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Standard stable Horikawa surfaces

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Abstract

We consider the stable compactification $\bar{\mathfrak{H}}$ of the moduli space of Horikawa surfaces with $K_X^2 = 2p_g(X) - 4$. When $K_X^2 = 8\ell$ we show that the closures of the two components $\mathfrak{H}^{\mathrm{I}}$ and $\mathfrak{H}^{\mathrm{II}}$ of the Gieseker moduli space intersect, for $\ell > 2$, in a divisor parametrising explicitly described semi-smooth surfaces. With growing K_X^2 we find an increasing number of generically non-reduced irreducible components in the same connected component of the moduli space of stable surfaces.

1. Introduction

For complex minimal surfaces of general type, we have the classical Noether inequality $K_X^2 \geqslant 2p_g - 4$; see [BHP+04, Chapter VII]. In 1976 Horikawa classified surfaces satisfying equality in the first of a series of seminal papers [Hor76]; in his honour, these surfaces are now called Horikawa surfaces.

Let us denote by $\mathfrak{H}_{2k}$ the Gieseker moduli space of Horikawa surfaces with $K_X^2 = 2k$. Then Horikawa showed that $\mathfrak{H}_{2k}$ is irreducible unless $2k \equiv 0 \bmod 8$, in which case $\mathfrak{H}_{2k} = \mathfrak{H}_{2k}^{\mathrm{I}} \sqcup \mathfrak{H}_{2k}^{\mathrm{II}}$ has two connected components of the same dimension. For $2k \equiv 8 \bmod 16$ he showed that the two components parametrise non-diffeomorphic surfaces, but ever since it has remained open whether $\mathfrak{H}_{16\ell}^{\mathrm{I}}$ and $\mathfrak{H}_{16\ell}^{\mathrm{II}}$ parametrise diffeomorphic surfaces [Aur06, FS97, LP11].

Nowadays, the Gieseker moduli space $\mathfrak{H}_{2k}$ embeds into a natural compactification $\bar{\mathfrak{H}}_{2k}$, the moduli space of stable Horikawa surfaces, which parametrises stable surfaces with the same Hilbert polynomial (see [Kol13a, Kol23]). The starting point of the present work was the question of whether the closures $\bar{\mathfrak{H}}_{16\ell}^{\mathrm{I}}$ and $\bar{\mathfrak{H}}_{16\ell}^{\mathrm{II}}$ intersect inside $\bar{\mathfrak{H}}_{16\ell}$.

Theorem A. *The intersection of $\bar{\mathfrak{H}}_{8\ell}^{\mathrm{I}}$ and $\bar{\mathfrak{H}}_{8\ell}^{\mathrm{II}}$ for $\ell > 1$ contains a divisor $\mathfrak{D}$ parametrising explicitly described non-normal (but semi-smooth) surfaces. The intersection is not normal crossing at the general point of $\mathfrak{D}$.*

We will prove this result in Theorem 3.4 and Corollary 5.12. Horikawa surfaces with $K_X^2 = 8$ are a bit of an exception because not all such surfaces are double covers of Hirzebruch surfaces. We show in Section 3.3 that $\bar{\mathfrak{H}}_8^{\mathrm{I}}$ and $\bar{\mathfrak{H}}_8^{\mathrm{II}}$ intersect but do not have as good a control over the intersection locus.

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While Theorem A gives us an explicit description of how to move from one component to the other, the standard tools in 4-manifold topology do not seem to be able to control the resulting surgery, so that the answer to the diffeomorphism question posed above remains elusive for now.

The ingredients in the proof of Theorem A are some abstract deformation theory and explicit toric constructions. The latter is a lower-dimensional version of the scrollar deformations used by Coughlan and Pignatelli to study canonical threefolds of general type on the (three-dimensional) Noether line [CP23]. It is remarkable that also in the case of threefolds, every eighth instance of the moduli space has two irreducible components. In contrast to Theorem A, these components intersect in a locus parametrising threefolds with canonical singularities.

The methods used to prove Theorem A lead us to consider more general stable Horikawa surfaces. Surprisingly, we find with growing K_X^2 a tail of trailing irreducible components in the moduli space.

Theorem B. *Let $k \geqslant 5$. The connected component of $\bar{\mathfrak{H}}_{2k}$ containing classical Horikawa surfaces satisfies*

$$\bar{\mathfrak{H}}_{2k} \supset \mathfrak{H}_{2k} \cup \bigcup_{\substack{k>m>\frac{1}{2}(k+4)\\ m\equiv k \bmod 2}} \bar{\mathfrak{H}}_{2k}^{(m)},$$

where the $\bar{\mathfrak{H}}_{2k}^{(m)}$ are generically non-reduced, irreducible components of dimension $5k+4m+19 > \dim \mathfrak{H}_{2k}$.

We illustrate the phenomenon schematically in Figures 1 and 2. The proof of Theorem B can be found in Section 5.3.

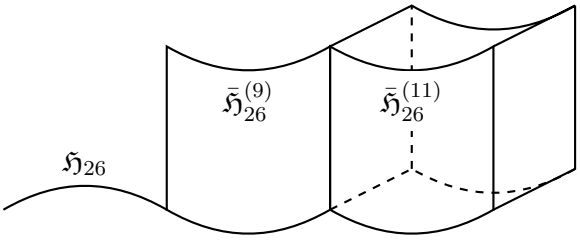


Figure 1. Standard components in $\bar{\mathfrak{H}}_{26}$

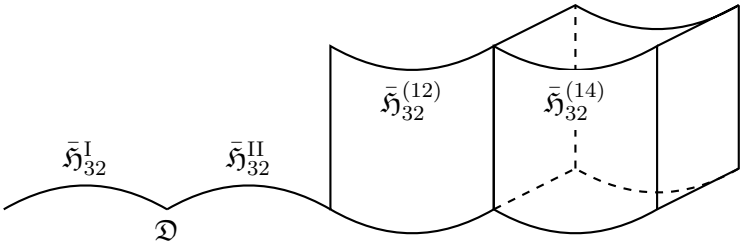


Figure 2. Standard components in $\bar{\mathfrak{H}}_{32}$

2. Standard stable Horikawa surfaces

We work over the complex numbers. General references for the notions related to stable surfaces and their moduli are [Kol13a, Kol13b, Kol23]. All necessary information about double covers in this context can be found in [AP12].

Let $\mathbb{F}_m$ be a Hirzebruch surface. We denote the negative curve by σ_∞, so $\sigma_\infty^2 = -m$, and the class of a fibre by Γ. We also fix a section disjoint from σ_∞, namely $\sigma_0 \in |\sigma_\infty + m\Gamma|$; in the toric model introduced later, σ_0 can be chosen to be invariant.

Definition 2.1. A standard stable Horikawa surface of type (m) is a double cover

$$f\colon X \to \mathbb{F}_m$$

branched over $B \in |6\sigma_\infty + 2a\Gamma|$ such that X has semi-log-canonical singularities and K_X is ample. We call it a classical Horikawa surface if X has at most canonical singularities.

Lemma 2.2. *A standard stable Horikawa surface of type (m) exists for $a > 2m+2$ and satisfies*

$$K_X^2 = 4a - 6m - 8 \quad \text{and} \quad p_g(X) = 2a - 3m - 2\,,$$

so $K_X^2 = 2p_g(X) - 4$. In addition:

(i) *If $2a \geqslant 6m$, then the linear system has no base points and the general branch divisor is smooth and connected.*

(ii) *If $6m > 2a \geqslant 5m$, then $B = \sigma_\infty + B'$ and B' moves in a base-point free linear system with $\sigma_\infty \cdot B' = 2a - 5m$.*
In particular, the general branch divisor is smooth and disconnected for $2a = 5m$.

(iii) *If $5m > 2a > 4m + 4$, then the general branch divisor is $B = 2\sigma_\infty + B'$ with B' in the base-point free linear system $|4\sigma_0 + 2(a-2m)\Gamma|$. In this case, the general X is non-normal with normal crossing singularities at the general point of $f^{-1}(\sigma_\infty)$ and $2(a-2m)$ pinch points.*

In particular, a classical Horikawa surface of type (m) with $K_X^2 = 2k$ exists if and only if $m \leqslant (k+4)/2$ and m and k have the same parity.

Remark 2.3. Note that for $m = 0, 1, 2, 3$ only the first case can occur, while for $m = 4, 5, 6$ only the first two cases can occur. For $m \geqslant 7$ all three cases are possible.

Also note that K_X^2 is always even, and it is divisible by 4 if and only if the type (m) is even.

Proof. The canonical divisor of X is

$$\begin{aligned} K_X &= f^*\left(K_{\mathbb{F}_m} + \tfrac{1}{2}B\right) \\ &= f^*(-2\sigma_\infty - (m+2)\Gamma + 3\sigma_\infty + a\Gamma) \\ &= f^*(\sigma_\infty + (a-m-2)\Gamma)\,, \end{aligned}$$

and this divisor is ample if and only if it is positive on the pullback of σ_∞, which holds if and only if $a > 2m+2$. Then

$$K_X^2 = 2(\sigma_\infty + (a-m-2)\Gamma)^2 = -2m + 4(a-m-2) = 4a - 6m - 8\,,$$
$$p_g(X) = h^0(\sigma_\infty + (a-m-2)\Gamma) = h^0(\mathcal{O}_{\mathbb{P}^1}(a-m-2)) + h^0(\mathcal{O}_{\mathbb{P}^1}(a-m-2)) = 2a - 3m - 2\,.$$

The rest of the claims rely on a standard computation on Hirzebruch surfaces, determining how often a particular linear system has to contain the negative section. □

We denote the subset of the moduli space of stable Horikawa surfaces parametrising standard stable Horikawa surfaces with $K_X^2 = 2k$ of type (m) by $\bar{\mathfrak{H}}_{2k}^{(m)} \subset \bar{\mathfrak{H}}_{2k}$ and the subset of classical Horikawa surfaces of type (m) by $\mathfrak{H}_{2k}^{(m)} = \bar{\mathfrak{H}}_{2k}^{(m)} \cap \mathfrak{H}_{2k}$.

The moduli space of stable surfaces also carries a natural scheme structure, and if $\bar{\mathfrak{H}}_{2k}^{(m)}$ forms an open subset of an irreducible component of $\bar{\mathfrak{H}}_{2k}$, then we consider it with this scheme structure. Otherwise, we just consider it as a reduced subscheme of $\bar{\mathfrak{H}}_{2k}$.

