O}_X$*-modules* on a k-scheme X with an action $\mu\colon G \times X \to X$ of an algebraic k-group G is a pair $(\mathcal{F}, \theta)$ consisting of a coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ and an isomorphism $\theta\colon \mu^*\mathcal{F} \xrightarrow{\sim} \mathrm{p}_2^*\mathcal{F}$ of coherent sheaves of $\mathcal{O}_{G\times X}$-modules, called a G*-linearization* of $\mathcal{F}$, that satisfies the cocycle relation $(m_G \times \mathrm{id}_X)^*\theta = \mathrm{p}_{23}^*\theta \circ (\mathrm{id}_G \times\mu)^*\theta$ on $G \times G \times X$, where $m_G\colon G \times G \to G$ is the group law on G and where $\mathrm{p}_{23}\colon G \times G \times X \to G \times X$ is the projection onto the last two factors. In particular, when G acts trivially on X, a G-linearization of $\mathcal{F}$ is the same as a homomorphism from G into the group $\mathrm{Aut}_X(\mathcal{F})$ of $\mathcal{O}_X$-module automorphisms of $\mathcal{F}$. Two G-linearizations θ and θ' of $\mathcal{F}$ are called *equivalent* if there exists an $\mathcal{O}_X$-module automorphism φ of $\mathcal{F}$ such that $\mathrm{p}_2^*\varphi \circ \theta' = \theta \circ \mu^*\varphi$.

Lemma 2.2. *Let* X *be a normal* k*-variety endowed with an action of a vector* k*-group* $\mathbb{U}$*. Then every coherent invertible sheaf of* $\mathcal{O}_X$*-modules* $\mathcal{L}$ *admits a* $\mathbb{U}$*-linearization* $\theta_{\mathcal{L}}$ *unique up to equivalence.*

Proof. This follows from [Bri15, Lemma 2.13] and the fact that since $\mathbb{U} \cong \mathbb{A}_k^n$, for every normal k-variety X, the pullback homomorphisms $\mathrm{p}_X^*\colon H^i(X, \mathcal{O}_X^*) \to H^i(X \times \mathbb{U}, \mathcal{O}_{X\times\mathbb{U}}^*)$, $i = 0, 1$, are isomorphisms. □

2.3.2 *Vector group structures and vector group varieties*

Definition 2.3. A *vector group variety* is a k-variety X endowed with an effective action μ: $\mathbb{U} \times X \to X$ of a vector k-group $\mathbb{U}$ which has a Zariski dense open orbit U_X. The action μ is said to define a *vector group structure* on X. Two vector group structures $\mu\colon \mathbb{U} \times X \to X$ and $\mu'\colon \mathbb{U}' \times X \to X$ on X are said to have the same equivalence class if there exist an isomorphism of k-groups $\alpha\colon \mathbb{U}' \to \mathbb{U}$ and a k-automorphism φ of X such that $\varphi \circ \mu' = \mu \circ (\alpha \times \varphi)$.

Lemma 2.4. *Let* X *be a vector group variety with open orbit* U_X*. Then* U_X *is a trivial* $\mathbb{U}$*-torsor. In particular,* U_X *contains a* k*-point of* X*.*

Proof. Since the action of $\mathbb{U}$ is effective and U_X is Zariski dense, the morphism $(\mu, \mathrm{p}_2)\colon \mathbb{U}\times_k U_X \to U_X \times U_X$ is an isomorphism; that is, U_X endowed with the induced action of $\mathbb{U}$ is a $\mathbb{U}$-torsor. The conclusion then follows from the additive form of Hilbert's Theorem 90, which asserts that every such torsor is trivial. □

Example 2.5. Since all orbits of unipotent group actions on a quasi-affine k-variety are closed [Ros61, Theorem 2], Lemma 2.4 implies that a quasi-affine vector group variety X is a trivial $\mathbb{U}$-torsor. In particular, $\mathbb{A}_k^n$ is the unique quasi-affine k-variety with a $\mathbb{G}_{a,k}^n$-structure, and this structure is unique up to isomorphism. On the other hand, by Sumihiro's equivariant completion [Sum74, Theorem 3], every normal vector group k-variety X admits a $\mathbb{U}$-equivariant open immersion $j\colon X \hookrightarrow \bar{X}$ into a complete vector group k-variety $\bar{X}$. When X is smooth, the existence of equivariant resolution of singularities [Kol07, Theorem 3.36 and Proposition 3.9.1] implies in addition that $\bar{X}$ can be chosen to be smooth and such that $\bar{X} \setminus j(X)$ is the support of a $\mathbb{U}$-stable smooth normal crossing divisor on $\bar{X}$.

Proposition 2.6. *Let* X *be a* k*-variety endowed with a vector group structure* $\mu\colon \mathbb{U} \times X \to X$*, let* $f\colon X \to Y$ *be a proper morphism to a* k*-variety* Y *such that* $f_*\mathcal{O}_X = \mathcal{O}_Y$*, and let* $i\colon F \hookrightarrow X$

be the scheme-theoretic fibre of f over a k-point y_0 of Y in the image by f of the open $\mathbb{U}$-orbit U_X. Then there exists an extension of vector k-groups $0 \to \mathbb{U}' \xrightarrow{a} \mathbb{U} \xrightarrow{b} \bar{\mathbb{U}} \to 0$ such that the following hold:

(i) *The variety Y is endowed with a vector group structure $\mu_Y\colon \bar{\mathbb{U}} \times_k Y \to Y$ with open $\bar{\mathbb{U}}$-orbit U_Y, and $f\colon X \to Y$ is equivariant with respect to the homomorphism $b\colon \mathbb{U} \to \bar{\mathbb{U}}$.*

(ii) *The scheme F is a k-variety endowed with a vector group structure $\mu_F\colon \mathbb{U}' \times_k F \to F$, and the closed immersion $i\colon F \hookrightarrow X$ is equivariant with respect to the homomorphism $a\colon \mathbb{U}' \to \mathbb{U}$.*

(iii) *Given any section $c\colon \bar{\mathbb{U}} \to \mathbb{U}$ of $b\colon \mathbb{U} \to \bar{\mathbb{U}}$, the morphism*

$$j\colon \mu \circ (c \times i) \circ \left(\mu_Y^{-1}(\cdot, y_0) \times \mathrm{id}_F\right)\colon U_Y \times F \overset{\cong}{\to} \bar{\mathbb{U}} \times F \to \mathbb{U} \times X \to X$$

is a $\mathbb{U}' \times \bar{\mathbb{U}}$-equivariant open immersion with image $f^{-1}(U_Y)$.

Proof. By Blanchard's lemma [Bri17, Theorem 7.2.1], there exists a unique action $\nu\colon \mathbb{U} \times Y \to Y$ such that f is $\mathbb{U}$-equivariant. Let $\mathbb{U}' \subset \mathbb{U}$ be the stabilizer of y_0, and let $\bar{\mathbb{U}} = \mathbb{U}/\mathbb{U}'$. Since y_0 belongs to $f(U_X)$, the $\mathbb{U}$-orbit of y_0 is a constructible set which is not contained in any proper closed subset of Y. It thus contains a Zariski dense open subset of Y, hence, being homogeneous under the action of $\mathbb{U}$, is a Zariski dense open subset of Y. This implies that $\mathbb{U}'$ acts trivially on Y and that the induced action $\mu_Y\colon \bar{\mathbb{U}} \times Y \to Y$ of $\bar{\mathbb{U}}$ is a vector group structure on Y with the property that $f\colon X \to Y$ is equivariant with respect to the k-group homomorphism $b\colon \mathbb{U} \to \bar{\mathbb{U}}$. This proves assertion (i). The closed subscheme F is $\mathbb{U}'$-stable, with $\mathbb{U}'$-action $\mu_F\colon \mathbb{U}' \times F \to F$ induced by μ. Given a section $c\colon \bar{\mathbb{U}} \to \mathbb{U}$ of b, we have the cartesian square of $\bar{\mathbb{U}}$-equivariant morphisms

$$\begin{array}{ccc} \bar{\mathbb{U}} \times F & \xrightarrow{\mu\circ(c\times i)} & X \\ {\scriptstyle \mathrm{id}_{\bar{\mathbb{U}}} \times f}\downarrow & & \downarrow{\scriptstyle f} \\ \bar{\mathbb{U}} \times \{y_0\} & \xrightarrow{\mu_Y(\cdot, y_0)} & Y, \end{array}$$

where $\bar{\mathbb{U}}$ acts on $\bar{\mathbb{U}} \times F$ and $\bar{\mathbb{U}} \times \{y_0\}$ by translations on the first factor and on X by $\mu \circ (c \times \mathrm{id}_X)$. For every $v \in \bar{\mathbb{U}}(k)$, the morphism $\mu \circ (c(v) \times \mathrm{id}_X)$ is an automorphism of X which maps F isomorphically onto the scheme-theoretic fibre of f over the point $\mu_Y(v, y_0)$. Since $\mu_Y(\cdot, y_0)\colon \bar{\mathbb{U}} \times \{y_0\} \to Y$ is an open immersion with image U_Y, it follows that $\mu \circ (c \times i)$ is an open immersion with image $f^{-1}(U_Y)$. Furthermore, $\mu \circ (c \times i)$ is equivariant for the isomorphism $h = (c, a)\colon \bar{\mathbb{U}} \times \mathbb{U}' \to \mathbb{U}$ with respect to the product action of $\bar{\mathbb{U}} \times \mathbb{U}'$ on $\bar{\mathbb{U}} \times F$ and the $\mathbb{U}$-action on X. Assertion (iii) follows. Finally, assertion (ii) follows from the observation that the intersection of the inverse image of $U_X \subset f^{-1}(U_Y)$ by $\mu \circ (c \times i)$ with $\{0\} \times F$ is a Zariski dense $\mathbb{U}'$-stable open subset U_F of F which is a principal homogeneous space of $\mathbb{U}'$. □

Recall that a coherent locally free sheaf $\mathcal{E}$ on a k-variety X is called *simple* if its only endomorphisms are scalar homotheties. Lemma 2.2 and Proposition 2.6 imply the following result.

Corollary 2.7. *For a simple coherent locally free sheaf $\mathcal{E}$ of rank $r \geqslant 2$ on a normal k-variety X, the total space of the projective bundle $\pi\colon \mathbb{P}_X(\mathcal{E}) \to X$ does not admit a vector group structure.*

Proof. Assume that $\mathbb{P}_X(\mathcal{E})$ admits a vector group structure given by the action of a vector group $\mathbb{U}$. By Lemma 2.2, the invertible sheaf $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1)$ is canonically $\mathbb{U}$-linearized. By Proposition 2.6, the morphism $\pi\colon \mathbb{P}_X(\mathcal{E}) \to X$ is $\mathbb{U}$-equivariant for a uniquely determined $\mathbb{U}$-action on X, which factors through an effective action of a non-trivial quotient $\bar{\mathbb{U}} = \mathbb{U}/\mathbb{U}'$ defining a vector group structure on X. Since π is $\mathbb{U}$-equivariant, $\mathcal{E} \cong \pi_*\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1)$ is endowed with an induced $\mathbb{U}$-linearization for the action on X, hence with a linearization for the trivial action of the positive-dimensional vector group $\mathbb{U}'$. The latter is determined by some group homomorphism $\mathbb{U}' \to \mathrm{Aut}_X(\mathcal{E})$ which is injective by Proposition 2.6. But this is impossible since $\mathrm{Aut}_X(\mathcal{E}) \cong \mathbb{G}_{m,X}$ by hypothesis. $\square$

3. Linear sections of G(2, 5) and their rational linear projections

In this section, we collect results concerning linear Schubert subvarieties of dimension greater than or equal to 2 of the Grassmannian $\mathrm{G}(2,5)$ of 2-dimensional k-vector subspaces of $k^{\oplus 5}$ and of its smooth linear sections in the Plücker embedding. We then review the description of the rational maps given by projections with respect to these linear Schubert subvarieties. Over algebraically closed fields, all this material is classical; see for example [Don77, Fuj81, PVdV99, Tod30].

3.1 Linear Schubert subvarieties and hyperplane sections of G(2, 5)

We put $G = \mathbb{G}_k(V^\vee, 2) \cong G(2,5)$ for some fixed 5-dimensional k-vector space V. We denote by $0 \to \mathcal{S} \to V_G^\vee \to \mathcal{Q} \to 0$ the universal sequence on G and by $j_P\colon G \hookrightarrow \mathbb{P}_k(\Lambda^2 V^\vee)$ the Plücker embedding. For any algebraic extension k' of k, we interpret k'-points of $G_{k'}$ either as 2-dimensional k'-vector subspaces $E \subset V_{k'}$ or as their corresponding lines $\mathbb{P}_{k'}(E^\vee)$ in $\mathbb{P}_{k'}(V_{k'}^\vee)$. For concrete examples, we fix the following coordinate convention.

Notation 3.1. For a chosen basis $e_1, \ldots, e_5$ of V with dual basis $e_1^\vee, \ldots, e_5^\vee$, we identify $\mathbb{P}_k(V^\vee)$ with $\mathbb{P}_k^4$ and $\mathrm{G}(2,5)$ with the closed subvariety of $\mathbb{P}_k(\Lambda^2 V^\vee) = \mathbb{P}_k^9$ endowed with the Plücker coordinates $w_{ij} = e_i^\vee \wedge e_j^\vee$, $1 \leqslant i < j \leqslant 5$, defined by the equations

$$\begin{cases} w_{12}w_{34} - w_{13}w_{24} + w_{14}w_{23} = 0\,, \\ w_{12}w_{35} - w_{13}w_{25} + w_{15}w_{23} = 0\,, \\ w_{12}w_{45} - w_{14}w_{25} + w_{15}w_{24} = 0\,, \\ w_{13}w_{45} - w_{14}w_{35} + w_{15}w_{34} = 0\,, \\ w_{23}w_{45} - w_{24}w_{35} + w_{25}w_{34} = 0\,. \end{cases}$$

3.1.1 *Solids and planes in* $\mathrm{G}(2,5)$. We consider the following linear Schubert subvarieties of G.

Definition 3.2. Let $\{V_1 \subset V_3 \subset V_4\}$ be a partial flag of k-vector subspaces of V, with $\dim_k V_i = i$:

- The $\sigma_{3,0}$-*solid* $\sigma_{3,0}(V_1) \cong \mathbb{P}_k\big((V/V_1)^\vee\big)$ associated with V_1 is the zero scheme of the homomorphism $V_{1,G} \hookrightarrow V_G \to \mathcal{S}^\vee$. Its intersection $\sigma_{3,1}(V_1 \subset V_4) \cong \mathbb{P}_k\big((V_4/V_1)^\vee\big)$ with the zero scheme of the homomorphism $(V/V_4)_G^\vee \hookrightarrow V_G^\vee \to \mathcal{Q}$ is called the $\sigma_{3,1}$-*plane* associated with $\{V_1 \subset V_4\}$.
- The $\sigma_{2,2}$-*plane* $\sigma_{2,2}(V_3) = \mathbb{G}_k\big(V_3^\vee, 2\big) \cong \mathbb{P}_k(V_3)$ associated with V_3 is the zero scheme of the homomorphism $(V/V_3)_G^\vee \hookrightarrow V_G^\vee \to \mathcal{Q}$.

