



Class field theory, Hasse principles and Picard–Brauer duality for two-dimensional local rings

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ABSTRACT

We draw concrete consequences from our arithmetic duality for two-dimensional local rings with perfect residue field. These consequences include class field theory, Hasse principles for coverings and K_2 and a duality between divisor class groups and Brauer groups. To obtain these, we analyze the ind-pro-algebraic group structures on arithmetic cohomology obtained earlier and prove some finiteness properties about them.

1. Introduction

1.1 Aim of the paper

Let A be a two-dimensional normal complete noetherian local ring of mixed characteristic $(0, p)$ with perfect residue field F . In [Sai86, Sai87], Saito gives arithmetic duality and class field theory for A when F is finite. We refined this theory in our recent work [Suz24] when F is a general perfect field, by giving ind-pro-algebraic group structures for arithmetic cohomology attached to A and taking these structures into account. However, our duality was on the derived category level, and we could not deduce concrete consequences due to a lack of precise understanding of the structure of each cohomology object.

In this paper, we will prove some finiteness (or admissibility) properties of our ind-pro-algebraic group structures. This allows us to deduce concrete duality statements for each cohomology object. The statements we deduce include class field theory, Hasse principles for coverings and K_2 and a duality between divisor class groups and Brauer groups. The key inputs are the relatively perfect unipotent group structures on p -adic nearby cycle sheaves [BS24], the finite generation of the pro- p fundamental group of the punctured spectrum [Suz23] and Lipman’s group scheme structure for the divisor class group [Lip69, Lip76].

Along the way, we also prove a duality for relatively perfect unipotent groups (especially, wound ones) over fields with finite p -bases, which may be of independent interest. When F is finite, we explain how to recover Saito’s duality and its local counterpart, namely Kato’s two-dimensional local class field theory [Kat79]. Our duality actually gives locally compact (and, in particular, Hausdorff!) topologies for the groups appearing in these theories.

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1.2 Main results

We formulate our results. Let F be a perfect field of characteristic $p > 0$. Let Alg/F be the category of perfections (inverse limits along the Frobenius morphism) of commutative algebraic groups over F with group scheme morphisms [Ser60]. Let $\text{Alg}_u/F \subset \text{Alg}/F$ be the full subcategory of perfections of unipotent groups. Let PAlg_u/F and IAlg_u/F be its pro-category and ind-category, respectively. Let IPAlg_u/F be the ind-category of PAlg_u/F . For $G \in \text{Alg}/F$, denote its identity component by G^0 and set $\pi_0 G = G/G^0$. The functors $G \mapsto G^0, \pi_0 G$ naturally extend to IPAlg_u/F . An object $G \in \text{IPAlg}_u/F$ is said to be connected if $\pi_0 G = 0$. As in [Suz24, Definition 3.1.1], we define the following class of objects of IPAlg_u/F with nice finiteness (or admissibility) properties.

DEFINITION 1.1. Define $\mathcal{W}_F \subset \text{IPAlg}_u/F$ to be the full subcategory of objects G admitting a filtration $G \supset G^0 \supset G' \supset 0$ such that

- $G/G^0 = \pi_0 G$ is finite étale (p -primary);
- G^0/G' can be written as a direct limit $\varinjlim_{n \geq 1} G''_n$, where each $G''_n \in \text{Alg}_u/F$ is connected and each transition morphism $G''_n \rightarrow G''_{n+1}$ is injective;
- G' can be written as an inverse limit $\varprojlim_{n \geq 1} G'_n$, where each $G'_n \in \text{Alg}_u/F$ is connected and each transition morphism $G'_{n+1} \rightarrow G'_n$ is surjective with connected kernel.

What is nice about $\mathcal{W}_F$ is that the full subcategory of $\mathcal{W}_F$ of connected groups has a canonical contravariant autoequivalence [Suz24, Proposition 3.1.7] that maps the perfection of the additive group to itself. We call it Serre duality for connected groups in $\mathcal{W}_F$.

Let A be a two-dimensional normal complete noetherian local ring of mixed characteristic with maximal ideal $\mathfrak{m}$ whose residue field is the above F . Let K be its fraction field. Let P be the set of height one prime ideals of A . Set $X = \text{Spec } A \setminus \{\mathfrak{m}\}$. Let U be a dense open subscheme of X . Let $\mathbb{Z}/p^n\mathbb{Z}(r) \in D(U_{\text{et}})$ be the Bloch cycle complex mod p^n in the étale topology if $r \geq 0$, and let $\mathbb{Z}/p^n\mathbb{Z}(r) \in D(U_{\text{et}})$ be the extension-by-zero of the usual Tate twist $\mathbb{Z}/p^n\mathbb{Z}(r)$ on $U \cap \text{Spec } A[1/p]$ if $r < 0$. Denote the étale cohomology functor by $H^q(U, \cdot)$, and let $H_c^q(U, \cdot) = H^q(X, j_! \cdot)$, where $j: U \hookrightarrow X$ is the inclusion. By [Suz24, Theorem 1.3.1], we have canonical objects $\mathbf{H}^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r))$ and $\mathbf{H}_c^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r))$ of IPAlg_u/F for all integers $n \geq 1, q$ and r , which are algebraic structures on $H^q(U, \mathbb{Z}/p^n\mathbb{Z}(r))$ and $H_c^q(U, \mathbb{Z}/p^n\mathbb{Z}(r))$. Note that when $U = X$, there is no difference between $\mathbf{H}^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r))$ and $\mathbf{H}_c^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r))$. The structure result for these groups is as follows.

THEOREM 1.2. We have $\mathbf{H}^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r)), \mathbf{H}_c^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r)) \in \mathcal{W}_F$. They are zero unless $0 \leq q \leq 3$. Their structures are described as follows:

(i) Assume $U = X$. Then we have the following table:

$\mathbf{H}^q(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z}(r))$	$r = 0$	$r = 1$	$r = 2$
$q = 0$	$\mathbb{Z}/p^n\mathbb{Z}$	finite	finite
$q = 1$	finite	pro-alg	pro-alg
$q = 2$	ind-alg	alg	pro-alg
$q = 3$	ind-alg	ind-alg	$\mathbb{Z}/p^n\mathbb{Z}$

Here finite means finite étale, alg means in Alg_u/F , pro-alg means in PAlg_u/F and ind-alg means in IAlg_u/F .

(ii) Assume $U \subset \operatorname{Spec} A[1/p]$. Then we have the following table:

$\mathbf{H}^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r))$	$\forall r$	$\mathbf{H}_c^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r))$	$\forall r$
$q = 0$	<i>finite</i>	$q = 0$	0
$q = 1$	<i>pro-alg</i>	$q = 1$	<i>finite</i>
$q = 2$	<i>pro-alg</i>	$q = 2$	<i>ind-alg</i>
$q = 3$	0	$q = 3$	<i>ind-alg</i>

(iii) In general, we have the following tables:

$\mathbf{H}^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r))$	$r \leq 0$	$r = 1$	$r \geq 2$
$q = 0$	<i>finite</i>	<i>finite</i>	<i>finite</i>
$q = 1$	<i>pro-alg</i>	<i>pro-alg</i>	<i>pro-alg</i>
$q = 2$	<i>general</i>	<i>pro-alg</i>	<i>pro-alg</i>
$q = 3$	<i>ind-alg</i>	<i>ind-alg</i>	<i>finite</i>

$\mathbf{H}_c^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r))$	$r \leq 0$	$r = 1$	$r \geq 2$
$q = 0$	<i>finite</i>	<i>finite</i>	<i>finite</i>
$q = 1$	<i>finite</i>	<i>pro-alg</i>	<i>pro-alg</i>
$q = 2$	<i>ind-alg</i>	<i>ind-alg</i>	<i>general</i>
$q = 3$	<i>ind-alg</i>	<i>ind-alg</i>	<i>ind-alg</i>

Here *general* means a general object of $\mathcal{W}_F$.

Note that the apparently missing group $\mathbf{H}^q(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z}(r))$ for $r < 0$ (respectively, $r > 2$) is isomorphic to $\mathbf{H}_c^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r))$ (respectively, $\mathbf{H}^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r))$) with $U = \operatorname{Spec} A[1/p]$, so it is also covered in the tables.

In particular, the group $\mathbf{H}^1(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z})$ is finite étale. We will first prove that this group is in Alg_u/F . That it is finite étale then follows from [Suz23]. This implies the above structure result $\mathbf{H}^3(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z}(2)) \cong \mathbb{Z}/p^n\mathbb{Z}$ by duality (see Theorem 1.4 below). The group $H^3(X, \mathbb{Z}/p^n\mathbb{Z}(2))$ is the group “ $SK_1(X)$ ” (see [Blo81, Sai86]) mod p^n when F is algebraically closed. Hence we obtain the following.

COROLLARY 1.3. Assume that F is algebraically closed. Then the sequence

$$K_2(K)/p^n K_2(K) \rightarrow \bigoplus_{\mathfrak{p} \in P} \kappa(\mathfrak{p})^\times / \kappa(\mathfrak{p})^{\times p^n} \rightarrow \mathbb{Z}/p^n\mathbb{Z} \rightarrow 0$$

is exact, where the first map consists of the tame symbols to the residue fields at $\mathfrak{p}$ and the second map is the sum of the normalized valuation maps.

By Levine [Lev88, Theorem 2.1], the group $SK_1(X)$ is isomorphic to the Grothendieck group of the category of A -modules of finite length and finite projective dimension. Corollary 1.3 says that the kernel of the “length” map $SK_1(X) \rightarrow \mathbb{Z}$ studied by Srinivas [Sri87], [Sri96, Chapter 9] is p -divisible. It is relatively easy to prove that $\operatorname{Ker}(SK_1(X) \rightarrow \mathbb{Z})$ is l -divisible for any prime $l \neq p$.¹ Therefore, this kernel is divisible. For an equal-characteristic A (normal and two-dimensional), this divisibility is proved in Srinivas’s unpublished notes [Sri85] by a global method.

The part $\mathbf{H}^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r)), \mathbf{H}_c^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r)) \in \mathcal{W}_F$ of Theorem 1.2 allows us to deduce concrete duality statements from the abstract duality statement of [Suz24]. The result is as follows.

¹Use the fact that $H^3(X, \mathbb{Z}/l\mathbb{Z}(2)) \cong H_m^4(A, \mathbb{Z}/l\mathbb{Z}(2)) \cong \mathbb{Z}/l\mathbb{Z}$, see [ILO14, Exposé XVII, Corollaire 3.4.1.6], in place of our result $H^3(X, \mathbb{Z}/p\mathbb{Z}(2)) \cong \mathbb{Z}/p\mathbb{Z}$ above.

THEOREM 1.4. We have a Serre duality

$$\mathbf{H}^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r))^0 \leftrightarrow \mathbf{H}_c^{4-q}(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(2-r))^0$$

of connected groups in $\mathcal{W}_F$ and a Pontryagin duality

$$\pi_0 \mathbf{H}^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r)) \leftrightarrow \pi_0 \mathbf{H}_c^{3-q}(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(2-r))$$

of finite étale groups over F .

Theorems 1.2 and 1.4 will be proved together in Section 8.

As a particular example, we have a Pontryagin duality

$$\mathbf{H}^1(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z}) \leftrightarrow \pi_0 \mathbf{H}^2(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z}(2)),$$

which is unramified class field theory in this context. The Serre duality

$$\mathbf{H}^1(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z})^0 \leftrightarrow \mathbf{H}_c^3(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(2))^0$$

should look more similar to Saito's class field theory [Sai87] since $H_c^3(U, \mathbb{Z}/p^n\mathbb{Z}(2))$ is the K_2 -idèle class group of U mod p^n when F is algebraically closed.

We now state Hasse principles in this context. Assume for the moment that F is algebraically closed. Let $H_{\text{cs}}^1(X, \mathbb{Z}/p^n\mathbb{Z}) \subset H^1(X, \mathbb{Z}/p^n\mathbb{Z})$ be the subgroup consisting of coverings of X completely split at all closed points of X . Let

$$\pi_0 \mathbf{H}^2(\mathbf{K}, \mathbb{Z}/p^n\mathbb{Z}(2)) := \varinjlim_U \pi_0 \mathbf{H}^2(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(2)),$$

where the direct limit is over all dense open subschemes $U \subset X$. For each $\mathfrak{p} \in P$, let $\hat{K}_{\mathfrak{p}}$ be the fraction field of the completion of the localization of A at $\mathfrak{p}$. We have a natural map

$$\pi_0 \mathbf{H}^2(\mathbf{K}, \mathbb{Z}/p^n\mathbb{Z}(2)) \rightarrow \bigoplus_{\mathfrak{p} \in P} \pi_0 \mathbf{H}^2(\hat{\mathbf{K}}_{\mathfrak{p}}, \mathbb{Z}/p^n\mathbb{Z}(2)) \cong \bigoplus_{\mathfrak{p} \in P} \mathbb{Z}/p^n\mathbb{Z},$$

where $\mathbf{H}^2(\hat{\mathbf{K}}_{\mathfrak{p}}, \mathbb{Z}/p^n\mathbb{Z}(2))$ is the algebraic structure on the Galois cohomology $H^2(\hat{K}_{\mathfrak{p}}, \mathbb{Z}/p^n\mathbb{Z}(2))$; see [Suz24, Theorem 6.4.2(1)].

THEOREM 1.5. Assume that F is algebraically closed.

(i) The sequence

$$\pi_0 \mathbf{H}^2(\mathbf{K}, \mathbb{Z}/p^n\mathbb{Z}(2)) \rightarrow \bigoplus_{\mathfrak{p} \in P} \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\text{sum}} \mathbb{Z}/p^n\mathbb{Z} \rightarrow 0$$

is exact, and the kernel of the first map is Pontryagin dual to $H_{\text{cs}}^1(X, \mathbb{Z}/p^n\mathbb{Z})$.

(ii) Let $\mathfrak{X} \rightarrow \text{Spec } A$ be a resolution of singularities such that $\mathfrak{X} \times_A A/pA \subset \mathfrak{X}$ is supported on a strict normal crossing divisor [Sta22, Tag 0BIC]. Let Y be the reduced part of $\mathfrak{X} \times_A F$. Then $H_{\text{cs}}^1(X, \mathbb{Z}/p^n\mathbb{Z}) \cong H^1(Y, \mathbb{Z}/p^n\mathbb{Z})$.

(iii) Let $Y_1, \dots, Y_m$ be the irreducible components of Y . Let Γ be the dual graph of Y . Then we have an exact sequence

$$0 \rightarrow H^1(\Gamma, \mathbb{Z}/p^n\mathbb{Z}) \rightarrow H^1(Y, \mathbb{Z}/p^n\mathbb{Z}) \rightarrow \bigoplus_i H^1(Y_i, \mathbb{Z}/p^n\mathbb{Z}) \rightarrow 0.$$

Saito gives this type of Hasse principle for finite F in [Sai87]. We will actually prove Theorem 1.5 for not-necessarily algebraically closed F , where the Galois actions will be taken into

account. The finite étale property of $\mathbf{H}^1(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z})$ is crucial in deducing Theorem 1.5 from Theorem 1.4.

We return to not-necessarily algebraically closed F to state a duality between the divisor class group and the Brauer group, which is a refinement of Saito's duality [Sai86]. Let $\mathbf{Pic}_X \in \mathbf{Alg}/F$ be the perfection of Lipman's group scheme structure on $\mathrm{Pic}(X)$ with identity component $\mathbf{Pic}_X^0$. Let $\mathbf{Pic}_{X,\mathrm{sAb}} \subset \mathbf{Pic}_X$ be the maximal semi-abelian part, and set $\mathbf{Pic}_X^0/\mathrm{sAb} = \mathbf{Pic}_X^0/\mathbf{Pic}_{X,\mathrm{sAb}}$. Define

$$\mathbf{Br}_X[p^\infty] := \varinjlim_{n \geq 1} \mathbf{H}^2(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z}(1)).$$

Let $\overline{A}$ be the completed unramified extension of A corresponding to $\overline{F}$, and set $\overline{X} = X \times_A \overline{A}$.

THEOREM 1.6. (i) *The group of $\overline{F}$ -points of $\mathbf{Br}_X[p^\infty]$ is $\mathrm{Br}(\overline{X})[p^\infty]$.*

(ii) *We have $\mathbf{Br}_X[p^\infty]^0 \in \mathbf{Alg}_u/F$, and the object $\pi_0(\mathbf{Br}_X[p^\infty])$ is of cofinite type.*

(iii) *We have a Serre duality*

$$\mathbf{Pic}_X^0/\mathrm{sAb} \leftrightarrow \mathbf{Br}_X[p^\infty]^0$$

of connected groups in $\mathbf{Alg}_u/F$ and a Pontryagin duality

$$T_p \mathbf{Pic}_{X,\mathrm{sAb}}^0 \leftrightarrow \pi_0(\mathbf{Br}_X[p^\infty]),$$

where T_p denotes the p -adic Tate module.

In particular, the dimension of $\mathbf{Br}_X[p^\infty]$ is equal to the dimension of the unipotent part of $\mathbf{Pic}_X$.

We give consequences for finite F . In this case, by [Suz20b, Proposition 10.3(b)], the group of F -valued points of any object of (the bounded derived category of) $\mathbf{IPAlg}_u/F$ can be equipped with a canonical structure as an ind-profinite group. Applying this process to (the derived categorical versions of) $\mathbf{H}^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r))$ and $\mathbf{H}_c^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r))$, we obtain ind-profinite group structures on $H^q(U, \mathbb{Z}/p^n\mathbb{Z}(r))$ and $H_c^q(U, \mathbb{Z}/p^n\mathbb{Z}(r))$.

THEOREM 1.7. *Assume that F is finite.*

(i) *The groups $H^q(U, \mathbb{Z}/p^n\mathbb{Z}(r))$ and $H_c^q(U, \mathbb{Z}/p^n\mathbb{Z}(r))$ are locally compact (and, in particular, Hausdorff).*

(ii) *We have a Pontryagin duality*

$$H^q(U, \mathbb{Z}/p^n\mathbb{Z}(r)) \leftrightarrow H_c^{4-q}(U, \mathbb{Z}/p^n\mathbb{Z}(2-r)).$$

We also prove a variant of this for local fields $\hat{K}_{\mathfrak{p}}$ at height one primes $\mathfrak{p} \in P$. This gives a locally compact group structure on $H^q(\hat{K}_{\mathfrak{p}}, \mathbb{Z}/p^n\mathbb{Z}(r))$ and shows its duality. The Hausdorff property of $H^2(\hat{K}_{\mathfrak{p}}, \mathbb{Z}/p^n\mathbb{Z}(2)) \cong K_2(\hat{K}_{\mathfrak{p}})/p^n K_2(\hat{K}_{\mathfrak{p}})$ is claimed in [Fes02, Section 4.3] (see also [Fes02, Theorem 4.7] and [Fes00, Section 6.6, Theorem 3]). But these references omit the relevant calculations of Vostokov's pairing, and this pairing is even undefined for the case $p = 2$. We do not use the results of [Fes00, Fes02].

1.3 Outline of the proofs

To prove Theorems 1.2 and 1.4, we first need to prove corresponding statements for the local field $\hat{K}_{\mathfrak{p}}$ (and its ring of integers) at each $\mathfrak{p} \in P$. We have nothing to add to [Suz24, Section 7] if $\mathfrak{p}$ does not divide p . For $\mathfrak{p}$ dividing p , what we need is already given in [Suz24, Proposition 6.4.1

and the proof of Proposition 6.5.2] if $n = 1$. The same proof does not work for $n \geq 2$ since the graded pieces of p -adic nearby cycles $R^q\Psi\mathbb{Z}/p^n\mathbb{Z}(r)$ are not completely determined in this case. A simple dévissage on n does not work either since $\mathcal{W}_F$ is not an abelian category. What we know is that $R^q\Psi\mathbb{Z}/p^n\mathbb{Z}(r)$ has the structure of a relatively perfect unipotent group over $\kappa(\mathfrak{p})$ (see [BS24]). For a relatively perfect unipotent group G , in Section 4, we show that $H^s(\kappa(\mathfrak{p}), G)$ for each $s \in \mathbb{Z}$ has the structure of an object of $\mathcal{W}_F$. To prove this statement for $s = 0$, we use Néron models of wound unipotent groups and their Greenberg transforms. The statement for $s = 1$ is reduced to the statement for $s = 0$ by the duality for relatively perfect unipotent groups given in Sections 3 and 4. Based on this result, we prove the desired result for $\hat{K}_{\mathfrak{p}}$ in Section 5.

We then prove that $\mathbf{H}^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r)) \in \mathrm{PAlg}_u/F$ and $\mathbf{H}_c^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r)) \in \mathrm{IAlg}_u/F$ for $U \subset \mathrm{Spec} A[1/p]$ in Section 7. These statements do not immediately imply that these groups are in $\mathcal{W}_F$, but they are the best we can have at this point. To prove these statements, we show the corresponding statements for curves, tubular neighborhoods and two-dimensional regular local rings in Sections 6 and 7. Applying them to a resolution of singularities of A , we get the desired statements for U .