Proposition 2.4. *The moduli spaces $\bar{\mathfrak{H}}_{2k}^{(m)}$ of standard stable Horikawa surfaces of type (m) are irreducible, and*

$$\dim \bar{\mathfrak{H}}_{2k}^{(m)} = \begin{cases} 7k+28\,, & m = 0\,, \\ 7k+29-m\,, & \tfrac{1}{3}(k+4) \geqslant m > 0\,, \\ 6k+2m+24\,, & \tfrac{1}{2}(k+4) \geqslant m > \tfrac{1}{3}(k+4)\,, \\ 5k+4m+19\,, & k > m > \tfrac{1}{2}(k+4)\,. \end{cases}$$

Proof. Since the Picard group of a Hirzebruch surface does not contain 2-torsion, the double cover is determined by its branch divisor. The condition that a double cover has semi-log-canonical singularities is open; cf. [Ant20, Lemma 3.2]. Therefore, a complete family of standard Horikawa surfaces of type (m) is parametrised by an open subset of the linear system $|6\sigma_\infty + 2a\Gamma|$ on $\mathbb{F}_m$, where $2k = 4a - 6m - 8$, so its image in the moduli space is irreducible as well.

Determining the dimension of the linear system is a straightforward cohomology computation:

$$\begin{aligned} h^0(\mathbb{F}_m, 6\sigma_\infty + 2a\Gamma) &= \sum_{i=0}^{6} h^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-im+2a)\big) \\ &= \sum_{i=0}^{4}(-im+2a+1) + \sum_{i=5}^{6} h^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-im+2a)\big)\,, \end{aligned}$$

and since $2a > 4m+4$ and $2a = k+3m+4$,

$$\begin{aligned} &= (10a - 10m + 5) + h^0(\mathcal{O}_{\mathbb{P}^1}(2a-5m)) + h^0(\mathcal{O}_{\mathbb{P}^1}(2a-6m)) \\ &= \begin{cases} 14a-21m+7\,, & 2a \geqslant 6m\,, \\ 12a-15m+6\,, & 6m > 2a \geqslant 5m\,, \\ 10a-10m+5\,, & 5m > 2a > 4m+4 \end{cases} \\ &= \begin{cases} 7k+35\,, & k \geqslant 3m-4\,, \\ 6k+3m+30\,, & 3m-4 > k \geqslant 2m-4\,, \\ 5k+5m+25 & 2m-4 > k > m\,. \end{cases} \end{aligned}$$

Then, because $h_0(\mathcal{T}_{\mathbb{F}_m}) = m+5$ for $m \geqslant 1$ and $h^0(\mathcal{T}_{\mathbb{P}^1\times\mathbb{P}^1}) = 6$, as computed in [Hor76] or [Ser06, Appendix B], we have

$$\dim \bar{\mathfrak{H}}_{2k}^{(m)} = \dim |B| - \dim \operatorname{Aut} \mathbb{F}_m = \begin{cases} 7k+28\,, & m = 0\,, \\ 7k+29-m\,, & \tfrac{1}{3}(k+4) \geqslant m > 0\,, \\ 6k+2m+24\,, & \tfrac{1}{2}(k+4) \geqslant m > \tfrac{1}{3}(k+4)\,, \\ 5k+4m+19\,, & k > m > \tfrac{1}{2}(k+4) \end{cases}$$

as claimed. □

With the above notation, we can phrase some of Horikawa's original results as follows, see also [BHP+04, Section VII.9].

Theorem 2.5 (Horikawa [Hor76]). *Let $\mathfrak{H}_{2k}$ be the moduli space of (classical) Horikawa surfaces with $K_X^2 = 2k = 2p_g(X) - 4$ for $k \neq 1, 4$.*

(i) *If k is odd, then*

$$\mathfrak{H}_{2k} = \bigcup_{d=0}^{\lfloor \frac{1}{4}(k+2) \rfloor} \mathfrak{H}_{2k}^{(2d+1)}$$

is irreducible of dimension $7k + 28$.

(ii) *If k is even and $2k \not\equiv 0 \bmod 8$, then*

$$\mathfrak{H}_{2k} = \bigcup_{d=0}^{\lfloor \frac{1}{4}(k+4) \rfloor} \mathfrak{H}_{2k}^{(2d)}$$

is irreducible of dimension $7k + 28$.

(iii) *If $2k \equiv 0 \bmod 8$, then $\mathfrak{H}_{2k} = \mathfrak{H}_{2k}^{\mathrm{I}} \sqcup \mathfrak{H}_{2k}^{\mathrm{II}}$ has two connected and irreducible components, both of dimension $7k + 28$: the general component*

$$\mathfrak{H}_{2k}^{\mathrm{I}} = \bigcup_{d=0}^{\frac{1}{4}k} \mathfrak{H}_{2k}^{(2d)}$$

and the special component

$$\mathfrak{H}_{2k}^{\mathrm{II}} = \mathfrak{H}_{2k}^{\left(\frac{1}{2}(k+4)\right)}.$$

If $2k \equiv 8 \bmod 16$, then smooth surfaces in the respective components are not diffeomorphic.

Horikawa surfaces with $K_X^2 = 2$ and $p_g(X) = 3$ are double covers of the projective plane branched over a sufficiently general octic; some information on their stable degenerations can be found in [Ant20]. The case $K_X^2 = 8$ will be discussed briefly in Section 3.3.

In the following we want to investigate how the subsets $\bar{\mathfrak{H}}_{2k}^{(m)}$ interact inside $\bar{\mathfrak{H}}_{2k}$. We focus particularly on the closures $\bar{\mathfrak{H}}^{\mathrm{I}}$ and $\bar{\mathfrak{H}}^{\mathrm{II}}$ of the special and general components in the cases where K_X^2 is divisible by 8. As a byproduct, we will actually re-prove most of Theorem 2.5.

For later use we also note the following.

Corollary 2.6. *Fix $K_X^2 = 2k > 8$. Then the dimensions of the non-classical subsets $\bar{\mathfrak{H}}_{2k}^{(m)}$ are strictly increasing:*

(i) *If k is odd, then*

$$\dim \mathfrak{H}_{2k} < \dim \bar{\mathfrak{H}}_{2k}^{\left(2\lfloor \frac{1}{4}(k+2) \rfloor + 3\right)} < \cdots < \dim \bar{\mathfrak{H}}_{2k}^{(k-2)}.$$

(ii) *If k is even, then*

$$\dim \mathfrak{H}_{2k} < \dim \bar{\mathfrak{H}}_{2k}^{\left(2\lfloor \frac{1}{4}(k+4) \rfloor + 2\right)} < \cdots < \dim \bar{\mathfrak{H}}_{2k}^{(k-2)}.$$

Proof. From Lemma 2.2 one can check that the listed spaces are exactly the ones containing no classical Horikawa surfaces. The rest follows by comparing their dimensions, computed in Proposition 2.4, with $\dim \mathfrak{H}_{2k} = \dim \bar{\mathfrak{H}}_{2k}^{(0)}$. □

3. Connecting $\mathfrak{H}^{\mathrm{I}}$ and $\mathfrak{H}^{\mathrm{II}}$

Recall, for example from [CLS11, Section 5.2], that for any integer α, the $\mathbb{Z}^2$-graded ring, with variables and weights

$$\begin{pmatrix} t_0 & t_1 & x_0 & x_1 \\ 1 & 1 & \alpha & \alpha - m \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and irrelevant ideal $(t_0, t_1) \cap (x_0, x_1)$, is the Cox ring of the Hirzebruch surface $\mathbb{F}_m$. The negative section σ_∞ is given by $\{x_1 = 0\}$, the positive section σ_0 by $\{x_0 = 0\}$, and the fibres by the vanishing of linear polynomials $\{f_1(t_0, t_1) = 0\}$.

3.1 Horikawa surfaces in weighted projective bundles

The fibration on $\mathbb{F}_m$ induces a pencil of genus two curves on any standard stable Horikawa surface X of type (m). Since a genus two curve can be canonically embedded in $\mathbb{P}(1,1,3)$, we can thus describe such a surface X with $K_X^2 = 2k$ as a hypersurface in a toric variety $T_{m,k}$, which is a $\mathbb{P}(1,1,3)$ bundle over $\mathbb{P}^1$ that varies depending on m and k.

In this section, we suppose that k is even (and therefore m as well by Lemma 2.2), and so $K_X^2 \equiv 0 \bmod 4$. We will treat the case where k is odd in Section 4.2. To simplify exposition, we let $m = 2d$ and $k = 2n$. Then the surfaces we want to describe are hypersurfaces in the toric threefold $T_{2d,2n}$ (defined for $0 \leqslant 2d \leqslant n+2$) given by

$$\begin{pmatrix} t_0 & t_1 & x_0 & x_1 & z \\ 1 & 1 & d-n-2 & -d-n-2 & -2(n+2) \\ 0 & 0 & 1 & 1 & 3 \end{pmatrix}$$

with irrelevant ideal $(t_0, t_1) \cap (x_0, x_1, z)$. The surface X arises as a sufficiently general hypersurface of bidegree $\binom{-4(n+2)}{6}$, and as such is defined by a polynomial $z^2 + f(x_0, x_1, t_0, t_1)$, where we eliminate the linear term in z by completing the square. Only the first entry of the degree vector is relevant to determine which monomials appear in $f(x_0, x_1, t_0, t_1)$; denoting it by $\deg_1$, we have for example $\deg_1(z^2) = -4(n+2)$.

Let us consider three examples corresponding to the lowest and highest possible values of d, in terms of n.

Example 3.1. If $d = 0$, then the matrix of weights becomes

$$\begin{pmatrix} t_0 & t_1 & x_0 & x_1 & z \\ 1 & 1 & -n-2 & -n-2 & -2(n+2) \\ 0 & 0 & 1 & 1 & 3 \end{pmatrix}.$$

In this case, the monomials appearing in f are of the form $x_0^r x_1^{6-r} g_{2(n+2)}(t_0, t_1)$, which are bihomogeneous when considered in the usual grading. We thus recognize X as a double cover of $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$.

The case of particular interest is the following, which for a choice of coefficients gives a key example of a singular stable Horikawa surface in $\bar{\mathfrak{H}}^{\mathrm{II}}_{2k}$.