The above subschemes are linear subspaces of G in the Plücker embedding $G \subset \mathbb{P}_k\big(\Lambda^2 V^\vee\big)$, given respectively as the intersections $G \cap \mathbb{P}_k\big(\Lambda^2 V^\vee/\Lambda^2(V/V_1)^\vee\big)$, $\sigma_{3,0}(V_1) \cap \mathbb{P}_k\big(\Lambda^2 V_4^\vee\big)$ and $G \cap \mathbb{P}_k\big(\Lambda^2 V_3^\vee\big)$. It follows from [Tod30] that they are the only linear k-subspaces of dimension greater than or equal to 2 contained in G.[1] Geometrically, the closed points of $\sigma_{3,0}(V_1)_{\bar{k}}$ and $\sigma_{3,1}(V_1 \subset V_4)_{\bar{k}}$ correspond, respectively, to lines in $\mathbb{P}_{\bar{k}}(V_{\bar{k}}^\vee)$ passing through the point $\mathbb{P}_{\bar{k}}(V_{1,\bar{k}}^\vee)$ and the subset of those which are contained in the subspace $\mathbb{P}_k(V_4^\vee)$. The closed points of $\sigma_{2,2}(V_2)_{\bar{k}}$ correspond to lines contained in the subspace $\mathbb{P}_{\bar{k}}(V_{3,\bar{k}}^\vee)$ of $\mathbb{P}_{\bar{k}}(V_{\bar{k}}^\vee)$.

Remark 3.3. The conormal sheaves in G of a solid $\Pi = \sigma_{3,0}(V_1)$ and of a plane $\Xi = \sigma_{2,2}(V_3)$ are canonically isomorphic to $\Omega^1_\Pi(1) \otimes V_{1,\Pi}$ and to $\Omega^1_\Xi(1) \otimes (V/V_3)^\vee_\Xi$, respectively. Indeed, since Π is the zero scheme of $V_{1,G} \to \mathcal{S}^\vee$, its conormal sheaf $\mathcal{C}_{\Pi/G}$ is isomorphic to $(\mathcal{S} \otimes V_{1,G})|_\Pi \cong \mathcal{S}|_\Pi \otimes V_{1,\Pi}$, and the canonical isomorphism $\mathcal{S}|_\Pi \cong \Omega^1_\Pi(1)$ is given by the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^1_\Pi(1) & \to & (V/V_1)^\vee_\Pi & \to & \mathcal{O}_\Pi(1) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{S}|_\Pi & \longrightarrow & V^\vee_\Pi & \longrightarrow & \mathcal{Q}|_\Pi & \to & 0 \\
 & & & \searrow^{0} & \downarrow & & \downarrow & & \\
 & & & & V^\vee_{1,\Pi} & = & V^\vee_{1,\Pi} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0\,. & &
\end{array}
$$

Similarly, since Ξ is a codimension 4 local complete intersection equal to the zero scheme of the homomorphism $(V/V_3)^\vee_G \to \mathcal{Q}$, the conormal sheaf $\mathcal{C}_{\Xi/G}$ is canonically isomorphic to the sheaf $\big(\mathcal{Q}^\vee \otimes (V/V_3)^\vee_G\big)|_\Xi \cong \big(\mathcal{Q}^\vee|_\Xi\big) \otimes (V/V_3)^\vee_\Xi$. Moreover, since $\mathcal{Q}|_\Xi$ equals the universal quotient sheaf on $\Xi \cong \mathbb{G}_k(V_3^\vee, 2)$, the canonical isomorphism $\mathbb{G}_k(V_3^\vee, 2) \cong \mathbb{P}_k(V_3)$ gives the identification $\mathcal{Q}^\vee|_\Xi \cong \Omega^1_\Xi(1)$.

Notation 3.4. Given a d-dimensional k-vector subspace $V_d \subset V$ with $1 \leqslant d \leqslant 4$, we denote by Δ_{V_d} the stabilizer in $\mathrm{GL}_k(V^\vee)$ of the subspace $(V/V_d)^\vee \subset V^\vee$. The choice of a splitting $V \cong V_d \oplus (V/V_d)$ identifies this k-group scheme with that consisting of block-matrices of the form

$$M(A_{5-d}, A_d, U) = \begin{pmatrix} A_{5-d} & U \\ 0 & A_d \end{pmatrix} \in \mathrm{GL}_k\big((V/V_d)^\vee \oplus V_d^\vee\big) \tag{3.1}$$

with $A_{5-d} \in \mathrm{GL}_k\big((V/V_d)^\vee\big)$, $A_d \in \mathrm{GL}_k\big(V_d^\vee\big)$ and $U \in \mathrm{Hom}_k\big(V_d^\vee, (V/V_d)^\vee\big)$. Under the isomorphism $\mathrm{PGL}_k\big(V^\vee\big) \cong \mathrm{Aut}_k(G)$ of (2.1), the associated subgroups $\mathrm{P}\Delta_{V_d}$ of $\mathrm{PGL}_k\big(V^\vee\big)$ correspond to the stabilizer subgroups of the solid $\sigma_{3,0}(V_1)$ if $d = 1$ and of the plane $\sigma_{2,2}(V_3)$ if $d = 3$.

3.1.2 *Smooth hyperplane sections.* By a hyperplane section of G, we mean the zero scheme $Z_{\langle s \rangle}$ of a non-zero global section $s \in H^0(G, \Lambda^2\mathcal{Q}) = \Lambda^2 V^\vee$, equivalently the intersection of G in the Plücker embedding with the hyperplane $\mathbb{P}_k\big(\Lambda^2 V^\vee/\langle s \rangle\big)$ of $\mathbb{P}_k\big(\Lambda^2 V^\vee\big)$. Denote by $\tilde{s}\colon V \to V^\vee$ the k-linear homomorphism corresponding to the form s under the canonical isomorphism $\mathrm{Hom}_k(V \otimes V, k) \cong \mathrm{Hom}_k\big(V, V^\vee\big)$. The fivefold $Z_{\langle s \rangle}$ is the *isotropic Grassmannian* $\mathrm{I}_s\mathbb{G}_k\big(V^\vee, 2\big)$: a closed point $E \subset V_{\bar{k}}$ of $G_{\bar{k}}$ belongs to $(Z_{\langle s \rangle})_{\bar{k}}$ if and only if the homomorphism $\tilde{s}_{\bar{k}}|_E\colon E \to V_{\bar{k}} \to V_{\bar{k}}^\vee$ has image contained in $(V_{\bar{k}}/E)^\vee$, hence if and only if E is $s_{\bar{k}}$-isotropic. Considering the conormal sequence

$$0 \to \mathcal{C}_{Z_{\langle s \rangle}/G} = \Lambda^2\mathcal{Q}^\vee|_{Z_{\langle s \rangle}} \otimes_{\mathcal{O}_{Z_{\langle s \rangle}}} \langle s \rangle_{Z_{\langle s \rangle}} \xrightarrow{d} \Omega^1_{G/k}|_{Z_{\langle s \rangle}} = \mathcal{H}om(\mathcal{Q}|_{Z_{\langle s \rangle}}, \mathcal{S}|_{Z_{\langle s \rangle}}) \to \Omega^1_{Z_{\langle s \rangle}/k} \to 0\,,$$

[1] The two types of planes in G are, respectively, called planes of the first and second system in [Tod30], and planes of vertex type and of non-vertex type in [Fuj81].

we see that $Z_{\langle s\rangle,\bar{k}}$ is smooth at $E \subset V_{\bar{k}}$ if and only if the map

$$d|_E \colon \Lambda^2 E \subset \mathrm{Hom}_k\big(E^\vee \otimes E^\vee, k\big) \cong \mathrm{Hom}_k\big(E^\vee, E\big) \to \mathrm{Hom}_{\bar{k}}\big(E^\vee, (V_{\bar{k}}/E)^\vee\big)\,, \quad f \mapsto \tilde{s}_{\bar{k}}|_E \circ f$$

is non-zero, hence if and only if $E \not\subset \mathrm{Ker}(\tilde{s}_{\bar{k}})$. In particular, $Z_{\langle s\rangle}$ is smooth if and only if $\tilde{s}$ has rank 4. This yields a functorial correspondence between k-points $\langle s\rangle^\vee$ of $\mathbb{P}_k\big(\Lambda^2 V\big) \setminus \mathbb{G}_k(V,2)$ and smooth hyperplane sections $Z_{\langle s\rangle}$ of G from which we get, in particular, that the action of $\mathrm{Aut}_k(G)(k) \cong \mathrm{PGL}_5(k)$ on the set of smooth hyperplane sections $Z_{\langle s\rangle}$ of G is transitive.

Given a smooth hyperplane section $Z_{\langle s\rangle}$, the s-orthogonal $V^\perp = \mathrm{Ker}(\tilde{s})$ of V has dimension 1. We put $\bar{V} = V/V^\perp$ and denote by $\bar{s} \in \Lambda^2 \bar{V}^\vee$ the symplectic form on $\bar{V}$ induced by s. Letting $\bar{G} = \mathbb{G}_k\big(\bar{V}^\vee, 2\big)$ with universal sequence $0 \to \mathcal{S}_{\bar{G}} \to \bar{V}^\vee_{\bar{G}} \to \mathcal{Q}_{\bar{G}} \to 0$, the zero scheme $Q_{\langle\bar{s}\rangle} \subset \bar{G}$ of $\bar{s}$ is the *Lagrangian Grassmannian* $\mathrm{L}_{\bar{s}}\mathbb{G}_k\big(\bar{V}^\vee, 2\big)$ whose k-points are the maximal $\bar{s}$-isotropic subspaces of $\bar{V}$. The Plücker embedding $\bar{G} \hookrightarrow \mathbb{P}_k\big(\Lambda^2 \bar{V}^\vee\big)$ induces a closed immersion of $Q_{\langle\bar{s}\rangle}$ in $\mathbb{P}_k\big(\Lambda^2\bar{V}^\vee/\langle\bar{s}\rangle\big)$ as the zero scheme of the non-degenerate quadratic form $\bar{q}$ associated with the symmetric bilinear form $\bar{b} \in \mathrm{Sym}^2\big(\Lambda^2\bar{V}^\vee/\langle\bar{s}\rangle\big)$ induced by the bilinear form $b\colon \Lambda^2\bar{V} \otimes \Lambda^2\bar{V} \to k$, $\bar{u}_1 \wedge \bar{v}_1 \otimes \bar{u}_2 \wedge \bar{v}_2 \mapsto (\bar{s} \wedge \bar{s})(\bar{u}_1 \wedge \bar{v}_1 \wedge \bar{u}_2 \wedge \bar{v}_2)$.

Lemma 3.5. *With the notation above, the following hold:*

(i) *Every smooth hyperplane section $Z_{\langle s\rangle}$ of G contains a unique solid $\Pi_{\langle s\rangle} = \sigma_{3,0}(V^\perp)$, and conversely every solid of G is contained in a (non-unique) smooth hyperplane section of G.*

(ii) *A plane $\sigma_{2,2}(V_3)$ of G is contained in a smooth hyperplane section $Z_{\langle s\rangle}$ if and only if $V^\perp \subset V_3$ and $V_3/V^\perp$ is $\bar{s}$-isotropic. In other words, the $\sigma_{2,2}$-planes in $Z_{\langle s\rangle}$ are in one-to-one correspondence with the k-points of $Q_{\langle\bar{s}\rangle}$, and they intersect the unique solid $\Pi_{\langle s\rangle}$ of $Z_{\langle s\rangle}$ along lines.*

Proof. A solid $\sigma_{3,0}(V_1)$ of G is contained in $Z_{\langle s\rangle}$ if and only if $V = V_1^\perp$, hence, since by hypothesis $V^\perp$ is 1-dimensional, if and only if $V_1 = V^\perp$. Conversely, for every 1-dimensional k-vector subspace V_1 of V, any choice of symplectic form $\bar{s} \in \Lambda^2(V/V_1)^\vee$ on the 4-dimensional k-space V/V_1 determines, through the inclusion $\Lambda^2(V/V_1)^\vee \subset \Lambda^2 V^\vee$, a skew-symmetric form $s \in \Lambda^2 V^\vee$ with $\mathrm{Ker}(\tilde{s}) = V_1$, hence a corresponding smooth hyperplane section $Z_{\langle s\rangle}$ of G containing $\sigma_{3,0}(V_1)$. A plane $\sigma_{2,2}(V_3)$ of G is contained in $Z_{\langle s\rangle}$ if and only if every 2-dimensional k-vector subspace $E \subset V_3$ is s-isotropic, which holds if and only if V_3 is s-isotropic. This property is equivalent to the fact that V_3 contains $V^\perp$ and $\bar{V}_3 = V_3/V^\perp$ is an $\bar{s}$-isotropic subspace of $\bar{V}$. Every $\sigma_{2,2}$-plane with this property intersects $\Pi_{\langle s\rangle} \cong \mathbb{P}_k\big(\bar{V}^\vee\big)$ along the line $\mathbb{P}_k\big(\bar{V}_3^\vee\big)$. □

Remark 3.6. For a smooth hyperplane section $Z = Z_{\langle s\rangle}$ of G, the exact sequence

$$0 \to \mathcal{C}_{Z/G}|_\Pi \cong \Lambda^2\mathcal{Q}^\vee|_\Pi \xrightarrow{\bar{s}} \mathcal{C}_{\Pi/G} \cong \Omega^1_\Pi(1) \to \mathcal{C}_{\Pi/Z} \to 0$$

identifies the conormal sheaf $\mathcal{C}_{\Pi/Z}$ of the unique solid $\Pi = \Pi_{\langle s\rangle} \cong \mathbb{P}_k\big(\bar{V}^\vee\big)$ of $Z_{\langle s\rangle}$ with the null-correlation rank 2 locally sheaf $\mathcal{N}_{\langle\bar{s}\rangle}$ associated with the symplectic form $\bar{s} \in \Lambda^2\bar{V}^\vee$, that is, the sheaf whose fibre over a closed point $\ell \subset \bar{V}_{\bar{k}}$ of $\mathbb{P}_{\bar{k}}(\bar{V}^\vee_{\bar{k}})$ is the quotient $\ell^\perp/\ell$, where $\ell^\perp$ is the $\bar{s}_{\bar{k}}$-orthogonal of ℓ.

3.1.3 *Smooth linear sections of codimension* 2. A codimension 2 linear section of G is the zero scheme W_L of a homomorphism $L_G \to \Lambda^2\mathcal{Q}$ determined by a 2-dimensional k-vector subspace $L \subset H^0\big(G, \Lambda^2\mathcal{Q}\big) = \Lambda^2 V^\vee$, equivalently the intersection of G in the Plücker embedding with the codimension 2 linear subspace $\mathbb{P}_k\big(\Lambda^2 V^\vee/L\big)$ of $\mathbb{P}_k\big(\Lambda^2 V^\vee\big)$. A closed point $E \subset V_{\bar{k}}$ of $G_{\bar{k}}$ belongs to W_L if and only if $s_{\bar{k}}|_E = 0$ for every skew-symmetric form s in L and, arguing as in § 3.1.2, we see from the conormal sequence

$$\mathcal{C}_{W_L/G} = \Lambda^2 \mathcal{Q}^\vee|_{W_L} \otimes L_{W_L} \xrightarrow{d} \Omega^1_{G/k}|_{W_L} = \mathcal{H}om(\mathcal{Q}|_{W_L}, \mathcal{S}|_{W_L}) \to \Omega^1_{W_L/k} \to 0$$

that $W_{L,\bar{k}}$ is smooth at a closed point $E \subset V_{\bar{k}}$ if and only if $E \not\subset \mathrm{Ker}(\tilde{s}_{\bar{k}})$ for all $s \in L \setminus \{0\}$. In particular, W_L is smooth if and only if the line $\mathbb{P}_k(L^\vee) \subset \mathbb{P}_k(\Lambda^2 V)$ is contained in $\mathbb{P}_k(\Lambda^2 V) \setminus \mathbb{G}_k(V, 2)$. This yields a natural correspondence between smooth codimension 2 linear sections W_L of G and k-points of the open subset of $\mathbb{G}_k(\Lambda^2 V, 2)$ parametrizing such lines, which is a homogeneous space under the action of $\mathrm{PGL}_5(k)$.