Now the above results for $\hat{K}_{\mathfrak{p}}$ and $U \subset \mathrm{Spec} A[1/p]$ are sufficient, in Section 8, to deduce Theorems 1.2 and 1.4 for general U by somewhat heavy homological algebra in IPAlg_u/F . The lemmas needed for this homological algebra are provided in Section 2.

This being done, it is relatively straightforward to deduce Theorem 1.5 in Section 9. The proof of Theorem 1.6, on the other hand, is a bit more complicated since we need to deal with the non-torsion coefficient sheaf $\mathbf{G}_m$ and direct/inverse limits in n . Sections 10 and 11 are devoted to this.

Finally, in Section 12, we deduce consequences for the case of finite F . In this case, for any connected group G in $\mathcal{W}_F$, the group of rational points $G(F)$ has a canonical structure of a locally compact group. We use the pro-étale site of a point $*_{\mathrm{proet}}$ to show this, just as we did in [Suz20b, Section 10] and [Suz20a, Section 4.2]. This proves Theorem 1.7.

2. Notation and homological algebra of ind-pro-algebraic groups

Throughout this paper, let F be a perfect field of characteristic $p > 0$. We generally follow the notation of [Suz24], especially [Suz24, Section 2]. We first briefly recall some of the notation there. We then give several facts on homological algebra of ind-pro-algebraic groups. They will be used later on to show that the ind-pro-algebraic groups at hand belong to the category $\mathcal{W}_F$.

All sites in this paper are defined by given pretopologies. For a site S , let $\mathrm{Ab}(S)$ be the category of sheaves of abelian groups on S , let $\mathrm{Ch}(S)$ be the category of complexes in $\mathrm{Ab}(S)$ (in the cohomological grading) and $D(S)$ its derived category. Let

$$\Lambda_n = \mathbb{Z}/p^n\mathbb{Z}, \quad \Lambda_\infty = \mathbb{Q}_p/\mathbb{Z}_p, \quad \Lambda = \Lambda_1 = \mathbb{Z}/p\mathbb{Z}.$$

Set $\otimes = \otimes_{\mathbb{Z}}$. Let $\mathbf{Hom}_S$ be the sheaf-Hom functor for $\mathrm{Ab}(S)$ with right derived functors $\mathbf{Ext}_S^q$ and $R\mathbf{Hom}_S$. For objects $G, H, K \in D(S)$, we say that a morphism $G \otimes^L H \rightarrow K$ is a perfect pairing if $G \rightarrow R\mathbf{Hom}_S(H, K)$ and $H \rightarrow R\mathbf{Hom}_S(G, K)$ are both isomorphisms. A premorphism of sites $f: S' \rightarrow S$ is a functor f^{-1} from the underlying category of S to the underlying category of S' that sends covering families to covering families such that $f^{-1}(Y \times_X Z) \xrightarrow{\sim} f^{-1}Y \times_{f^{-1}X} f^{-1}Z$ whenever $Y \rightarrow X$ appears in a covering family. Let $f^*: \mathrm{Ab}(S) \rightarrow \mathrm{Ab}(S')$ be the pullback functor for sheaves of abelian groups and $Lf^*: D(S) \rightarrow D(S')$ its left derived functor. We say that an object $G \in D(S)$ is f -acyclic if $G \xrightarrow{\sim} Rf_*Lf^*G$.

We recall the ind-rational pro-étale site $\mathrm{Spec} F_{\mathrm{proet}}^{\mathrm{indrat}}$ from [Suz20b]. An F -algebra is said to be rational if it is a finite direct product of perfections (direct limits along the Frobenius morphism) of finitely generated fields over F . An ind-rational F -algebra is a direct limit of a filtered direct system consisting of rational F -algebras. Let F^{indrat} be the category of ind-rational F -algebras with F -algebra homomorphisms. Then $\mathrm{Spec} F_{\mathrm{proet}}^{\mathrm{indrat}}$ is (the opposite category of) the category F^{indrat} equipped with the pro-étale topology. We write $\mathrm{Ab}(F_{\mathrm{proet}}^{\mathrm{indrat}}) = \mathrm{Ab}(\mathrm{Spec} F_{\mathrm{proet}}^{\mathrm{indrat}})$ and use similar notation elsewhere, such as $\mathbf{Hom}_{F_{\mathrm{proet}}^{\mathrm{indrat}}}$.

We recall the perfect artinian étale site $\mathrm{Spec} F_{\mathrm{et}}^{\mathrm{perar}}$ from [Suz21]. Let F^{perar} be the category of perfect artinian F -algebras (or, equivalently, finite direct products of perfect field extensions of F) with F -algebra homomorphisms. Then $\mathrm{Spec} F_{\mathrm{et}}^{\mathrm{perar}}$ is the category F^{perar} endowed with the étale topology.

We recall the perfect pro-fppf site $\mathrm{Spec} F_{\mathrm{proppf}}^{\mathrm{perf}}$ from [Suz22]. A perfect F -algebra homomorphism $R \rightarrow S$ is said to be flat of finite presentation (in the perfect algebra sense) if S is the perfection of a flat R -algebra of finite presentation. It is said to be flat of ind-finite presentation if S is the direct limit of a filtered direct system of perfect R -algebras flat of finite presentation. Define $\mathrm{Spec} F_{\mathrm{proppf}}^{\mathrm{perf}}$ to be the category of perfect F -algebras where a covering of an object R is a finite family $\{R_i\}$ of R -algebras flat of ind-finite presentation such that $\prod_i R_i$ is faithfully flat over R .

In any of these sites, if F_0 is a finite extension of F , let $\mathfrak{R}_{F_0/F}$ be the Weil restriction functor for F_0/F (which is the pushforward functor by the morphism of sites defined by the base change functor for $\mathrm{Spec} F_0 \rightarrow \mathrm{Spec} F$).

We recall the functor $\acute{a}: D(F_{\mathrm{et}}^{\mathrm{perar}}) \rightarrow D(F_{\mathrm{proet}}^{\mathrm{indrat}})$ from [Suz21]. Let

$$\mathrm{Spec} F_{\mathrm{proppf}}^{\mathrm{perf}} \xrightarrow{f} \mathrm{Spec} F_{\mathrm{proet}}^{\mathrm{indrat}} \xrightarrow{g} \mathrm{Spec} F_{\mathrm{et}}^{\mathrm{perar}}$$

be the premorphisms defined by the inclusion functors on the underlying categories. Let $h = g \circ f$. Then $\acute{a} = Rf_* Lh^*$.

Recall that the Yoneda functors $\mathcal{W}_F \rightarrow \mathrm{Ab}(F_{\mathrm{et}}^{\mathrm{perar}})$ and $\mathrm{Alg}/F \rightarrow \mathrm{Ab}(F_{\mathrm{et}}^{\mathrm{perar}})$ are fully faithful [Suz21, Proposition 7.1]. As in [Suz24, Definition 3.1.3], define $\langle \mathcal{W}_F \rangle_{F_{\mathrm{et}}^{\mathrm{perar}}}$ to be the smallest full triangulated subcategory of $D(F_{\mathrm{et}}^{\mathrm{perar}})$ closed under direct summands containing objects of $\mathcal{W}_F$ placed in degree zero. We say that an object $G \in \mathrm{IPAlg}_u/F$ is pro-algebraic or ind-algebraic if it is in the subcategory PAlg_u/F or IAlg_u/F , respectively.

PROPOSITION 2.1. *Let $q \in \mathbb{Z}$ and $G \in \langle \mathcal{W}_F \rangle_{F_{\mathrm{et}}^{\mathrm{perar}}}$. Set $H = \acute{a}G$. Assume $H^q G \in \mathcal{W}_F$ and $H^q H \in \mathcal{W}_F$. Then $H^q G \cong H^q H$.*

Proof. By [Suz24, Proposition 3.1.4], we have $R\Gamma(F'_{\mathrm{et}}, G) \cong R\Gamma(F'_{\mathrm{proet}}, H)$ for all $F' \in F^{\mathrm{perar}}$. In particular, we have $(H^q G)(\overline{F'}) \cong (H^q H)(\overline{F'})$ for any field $F' \in F^{\mathrm{perar}}$ and its algebraic closure $\overline{F'}$ compatible with the actions of $\mathrm{Gal}(\overline{F'}/F')$. The assumption $H^q G \in \mathcal{W}_F$ implies that the $\mathrm{Gal}(\overline{F'}/F')$ -invariant part of $(H^q G)(\overline{F'})$ is $(H^q G)(F')$. The assumption $G \in \langle \mathcal{W}_F \rangle_{F_{\mathrm{et}}^{\mathrm{perar}}}$ implies that $H \in D^b(\mathrm{IPAlg}_u/F)$ by [Suz24, Proposition 3.1.5] and hence $H^q H \in \mathrm{IPAlg}_u/F$. Therefore, the $\mathrm{Gal}(\overline{F'}/F')$ -invariant part of $(H^q H)(\overline{F'})$ is $(H^q H)(F')$. Thus $(H^q G)(F') \cong (H^q H)(F')$ for all $F' \in F^{\mathrm{perar}}$. Since $\mathcal{W}_F$ is a full subcategory of $\mathrm{Ab}(F_{\mathrm{et}}^{\mathrm{perar}})$, we obtain an isomorphism $H^q G \cong H^q H$. $\square$

PROPOSITION 2.2. *Let $G \in \mathrm{IAlg}_u/F$. Then any subobject and quotient object of G in IPAlg_u/F is in IAlg_u/F .*

Proof. Let $H \hookrightarrow G$ be a subobject in IPAlg_u/F . Write $H = \varinjlim_\lambda H_\lambda$ with $H_\lambda \in \mathrm{PAlg}_u/F$. Then for any λ , the composite morphism $H_\lambda \rightarrow H \hookrightarrow G$ is a morphism from an object of PAlg_u/F to an object of IAlg_u/F . Hence it factors through an object of Alg_u/F . Hence $H_\lambda \rightarrow H$ factors through an object of Alg_u/F . This implies $H \in \mathrm{IAlg}_u/F$, thus $G/H \in \mathrm{IAlg}_u/F$. $\square$

PROPOSITION 2.3. *Let $G \twoheadrightarrow H$ be a surjection in IPAlg_u/F . Assume $G \in \mathcal{W}_F$ and $H \in \mathrm{PAlg}_u/F$. Then $H \in \mathcal{W}_F$.*

Proof. First assume that G is pro-algebraic. We may assume G connected. Let K be the kernel of $G \twoheadrightarrow H$. Write G as the inverse limit of a surjective system of connected groups $G_n \in \mathrm{Alg}_u/F$ with connected $\mathrm{Ker}(G_{n+1} \twoheadrightarrow G_n)$. Let H_n be the cokernel of the composite $K \hookrightarrow G \twoheadrightarrow G_n$. Then H is the inverse limit of the surjective system of the connected unipotent groups H_n with connected $\mathrm{Ker}(H_{n+1} \twoheadrightarrow H_n)$.

For the general case, write G as a filtered union of pro-algebraic subgroups $G_m \in \mathcal{W}_F$. Since $H \in \mathrm{PAlg}_u/F$, we know that the composite $G_m \hookrightarrow G \twoheadrightarrow H$ is surjective for some m , which reduces the statement to the first case. $\square$

PROPOSITION 2.4. *Let $\varphi: G \rightarrow H$ be a morphism in IPAlg_u/F . Assume $G, H \in \mathcal{W}_F$. Then $\mathrm{Im}(\varphi) \in \mathcal{W}_F$.*

Proof. We may assume that G and H are connected. Write G as a direct limit $\varinjlim_n G_n$ with $G_n \in \mathcal{W}_F$ connected pro-algebraic and $G_{n+1}/G_n \in \mathrm{Alg}_u/F$. The restriction of φ to G_n factors through some pro-algebraic subgroup of H . Hence $\varphi(G_n) \in \mathrm{PAlg}_u/F$, so $\varphi(G_n) \in \mathcal{W}_F$ by Proposition 2.3. It is connected. The group $\varphi(G_{n+1})/\varphi(G_n)$ is a quotient of $G_{n+1}/G_n \in \mathrm{Alg}_u/F$; hence $\varphi(G_{n+1})/\varphi(G_n) \in \mathrm{Alg}_u/F$. Thus $\mathrm{Im}(\varphi) = \varinjlim_n \varphi(G_n)$ is in $\mathcal{W}_F$. $\square$

PROPOSITION 2.5. *Let $H \hookrightarrow G$ be an injection in IAlg_u/F . If $G \in \mathcal{W}_F$, then $H^0 \in \mathcal{W}_F$. Hence if $G \in \mathcal{W}_F$ and $\pi_0(H)$ is finite, then $H \in \mathcal{W}_F$.*

Proof. We may assume that G and H are connected. First assume $G \in \mathrm{Alg}_u/F$. Write H as a filtered direct limit of connected $H_\lambda \in \mathrm{Alg}_u/F$. Then $\mathrm{Im}(H_\lambda \rightarrow H) \cong \mathrm{Im}(H_\lambda \rightarrow G)$ is connected in Alg_u/F . As $G \in \mathrm{Alg}_u/F$, this implies that the system of subgroups $\mathrm{Im}(H_\lambda \rightarrow H)$ of G is stationary, meaning that $\mathrm{Im}(H_\lambda \rightarrow H) = H$ for some λ . Hence $H \in \mathrm{Alg}_u/F$.

For the general case, write G as the direct limit of an injective system of connected groups $G_n \in \mathrm{Alg}_u/F$. Then $(H \cap G_n)^0 \in \mathrm{Alg}_u/F$ by the first case. Hence H is the direct limit of the injective system of the connected groups $(H \cap G_n)^0 \in \mathrm{Alg}_u/F$. $\square$

PROPOSITION 2.6. *Let $0 \rightarrow K \rightarrow G \rightarrow H \rightarrow 0$ be an exact sequence in IPAlg_u/F . Assume $G \in \mathcal{W}_F$ and that H is ind-algebraic in $\mathcal{W}_F$. Then $K^0 \in \mathcal{W}_F$ and $\pi_0(K)$ is étale (that is, in IAlg_u/F).*

Proof. Let $G' \subset G^0$ be connected pro-algebraic in $\mathcal{W}_F$ such that $G^0/G' \in \mathcal{W}_F$ is ind-algebraic. Then the image of G' in H is in Alg_u/F . Therefore, $G' \cap K$ is pro-algebraic in $\mathcal{W}_F$. In the exact sequence

$$0 \rightarrow K/(G' \cap K) \rightarrow G/(G' \cap K) \rightarrow H \rightarrow 0,$$

the second and third terms are ind-algebraic in $\mathcal{W}_F$. Hence the first term $K/(G' \cap K)$ is ind-algebraic and its identity component is in $\mathcal{W}_F$ by Proposition 2.5. $\square$

PROPOSITION 2.7. *Let*

$$\dots \rightarrow C^{n-1} \rightarrow A^n \rightarrow B^n \rightarrow C^n \rightarrow A^{n+1} \rightarrow \dots \quad (2.1)$$

be a long exact sequence in IPAlg_u/F . Assume $A^n \in \mathrm{PAlg}_u/F$, $B^n \in \mathcal{W}_F$ and $C^n \in \mathrm{IAlg}_u/F$ for all n . Then any term of (2.1) and the image of any morphism in (2.1) are in $\mathcal{W}_F$. The images of $A^n \rightarrow B^n$, $B^n \rightarrow C^n$ and $C^n \rightarrow A^{n+1}$ are pro-algebraic, ind-algebraic and in Alg_u/F , respectively, for all n .

Proof. Let D^n , E^n and F^n be the images of $A^n \rightarrow B^n$, $B^n \rightarrow C^n$ and $C^n \rightarrow A^{n+1}$, respectively. Choose a pro-algebraic subgroup $B'^n \in \mathcal{W}_F$ of B^n such that $B^n/B'^n \in \mathcal{W}_F$ is ind-algebraic and the morphism $A^n \rightarrow B^n$ factors through B'^n . Then D^n is the image of $A^n \rightarrow B'^n$, so it is in PAlg_u/F . We have $E^n, F^n \in \mathrm{IAlg}_u/F$ by Proposition 2.2. The exact sequence $0 \rightarrow F^n \rightarrow A^{n+1} \rightarrow D^{n+1} \rightarrow 0$ shows that F^n is also in PAlg_u/F , hence in Alg_u/F . We have an exact sequence

$$0 \rightarrow B'^n/D^n \rightarrow E^n \rightarrow B^n/B'^n \rightarrow 0.$$

Hence $B'^n/D^n \in \mathrm{Alg}_u/F$ by Proposition 2.2. As $B^n/B'^n \in \mathcal{W}_F$, we have $E^n \in \mathcal{W}_F$. As $B'^n \in \mathcal{W}_F$, we have $D^n \in \mathcal{W}_F$. The rest follows since $\mathcal{W}_F$ is closed under extensions. $\square$

A perfect pairing on the derived category level gives a duality for all cohomology objects as soon as the cohomology objects are shown to belong to $\mathcal{W}_F$.

PROPOSITION 2.8. *Let $G \otimes^L H \rightarrow \Lambda_\infty$ be a perfect pairing in $D(F_{\mathrm{proet}}^{\mathrm{indrat}})$ or $D(F_{\mathrm{et}}^{\mathrm{perar}})$. Assume that the $H^q G$ are in $\mathcal{W}_F$ for all q and is zero for all but finitely many q . Then the $H^q H$ are in $\mathcal{W}_F$ for all q and are zero for all but finitely many q , and the pairing induces a Serre duality*

$$(H^q G)^0 \leftrightarrow (H^{1-q} H)^0$$

of connected groups in $\mathcal{W}_F$ and a Pontryagin duality

$$\pi_0(H^q G) \leftrightarrow \pi_0(H^{-q} H)$$

of finite étale groups over F for all q .

Proof. This follows from [Suz24, Propositions 3.1.7 and 3.1.9]. $\square$

3. Unipotent groups in positive characteristic

In this section, we show that the derived category level duality statement by Kato [Kat86, Theorem 4.3(ii)] has a counterpart for each cohomology object when the base scheme is a field with a finite p -basis. The cohomology objects turn out to be relatively perfect unipotent groups studied in [BS24, Section 8]. Hence Kato's duality in this case can be interpreted as a duality for relatively perfect unipotent groups. Under this duality, split groups correspond to split groups and wound groups correspond to wound groups.

Let k be a field of characteristic $p > 0$ such that $[k : k^p] = p^r$ for some finite $r \geq 0$. Let $\mathrm{Spec} k_{\mathrm{RP}}$ be the relatively perfect site of k (see [Kat86, Section 2], [KS19, Section 2], [BS24, Definition 8.2]). It is the category of relatively perfect k -schemes (where relatively perfect means that the relative Frobenius morphism $Y \rightarrow Y^{(p)}$ over k is an isomorphism) with k -scheme morphisms endowed with the étale topology. The inclusion functor from the category of relatively perfect k -schemes to the category of all k -schemes admits a right adjoint [Kat86, Definition 1.8], called the relative perfection functor and denoted by $Y \mapsto Y^{\mathrm{RP}}$. Let $\mathrm{Alg}^{\mathrm{RP}}/k \subset \mathrm{Ab}(k_{\mathrm{RP}})$ be the full subcategory consisting of relative perfections of quasi-compact smooth group schemes over k . It is an abelian subcategory that is closed under extensions by [BS24, Propositions 8.7 and 8.12].

Recall from [DG70, Section IV.2, Definition 2.1] that a (commutative) group scheme G over k (not necessarily of finite type) is said to be *unipotent* if it is affine and for any closed subgroup scheme $0 \neq H \subset G$, there is a non-zero morphism $H \rightarrow \mathbf{G}_a$ of group schemes over k .

PROPOSITION 3.1. *An object of Alg^{RP}/k is unipotent if and only if it is the relative perfection of a quasi-compact smooth unipotent group scheme over k . Denote by $\text{Alg}_u^{\text{RP}}/k$ the full subcategory of Alg^{RP}/k consisting of objects satisfying these equivalent conditions. It is an abelian subcategory that is closed under extensions.*

Proof. Let $G \in \text{Alg}^{\text{RP}}/k$ be unipotent. Write it as the relative perfection of a quasi-compact smooth group scheme G_0 over k . The natural morphism $G \rightarrow G_0$ is faithfully flat by [BS24, Proposition 8.13]. Hence G_0 is unipotent by [DG70, Section IV.2, Proposition 2.3(a)]. Conversely, let G_0 be a quasi-compact smooth unipotent group scheme over k . Then $G = G_0^{\text{RP}}$ can be written as the inverse limit of a system $\cdots \rightarrow G_2 \rightarrow G_1 \rightarrow G_0$ of groups schemes over k , where each G_n is the Weil restriction along the p^n th power Frobenius morphism $\text{Spec } k \rightarrow \text{Spec } k$ of G_0 ; see [BS24, Section 2]. Since Weil restrictions of unipotent groups are unipotent, we know that G_n is unipotent. Hence $G = \varprojlim_n G_n$ is unipotent. The second statement follows from the fact that unipotent group schemes are closed under kernels, cokernels and extensions. $\square$

Let $D_0(k_{\text{RP}}) \subset D(k_{\text{RP}})$ be the smallest triangulated full subcategory containing $\mathbf{G}_a^{\text{RP}}$; see [Kat86, Definition 4.2.3].