Example 3.2. If n is even and $d = (n+2)/2$, then the weight matrix is

$$\begin{pmatrix} t_0 & t_1 & x_0 & x_1 & z \\ 1 & 1 & -\left(\frac{1}{2}n+1\right) & -\frac{3}{2}(n+2) & -2(n+2) \\ 0 & 0 & 1 & 1 & 3 \end{pmatrix},$$

and we see that $\deg_1 x_0^6 = -3n-6 > \deg_1 z^2 = \deg_1 x_0^5 x_1$. Since multiplying with a polynomial in t_0 and t_1 can only increase the degree, a general polynomial in the linear system is of the form

$$z^2 - x_1\big(\mu x_0^5 + \cdots\big)\,, \tag{3.1}$$

where μ is a non-zero constant. The branch divisor then contains $\sigma_\infty = \{x_1 = 0\}$ once, and the rest is disjoint from σ_∞ because the term $\big(\mu x_0^5 + \cdots\big)$ has x_0^5 with non-vanishing coefficient.

If we eliminate the variable z, the remaining matrix describes the Hirzebruch surface $\mathbb{F}_{n+2}$, and so stable hypersurfaces given by such equations lie in the special component $\bar{\mathfrak{H}}^{\mathrm{II}}_{4n}$.

Example 3.3. In the same ambient toric threefold and linear system as in Example 3.2, we now consider the hyperplane of surfaces defined by an equation as in (3.1), where the coefficient μ vanishes; that is, the polynomial is of the form

$$z^2 - x_1^2\big(x_0^4 g_{n+2}(t_0,t_1) + \cdots\big)\,. \tag{3.2}$$

The general such surface is stable: it is a double cover of $\mathbb{F}_{n+2}$ branched over $2\sigma_\infty + B'$, where B' is smooth and intersects the negative section transversally. Thus X is smooth outside the non-normal locus, which is the pullback of the negative section. Over the general point of the negative section, X has normal crossing singularities, that is, locally $x_1^2 - z^2$, and at the $n+2$ intersection points of σ_∞ and B', the surface X has pinch points, that is, locally $z^2 - tx_1^2$.

The closure of the locus of these surfaces forms a divisor $\mathfrak{D}$ in $\bar{\mathfrak{H}}^{(n+2)}_{4n} \subset \bar{\mathfrak{H}}^{\mathrm{II}}_{4n}$.

3.2 Connecting the general and the special component ($K_X^2 > 8$)

We assume in this section that $K_X^2 = 2k = 4n \equiv 0 \bmod 8$ and $n \geqslant 3$. Then by Theorem 2.5, the Gieseker moduli space $\mathfrak{H}_{4n} = \mathfrak{H}^{\mathrm{I}}_{4n} \sqcup \mathfrak{H}^{\mathrm{II}}_{4n}$ is the union of the two irreducible and connected components, each of dimension $14(n+2)$, which we call the general and the special component. We show that the closures of these components intersect in the stable compactification $\bar{\mathfrak{H}}_{4n}$.

Theorem 3.4. *If $K_X^2 = 4n \equiv 0 \bmod 8$ and $n \geqslant 3$, then the intersection $\bar{\mathfrak{H}}^{\mathrm{I}}_{4n} \cap \bar{\mathfrak{H}}^{\mathrm{II}}_{4n}$ contains the divisor $\mathfrak{D}$ containing the locus of semi-smooth double covers of $\mathbb{F}_{n+2}$ described in Example 3.3.*

The general surface in $\mathfrak{D}$ can be smoothed to a surface in $\mathfrak{H}^{\mathrm{II}}_{4n}$ using the parameter μ in Example 3.2. So it remains to show that it can be smoothed to the other component as well.

We start by constructing an explicit family over $\mathbb{A}^1_\lambda$, depending on a choice of two polynomials $p_0, p_1 \in R_{n/2+1)}$, with special fibre a singular surface described in Example 3.3 and general fibre a smooth surface of the type described in Example 3.1.

Consider the toric fourfold T given by

$$\begin{pmatrix} t_0 & t_1 & y_0 & x_0 & x_1 & z \\ 1 & 1 & -(\frac{1}{2}n+1) & -n-2 & -n-2 & -2(n+2) \\ 0 & 0 & 1 & 1 & 1 & 3 \end{pmatrix}$$

with irrelevant ideal $(t_0,t_1) \cap (y_0,x_0,x_1,z)$. Let $R = \mathbb{C}[t_0,t_1]$ with the usual grading.

Inside T let us first consider the family T_λ over $\mathbb{A}^1_\lambda$ of threefolds given by

$$\lambda y_0 = p_1 x_0 - p_0 x_1 \quad \text{with } p_i \in R_{(n/2+1)} \text{ and } \gcd(p_0,p_1) = 1\,. \tag{3.3}$$

If $\lambda \neq 0$, then we eliminate the variable y_0 and obtain $T_{0,2n}$, the threefold considered in Example 3.1.

If $\lambda = 0$, then we can introduce a new variable

$$y_1 = \frac{x_0}{p_0} = \frac{x_1}{p_1}\,, \quad \deg y_1 = \begin{pmatrix} -\frac{3}{2}(n+2) \\ 1 \end{pmatrix}$$

because the p_i cannot vanish simultaneously. The resulting equations

$$x_0 = p_0 y_1\,, \quad x_1 = p_1 y_1 \tag{3.4}$$

eliminate the variables x_0 and x_1, and we recover $T_{n+2,2n}$, the toric threefold from Example 3.2.

To define a family of hypersurfaces in T_λ that restricts in the desired fashion, we need to analyse which elements in the relevant linear system on the central fibre are global on the family. That is, we need to describe all polynomials in t_0, t_1, x_0, x_1, y_0 of bidegree $\binom{-4(n+2)}{6}$. These are sums of elements of the subspaces

$$y_0^{6-i} \cdot \mathsf{Sym}^i\langle x_0, x_1\rangle \cdot R_{(i-2)(\frac{1}{2}n+1)} \quad (i \geqslant 2) \tag{3.5}$$

because these have $\deg_1$ equal to $-(6-i)\cdot(n/2+1) - i\cdot 2(n/2+1) + (i-2)(n/2+1) = -4(n+2)$. Note that the cases $i = 0, 1$ do not occur because there are no polynomials of negative degree in R.

Now we let λ go to zero to see which monomials we get on the central fibre. To describe this, let $V = \langle p_0, p_1\rangle$. Then substituting (3.4) into (3.5), we obtain on the special fibre all equations of the form $z^2 - f$ with f in the subspace generated by

$$y_0^{6-i} y_1^i \cdot \mathsf{Sym}^i V \cdot R_{(i-2)(\frac{1}{2}n+1)} \subset y_0^{6-i} y_1^i \cdot R_{(2i-2)(\frac{1}{2}n+1)} \quad (i \geqslant 2).$$

Lemma 3.5. *Let $p_0, p_1 \in R_{n/2+1}$ be without common divisor, and let $i \geqslant 3$. Then the multiplication map*

$$\varphi_i^p \colon \mathsf{Sym}^i V \cdot R_{(i-2)(\frac{1}{2}n+1)} \to R_{(2i-2)(\frac{1}{2}n+1)}$$

is surjective.

Proof. Let $A = A' = \mathbb{P}^1$. We have maps

$$A' \xrightarrow{\pi=(p_0:p_1)} A \xrightarrow{v_i} \mathbb{P}^i\,,$$

where v_i is the Veronese embedding of $\mathbb{P}^1$ into $\mathbb{P}^i$ as a rational normal curve. Note that the $\mathcal{O}_A$-algebra $\pi_*\mathcal{O}_{A'}$ is a torsion-free coherent sheaf on $\mathbb{P}^1$, hence a vector bundle. The trace map splits off a trivial summand, and by Grothendieck's lemma (see for example [Huy05, Corollary 5.2.8]), the whole bundle is a direct sum of line bundles. Since the cohomology of $\pi_*\mathcal{O}_{A'}$ gives the cohomology of $\mathcal{O}_{A'}$, we have

$$\pi_*\mathcal{O}_{A'} = \mathcal{O}_A \oplus \mathcal{O}_A(-1)^{\oplus \frac{1}{2}n}\,.$$

We then have

$$\mathsf{Sym}^i V = \pi^* H^0(A, \mathcal{O}_A(i)) = \pi^* v_i^* H^0(\mathbb{P}^i, \mathcal{O}_{\mathbb{P}^i}(1)) \quad \text{and} \quad R_{\alpha(\frac{1}{2}n+1)} = H^0(A', \pi^*\mathcal{O}_A(\alpha))\,.$$

Restricting the Euler sequence of $\mathbb{P}^i$ to the rational normal curve and pulling back to A', we get

$$0 \to \pi^* v_i^*(\Omega_{\mathbb{P}^i}(1)) \otimes \pi^*\mathcal{O}_A(i-2) \to \mathsf{Sym}^i V \otimes \pi^*\mathcal{O}_A(i-2) \to \pi^*\mathcal{O}_A(2i-2) \to 0$$

and find by taking global sections that the cokernel of the multiplication map is

$$\begin{aligned} \operatorname{coker}\varphi_i^p &= H^1(A', \pi^* v_i^*(\Omega_{\mathbb{P}^i}(1)) \otimes \pi^*\mathcal{O}_A(i-2)) \\ &= H^1(A, v_i^*(\Omega_{\mathbb{P}^i}(1)) \otimes \mathcal{O}_A(i-2) \otimes \pi_*\mathcal{O}_{A'}) \\ &= H^1\big(A, v_i^*(\Omega_{\mathbb{P}^i}) \otimes \mathcal{O}_A(2i-2) \otimes \big(\mathcal{O}_A \oplus \mathcal{O}_A(-1)^{\oplus \frac{1}{2}n}\big)\big)\,, \end{aligned}$$

which by [BE95, Proposition 5A.2] becomes

$$\begin{aligned}\operatorname{coker}\varphi_i^p &= H^1\big(A, \big(\mathcal{O}_A(-i-1)^{\oplus i}\big) \otimes \mathcal{O}_A(2i-2) \otimes \big(\mathcal{O}_A \oplus \mathcal{O}_A(-1)^{\oplus \frac{1}{2}n}\big)\big)\\ &= H^1\big(A, \mathcal{O}_A(i-3)^{\oplus i}\big) \oplus H^1\big(A, \mathcal{O}_A(i-4)^{\oplus \frac{1}{2}in}\big)\\ &= 0 \qquad (\text{for } i \geqslant 3)\,.\end{aligned}$$

Note that for $i = 2$ it is quite obvious that the map is not onto. □

Theorem 3.4 now follows immediately from the slightly more precise result.