Lemma 3.7. *A smooth linear section W_L of G does not contain any $\sigma_{3,0}$-solid of G. It contains a unique $\sigma_{2,2}$-plane $\Xi_L = \sigma_{2,2}(V_{3,L})$, where $V_{3,L} \subset V$ is the linear span of the kernels of the linear maps $\tilde{s}$, $s \in L \setminus \{0\}$.*

Proof. By the adjunction formula, a solid Π contained in W_L would have normal sheaf isomorphic to $\mathcal{O}_\Pi(-1)$, and hence $\mathcal{O}_{W_L}(\Pi)$ would be an invertible sheaf of degree 1 on W_L. This is impossible since $\mathrm{Pic}(W_L) \cong \mathbb{Z}$ is generated by an ample invertible sheaf of degree 5. A plane $\sigma_{2,2}(V_3)$ of G is contained in W_L if and only if V_3 contains the linear span of the kernels of the linear maps $\tilde{s}$ corresponding to the forms $s \in L \setminus \{0\}$. The image of $\mathbb{P}_k(L^\vee) \subset \mathbb{P}_k(\Lambda^2 V) \setminus \mathbb{G}_k(V, 2)$ by the morphism $\mathbb{P}_k(\Lambda^2 V) \setminus \mathbb{G}_k(V, 2) \to \mathbb{G}_k(V, 4)$ given by the complete linear system of quadrics containing $\mathbb{G}_k(V, 2)$ is a smooth conic C_L in $\mathbb{G}_k(V, 4) \cong \mathbb{P}_k(V^\vee)$ whose k-points are the kernels $\mathrm{Ker}(\tilde{s}) \subset V$ of the elements $s \in L \setminus \{0\}$. Letting $V_{3,L} \subset V$ be the unique 3-dimensional k-vector subspace such that $\mathbb{P}_k(V_{3,L}^\vee)$ contains C_L, we conclude that $\sigma_{2,2}(V_{3,L})$ is the unique $\sigma_{2,2}$-plane in W_L. □

3.2 Projections from $\sigma_{3,0}$-solids

Let $V_1 \subset V$ be a 1-dimensional k-vector subspace, let $p\colon V \to \bar{V} = V/V_1$ be the quotient morphism, and let $\bar{G} = \mathbb{G}_k(\bar{V}^\vee, 2)$ with universal sequence $0 \to \mathcal{S}_{\bar{G}} \to \bar{V}^\vee_{\bar{G}} \to \mathcal{Q}_{\bar{G}} \to 0$. Let $Z_{\langle s \rangle} \subset G$ be a smooth hyperplane section determined by the image $s \in \Lambda^2 V^\vee$ of a symplectic form $\bar{s} \in \Lambda^2 \bar{V}^\vee$, and let $Q_{\langle \bar{s} \rangle} \subset \bar{G}$ be the zero scheme of $\bar{s}$. Let $\Pi = \sigma_{3,0}(V_1) = G \cap \mathbb{P}_k(\Lambda^2 V^\vee / \Lambda^2 (\bar{V})^\vee)$ be the solid of G contained in $Z_{\langle s \rangle}$ determined by V_1. The *projection of G from the solid* Π is the dominant rational map

$$\pi_\Pi\colon G = \mathbb{G}_k(V^\vee, 2) \dashrightarrow \bar{G} = \mathbb{G}_k(\bar{V}^\vee, 2) \tag{3.2}$$

given by the restriction to G of the linear projection $\mathbb{P}_k(\Lambda^2 V^\vee) \dashrightarrow \mathbb{P}_k(\Lambda^2 \bar{V}^\vee)$. The morphism $\pi_\Pi|_{G \setminus \Pi}$ maps a k-point $E \subset V$ of G not containing V_1 to the k-point $p(E)$ of $\bar{G}$. Conversely, the closure in G of a fibre of π_Π over a k-point $\bar{E} \subset \bar{V}$ of $\bar{G}$ is the plane $\sigma_{2,2}(p^{-1}(\bar{E}))$ of G.

The restriction $\pi_\Pi\colon Z_{\langle s \rangle} \dashrightarrow Q_{\langle \bar{s} \rangle}$ of π_Π to $Z_{\langle s \rangle}$ is called the *projection of $Z_{\langle s \rangle}$ from its solid* $\Pi = \Pi_{\langle s \rangle}$. The restriction $\pi_\Pi|_{Z_{\langle s \rangle} \setminus \Pi}$ maps a k-point of $Z_{\langle s \rangle}$ represented by an s-isotropic k-point $E \subset V$ of G not containing V_1 to the k-point of $Q_{\langle \bar{s} \rangle}$ represented by the $\bar{s}$-isotropic k-point $p(E)$ of $\bar{G}$.

To state the next result, we put $(X_6, \mathbf{Q}_4, \mathcal{E}_6) = (G, \bar{G}, \mathcal{Q}^\vee_{\bar{G}})$ and $(X_5, \mathbf{Q}_3, \mathcal{E}_5) = (Z_{\langle s \rangle}, Q_{\langle \bar{s} \rangle}, \mathscr{S})$, where $\mathscr{S} = \mathcal{Q}^\vee_{\bar{G}}|_{Q_{\langle \bar{s} \rangle}}$ is the spinor locally free sheaf of rank 2 on the quadric threefold $Q_{\langle \bar{s} \rangle} \subset \bar{G}$; see for example [Ott88].

Proposition 3.8. *For $i = 5, 6$, let $Y_\Pi \subset X_i \times \mathbf{Q}_{i-2}$ be the graph of π_Π with projections $\mathrm{p}_{X_i} \colon Y_\Pi \to X_i$, and let $\mathrm{p}_{\mathbf{Q}_{i-2}} \colon Y_\Pi \to \mathbf{Q}_{i-2}$. Then we have the Sarkisov link*

$$\begin{array}{ccccc} \mathbb{P}_\Pi(\mathcal{C}_{\Pi/X_i}) \cong \mathbb{P}_{\mathbf{Q}_{i-2}}(\mathcal{E}_i) & \hookrightarrow & Y_\Pi & \xrightarrow[\mathrm{p}_{\mathbf{Q}_{i-2}}]{\cong} & \mathbb{P}_{\mathbf{Q}_{i-2}}(\mathcal{E}_i \oplus \mathcal{O}_{\mathbf{Q}_{i-2}}) \\ \downarrow & & {\scriptstyle \mathrm{p}_{X_i}} \swarrow \quad \searrow & & \downarrow \\ \Pi & \hookrightarrow & X_i & \overset{\pi_\Pi}{\dashrightarrow} & \mathbf{Q}_{i-2}\,, \end{array} \tag{3.3}$$

where $\mathrm{p}_{X_i} \colon Y_\Pi \to X_i$ is the contraction of the subbundle $\mathbb{P}_{\mathbf{Q}_{i-2}}(\mathcal{E}_i) \subset \mathbb{P}_{\mathbf{Q}_{i-2}}(\mathcal{E}_i \oplus \mathcal{O}_{\mathbf{Q}_{i-2}})$ onto Π.

Proof. The projection $\mathrm{p}_{\bar{G}} \colon Y_\Pi \to \bar{G}$ is isomorphic to the projective bundle $\mathbb{G}_{\bar{G}}(\mathcal{F}, 2) \cong \mathbb{P}_{\bar{G}}(\mathcal{F}^\vee) \to \bar{G}$, where $\mathcal{F}$ is the cokernel of the injective homomorphism $\mathcal{S}_{\bar{G}} \to \bar{V}^\vee_{\bar{G}} \to V^\vee_{\bar{G}}$. The latter is locally free of rank 3, isomorphic to an extension of $V^\vee_{1,\bar{G}}$ by $\mathcal{Q}_{\bar{G}}$, hence to $\mathcal{Q}_{\bar{G}} \oplus \mathcal{O}_{\bar{G}}$ due to the vanishing of $H^1(\bar{G}, \mathcal{Q}_{\bar{G}})$, and the projection $\mathrm{p}_G \colon Y_\Pi \to G$ contracts the projective subbundle $\mathbb{P}_{\bar{G}}(\mathcal{Q}^\vee{}_{\bar{G}}) \subset \mathbb{P}_{\bar{G}}(\mathcal{Q}^\vee{}_{\bar{G}} \oplus \mathcal{O}_{\bar{G}})$ to Π. In particular, this identifies $\mathbb{P}_{\bar{G}}(\mathcal{Q}^\vee_{\bar{G}})$ with the exceptional divisor $\mathbb{P}_\Pi(\mathcal{C}_{\Pi/G})$ of the blow-up of Π in G. The corresponding diagram for the smooth hyperplane section $Z_{\langle s \rangle}$ follows immediately by restriction. □

Example 3.9. With Notation 3.1, the kernel $V^\perp$ of the linear map $\tilde{s}$ associated with the skew-symmetric form $s = e_1^\vee \wedge e_3^\vee - e_2^\vee \wedge e_4^\vee$ equals $\langle e_5 \rangle$. The associated hyperplane section $Z_{\langle s \rangle} = G \cap \{w_{13} - w_{24} = 0\}$ is the smooth fivefold in $\mathbb{P}^8_k \subset \mathbb{P}^9_k$ with coordinates w_{ij}, $(i,j) \neq (2,4)$, defined by the equations

$$\begin{cases} w_{12}w_{34} - w_{13}^2 + w_{14}w_{23} = 0\,, \\ w_{12}w_{35} - w_{13}w_{25} + w_{15}w_{23} = 0\,, \\ w_{12}w_{45} - w_{14}w_{25} + w_{13}w_{15} = 0\,, \\ w_{13}w_{45} - w_{14}w_{35} + w_{15}w_{34} = 0\,, \\ w_{23}w_{45} - w_{13}w_{35} + w_{25}w_{34} = 0\,. \end{cases}$$

Putting $\bar{V} = V/\langle e_5 \rangle$, the image of G by the projection

$$\mathbb{P}_k(\Lambda^2 V^\vee) = \mathbb{P}^9_k \dashrightarrow \mathbb{P}^5_k = \mathbb{P}_k(\Lambda^2 \bar{V}^\vee)\,, \quad [w_{ij}]_{1 \leqslant i < j \leqslant 5} \mapsto [w_{12} : w_{13} : w_{14} : w_{23} : w_{24} : w_{34}]$$

from the solid $\Pi = \sigma_{3,0}(V^\perp) = \{w_{12} = w_{13} = w_{14} = w_{23} = w_{24} = w_{34} = 0\}$ is the smooth quadric fourfold $\bar{G} = \mathbb{G}_k(\bar{V}^\vee, 2) = \{\overline{w}_{12}\overline{w}_{34} - \overline{w}_{13}\overline{w}_{24} + \overline{w}_{14}\overline{w}_{23} = 0\}$ in $\mathbb{P}^5_k$ with Plücker coordinates $\bar{w}_{ij} = \bar{e}_i^\vee \wedge \bar{e}_j^\vee$, $1 \leqslant i < j \leqslant 4$, where $\bar{e}_i$ denotes the image of e_i in $\bar{V}$. The image of $Z_{\langle s \rangle}$ by this projection is the smooth quadric threefold $Q_{\langle \bar{s} \rangle} = \{\overline{w}_{12}\overline{w}_{34} - \overline{w}_{13}^2 + \overline{w}_{14}\overline{w}_{23} = 0\}$ in $\mathbb{P}^4_k \subset \mathbb{P}^5_k$ with coordinates $\bar{w}_{ij}$, $(i,j) \neq (2,4)$.

With the notation above, let $\mathrm{Sp}_k(\bar{V}^\vee, \bar{s})$ be the symplectic group of the symplectic form $\bar{s} \in \Lambda^2 \bar{V}^\vee$, and let $\mathrm{PSp}_k(\bar{V}^\vee, \bar{s})$ be its image in $\mathrm{PGL}_k(\bar{V}^\vee)$. We infer the following description.

Corollary 3.10. *There exists a split exact sequence of k-group schemes*

$$\begin{aligned} 1 &\to \mathrm{Aut}_k(Z_{\langle s \rangle}, \Pi_{\langle s \rangle})_0 \cong \mathbb{G}^4_{a,k} \rtimes \mathbb{G}_{m,k} \to \mathrm{Aut}_k(Z_{\langle s \rangle}, \Pi_{\langle s \rangle}) = \mathrm{Aut}_k(Z_{\langle s \rangle}) \\ &\to \mathrm{Aut}_k(Q_{\langle s \rangle}) \cong \mathrm{PSp}_k(\bar{V}^\vee, \bar{s}) \to 1\,, \end{aligned}$$

where the group $\mathrm{Aut}_k(Z_{\langle s \rangle}, \Pi_{\langle s \rangle})_0$ is the kernel of the restriction homomorphism $\mathrm{Aut}_k(Z_{\langle s \rangle}, \Pi_{\langle s \rangle}) \to \mathrm{Aut}_k(\Pi_{\langle s \rangle})$. Moreover, up to the choice of a splitting $V \cong \bar{V} \oplus V^\perp$, the group $\mathrm{Aut}_k(Z_{\langle s \rangle})$ is the image by the restriction homomorphism $\mathrm{Aut}_k(G, Z_{\langle s \rangle}) \to \mathrm{Aut}_k(Z_{\langle s \rangle})$ of the subgroup

$$\left\{ \begin{pmatrix} \mathrm{Sp}_k(\bar{V}^\vee, \bar{s}) & \mathrm{Hom}_k\big((V^\perp)^\vee, \bar{V}^\vee\big) \\ 0 & \mathbb{G}_{m,k} \end{pmatrix} \right\} \Big/ \{\pm \mathrm{Id}\}$$

of $\mathrm{PGL}_k(V^\vee)$*, seen as a subgroup of* $\mathrm{Aut}_k(G)$ *via the isomorphism* $\Phi\colon \mathrm{PGL}_k(V^\vee) \to \mathrm{Aut}_k(G)$ *of* § 2.2.2*.*

Proof. Since, by Lemma 3.5, the object $\Pi = \Pi_{\langle s\rangle}$ is the unique solid contained in $Z = Z_{\langle s\rangle}$, we have $\mathrm{Aut}_k(Z) = \mathrm{Aut}_k(Z, \Pi)$. The action of $\mathrm{Aut}_k(Z, \Pi)$ lifts to the blow-up $\mathrm{p}_Z\colon Y \to Z$ of Π, and since the fibres of the projection $\pi_\Pi\colon Z \dashrightarrow Q_{\langle \bar{s}\rangle}$ of Z from Π over k-points of $Q_{\langle \bar{s}\rangle}$ meet Π along lines, $\mathrm{Aut}_k(Z, \Pi)_0$ equals the kernel of the homomorphism $\mathrm{Aut}_k(Z, \Pi) \to \mathrm{Aut}_k(Q_{\langle \bar{s}\rangle})$ induced by $\mathrm{p}_{Q_{\langle \bar{s}\rangle}}\colon Y \to Q_{\langle \bar{s}\rangle}$. Let $p\colon \mathrm{Aut}_k(Z, \Pi) \to B := \mathrm{Aut}_k(Z, \Pi)/\mathrm{Aut}_k(Z, \Pi)_0$ be the quotient morphism, and let $\gamma\colon B \to \mathrm{Aut}_k(Q_{\langle \bar{s}\rangle})$ be the induced injective homomorphism. Let $\Delta_{V^\perp} \subset \mathrm{GL}_k(V^\vee)$ be the stabilizer of $\bar{V}^\vee \subset V^\vee$; see Notation 3.4. Let $\Delta_{V^\perp,0} \cong \mathbb{G}_{a,k}^4 \rtimes \mathbb{G}_{m,k}$ be its normal subgroup consisting of matrices $M(\mathrm{id}_4, \lambda, U)$, and let $S_{V^\perp}$ be its subgroup consisting of matrices $M(A_4, \pm 1, 0)$ with $A_4 \in \mathrm{Sp}_k(\bar{V}^\vee, \bar{s})$. The image of $\mathrm{P}\Delta_{V^\perp} \subset \mathrm{PGL}_k(V^\vee)$ by $\Phi\colon \mathrm{PGL}_k(V^\vee) \to \mathrm{Aut}_k(G)$ is contained in the stabilizer $\mathrm{Aut}_k(G, (Z, \Pi))$ of the pair (Z, Π). A direct verification shows that the homomorphism $j\colon \mathrm{P}\Delta_{V^\perp} \to \mathrm{Aut}_k(Z)$ obtained by composing with the restriction homomorphism $\mathrm{Aut}_k(G, Z) \to \mathrm{Aut}_k(Z)$ is injective and maps $\mathrm{P}\Delta_{V^\perp,0} \cong \Delta_{V^\perp,0}$ isomorphically onto $\mathrm{Aut}_k(Z, \Pi)_0$. Letting $\bar{q} \in \mathrm{Sym}^2(\Lambda^2\bar{V}^\vee/\langle \bar{s}\rangle)$ be the quadratic form associated with the symplectic form $\bar{s}$ (see § 3.1.2), the conclusion then follows from the fact that the restriction of the composition

$$\mathrm{P}\Delta_{V^\perp} \xrightarrow{j} \mathrm{Aut}_k(Z, \Pi) \xrightarrow{p} B \xrightarrow{\gamma} \mathrm{Aut}_k(Q_{\langle \bar{s}\rangle}) = \mathrm{PO}_k(\Lambda^2\bar{V}^\vee/\langle s\rangle, \bar{q}) = \mathrm{SO}_k(\Lambda^2\bar{V}^\vee/\langle s\rangle, \bar{q})$$

to the subgroup $\mathrm{P}S_{V^\perp} \cong \mathrm{PSp}_k(\bar{V}^\vee, \bar{s})$ is an isomorphism onto its image $\mathrm{SO}_k(\Lambda^2\bar{V}^\vee/\langle s\rangle, \bar{q})$. □

3.3 Projections from $\sigma_{2,2}$-planes

Let $V_3 \subset V$ be a 3-dimensional k-vector subspace, let $K = \Lambda^2 V/\Lambda^2 V_3$, and let $\langle s\rangle \subset K^\vee$ and $L \subset K^\vee$ be, respectively, a 1-dimensional and a 2-dimensional linear subspace of skew-symmetric bilinear forms on V whose non-zero elements all have maximal rank. These data determine a plane $\Xi = \sigma_{2,2}(V_3) = G \cap \mathbb{P}_k(\Lambda^2 V_3^\vee)$ of G, a smooth hyperplane section $Z_{\langle s\rangle}$ of G and a smooth codimension 2 linear section W_L of G; both $Z_{\langle s\rangle}$ and W_L contain Ξ.