PROPOSITION 3.2. *The subcategory $D_0(k_{\text{RP}}) \subset D(k_{\text{RP}})$ agrees with the full subcategory of $D^b(k_{\text{RP}})$ consisting of objects G with $H^q G \in \text{Alg}_u^{\text{RP}}/k$ for all q .*

Proof. The latter category is triangulated by [Sta22, Tag 06UQ] and hence contains $D_0(k_{\text{RP}})$. For the opposite inclusion, it is enough to show that $\text{Alg}_u^{\text{RP}}/k \subset D_0(k_{\text{RP}})$. Let $G \in \text{Alg}_u^{\text{RP}}/k$ be the relative perfection of a quasi-compact smooth unipotent group scheme G_0 . By [BS24, Propositions 8.13 and 8.14], we know that $\pi_0(G) \cong \pi_0(G_0)$ is a finite étale group scheme over k . The Artin–Schreier sequence shows that $\Lambda \in D_0(k_{\text{RP}})$ and hence $\pi_0(G) \in D_0(k_{\text{RP}})$. Therefore, we may assume that G (and hence G_0) is connected. Then G_0 admits a finite filtration whose graded pieces are smooth connected group schemes killed by p by [CGP15, Corollary B.3.3]. A short exact sequence of smooth group schemes remains exact after relative perfection by [BS24, Proposition 8.8]. Hence we may assume that G_0 is killed by p . Then there exists an exact sequence $0 \rightarrow G_0 \rightarrow \mathbf{G}_a^{\dim G_0 + 1} \rightarrow \mathbf{G}_a \rightarrow 0$ by [CGP15, Proposition B.1.13]. Hence $G \in D_0(k_{\text{RP}})$. $\square$

We say that $G \in \text{Alg}_u^{\text{RP}}/k$ is *wound* if any morphism $\mathbf{G}_a^{\text{RP}} \rightarrow G$ is zero. This is equivalent to G being the relative perfection of a wound smooth unipotent group scheme G_0 over k in the sense that any morphism $\mathbf{G}_a \rightarrow G_0$ is zero [CGP15, Definition B.2.1]. Here we allow disconnected groups. Any p -primary finite étale group is wound in our sense. We say that $G \in \text{Alg}_u^{\text{RP}}/k$ is *split* if it admits a finite filtration whose graded pieces are isomorphic to $\mathbf{G}_a^{\text{RP}}$. By [CGP15, Theorem B.3.4], any $G \in \text{Alg}_u^{\text{RP}}/k$ admits a unique maximal split subgroup G_{split} , and G/G_{split} is wound.

For $n \geq 1$, let $\nu_n(r) \in \text{Ab}(k_{\text{RP}})$ be the dlog part of the Hodge–Witt sheaf $W_n \Omega_k^r$; see [Kat86, Section 4]. Let $\nu_\infty(r)$ be the direct limit of $\nu_n(r)$ over $n \geq 1$. We have

$$R\mathbf{Hom}_{k_{\text{RP}}}(\cdot, \nu_\infty(r)) \cong \varinjlim_n R\mathbf{Hom}_{k_{\text{RP}}}(\cdot, \nu_n(r))$$

on $D_0(k_{\text{RP}})$ by the same proof as that of [BS24, Proposition 8.10]. Recall from [Kat86, Theorem 4.3] that the endofunctor $R\mathbf{Hom}_{k_{\text{RP}}}(\cdot, \nu_\infty(r))$ on $D(k_{\text{RP}})$ restricts to a contravariant autoequivalence on the subcategory $D_0(k_{\text{RP}})$ with inverse itself.

PROPOSITION 3.3. *Let $G \in \text{Alg}_u^{\text{RP}}/k$. Then the object $\mathbf{Hom}_{k_{\text{RP}}}(G, \nu_\infty(r)) \in \text{Alg}_u^{\text{RP}}/k$ is wound, the object $\mathbf{Ext}_{k_{\text{RP}}}^1(G, \nu_\infty(r)) \in \text{Alg}_u^{\text{RP}}/k$ is split, and $\mathbf{Ext}_{k_{\text{RP}}}^q(G, \nu_\infty(r)) = 0$ for all $q \geq 2$.*

Proof. We have $R\mathbf{Hom}_{k_{\text{RP}}}(G, \nu_\infty(r)) \in D_0(k_{\text{RP}})$ by [Kat86, Theorem 4.3(ii)]. Hence by Proposition 3.2, we have $\mathbf{Ext}_{k_{\text{RP}}}^q(G, \nu_\infty(r)) \in \text{Alg}_u^{\text{RP}}/k$ for all q . Also, $\mathbf{Ext}_{k_{\text{RP}}}^q(G, \nu_\infty(r)) = 0$ for all $q \geq 2$ by [Kat86, Theorem 3.2]. Since $\mathbf{Hom}_{k_{\text{RP}}}(\mathbf{G}_a^{\text{RP}}, \nu_\infty(r)) = 0$ by [Kat86, Theorem 3.2(ii)], any morphism $\mathbf{G}_a^{\text{RP}} \rightarrow \mathbf{Hom}_{k_{\text{RP}}}(G, \nu_\infty(r))$ is zero. Therefore, $\mathbf{Hom}_{k_{\text{RP}}}(G, \nu_\infty(r))$ is wound. To show the splitness of $\mathbf{Ext}_{k_{\text{RP}}}^1(G, \nu_\infty(r)) \in \text{Alg}_u^{\text{RP}}/k$, note that a quotient of a split group is split and an extension of split groups is split. Hence again by [CGP15, Corollary B.3.3], we may assume $pG = 0$. Take an exact sequence $0 \rightarrow G_0 \rightarrow \mathbf{G}_a^{\dim G_0 + 1} \rightarrow \mathbf{G}_a \rightarrow 0$ as above. We have $\mathbf{Ext}_{k_{\text{RP}}}^1(\mathbf{G}_a^{\text{RP}}, \nu_\infty(r)) \cong \Omega_k^r \cong \mathbf{G}_a^{\text{RP}}$ by [Kat86, Theorem 3.2(ii)]. Therefore, applying $\mathbf{Ext}_{k_{\text{RP}}}^1(\cdot, \nu_\infty(r))$ to $G_0 \hookrightarrow \mathbf{G}_a^{\dim G_0 + 1}$ gives a surjection from $(\mathbf{G}_a^{\text{RP}})^{\dim G_0 + 1}$ to $\mathbf{Ext}_{k_{\text{RP}}}^1(G, \nu_\infty(r))$. Thus $\mathbf{Ext}_{k_{\text{RP}}}^1(G, \nu_\infty(r))$ is split. $\square$

PROPOSITION 3.4. *Let $G \in \text{Alg}_u^{\text{RP}}/k$. If G is split, then $\mathbf{Hom}_{k_{\text{RP}}}(G, \nu_\infty(r)) = 0$, and if G is wound, then $\mathbf{Ext}_{k_{\text{RP}}}^1(G, \nu_\infty(r)) = 0$.*

Proof. The first statement is [Kat86, Theorem 3.2(ii)]. Assume that G is wound. By Proposition 3.3, the isomorphism

$$G \xrightarrow{\sim} R\mathbf{Hom}_{k_{\text{RP}}}(R\mathbf{Hom}_{k_{\text{RP}}}(G, \nu_\infty(r)), \nu_\infty(r))$$

induces an exact sequence

$$0 \rightarrow \mathbf{Ext}_{k_{\text{RP}}}^1(\mathbf{Ext}_{k_{\text{RP}}}^1(G, \nu_\infty(r)), \nu_\infty(r)) \rightarrow G \rightarrow \mathbf{Hom}_{k_{\text{RP}}}(\mathbf{Hom}_{k_{\text{RP}}}(G, \nu_\infty(r)), \nu_\infty(r)) \rightarrow 0.$$

The first term in this sequence is split by Proposition 3.3 and is hence zero. This implies that the split group $\mathbf{Ext}_{k_{\text{RP}}}^1(G, \nu_\infty(r))$ is zero. $\square$

PROPOSITION 3.5. *Let $G \in \text{Alg}_u^{\text{RP}}/k$. If G is wound, then $\mathbf{Hom}_{k_{\text{RP}}}(G, \nu_\infty(r)) \in \text{Alg}_u^{\text{RP}}/k$ is wound and*

$$G \cong \mathbf{Hom}_{k_{\text{RP}}}(\mathbf{Hom}_{k_{\text{RP}}}(G, \nu_\infty(r)), \nu_\infty(r)).$$

If G is split, then $\mathbf{Ext}_{k_{\text{RP}}}^1(G, \nu_\infty(r)) \in \text{Alg}_u^{\text{RP}}/k$ is split and

$$G \cong \mathbf{Ext}_{k_{\text{RP}}}^1(\mathbf{Ext}_{k_{\text{RP}}}^1(G, \nu_\infty(r)), \nu_\infty(r)).$$

Proof. This follows from Propositions 3.3 and 3.4. $\square$

This gives a duality for wound groups and for split groups in $\text{Alg}_u^{\text{RP}}/k$. For example, for any $n \geq 1$ and $0 \leq s \leq r$, the groups $\nu_n(s)$ and $\nu_n(r - s)$ are wound groups dual to each other by [Kat86, Theorem 4.3]. Note that $\nu_n(0) = \mathbb{Z}/p^n\mathbb{Z}$ is discrete, while one can check that its dual $\nu_n(r)$ is connected if $r > 0$. On the other hand, for a finite-dimensional k -vector group V , its dual in the above sense (that is, $\mathbf{Ext}_{k_{\text{RP}}}^1(V, \nu_\infty(r))$) is given by the vector space dual V^* tensored with Ω_k^r by [Kat86, Theorem 3.2(ii)].

4. Duality for unipotent groups over local fields of positive characteristic

Recall that F is a fixed perfect field of characteristic $p > 0$. Let k be a complete discrete valuation field of characteristic p with residue field F . In this section, we give a duality for cohomology

of k with coefficients in relatively perfect unipotent groups. Compare it with Rosengarten's duality [Ros23].

Let $\mathcal{O}_k$ be the ring of integers of k with maximal ideal $\mathfrak{p}_k$. The functor

$$F' \mapsto \mathbf{k}(F') = (F' \hat{\otimes}_F \mathcal{O}_k) \otimes_{\mathcal{O}_k} k$$

(where $F' \hat{\otimes}_F \mathcal{O}_k$ means the inverse limit of $F' \otimes_F \mathcal{O}_k/\mathfrak{p}_k^n$ over n) defines a premorphism of sites

$$\pi_{k,\text{RP}}: \text{Spec } k_{\text{RP}} \rightarrow \text{Spec } F_{\text{et}}^{\text{perar}}.$$

PROPOSITION 4.1. *Let $G \in \text{Alg}_u^{\text{RP}}/k$. Then $\pi_{k,\text{RP},*}G \in \mathcal{W}_F$. It is pro-algebraic if G is wound.*

Proof. We have $R^q\pi_{k,\text{RP},*}G_{\text{split}} = 0$ for $q \geq 1$. Hence we have an exact sequence

$$0 \rightarrow \pi_{k,\text{RP},*}G_{\text{split}} \rightarrow \pi_{k,\text{RP},*}G \rightarrow \pi_{k,\text{RP},*}(G/G_{\text{split}}) \rightarrow 0$$

in $\text{Ab}(F_{\text{et}}^{\text{perar}})$. Since $\pi_{k,\text{RP},*}\mathbf{G}_a^{\text{RP}} \cong \mathbf{k} \cong \mathbf{G}_a^{\mathbb{N}} \oplus \mathbf{G}_a^{\oplus \mathbb{N}} \in \mathcal{W}_F$, we have $\pi_{k,\text{RP},*}G_{\text{split}} \in \mathcal{W}_F$. Let $G' = G/G_{\text{split}}$. Since it is wound, it admits a Néron finite-type model $\mathcal{G}'$ over $\mathcal{O}_k$ by [BLR90, Section 10.2, Theorem 1]. For any $F' \in F^{\text{perar}}$, we have

$$(\pi_{k,\text{RP},*}G')(F') \cong \mathcal{G}'(F' \hat{\otimes}_F \mathcal{O}_k).$$

Therefore, $\pi_{k,\text{RP},*}G'$ is represented by the perfect Greenberg transform of $\mathcal{G}'$; see [BG18, Section 14, (74)]. By [BG18, Proposition 11.1], the natural morphism $\pi_{k,\text{RP},*}G' \rightarrow \mathcal{G}'_F$ to the special fiber of $\mathcal{G}'$ is surjective, and its kernel is pro-algebraic in $\mathcal{W}_F$. Thus $\pi_{k,\text{RP},*}G' \in \mathcal{W}_F$. Hence $\pi_{k,\text{RP},*}G \in \mathcal{W}_F$. $\square$

PROPOSITION 4.2. *Let $G \in \text{IPAlg}_u/F$ be such that $\pi_0(G)$ is étale. Assume*

$$\varinjlim_{\lambda} G(F'_{\lambda}) \xrightarrow{\sim} G(\varinjlim_{\lambda} F'_{\lambda})$$

for any filtered direct system $\{F'_{\lambda}\}$ of perfect fields over F . Then $G \in \text{IPAlg}_u/F$.

Proof. The assumption and conclusion are satisfied if G is étale. Hence we may assume that G is connected. Then G can be written as the direct limit of a filtered direct system $\{G_{\mu}\}$ of connected groups in PAlg_u/F . Hence we need to show that any morphism $H \rightarrow G$ from a connected group $H \in \text{PAlg}_u/F$ factors through an object of Alg_u/F . Let ξ_H be the generic point of H . The same argument as in the proof of [Suz20b, Lemma 3.4.4] applied to this situation gives the result. $\square$

PROPOSITION 4.3. *Let $G \in \text{Alg}_u^{\text{RP}}/k$. Then $R^1\pi_{k,\text{RP},*}G \in \text{IPAlg}_u/F$ is connected, and we have $R^q\pi_{k,\text{RP},*}G = 0$ for $q \geq 2$.*

Proof. As in the proof of Proposition 3.2, we may assume that G is wound and that there exists an exact sequence $0 \rightarrow G \rightarrow \mathbf{G}_a^{\text{RP}n} \rightarrow \mathbf{G}_a^{\text{RP}} \rightarrow 0$, where $n = \dim(G) + 1$. We have $R^q\pi_{k,\text{RP},*}\mathbf{G}_a^{\text{RP}} = 0$ for $q \geq 1$. Hence $R^q\pi_{k,\text{RP},*}G = 0$ for $q \geq 2$, and we have an exact sequence

$$0 \rightarrow \pi_{k,\text{RP},*}G \rightarrow \mathbf{k}^n \rightarrow \mathbf{k} \rightarrow R^1\pi_{k,\text{RP},*}G \rightarrow 0$$

in $\text{Ab}(F_{\text{et}}^{\text{perar}})$. The morphism $\pi_{k,\text{RP},*}G \rightarrow \mathbf{k}^n$ is injective in IPAlg_u/F . Let H be its cokernel in IPAlg_u/F , which is in $\mathcal{W}_F$ by Propositions 4.1 and 2.7. Then, since the pro-étale cohomology with coefficients in an object of $\mathcal{W}_F$ is isomorphic to the étale cohomology by [Suz24, Proposition 3.1.2], this cokernel H represents the cokernel of $\pi_{k,\text{RP},*}G \rightarrow \mathbf{k}^n$ in $\text{Ab}(F_{\text{et}}^{\text{perar}})$. The same argument applied to $H \hookrightarrow \mathbf{k}$ shows that $R^1\pi_{k,\text{RP},*}G \in \text{IPAlg}_u/F$. It is connected since $\mathbf{k}$ is connected. For any filtered direct system $\{F'_{\lambda}\}$ of perfect fields over F , we have

$$\varinjlim_{\lambda} H^1(\mathbf{k}(F'_{\lambda}), G) \xrightarrow{\sim} H^1(\varinjlim_{\lambda} \mathbf{k}(F'_{\lambda}), G) \xrightarrow{\sim} H^1(\mathbf{k}(\varinjlim_{\lambda} F'_{\lambda}), G)$$

by [GGM14, Proposition 3.5.3(2)]. This implies that $R^1\pi_{k,\text{RP},*}G \in \text{IPAlg}_u/F$ satisfies the assumption of Proposition 4.2. Thus $R^1\pi_{k,\text{RP},*}G \in \text{IALg}_u/F$. $\square$

We have

$$R^q\pi_{k,\text{RP},*\nu_\infty(1)} \cong \begin{cases} \mathbf{k}^\times \otimes_{\mathbb{Z}} \Lambda_\infty & \text{if } q = 0, \\ 0 & \text{otherwise} \end{cases}$$

by [Suz24, Proposition 6.1.6]. With the valuation map $\mathbf{k}^\times \twoheadrightarrow \mathbb{Z}$, this gives a canonical morphism

$$R\pi_{k,\text{RP},*\nu_\infty(1)} \rightarrow \Lambda_\infty$$

in $D(F_{\text{et}}^{\text{perar}})$.

PROPOSITION 4.4. *Let $G \in D_0(k_{\text{RP}})$, and set $H = R\mathbf{Hom}_{k_{\text{RP}}}(G, \nu_\infty(1))$. Then the composite*

$$R\pi_{k,\text{RP},*}G \otimes^L R\pi_{k,\text{RP},*}H \rightarrow R\pi_{k,\text{RP},*\nu_\infty(1)} \rightarrow \Lambda_\infty \quad (4.1)$$

is a perfect pairing in $D(F_{\text{et}}^{\text{perar}})$.

Proof. We may assume $G = \mathbf{G}_a^{\text{RP}}$ and hence $H = \Omega_k^1[-1]$. The proof in this case is analogous (or can be reduced) to that of [Suz20b, Proposition 5.2.2.4]. More explicitly, we have $R\pi_{k,\text{RP},*}\mathbf{G}_a^{\text{RP}} \cong \mathbf{k}$ and $R\pi_{k,\text{RP},*}\Omega_k^1 \cong \Omega_{\mathbf{k}}^1$, where $\Omega_{\mathbf{k}}^1 = \Omega_k^1 \otimes_k \mathbf{k}$. The multiplication pairing $k \times \Omega_k^1 \rightarrow \Omega_k^1$ followed by the residue map $\Omega_k^1 \twoheadrightarrow F$ is a perfect pairing of Tate vector spaces over F . Hence [Suz24, Propositions 3.4.5 and 3.4.4] imply the result. $\square$

PROPOSITION 4.5. *Let $G \in \text{Alg}_u^{\text{RP}}/k$ be wound and set $H = \mathbf{Hom}_{k_{\text{RP}}}(G, \nu_\infty(1))$. Then the paring (4.1) induces isomorphisms*

$$\begin{aligned} \pi_0(\pi_{k,\text{RP},*}G) &\cong \mathbf{Hom}_{F_{\text{et}}^{\text{perar}}}(\pi_0(\pi_{k,\text{RP},*}H), \Lambda_\infty), \\ R^1\pi_{k,\text{RP},*}G &\cong \mathbf{Ext}_{F_{\text{et}}^{\text{perar}}}^1((\pi_{k,\text{RP},*}H)^0, \Lambda_\infty). \end{aligned}$$

Proof. This follows from the connectedness of $R^1\pi_{k,\text{RP},*}H$ (see Proposition 4.3). $\square$

PROPOSITION 4.6. *Let $G \in \text{Alg}_u^{\text{RP}}/k$. Then $R^1\pi_{k,\text{RP},*}G \in \mathcal{W}_F$.*

Proof. We may assume that G is wound. Let $H = \mathbf{Hom}_{k_{\text{RP}}}(G, \nu_\infty(1))$. Since $\pi_{k,\text{RP},*}H \in \mathcal{W}_F$ by Proposition 4.1, we have $\mathbf{Ext}_{F_{\text{et}}^{\text{perar}}}^1((\pi_{k,\text{RP},*}H)^0, \Lambda_\infty) \in \mathcal{W}_F$ by [Suz24, Proposition 3.1.7(2)]. With Proposition 4.5, we get the result. $\square$

Set

$$R\mathbf{\Gamma}(\mathbf{k}, G) = \acute{a}R\pi_{k,\text{RP},*}G \in D(F_{\text{proet}}^{\text{indrat}})$$

for $G \in \text{Alg}_u^{\text{RP}}$ and $\mathbf{H}^q = H^q R\mathbf{\Gamma}$ for $q \in \mathbb{Z}$ and $\mathbf{\Gamma} = \mathbf{H}^0$.

PROPOSITION 4.7. *Let $G \in \text{Alg}_u^{\text{RP}}/k$.*

- (i) *Assume that G is split. Set $H = \mathbf{Ext}_{k_{\text{RP}}}^1(G, \nu_\infty(1))$. Then $\mathbf{H}^q(\mathbf{k}, G) = 0$ unless $q = 0$. We have a Serre duality*

$$\mathbf{\Gamma}(\mathbf{k}, G) \leftrightarrow \mathbf{\Gamma}(\mathbf{k}, H)$$

of connected groups in $\mathcal{W}_F$.