Proposition 3.6. *Let X_0 be a hypersurface as in Example 3.3 such that in equation (3.2), the polynomial $g_{2(n/2+1)}$ equals p_0p_1, where $p_i \in R_{n/2+1}$ and $\gcd(p_0, p_1) = 1$. Then X_0 can be smoothed to a Horikawa surface in $\mathfrak{H}^{\mathrm{I}}_{2n}$.*

Proof. By assumption and Lemma 3.5, we can choose monomials such that the surface defined in the toric fourfold T by the equations

$$\lambda y_0 = p_1x_0 - p_0x_1 \quad \text{and} \quad z^2 + y_0^4x_0x_1 + \big(\text{terms not divisible by } y_0^4\big) = 0$$

degenerates over $\mathbb{A}^1_\lambda$ to X_0 in the family described above. After possibly adding λ times a general polynomial, the general fibre will be smooth as claimed. □

3.3 Connecting $\mathfrak{H}^{\mathrm{I}}_8$ and $\mathfrak{H}^{\mathrm{II}}_8$

Here we consider the last remaining case, Horikawa surfaces with $K_X^2 = 8$ and $p_g(X) = 6$. There are two new constructions here:

Type (∞)**.** The surface X is a double cover of the projective plane branched over a smooth curve of degree ten. A quick dimension count shows that this family $\mathfrak{H}^{(\infty)}_8$ has dimension $\dim|\mathcal{O}_{\mathbb{P}^2}(10)| - \dim \operatorname{Aut}(\mathbb{P}^2) = 57$.

Type $(4')$**.** The surface X is a double cover of the cone over a rational normal curve of degree four branched over the vertex and a quintic section. Thus we can realise X as a (sufficiently general) hypersurface of degree 20 in weighted projective space $\mathbb{P}(1, 1, 4, 10)$.

The main difference between this and the general case is that $K_{\mathbb{F}_4} + \frac{1}{2}(6\sigma_\infty + 20\Gamma)$ is not ample on $\mathbb{F}_4$ but only big and nef and contracts the negative section. So the double cover of $\mathbb{F}_4$ branched over a smooth $B \in |6\sigma_\infty + 20\Gamma|$ gives a smooth minimal Horikawa surface, but the preimage of the negative section is a (-2)-curve, which we need to contract to get the canonical model.

Arguing as in Proposition 2.4, we see that the family $\mathfrak{H}^{(4')}_8$ is of dimension 56.

By [Hor76, Section 4] again the moduli space has two connected components, namely

$$\mathfrak{H}^{\mathrm{I}}_8 = \mathfrak{H}^{(0)}_8 \sqcup \mathfrak{H}^{(2)}_8 \quad \text{and} \quad \mathfrak{H}^{\mathrm{II}}_8 = \mathfrak{H}^{(\infty)}_8 \sqcup \mathfrak{H}^{(4')}_8\,.$$

The main point is that every surface of type $(4')$ can be deformed to one of type (∞) by taking a suitable double cover of the $\mathbb{Q}$-Gorenstein smoothing of $\mathbb{P}(1, 1, 4)$ to $\mathbb{P}^2$.

Note that in this case, the dimensions of the two components are different, namely $\dim \mathfrak{H}^{\mathrm{I}}_8 = 56$ and $\dim \mathfrak{H}^{\mathrm{II}}_8 = 57$.

We will now prove the following.

Proposition 3.7. *The closures $\bar{\mathfrak{H}}^{\mathrm{I}}_8$ and $\bar{\mathfrak{H}}^{\mathrm{II}}_8$ intersect in $\bar{\mathfrak{H}}_8$.*

Remark 3.8. Arguing more carefully as in Section 3.2, one can be a bit more precise: the intersection of the two 56-dimensional subsets $\bar{\mathfrak{H}}_8^{(0)}$ and $\bar{\mathfrak{H}}_8^{(4')}$ contains a subset $\mathfrak{D}$ of dimension 55, which is therefore a divisor in $\bar{\mathfrak{H}}_8^{\mathrm{I}}$, and a subset of codimension two in $\bar{\mathfrak{H}}_8^{\mathrm{II}}$. We illustrate this in Figure 3.

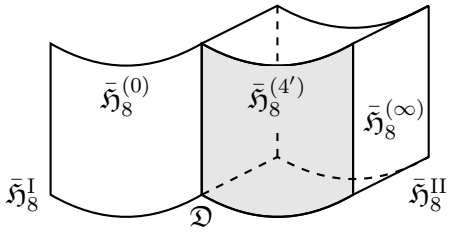


Figure 3. Standard strata in $\bar{\mathfrak{H}}_8$

Proof. The aim is to construct a surface in the closure of both components. In contrast to the previous section, this surface will be normal with one elliptic singularity.

Consider the toric fourfold T given by

$$\begin{pmatrix} t_0 & t_1 & y_0 & x_0 & x_1 & z \\ 1 & 1 & 4 & 2 & 2 & 10 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{pmatrix}$$

with irrelevant ideal $(t_0, t_1) \cap (y_0, x_0, x_1, z)$. Let $R = \mathbb{C}[t_0, t_1]$ with the usual grading.

Inside T let us consider the family of threefolds T_λ over $\mathbb{A}^1_\lambda$ given by

$$\lambda y_0 = p_1 x_0 - p_0 x_1 \quad \text{with } p_0, p_1 \in R_2 \text{ and } p_0 p_1 \text{ without multiple zeros},$$

which we intersect with a sufficiently general hypersurface of bidegree $\binom{20}{6}$ given by

$$z^2 + x_0 x_1 y_0^4 + \text{lower-order terms in } y_0 + \lambda g(x_0, x_1, t_0, t_1) = 0$$

to get a family of surfaces X_λ. If $\lambda \neq 0$, then we eliminate the variable y_0 and find, up to a change of basis in the weight matrix, the threefold $T_\lambda \cong T_{0,4}$ defined at the beginning of Section 3.1. Thus X_λ is a Horikawa surface of type (0) for $\lambda \neq 0$.

Now let us consider the central fibre $\lambda = 0$. We introduce a new variable

$$y_1 = \frac{x_0}{p_0} = \frac{x_1}{p_1}, \quad \deg y_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

because the p_i cannot vanish simultaneously. The resulting equations $x_0 = p_0 y_1$ and $x_1 = p_1 y_1$ eliminate the variables x_0 and x_1, and we get the toric threefold T_0 with weights

$$\begin{pmatrix} t_0 & t_1 & y_0 & y_1 & z \\ 1 & 1 & 4 & 0 & 10 \\ 0 & 0 & 1 & 1 & 3 \end{pmatrix},$$

in which the equation for X_0 becomes

$$z^2 + p_0 p_1 y_0^4 y_1^2 + (\text{lower-order terms in } y_0) = 0\,.$$

Translating back to geometry, that is, projecting to the first four coordinates, we find that $\pi\colon X_0 \to \mathbb{F}_4$ is a double cover branched over $2\sigma_\infty + B'$, where B' intersects σ_∞ transversally in four points because $p_0 p_1$ has no multiple zeros.

Note that $K_{X_0} = \pi^*(\sigma_\infty + 4\Gamma)$ is not ample because it is trivial on the non-normal locus, which is the preimage of σ_∞. To understand the (log)-canonical model, let us take the normalisation,

$$\begin{array}{ccc} \bar{X}_0 & \xrightarrow{\quad\nu\quad} & X_0 \\ & {\scriptstyle\bar{\pi}}\searrow \quad \swarrow{\scriptstyle\pi} & \\ \text{branched over } B' & & \text{branched over } 2\sigma_\infty + B' \\ & \mathbb{F}_4\,. & \end{array}$$

Then $E = \bar{\pi}^{-1}(\sigma_\infty)$ is a smooth elliptic curve with $E^2 = -8$ and $\nu^* K_{X_0} = K_{\bar{X}_0} + E$. Since all sections of $m(K_{\bar{X}_0} + E)$ are constant on E, we have $\pi^*\colon H^0(mK_{X_0}) \cong H^0(m(K_{\bar{X}_0} + E))$, and the log-canonical model of X_0 is the log-canonical model of $(\bar{X}_0, E)$. The result is the surface Y_0, where we contract the elliptic curve E to an elliptic singularity.

On the level of the ambient toric threefold T_0, this corresponds to eliminating the variable y_1, that is, the rational projection

$$T_0 \dashrightarrow \mathbb{P}(1,1,4,10)\,.$$

The resulting hypersurface Y_0 can be deformed to a general such hypersurface, which is a Horikawa surface of type $(4')$. Thus the surface Y_0 is in the closure of both $\mathfrak{H}_8^{\mathrm{I}}$ and $\mathfrak{H}_8^{\mathrm{II}}$. □

4. Connecting adjacent (non-classical) components

In this section we use the toric construction described above to show that all standard stable Horikawa surfaces are contained in the same connected component of $\bar{\mathfrak{H}}_{2k}$. The details of the proof depend on the parity of k.

4.1 Connecting $\bar{\mathfrak{H}}_{4n}^{(2d)}$ and $\bar{\mathfrak{H}}_{4n}^{(2d+2)}$

We begin by assuming that k is even and take $k = 2n$.

Proposition 4.1. *The subsets $\bar{\mathfrak{H}}_{4n}^{(2d)}$ and $\bar{\mathfrak{H}}_{4n}^{(2d+2)}$ are in the same connected component of $\bar{\mathfrak{H}}_{4n}$.*

Proof. Note that the closure of the Gieseker moduli space of classical Horikawa surfaces is connected by Horikawa's Theorem 2.5 and by Section 3, so we may assume that $\bar{\mathfrak{H}}_{4n}^{(2d+2)}$ is a component containing only non-classical surfaces; that is, $n \leqslant 2d < 2n - 2$.

Since both subsets are connected, it suffices to exhibit a (Gorenstein, hence $\mathbb{Q}$-Gorenstein) family of standard stable Horikawa surfaces, where the general fibre is of type $(2d)$ and a special fibre is of type $(2d+2)$.

Consider for $n \leqslant 2d < 2n - 2$ the toric fourfold T given by

$$\begin{pmatrix} t_0 & t_1 & y_0 & x_0 & x_1 & z \\ 1 & 1 & d-n-1 & d-n-2 & -d-n-2 & -2(n+2) \\ 0 & 0 & 1 & 1 & 1 & 3 \end{pmatrix}$$

with irrelevant ideal $(t_0, t_1) \cap (y_0, x_0, x_1, z)$. Let $R = \mathbb{C}[t_0, t_1]$ with the usual grading.