- The *projection of* G *from the plane* Ξ is the birational map

$$\pi_\Xi\colon G = \mathbb{G}_k(V^\vee, 2) \dashrightarrow \mathbb{P}_k(K^\vee) \tag{3.4}$$

induced by the restriction to G of the linear projection $\mathbb{P}_k(\Lambda^2 V^\vee) \dashrightarrow \mathbb{P}_k(K^\vee)$. The morphism $\pi_\Xi|_{G\setminus\Xi}$ maps a k-point $E \subset V$ of G not contained in V_3 to the image of $\Lambda^2 E \subset \Lambda^2 V$ by the quotient homomorphism $\Lambda^2 V \to K$. Let $Z_\Xi = G \cap \mathbb{P}_k(\Lambda^2 V^\vee/\Lambda^2(V/V_3)^\vee)$ and $H_G = \mathbb{P}_k(K^\vee/\Lambda^2(V/V_3)^\vee)$ be the hyperplane sections of G and $\mathbb{P}_k(K^\vee)$ determined by the 1-dimensional k-vector subspace $\Lambda^2(V/V_3)^\vee$ of $K^\vee \subset \Lambda^2 V^\vee$. Then the image S_G of the rational map $\pi_\Xi|_{Z_\Xi}\colon Z_\Xi \dashrightarrow H_G$ equals that of the Segre embedding

$$s_{1,1}\colon \mathbb{P}_k(V_3^\vee) \times \mathbb{P}_k((V/V_3)^\vee) \hookrightarrow \mathbb{P}_k(V_3^\vee \otimes_k (V/V_3)^\vee) \cong \mathbb{P}_k(K^\vee/\Lambda^2(V/V_3)^\vee)\,.$$

A k-point $E \subset V$ of G belongs to Z_Ξ if and only if $E \cap V_3 \neq \{0\}$, and the closure in G of the fibre of $\pi_\Xi|_{Z_\Xi\setminus\Xi}$ over the image by $s_{1,1}$ of a k-point $(V_1 \subset V_3, V_4/V_3 \subset V/V_3)$ of $\mathbb{P}_k(V_3^\vee) \times \mathbb{P}_k((V/V_3)^\vee)$ is the $\sigma_{3,1}$-plane $\sigma_{3,1}(V_1 \subset V_4)$.

- Since $\langle s\rangle \subset K^\vee$, on $Z_{\langle s\rangle}$, the projection of G from Ξ restricts to the birational map

$$\pi_\Xi\colon Z_{\langle s\rangle} \dashrightarrow \mathbb{P}_k(K^\vee/\langle s\rangle)$$

defined by the complete linear system of hyperplane sections of $Z_{\langle s\rangle}$ containing Ξ, called the *projection of* $Z_{\langle s\rangle}$ *from the plane* Ξ. Letting $H_{Z_{\langle s\rangle}} = H_G \cap \mathbb{P}_k(K^\vee/\langle s\rangle)$, the image of the induced rational map $\pi_\Xi|_{Z_\Xi\cap Z_{\langle s\rangle}}\colon Z_\Xi\cap Z_{\langle s\rangle} \dashrightarrow H_{Z_{\langle s\rangle}}$ is the smooth cubic scroll $S_{Z_{\langle s\rangle}} = S_G\cap\mathbb{P}_k\big(K^\vee/\langle s\rangle\big)$ in $\mathbb{P}_k\big(K^\vee/\langle s\rangle\big)$.

- Since $L\subset K^\vee$, on W_L, the projection of G from Ξ restricts to the birational map

$$\pi_\Xi\colon W_L \dashrightarrow \mathbb{P}_k\big(K^\vee/L\big) \tag{3.5}$$

defined by the complete linear system of hyperplane sections of W_L containing Ξ, called the *projection of W_L from the plane Ξ*. Putting $H_{W_L} = H_G\cap\mathbb{P}_k\big(K^\vee/L\big)$, the image of the induced rational map $\pi_\Xi|_{Z_\Xi\cap W_L}\colon Z_\Xi\cap W_L \dashrightarrow H_{W_L}$ is the smooth rational cubic curve $S_{W_L} = S_G\cap\mathbb{P}_k\big(K^\vee/L\big)$ in $\mathbb{P}_k(K^\vee/L)$.

To state the next result, we put $(X_6,\mathbf{P}_6) = \big(G,\mathbb{P}_k\big(K^\vee\big)\big)$, $(X_5,\mathbf{P}_5) = \big(Z_{\langle s\rangle},\mathbb{P}_k\big(K^\vee/\langle s\rangle\big)\big)$ and $(X_4,\mathbf{P}_4) = \big(W_L,\mathbb{P}_k\big(K^\vee/L\big)\big)$. We denote by $Y_{\Xi,i}\subset X_i\times\mathbf{P}_i$ the graph of $\pi_{\Xi,i} = \pi_\Xi\colon X_i\dashrightarrow\mathbf{P}_i$, $i=4,5,6$.

Proposition 3.11. *With the notation above, for $i=4,5,6$, we have the Sarkisov link*

$$\begin{array}{ccccccccc}
\mathrm{Bl}_{S_{X_i}} H_{X_i}\cong\mathbb{P}_\Xi(\mathcal{C}_{\Xi/X_i}) & \hookrightarrow & & Y_{\Xi,i} & & \hookleftarrow & & & \mathrm{Bl}_\Xi(Z_\Xi\cap X_i) \\
\downarrow & & \swarrow{\scriptstyle \mathrm{p}_{X_i}} & & \searrow{\scriptstyle \mathrm{p}_2} & & & & \downarrow \\
\Xi & \hookrightarrow & Z_\Xi\cap X_i \hookrightarrow X_i & \underset{\Phi_i}{\overset{\pi_{\Xi,i}}{\rightleftarrows}} & \mathbf{P}_i & \hookleftarrow & H_{X_i} & \hookleftarrow & S_{X_i}\,,
\end{array} \tag{3.6}$$

where $\mathrm{p}_{X_i}\colon Y_{\Xi,i}\to X_i$ is the blow-up of Ξ, $\mathrm{Bl}_\Xi(Z_\Xi\cap X_i)$ is the proper transform of $Z_\Xi\cap X_i$, $\mathrm{p}_2\colon Y_{\Xi,i}\to\mathbf{P}_i$ is the blow-up of S_{X_i} and $\mathrm{Bl}_{S_{X_i}} H_{X_i}$ is the proper transform of $H_{X_i}\subset\mathbf{P}_i$. The birational inverse Φ_i of $\pi_{\Xi,i}$ is given by the complete linear system of quadrics in $\mathbf{P}_i$ containing S_{X_i}.

Proof. All the properties follow from [Tod30] and the description above. □

Example 3.12. With Notation 3.1, let $V_3 = \langle e_3,e_4,e_5\rangle$, let

$$\Xi = \sigma_{2,2}(V_3) = \{w_{12}=w_{13}=w_{14}=w_{15}=w_{23}=w_{24}=w_{25}=0\}$$

be the associated plane of G, and let

$$\mathbb{P}_k\big(\Lambda^2V^\vee\big) = \mathbb{P}^9_k \dashrightarrow \mathbb{P}^6_k = \mathbb{P}_k\big(K^\vee\big)\,,\quad [w_{ij}]_{1\leqslant i<j\leqslant 5}\mapsto[w_{12}:w_{13}:w_{14}:w_{15}:w_{23}:w_{24}:w_{25}]$$

be the associated linear projection. The skew-symmetric forms $s = e_1^\vee\wedge e_3^\vee - e_2^\vee\wedge e_4^\vee$ and $s' = e_1^\vee\wedge e_4^\vee - e_2^\vee\wedge e_5^\vee$ generate a subspace $L\subset\Lambda^2V^\vee$ whose non-zero elements all have rank 4. The associated smooth linear section $W_L = G\cap\{w_{13}-w_{24}=0\}\cap\{w_{14}-w_{25}=0\}$ is the smooth fourfold in $\mathbb{P}^7_k\subset\mathbb{P}^9_k$ with coordinates w_{ij}, $(i,j)\neq(2,4),(2,5)$ defined by the equations

$$\begin{cases}
w_{12}w_{34} - w_{13}^2 + w_{14}w_{23} = 0\,,\\
w_{12}w_{35} - w_{13}w_{14} + w_{15}w_{23} = 0\,,\\
w_{12}w_{45} - w_{14}^2 + w_{13}w_{15} = 0\,,\\
w_{13}w_{45} - w_{14}w_{35} + w_{15}w_{34} = 0\,,\\
w_{23}w_{45} - w_{13}w_{35} + w_{14}w_{34} = 0\,.
\end{cases}$$

The smooth hyperplane section $Z_{\langle s\rangle} = G\cap\{w_{13}-w_{24}=0\}$ of Example 3.9 and the above smooth linear section W_L both contain Ξ. The images of $Z_\Xi = G\cap\{w_{12}=0\}$, $Z_\Xi\cap Z_{\langle s\rangle}$ and $Z_\Xi\cap W_L$ by the projection π_Ξ and its successive restrictions are the smooth threefold $S_G\cong\mathbb{P}^1_k\times\mathbb{P}^2_k$, the smooth rational cubic surface $S_{Z_{\langle s\rangle}}\cong\mathbb{P}_{\mathbb{P}^1_k}\big(\mathcal{O}_{\mathbb{P}^1_k}(1)\oplus\mathcal{O}_{\mathbb{P}^1_k}(2)\big)\cong\mathbb{F}_1$ and the smooth rational normal curve $S_{W_L} \cong \mathbb{P}^1_k$ with equations

$$\begin{cases} -w_{13}w_{24} + w_{14}w_{23} = 0\,, \\ -w_{13}w_{25} + w_{15}w_{23} = 0\,, \\ -w_{14}w_{25} + w_{15}w_{24} = 0\,, \end{cases} \quad \begin{cases} -w_{13}^2 + w_{14}w_{23} = 0\,, \\ -w_{13}w_{25} + w_{15}w_{23} = 0\,, \\ -w_{14}w_{25} + w_{15}w_{13} = 0\,, \end{cases} \quad \text{and} \quad \begin{cases} -w_{13}^2 + w_{14}w_{23} = 0\,, \\ -w_{13}w_{14} + w_{15}w_{23} = 0\,, \\ -w_{14}^2 + w_{15}w_{13} = 0 \end{cases}$$

in $\mathbb{P}^5_k$, $\mathbb{P}^4_k$ and $\mathbb{P}^3_k$, respectively.

Remark 3.13. By a result attributed to Weil [Wei57], the smooth varieties $\mathbb{P}^1_k \times \mathbb{P}^2_k \subset \mathbb{P}^5_k$, $\mathbb{F}_1 \subset \mathbb{P}^4_k$ and the rational normal curve $\mathbb{P}^1_k \subset \mathbb{P}^3_k$ are the only smooth cubics which are not hypersurfaces.

For a smooth codimension 2 linear section W_L of G with unique $\sigma_{2,2}$-plane $\Xi_L = \sigma_{2,2}(V_{3,L})$, we infer the following description of $\mathrm{Aut}_k(W_L)$.

Corollary 3.14. *There exists a split exact sequence of k-group schemes*

$$\begin{aligned} 1 \to \mathrm{Aut}_k(W_L, \Xi_L)_0 \cong \mathbb{G}^4_{a,k} \rtimes \mathbb{G}_{m,k} &\to \mathrm{Aut}_k(W_L, \Xi_L) = \mathrm{Aut}_k(W_L) \\ &\to \mathrm{Aut}_k(S_{W_L}) \cong \mathrm{PGL}_k(L) \to 1\,, \end{aligned}$$

where $\mathrm{Aut}_k(W_L, \Xi_L)_0$ *is the kernel of the restriction homomorphism* $\mathrm{Aut}_k(W_L, \Xi_L) \to \mathrm{Aut}_k(\Xi_L)$.

Proof. Since, by Lemma 3.7, the plane $\Xi = \Xi_L$ is the unique $\sigma_{2,2}$-plane contained in W_L and since the intersection $Z_\Xi \cap W_L$ is the union of all the $\sigma_{3,1}$-planes contained in W_L, we have $\mathrm{Aut}_k(W_L) = \mathrm{Aut}_k(W_L, (Z_\Xi, \Xi))$. Since the $\sigma_{3,1}$-planes of W_L intersect $\Xi \cong \mathbb{P}_k(V_{3,L})$ along the smooth conic $C_{L^\vee}$ dual to $C_L \cong \mathbb{P}_k(L^\vee) \hookrightarrow \mathbb{P}_k(V_{3,L}^\vee)$ (see the proof of Lemma 3.7), the image of the restriction homomorphism $\mathrm{Aut}_k(W_L, \Xi_L) \to \mathrm{Aut}_k(\Xi_L)$ is contained in the subgroup

$$\mathrm{Aut}(\Xi_L, C_{L^\vee}) \cong \mathrm{Aut}_k(C_{L^\vee}) \cong \mathrm{PGL}_k(L)\,.$$

Since, on the other hand, the $\sigma_{3,1}$-planes of W_L are the closures of the fibres of $\pi_\Xi \colon W_L \setminus \Xi \to \mathbb{P}_k(K^\vee/L)$ over the k-points of S_{W_L}, the projection $\pi_\Xi \colon W_L \dashrightarrow \mathbb{P}_k(K^\vee/L)$ induces an isomorphism of k-group schemes $\mathrm{Aut}_k(W_L, (Z_\Xi, \Xi)) \cong \mathrm{Aut}_k(\mathbb{P}_k(K^\vee/L), (H_{W_L}, S_{W_L}))$ which maps $\mathrm{Aut}_k(W_L, \Xi_L)_0$ isomorphically onto the kernel $\mathbb{G}^4_{a,k} \rtimes \mathbb{G}_{m,k}$ of the restriction homomorphism $\mathrm{Aut}_k(\mathbb{P}_k(K^\vee/L), (H_{W_L}, S_{W_L})) \to \mathrm{Aut}_k(S_{W_L})$. The latter homomorphism is a split surjection which identifies $\mathrm{Aut}_k(S_{W_L})$ with $\mathrm{Aut}_k(H_{W_L}, S_{W_L})$. □

4. Smooth quintic del Pezzo varieties with vector group structures

Recall that a smooth quintic del Pezzo k-variety of dimension $n \in \{2, \dots, 6\}$ is a k-form X of a smooth section of the Grassmannian $\mathrm{G}(2,5) \subset \mathbb{P}^9_k$ by a linear subspace of dimension $6-n$. For $n \leqslant 3$, the automorphism group of $X_{\bar{k}}$ is too small to allow the existence of a vector subgroup structure on $X_{\bar{k}}$. In this section, we consider the case of smooth quintic del Pezzo k-varieties of dimension 4, 5 and 6.