- (ii) *Assume that G is wound. Set $H = \mathbf{Hom}_{k_{\text{RP}}}(G, \nu_\infty(1))$. Then $\mathbf{H}^q(\mathbf{k}, G) = 0$ unless $0 \leq q \leq 1$. We have a Pontryagin duality*

$$\pi_0\mathbf{\Gamma}(\mathbf{k}, G) \leftrightarrow \pi_0\mathbf{\Gamma}(\mathbf{k}, H) \quad (4.2)$$

of finite étale groups over F and a Serre duality

$$\mathbf{H}^1(\mathbf{k}, G) \leftrightarrow \Gamma(\mathbf{k}, H)^0 \quad (4.3)$$

of connected groups in $\mathcal{W}_F$.

Proof. This follows from Propositions 2.8, 4.1, 4.3, 4.4, 4.5 and 4.6. $\square$

The dualities (4.2) and (4.3) above are unipotent group analogs of Grothendieck's and Shafarevich's duality conjectures, respectively, for abelian varieties, proved in [Suz20b] after Grothendieck, Werner, Bégueri, Bester, Bertapelle and others.

5. Two-dimensional local fields

Let k be as in Section 4. Let K be a complete discrete valuation field of characteristic zero with residue field k . Let $\mathcal{O}_K$ be its ring of integers. In this section, we show that the derived category level duality statement for K in [Suz24, Section 6] has a counterpart for each cohomology object. The key point is to show that the nearby cycle functor from K to k takes values in relatively perfect unipotent groups, which allows us to apply the results of Section 4 to these groups. There are some technicalities to deal with about the choice of sites. Below we identify $\mathbb{Z}/p^n\mathbb{Z}(r)$ with the p -adic étale Tate twist $\mathfrak{T}_n(r)$ (see [Sch94, Gei04, Sat07, Sat13]).

Let $\mathrm{Spec} k_{\mathrm{RPS}}$ be the relatively perfectly smooth site of k (see [KS19, Section 2]). It is the category of relatively perfectly smooth k -schemes (that is, k -schemes Zariski locally isomorphic to relative perfection of smooth k -schemes) with k -scheme morphisms endowed with the étale topology. Let $D_0(k_{\mathrm{RPS}}) \subset D(k_{\mathrm{RPS}})$ be the smallest triangulated subcategory containing $\mathbf{G}_a^{\mathrm{RP}}$. Let $\alpha: \mathrm{Spec} k_{\mathrm{RP}} \rightarrow \mathrm{Spec} k_{\mathrm{RPS}}$ be the premorphism defined by the inclusion functor on the underlying categories. Its (exact) pushforward functor gives an equivalence $D_0(k_{\mathrm{RP}}) \xrightarrow{\sim} D_0(k_{\mathrm{RPS}})$ (see [KS19, Corollary 2.3]). We identify these equivalent categories. For $G \in D_0(k_{\mathrm{RP}})$ (or $D_0(k_{\mathrm{RPS}})$), the natural morphism gives an isomorphism

$$\alpha_* R\mathbf{Hom}_{k_{\mathrm{RP}}}(G, \nu_\infty(1)) \xrightarrow{\sim} R\mathbf{Hom}_{k_{\mathrm{RPS}}}(G, \nu_\infty(1))$$

by [KS19, Proposition 2.2].

Let $\mathrm{Spec} \mathcal{O}_{K,\mathrm{RPS}}$ be relatively perfectly smooth site of $\mathcal{O}_K$ (see [KS19, Section 3]), whose objects are flat $\mathcal{O}_K$ -algebras R such that $R \otimes_{\mathcal{O}_K} k$ is relatively perfectly smooth over k . The topology of $\mathrm{Spec} \mathcal{O}_{K,\mathrm{RPS}}$ is the étale topology. Let $\mathrm{Spec} K_{\mathrm{Et}}$ be the big étale site of K . By [KS19, Proposition 3.2, (3.1)], the base change functors define morphisms of sites

$$\mathrm{Spec} K_{\mathrm{Et}} \xrightarrow{j_{\mathrm{RPS}}} \mathrm{Spec} \mathcal{O}_{K,\mathrm{RPS}} \xleftarrow{i_{\mathrm{RPS}}} \mathrm{Spec} k_{\mathrm{RPS}}.$$

Let $R\Psi_{\mathrm{RPS}} = i_{\mathrm{RPS}}^* Rj_{\mathrm{RPS},*}$ and $R^q\Psi_{\mathrm{RPS}} = i_{\mathrm{RPS}}^* R^q j_{\mathrm{RPS},*}$ for $q \in \mathbb{Z}$. For $n \geq 1$ and $r \geq 0$, we have a canonical morphism

$$R^r\Psi_{\mathrm{RPS}}\Lambda_n(r) \rightarrow \nu_n(r-1)$$

in $\mathrm{Ab}(k_{\mathrm{RPS}})$ by [KS19, Proposition 5.1] (where we set $\nu_n(-1) = 0$). Hence we have a canonical morphism

$$\tau_{\leq r} Rj_{\mathrm{RPS},*}\Lambda_n(r) \rightarrow i_{\mathrm{RPS},*}\nu_n(r-1) \quad (5.1)$$

in $D(\mathcal{O}_{K,\mathrm{RPS}})$. Let $\mathfrak{T}_n(r) \in D(\mathcal{O}_{K,\mathrm{RPS}})$ be its canonical mapping fiber. For $r < 0$, let $\mathfrak{T}_n(r) = j_{\mathrm{RPS},!}\Lambda_n(r)$. We have a canonical natural transformation

$$i_{\mathrm{RPS}}^* \rightarrow i_{\mathrm{RPS}}^* Rj_{\mathrm{RPS},*} j_{\mathrm{RPS}}^* (: D(\mathcal{O}_{K,\mathrm{RPS}}) \rightarrow D(k_{\mathrm{RPS}})).$$

Since j_{RPS}^* sends $\mathbf{K}$ -injectives to $\mathbf{K}$ -injectives, we have its canonical mapping fiber, which we denote by $Ri_{\text{RPS}}^!$, so that we have a canonical distinguished triangle

$$Ri_{\text{RPS}}^! G \rightarrow i_{\text{RPS}}^* G \rightarrow R\Psi_{\text{RPS}} j_{\text{RPS}}^* G$$

in $D(k_{\text{RPS}})$ for $G \in D(\mathcal{O}_{K,\text{RPS}})$ functorially. In particular, we have a distinguished triangle

$$i_{\text{RPS}}^* \mathfrak{T}_n(r) \rightarrow R\Psi_{\text{RPS}} \Lambda_n(r) \rightarrow Ri_{\text{RPS}}^! \mathfrak{T}_n(r)[1].$$

PROPOSITION 5.1. *Let $n \geq 1$ and $r \in \mathbb{Z}$. Then the objects $R\Psi_{\text{RPS}} \Lambda_n(r)$, $i_{\text{RPS}}^* \mathfrak{T}_n(r)$ and $Ri_{\text{RPS}}^! \mathfrak{T}_n(r)$ are in $D_0(k_{\text{RPS}})$.*

Proof. It is enough to treat $R\Psi_{\text{RPS}} \Lambda_n(r)$ and $i_{\text{RPS}}^* \mathfrak{T}_n(r)$. We may assume $n = 1$. We may also assume that K contains a fixed primitive p th root of unity ζ_p . We identify all twists $\Lambda(r)$ with Λ using ζ_p . Then for any $q \in \mathbb{Z}$, the sheaf $R^q \Psi_{\text{RPS}} \Lambda$ has a finite filtration whose graded pieces are either $\mathbf{G}_a^{\text{RP}}$, Ω_k^1 , $\mathbf{G}_a^{\text{RP}}/(\mathbf{G}_a^{\text{RP}})^p$, Λ or $\nu(1)$ by [KS19, Proposition 6.1]. All these graded pieces are in $\text{Alg}_u^{\text{RP}}/k$. Hence $R^q \Psi_{\text{RPS}} \Lambda \in \text{Alg}_u^{\text{RP}}/k$ and so $R\Psi_{\text{RPS}} \Lambda \in D_0(\text{Alg}_u^{\text{RP}}/k)$. We have

$$H^q i_{\text{RPS}}^* \mathfrak{T}(r) \cong \begin{cases} R^q \Psi_{\text{RPS}} \Lambda & \text{if } q < r, \\ \text{Ker}(R^r \Psi_{\text{RPS}} \Lambda \rightarrow \nu(r-1)) & \text{if } q = r, \\ 0 & \text{if } q > r. \end{cases}$$

Hence it is in $\text{Alg}_u^{\text{RP}}/k$ for all q . $\square$

PROPOSITION 5.2. *Let $n \geq 1$ and $q, s, r \in \mathbb{Z}$. Let $G \in \text{Alg}_u^{\text{RP}}/k$ be either $R^q \Psi_{\text{RPS}} \Lambda_n(r)$, $H^q i_{\text{RPS}}^* \mathfrak{T}_n(r)$ or $H^q Ri_{\text{RPS}}^! \mathfrak{T}_n(r)$. Then $R^s \pi_{k,\text{RP},*} G \in \mathcal{W}_F$.*

Proof. This follows from Propositions 4.1, 4.6 and 5.1. $\square$

Recall from [Kat82, Section 1, Definition 1] and [Suz24, Section 2.3] that for a relatively perfect k -algebra R , its Kato canonical lifting over $\mathcal{O}_K$ is a unique complete flat $\mathcal{O}_K$ -algebra S equipped with an isomorphism $S \otimes_{\mathcal{O}_K} k \cong R$ over k .

For $F' \in F^{\text{perar}}$, let $\mathbf{O}_K(F')$ be the Kato canonical lifting of $\mathbf{k}(F')$ over $\mathcal{O}_K$. Define $\mathbf{K}(F') = \mathbf{O}_K(F') \otimes_{\mathcal{O}_K} K$. We recall the relative sites $\text{Spec } \mathbf{K}_{\text{et}}$, $\text{Spec } \mathbf{O}_{K,\text{et}}$ and $\text{Spec } \mathbf{k}_{\text{et}}$ from [Suz24, Sections 4.1 and 5.1]. The underlying category of $\text{Spec } \mathbf{k}_{\text{et}}$ consists of pairs (k', F') , where $F' \in F^{\text{perar}}$ and k' is an étale $\mathbf{k}(F')$ -algebra. Morphisms are morphisms between morphisms $(\mathbf{k}(F') \rightarrow k') \rightarrow (\mathbf{k}(F'') \rightarrow k'')$. A covering is a finite family $\{(k'_i, F'_i)\}$ such that $F' \rightarrow F'_i$ is étale for all i and $k' \rightarrow \prod_i k'_i$ is faithfully flat étale. The sites $\text{Spec } \mathbf{K}_{\text{et}}$ and $\text{Spec } \mathbf{O}_{K,\text{et}}$ are defined similarly using the functors $\mathbf{K}$ and $\mathbf{O}_K$ instead.

We have morphisms of sites

$$\begin{array}{ccccc} \text{Spec } \mathbf{K}_{\text{et}} & \xrightarrow{j} & \text{Spec } \mathbf{O}_{K,\text{et}} & \xleftarrow{i} & \text{Spec } \mathbf{k}_{\text{et}} \\ \pi_{\mathbf{K}} \downarrow & & \pi_{\mathbf{O}_K} \downarrow & & \pi_{\mathbf{k}} \downarrow \\ \text{Spec } F_{\text{et}}^{\text{perar}} & \xlongequal{\quad} & \text{Spec } F_{\text{et}}^{\text{perar}} & \xlongequal{\quad} & \text{Spec } F_{\text{et}}^{\text{perar}}, \end{array}$$

where the vertical morphisms are defined by the functors $F' \mapsto (\mathbf{K}(F'), F')$, $(\mathbf{O}_K(F'), F')$ and $(\mathbf{k}(F'), F')$ and the upper horizontal morphisms by $(A', F') \mapsto (A' \otimes_{\mathcal{O}_K} K, F')$ and $(A', F') \mapsto (A' \otimes_{\mathcal{O}_K} k, F')$ (where A' is an étale $\mathbf{O}_K(F')$ -algebra).

Let $R\Psi = i^* Rj_*$. For $r \in \mathbb{Z}$ and $n \geq 1$, the objects $\Lambda_n(r)$ and $\mathfrak{T}_n(r)$ are naturally viewed as objects of $\text{Ab}(\mathbf{K}_{\text{et}})$ and $D(\mathbf{O}_{K,\text{et}})$, respectively. For $q \in \mathbb{Z}$, the sheaves $R^q \pi_{\mathbf{K},*} \Lambda_n(r)$ and

$R^q \pi_{\mathbf{O}_K, *} \mathfrak{T}_n(r)$ are the étale sheafifications of the presheaves

$$F' \mapsto H^q(\mathbf{K}(F'), \Lambda_n(r)), \quad H^q(\mathbf{O}_K(F'), \mathfrak{T}_n(r)).$$

The object $R\pi_{\mathbf{O}_K, *} \mathfrak{T}_n(r)$ is defined as $R\pi_{\mathbf{K}, *} Ri^! \mathfrak{T}_n(r)$, where $i^!$ is the right adjoint of i_* (see [Suz24, Section 4.3] for more details).

PROPOSITION 5.3. *Let $G \in \text{Ab}(\mathbf{k}_{\text{et}})$ be torsion. Then $R^q \pi_{\mathbf{K}, *} G = 0$ for all $q \geq 2$.*

Proof. The sheaf $R^q \pi_{\mathbf{K}, *} G \in \text{Ab}(F_{\text{et}}^{\text{perar}})$ is the étale sheafification of the presheaf $F' \mapsto H^q(\mathbf{k}(F'), G)$, where G is restricted to the small étale site of $\mathbf{k}(F')$. Let $F' \in F^{\text{perar}}$ be a field and $\overline{F'}$ its algebraic closure. Then the direct limit of $\mathbf{k}(F'')$ over all finite subextensions F'' of $\overline{F'}/F'$ is a henselian discrete valuation field with residue field $\overline{F'}$. Such a field has cohomological dimension one. This implies the result. $\square$

PROPOSITION 5.4. *Let $F' \in F^{\text{perar}}$. Then the base change functor defines a morphism of sites $\text{Spec } \mathbf{k}(F')_{\text{RPS}} \rightarrow \text{Spec } k_{\text{RPS}}$.*

Proof. The only point to check is the exactness of the pullback functor for sheaves of sets. It is enough to show that $\mathbf{k}(F')$ is a filtered direct limit of relatively perfectly smooth k -algebras. We may assume that F' is a field. The ring $F' \hat{\otimes}_F \mathcal{O}_k$ is regular over $\mathcal{O}_k$ by [BLR90, Section 3.6, Lemma 2]. Hence $\mathbf{k}(F')$ is a filtered direct limit of smooth k -algebras by the paragraph before [BLR90, Section 3.6, Lemma 2]. Applying the relative perfectness, we know that $\mathbf{k}(F')$ is a filtered direct limit of relatively perfectly smooth k -algebras since $\mathbf{k}(F')$ is relatively perfect over k . $\square$

PROPOSITION 5.5. *Let $q, s, r \in \mathbb{Z}$ and $n \geq 1$. Let $G \in \text{Ab}(\mathbf{k}_{\text{et}})$ be either $R^q \Psi \Lambda_n(r)$, $H^q i^* \mathfrak{T}_n(r)$ or $H^q Ri^! \mathfrak{T}_n(r)$. Then $R^s \pi_{\mathbf{K}, *} G \in \mathcal{W}_F$.*

Proof. Let $G' \in \text{Alg}_{\text{su}}^{\text{RP}}/k$ be either $R^q \Psi_{\text{RPS}} \Lambda_n(r)$, $H^q i_{\text{RPS}}^* \mathfrak{T}_n(r)$ or $H^q Ri_{\text{RPS}}^! \mathfrak{T}_n(r)$, correspondingly. By Proposition 5.2, it is enough to show that $R^s \pi_{\mathbf{K}, *} G \cong R^s \pi_{k, \text{RPS}, *} G'$. They are the sheafifications of the presheaves

$$F' \mapsto H^s(\mathbf{k}(F'), G), \quad H^s(\mathbf{k}(F'), G' \times_k \mathbf{k}(F')),$$

where the G in the first presheaf is the restriction of G to the small étale site of $\mathbf{k}(F')$ and the $G' \times_k \mathbf{k}(F')$ in the second presheaf is the fiber product of schemes ($G' \in \text{Alg}_{\text{su}}^{\text{RP}}/k$ being representable). Hence it is enough to show that the group scheme $G' \times_k \mathbf{k}(F')$ as a sheaf on $\mathbf{k}(F')_{\text{et}}$ is isomorphic to the restriction of G to $\mathbf{k}(F')_{\text{et}}$. We may assume that F' is a field. The base change functors define morphisms of sites

$$\text{Spec } \mathbf{K}(F')_{\text{Et}} \xrightarrow{j_{\text{RPS}, F'}} \text{Spec } \mathbf{O}_K(F')_{\text{RPS}} \xleftarrow{i_{\text{RPS}, F'}} \text{Spec } \mathbf{k}(F')_{\text{RPS}}.$$

Let $R\Psi_{\text{RPS}, F'} = i_{\text{RPS}, F'}^* Rj_{\text{RPS}, F', *}.$ Let

$$\begin{array}{ccccc} \text{Spec } \mathbf{K}(F')_{\text{Et}} & \xrightarrow{j_{\text{RPS}, F'}} & \text{Spec } \mathbf{O}_K(F')_{\text{RPS}} & \xleftarrow{i_{\text{RPS}, F'}} & \text{Spec } \mathbf{k}(F')_{\text{RPS}} \\ g_K \downarrow & & g_{\mathcal{O}_K} \downarrow & & g_k \downarrow \\ \text{Spec } K_{\text{Et}} & \xrightarrow{j_{\text{RPS}}} & \text{Spec } \mathcal{O}_{K, \text{RPS}} & \xleftarrow{i_{\text{RPS}}} & \text{Spec } k_{\text{RPS}} \end{array}$$

be the premorphisms of sites defined by base change functors. Let $G'' \in \text{Ab}(\mathbf{k}(F')_{\text{RPS}})$ be either $R^q \Psi_{\text{RPS}, F'} \Lambda_n(r)$, $H^q i_{\text{RPS}, F'}^* \mathfrak{T}_n(r)$ or $H^q Ri_{\text{RPS}, F'}^! \mathfrak{T}_n(r)$, correspondingly. Then the restriction of G'' to $\mathbf{k}(F')_{\text{et}}$ is the restriction of G to $\mathbf{k}(F')_{\text{et}}$. Hence it is enough to show that $G' \times_k \mathbf{k}(F') \cong g_k^* G''$ is isomorphic to G'' . We have an isomorphism of functors

$$g_k^* i_{\text{RPS}}^* \cong i_{\text{RPS}, F'}^* Lg_{\mathcal{O}_K}^* (: D(\mathcal{O}_{K, \text{RPS}}) \rightarrow D(\mathbf{k}(F')_{\text{RPS}}))$$

and a natural morphism of functors

$$Lg_{\mathcal{O}_K}^* Rj_{\mathrm{RPS},*} \rightarrow Rj_{\mathrm{RPS},F',*} g_K^* (: D(K_{\mathrm{Et}}) \rightarrow D(\mathbf{O}_K(F')_{\mathrm{RPS}}))$$

by [Suz21, Proposition 2.6]. Combining them, we have a morphism of functors

$$g_K^* R\Psi_{\mathrm{RPS}} \rightarrow R\Psi_{\mathrm{RPS},F'} g_K^* (: D(K_{\mathrm{Et}}) \rightarrow D(\mathbf{k}(F')_{\mathrm{RPS}})).$$

This induces a morphism

$$g_K^* R\Psi_{\mathrm{RPS}} \Lambda_n(r) \rightarrow R\Psi_{\mathrm{RPS},F'} \Lambda_n(r). \quad (5.2)$$

For $n = 1$, the explicit calculations of the graded pieces of $R^q \Psi_{\mathrm{RPS}} \Lambda(r)$ (see [Suz24, (6.3.1), (6.3.2)]) when $\zeta_p \in K$ shows that the morphism

$$g_K^* R^q \Psi_{\mathrm{RPS}} \Lambda(r) \rightarrow R^q \Psi_{\mathrm{RPS},F'} \Lambda(r)$$

is an isomorphism (with $\zeta_p \in K$ or not) for all q . Hence (5.2) is an isomorphism for general n . This settles the case $G' = R^q \Psi_{\mathrm{RPS}} \Lambda(r)$ and $G'' = R^q \Psi_{\mathrm{RPS},F'} \Lambda(r)$. We have a commutative diagram

$$\begin{array}{ccc} g_K^* R^r \Psi_{\mathrm{RPS}} \Lambda_n(r) & \longrightarrow & g_K^* \nu_n(r-1) \\ \downarrow & & \downarrow \\ R^r \Psi_{\mathrm{RPS},F'} \Lambda_n(r) & \longrightarrow & \nu_n(r-1) \end{array}$$

in $\mathrm{Ab}(\mathbf{k}(F')_{\mathrm{RPS}})$ by the construction of the Kato boundary map. The vertical morphisms are isomorphisms. Hence they induce an isomorphism of triangles

$$\begin{array}{ccccc} g_K^* i_{\mathrm{RPS}}^* \mathfrak{T}_n(r) & \longrightarrow & g_K^* \tau_{\leq r} R\Psi_{\mathrm{RPS}} \Lambda_n(r) & \longrightarrow & g_K^* \nu_n(r-1) \\ \parallel & & \parallel & & \parallel \\ i_{\mathrm{RPS},F'}^* \mathfrak{T}_n(r) & \longrightarrow & \tau_{\leq r} R\Psi_{\mathrm{RPS},F'} \Lambda_n(r) & \longrightarrow & \nu_n(r-1). \end{array}$$