Inside T let us first consider the family of threefolds T_λ over $\mathbb{A}^1_\lambda$ given by

$$\lambda y_0 = p_1 x_0 - p_0 x_1 \quad \text{with } p_1 \in R_1,\ p_0 \in R_{2d+1} \text{ and } \gcd(p_0, p_1) = 1\,.$$

If $\lambda \neq 0$, then we eliminate the variable y_0 and find the threefold $T_{2d,2n}$ defined at the beginning of Section 3.1.

If $\lambda = 0$, then we can introduce a new variable

$$y_1 = \frac{x_0}{p_0} = \frac{x_1}{p_1}, \quad \deg y_1 = \begin{pmatrix} -d-n-3 \\ 1 \end{pmatrix},$$

because the p_i cannot vanish simultaneously. The resulting equations

$$x_0 = p_0 y_1\,, \quad x_1 = p_1 y_1 \tag{4.1}$$

eliminate the variables x_0 and x_1, and we get $T_{2d+2,2n}$.

To prove that $\bar{\mathfrak{H}}^{2d}_{4n}$ meets $\bar{\mathfrak{H}}^{2d+2}_{4n}$, we need an equation of bidegree $\binom{-4(n+2)}{6}$ on T of the form $z^2 + f(t_0, t_1, y_0, x_0, x_1)$ that defines a stable surface if we set $\lambda = 0$. Writing

$$\begin{pmatrix} -4(n+2) \\ 6 \end{pmatrix} = \begin{pmatrix} 2n-2d \\ 0 \end{pmatrix} + 2\begin{pmatrix} -d-n-2 \\ 1 \end{pmatrix} + 4\begin{pmatrix} d-n-1 \\ 1 \end{pmatrix},$$

we can set

$$f(t_0, t_1, y_0, x_0, x_1) = g_{2n-2d}(t_0, t_1) \cdot x_1^2 \cdot \prod_{i=1}^{4} (y_0 + a_i x_0 + b_i x_1))$$

for general $a_i \in R_1$ and $b_i \in R_{2d+1}$.

Upon intersection with T_λ for $\lambda = 0$, we obtain

$$f(t_0, t_1, y_0, p_0 y_1, p_1 y_1) = g_{2n-2d}(t_0, t_1) p_1^2 y_1^2 \cdot \prod_{i=1}^{4} (y_0 + (a_i p_0 + b_i p_1) y_1))\,.$$

For general choices (for example, choosing $p_0 = t_0^{2d+1}$, $p_1 = t_1$, and generic a_i, b_i), this equation defines a stable surface because the branch curve B is the union of $2n - 2d$ fibres, twice the negative section, twice a fibre, and four sufficiently general sections in $|\sigma_0|$ that are disjoint from σ_∞; thus $\left(\mathbb{F}_{2d+2}, \frac{1}{2}B\right)$ is a log-canonical pair; cf., for example, [AP12].

More concretely, the double cover (generically) has $16(d+n+2)$ A_1 singularities, normal crossing singularities over the general point of the double locus, $2n - 2d + 4$ pinch points, and a degenerate cusp with local equation $z^2 + x^2 t^2$ over the point where $2\sigma_\infty$ meets the double fibre. This proves the claim. □

4.2 Connecting $\bar{\mathfrak{H}}^{(2d+1)}_{4n-2}$ and $\bar{\mathfrak{H}}^{(2d-1)}_{4n-2}$

We suppose that $k = K_X^2/2$ is odd and take $k = 2n - 1$. Then by Lemma 2.2 the type (m) has to be odd as well.

To connect the components of standard stable Horikawa surfaces of odd type, we follow the same strategy employed above: we realise the individual surfaces as hypersurfaces of a toric threefold and then connect these constructions inside a toric fourfold.

Example 4.2. Consider for $0 < d < n$ the toric threefold $T_{2d-1,2n-1}$ described via

$$\begin{pmatrix} t_0 & t_1 & x_0 & x_1 & z \\ 1 & 1 & d-n-1 & -d-n & -2n \\ 0 & 0 & 1 & 1 & 3 \end{pmatrix}$$

with irrelevant ideal $(t_0, t_1) \cap (x_0, x_1, z)$. A general equation of bidegree $\binom{-4n}{6}$ without linear term in z is of the form

$$z^2 + \sum_{i=0}^{6} g_{2n+6d-i(2d-1)} x_0^i x_1^{6-i}$$

and thus defines a double cover of $\mathbb{F}_{2d-1}$ branched over a curve in $|6\sigma_\infty + (2n+6d)\Gamma|$. By Lemma 2.2 we can describe all standard stable Horikawa surfaces of type $(2d-1)$ with $K_X^2 = 2(2n-1)$ in this way.

Proposition 4.3. *The subsets $\bar{\mathfrak{H}}^{(2d+1)}_{4n-2}$ and $\bar{\mathfrak{H}}^{(2d-1)}_{4n-2}$ are in the same connected component of $\bar{\mathfrak{H}}_{4n+2}$.*

Proof. We exhibit a ($\mathbb{Q}$-)Gorenstein family of standard stable Horikawa surfaces, where the general fibre is of type $(2d-1)$ and a special fibre is of type $(2d+1)$.

Consider for $n+2 \leqslant 2d < 2n+2$ the toric fourfold T given by

$$\begin{pmatrix} t_0 & t_1 & y_0 & x_0 & x_1 & z \\ 1 & 1 & d-n & d-n-1 & -d-n & -2n \\ 0 & 0 & 1 & 1 & 1 & 3 \end{pmatrix}$$

with irrelevant ideal $(t_0,t_1) \cap (y_0,x_0,x_1,z)$. Let $R = \mathbb{C}[t_0,t_1]$ with the usual grading.

Inside T let us consider the family of threefolds T_λ over $\mathbb{A}^1_\lambda$ given by

$$\lambda y_0 = p_1 x_0 - p_0 x_1 \quad \text{with } p_1 \in R_1,\ p_0 \in R_{2d} \text{ and } \gcd(p_0,p_1) = 1\,.$$

If $\lambda \neq 0$, then we eliminate the variable y_0 and find the toric threefold $T_{2d-1,2n-1}$ from Example 4.2.

If $\lambda = 0$, then we can introduce a new variable

$$y_1 = \frac{x_0}{p_0} = \frac{x_1}{p_1}\,, \quad \deg y_1 = \begin{pmatrix} -d-n-1 \\ 1 \end{pmatrix},$$

because the p_i cannot vanish simultaneously. The resulting equations

$$x_0 = p_0 y_1\,, \quad x_1 = p_1 y_1$$

eliminate the variables x_0 and x_1, and we find the threefold $T_{2d+1,2n-1}$ with weights

$$\begin{pmatrix} t_0 & t_1 & y_0 & y_1 & z \\ 1 & 1 & d-n & -d-n-1 & -2n \\ 0 & 0 & 1 & 1 & 3 \end{pmatrix}.$$

To prove that $\bar{\mathfrak{H}}^{2d-1}_{4n-2}$ meets $\bar{\mathfrak{H}}^{2d+1}_{4n-2}$, we need an equation of bidegree $\binom{-4n}{6}$ on T of the form $z^2 + f(t_0,t_1,y_0,x_0,x_1)$ that defines a stable surface if we set $\lambda = 0$. Writing

$$\begin{pmatrix} -4n \\ 6 \end{pmatrix} = \begin{pmatrix} 2n-2d \\ 0 \end{pmatrix} + 2\begin{pmatrix} -d-n \\ 1 \end{pmatrix} + 4\begin{pmatrix} d-n \\ 1 \end{pmatrix},$$

we can set

$$f(t_0,t_1,y_0,x_0,x_1) = g_{2n-2d}(t_0,t_1)\cdot x_1^2 \cdot \prod_{i=1}^{4}(y_0 + a_i x_0 + b_i x_1))$$

for general $a_i \in R_1$ and $b_i \in R_{2d}$ and $g_{2n-2d} \in R_{2n-2d}$, because $n > d$.

Upon intersection with T_λ for $\lambda = 0$, we obtain

$$f(t_0,t_1,y_0,p_0y_1,p_1y_1) = g_{2n-2d}(t_0,t_1)p_1^2y_1^2 \cdot \prod_{i=1}^{4}(y_0 + (a_ip_0 + b_ip_1)y_1))\,,$$

which defines a stable surface for sufficiently general choices, as in the proof of Proposition 4.1. □

5. Infinitesimal deformations of standard stable Horikawa surfaces

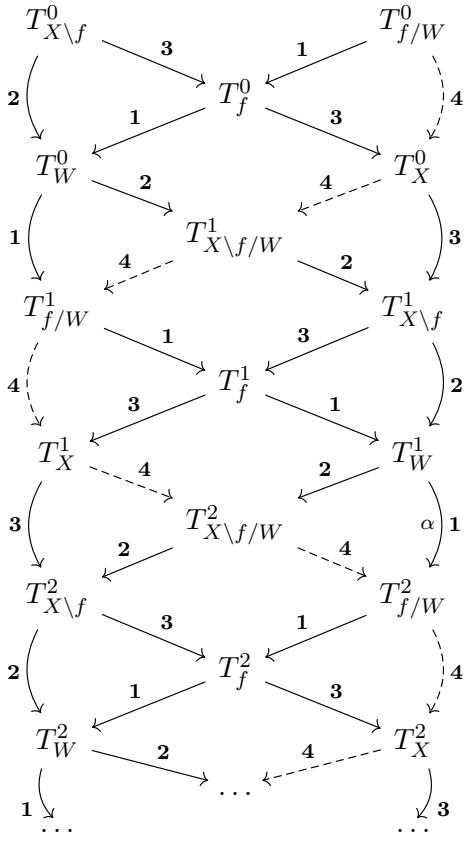
We start by considering some consequences of the general theory of deformations of maps (see for example [GLS07, Appendix C]), which we will then apply to Horikawa surfaces. In full generality, the information of Def_A, deformations of an object A, is encoded in some cotangent complex $\mathbb{L}_A^\bullet$ and the associated cohomology groups T_A^i and sheaves $\mathcal{T}_A^i$. More concretely, for a finite morphism

$$f\colon X \to W\,,$$

we consider

Def_X	Deformations of X,
Def_W	Deformations of W,
Def_f	Deformations of the map f possibly varying both X and W,
$\mathsf{Def}_{f/W}$	Deformations of the map f preserving W,
$\mathsf{Def}_{X\setminus f}$	Deformations of the map f preserving X,
$\mathsf{Def}_{X\setminus f/W}$	Deformations of the map f preserving W and X.