4.1 Toric vector groups structures on linear sections of G(2, 5)

Let V be a k-vector space of dimension 5 and, with the notation introduced in § 3.3, let $X_6 = \mathbb{G}_k(V^\vee, 2)$, let $V_3 \subset V$ be a 3-dimensional k-vector subspace with associated plane $\Xi = \sigma_{2,2}(V_3)$ of X_6, and let $K_6 = \Lambda^2 V/\Lambda^2 V_3$. Let $\langle s \rangle \subset K_6^\vee$ and $L \subset K_6^\vee$ be, respectively, a 1-dimensional and a 2-dimensional linear subspace of skew-symmetric bilinear forms on V whose non-zero elements all have maximal rank. Put $K_5^\vee = K_6^\vee/\langle s \rangle$ and $K_4^\vee = K_6^\vee/L$, and let $X_5 = Z_{\langle s \rangle}$ and $X_4 = W_L$ be the smooth linear sections of X_6 containing Ξ defined by $\langle s \rangle$ and L, respectively.

Let $F_i = \mathrm{Hom}_k\big(K_i^\vee/\Lambda^2(V/V_3)^\vee, \Lambda^2(V/V_3)^\vee\big) \cong k^{\oplus i}$, $i = 4,5,6$, and let $\mathbb{V}_k\big(F_i^\vee\big)$ be the associated vector group. We derive from the exact sequence

$$0 \to \Lambda^2(V/V_3)^\vee \xrightarrow{a} K_i^\vee \xrightarrow{b} K_i^\vee/\Lambda^2(V/V_3)^\vee \to 0\,, \quad i = 4,5,6\,,$$

a faithful homomorphism of k-group schemes $\mathbb{V}_k\big(F_i^\vee\big) \to \mathrm{GL}_k\big(K_i^\vee\big)$, $f \mapsto \mathrm{id}_{K_i^\vee} + a \circ f \circ b$ corresponding to a $\mathbb{V}_k\big(F_i^\vee\big)$-action on $\mathbb{P}_k\big(K_i^\vee\big)$ restricting to the trivial action on the invariant hyperplane $H_{X_i} = \mathbb{P}_k\big(K_i^\vee/\Lambda^2(V/V_3)^\vee\big)$ and having $\mathbb{P}_k\big(K_i^\vee\big) \setminus H_{X_i}$ as an open orbit. By Proposition 3.11, this action lifts through the birational projection $\pi_\Xi\colon X_i \dashrightarrow \mathbb{P}_k\big(K_i^\vee\big)$ from the plane Ξ to a $\mathbb{V}_k\big(F_i^\vee\big)$-action on X_i with open orbit $X_i \setminus (Z_\Xi \cap X_i)$ and with trivial restriction to Ξ. As a consequence, we obtain the following.

Proposition 4.1. *Every smooth section of* $\mathrm{G}(2,5) \subset \mathbb{P}_k^9$ *by a linear subspace of codimension at most* 3 *admits a vector group structure.*

Example 4.2. With the notation of Example 3.12, let $V_3 = \langle e_3, e_4, e_5\rangle$ with associated linear projection

$$\mathbb{P}_k^9 \dashrightarrow \mathbb{P}_k^6\,, \quad [w_{ij}]_{1\leqslant i<j\leqslant 5} \mapsto [w_{12} : w_{13} : w_{14} : w_{15} : w_{23} : w_{24} : w_{25}],$$

let $s = e_1^\vee \wedge e_3^\vee - e_2^\vee \wedge e_4^\vee \in K_6^\vee$ and $L \subset K_6^\vee$ be the subspace generated by s and $s' = e_1^\vee \wedge e_4^\vee - e_2^\vee \wedge e_5^\vee$.

- For the basis $t_{ij} = \big(e_i^\vee \wedge e_j^\vee\big) \otimes (\bar{e}_1 \wedge \bar{e}_2)$, $i = 1,2$, $j = 3,4,5$, of $F_6^\vee$, the corresponding action of $\mathbb{V}_k\big(F_6^\vee\big) \cong \mathrm{Spec}(k[t_{ij}])$ on $\mathbb{P}_k\big(K_6^\vee\big) = \mathbb{P}_k^6$ with open orbit $\mathbb{P}_k^6 \setminus \{w_{12} = 0\}$ is the "toric" $\mathbb{G}_a^6$-structure on $\mathbb{P}_k^6$ given by $w_{12} \mapsto w_{12}$ and $w_{ij} \mapsto w_{ij} + t_{ij}w_{12}$ for $(i,j) \neq (1,2)$.[2] Its lift to $X_6 = G$ is the restriction of the $\mathbb{V}_k\big(F_6^\vee\big)$-action on $\mathbb{P}_k\big(\Lambda^2 V^\vee\big)$ induced by the second exterior power of the representation

$$\rho_6\colon \mathbb{V}_k\big(F_6^\vee\big) \to \mathrm{GL}_k\big(V^\vee\big)\,, \quad (t_{13}, t_{23}, t_{14}, t_{24}, t_{15}, t_{25}) \mapsto \begin{pmatrix} 1 & 0 & -t_{23} & -t_{24} & -t_{25} \\ 0 & 1 & t_{13} & t_{14} & t_{15} \\ \vdots & \ddots & 1 & 0 & 0 \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}. \quad (4.1)$$

Comparing with the classification in [ST66, § IV.3.3], we see that the image of ρ_6 is one of the two maximal abelian unipotent subgroups of GL_5 corresponding to cases N_2 and N_3. The other one, given by the representation dual to ρ_6, corresponds to a vector group structure on $\mathbb{G}_k(V,2) \cong \mathbb{G}_k\big(V^\vee, 3\big)$.

- The corresponding action of $\mathbb{V}_k\big(F_5^\vee\big) \cong \mathrm{Spec}(k[t_{13}, t_{23}, t_{14}, t_{15}, t_{25}])$ on $\mathbb{P}_k\big(K_5^\vee\big) = \mathbb{P}_k^5$ with open orbit $\mathbb{P}_k^5 \setminus \{w_{12} = 0\}$ is the toric $\mathbb{G}_a^5$-structure on $\mathbb{P}_k^5$ given by $w_{12} \mapsto w_{12}$ and $w_{ij} \mapsto w_{ij} + t_{ij}w_{12}$ for $(i,j) \neq (1,2), (2,4)$. Its lift to $X_5 = Z_{\langle s\rangle}$ is the restriction of the $\mathbb{V}_k\big(F_5^\vee\big)$-action on $\mathbb{P}_k\big(\Lambda^2 V^\vee\big)$ preserving $Z_{\langle s\rangle} = G \cap \{w_{13} - w_{24} = 0\}$ induced by the second exterior power of

[2] By [AR17], every complete toric variety admitting a vector group structure has such a structure which is normalized by the torus. We used the term "toric" here to indicate the fact that the given $\mathbb{G}_a^6$-action is normalized by the toric structure on $\mathbb{P}_k^6$.

the representation

$$\rho_5\colon \mathbb{V}_k\big(F_5^\vee\big) \to \mathrm{GL}_k\big(V^\vee\big)\,, \quad (t_{13},t_{23},t_{14},t_{15},t_{25}) \mapsto \begin{pmatrix} 1 & 0 & -t_{23} & -t_{13} & -t_{25} \\ 0 & 1 & t_{13} & t_{14} & t_{15} \\ \vdots & \ddots & 1 & 0 & 0 \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}. \tag{4.2}$$

The stabilizer $\mathrm{Stab}\big(\langle e_5^\vee\rangle\big)$ of the subspace $\langle e_5^\vee\rangle$ is the subgroup $\mathrm{Spec}(k[t_{15},t_{25}])$ of $\mathbb{V}_k\big(F_5^\vee\big)$. The induced action of the vector group $\mathbb{V}_k\big(F_5^\vee\big)/\,\mathrm{Stab}\big(\langle e_5^\vee\rangle\big) \cong \mathrm{Spec}(k[\bar{t}_{13},\bar{t}_{23},\bar{t}_{14}])$ on $\bar{V}^\vee = (V/\langle e_5\rangle)^\vee$ endowed with the basis dual to that determined by the images $\bar{e}_i$ of the e_i, $i=1,\ldots,4$, is given by the representation

$$\bar{\rho}_5\colon \mathbb{V}_k\big(F_5^\vee\big)/\,\mathrm{Stab}\big(\langle e_5^\vee\rangle\big) \to \mathrm{Sp}_4\big(\bar{V}^\vee,\bar{s}\big)\,, \quad (\bar{t}_{13},\bar{t}_{23},\bar{t}_{14}) \mapsto \begin{pmatrix} 1 & 0 & -\bar{t}_{23} & -\bar{t}_{13} \\ 0 & 1 & \bar{t}_{13} & \bar{t}_{14} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.3}$$

where $\bar{s} = \bar{e}_1^\vee \wedge \bar{e}_3^\vee - \bar{e}_2^\vee \wedge \bar{e}_4^\vee$ is the symplectic form on $\bar{V}$ induced by s.

• The corresponding action of $\mathbb{V}_k\big(F_4^\vee\big) \cong \mathrm{Spec}(k[t_{13},t_{23},t_{14},t_{15},])$ on $\mathbb{P}_k\big(K_4^\vee\big) = \mathbb{P}_k^4$ with open orbit $\mathbb{P}_k^4 \setminus \{w_{12}=0\}$ is the toric $\mathbb{G}_a^4$-structure on $\mathbb{P}_k^4$ given by $w_{12} \mapsto w_{12}$ and $w_{ij} \mapsto w_{ij} + t_{ij}w_{12}$ for $(i,j) \neq (1,2),(2,4),(2,5)$. Its lift to $X_4 = W_L$ is the restriction of the $\mathbb{V}_k\big(F_4^\vee\big)$-action on $\mathbb{P}_k\big(\Lambda^2 V^\vee\big)$ preserving $W_L = G \cap \{w_{13} - w_{24} = w_{14} - w_{25} = 0\}$ induced by the second exterior power of the representation

$$\rho_4\colon \mathbb{V}_k\big(F_4^\vee\big) \to \mathrm{GL}_k\big(V^\vee\big)\,, \quad (t_{13},t_{23},t_{14},t_{15}) \mapsto \begin{pmatrix} 1 & 0 & -t_{23} & -t_{13} & -t_{14} \\ 0 & 1 & t_{13} & t_{14} & t_{15} \\ \vdots & \ddots & 1 & 0 & 0 \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}. \tag{4.4}$$

Let Ξ be the plane $\sigma_{2,2}(V_3) \subset X_n$, and let H_n be the hyperplane section $\{w_{12}=0\}$ of X_n. Putting $S_6 = S_G \cong \mathbb{P}_k^1 \times \mathbb{P}_k^2$, $S_5 = S_{Z_{\langle s\rangle}} \cong \mathbb{F}_1$ and $S_4 = S_{W_L} \cong \mathbb{P}_k^1$, the varieties X_n, $n=4,5,6$, have an equivariant stratification $\Xi \subset H_n \subset X_n$ with respect to the $\mathbb{G}_a^n$-action constructed above for which $X_n \setminus H_n$ is the open orbit and Ξ is the fixed locus and for which the restriction $H_n \setminus \Xi \to S_n$ of the projection from Ξ is a Zariski locally trivial $\mathbb{A}^2$-bundle whose closed fibres are the orbits of the induced action $\mathbb{G}_a^n$-action on $H_n \setminus \Xi$.

4.2 Proof of Theorem 1.1

We now proceed to the proof of Theorem 1.1; each case $n \in \{4,5,6\}$ is treated separately in the next subsections.

4.2.1 *Proof of Theorem 1.1 for sixfolds.* A smooth quintic del Pezzo sixfold X is a k-form of the Grassmannian $\mathrm{G}(2,5)$. Recall [BS64] that isomorphism classes of k-forms of a projective k-variety X are in one-to-one correspondence with the elements of the Galois cohomology set $H^1(\Gamma,\mathrm{Aut}_{\bar{k}}(X_{\bar{k}}))$ of continuous Galois 1-cocycles $\gamma\colon \Gamma = \mathrm{Gal}(\bar{k}/k) \to \mathrm{Aut}_{\bar{k}}(X_{\bar{k}})$, where $\mathrm{Aut}_{\bar{k}}(X_{\bar{k}})$ is endowed with the discrete topology and the natural action of Γ by conjugation. Since $\mathrm{Aut}_{\bar{k}}(\mathrm{G}(2,5)_{\bar{k}}) \cong \mathrm{PGL}_5(\bar{k}) = \mathrm{Aut}_{\bar{k}}\big(\mathbb{P}_{\bar{k}}^4\big)$ (see § 2.2), isomorphism classes of k-forms of $\mathrm{G}(2,5)$ are in one-to-one correspondence with isomorphism classes of k-forms P of $\mathbb{P}^4_k$. In view of Lemma 2.4, the existence of a k-rational point is a necessary condition for the existence of a vector group structure on X. Conversely, for a k-form X of $\mathrm{G}(2,5)$ containing a k-rational point, the corresponding k-form P of $\mathbb{P}^4_k$ contains a closed subvariety C defined over k whose base extension $C_{\bar{k}}$ is a line in $P_{\bar{k}} \cong \mathbb{P}^4_{\bar{k}}$. Since the restriction of a canonical divisor K_P of P to C has odd degree, C is the trivial k-form of $\mathbb{P}^1_k$, and hence P is the trivial k-form of $\mathbb{P}^4_k$. This implies in turn that X is isomorphic to $\mathrm{G}(2,5)$. By Proposition 4.1, the Grassmannian $\mathrm{G}(2,5)$ admits at least one vector group structure.

The following proposition completes the proof of Theorem 1.1 in the case $n=6$.