This settles the case $G' = H^q i_{\mathrm{RPS}}^* \mathfrak{T}_n(r)$ and $G'' = H^q i_{\mathrm{RPS},F'}^* \mathfrak{T}_n(r)$. This in turn induces an isomorphism of triangles

$$\begin{array}{ccccc} g_K^* i_{\mathrm{RPS}}^* \mathfrak{T}_n(r) & \longrightarrow & g_K^* R\Psi_{\mathrm{RPS}} \Lambda_n(r) & \longrightarrow & g_K^* Ri_{\mathrm{RPS}}^! \mathfrak{T}_n(r)[1] \\ \parallel & & \parallel & & \parallel \\ i_{\mathrm{RPS},F'}^* \mathfrak{T}_n(r) & \longrightarrow & R\Psi_{\mathrm{RPS},F'} \Lambda_n(r) & \longrightarrow & Ri_{\mathrm{RPS},F'}^! \mathfrak{T}_n(r)[1], \end{array}$$

settling the remaining case. □

PROPOSITION 5.6. *Let $q, r \in \mathbb{Z}$ and $n \geq 1$. Then the objects $R^q \pi_{\mathbf{K},*} \Lambda_n(r)$, $R^q \pi_{\mathbf{O}_K,*} \mathfrak{T}_n(r)$ and $R^{q+1} \pi_{\mathbf{O}_K,!} \mathfrak{T}_n(r)$ are in $\mathcal{W}_F$. They are zero unless $q = 0, 1, 2$.*

Proof. Let $G \in D(\mathbf{k}_{\mathrm{et}})$ be either $R\Psi \Lambda_n(r)$, $i^* \mathfrak{T}_n(r)$ or $Ri^! \mathfrak{T}_n(r)$. By Proposition 5.3, we have an exact sequence

$$0 \rightarrow R^1 \pi_{\mathbf{K},*} H^{q-1} G \rightarrow R^q \pi_{\mathbf{K},*} G \rightarrow \pi_{\mathbf{K},*} H^q G \rightarrow 0$$

in $\mathrm{Ab}(F_{\mathrm{et}}^{\mathrm{perar}})$. The first and third terms are in $\mathcal{W}_F$ by Proposition 5.5. Hence so is the second. The vanishing for $q \neq 0, 1, 2$ follows from [Suz24, Propositions 6.2.1 and 6.5.1]. □

PROPOSITION 5.7. *Let $r, q \in \mathbb{Z}$ and $n \geq 1$.*

(i) *The perfect pairing*

$$R\pi_{\mathbf{K},*} \Lambda_n(r) \otimes^L R\pi_{\mathbf{K},*} \Lambda_n(2-r) \rightarrow \Lambda_{\infty}[-2]$$

in [Suz24, Proposition 6.2.2] induces a Pontryagin duality

$$\pi_0(R^q \pi_{\mathbf{K},*} \Lambda_n(r)) \leftrightarrow \pi_0(R^{2-q} \pi_{\mathbf{K},*} \Lambda_n(2-r))$$

of finite étale group schemes over F and a Serre duality

$$(R^q \pi_{\mathbf{K},*} \Lambda_n(r))^0 \leftrightarrow (R^{3-q} \pi_{\mathbf{K},*} \Lambda_n(2-r))^0$$

of connected groups in $\mathcal{W}_F$.

(ii) The perfect pairing

$$R\pi_{\mathbf{O}_K,*} \mathfrak{T}_n(r) \otimes^L R\pi_{\mathbf{O}_K,!} \mathfrak{T}_n(2-r) \rightarrow \Lambda_\infty[-3]$$

in [Suz24, Proposition 6.5.1] induces a Pontryagin duality

$$\pi_0(R^q \pi_{\mathbf{O}_K,*} \mathfrak{T}_n(r)) \leftrightarrow \pi_0(R^{3-q} \pi_{\mathbf{O}_K,!} \mathfrak{T}_n(2-r))$$

of finite étale group schemes over F and a Serre duality

$$(R^q \pi_{\mathbf{O}_K,*} \mathfrak{T}_n(r))^0 \leftrightarrow (R^{4-q} \pi_{\mathbf{O}_K,!} \mathfrak{T}_n(r))^0$$

of connected groups in $\mathcal{W}_F$.

Proof. This follows from Propositions 2.8 and 5.6. □

PROPOSITION 5.8. *Let $n \geq 1$. Then*

- (i) $\pi_{\mathbf{O}_K,*} \Lambda_n \cong \Lambda_n$ is finite, $R^1 \pi_{\mathbf{O}_K,*} \Lambda_n$ is connected ind-algebraic, and $R^2 \pi_{\mathbf{O}_K,*} \Lambda_n = 0$;
- (ii) $\pi_{\mathbf{O}_K,*} \mathfrak{T}_n(1)$ is finite, and $R^2 \pi_{\mathbf{O}_K,*} \mathfrak{T}_n(1) = 0$;
- (iii) $\pi_{\mathbf{O}_K,*} \mathfrak{T}_n(2)$ is finite;
- (iv) $R^1 \pi_{\mathbf{O}_K,!} \mathfrak{T}_n(r) = 0$ for all r ;
- (v) $R^2 \pi_{\mathbf{O}_K,!} \mathfrak{T}_n(1)$ is finite;
- (vi) $R^2 \pi_{\mathbf{O}_K,!} \mathfrak{T}_n(2) = 0$, and $R^3 \pi_{\mathbf{O}_K,*} \mathfrak{T}_n(2)$ is pro-algebraic.

Proof. Items (i), (ii) and (iii) follow by applying étale sheafification to the descriptions of the Zariski sheaves in the proof of [Suz24, Proposition 6.5.2]. We have $\pi_{\mathbf{O}_K,*} \mathfrak{T}_n(r) \xrightarrow{\sim} \pi_{\mathbf{K},*} \Lambda_n(r)$ for all r . The morphisms $R^q \pi_{\mathbf{O}_K,*} \mathfrak{T}_n(r) \rightarrow R^q \pi_{\mathbf{K},*} \Lambda_n(r)$ are injective for all q and r by the injectivity stated in the last paragraph of the proof of [Suz24, Proposition 6.5.2]. These imply item (iv). The rest follow from Proposition 5.7 by duality. □

Recall from [Suz24, paragraph after Proposition 4.3.2] that we define

$$\begin{aligned} R\Gamma(\mathbf{K}, \cdot) &= \acute{a}R\pi_{\mathbf{K},*} : D(\mathbf{K}_{\text{et}}) \rightarrow D(F_{\text{proet}}^{\text{indrat}}), \\ R\Gamma(\mathbf{O}_K, \cdot) &= \acute{a}R\pi_{\mathbf{O}_K,*} : D(\mathbf{O}_{K,\text{et}}) \rightarrow D(F_{\text{proet}}^{\text{indrat}}), \\ R\Gamma_c(\mathbf{O}_K, \cdot) &= \acute{a}R\pi_{\mathbf{O}_K,!} : D(\mathbf{O}_{K,\text{et}}) \rightarrow D(F_{\text{proet}}^{\text{indrat}}) \end{aligned}$$

and $\mathbf{H}^q = H^q R\Gamma$, $\mathbf{H}_c^q = H^q R\Gamma_c$. If $G \in D^b(\mathbf{K}_{\text{et}})$ or $D^b(\mathbf{O}_{K,\text{et}})$ is concentrated in non-negative degrees and $R^q \pi_{\mathbf{K},*} G$ is in $\mathcal{W}_F$ for all q and is zero for large enough q , then we occasionally write $\Gamma(\mathbf{K}, G) = \mathbf{H}^0(\mathbf{K}, G)$ or $\Gamma(\mathbf{O}_K, G) = \mathbf{H}^0(\mathbf{O}_K, G)$, respectively.

PROPOSITION 5.9. *Let $q, r \in \mathbb{Z}$ and $n \geq 1$. Then*

- (i) $\mathbf{H}^q(\mathbf{K}, \Lambda_n(r))$, $\mathbf{H}^q(\mathbf{O}_K, \mathfrak{T}_n(r))$ and $\mathbf{H}_c^q(\mathbf{O}_K, \mathfrak{T}_n(r))$ are in $\mathcal{W}_F$ for all q and r ;
- (ii) $\Gamma(\mathbf{K}, \Lambda_n(r))$ is finite, and $\mathbf{H}^q(\mathbf{K}, \Lambda_n(r)) = 0$ for all $q \neq 0, 1, 2$ and all r ;

- (iii) $\Gamma(\mathbf{O}_K, \Lambda_n) \cong \Lambda_n$ is finite, $\mathbf{H}^1(\mathbf{O}_K, \Lambda_n)$ is connected ind-algebraic, and $\mathbf{H}^q(\mathbf{O}_K, \Lambda_n) = 0$ for all $q \neq 0, 1$;
- (iv) $\mathbf{H}_c^q(\mathbf{O}_K, \Lambda_n) = 0$ for all $q \neq 2, 3$;
- (v) $\Gamma(\mathbf{O}_K, \mathfrak{T}_n(1))$ is finite, and $\mathbf{H}^q(\mathbf{O}_K, \mathfrak{T}_n(1)) = 0$ for all $q \neq 0, 1$;
- (vi) $\mathbf{H}_c^2(\mathbf{O}_K, \mathfrak{T}_n(1))$ is finite, and $\mathbf{H}_c^q(\mathbf{O}_K, \mathfrak{T}_n(1)) = 0$ for all $q \neq 2, 3$;
- (vii) $\Gamma(\mathbf{O}_K, \mathfrak{T}_n(2))$ is finite, and $\mathbf{H}^q(\mathbf{O}_K, \mathfrak{T}_n(2)) = 0$ for all $q \neq 0, 1, 2$;
- (viii) $\mathbf{H}_c^3(\mathbf{O}_K, \mathfrak{T}_n(2))$ is pro-algebraic, and $\mathbf{H}_c^q(\mathbf{O}_K, \mathfrak{T}_n(2)) = 0$ for all $q \neq 3$.

Proof. This follows from Propositions 2.1, 5.6 and 5.8. □

PROPOSITION 5.10. Let $r, q \in \mathbb{Z}$ and $n \geq 1$.

(i) The perfect pairing

$$R\Gamma(\mathbf{K}, \Lambda_n(r)) \otimes^L R\Gamma(\mathbf{K}, \Lambda_n(2-r)) \rightarrow \Lambda_\infty[-2]$$

in [Suz24, Proposition 6.2.2] induces a Pontryagin duality

$$\pi_0(\mathbf{H}^q(\mathbf{K}, \Lambda_n(r))) \leftrightarrow \pi_0(\mathbf{H}^{2-q}(\mathbf{K}, \Lambda_n(2-r)))$$

of finite étale group schemes over F and a Serre duality

$$\mathbf{H}^q(\mathbf{K}, \Lambda_n(r))^0 \leftrightarrow \mathbf{H}^{3-q}(\mathbf{K}, \Lambda_n(2-r))^0$$

of connected groups in $\mathcal{W}_F$.

(ii) The perfect pairing

$$R\Gamma(\mathbf{O}_K, \mathfrak{T}_n(r)) \otimes^L R\Gamma_c(\mathbf{O}_K, \mathfrak{T}_n(2-r)) \rightarrow \Lambda_\infty[-3]$$

in [Suz24, Proposition 6.5.1] induces a Pontryagin duality

$$\pi_0(\mathbf{H}^q(\mathbf{O}_K, \mathfrak{T}_n(r))) \leftrightarrow \pi_0(\mathbf{H}_c^{3-q}(\mathbf{O}_K, \mathfrak{T}_n(2-r)))$$

of finite étale group schemes over F and a Serre duality

$$\mathbf{H}^q(\mathbf{O}_K, \mathfrak{T}_n(r))^0 \leftrightarrow \mathbf{H}_c^{4-q}(\mathbf{O}_K, \mathfrak{T}_n(2-r))^0$$

of connected groups in $\mathcal{W}_F$.

Proof. This follows from Proposition 5.7. □

Here is a summary of the above finiteness results:

$\mathbf{H}^q(\mathbf{K}, \Lambda_n(r))$	$\forall r$	$\mathbf{H}^q(\mathbf{O}_K, \mathfrak{T}_n(r))$	$r = 0$	$r = 1$	$r = 2$	
$q = 0$	finite	$q = 0$	Λ_n	finite	finite	
$q = 1$	general	$q = 1$	ind-alg	general	general	
$q = 2$	general	$q = 2$	0	0	general	
			$\mathbf{H}_c^q(\mathbf{O}_K, \mathfrak{T}_n(r))$	$r = 0$	$r = 1$	$r = 2$
			$q = 2$	general	finite	0
			$q = 3$	general	general	pro-alg

These groups are zero for (q, r) outside the ranges in the tables. By Proposition 5.10 (or by a direct calculation), we have

$$\pi_0 \mathbf{H}_c^3(\mathbf{O}_K, \mathfrak{T}_n(2)) \cong \Lambda_n. \quad (5.3)$$

6. Curves and their tubular neighborhoods

In this section, we show that the ind-pro-algebraic group structures on cohomology of smooth affine curves over F and their p -adic tubular neighborhoods constructed in [Suz24, Section 9] belong to $\mathcal{W}_F$.

Let $V = \operatorname{Spec} B$ be a smooth affine geometrically connected curve over F . Let Y be the smooth compactification of V , and set $T = Y \setminus V$. We briefly recall the constructions in [Suz24, Section 9.1]. For $F' \in F^{\text{perar}}$, we set $\mathbf{B}(F') = F' \otimes_F B$. The functor $\mathbf{B}$ defines a site $\operatorname{Spec} \mathbf{B}_{\text{et}}$ and a morphism of sites

$$\pi_{\mathbf{B}}: \operatorname{Spec} \mathbf{B}_{\text{et}} \rightarrow \operatorname{Spec} F_{\text{et}}^{\text{perar}}$$

by the method in Section 5. That is, $\operatorname{Spec} \mathbf{B}_{\text{et}}$ is the category of pairs (B', F') (where $F' \in F^{\text{perar}}$ and B' is an étale $\mathbf{B}(F')$ -algebra) endowed with the étale topology. For $x \in T$, let $\hat{\mathcal{O}}_{k_x}$ be the complete local ring of Y at x . Let $\hat{k}_x$ and F_x be its fraction field and residue field, respectively. As in Section 5, we have a functor

$$\hat{\mathbf{k}}_{x,0}(F'_x) = (F'_x \hat{\otimes}_{F_x} \hat{\mathcal{O}}_{k_x}) \otimes_{\hat{\mathcal{O}}_{k_x}} \hat{k}_x$$

in $F'_x \in F_x^{\text{perar}}$. Let $\hat{\mathbf{k}}_x$ be the Weil restriction $\mathfrak{R}_{F_x/F} \hat{\mathbf{k}}_{x,0}$, so

$$\hat{\mathbf{k}}_x(F') = \hat{\mathbf{k}}_{x,0}(F' \otimes_F F_x) = (F' \hat{\otimes}_F \hat{\mathcal{O}}_{k_x}) \otimes_{\hat{\mathcal{O}}_{k_x}} \hat{k}_x$$

for $F' \in F^{\text{perar}}$. By the method in Section 5, the functor $\hat{\mathbf{k}}_x$ defines a site $\operatorname{Spec} \hat{\mathbf{k}}_x$ and a morphism of sites

$$\pi_{\hat{\mathbf{k}}_x}: \operatorname{Spec} \hat{\mathbf{k}}_x \rightarrow \operatorname{Spec} F_{\text{et}}^{\text{perar}}.$$

Let

$$\pi_{\mathbf{k}_x/\mathbf{B}}: \operatorname{Spec} \hat{\mathbf{k}}_{x,\text{et}} \rightarrow \operatorname{Spec} \mathbf{B}_{\text{et}}$$

be the morphism defined by the functor $(B', F') \mapsto (B' \otimes_{\mathbf{B}(F')} \hat{\mathbf{k}}_x(F'), F')$ (where $F' \in F^{\text{perar}}$ and B' is an étale $\mathbf{B}(F')$ -algebra).

A B -module M can naturally be viewed as a sheaf $(B', F') \mapsto B' \otimes_B M$ on $\operatorname{Spec} \mathbf{B}_{\text{et}}$. A similar process exists for viewing a $\hat{k}_x$ -module as a sheaf on $\operatorname{Spec} \hat{\mathbf{k}}_{x,\text{et}}$. Hence the sheaf $\nu(1)$ (the dlog part of Ω^1) can also be viewed as a sheaf on $\operatorname{Spec} \mathbf{B}_{\text{et}}$ or $\operatorname{Spec} \hat{\mathbf{k}}_{x,\text{et}}$.

Let $G \in \operatorname{Ab}(\mathbf{B}_{\text{et}})$ and $G_x \in \operatorname{Ab}(\hat{\mathbf{k}}_x)$ for each $x \in T$ be sheaves and $G \rightarrow \pi_{\hat{\mathbf{k}}_x/\mathbf{B},*} G_x$ a morphism in $\operatorname{Ab}(\mathbf{B}_{\text{et}})$. The latter induces a morphism

$$R\pi_{\mathbf{B},*} G \rightarrow R\pi_{\hat{\mathbf{k}}_x,*} G_x$$

in $D(F_{\text{et}}^{\text{perar}})$. Then the object $R\pi_{\mathbf{B},!} G \in D(F_{\text{et}}^{\text{perar}})$ in [Suz24, Section 9.1] is a canonical object fitting in a distinguished triangle²

$$R\pi_{\mathbf{B},!} G \rightarrow R\pi_{\mathbf{B},*} G \rightarrow \bigoplus_{x \in T} R\pi_{\hat{\mathbf{k}}_x,*} G_x.$$

For $q \in \mathbb{Z}$, set $R^q \pi_{\mathbf{B},!} G = H^q R\pi_{\mathbf{B},!} G$. For $G = \Lambda$ or $\nu(1)$, we take G_x to be its natural counterpart over $\operatorname{Spec} \hat{\mathbf{k}}_x$. For $G = M$ a B -module, we take $G_x = M \otimes_B \hat{k}_x$. The morphism $G \rightarrow \pi_{\hat{\mathbf{k}}_x/\mathbf{B},*} G_x$ is the natural one.

²Actually it is denoted by $R\bar{\pi}_{\mathbf{B},!} G$ in [Suz24, Section 9.1]. The bar in $\bar{\pi}$ indicates that it is defined by “fibered sites.” Since we do not get into the details of fibered sites in this paper, we will not need or use this notation. We have $R\pi_{\mathbf{B},*} G \cong R\bar{\pi}_{\mathbf{B},*} G$ and $R\pi_{\hat{\mathbf{k}}_x,*} G_x \cong R\bar{\pi}_{\hat{\mathbf{k}}_x,*} G_x$ by construction.

PROPOSITION 6.1. *Let $q \in \mathbb{Z}$. Let G be either Λ , $\nu(1)$ or a finite projective B -module. Then $R^q \pi_{\mathbf{B},*} G \in \mathcal{W}_F$ is ind-algebraic and $R^{q+1} \pi_{\mathbf{B},!} G \in \mathcal{W}_F$ is pro-algebraic. They are zero unless $q = 0, 1$.*

Proof. This follows from [Suz24, Proposition 9.1.5] and its proof. $\square$

Next, let A be a ring. Assume all of the following:

- (i) The ring A is a two-dimensional regular integral domain.
- (ii) The ring A contains a primitive p th root of unity ζ_p .
- (iii) The radical I of the ideal (p) of A is principal.
- (iv) The quotient $B := A/I$ is a one-dimensional geometrically connected smooth algebra over F .
- (v) The pair (A, I) is complete.