The corresponding tangent cohomology groups (or sheaves) are intertwined in the cotangent braid of Buchweitz (cf. [GLS07, Appendix C, p. 446]), shown in Figure 4.



(5.1)

Figure 4. The cotangent braid of Buchweitz

Remark 5.1. All spaces that we consider are either smooth or local complete intersections, so that for deformation purposes it would be enough to work with the sheaf of Kähler differentials.

But the deformations of f without fixing source or target need another calibre of theory, which is the reason why we use the general machinery.

Very similar problems have been considered previously, for example in [CvS06, GGP10, Man97], but we find it most transparent to start from scratch.

Lemma 5.2. *Assume that W is smooth and that*

(i) $H^0(W, \mathcal{T}_W) \to H^0(W, f_* f^* \mathcal{T}_W)$ *is an isomorphism,*

(ii) $H^1(W, \mathcal{T}_W) \to H^1(W, f_* f^* \mathcal{T}_W)$ *is an isomorphism,*

(iii) $H^2(W, \mathcal{T}_W) = 0$.

Then the natural forgetful maps $T^i_f \to T^i_X$ are isomorphisms for $i \leqslant 2$ and $\mathsf{Def}_f \cong \mathsf{Def}_X$. In other words, every deformation of X is induced by a deformation of the map f in a unique way.

Proof. From sequence **3** in (5.1), we see that it is enough to show that $T^i_{X\setminus f} = 0$ for $i = 0, 1, 2$. Using sequence **2** from (5.1) and the isomorphisms $T^i_W = H^i(W, \mathcal{T}_W)$ and $T^i_{X\setminus f/W} = H^{i-1}(W, f_* f^* \mathcal{T}_W)$ from [Ser06, Proposition 3.4.2], the result follows. □

Now we work out more specifically some groups and maps in the cotangent braid in the case of a double cover $f\colon X \to W$ branched over a divisor B. We will always assume that W is smooth and now recall the standard theory from [BHP+04, Section I.17]: we have $f_* \mathcal{O}_X = \mathcal{O}_W \oplus \mathcal{L}^{-1}$ for some line bundle $\mathcal{L}$ such that $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_W(B)$. Consider the geometric line bundle $\pi\colon |\mathcal{L}| = \underline{\mathsf{Spec}}_W\big(\mathsf{Sym}^\bullet \mathcal{L}^{-1}\big) \to W$, and denote the tautological section in $\pi^*\mathcal{L}$ by z. If σ_B is the section defining the branch locus, then X is the divisor in $|\mathcal{L}|$ defined by the section $\sigma_X = z^2 - \pi^*\sigma_B$ of $\pi^*\mathcal{O}_W(B)$. In particular, as a hypersurface in a smooth variety, X is a local complete intersection.

On X the section $\sigma_R = z$ is a square root of $\pi^*\sigma_B$ and as such defines the ramification divisor R.

Proposition 5.3. *Let $f\colon X \to W$ be a double cover of a smooth variety. In the notation above the following hold:*

(i) *We have $\mathcal{T}^i_{f/W} = 0$ for $i \neq 1$ and $\mathcal{T}^1_{f/W} \cong \mathcal{O}_X(f^*B)|_R \cong \mathcal{O}_B(B)$ (under the identification $f\colon R \cong B$).*

(ii) *If $h^1(\mathcal{O}_W) = 0$, then*

$$T^i_{f/W} = H^{i-1}(X, \mathcal{T}^1_{f/W}) = \begin{cases} 0\,, & i \neq 1, 2\,, \\ H^0(X, \mathcal{O}_X(B))/\langle \sigma_B \rangle\,, & i = 1\,, \\ H^1(B, \mathcal{O}_B(B))\,, & i = 2\,. \end{cases}$$

In other words, infinitesimal deformations of the map with fixed target W are exactly given by deformations of the branch divisor.

(iii) *The map $T^1_W \to T^2_{f/W}$ in sequence **4** in the cotangent braid (5.1) is, under the identification*

$$\alpha\colon H^1(\mathcal{T}_W) = T^1_W \to T^2_{f/W} = H^1(B, \mathcal{O}_B(B))\,, \tag{5.2}$$

induced by the map of sheaves $\alpha\colon \mathcal{T}_W \to \mathcal{O}_B(B)$ described in the following way: for a local vector field ξ, the section $\alpha(\xi)$ is the restriction to B of the derivative of the equation of B in direction ξ. That is,

$$\alpha(\xi) = (\xi \lrcorner d\sigma_B)\,|_B = \xi(\sigma_B)|_B\,.$$

Proof. Note that the sheaf of relative differentials of the line bundle $\pi\colon |\mathcal{L}| \to W$ is naturally $\omega_{|\mathcal{L}|/W} = \mathcal{L}^{-1}$. Thus we can consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & f^*\Omega_W & = & f^*\Omega_W & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{N}^*_{X/|\mathcal{L}|} & \xrightarrow{\cdot d\sigma_X} & \Omega_{|\mathcal{L}|}|_X & \longrightarrow & \Omega_X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & f^*\mathcal{O}_W(-B) & \xrightarrow{\cdot 2z} & f^*\mathcal{L}^{-1} & \longrightarrow & \Omega_{X/W} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0\,, & &
\end{array} \tag{5.3}$$

where the second row is the usual conormal sequence for a divisor and the first map is given by multiplication with the derivative of the section σ_X as indicated. In the third row we only look at the differential in fibre direction; therefore, under the identification $\omega_{|\mathcal{L}|/W} = \mathcal{L}^{-1}$, the first map becomes multiplication with $2z = 2\sigma_R$.

Note that because X is a local complete intersection and because $\mathbb{L}_{X\setminus f/W} = f^*\Omega_W$ as in the proof of Lemma 5.2, applying $\mathrm{Hom}(-, \mathcal{O}_X)$ to the third column of (5.3) gives sequence **4** in the cotangent braid (5.1).

Applying $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(-, \mathcal{O}_X)$ to the last row, we get a short exact sequence

$$0 \to f^*\mathcal{L} \xrightarrow{\cdot\sigma_R} \mathcal{O}_X(f^*B) \to \mathcal{T}^1_{f/W} \to 0\,, \tag{5.4}$$

so

$$\mathcal{T}^i_{f/W} = 0 \quad \text{for } i \neq 1 \qquad \text{and} \qquad \mathcal{T}^1_{f/W} \cong \mathcal{O}_X(f^*B)|_R \cong \mathcal{O}_B(B)$$

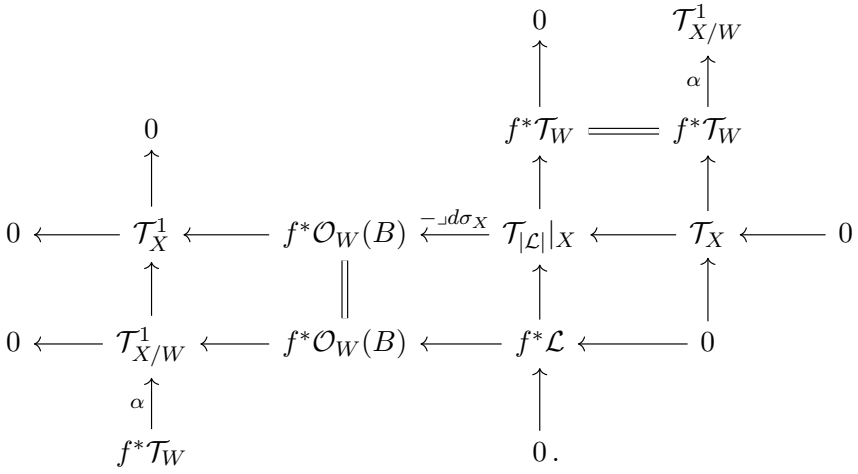
(under the identification $f\colon R \cong B$).

If we push forward to W and then take cohomology, then multiplication with the equation of the ramification divisor on X exchanges the invariant and anti-invariant subspaces of cohomology groups, so we can identify

$$\begin{array}{ccc}
H^i(X, f^*\mathcal{L}) & \xrightarrow{\qquad\sigma_R\qquad} & H^i(X, f^*\mathcal{O}_W(B)) \\
\| & & \| \\
H^i(W, \mathcal{L}) \oplus H^i(W, \mathcal{O}_W) & \xrightarrow{\left(\begin{smallmatrix} 0 & \sigma_B \\ 1 & 0 \end{smallmatrix}\right)} & H^i(W, \mathcal{O}_W(B)) \oplus H^i\big(W, \mathcal{L}^{-1}(B)\big)\,.
\end{array}$$

Cancelling the components on which the map is an isomorphism and noting that $H^1(\mathcal{O}_W) = 0$ by assumption, we see that the long exact sequence associated with (5.4) and the local-to-global Ext sequence give claim (ii).

To identify the map α in sequence **1** in (5.1), we apply $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(-\mathcal{O}_X)$ to (5.3) and obtain



Chasing through the diagram shows that the map α is defined as follows: Given a (local) vector field ξ on W, we can choose any lift $\tilde{\xi}$ to a vector field on $|\mathcal{L}|$. Then $\alpha(\xi) = \tilde{\xi} \lrcorner d\sigma_X$ projected to $\mathcal{T}^1_{X/W}$, where σ_X is the equation defining X and $\lrcorner$ is the contraction of 1-forms with vector fields. This gives the claimed map. □

Remark 5.4. To summarize, sequence **1** in the cotangent braid in the situation of a double cover is associated with the dual of the residue sequence $0 \to \Omega_W \to \Omega_W(\log B) \to \mathcal{O}_B \to 0$, which is also explained in [CvS06].

5.1 Cohomology computations

We will now apply the results of the previous section to standard stable Horikawa surfaces.

Proposition 5.5. *Let $W = \mathbb{F}_m$ be a Hirzebruch surface and $f\colon X \to W$ be a double cover branched over any divisor B in $|6\sigma_\infty + 2a\Gamma|$ with $a > 2m+2$. Then the natural map $\mathsf{Def}_f \to \mathsf{Def}_X$ is an isomorphism.*

Proof. We need to check the conditions of Lemma 5.2. First note that $H^2(W, \mathcal{T}_W) = 0$ by, for example, [Ser06, Appendix B] or the computations done below. For the other two conditions, we follow the proof of [Hor76, Lemma 2.3].