Proposition 4.3. *The Grassmannian* $\mathrm{G}(2,5)$ *has a unique class of vector group structures.*

Proof. Write $\mathrm{G}(2,5) = \mathbb{G}_k(V^\vee,2) = G$ for some 5-dimensional k-vector space V. Through the isomorphism $\mathrm{PGL}_k(V^\vee) \cong \mathrm{Aut}_k(G)$ of (2.1), a vector group structure on G is given by the projective representation associated with a faithful representation $\rho\colon \mathbb{U} \to \mathrm{GL}_k(V^\vee)$ of a vector group $\mathbb{U}$. Let $V_1 \subset V$ be a 1-dimensional k-vector subspace of V that is invariant for the representation dual to ρ, its existence being guaranteed by Lemma 2.1(iii). Since the action of $\mathrm{GL}_k(V)$ on such 1-dimensional subspaces is transitive, up to the choice of a basis of V as in Notation 3.1 and up to changing ρ to its conjugate by a suitable automorphism of G, we henceforth assume without loss of generality that $V_1 = \langle e_5\rangle$ and put $\bar{V} = V/V_1 \cong \langle e_1,\dots,e_4\rangle$. The image of ρ is then contained in the stabilizer Δ_{V_1} of the subspace $\bar{V}^\vee \subset V^\vee$; see Notation 3.4. Let $\mathbb{U}'$ be the kernel of the induced representation $\mathbb{U} \to \mathrm{GL}_k(\bar{V}^\vee)$, and let $\bar{\rho}\colon \bar{\mathbb{U}} = \mathbb{U}/\mathbb{U}' \to \mathrm{GL}_k(\bar{V}^\vee)$ be the induced faithful representation. With the notation of § 3.2 and Example 3.9, the projection

$$\pi_\Pi\colon G \dashrightarrow \bar{G} = \{\overline{w}_{12}\overline{w}_{34} - \overline{w}_{13}\overline{w}_{24} + \overline{w}_{14}\overline{w}_{23} = 0\} \subset \mathbb{P}^5_k$$

from the solid $\Pi = \sigma_{3,0}(V_1)$ is then $\mathbb{U}$-equivariant for the action of $\mathbb{U}$ on $\bar{G}$ factoring through the action of $\bar{\mathbb{U}}$ determined under the isomorphism $\mathrm{Aut}_k(\bar{G}) \cong \mathrm{PGL}_k(\bar{V}^\vee)$ by the projective representation induced by $\bar{\rho}$. The $\mathbb{U}$-action on G lifts to a vector group structure on the blow-up $Y_\Pi \to G$ of Π, and, by Proposition 2.6 applied to the induced morphism $\mathrm{p}_{\bar{G}}\colon Y_\Pi \to \bar{G}$, the $\bar{\mathbb{U}}$-action on $\bar{G}$ defines a vector group structure on it. In particular, $\bar{\mathbb{U}}$ is 4-dimensional, say $\bar{\mathbb{U}} = \mathrm{Spec}(k[\bar{t}_{13},\bar{t}_{23},\bar{t}_{14},\bar{t}_{24}])$. By [Sha09], the variety $\bar{G}$ admits a unique vector group structure given up to isomorphism by the projective representation associated with the representation

$$\bar{\mathbb{U}} \to \mathrm{GL}_k(\bar{V}^\vee)\,, \quad \bar{t} = (\bar{t}_{ij})_{i=1,2,j=3,4} \mapsto A(\bar{t}) = \begin{pmatrix} 1 & 0 & -\bar{t}_{23} & -\bar{t}_{24} \\ 0 & 1 & \bar{t}_{13} & \bar{t}_{14} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Write $\mathbb{U}' = \mathrm{Spec}(k[s_1,s_2])$ and $\mathbb{U} \cong \mathrm{Spec}(k[\bar{t}][s_1,s_2]) \cong \bar{\mathbb{U}} \times \mathbb{U}'$, and put $u = (\bar{t}, s_1, s_2)$. Then, with Notation 3.4, the representation $\rho\colon \mathbb{U} \to \Delta_{V_1}$ lifting $\bar{\rho}\colon \bar{\mathbb{U}} \to \mathrm{GL}_k(\bar{V}^\vee)$ has the form $u \mapsto M(u) = M(A(\bar{t}), 1, {}^tL(u))$ for some row matrix $L(u) = (f_i(u))_{i=1,\dots,4}$ of elements of $k[u]$ such that $M(u+u') = M(u)M(u') = M(u)M(u') = M(u'+u)$. By direct computation, we see that this identity implies that $f_3 = f_4 = 0$ and that f_1 and f_2 are linear polynomials. Moreover, since ρ is injective, we must have $k[\bar{t}, f_1, f_2] = k[\bar{t}, s_1, s_2]$. Denoting $\bar{t}_{ij}$, f_1 and f_2 anew by t_{ij}, $-t_{25}$ and t_{15} then identifies ρ with the representation (4.1) in Example 4.2. □

4.2.2 *Proof of Theorem 1.1 for fivefolds.* The following proposition establishes the first part of the assertion of Theorem 1.1 for $n=5$.

Proposition 4.4. *A smooth quintic del Pezzo fivefold is isomorphic to a smooth hyperplane section of* $\mathrm{G}(2,5)$*, and all these sections are isomorphic.*

Proof. Let X be smooth quintic del Pezzo fivefold. By [Fuj81], the base change $X_{\bar{k}}$ is isomorphic to a smooth hyperplane section $Z_{\langle s\rangle}$ of $G=\mathbb{G}_{\bar{k}}(V^\vee,2)$ for some 5-dimensional $\bar{k}$-vector space V. By Lemma 3.5, the fivefold X contains a unique 3-dimensional subscheme Π whose base extension $\Pi_{\bar{k}}$ to $\bar{k}$ is a $\sigma_{3,0}$-solid of $X_{\bar{k}}$. Let $\mathcal{I}_{\Pi_{\bar{k}}}\subset\mathcal{O}_{X_{\bar{k}}}$ be the ideal sheaf of $\Pi_{\bar{k}}$, let $\mathcal{O}_{X_{\bar{k}}}(1)=\Lambda^2\mathcal{Q}|_{X_{\bar{k}}}$, and consider the projection

$$\pi_{\Pi_{\bar{k}}}\colon X_{\bar{k}}=Z_{\langle s\rangle}\dashrightarrow Q_{\langle\bar{s}\rangle}\subset\mathbb{P}_{\bar{k}}\big(H^0(X_{\bar{k}},\mathcal{I}_{\Pi_{\bar{k}}}(1))\big)\cong\mathbb{P}^4_{\bar{k}}$$

from $\Pi_{\bar{k}}$. Since the action of the Galois group $\Gamma=\mathrm{Gal}(\bar{k}/k)$ on $X_{\bar{k}}$ maps smooth hyperplane sections of $X_{\bar{k}}$ to smooth hyperplane sections, the projective space $\mathbb{P}_{\bar{k}}\big(H^0(X_{\bar{k}},\mathcal{I}_{\Pi_{\bar{k}}}(1))\big)$ inherits a natural continuous linear Galois action of Γ. By Galois descent for quasi-projective varieties and rational maps between these, the map $\pi_{\Pi_{\bar{k}}}$ thus descends to a rational map $\pi_\Pi\colon X\dashrightarrow Q\subset P_4$ whose image is a k-form Q of $Q_{\langle\bar{s}\rangle}$ in a k-form P_4 of $\mathbb{P}^4_{\bar{k}}$. The divisor $-K_{P_4}-2Q$ being defined over k and of degree 1, P_4 is the trivial form of $\mathbb{P}^4_k$. Let $\mathrm{p}_X\colon Y\to X$ be the blow-up of Π. Then, by Proposition 3.8, the induced morphism $\mathrm{p}_Q\colon Y\to Q$ is an étale locally trivial $\mathbb{P}^2$-bundle whose base extension to $\bar{k}$ is isomorphic to $\mathbb{P}_{Q_{\langle s\rangle}}(\mathscr{S}\oplus\mathcal{O}_{Q_{\langle\bar{s}\rangle}})$, where $\mathscr{S}$ denotes the spinor sheaf on $Q_{\langle\bar{s}\rangle}$. Furthermore, the restriction $\mathrm{p}_Q\colon E\to Q$ of p_Q to the exceptional divisor of p_X is an étale locally trivial $\mathbb{P}^1$-subbundle of $\mathrm{p}_Q\colon Y\to Q$, whose base extension to $\bar{k}$ is isomorphic to the subbundle $\mathbb{P}_{Q_{\langle\bar{s}\rangle}}(\mathscr{S})$ of $\mathbb{P}_{Q_{\langle\bar{s}\rangle}}(\mathscr{S}\oplus\mathcal{O}_{Q_{\langle\bar{s}\rangle}})$. Thus, considering the direct image by p_Q of the exact sequence $0\to\mathcal{O}_Y\to\mathcal{O}_Y(E)\to\mathcal{O}_E(E)\to0$ on Y, we conclude that $\mathrm{p}_Q\colon Y\to Q$ is isomorphic to the $\mathbb{P}^2$-bundle $\mathbb{P}_Q(\mathcal{E})$, where $\mathcal{E}=(\mathrm{p}_Q)_*\mathcal{O}_Y(E)$ is a locally free sheaf of rank 3 on Q. Furthermore, $\mathscr{S}_Q=(\mathrm{p}_Q)_*\mathcal{O}_E(E)$ is a locally free sheaf of rank 2 whose base extension to $\bar{k}$ is isomorphic to the spinor sheaf $\mathscr{S}$ on $Q_{\bar{k}}\cong Q_{\langle\bar{s}\rangle}$. Since $H^0\big(Q_{\langle\bar{s}\rangle},\mathscr{S}^\vee\big)\cong H^0\big(Q,\mathscr{S}_Q^\vee\big)\otimes_k\bar{k}$ by flat base change and since $\mathscr{S}^\vee$ is globally generated by sections whose zero schemes are lines of $Q_{\langle\bar{s}\rangle}\subset\mathbb{P}^4_{\bar{k}}$ (see for example [Ott88]), we infer that Q contains a line of $P_4\cong\mathbb{P}^4_k$. A smooth quadric of $\mathbb{P}^4_k$ containing a line being unique up to isomorphism and equal to a hyperplane section of the smooth quadric $\mathrm{G}(2,4)\subset\mathbb{P}^5_k$, we conclude by reading the Sarkisov link of Proposition 3.11 backwards that X is isomorphic to a smooth hyperplane section of $\mathrm{G}(2,5)$ over k. The transitivity of the action of $\mathrm{Aut}_k(G)(k)$ on the set of such smooth sections, see § 3.1.2, implies that they are all isomorphic. □

Since by Proposition 4.1 every smooth hyperplane section of $\mathrm{G}(2,5)\subset\mathbb{P}^9_k$ admits a vector group structure, the following proposition completes the proof of Theorem 1.1 in the case $n=5$.

Proposition 4.5. *A smooth hyperplane section of* $\mathrm{G}(2,5)$ *has a unique class of vector group structures.*

Proof. Write $\mathrm{G}(2,5)=\mathbb{G}_k(V^\vee,2)=G$ for some 5-dimensional k-vector space V. The action of $\mathrm{Aut}_k(G)$ on the set of smooth hyperplane sections being transitive, we are reduced without loss of generality to proving that the smooth hyperplane section $Z_{\langle s\rangle}$ associated with the skew-symmetric form $s=e_1^\vee\wedge e_3^\vee-e_2^\vee\wedge e_4^\vee$ with $V^\perp=\langle e_5\rangle$ considered in Example 4.2 admits a unique class of vector group structures. Under the isomorphism of Corollary 3.10, a vector group structure on $Z_{\langle s\rangle}$ is given by a certain faithful representation $\rho\colon\mathbb{U}\to\mathrm{GL}_k(V^\vee)$ of a vector group $\mathbb{U}$ with image contained in the subgroup of the stabilizer $\Delta_{V^\perp}$ of the subspace $\bar{V}^\vee\subset V^\vee$ consisting of matrices of the form

$$M(A_4,\lambda,U)=\begin{pmatrix}A_4 & U\\ 0 & \lambda\end{pmatrix}$$

with

$$A_4 \in \mathrm{Sp}_4\big(\bar{V}^\vee, \bar{s}\big), \quad U \in \mathrm{Hom}_k\big(\big(V^\perp\big)^\vee, \bar{V}^\vee\big), \quad \lambda \in \mathrm{GL}_k\big(\big(V^\perp\big)^\vee\big) = \mathbb{G}_{m,k}.$$

Let $\mathbb{U}'$ be the kernel of the induced representation $\mathbb{U} \to \mathrm{Sp}_4\big(\bar{V}^\vee, \bar{s}\big)$, and let $\bar{\rho}\colon \bar{\mathbb{U}} = \mathbb{U}/\mathbb{U}' \to \mathrm{Sp}_4\big(\bar{V}^\vee, \bar{s}\big)$ be the induced injective homomorphism. With the notation of Section 3.2 and Example 3.9, the projection

$$\pi_\Pi\colon Z_{\langle s\rangle} \dashrightarrow Q_{\langle \bar{s}\rangle} = \big\{\overline{w}_{12}\overline{w}_{34} - \overline{w}_{13}^2 + \overline{w}_{14}\overline{w}_{23} = 0\big\}$$

from the solid $\Pi = \sigma_{3,0}\big(V^\perp\big)$ is then $\mathbb{U}$-equivariant for the action of $\mathbb{U}$ on $Q_{\langle \bar{s}\rangle}$ factoring through the action of $\bar{\mathbb{U}}$ determined under the isomorphism $\mathrm{Aut}_k(Q_{\langle \bar{s}\rangle}) \cong \mathrm{PSp}_k\big(\bar{V}^\vee, \bar{s}\big)$ by the projective representation induced by $\bar{\rho}$. The $\mathbb{U}$-action on $Z_{\langle s\rangle}$ lifts to a vector group structure on the blow-up $Y_\Pi \to Z_{\langle s\rangle}$ of Π, and, by Proposition 2.6 applied to the induced morphism $\mathrm{p}_{Q_{\langle \bar{s}\rangle}}\colon Y_\Pi \to Q_{\langle \bar{s}\rangle}$, the $\bar{\mathbb{U}}$-action on $Q_{\langle \bar{s}\rangle}$ defines a vector group structure on it. In particular, $\bar{\mathbb{U}}$ is 3-dimensional, say $\bar{\mathbb{U}} = \mathrm{Spec}(k[\bar{t}_{13}, \bar{t}_{23}, \bar{t}_{24}])$. By [Sha09], the quadratic threefold $Q_{\langle \bar{s}\rangle}$ admits a unique vector group structure up to isomorphism, which, with our choice of coordinates, is induced by the representation $\bar{\rho}_5\colon \mathbb{V}_k\big(F_5^\vee\big)/\,\mathrm{Stab}\big(\langle e_5^\vee\rangle\big) \to \mathrm{Sp}_4\big(\bar{V}^\vee, \bar{s}\big)$ of (4.3) in Example 4.2. Writing $\mathbb{U}' = \mathrm{Spec}(k[s_1, s_2])$, $\mathbb{U} \cong \mathrm{Spec}(k[\bar{t}][s_1, s_2]) \cong \bar{\mathbb{U}} \times \mathbb{U}'$ and $u = (\bar{t}, s_1, s_2)$, the same argument as in the proof of Proposition 4.3 implies that the lift $\rho\colon \mathbb{U} \to \Delta_{V^\perp} \subset \mathrm{GL}_k\big(V^\vee\big)$ of $\bar{\rho}\colon \bar{\mathbb{U}} \to \mathrm{Sp}_4\big(\bar{V}^\vee, \bar{s}\big)$ has the form $u \mapsto M(u) = M\big(A(\bar{t}), 1, {}^tL(s)\big)$, where $L(s) = (f_1(s), f_2(s), 0, 0)$ is a row matrix of linear elements of $k[s]$ with the property that $k[\bar{t}, f_1, f_2] = k[\bar{t}, s_1, s_2]$. Denoting the $\bar{t}_{ij}$, f_1 and f_2 anew by t_{ij}, $-t_{25}$ and t_{15}, respectively, we get that $\rho\colon \mathbb{U} \to \mathrm{GL}_k\big(V^\vee\big)$ is the representation (4.2) of Example 4.2. □

4.2.3 *Proof of Theorem 1.1 for fourfolds.* The following proposition completes the proof of Theorem 1.1.