Set $R = A[1/p]$. We apply the notation above for $V = \text{Spec } B$. We briefly recall the constructions in [Suz24, Section 9.2]. For $F' \in F^{\text{perar}}$, let $\hat{\mathbf{A}}(F')$ be the Kato canonical lifting of the relatively perfect B -algebra $\mathbf{B}(F')$ to A . Set $\hat{\mathbf{R}}(F') = \hat{\mathbf{A}}(F') \otimes_A R$. This functor defines a site $\text{Spec } \hat{\mathbf{R}}_{\text{et}}$ and a morphism of sites $\pi_{\hat{\mathbf{R}}}: \text{Spec } \hat{\mathbf{R}}_{\text{et}} \rightarrow \text{Spec } F_{\text{et}}^{\text{perar}}$ by the method in Section 5. For $x \in T$, let $\hat{\mathcal{O}}_{\eta_x}$ be the Kato canonical lifting of the relatively perfect B -algebra $\hat{k}_x$ to A . Let $\hat{K}_{\eta_x}$ be its fraction field, which is a complete discrete valuation field with residue field $\hat{k}_x$. Let $\hat{\mathbf{K}}_{\eta_x,0}(F'_x)$ be the functor in $F'_x \in F_x^{\text{perar}}$ defined in Section 5 for $\hat{K}_{\eta_x}$. Let $\hat{\mathbf{K}}_{\eta_x} = \mathfrak{R}_{F_x/F} \hat{\mathbf{K}}_{\eta_x,0}$. Then we have a site $\text{Spec } \hat{\mathbf{K}}_{\eta_x,\text{et}}$ and a morphism of sites $\pi_{\hat{\mathbf{K}}_{\eta_x}}: \text{Spec } \hat{\mathbf{K}}_{\eta_x,\text{et}} \rightarrow \text{Spec } F_{\text{et}}^{\text{perar}}$ by the method in Section 5. The object $R\pi_{\hat{\mathbf{K}}_{\eta_x},*} \Lambda \in D(F_{\text{et}}^{\text{perar}})$ is the Weil restriction $\mathfrak{R}_{F_x/F}$ of $R\pi_{\hat{\mathbf{K}}_{\eta_x,0},*} \Lambda \in D(F_{x,\text{et}}^{\text{perar}})$. We have a canonical morphism

$$R\pi_{\hat{\mathbf{R}},*} \Lambda \rightarrow R\pi_{\hat{\mathbf{K}}_{\eta_x},*} \Lambda$$

in $D(F_{\text{et}}^{\text{perar}})$. Then the object $R\pi_{\hat{\mathbf{R}},!} \Lambda \in D(F_{\text{et}}^{\text{perar}})$ in [Suz24, Section 9.2] is a canonical object fitting in a distinguished triangle

$$R\pi_{\hat{\mathbf{R}},!} \Lambda \rightarrow R\pi_{\hat{\mathbf{R}},*} \Lambda \rightarrow \bigoplus_{x \in T} R\pi_{\hat{\mathbf{K}}_{\eta_x},*} \Lambda.$$

For $q \in \mathbb{Z}$, set $R^q \pi_{\hat{\mathbf{R}},!} \Lambda = H^q R\pi_{\hat{\mathbf{R}},!} \Lambda$.

PROPOSITION 6.2. *Let $q \in \mathbb{Z}$. Then $R^q \pi_{\hat{\mathbf{R}},*} \Lambda \in \mathcal{W}_F$ is ind-algebraic and $R^{q+1} \pi_{\hat{\mathbf{R}},!} \Lambda \in \mathcal{W}_F$ is pro-algebraic. They are zero unless $q = 0, 1, 2$.*

Proof. By [Suz24, Section 9.3, Propositions 9.2.4 and 9.3.1–3], there are spectral sequences

$$E_2^{ij} = R^i \pi_{\mathbf{B},*} H^j \mathcal{E} \Rightarrow R^{i+j} \pi_{\hat{\mathbf{R}},*} \Lambda, \quad E_2^{ij} = R^i \pi_{\mathbf{B},!} H^j \mathcal{E} \Rightarrow R^{i+j} \pi_{\hat{\mathbf{R}},!} \Lambda,$$

where $H^j \mathcal{E} \in \text{Ab}(\mathbf{B}_{\text{et}})$ (and, implicitly, $H^j \mathcal{E}_x \in \text{Ab}(\hat{\mathbf{k}}_{x,\text{et}})$ and a morphism $H^j \mathcal{E} \rightarrow \pi_{\hat{\mathbf{k}}_x/\mathbf{B},*} H^j \mathcal{E}_x$) is a certain sheaf admitting a finite filtration with graded pieces isomorphic to Λ , $\nu(1)$ or a finite projective B -module (and this filtration is compatible at all $x \in T$). By [Suz24, Proposition 9.5.3], we have $R^3 \pi_{\hat{\mathbf{R}},*} \Lambda = 0$. Also, $R^3 \pi_{\hat{\mathbf{K}}_{\eta_x},*} \Lambda = 0$ by Proposition 5.6. With Proposition 5.3, the result follows by writing down the above (degenerate) spectral sequences and applying Propositions 6.1 and 5.6. $\square$

Set

$$R\Gamma(\hat{\mathbf{R}}, \Lambda) = \acute{a} R\pi_{\hat{\mathbf{R}},\Lambda} \Lambda \in D(F_{\text{proet}}^{\text{indrat}}), \quad R\Gamma_c(\hat{\mathbf{R}}, \Lambda) = \acute{a} R\pi_{\hat{\mathbf{R}},!} \Lambda \in D(F_{\text{proet}}^{\text{indrat}})$$

and $\mathbf{H}^q = H^q R\Gamma$, $\mathbf{H}_c^q = H^q R\Gamma_c$. The object $R\Gamma(\hat{\mathbf{K}}_{\eta_x}, \Lambda) = \acute{a}R\pi_{\hat{\mathbf{K}}_{\eta_x}} \Lambda$ of $D(F_{\text{proet}}^{\text{indrat}})$ is the Weil restriction $\mathfrak{R}_{F_x/F}$ of $R\Gamma(\hat{\mathbf{K}}_{\eta_x,0}, \Lambda) \in D(F_{x,\text{proet}}^{\text{indrat}})$.

PROPOSITION 6.3. *Let $q \in \mathbb{Z}$. Then $\mathbf{H}^q(\hat{\mathbf{R}}, \Lambda) \in \mathcal{W}_F$ is ind-algebraic and $\mathbf{H}_c^{q+1}(\hat{\mathbf{R}}, \Lambda) \in \mathcal{W}_F$ is pro-algebraic. They are zero unless $q = 0, 1, 2$.*

Proof. This follows from Propositions 2.1 and 6.2. $\square$

7. Two-dimensional local rings: Preliminaries

For the rest of the paper, let A be a two-dimensional normal noetherian complete local ring of mixed characteristic with residue field F .

In this section, we first show that the ind-pro-algebraic group structures on cohomology of punctured spectra of A constructed in [Suz24] belong to $\mathcal{W}_F$ if A is “resolved enough.” We then show a slightly weaker property for a general A as preparation for the next section. This is done by taking a resolution of singularities of A and combining the statements already proved for tubular neighborhoods of the smooth part of the exceptional divisor (see Section 6) and the local rings of the resolution at closed points.

Let K be the fraction field of A . Let $X = \text{Spec } A \setminus \{\mathfrak{m}\}$, where $\mathfrak{m}$ is the maximal ideal of A . Let P be the set of height one primes ideals of A . For $\mathfrak{p} \in P$, let $\hat{A}_{\mathfrak{p}}$ be the complete local ring of A at $\mathfrak{p}$ and $\hat{\mathbf{K}}_{\mathfrak{p}}$ and $\kappa(\mathfrak{p})$ its fraction field and residue field, respectively. Let $F_{\mathfrak{p}}$ be the residue field of $\kappa(\mathfrak{p})$. Let $S \subset P$ be a finite subset. Set $U_S = X \setminus S$.

We briefly recall the constructions in [Suz24, Section 10.1]. The ring A has a canonical $W(F)$ -algebra structure [DG70, Section V.4, Theorem 2.1]. For $F' \in F^{\text{perar}}$, let

$$\mathbf{A}(F') = W(F') \hat{\otimes}_{W(F)} A = \varprojlim_n (W(F') \otimes_{W(F)} A/\mathfrak{m}^n),$$

$$\mathbf{U}_S(F') = U_S \times_{\text{Spec } A} \text{Spec } \mathbf{A}(F').$$

Let $\mathbf{U}_{S,\text{et}}$ be the category of pairs (U', F') (where $F' \in F^{\text{perar}}$ and U' is an étale $\mathbf{U}_S(F')$ -scheme) endowed with the étale topology. The functor $F' \mapsto (\mathbf{U}_S(F'), F')$ defines a morphism of sites

$$\pi_{\mathbf{U}_S}: \mathbf{U}_{S,\text{et}} \rightarrow \text{Spec } F_{\text{et}}^{\text{perar}}.$$

Let $\lambda_S: \mathbf{U}_{S,\text{et}} \rightarrow \mathbf{X}_{\text{et}}$ be the morphism defined by $(X', F') \mapsto (X' \times_{\mathbf{X}(F')} \mathbf{U}_S(F'), F')$. Let $R\pi_{\mathbf{U}_S,!} = R\pi_{\mathbf{X},*} \lambda_{S,!}$. Define

$$R\Gamma(\mathbf{U}_S, \cdot) = \acute{a}R\pi_{\mathbf{U}_S,*}: D(\mathbf{U}_{S,\text{et}}) \rightarrow D(F_{\text{proet}}^{\text{indrat}}),$$

$$R\Gamma_c(\mathbf{U}_S, \cdot) = \acute{a}R\pi_{\mathbf{U}_S,!}: D(\mathbf{U}_{S,\text{et}}) \rightarrow D(F_{\text{proet}}^{\text{indrat}})$$

and $\mathbf{H}^q = H^q R\Gamma$, $\mathbf{H}_c^q = H^q R\Gamma_c$. If $G \in D^b(\mathbf{U}_{S,\text{et}})$ is concentrated in non-negative degrees and $R^q\pi_{\mathbf{U}_S,*}G$ is in $\mathcal{W}_F$ for all q and is zero for large enough q , then we occasionally write $\Gamma(\mathbf{U}_S, G) = \mathbf{H}^0(\mathbf{U}_S, G)$. If $U_S = \text{Spec } R$ is affine, then $\pi_{\mathbf{U}_S}$, $R\Gamma(\mathbf{U}_S, \cdot)$ and $R\Gamma_c(\mathbf{U}_S, \cdot)$ are also denoted by $\pi_{\mathbf{R}}$, $R\Gamma(\mathbf{R}, \cdot)$ and $R\Gamma_c(\mathbf{R}, \cdot)$, respectively.

For $\mathfrak{p} \in P$ and $F' \in F^{\text{perar}}$, let $\hat{\mathbf{A}}_{\mathfrak{p}}(F')$ be the ring $\mathbf{A}(F') \otimes_A \hat{A}_{\mathfrak{p}}$ completed with respect to the ideal $\mathbf{A}(F') \otimes_A \mathfrak{p}\hat{A}_{\mathfrak{p}}$. Let

$$\kappa(\mathfrak{p})(F') = \hat{\mathbf{A}}_{\mathfrak{p}}(F') \otimes_{\hat{A}_{\mathfrak{p}}} \kappa(\mathfrak{p}), \quad \hat{\mathbf{K}}_{\mathfrak{p}}(F') = \hat{\mathbf{A}}_{\mathfrak{p}}(F') \otimes_{\hat{A}_{\mathfrak{p}}} \hat{K}_{\mathfrak{p}}.$$

If $\mathfrak{p}$ divides p , then these functors are the Weil restrictions $\mathfrak{R}_{F_{\mathfrak{p}}/F}$ of the functors defined in

Section 5 for $\hat{K}_{\mathfrak{p}}$. We have sites and morphisms of sites

$$\begin{array}{ccccc} \mathrm{Spec} \hat{K}_{\mathfrak{p},\mathrm{et}} & \xrightarrow{\lambda_{\mathfrak{p}}} & \mathrm{Spec} \hat{A}_{\mathfrak{p},\mathrm{et}} & \xleftarrow{i_{\mathfrak{p}}} & \mathrm{Spec} \kappa(\mathfrak{p})_{\mathrm{et}} \\ \pi_{\hat{K}_{\mathfrak{p}}} \downarrow & & \pi_{\hat{A}_{\mathfrak{p}}} \downarrow & & \pi_{\kappa(\mathfrak{p})} \downarrow \\ \mathrm{Spec} F_{\mathrm{et}}^{\mathrm{perar}} & \xlongequal{\quad} & \mathrm{Spec} F_{\mathrm{et}}^{\mathrm{perar}} & \xlongequal{\quad} & \mathrm{Spec} F_{\mathrm{et}}^{\mathrm{perar}} \end{array}$$

as in Section 5. Let $R\Gamma(\hat{K}_{\mathfrak{p}}, \cdot) = \acute{a}R\pi_{\hat{K}_{\mathfrak{p}},*}$ and so on.

For any $n \geq 1$, $r \geq 0$ and $\mathfrak{p} \in P$ dividing p , we have the sheaf $\nu_n(r)$ on $\mathrm{Spec} \kappa(\mathfrak{p})_{\mathrm{et}}$. Setting S to be the set of primes dividing p , we have a morphism

$$\tau_{\leq r} R\lambda_{S,*} \Lambda_n(r) \rightarrow i_{\mathfrak{p},*} \nu_n(r-1)$$

in $D(\mathbf{X}_{\mathrm{et}})$ by the restriction of the morphism (5.1) (where we set $\nu_n(-1) = 0$). Let $\mathfrak{T}_n(r) \in D(\mathbf{X}_{\mathrm{et}})$ to be the canonical mapping cone of the resulting morphism

$$\tau_{\leq r} R\lambda_{S,*} \Lambda_n(r) \rightarrow \bigoplus_{\mathfrak{p} \in S} i_{\mathfrak{p},*} \nu_n(r-1).$$

For $r < 0$, we set $\mathfrak{T}_n(r) = \lambda_{S,!} \Lambda_n(r)$. For a general $S \subset P$ and $r \in \mathbb{Z}$, we obtain an object $\mathfrak{T}_n(r) \in D(\mathbf{U}_{S,\mathrm{et}})$ by the restriction of $\mathfrak{T}_n(r) \in D(\mathbf{X}_{\mathrm{et}})$.

PROPOSITION 7.1. *Assume all of the following:*

- (i) *The ring A is regular.*
- (ii) *The ring A contains a fixed primitive p th root of unity ζ_p .*
- (iii) *Either of the following holds:*
 - (a) *The ideal (p) is divisible by exactly one prime ideal $\mathfrak{p}$ and $A/\mathfrak{p}$ is regular.*
 - (b) *The ideal (p) is divisible by exactly two prime ideals $\mathfrak{p}_{\alpha}$ and $\mathfrak{p}_{\beta}$, and $\mathfrak{p}_{\alpha} + \mathfrak{p}_{\beta} = \mathfrak{m}$, and both $A/\mathfrak{p}_{\alpha}$ and $A/\mathfrak{p}_{\beta}$ are regular.*

Set $R = A[1/p]$ and write $\mathrm{Spec} R = U_S$ (so S is $\{\mathfrak{p}\}$ in case (a) and $\{\mathfrak{p}_{\alpha}, \mathfrak{p}_{\beta}\}$ in case (b)). Let $q \in \mathbb{Z}$.

Then $R^q \pi_{\mathbf{R},*} \Lambda \in \mathcal{W}_F$ is pro-algebraic and $R^{q+1} \pi_{\mathbf{R},!} \Lambda \in \mathcal{W}_F$ is ind-algebraic. They are zero unless $q = 0, 1, 2$. We have $\pi_{\mathbf{R},*} \Lambda \cong \Lambda$.

Proof. The statement for $R^q \pi_{\mathbf{R},*} \Lambda$ follows from [Suz24, Proposition 10.3.5]. We have a perfect pairing

$$R\pi_{\mathbf{R},*} \Lambda \otimes^L R\pi_{\mathbf{R},!} \Lambda \rightarrow \Lambda_{\infty}[-3]$$

in $D(F_{\mathrm{et}}^{\mathrm{perar}})$ by [Suz24, Proposition 10.1.5]. Hence Proposition 2.8 implies the statement for $R^{q+1} \pi_{\mathbf{R},!} \Lambda$. $\square$

PROPOSITION 7.2. *Under the assumptions of Proposition 7.1, let $q \in \mathbb{Z}$. Then $\mathbf{H}^q(\mathbf{R}, \Lambda) \in \mathcal{W}_F$ is pro-algebraic and $\mathbf{H}_c^{q+1}(\mathbf{R}, \Lambda) \in \mathcal{W}_F$ is ind-algebraic. They are zero unless $q = 0, 1, 2$. We have $\mathbf{\Gamma}(\mathbf{R}, \Lambda) \cong \Lambda$.*

Proof. This follows from Propositions 2.1 and 7.1. $\square$

Let $\pi_{\mathfrak{X}/A}: \mathfrak{X} \rightarrow \mathrm{Spec} A$ be a resolution of singularities such that $Y \cup \bar{S} \subset \mathfrak{X}$ is supported on a strict normal crossing divisor, where Y is the reduced part of $\mathfrak{X} \times_A F$ and $\bar{S}$ is the (reduced) closure of S in $\mathfrak{X}$. We briefly recall the constructions in [Suz24, Sections 10.4 and 10.5].

Let Y_0 (respectively, Y_1) be the set of closed (respectively, generic) points of Y . For $\eta \in Y_1$, let Y_η be the closure of η in Y , and set $\eta_0 = Y_\eta \cap Y_0$. Let F_η be the constant field of Y_η . Let $\hat{\mathcal{O}}_{K_\eta}$ be the complete local ring of $\mathfrak{X}$ at η and K_η its fraction field. For $x \in Y_0$, let $\hat{A}_x$ be the complete local ring of $\mathfrak{X}$ at x with residue field F_x , and let Y_1^x be the set of height one primes of $\hat{A}_x$ lying over some element of Y_1 (via the morphism $\text{Spec } \hat{A}_x \rightarrow \mathfrak{X}$). For $\eta \in Y_1$ and $x \in \eta_0$, there is a unique $\eta_x \in Y_1^x$ lying over η . Let $\hat{K}_{\eta_x}$ be the complete local field of $\hat{A}_x$ at η_x , $\hat{\mathcal{O}}_{K_{\eta_x}}$ its ring of integers and $\hat{\kappa}(\eta_x)$ its residue field. For each $x \in Y_0$, let $\hat{B}_x = \mathcal{O}(\text{Spec } \hat{A}_x \times_{\mathfrak{X}} Y)$, $\hat{R}_x = \mathcal{O}(\text{Spec } \hat{A}_x \times_{\mathfrak{X}} X)$ and $\hat{R}_{x,S} = \mathcal{O}(\text{Spec } \hat{A}_x \times_{\mathfrak{X}} U_S)$.

We will take completed neighborhoods of small enough affine opens of irreducible components of Y . For each $\eta \in Y_1$, choose an affine open neighborhood $W_\eta \subset \mathfrak{X}$ of η small enough so that

- $W_\eta \cap Y_{\eta'} = \emptyset$ for any $\eta' \in Y_1 \setminus \{\eta\}$,
- W_η does not contain the specialization of any element of S in Y and
- $W_\eta \cap Y \subset W_\eta$ is a principal divisor.

Set $T = Y \setminus \bigcup_{\eta \in Y_1} (W_\eta \cap Y)$ and $B_{\eta,T} = \mathcal{O}(W_\eta \cap Y)$. For each $\eta \in Y_1$, define $\hat{A}_{\eta,T}$ to be the completion of $\mathcal{O}(W_\eta)$ with respect to the ideal $\mathfrak{m}\mathcal{O}(W_\eta)$. Write $\text{Spec } \hat{R}_{\eta,T} = \text{Spec } \hat{A}_{\eta,T} \setminus \text{Spec } B_{\eta,T}$. For any $\eta \in Y_1$ and $x \in \eta_0$, we have a canonical A -algebra homomorphism $\hat{A}_{\eta,T} \rightarrow \hat{\mathcal{O}}_{K_{\eta_x}}$ inducing the natural inclusion map $B_{\eta,T} \hookrightarrow \hat{\kappa}(\eta_x)$ on the quotients.

We can apply the construction in this section for $\hat{A}_x$ and the construction in Section 6 for $\hat{A}_{\eta,T}$. With Weil restrictions $\mathfrak{R}_{F_x/F}$ and $\mathfrak{R}_{F_\eta/F}$, we have objects $R\pi_{\hat{\mathbf{R}}_{x,S},*}\Lambda \in D(F_{\text{et}}^{\text{perar}})$, $R\Gamma_c(\hat{\mathbf{R}}_{\eta,T}, \Lambda) \in D(F_{\text{proet}}^{\text{indrat}})$ and so on.