First note that since $f_* f^* \mathcal{T}_W = \mathcal{T}_W \otimes f_* \mathcal{O}_X = \mathcal{T}_W \oplus \mathcal{T}_W(-3\sigma_\infty - a\Gamma)$, it is enough to show that $H^i(\mathcal{T}_W(-3\sigma_\infty - a\Gamma)) = 0$ for $i = 0, 1$.

In the relative tangent sequence for the fibration $\pi\colon \mathbb{F}_m \to \mathbb{P}^1$,

$$0 \to \mathcal{T}_{\mathbb{F}_m/\mathbb{P}^1} \to \mathcal{T}_{\mathbb{F}_m} \to \pi^* \mathcal{T}_{\mathbb{P}^1} \to 0\,,$$

we can identify $\pi^*\mathcal{T}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{F}_m}(2\Gamma)$ and $\mathcal{T}_{\mathbb{F}_m/\mathbb{P}^1} = \mathcal{O}_{\mathbb{F}_m}(2\sigma_\infty + m\Gamma)$. Twisting with $(-3\sigma_\infty - a\Gamma)$, we immediately get

$$\begin{aligned} h^0(-\sigma_\infty + (m-a)\Gamma) &= 0\,, \\ h^1(-\sigma_\infty + (m-a)\Gamma) &= 0\,, \\ h^0(-3\sigma_\infty + (2-a)\Gamma) &= 0\,. \end{aligned}$$

Furthermore,

$$\begin{aligned} h^1(-3\sigma_\infty + (2-a)\Gamma) &= h^1(K_W - \sigma_\infty + (4+m-a)\Gamma) \\ &= h^1(\sigma_\infty + (a-4-m)\Gamma) \\ &= h^1(\mathcal{O}_{\mathbb{P}^1}(a-4-m)) + h^1(\mathcal{O}_{\mathbb{P}^1}(a-4-2m)) \\ &= 0 \end{aligned}$$

because by assumption $a > 2m+2$, that is, $a-4-2m > -2$. The required vanishing follows from the long exact cohomology sequence. □

In the situation of standard Horikawa surfaces, we want to compute the map (5.2) explicitly, thereby taking control over sequence **1** in the cotangent braid (5.1).

Fix $m \geqslant 0$, and consider $B \in |6\sigma_\infty + 2a\Gamma|$ with $a > 2m+2$ on $W = \mathbb{F}_m$.

Lemma 5.6. *If $B = k\sigma_\infty + B'$ with $k \in \{0,1,2\}$, then the inclusions $k\sigma_\infty \hookrightarrow B \hookrightarrow W$ induce isomorphisms*

$$H^1(W, \mathcal{O}_W(B)) \cong H^1(B, \mathcal{O}_B(B)) \cong H^1(k\sigma_\infty, \mathcal{O}_{k\sigma_\infty}(B))\,,$$

and the dimension of this group is

$$h^1(W, \mathcal{O}_W(6\sigma_\infty + 2a\Gamma)) = \begin{cases} 0\,, & 2a \geqslant 6m-1\,, \\ 6m-2a-1\,, & 6m-1 > 2a \geqslant 5m-1\,, \\ 11m-4a-2\,, & 5m-1 > 2a > 4m+4\,. \end{cases}$$

Under the given conditions, the group vanishes if $m \leqslant 4$.

Proof. The first isomorphism follows from the restriction sequence and $H^1(\mathcal{O}_W) = H^2(\mathcal{O}_W) = 0$. Using $2a > 4m+4$, the dimensions are computed as

$$\begin{aligned} h^1(W, \mathcal{O}_W(6\sigma_\infty + 2a\Gamma)) &= \sum_{i=0}^{6} h^1\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2a-im)\big) \\ &= h^1\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2a-5m)\big) + h^1\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2a-6m)\big) \\ &= \begin{cases} 0\,, & 2a \geqslant 6m-1\,, \\ 6m-2a-1\,, & 6m-1 > 2a \geqslant 5m-1\,, \\ 11m-4a-2\,, & 5m-1 > 2a > 4m+4\,. \end{cases} \end{aligned}$$

Now assume $B = k\sigma_\infty + B'$ with $k = 1$ or $k = 2$. Then there is an exact sequence

$$0 \to \mathcal{O}_{B'}(B') \to \mathcal{O}_B(B) \to \mathcal{O}_{k\sigma_\infty}(B) \to 0\,.$$

The corresponding long exact cohomology sequence gives an isomorphism between $H^1(B, \mathcal{O}_B(B))$ and $H^1(k\sigma_\infty, \mathcal{O}_{k\sigma_\infty}(B))$.

Note that Lemma 2.2 ensures that B contains σ_∞ or even $2\sigma_\infty$ when the cohomology group $H^1(W, \mathcal{O}_W(B))$ is non-zero. □

To compute the map (5.2) explicitly, we want explicit Čech cohomology descriptions of the relevant groups. We will assume $m > 0$ since otherwise the cohomology groups we are interested in vanish anyway. We closely follow [Ser06, Appendix B]. Let us set up our notation starting

from the toric model of the Hirzebruch surface given by

$$\begin{pmatrix} t_0 & t_1 & x_0 & x_1 \\ 1 & 1 & 0 & -m \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

with projection $\pi\colon \mathbb{F}_m \to \mathbb{P}^1$. Let

$$\begin{aligned} \tau &= \frac{t_0}{t_1}\,, & \xi &= \frac{x_1 t_1^m}{x_0}\,, \\ \tau' &= \tau^{-1}\,, & \xi' &= \frac{x_1 t_0^m}{x_0} = \xi\tau^m \end{aligned} \tag{5.5}$$

and $R = \mathsf{Spec}[\tau, \tau^{-1}]$, considered as a $\mathbb{Z}$-graded ring. Then with $U = \{t_1 \neq 0\}$ and $U' = \{t_0 \neq 0\}$, we have $\mathbb{P}^1 = U \cup U'$. Consider

$$\begin{aligned} V = \pi^{-1}(U) \supset V_0 &:= \mathsf{Spec}\,\mathbb{C}[\tau, \xi]\,, \\ V' = \pi^{-1}(U') \supset V_0' &:= \mathsf{Spec}\,\mathbb{C}[\tau', \xi']\,. \end{aligned}$$

We will compute Čech cohomology with respect to the covering $W = V \cup V'$ but represent sections by their restrictions to the affine subsets V_0, respectively V_0'. On V_0 the curve B is defined by an equation

$$\sigma_B^0 = g_0 + g_1\xi + g_2\xi^2 + \cdots + g_6\xi^6$$

for certain $g_i \in \mathbb{C}[\tau]$. We may assume for simplicity that we see all zeros of the coefficients of σ_B in $V \cap V'$; that is, $\deg g_i = 2a - 6(m-i)$ and g_i has a non-zero constant term unless it vanishes identically. Note that $\deg g_0 = 2a - 6m = B \cdot \sigma_\infty$.

Lemma 5.7. *For any $B \in |6\sigma_\infty + 2a\Gamma|$ with $2a > 4m + 4$, with respect to the above covering, we have*

$$\check{H}^1(W, \mathcal{O}_W(B)) = \frac{1}{\sigma_B^0}\left(1\frac{R}{R_{\geqslant 0} + R_{\leqslant 2a-6m}} + \xi\frac{R}{R_{\geqslant 0} + R_{\leqslant 2a-5m}}\right).$$

Proof. To compute Čech cohomology, we need to describe the section on V and V' and then compare the descriptions over $V \cap V'$. For this note that, by the relation between the coordinates, we can describe the equation for B on V_0' as

$$\sigma_B^{0\,\prime} = \sigma_B^0 \cdot \tau^{6m-2a}\,,$$

so that by (5.5),

$$\Gamma(V', \mathcal{O}_W(B)) = \frac{1}{\sigma_B^{0\,\prime}}\mathbb{C}[\tau']\langle 1, \ldots, \xi'^6\rangle = \frac{\tau^{2a-6m}}{\sigma_B^0} R_{\leqslant 0}\langle 1, \tau^m\xi, \ldots, \tau^m\xi^6\rangle\,.$$

Then using $2a > 4m + 4$, we compute

$$\begin{aligned} \check{H}^1(W, \mathcal{O}_W(B)) &= \Gamma(V \cap V', \mathcal{O}_W(B))/\left(\Gamma(V, \mathcal{O}_W(B)) + \Gamma(V', \mathcal{O}_W(B))\right) \\ &= \frac{1}{\sigma_B^0}\left(1 \cdot R/\left(R_{\geqslant 0} + R_{\leqslant 2a-6m}\right) + \xi \cdot R/\left(R_{\geqslant 0} + R_{\leqslant 2a-5m}\right)\right) \end{aligned}$$

as claimed. □

Lemma 5.8. *This covering computes $H^1(W, \mathcal{T}_W)$, and explicitly we have*

$$\check{H}^1(W, \mathcal{T}_W) = \frac{R \cdot \frac{\partial}{\partial \xi}}{R_{\geqslant 0} \cdot \frac{\partial}{\partial \xi} + R_{\leqslant 0} \cdot \tau^{-m} \frac{\partial}{\partial \xi}} \cong \left\langle \tau^{-1} \frac{\partial}{\partial \xi}, \dots, \tau^{-(m-1)} \frac{\partial}{\partial \xi} \right\rangle,$$

a vector space of dimension $m-1$.