Proposition 4.6. *A smooth quintic del Pezzo fourfold is isomorphic to a smooth section of $\mathrm{G}(2,5) \subset \mathbb{P}^9_k$ by a linear subspace of codimension 2. Furthermore, all such sections are isomorphic and admit exactly one class of vector group structures.*

Proof. Let X be smooth quintic del Pezzo fourfold. By [Fuj81], we can assume that $X_{\bar{k}}$ is a smooth section W_L of $\mathrm{G}(2,5)_{\bar{k}}$ by a linear subspace of codimension 2. By Lemma 3.7, the variety X contains a unique 2-dimensional subscheme Ξ whose base extension $\Xi_{\bar{k}}$ to $\bar{k}$ is a $\sigma_{2,2}$-plane of W_L. Let $\mathcal{I}_{\Xi_{\bar{k}}} \subset \mathcal{O}_{X_{\bar{k}}}$ be the ideal sheaf of $\Xi_{\bar{k}}$, let $\mathcal{O}_{X_{\bar{k}}}(1) = \Lambda^2\mathcal{Q}|_{X_{\bar{k}}}$, and consider the projection

$$\pi_{\Xi_{\bar{k}}}\colon X_{\bar{k}} = W_L \dashrightarrow \mathbb{P}_{\bar{k}}\big(H^0(X_{\bar{k}}, \mathcal{I}_{\Xi_{\bar{k}}}(1))\big) \cong \mathbb{P}^4_{\bar{k}}$$

from $\Xi_{\bar{k}}$ as in § 3.3. The projective space $\mathbb{P}_{\bar{k}}\big(H^0(X_{\bar{k}}, \mathcal{I}_{\Xi_{\bar{k}}}(1))\big)$ inherits a natural continuous linear action of the Galois group $\Gamma = \mathrm{Gal}\big(\bar{k}/k\big)$, which stabilizes the hyperplane corresponding to the unique section of $\mathcal{I}_{\Xi_{\bar{k}}}(1)$ whose zero scheme is the special hyperplane section $Z_{\Xi_{\bar{k}}} \cap X_{\bar{k}}$ of $X_{\bar{k}}$ spanned by the $\sigma_{3,1}$-planes of $X_{\bar{k}}$. In particular, the divisor $Z_{\Xi_{\bar{k}}} \cap X_{\bar{k}}$ is defined over k, say $Z_{\Xi_{\bar{k}}} \cap X_{\bar{k}} = (Z_\Xi)_{\bar{k}}$ for some geometrically irreducible divisor $Z_\Xi \subset X$ containing Ξ. In view of Proposition 3.11, the projection $\pi_{\Xi_{\bar{k}}}$ thus descends to a birational map $\pi_\Xi\colon X \dashrightarrow P_4$ with image equal to a k-form P_4 of $\mathbb{P}^4_k$, which contracts Z_Ξ onto a smooth curve C contained in a hypersurface $P_3 \subset P_4$ such that the triple (P_4, P_3, C) is a k-form of the triple $(\mathbb{P}^4_k, \mathbb{P}^3_k, C_3)$, where $\mathbb{P}^3_k$ is a hyperplane of $\mathbb{P}^4_k$ and $C_3 \subset \mathbb{P}^3_k$ is a smooth rational cubic curve. Thus, P_4 and P_3 are trivial forms of $\mathbb{P}^4_k$ and $\mathbb{P}^3_k$, respectively. Since the intersection of C with a hyperplane of $\mathbb{P}^3_k$ is a divisor of degree 3 on C, it follows that $C \cong \mathbb{P}^1_k$. Reading the Sarkisov link of Proposition 3.11 backwards, we conclude that X is isomorphic to a smooth section of $\mathrm{G}(2,5)$ by a linear subspace of codimension 2. The uniqueness up to isomorphism follows from the transitivity of the action $\mathrm{PGL}_5(k)$ on triples $(\mathbb{P}^4_k, \mathbb{P}^3_k, C_3)$, where $C_3 \subset \mathbb{P}^3_k$ is a smooth rational cubic. Proposition 4.1 implies that X admits a vector group structure. To show the uniqueness, we observe that the unique $\sigma_{2,2}$-plane Ξ is stable under any faithful action of a unipotent abelian group $\mathbb{U}$ on X defining a vector group structure on X. This vector group structure lifts to the blow-up $\mathrm{p}_\Xi \colon Y \to X$ of Ξ and then, by Proposition 2.6, descends via the contraction $\mathrm{p}_2 \colon Y \to \mathbb{P}^4_k$ of the proper transform of Z_Ξ to a faithful $\mathbb{U}$-action on $\mathbb{P}^4_k$ defining a vector group structure, for which the pair $(\mathbb{P}^3_k, C)$ is globally $\mathbb{U}$-stable. By the classification of vector group structures on $\mathbb{P}^4_k$, see [HT99] or [HM20, Corollary 3.6], the unique class of vector group structures with this property is the toric $\mathbb{G}^4_a$-structure on $\mathbb{P}^4_k$ described in Example 4.2. □

Remark 4.7. A byproduct of Proposition 4.6 is that among the four non-isomorphic compactifications of $\mathbb{A}^4_{\mathbb{C}}$ into the smooth quintic del Pezzo fourfold classified by Prokhorov [Pro94, Theorem 3.1], only case (i) can be endowed with a vector group structure making it an *equivariant* compactification of $\mathbb{G}^4_{a,\mathbb{C}}$.

Remark 4.8. By [DK19, § 2.2], smooth sections of $\mathrm{G}(2,5) \subset \mathbb{P}^9_k$ by linear subspaces of codimension 3 in general have non-trivial k-forms, whose isomorphism classes are parametrized by equivalence classes of non-degenerate ternary quadratic forms over k. In contrast, Propositions 4.4 and 4.6 imply that smooth del Pezzo quintics of dimension $n = 4, 5$ do not have non-trivial k-forms. Since these are compactifications of $\mathbb{A}^n_k$, one deduces from the techniques in loc. cit. that a proper morphism $f \colon X \to Y$ between normal varieties over an algebraically closed field of characteristic zero whose general closed fibres are smooth quintic del Pezzo varieties of dimension $n = 4, 5$ contains a vertical $\mathbb{A}^n$-cylinder in the sense of [DK19]. Similarly, a proper morphism $f \colon X \to Y$ whose general closed fibres are isomorphic to $\mathrm{G}(2,5)$ and which has a rational section contains a vertical $\mathbb{A}^6$-cylinder.

5. Vector group structures on terminal quintic del Pezzo threefolds and canonical quintic del Pezzo surfaces

Classifying vector group structures on all del Pezzo quintics, including singular ones, is a challenging problem. For instance, many families of singular del Pezzo quintics of dimension $n = 3, 4, 5$ endowed with a vector group structure can be constructed as hyperplanes sections of $G = \mathrm{G}(2,5)$ with respect to the Plücker embedding containing a $\sigma_{2,2}$-plane Ξ of G (see § 3.3 for the notation) which is fixed by the $\mathbb{G}^6_{a,k}$-structure on G corresponding under the birational projection $\pi_\Xi \colon G = \mathrm{G}(2,5) \dashrightarrow \mathbb{P}^6_k = \mathbb{P}(H^0(G, \mathcal{I}_\Xi(1)))$ to the toric $\mathbb{G}^6_{a,k}$-structure on $\mathbb{P}^6_k$ described in Example 4.2. Namely, every linear subspace L of codimension $m = 1, 2, 3$ of $\mathbb{P}^6_k$ which intersects the complement $H_\infty = \{w_{12} = 0\}$ of the open orbit transversely has stabilizer isomorphic to $\mathbb{G}^m_{a,k}$, and the induced effective action of the quotient group $\mathbb{G}^{6-m}_{a,k}$ on L defines a vector group structure on it. The proper transform of L by π_Ξ is then a linear section X_L of G of dimension $6-m$. This section is singular in general, endowed with a vector structure for which the inclusion $X_L \hookrightarrow G$ is equivariant for the induced action of a subgroup $\mathbb{G}^{6-m}_{a,k}$ of the group $\mathbb{G}^6_{a,k}$ which acts with an open orbit on G.

Here, we consider mildly singular del Pezzo quintics of dimension 3 and 2, a case of special interest due to the fact that no smooth del Pezzo quintics in these dimensions admit a vector group structure.

5.1 Vector group structures on terminal quintic del Pezzo threefolds

Over algebraically closed fields of characteristic zero, non-smooth terminal quintic del Pezzo threefolds are classified in [Pro13]. By [Pro13, Corollary 5.3], the main invariant of such a threefold X is the rank $r(X) = \mathrm{rk}_{\mathbb{Z}}\mathrm{Cl}(X)$ of its divisor class group, which is either 2, 3 or 4. Furthermore, all the singularities of X are ordinary double points, and, by [Pro13, Corollary 8.3.1], the number of such nodes equals $r(X) - 1$.

Lemma 5.1. *A terminal quintic del Pezzo threefold X over k whose base extension $X_{\bar{k}}$ to $\bar{k}$ has one or two nodes does not admit a vector group structure.*

Proof. Since it is enough to show that $X_{\bar{k}}$ does not admit any vector group structure, we henceforth assume $k = \bar{k}$. Assume that X has a vector group structure. Then this structure lifts to a vector group structure on a small $\mathbb{Q}$-factorialization $\xi\colon \hat{X} \to X$ of X. If X has a unique node, then, by [JP08, Theorem 3.6] or [Pro13, Theorem 5.2], the threefold $\hat{X}$ is isomorphic to the total space of a $\mathbb{P}^1$-bundle $\pi\colon \hat{X} \cong \mathbb{P}_{\mathbb{P}^2_k}(\mathcal{E}) \to \mathbb{P}^2_k$ for some stable, hence simple, locally free sheaf $\mathcal{E}$ of rank 2 on $\mathbb{P}^2_k$. We would thus obtain a vector group structure on $\mathbb{P}_{\mathbb{P}^2_k}(\mathcal{E})$, which is impossible by Corollary 2.7. If X has two nodes, then, by [Pro13, Theorem 8.1 and Corollary 8.1.1], the threefold $\hat{X}$ is isomorphic to the blow-up $\sigma\colon \hat{X} \to \tilde{X}$ of the total space of the projective bundle $\pi\colon \tilde{X} = \mathbb{P}_{\mathbb{P}^2_k}(\mathcal{T}_{\mathbb{P}^2_k}(-1)) \to \mathbb{P}^2_k$ at a point. By Proposition 2.6, the vector group structure on $\hat{X}$ descends to a vector group structure on $\tilde{X}$. But again, this is impossible since $\mathcal{T}_{\mathbb{P}^2_k}(-1)$ is a simple sheaf. □

We now consider the case of terminal quintic del Pezzo threefolds whose base extensions to $\bar{k}$ possess exactly three nodes, which we henceforth call *trinodal quintic del Pezzo threefolds* for short.

Proposition 5.2. *A trinodal quintic del Pezzo threefold X_3 is isomorphic to a section of $\mathrm{G}(2,5) \subset \mathbb{P}^9_k$ by a linear subspace of codimension 3. It contains a unique $\sigma_{2,2}$-plane Ξ of $\mathrm{G}(2,5)$, and the projection $\pi_\Xi\colon \mathrm{G}(2,5) \dashrightarrow \mathbb{P}^6_k$ from Ξ induces the Sarkisov link*

$$\begin{array}{ccccccccc}
\mathrm{Bl}_{S_{X_3}} H_{X_3} & \hookrightarrow & & \mathrm{Bl}_\Xi X_3 \cong \mathrm{Bl}_{S_{X_3}} \mathbb{P}^3_k & & & & & \\
\downarrow & & \swarrow^{\mathrm{p}_{X_3}} & & \searrow^{\mathrm{p}_2} & & & & \\
\Xi & \hookrightarrow & X_3 & \underset{\Phi}{\overset{\pi_\Xi}{\rightleftarrows}} & & \mathbb{P}^3_k & \hookleftarrow H_{X_3} & \hookleftarrow & S_{X_3}\,,
\end{array} \tag{5.1}$$

where $\mathrm{p}_{X_3}\colon \mathrm{Bl}_\Xi X_3 \to X_3$ is the blow-up of Ξ, $\mathrm{p}_2\colon \mathrm{Bl}_\Xi X_3 \to \mathbb{P}^3_k$ is the blow-up of a smooth 0-dimensional subscheme S_{X_3} of length 3 of $\mathbb{P}^3_k$ not contained in a line, H_{X_3} is the unique hyperplane of $\mathbb{P}^3_k$ containing S_{X_3} and $\mathrm{Bl}_{S_{X_3}} H_{X_3}$ is its proper transform. Furthermore, the birational inverse Φ of π_Ξ is given by the complete linear system of quadrics of $\mathbb{P}^3_k$ containing S_{X_3}.

Proof. By [Pro13, Theorem 7.1], the base change $X_{3,\bar{k}}$ is isomorphic to a threefold obtained from $\mathbb{P}^3_{\bar{k}}$ as the blow-up $\sigma\colon Y \to \mathbb{P}^3_{\bar{k}}$ of three non-colinear closed points, say p_1, p_2, p_3, followed by the contraction $\xi\colon Y \to X_{3,\bar{k}}$ of the proper transforms by σ of the lines in $\mathbb{P}^3_{\bar{k}}$ passing through p_i and p_j, $1 \leqslant i < j \leqslant 3$, to the nodes of $X_{3,\bar{k}}$. The class group of $X_{3,\bar{k}}$ is freely generated by the classes P of the proper transform in X of the unique hyperplane $H \subset \mathbb{P}^3_{\bar{k}}$ containing the points p_i and of the images F_i, $1 \leqslant i \leqslant 3$, of the exceptional divisors of σ. It follows from [Pro13, Theorem 7.2] that P and the F_i are the only planes contained in $X_{3,\bar{k}}$. Their union is thus defined over k, and since P is the only plane among these which fully contains the singular locus of $X_{3,\bar{k}}$, it is defined over k as well, say $P = \Xi_{\bar{k}}$ for some closed subscheme Ξ of X_3. This implies in turn that the union of the F_i is defined over k, say $\bigcup F_i = F_{\bar{k}}$ for some closed subscheme F of X_3. Since the divisor $(\Xi+F)_{\bar{k}} = P+\sum_{i=1}^{3} F_i$ is linearly equivalent to the proper transform in $X_{3,\bar{k}}$ of any hyperplane $H' \subset \mathbb{P}^3_{\bar{k}}$ not passing through the points blown-up, it is Cartier. The invertible sheaf $\mathcal{O}_{X_{3,\bar{k}}}\big(P+\sum_{i=1}^{3} F_i\big)$ is then the base extension to $\bar{k}$ of the invertible sheaf $\mathcal{O}_{X_3}(1) := \mathcal{O}_{X_3}(\Xi + F)$. Moreover, letting $\mathcal{I} \subset \mathcal{O}_{X_3}$ be the ideal sheaf of the singular locus of X_3, the rational map $\sigma \circ \xi^{-1}\colon X_{3,\bar{k}} \dashrightarrow \mathbb{P}^3_{\bar{k}}$ is the base extension of the birational map $\pi\colon X_3 \dashrightarrow \mathbb{P}\big(H^0(X_3, \mathcal{I}(1))\big) \cong \mathbb{P}^3_k$. The latter maps Ξ to a hyperplane $H_{X_3} \subset \mathbb{P}^3_k$ and contracts F to a smooth closed subscheme $S_{X_3} \subset H_{X_3}$ of length 3 whose base extension to $\bar{k}$ equals the union of the points p_i. Since S_{X_3} is not contained in a line, there exists a smooth rational cubic curve $S_{X_4} \subset \mathbb{P}^3_k$ whose scheme-theoretic intersection with H_{X_3} equals S_{X_3}. Considering $\mathbb{P}^3_k$ as a hyperplane H_{X_4} of $\mathbb{P}^4_k$, it follows from Proposition 3.11 that the image of the rational map $\mathbb{P}^4_k \dashrightarrow \mathbb{P}^7_k$ given by the complete linear system of quadrics containing $S_{X_4} \subset H_{X_4}$ is a smooth quintic del Pezzo fourfold X_4 which contains X_3 as a hyperplane section and which has the proper transform Ξ of H_{X_4} as its unique $\sigma_{2,2}$-plane. By construction, the birational map $\pi\colon X_3 \dashrightarrow \mathbb{P}^3_k$ then coincides with the restriction to X_3 of the projection $\pi_\Xi\colon X_4 \dashrightarrow \mathbb{P}^4_k$ from Ξ, which completes the proof. □

COROLLARY 5.3. *Isomorphism classes of trinodal quintic del Pezzo threefolds are in one-to-one correspondence with* $\mathrm{PGL}_2(k)$*-orbits of smooth zero-dimensional subschemes of* $\mathbb{P}^1_k$ *of length* 3. *Furthermore, every such threefold admits a unique class of vector group structures.*

Proof. By Proposition 5.2, two trinodal quintic del Pezzo threefolds X_3 and X_3' are isomorphic if and only if there exists an automorphism of $\mathbb{P}^3_k$ which maps the pair (H_{X_3}, S_{X_3}) onto the pair $(H_{X_3'}, S_{X_3'})$. Being of length 3 and not contained in a line, the schemes S_{X_3} and $S_{X_3'}$ are contained in smooth k-rational conics C_{X_3} and $C_{X_3'}$ of H_{X_3} and $H_{X_3'}$, respectively. Since $\mathrm{Aut}_k(\mathbb{P}^3_k)$ acts transitively on pairs (H, C) consisting of a hyperplane H of $\mathbb{P}^3_k$ and a smooth k-rational conic $C \cong \mathbb{P}^1_k$ in it and since for such pairs the restriction homomorphism $\mathrm{Aut}_k(H, C) \to \mathrm{Aut}_k(C)$ is an isomorphism, we conclude that X_3 and X_3' are isomorphic if and only if there exists an isomorphism $\varphi\colon C_{X_3} \to C_{X_3'}$ which maps S_{X_3} onto $S_{X_3'}$. This holds if and only if S_{X_3} and $\varphi^{-1}(S_{X_3'})$ belong to the same orbit of the action of $\mathrm{Aut}_k(C_{X_3})(k) \cong \mathrm{PGL}_2(k)$.