PROPOSITION 7.3. *Assume that S is the set of height one primes dividing p and that $\zeta_p \in A$. Let $q \in \mathbb{Z}$. Then $\mathbf{H}^q(\mathbf{U}_S, \Lambda) \in \text{PAlg}_u/F$ and $\mathbf{H}_c^{q+1}(\mathbf{U}_S, \Lambda) \in \text{IAlg}_u/F$. They are zero unless $q = 0, 1, 2, 3$ and $q = 0, 1, 2$, respectively. We have $\Gamma(\mathbf{U}_S, \Lambda) \cong \Lambda$.*

Proof. We have distinguished triangles

$$\begin{aligned} \bigoplus_{\eta \in Y_1} R\Gamma_c(\hat{\mathbf{R}}_{\eta,T}, \Lambda) &\rightarrow R\Gamma(\mathbf{U}_S, \Lambda) \rightarrow \bigoplus_{x \in T} R\Gamma(\hat{\mathbf{R}}_{x,S}, \Lambda), \\ \bigoplus_{x \in T} R\Gamma_c(\hat{\mathbf{R}}_{x,S}, \Lambda) &\rightarrow R\Gamma_c(\mathbf{U}_S, \Lambda) \rightarrow \bigoplus_{\eta \in Y_1} R\Gamma(\hat{\mathbf{R}}_{\eta,T}, \Lambda) \end{aligned}$$

in $D(F_{\text{proet}}^{\text{indrat}})$ by the diagram in the proof of [Suz24, Proposition 10.5.5]. (The objects $R\bar{\pi}_{\hat{\mathbf{U}}_{\hat{T},*}}\Lambda$ and $R\bar{\pi}_{\hat{\mathbf{U}}_{\hat{T},!}}\Lambda$ in the proof of [Suz24, Proposition 10.5.5] are isomorphic to $R\pi_{\mathbf{U}_S,*}\Lambda$ and $R\bar{\pi}_{\mathbf{U}_S,!}\Lambda$ ($= R\pi_{\mathbf{U}_S,!}\Lambda$ in our notation), respectively, by [Suz24, Propositions 10.4.1 and 10.5.1]. After applying $\acute{a}$, they become $R\Gamma(\mathbf{U}_S, \Lambda)$ and $R\Gamma_c(\mathbf{U}_S, \Lambda)$, respectively.) Hence Propositions 6.3 and 7.2 imply the result. $\square$

PROPOSITION 7.4. *Assume that S contains all height one primes dividing p . Let $n \geq 1$ and $r \in \mathbb{Z}$. Then $\mathbf{H}^q(\mathbf{U}_S, \Lambda_n(r)) \in \text{PAlg}_u/F$ and $\mathbf{H}_c^{q+1}(\mathbf{U}_S, \Lambda_n(r)) \in \text{IAlg}_u/F$. They are zero unless $q = 0, 1, 2, 3$ and $q = 0, 1, 2$, respectively. The group $\Gamma(\mathbf{U}_S, \Lambda_n(r))$ is finite.*

Proof. The case $n \geq 1$ is reduced to the case $n = 1$. Let $S' \subset P$ be a finite subset containing S . Let $U_{S'} = X \setminus S'$. Then the statement for U_S is equivalent to the statement for $U_{S'}$. Indeed, we

have distinguished triangles

$$\bigoplus_{\mathfrak{p} \in S' \setminus S} R\Gamma_c(\hat{\mathbf{A}}_{\mathfrak{p}}, \Lambda(r)) \rightarrow R\Gamma(\mathbf{U}_S, \Lambda(r)) \rightarrow R\Gamma(\mathbf{U}_{S'}, \Lambda(r)),$$

$$R\Gamma_c(\mathbf{U}_{S'}, \Lambda(r)) \rightarrow R\Gamma_c(\mathbf{U}_S, \Lambda(r)) \rightarrow \bigoplus_{\mathfrak{p} \in S' \setminus S} R\Gamma(\hat{\mathbf{A}}_{\mathfrak{p}}, \Lambda(r))$$

in $D(F_{\text{proet}}^{\text{indrat}})$; see the diagram in [Suz24, Section 10.6]. For any $\mathfrak{p} \in P$ not dividing p , we have

$$R\Gamma_c(\hat{\mathbf{A}}_{\mathfrak{p}}, \Lambda(r)) \cong R\Gamma(\kappa(\mathfrak{p}), \Lambda(r-1))[-2],$$

$$R\Gamma(\hat{\mathbf{A}}_{\mathfrak{p}}, \Lambda(r)) \cong R\Gamma(\kappa(\mathfrak{p}), \Lambda(r)),$$

which are objects of $D^b(\text{Alg}_u/F)$ concentrated in degrees 2, 3, and in degrees 0, 1, respectively, by [Suz24, Theorem 7.2.4]. This implies the equivalence. Now a standard dévissage argument [Suz24, Section 10.6] reduces the general case to Proposition 7.3. $\square$

We will later show in Proposition 8.4 that the groups in Proposition 7.4 are actually in $\mathcal{W}_F$.

8. Finiteness and duality

We keep the notation of Section 7. In this section, we prove Theorems 1.2 and 1.4. We first treat the case $S = \emptyset$ (where $U = U_S$) since this case admits stronger finiteness properties than the general case and hence is easier. Based on it, we then treat the case where $S \subset P$ contains all prime ideals dividing p . Finally, we treat the general case.

We first recall the perfect pairing

$$R\Gamma(\mathbf{U}_S, \mathfrak{T}_n(r)) \otimes^L R\Gamma_c(\mathbf{U}_S, \mathfrak{T}_n(2-r)) \rightarrow \Lambda_{\infty}[-3] \quad (8.1)$$

in $D(F_{\text{proet}}^{\text{indrat}})$ from [Suz24, Theorem 10.6.1].

PROPOSITION 8.1. *Let $n \geq 1$. Then $\Gamma(\mathbf{X}, \Lambda_n) \cong \Lambda_n$ and $\mathbf{H}^1(\mathbf{X}, \Lambda_n)$ are finite, $\mathbf{H}^2(\mathbf{X}, \Lambda_n)$ and $\mathbf{H}^3(\mathbf{X}, \Lambda_n)$ are ind-algebraic in $\mathcal{W}_F$ and $\mathbf{H}^q(\mathbf{X}, \Lambda_n) = 0$ for $q \neq 0, 1, 2, 3$.*

Proof. Let S be the set of height one primes dividing p . By Proposition 5.9(iv), we have $\Gamma(\mathbf{X}, \Lambda_n) \simeq \Gamma(\mathbf{U}_S, \Lambda_n) \cong \Lambda_n$. Consider the distinguished triangle

$$R\Gamma_c(\mathbf{U}_S, \Lambda_n) \rightarrow R\Gamma(\mathbf{X}, \Lambda_n) \rightarrow \bigoplus_{\mathfrak{p} \in S} R\Gamma(\hat{\mathbf{A}}_{\mathfrak{p}}, \Lambda_n)$$

in $D(F_{\text{proet}}^{\text{indrat}})$. The first and third terms are in $D^b(\text{IAlg}_u/F)$ by Propositions 7.4 and 5.9(iii), respectively. Hence so is the second. The same propositions show $\mathbf{H}^q(\mathbf{X}, \Lambda_n) = 0$ for $q \neq 0, 1, 2, 3$. In the distinguished triangle

$$R\Gamma(\mathbf{X}, \Lambda_n) \rightarrow R\Gamma(\mathbf{U}_S, \Lambda_n) \rightarrow \bigoplus_{\mathfrak{p} \in S} R\Gamma_c(\hat{\mathbf{A}}_{\mathfrak{p}}, \Lambda_n)[1], \quad (8.2)$$

the second term is in $D^b(\text{PAlg}_u/F)$ by Proposition 7.4, and the cohomologies of the third term are in $\mathcal{W}_F$ by Proposition 5.9(i). Therefore, we may apply Proposition 2.7 to the long exact sequence associated with (8.2). This implies that $\mathbf{H}^q(\mathbf{X}, \Lambda_n) \in \mathcal{W}_F$ for all q . Since $\mathbf{H}_c^1(\hat{\mathbf{A}}_{\mathfrak{p}}, \Lambda_n) = 0$, we have an exact sequence

$$0 \rightarrow \mathbf{H}^1(\mathbf{X}, \Lambda_n) \rightarrow \mathbf{H}^1(\mathbf{U}_S, \Lambda_n) \rightarrow \bigoplus_{\mathfrak{p} \in S} \mathbf{H}_c^2(\hat{\mathbf{A}}_{\mathfrak{p}}, \Lambda_n).$$

All the terms are in $\mathcal{W}_F$, with the first ind-algebraic and the second pro-algebraic. Therefore, $\mathbf{H}^1(\mathbf{X}, \Lambda_n) \in \text{Alg}_u/F$. By [Suz23, Theorem 1.2], we know that the group $\mathbf{H}^1(\mathbf{X}, \Lambda_n)(\overline{F}) \cong H^1(\mathbf{X}(\overline{F}), \Lambda_n)$ is finite. This implies that $\mathbf{H}^1(\mathbf{X}, \Lambda_n)$ is finite. $\square$

PROPOSITION 8.2. *Let $n \geq 1$. Then $\Gamma(\mathbf{X}, \mathfrak{T}_n(2))$ and $\mathbf{H}^3(\mathbf{X}, \mathfrak{T}_n(2)) \cong \Lambda_n$ are finite, $\mathbf{H}^1(\mathbf{X}, \mathfrak{T}_n(2))$ and $\mathbf{H}^2(\mathbf{X}, \mathfrak{T}_n(2))$ are pro-algebraic in $\mathcal{W}_F$ and $\mathbf{H}^q(\mathbf{X}, \Lambda_n) = 0$ for $q \neq 0, 1, 2, 3$.*

Proof. This follows from Propositions 2.8 and 8.1 by the perfect pairing (8.1) with $S = \emptyset$ and $r = 0$. $\square$

PROPOSITION 8.3. *Let $n \geq 1$. Then $\Gamma(\mathbf{X}, \mathfrak{T}_n(1))$ is finite, $\mathbf{H}^1(\mathbf{X}, \mathfrak{T}_n(1))$ is pro-algebraic in $\mathcal{W}_F$, $\mathbf{H}^2(\mathbf{X}, \mathfrak{T}_n(1))$ is in Alg_u/F , $\mathbf{H}^3(\mathbf{X}, \mathfrak{T}_n(1))$ is ind-algebraic in $\mathcal{W}_F$ and $\mathbf{H}^q(\mathbf{X}, \mathfrak{T}_n(1)) = 0$ for $q \neq 0, 1, 2, 3$.*

Proof. The statement for $\Gamma(\mathbf{X}, \mathfrak{T}_n(1))$ is obvious. Let S be the set of primes dividing p . Consider the distinguished triangle

$$\bigoplus_{\mathfrak{p} \in S} R\Gamma_c(\hat{\mathbf{A}}_{\mathfrak{p}}, \mathfrak{T}_n(1)) \rightarrow R\Gamma(\mathbf{X}, \mathfrak{T}_n(1)) \rightarrow R\Gamma(\mathbf{U}_S, \Lambda_n(1)) \quad (8.3)$$

in $D(F_{\text{proet}}^{\text{indrat}})$. By Propositions 5.9 and 7.4, we know that $\mathbf{H}^q(\mathbf{X}, \mathfrak{T}_n(1)) \in \text{PAlg}_u/F$ for $q = 1, 2$ and that $\mathbf{H}^q(\mathbf{X}, \mathfrak{T}_n(1))$ is zero for $q \neq 0, 1, 2, 3$. Consider the distinguished triangle

$$R\Gamma_c(\mathbf{U}_S, \Lambda_n(1)) \rightarrow R\Gamma(\mathbf{X}, \mathfrak{T}_n(1)) \rightarrow \bigoplus_{\mathfrak{p} \in S} R\Gamma(\hat{\mathbf{A}}_{\mathfrak{p}}, \mathfrak{T}_n(1))$$

in $D(F_{\text{proet}}^{\text{indrat}})$. As $\mathbf{H}^2(\hat{\mathbf{A}}_{\mathfrak{p}}, \mathfrak{T}_n(1)) = 0$ by Propositions 5.9, this induces a distinguished triangle

$$\tau_{\leq 2} R\Gamma_c(\mathbf{U}_S, \Lambda_n(1)) \rightarrow \tau_{\leq 2} R\Gamma(\mathbf{X}, \mathfrak{T}_n(1)) \rightarrow \tau_{\leq 2} \bigoplus_{\mathfrak{p} \in S} R\Gamma(\hat{\mathbf{A}}_{\mathfrak{p}}, \mathfrak{T}_n(1)).$$

The first, second and third terms have cohomologies in IAlg_u/F , PAlg_u/F and $\mathcal{W}_F$, respectively. Applying Proposition 2.7, we know that $\mathbf{H}^1(\mathbf{X}, \mathfrak{T}_n(1)) \in \mathcal{W}_F$ and $\mathbf{H}^2(\mathbf{X}, \mathfrak{T}_n(1)) \in \text{Alg}_u/F$. Also, we have $\mathbf{H}_c^3(\mathbf{U}_S, \Lambda_n(1)) \xrightarrow{\sim} \mathbf{H}^3(\mathbf{X}, \mathfrak{T}_n(1))$, so $\mathbf{H}^3(\mathbf{X}, \mathfrak{T}_n(1))$ is in IAlg_u/F . The distinguished triangle (8.3) induces an exact sequence

$$\begin{aligned} \mathbf{H}^2(\mathbf{X}, \mathfrak{T}_n(1)) &\rightarrow \mathbf{H}^2(\mathbf{U}_S, \Lambda_n(1)) \rightarrow \bigoplus_{\mathfrak{p} \in S} \mathbf{H}_c^3(\hat{\mathbf{A}}_{\mathfrak{p}}, \mathfrak{T}_n(1)) \\ &\rightarrow \mathbf{H}^3(\mathbf{X}, \mathfrak{T}_n(1)) \rightarrow \mathbf{H}^3(\mathbf{U}_S, \Lambda_n(1)) \rightarrow 0, \end{aligned}$$

where the kernel of the first morphism is finite. These terms, from left to right, are in Alg_u/F , PAlg_u/F , $\mathcal{W}_F$, IAlg_u/F and PAlg_u/F . Applying Proposition 2.7, we see that $\mathbf{H}^3(\mathbf{X}, \mathfrak{T}_n(1)) \in \mathcal{W}_F$. $\square$

PROPOSITION 8.4. *Let $n \geq 1$ and $r \in \mathbb{Z}$. Assume that S contains all primes dividing p .*

- (i) *The group $\Gamma(\mathbf{U}_S, \Lambda_n(r))$ is finite, $\mathbf{H}^1(\mathbf{U}_S, \Lambda_n(r))$ and $\mathbf{H}^2(\mathbf{U}_S, \Lambda_n(r))$ are pro-algebraic in $\mathcal{W}_F$, and $\mathbf{H}^q(\mathbf{U}_S, \Lambda_n(r)) = 0$ for $q \neq 0, 1, 2$.*
- (ii) *The group $\mathbf{H}_c^1(\mathbf{U}_S, \Lambda_n(r))$ is finite, $\mathbf{H}_c^2(\mathbf{U}_S, \Lambda_n(r))$ and $\mathbf{H}_c^3(\mathbf{U}_S, \Lambda_n(r))$ are ind-algebraic in $\mathcal{W}_F$, and $\mathbf{H}_c^q(\mathbf{U}_S, \Lambda_n(r)) = 0$ for $q \neq 1, 2, 3$.*

Proof. That these groups are in $\mathcal{W}_F$ is already proved in the proofs of Propositions 8.1 and 8.3 if S is the set of all primes dividing p . The case of larger S can be reduced to this case. With

Proposition 7.4, what remains to show is $\mathbf{H}^3(\mathbf{U}_S, \Lambda_n(r)) = 0$. In the exact sequence

$$\bigoplus_{\mathfrak{p} \in S} \Gamma(\hat{\mathbf{A}}_{\mathfrak{p}}, \Lambda_n) \rightarrow \mathbf{H}_c^1(\mathbf{U}_S, \Lambda_n) \rightarrow \mathbf{H}^1(\mathbf{X}, \Lambda_n),$$

the first and third terms are finite étale, trivially and by Proposition 8.1, respectively. Therefore, $\mathbf{H}_c^1(\mathbf{U}_S, \Lambda_n)$ is finite étale. This implies that $\mathbf{H}_c^1(\mathbf{U}_S, \Lambda(r))$ is finite étale for all r (reduce to the case $\zeta_p \in A$) and hence that $\mathbf{H}_c^1(\mathbf{U}_S, \Lambda_n(r))$ is finite étale for all r and n . With this and by the perfect pairing (8.1) and Proposition 2.8, we know that $\mathbf{H}^3(\mathbf{U}_S, \Lambda_n(r)) = 0$. $\square$

Now we make no assumption on S .

PROPOSITION 8.5. *Let $n \geq 1$ and $r \leq 0$. Then $\Gamma(\mathbf{U}_S, \mathfrak{T}_n(r))$ is finite, $\mathbf{H}^1(\mathbf{U}_S, \mathfrak{T}_n(r))$ is pro-algebraic in $\mathcal{W}_F$, $\mathbf{H}^2(\mathbf{U}_S, \mathfrak{T}_n(r))$ is in $\mathcal{W}_F$, $\mathbf{H}^3(\mathbf{U}_S, \mathfrak{T}_n(r))$ is ind-algebraic in $\mathcal{W}_F$ and we have $\mathbf{H}^q(\mathbf{U}_S, \mathfrak{T}_n(r)) = 0$ for $q \neq 0, 1, 2, 3$.*

Proof. Let S_0 be the set of all primes dividing p . We may assume $S \subset S_0$. Let $U_0 = U_{S_0}$. The finiteness of $\Gamma(\mathbf{U}_S, \mathfrak{T}_n(r))$ is obvious. Denote the distinguished triangles

$$\begin{aligned} \bigoplus_{\mathfrak{p} \in S} R\Gamma_c(\hat{\mathbf{A}}_{\mathfrak{p}}, \mathfrak{T}_n(r)) &\rightarrow R\Gamma(\mathbf{X}, \mathfrak{T}_n(r)) \rightarrow R\Gamma(\mathbf{U}_S, \mathfrak{T}_n(r)), \\ \bigoplus_{\mathfrak{p} \in S_0 \setminus S} R\Gamma_c(\hat{\mathbf{A}}_{\mathfrak{p}}, \mathfrak{T}_n(r)) &\rightarrow R\Gamma(\mathbf{U}_S, \mathfrak{T}_n(r)) \rightarrow R\Gamma(\mathbf{U}_0, \mathfrak{T}_n(r)), \end{aligned}$$

by, respectively,

$$D \rightarrow C_X \rightarrow C_{U_S}, \quad E \rightarrow C_{U_S} \rightarrow C_{U_0}.$$

The morphism $H^1 C_{U_0} \rightarrow H^2 E$ is a morphism from a pro-algebraic group in $\mathcal{W}_F$ to an object of $\mathcal{W}_F$. Hence its kernel $H^1 C_{U_S}$ is pro-algebraic. Hence the morphism $H^1 C_{U_S} \rightarrow H^2 D$ is a morphism from a pro-algebraic group to an object of $\mathcal{W}_F$. Hence its image is pro-algebraic. The morphism $H^2 D \rightarrow H^2 C_X$ is a morphism from an object of $\mathcal{W}_F$ to an ind-algebraic group in $\mathcal{W}_F$. Hence its image is ind-algebraic. Therefore, $\text{Im}(H^1 C_{U_S} \rightarrow H^2 D), \text{Im}(H^2 D \rightarrow H^2 C_X) \in \mathcal{W}_F$ by Proposition 2.7. As $H^1 C_X$ is finite, we have $H^1 C_{U_S} \in \mathcal{W}_F$.

In the exact sequence

$$0 \rightarrow H^1 C_{U_S} \rightarrow H^1 C_{U_0} \rightarrow H^2 E,$$

the terms are in $\mathcal{W}_F$; the first and second terms are pro-algebraic. Hence $\text{Im}(H^1 C_{U_0} \rightarrow H^2 E)$ is pro-algebraic in $\mathcal{W}_F$ by Proposition 2.3. Hence $\text{Im}(H^2 E \rightarrow H^2 C_{U_S}) \in \mathcal{W}_F$ again by Proposition 2.3. As $H^2 C_X$ is ind-algebraic in $\mathcal{W}_F$, the object $\text{Im}(H^2 C_X \rightarrow H^2 C_{U_S})$ is ind-algebraic by Proposition 2.2.

The morphism $H^2 C_{U_0} \rightarrow H^3 E$ is a morphism from a pro-algebraic group in $\mathcal{W}_F$ to an object of $\mathcal{W}_F$. Hence $\text{Im}(H^2 C_{U_S} \rightarrow H^2 C_{U_0}) = \text{Ker}(H^2 C_{U_0} \rightarrow H^3 E)$ and $\text{Im}(H^2 C_{U_0} \rightarrow H^3 E)$ are pro-algebraic. Since the object $H^3 C_X$ is ind-algebraic in $\mathcal{W}_F$ and the morphism $H^3 C_X \rightarrow H^3 C_{U_S}$ is surjective, it follows that $H^3 C_{U_S}$ is ind-algebraic by Proposition 2.2. We have $H^3 E \in \mathcal{W}_F$. Applying Proposition 2.7 to the exact sequence

$$0 \rightarrow \text{Im}(H^2 C_{U_0} \rightarrow H^3 E) \rightarrow H^3 E \rightarrow H^3 C_{U_S} \rightarrow 0,$$

we know that the first and third terms in this sequence are in $\mathcal{W}_F$. Thus $H^3 C_{U_S}$ is ind-algebraic in $\mathcal{W}_F$.