Proof. The sheaf $\mathcal{T}_W$ does not have higher cohomology on V and V', so we can use this covering to compute cohomology. Again representing everything in the local coordinates given on V_0 and V_0', we get

$$\begin{aligned}
\Gamma(V, \mathcal{T}_W) &= \mathbb{C}[\tau] \left\langle \frac{\partial}{\partial \tau}, \frac{\partial}{\partial \xi}, \xi \frac{\partial}{\partial \xi}, \xi^2 \frac{\partial}{\partial \xi} \right\rangle, \\
\Gamma(V', \mathcal{T}_W) &= \mathbb{C}[\tau'] \left\langle \frac{\partial}{\partial \tau'}, \frac{\partial}{\partial \xi'}, \xi \frac{\partial}{\partial \xi'}, \xi'^2 \frac{\partial}{\partial \xi'} \right\rangle \\
&= \mathbb{C}[\tau^{-1}] \left\langle -\tau^2 \frac{\partial}{\partial \tau} + m\tau\xi \frac{\partial}{\partial \xi}, \tau^{-m} \frac{\partial}{\partial \xi}, \xi \frac{\partial}{\partial \xi}, \tau^m \xi^2 \frac{\partial}{\partial \xi} \right\rangle \\
&\subset \Gamma(V \cap V', \mathcal{T}_W),
\end{aligned}$$

where we get the formulas for the coordinate change by differentiating (5.5). It is straightforward to see that we get the basis of the quotient stated above.[1] □

Proposition 5.9. *If $B = \sigma_\infty + B'$ and $\sigma_\infty \not\leqslant B'$, then the map (5.2) is surjective, and it is an isomorphism if and only if $2a = 5m$.*

Proof. We have $2a \geqslant 5m$ by Lemma 2.2. We write the equation

$$\sigma_B^0 = \xi g_1 + (\text{higher-order terms in } \xi)$$

for some polynomial g_1 of degree $2a - 5m \geqslant 0$. Then we compute the map (5.2) using Lemma 5.6:

$$\check{H}^1(W, \mathcal{T}_W) \to \check{H}^1(\sigma_\infty, \mathcal{O}_{\sigma_\infty}(B)), \quad \tau^{-i} \frac{\partial}{\partial \xi} \mapsto g_1 \cdot \tau^{-i} \frac{1}{\sigma_B^0}.$$

Since we have chosen coordinates such that g_1 has a constant term, the claim follows from Lemmas 5.7, 5.6 and 5.8. □

Proposition 5.10. *If $B = 2\sigma_\infty + B'$ and $\sigma_\infty \not\leqslant B'$, then the map α from (5.2)*

(i) *is surjective if and only if $H^1(B, \mathcal{O}_B(B)) = 0$ if and only if $2a > 6m$,*
(ii) *is zero if and only if $2a \geqslant 5m$,*
(iii) *has rank $5m - 2a - 1 < m - 4$ for $4m + 4 < 2a < 5m$.*

Note that by Remark 2.3 the last case can only occur for $m \geqslant 7$, so that the inequalities make sense.

Proof. We have $2a > 4m + 4$ by Lemma 2.2. We write the equation

$$\sigma_B^0 = \xi^2 g_2 + (\text{higher-order terms in } \xi)$$

for some polynomial g_1 of degree $2a - 4m > 4$. By Lemma 5.6 we can compute the composition

$$\check{H}^1(W, \mathcal{T}_W) \xrightarrow{\alpha} H^1(B, \mathcal{O}_B(B)) \to H^1(2\sigma_\infty, \mathcal{O}_{2\sigma_\infty}(B)),$$

and the last group is simply the restriction of the one computed in Lemma 5.7 to $2\sigma_\infty$.

[1] In [Ser06, Appendix B] the result of this computation is stated with a missing inverse.

For the explicit representatives computed in Lemma 5.8, we get

$$\alpha\left(\tau^{-i}\frac{\partial}{\partial\xi}\right) = \tau^{-i}\cdot 2\xi\cdot g_2\frac{1}{\sigma_B^0}\Big|_{2\sigma_\infty}$$

Since we have chosen coordinates such that g_2 has a constant term, the map surjects on the subspace

$$\check{H}^1(W,\mathcal{T}_W) \twoheadrightarrow \xi\frac{R}{R_{\geqslant 0}+R_{\leqslant 2a-5m}} \hookrightarrow \check{H}^1(2\sigma_\infty,\mathcal{O}_{2\sigma_\infty}(B))\,.$$

Thus the map α is surjective if and only if $H^1(B,\mathcal{O}_B(B))=0$, which holds if and only if $2a>6m$. The other two items follow by counting the dimension of the spaces involved. □

5.2 Deformation-theoretic interpretations

We now interpret the computations of the previous section in our context. First recall that for a Hirzebruch surface $W=\mathbb{F}_m$ with $m>0$, we have

$$h^0(\mathcal{T}_W)=m+5\,,\quad h^1(\mathcal{T}_W)=m-1\,,\quad h^2(\mathcal{T}_W)=0\,,$$

deformations are unobstructed, and W deforms to $\mathbb{F}_{m-2k}$ for $k=1,\dots,\lfloor m/2\rfloor$. This is explained, for example, in [Cat13, Section 1.2, pp. 169–170], and a more precise description of the stratification of the universal deformation space has been given by Suwa [Suw73].

Now for a standard stable Horikawa surface, consider sequence **1** in the cotangent braid (5.1) and its more worldly incarnation derived from Propositions 5.5 and 5.3:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & T^0_W & \longrightarrow & T^1_{f/W} & \longrightarrow & T^1_f & \longrightarrow & T^1_W & \xrightarrow{\alpha} & T^2_{f/W} & \longrightarrow & T^2_f & \longrightarrow & T^2_W \\
 & & \| & & \| & & \| & & \| & & \| & & \| & & \| \\
0 & \longrightarrow & H^0(\mathcal{T}_W) & \longrightarrow & H^0(\mathcal{O}_B(B)) & \longrightarrow & T^1_X & \longrightarrow & H^1(\mathcal{T}_W) & \longrightarrow & H^1(\mathcal{O}_B(B)) & \longrightarrow & T^2_X & \longrightarrow & 0\,.
\end{array} \tag{5.6}$$

The first part of the following result re-proves some of the results in [Hor76]. Part of it can also be deduced directly from the concrete families constructed in Sections 3 and 4.

Proposition 5.11. *Let $f\colon X\to W$ be a standard stable Horikawa surface of type (m) with branch curve $B\in|6\sigma_\infty+2a\Gamma|$.*

(i) *If $2a\geqslant 6m$, then deformations of X are unobstructed and every deformation of W can be lifted to a deformation of X. In particular, X deforms to a Horikawa surface of type (0).*

(ii) *If $2\sigma_\infty\not<B$, that is, $2a\geqslant 5m$, then deformations of X are unobstructed. If $2a<5m$, then X deforms to a Horikawa surface of lower type.*

(iii) *Assume $2a=5m$ and write $B=k\sigma_\infty+B'$.*

  (a) *If $k=1$, then deformations of X are unobstructed and $\mathsf{Def}_X\cong\mathsf{Def}_{f/\mathbb{F}_m}$; that is, every small deformation is of type (m) again.*

  (b) *If $k=2$, then $\dim T^1_X=12a-15m-1$.*

(iv) *If $5m>2a>4m+4$ and $B=2\sigma_\infty+B'$, then the map $T^1_X\to T^1_W$ is non-trivial of rank $K_X^2/2-m+3=2a-4m-1\geqslant 4$.*

Proof. All items follow immediately from (5.6) in conjunction with Propositions 5.9 and 5.10, where we use that a deformation functor is unobstructed if $T^2=0$ and our knowledge of deformations of Hirzebruch surfaces. □

We can now complete the proof of Theorem A, giving information on the local structure $\bar{\mathfrak{H}}_{8k}$ at a surface as described in Example 3.2.

Corollary 5.12. *Suppose $K_X^2 = 8\ell$ with $\ell > 2$, and consider a surface X representing a general point in $\mathfrak{D} \subset \bar{\mathfrak{H}}^{\mathrm{I}} \cap \bar{\mathfrak{H}}^{\mathrm{II}}$ in the intersection of the special and general component. Then the point $[X]$ representing X in $\bar{\mathfrak{H}}_{k\ell}$ lies in exactly these two irreducible components, but $\bar{\mathfrak{H}}_{8\ell}$ is not normal crossing near $[X]$.*

Proof. We know that every deformation of X is again a standard stable Horikawa surface (Proposition 5.5) and that the type can only decrease, so X can only deform to one of these two components. If the moduli space were normal crossing at $[X]$, then we would have $\dim T_{[X]}\bar{\mathfrak{H}}_{k\ell} = \dim \mathfrak{H}^{\mathrm{I}}_{k\ell} + 1$, but by Proposition 5.11 and Theorem 2.5, we compute

$$\dim T_{[X]}\bar{\mathfrak{H}}_{k\ell} - \dim \mathfrak{H}^{\mathrm{I}}_{k\ell} = \dim T^1_X - \dim \mathfrak{H}^{\mathrm{I}}_{k\ell} + 1 + (m-2) > 1\,.$$

This proves the claim. □

5.3 Proof of Theorem B

We are now ready to prove Theorem B from the introduction.

By induction and Propositions 4.1 and 4.3, we know that all subsets $\bar{\mathfrak{H}}^{(m)}_{2k}$ are contained in the connected component of the moduli space containing $\mathfrak{H}_{2k}$.

So let us now consider a general surface X parametrised by $\bar{\mathfrak{H}}^{(m)}_{2k}$ for $2k > 2m > k+4$. In particular, we may assume that the reduction of $\bar{\mathfrak{H}}^{(m)}_{2k}$ is smooth at $[X]$.

Then by Lemma 2.2 the surface X is not classical but is a non-normal stable surface, and by Proposition 5.5, every deformation of X lifts to a deformation of the map $f\colon X \to \mathbb{F}_m$. By the deformation theory of Hirzebruch surfaces, the type cannot increase in a neighbourhood of X in $\bar{\mathfrak{H}}_{2k}$, so all deformations of X are contained in the union $\bigcup_{m' \leqslant m} \bar{\mathfrak{H}}^{(m')}_{2k}$. On the other hand, the union of all strata of smaller type has smaller dimension than $\bar{\mathfrak{H}}^{(m)}_{2k}$ by Corollary 2.6, so that the general X cannot lie in the closure of these components. In other words, all non-infinitesimal deformations of X are again of type (m), and $\bar{\mathfrak{H}}^{(m)}_{2k}$ forms (an open subset of) an irreducible component of the moduli space.

Endowing $\bar{\mathfrak{H}}^{(m)}_{2k}$ with the scheme structure defined by the moduli space of stable surfaces and noting that X is Gorenstein, so every deformation is admissible, we show that the component is generically non-reduced by computing the dimension of the tangent space at the general point X:

$$\dim T_{[X]}\bar{\mathfrak{H}}^{(m)}_{2k} = \dim T^1_X = \dim \bar{\mathfrak{H}}^{(m)}_{2k} + k - m + 3 > \dim \bar{\mathfrak{H}}^{(m)}_{2k} = \dim T_{[X]}\big(\bar{\mathfrak{H}}^{(m)}_{2k}\big)_{\mathrm{red}}$$

by Proposition 5.11(iv). This concludes the proof.

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