For the second assertion, since the singular locus of X_3 and the unique $\sigma_{2,2}$-plane Ξ of X_3 are stable under any vector group action on X_3, it follows from Proposition 2.6 applied to the birational morphism $\mathrm{Bl}_\Xi X \to \mathbb{P}^3_k$ that the Sarkisov link of Proposition 5.2 is equivariant for any vector group structure on X_3 and that the corresponding vector group structure on $\mathbb{P}^3_k$ stabilizes the non-linear closed subscheme S_{X_3} of length 3. By the classification [HT99] of vector group structures on $\mathbb{P}^3_k = \mathrm{Proj}_k(k[x_0, x_1, x_2, x_3])$, the unique class of vector group structures with this property is that of the toric $\mathbb{G}_a^3$-structure defined by $x_0 \mapsto x_0$ and $x_i \mapsto x_i + t_i x_0$, $1 \leqslant i \leqslant 3$. Conversely, this structure lifts to a vector group structure on the blow-up of S_{X_3}, which, by Proposition 2.6 again, in turn descends to a vector group structure on X_3. □

Remark 5.4. By Proposition 5.2 and Corollary 5.3, every trinodal quintic del Pezzo threefold contains the affine 3-space $\mathbb{A}^3_k$ as a Zariski open subset. In contrast, there exist in general k-forms of smooth quintic del Pezzo threefolds which do not contain $\mathbb{A}^3_k$; see [DK19, Theorem 12].

Example 5.5. With Notation 3.1, let $V_3 = \langle e_3, e_4, e_5\rangle$, and let $\Xi = \sigma_{2,2}(V_3)$ be the associated plane of $G = \mathrm{G}(2,5) \subset \mathbb{P}^9_k$ as in Example 3.12. For every $\beta \in k^*$, the linear section

$$X_3(\beta) = G \cap \{w_{13} - w_{24} = 0\} \cap \{\beta w_{14} - w_{25} = 0\} \cap \{\beta w_{15} + w_{23} = 0\}$$

is a trinodal quintic del Pezzo threefold containing Ξ, isomorphic to the subvariety in $\mathbb{P}^6_k$ with coordinates w_{ij}, $(i,j) \neq (2,3),(2,4),(2,5)$, defined by the equations

$$\begin{cases} w_{12}w_{34} - w_{13}^2 - \beta w_{14}w_{15} = 0\,, \\ w_{12}w_{35} - \beta w_{13}w_{14} - \beta w_{15}^2 = 0\,, \\ w_{12}w_{45} - \beta w_{14}^2 + w_{13}w_{15} = 0\,, \\ w_{13}w_{45} - w_{14}w_{35} + w_{15}w_{34} = 0\,, \\ -\beta w_{15}w_{45} - w_{13}w_{35} + \beta w_{14}w_{34} = 0\,. \end{cases}$$

Its singular locus $\mathrm{Sing}(X_3(\beta))$ is the closed subscheme of $\Xi \cong \mathrm{Proj}_k(k[w_{34}, w_{35}, w_{45}])$ with equations

$$\beta w_{34}w_{45} - w_{35}^2 = w_{34}w_{35} - \beta w_{45}^2 = w_{35}w_{45} - w_{34}^2 = 0\,.$$

Letting $\lambda, \epsilon \in \bar{k}$ be, respectively, a third root of β and a primitive third root of unity, the singular locus of $X_3(\beta)_{\bar{k}}$ is the union of the three closed points $\left[\lambda\epsilon^m : (\lambda\epsilon^m)^2 : 1\right]$, $0 \leqslant m \leqslant 2$, of Ξ. Thus, according to whether β is a cube in k^* or not and k^* contains a primitive third root of unity or not, $\mathrm{Sing}(X_3(\beta))$ consists of either a single closed point, or the union of a k-point and a single other closed point, or the union of three k-points. The image of Ξ by the restriction $X_3(\beta) \dashrightarrow \mathbb{P}^3_k = \mathrm{Proj}_k(k[w_{12}, w_{13}, w_{14}, w_{15}])$ of the projection from Ξ is the hyperplane $H_{X_3(\beta)} = \{w_{12} = 0\}$. The associated smooth zero-dimensional subscheme $S_{X_3(\beta)}$ of length 3 is the closed subscheme of $H_{X_3(\beta)}$ defined by the equations

$$\beta w_{14}w_{15} + w_{13}^2 = w_{13}w_{14} + w_{15}^2 = w_{13}w_{15} - \beta w_{14}^2 = 0\,.$$

The restriction to $X_3(\beta)$ of the action of the subgroup $\mathbb{U} \cong \mathbb{G}^3_{a,k}$ of the vector group $\mathbb{V}_k(F_6^\vee)$ of (4.1) in Example 4.2 defined by $t_{24} = t_{13}$, $t_{25} = \beta t_{14}$ and $t_{23} = -\beta t_{15}$ induces a vector group structure

$$\begin{cases} w_{12} \longmapsto w_{12}\,, \\ w_{13} \longmapsto w_{13} + t_{13}w_{12}\,, \\ w_{14} \longmapsto w_{14} + t_{14}w_{12}\,, \\ w_{15} \longmapsto w_{15} + t_{15}w_{12}\,, \\ w_{34} \longmapsto w_{34} + 2t_{13}w_{13} + \beta(t_{15}w_{14} + t_{14}w_{15}) + \left(t_{13}^2 + \beta t_{14}t_{15}\right)w_{12}\,, \\ w_{35} \longmapsto w_{35} + \beta(t_{14}w_{13} + t_{13}w_{14} + 2t_{15}w_{15}) + \beta\left(t_{13}t_{14} + t_{15}^2\right)w_{12}\,, \\ w_{45} \longmapsto w_{45} + (2\beta t_{14}w_{14} - t_{15}w_{13} - t_{13}w_{15}) + \left(\beta t_{14}^2 - t_{15}t_{13}\right)w_{12} \end{cases}$$

on $X_3(\beta)$. The base extension to $\bar{k}$ of the hyperplane section $\{w_{12} = 0\}$ of $X_3(\beta)$ consists of four irreducible components isomorphic to $\mathbb{P}^2_{\bar{k}}$: the plane $\Xi_{\bar{k}}$ and the images in $X_3(\beta)_{\bar{k}}$ of the exceptional divisors of the blow-up of $S_{X_3(\beta),\bar{k}}$, which intersect Ξ along the lines joining the three singular points of $X_3(\beta)_{\bar{k}}$. As in Example 4.2, we get a $\mathbb{G}_a^3$-equivariant stratification $\Xi \subset H_3(\beta) = \{w_{12} = 0\} \subset X_3(\beta)$ of $X_3(\beta)$ for which $X_3(\beta) \setminus H_3(\beta)$ is the open orbit and Ξ is the fixed locus and such that the restriction $H_3(\beta) \setminus \Xi \to S_{X_3(\beta)}$ of the projection from Ξ is the trivial $\mathbb{A}^2$-bundle $S_{X_3(\beta)} \times_k \mathbb{A}^2_k$ with an induced transitive $\mathbb{G}_a^3$-action on the second factor.

5.2 Vector group structures on canonical quintic del Pezzo surfaces

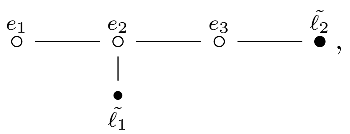
Del Pezzo surfaces with canonical singularities admitting a vector group structure are classified in [DL10]; see also [MS20]. For the sake of completeness, we record the following consequence of the classification in the quintic case.

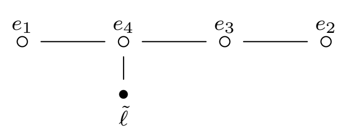
Proposition 5.6. *Up to isomorphism, there exist two quintic del Pezzo surfaces with canonical singularities which admit a vector group structure:*

(i) *a surface S with an A_3-singularity, whose neutral component $\mathrm{Aut}^0(S)$ of the automorphism group is isomorphic to $\mathbb{G}_{a,k}^2 \rtimes \mathbb{G}_{m,k}$ and which admits a unique class of vector group structures,*

(ii) *a surface S' with an A_4-singularity, whose neutral component $\mathrm{Aut}^0(S')$ of the automorphism group is isomorphic to $\mathbb{U}_3 \rtimes \mathbb{G}_{m,k}$, where $\mathbb{U}_3$ is a maximal unipotent subgroup of $\mathrm{PGL}_3(k)$, and which admits exactly two classes of vector group structures.*

Proof. All the properties but those concerning the actual number of equivalence classes of vector group structures are established in [DL10] The descriptions of the automorphism groups can be found in [MS20, Table 1], cases $5E$ and $5F$ for S and S', respectively. We briefly recall the principle of the argument in loc. cit. and explain how derive from it the equivalence classes of vector group structures. Equivalence classes of vector group structures on a del Pezzo surface S with canonical singularities are in one-to-one correspondence with those on its minimal desingularization $\tilde{S} \to S$, which is obtained from S by performing a finite sequence of successive blow-ups of singular loci of intermediate surfaces and normalizations. Indeed, a vector group structure stabilizes singular loci, hence canonically lifts to their blow-ups and, by the universal property, canonically lifts as well to normalizations. Conversely, Proposition 2.6 ensures that every vector group structure on $\tilde{S}$ descend to S. Here, the surfaces $\tilde{S}$ are weak del Pezzo surfaces of degree 5 whose base extensions to $\bar{k}$ are obtained from $\mathbb{P}^2_{\bar{k}}$ by performing certain finite sequences of blow-ups of closed points. In the two cases under consideration, the respective dual graphs of the unions of the (-1)-curves and (-2)-curves in $\tilde{S}_{\bar{k}}$ have the following structure:





in which the vertices $\bullet$ and $\circ$ correspond, respectively, to (-1)-curves which are the proper transforms of the lines in $S_{\bar{k}}$ and to (-2)-curves which are the exceptional divisors of the desingularization $\tilde{S}_{\bar{k}} \to S_{\bar{k}}$. Since these diagrams have no symmetries, all the curves displayed are defined over k, corresponding to irreducible smooth k-rational curves in $\tilde{S}$ with the same self-intersection numbers.

In the case of an A_3-singularity, the successive contractions of $\tilde{\ell}_2$, and then of the exceptional divisors e_3, e_2 and e_1, yield a birational morphism $\sigma\colon \tilde{S} \to \mathbb{P}^2_k = \mathrm{Proj}_k(k[u_0, u_1, u_2])$ which maps $\tilde{\ell}_1$ onto a line $\ell \subset \mathbb{P}^2_k$ and contracts $\tilde{\ell}_2 \cup e_1 \cup e_2 \cup e_3$ onto a k-point $p \in \ell$, say, up to composition by a suitable automorphism of $\mathbb{P}^2_k$, $\ell = \{u_2 = 0\}$ and $p = [1:0:0]$. A vector group structure on S and its canonical lift to $\tilde{S}$ being given, Proposition 2.6 implies the existence of a unique vector group structure on $\mathbb{P}^2_k$ for which $\sigma\colon \tilde{S} \to \mathbb{P}^2_k$ is equivariant. The latter stabilizes ℓ as well as the proper and infinitely near base points of σ^{-1}. By [HT99, Proposition 3.2], there are two classes of vector group structures on $\mathbb{P}^2_k$ fixing ℓ and p: the “toric” structure given by the $\mathbb{G}^2_{a,k}$-action $[u_0 : u_1 : u_2] \mapsto [u_0 + t_0u_2 : u_1 + t_1u_2 : u_2]$ and the “non-toric” one given by the $\mathbb{G}^2_{a,k}$-action $[u_0 : u_1 : u_2] \mapsto \big[u_0 + t_1u_1 + \big(\frac{1}{2}t_1^2 + t_0\big)u_2 : u_1 + t_1u_2 : u_2\big]$. A direct verification shows that the lift of the toric structure to the surface $\tilde{S}_1$ obtained from $\tilde{S}$ by contracting $\tilde{\ell}_2$ acts transitively on $e_3 \setminus e_2$, hence that this structure cannot be induced by a vector group structure on $\tilde{S}$. On the other hand, the lift to $\tilde{S}_1$ of the other structure fixes e_3 pointwise, hence is descended via the contraction of $\tilde{\ell}_2$ from a vector group structure on $\tilde{S}$. Thus, $\tilde{S}$, whence S, has a unique class of vector group structures.

In the case of an A_4-singularity, the successive contractions of $\tilde{\ell}$ and then of the exceptional divisors e_4, e_3 and e_2 yield birational morphism $\sigma\colon \tilde{S} \to \mathbb{P}^2_k$ which maps e_1 onto a line $\ell \subset \mathbb{P}^2_k$ and $\tilde{\ell} \cup e_4 \cup e_3 \cup e_2$ onto a k-point $p \in \ell$. Up to composing by a suitable automorphism of $\mathbb{P}^2_k$ as above, we again infer that a vector group structure on $\tilde{S}$ is equivalent to the lift via σ of one of the two equivalence classes of such structures on $\mathbb{P}^2_k$ described above. Noting that for both structures, the first three points blown-up by σ are fixed and that the lifts of these two structures to the resulting surface both fix e_4 pointwise, we conclude that both structures lift to $\tilde{S}$. These two structures in turn descend on S, showing that S has at most two equivalence classes of vector group structures. The conclusion follows from the observation that two so-constructed induced structures have non-isomorphic fixed point schemes, hence are not equivalent. □

Remark 5.7. With the notation of the proof of Proposition 5.6, in the case of the del Pezzo surface S with an A_3-singularity, the contractions of $\tilde{\ell}_1$, e_1, e_2 and $\tilde{\ell}_2$ yield another birational morphism $\sigma'\colon \tilde{S} \to \mathbb{P}^2_k$ which maps e_3 onto a line ℓ' and contracts $\tilde{\ell}_1 \cup e_2 \cup e_1$ and $\tilde{\ell}_2$ onto a pair of distinct k-points of ℓ. In contrast with the morphism $\sigma\colon \tilde{S} \to \mathbb{P}^2_k$ constructed in the proof of Proposition 5.6 which is equivariant with respect to the non-toric $\mathbb{G}^2_{a,k}$-structure on $\mathbb{P}^2_k$, the birational morphism σ' is equivariant with respect to the toric $\mathbb{G}^2_{a,k}$-structure on $\mathbb{P}^2_k$. The toric and non-toric structures on $\mathbb{P}^2_k$ thus become equivalent on S and hence are birationally conjugated on $\mathbb{P}^2_k$ by the birational automorphism $\sigma' \circ \sigma^{-1}$.

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