The morphism $H^3 D \rightarrow H^3 C_X$ is a morphism from an object of $\mathcal{W}_F$ to an ind-algebraic group in $\mathcal{W}_F$. Hence we have $\text{Im}(H^3 D \rightarrow H^3 C_X) \in \mathcal{W}_F$ by Proposition 2.5. Hence the morphism

$H^3D \twoheadrightarrow \text{Im}(H^3D \rightarrow H^3C_X)$ is a morphism from an object of $\mathcal{W}_F$ to an ind-algebraic group in $\mathcal{W}_F$. Therefore, Proposition 2.6 implies that its kernel $\text{Im}(H^2C_{U_S} \rightarrow H^3D)$ has identity component in $\mathcal{W}_F$ and étale component group.

Consider the exact sequences

$$\begin{aligned} 0 \rightarrow \text{Im}(H^2C_X \rightarrow H^2C_{U_S}) \rightarrow H^2C_{U_S} \rightarrow \text{Im}(H^2C_{U_S} \rightarrow H^3D) \rightarrow 0, \\ 0 \rightarrow \text{Im}(H^2E \rightarrow H^2C_{U_S}) \rightarrow H^2C_{U_S} \rightarrow \text{Im}(H^2C_{U_S} \rightarrow H^2C_{U_0}) \rightarrow 0. \end{aligned}$$

With what has been proved so far, the first sequence shows that $\pi_0(H^2C_{U_S})$ is étale, and the second sequence shows that $\pi_0(H^2C_{U_S})$ is profinite. Thus $\pi_0(H^2C_{U_S})$ is finite, and $\text{Im}(H^2C_{U_S} \rightarrow H^3D) \in \mathcal{W}_F$. The image of the composite

$$\text{Im}(H^2C_X \rightarrow H^2C_{U_S}) \hookrightarrow H^2C_{U_S} \twoheadrightarrow \text{Im}(H^2C_{U_S} \rightarrow H^2C_{U_0})$$

is $\text{Im}(H^2C_X \rightarrow H^2C_{U_0})$, which is in Alg_u/F by what was shown about (8.2) in the proof of Proposition 8.1. Hence, with what has been proved so far, we know that the induced morphism

$$\text{Im}(H^2C_{U_S} \rightarrow H^3D) \twoheadrightarrow \frac{\text{Im}(H^2C_{U_S} \rightarrow H^2C_{U_0})}{\text{Im}(H^2C_X \rightarrow H^2C_{U_0})}$$

is a surjection from an object of $\mathcal{W}_F$ onto a pro-algebraic group. Therefore, Proposition 2.3 implies that $\text{Im}(H^2C_{U_S} \rightarrow H^2C_{U_0})/\text{Im}(H^2C_X \rightarrow H^2C_{U_0})$ is in $\mathcal{W}_F$. Hence $\text{Im}(H^2C_{U_S} \rightarrow H^2C_{U_0}) \in \mathcal{W}_F$. With $\text{Im}(H^2E \rightarrow H^2C_{U_S}) \in \mathcal{W}_F$, we know that $H^2C_{U_S} \in \mathcal{W}_F$. $\square$

PROPOSITION 8.6. *Let $n \geq 1$ and $r \geq 2$. Then $\mathbf{H}_c^0(\mathbf{U}_S, \mathfrak{T}_n(r))$ is finite, $\mathbf{H}_c^1(\mathbf{U}_S, \mathfrak{T}_n(r))$ is pro-algebraic in $\mathcal{W}_F$, $\mathbf{H}_c^2(\mathbf{U}_S, \mathfrak{T}_n(r))$ is in $\mathcal{W}_F$, $\mathbf{H}_c^3(\mathbf{U}_S, \mathfrak{T}_n(r))$ is ind-algebraic in $\mathcal{W}_F$, and we have $\mathbf{H}_c^q(\mathbf{U}_S, \mathfrak{T}_n(r)) = 0$ for $q \neq 0, 1, 2, 3$.*

Proof. This follows from Proposition 8.5 by duality. $\square$

PROPOSITION 8.7. *Let $n \geq 1$ and $r \leq 0$. Then $\mathbf{H}_c^0(\mathbf{U}_S, \mathfrak{T}_n(r))$ and $\mathbf{H}_c^1(\mathbf{U}_S, \mathfrak{T}_n(r))$ are finite, $\mathbf{H}_c^2(\mathbf{U}_S, \mathfrak{T}_n(r))$ and $\mathbf{H}_c^3(\mathbf{U}_S, \mathfrak{T}_n(r))$ are ind-algebraic in $\mathcal{W}_F$, and $\mathbf{H}^q(\mathbf{U}_S, \mathfrak{T}_n(r)) = 0$ for $q \neq 0, 1, 2, 3$.*

Proof. The case $r < 0$ follows from Proposition 8.4. For $r = 0$, the statement follows from the distinguished triangle

$$R\Gamma_c(\mathbf{U}_S, \Lambda_n) \rightarrow R\Gamma(\mathbf{X}, \Lambda_n) \rightarrow \bigoplus_{\mathfrak{p} \in S} R\Gamma(\hat{\mathbf{A}}_{\mathfrak{p}}, \Lambda_n)$$

and Propositions 5.9 and 8.1. $\square$

PROPOSITION 8.8. *Let $n \geq 1$ and $r \geq 2$. Then $\Gamma(\mathbf{U}_S, \mathfrak{T}_n(r))$ and $\mathbf{H}^3(\mathbf{U}_S, \mathfrak{T}_n(r))$ are finite, $\mathbf{H}^1(\mathbf{U}_S, \mathfrak{T}_n(r))$ and $\mathbf{H}^2(\mathbf{U}_S, \mathfrak{T}_n(r))$ are pro-algebraic in $\mathcal{W}_F$, and $\mathbf{H}^q(\mathbf{U}_S, \mathfrak{T}_n(r)) = 0$ for $q \neq 0, 1, 2, 3$.*

Proof. This follows from the distinguished triangle

$$\bigoplus_{\mathfrak{p} \in S} R\Gamma_c(\hat{\mathbf{A}}_{\mathfrak{p}}, \mathfrak{T}_n(r)) \rightarrow R\Gamma(\mathbf{X}, \mathfrak{T}_n(r)) \rightarrow R\Gamma(\mathbf{U}_S, \mathfrak{T}_n(r))$$

and Propositions 5.9 and 8.1 (or from Proposition 8.7 by duality). $\square$

PROPOSITION 8.9. *Let $n \geq 1$. Then $\Gamma(\mathbf{U}_S, \mathfrak{T}_n(1))$ is finite, $\mathbf{H}^1(\mathbf{U}_S, \mathfrak{T}_n(1))$ and $\mathbf{H}^2(\mathbf{U}_S, \mathfrak{T}_n(1))$ are pro-algebraic in $\mathcal{W}_F$, $\mathbf{H}^3(\mathbf{U}_S, \mathfrak{T}_n(1))$ is ind-algebraic in $\mathcal{W}_F$, and $\mathbf{H}^q(\mathbf{U}_S, \mathfrak{T}_n(1)) = 0$ for $q \neq 0, 1, 2, 3$.*

Proof. Let S_0 be the set of all primes dividing p . We may assume $S \subset S_0$. Let $U_0 = U_{S_0}$. The finiteness of $\Gamma(\mathbf{U}_S, \mathfrak{T}_n(1))$ is obvious. Denote the distinguished triangles

$$\bigoplus_{\mathfrak{p} \in S} R\Gamma_c(\hat{\mathbf{A}}_{\mathfrak{p}}, \mathfrak{T}_n(1)) \rightarrow R\Gamma(\mathbf{X}, \mathfrak{T}_n(1)) \rightarrow R\Gamma(\mathbf{U}_S, \mathfrak{T}_n(1)),$$

$$\bigoplus_{\mathfrak{p} \in S_0 \setminus S} R\Gamma_c(\hat{\mathbf{A}}_{\mathfrak{p}}, \mathfrak{T}_n(1)) \rightarrow R\Gamma(\mathbf{U}_S, \mathfrak{T}_n(1)) \rightarrow R\Gamma(\mathbf{U}_0, \mathfrak{T}_n(1)),$$

by, respectively,

$$D \rightarrow C_X \rightarrow C_{U_S}, \quad E \rightarrow C_{U_S} \rightarrow C_{U_0}.$$

In the exact sequence

$$0 \rightarrow H^1 C_X \rightarrow H^1 C_{U_S} \rightarrow H^2 D,$$

the first term is pro-algebraic in $\mathcal{W}_F$ and the third term is finite. Hence $H^1 C_{U_S}$ is pro-algebraic in $\mathcal{W}_F$. Since $H^3 C_X$ is ind-algebraic in $\mathcal{W}_F$, the surjection $H^3 C_X \twoheadrightarrow H^3 C_{U_S}$ and Proposition 2.2 imply that $H^3 C_{U_S} \in \text{IAlg}_u/F$. The morphism $H^2 C_{U_0} \rightarrow H^3 E$ is a morphism from a pro-algebraic group in $\mathcal{W}_F$ to an object of $\mathcal{W}_F$. Hence its image and kernel are pro-algebraic. Applying Proposition 2.7 to the exact sequence

$$0 \rightarrow \text{Im}(H^2 C_{U_0} \rightarrow H^3 E) \rightarrow H^3 E \rightarrow H^3 C_{U_S} \rightarrow 0,$$

we know that $H^3 C_{U_S} \in \mathcal{W}_F$. In the exact sequence

$$H^2 E \rightarrow H^2 C_{U_S} \rightarrow \text{Ker}(H^2 C_{U_0} \rightarrow H^3 E) \rightarrow 0,$$

the first term is finite and the third term is pro-algebraic. Hence $H^2 C_{U_S}$ is pro-algebraic. In the exact sequence

$$H^2 C_X \rightarrow H^2 C_{U_S} \rightarrow H^3 D \rightarrow H^3 C_X,$$

the first term is in Alg_u/F , the second in PAlg_u/F , the third in $\mathcal{W}_F$ and the fourth is ind-algebraic in $\mathcal{W}_F$. Applying Proposition 2.7, we know that $H^2 C_{U_S} \in \mathcal{W}_F$. $\square$

PROPOSITION 8.10. *Let $n \geq 1$. Then $\Gamma_c(\mathbf{U}_S, \mathfrak{T}_n(1))$ is finite, $\mathbf{H}_c^1(\mathbf{U}_S, \mathfrak{T}_n(1))$ is pro-algebraic in $\mathcal{W}_F$, $\mathbf{H}_c^2(\mathbf{U}_S, \mathfrak{T}_n(1))$ and $\mathbf{H}_c^3(\mathbf{U}_S, \mathfrak{T}_n(1))$ are ind-algebraic in $\mathcal{W}_F$, and $\mathbf{H}_c^q(\mathbf{U}_S, \mathfrak{T}_n(1)) = 0$ for $q \neq 0, 1, 2, 3$.*

Proof. This follows from Proposition 8.9 by duality. $\square$

Theorem 1.2 follows from the above results. With Proposition 2.8, we obtain Theorem 1.4.

9. Hasse principles

In this section, we prove Theorem 1.5.

DEFINITION 9.1. Assume $F = \overline{F}$.

- (i) Let $U \subset X$ be a dense open subscheme. Let $\mathfrak{p} \in P \cap U$. A finite étale covering U'/U is said to be *completely split at $\mathfrak{p}$* if the $\kappa(\mathfrak{p})$ -scheme $U' \times_U \kappa(\mathfrak{p})$ is isomorphic to a disjoint union of copies of $\text{Spec } \kappa(\mathfrak{p})$.
- (ii) A finite étale covering X'/X is said to be *completely split* if it is completely split at all elements of P .

(iii) For $n \geq 1$, define $H_{\text{cs}}^1(X, \Lambda_n)$ to be the subgroup of $H^1(X, \Lambda_n)$ consisting of completely split coverings.

PROPOSITION 9.2. Assume $F = \overline{F}$ and that A satisfies the assumptions and condition (iii)(a) of Proposition 7.1. Write $\mathfrak{p} = (\pi)$ and $\mathfrak{m} = (\pi, t)$. Let $U = \text{Spec } A[1/p]$. Let $n \geq 1$, and let $U' \in H^1(U, \Lambda_n)$. If U' is completely split at the primes $(t^m - \pi)$ for all $m \geq 1$, then it is trivial.

Proof. We may assume $n = 1$. Identifying Λ with $\Lambda(1)$ over U , we want to show that the kernel of the natural map

$$\varphi: A[1/p]^\times / A[1/p]^{\times p} \rightarrow \prod_{m \geq 1} \kappa(t^m - \pi)^\times / \kappa(t^m - \pi)^{\times p}$$

is zero. This kernel is contained in $G := (1 + \mathfrak{m}) / (1 + \mathfrak{m})^p$. Let e_A be the absolute ramification index of $\hat{A}_{\mathfrak{p}}$, and set $f_A = pe_A / (p - 1)$. For non-negative integers c, d not both zero, let $G^{c,d}$ be the image of $1 + \pi^c t^d A$ in G . Then $G^{c,d} = 0$ if $c \geq f_A$. Also, $G^{c,d} \subset G^{c+1,d} + G^{c,d+1}$ if $c < f_A$ and $p \mid \gcd(c, d)$. Hence it is enough to show that $G^{c,d} \cap \text{Ker}(\varphi) \subset G^{c+1,d} + G^{c,d+1}$ if $c < f_A$ and $p \nmid \gcd(c, d)$. Let $m > d / (f_A - c)$ be an integer such that $p \nmid mc + d$. For a positive integer l , let G_m^l be the image of $1 + t^l A$ in $\kappa(t^m - \pi)^\times / \kappa(t^m - \pi)^{\times p}$. Since me_A is the absolute ramification index of $A / (t^m - \pi)$ and $p \nmid mc + d < mf_A$, we have $G_m^{mc+d} / G_m^{mc+d+1} \cong F$ via $1 + t^{mc+d} x \mapsto x \pmod{t}$. Hence the kernel of the composite $G^{c,d} \rightarrow G_m^{mc+d} \twoheadrightarrow F$ is contained in $G^{c+1,d} + G^{c,d+1}$. $\square$

As in Section 7, we take a resolution $\mathfrak{X} \rightarrow \text{Spec } A$ such that $\mathfrak{X} \times_A A/pA \subset \mathfrak{X}$ is supported on a strict normal crossing divisor. For any $n \geq 1$, the natural map $H^1(\mathfrak{X}, \Lambda_n) \rightarrow H^1(X, \Lambda_n)$ is injective, and the natural map $H^1(\mathfrak{X}, \Lambda_n) \rightarrow H^1(Y, \Lambda_n)$ is an isomorphism. We view $H^1(Y, \Lambda_n)$ as a subgroup of $H^1(X, \Lambda_n)$.

PROPOSITION 9.3. Assume $F = \overline{F}$. Let $n \geq 1$ and $X' \in H^1(X, \Lambda_n)$. Then X' is completely split if and only if $X' \in H^1(Y, \Lambda_n)$. In particular, the subgroup $H^1(Y, \Lambda_n)$ of $H^1(X, \Lambda_n)$ is independent of the resolution $\mathfrak{X}$.

Proof. We may assume $\zeta_p \in K$ by the natural exact sequences

$$\begin{aligned} 0 \rightarrow H_{\text{cs}}^1(X, \Lambda_n) \rightarrow H^1(K, \Lambda_n) \rightarrow \prod_{\mathfrak{p} \in P} H^1(K_{\mathfrak{p}}, \Lambda_n), \\ 0 \rightarrow H^1(Y, \Lambda_n) \rightarrow H^1(K, \Lambda_n) \rightarrow \prod_{\mathfrak{p} \in P} \frac{H^1(K_{\mathfrak{p}}, \Lambda_n)}{H^1(\kappa(\mathfrak{p}), \Lambda_n)} \times \prod_{\eta \in Y_1} \frac{H^1(K_{\eta}, \Lambda_n)}{H^1(\kappa(\eta), \Lambda_n)}. \end{aligned}$$

The “if” part is easy. Assume that X' is completely split. For any non-singular point $x \in Y_0$ that is not the specialization of any element of P dividing p , the covering $X' \times_X \hat{R}_x$ of $\text{Spec } \hat{R}_x$ is trivial by Propositions 9.2. Hence X' extends to a finite étale covering of $\mathfrak{X}$ minus finitely many closed points and hence to a finite étale covering of $\mathfrak{X}$ by the purity of branch locus [Sta22, Tag 0BMA]. $\square$

Define

$$\mathbf{H}_{\text{cs}}^1(\mathbf{X}, \Lambda_n) := \mathbf{H}^1(\mathbf{Y}, \Lambda_n),$$

the finite étale group scheme over F with group of $\overline{F}$ -points given by $H^1(\mathbf{Y}(\overline{F}), \Lambda_n)$, where $\mathbf{Y}(\overline{F}) = Y \times_F \overline{F}$. It is an F -subgroup scheme of $\mathbf{H}^1(\mathbf{X}, \Lambda_n)$ independent of $\mathfrak{X}$ by Proposition 9.3. Also define

$$\pi_0(\mathbf{H}^2(\mathbf{K}, \Lambda_n(2))) := \varinjlim_S \pi_0(\mathbf{H}^2(\mathbf{U}_S, \Lambda_n(2))),$$

where S runs through all finite subsets of P . It is an étale group over F . Applying π_0 to the exact sequences

$$\mathbf{H}^2(\mathbf{U}_S, \mathfrak{T}_n(2)) \rightarrow \bigoplus_{\mathfrak{p} \in S} \mathbf{H}_c^3(\hat{\mathbf{A}}_{\mathfrak{p}}, \mathfrak{T}_n(2)) \rightarrow \mathbf{H}^3(\mathbf{X}, \mathfrak{T}_n(2)) \rightarrow 0$$

for $S \subset P$ and using (5.3) for $\mathfrak{p}$ dividing p and [Suz24, Theorem 7.2.4] otherwise, induces an exact sequence

$$\pi_0(\mathbf{H}^2(\mathbf{K}, \Lambda_n(2))) \rightarrow \bigoplus_{\mathfrak{p} \in S} \mathfrak{R}_{F_{\mathfrak{p}}/F} \Lambda_n \rightarrow \Lambda_n \rightarrow 0. \quad (9.1)$$

PROPOSITION 9.4. *The kernel of the first morphism in (9.1) is canonically isomorphic to the Pontryagin dual of $\mathbf{H}_{\text{cs}}^1(\mathbf{X}, \Lambda_n)$.*

Proof. Let $S \subset P$ be finite and non-empty. The distinguished triangle

$$R\Gamma_c(\mathbf{U}_S, \Lambda_n) \rightarrow R\Gamma(\mathbf{X}, \Lambda_n) \rightarrow \bigoplus_{\mathfrak{p} \in S} R\Gamma(\hat{\mathbf{A}}_{\mathfrak{p}}, \Lambda_n)$$

induces an exact sequence

$$0 \rightarrow \Lambda_n \rightarrow \bigoplus_{\mathfrak{p} \in S} \mathfrak{R}_{F_{\mathfrak{p}}/F} \Lambda_n \rightarrow \mathbf{H}_c^1(\mathbf{U}_S, \Lambda_n) \rightarrow \mathbf{H}^1(\mathbf{X}, \Lambda_n) \rightarrow \bigoplus_{\mathfrak{p} \in S} \mathbf{H}^1(\kappa(\mathfrak{p}), \Lambda_n).$$

Hence we have an exact sequence

$$0 \rightarrow \Lambda_n \rightarrow \prod_{\mathfrak{p} \in P} \mathfrak{R}_{F_{\mathfrak{p}}/F} \Lambda_n \rightarrow \varprojlim_S \mathbf{H}_c^1(\mathbf{U}_S, \Lambda_n) \rightarrow \mathbf{H}_{\text{cs}}^1(\mathbf{X}, \Lambda_n) \rightarrow 0$$

of pro-finite-étale groups over F . Taking the Pontryagin dual and applying Theorem 1.4, we get the result. $\square$

Let Γ_Y be the dual graph of $\mathbf{Y}(\overline{F})$. It has a continuous action of $\text{Gal}(\overline{F}/F)$. Let $\mathbf{H}^1(\Gamma_Y, \Lambda_n)$ be the first cohomology of Γ_Y with coefficients in Λ_n . It is a finite étale group scheme over F .

PROPOSITION 9.5. *We have an exact sequence*

$$0 \rightarrow \mathbf{H}^1(\Gamma_Y, \Lambda_n) \rightarrow \mathbf{H}^1(\mathbf{Y}, \Lambda_n) \rightarrow \bigoplus_{\eta \in Y_1} \mathbf{H}^1(\mathbf{Y}_{\eta}, \Lambda_n) \rightarrow 0.$$

Proof. This follows from the distinguished triangle

$$R\Gamma(\mathbf{Y}, \Lambda_n) \rightarrow \bigoplus_{\eta \in Y_1} R\Gamma(\mathbf{Y}_{\eta}, \Lambda_n) \rightarrow \bigoplus_{x \in Y_{\text{sin}}} \mathfrak{R}_{F_x/F} \Lambda_n,$$

where Y_{sin} denotes the set of singular points of Y . $\square$

This finishes the proof of Theorem 1.5.

10. Lemmas on localization over resolutions

In the next section, we prove a finiteness statement about $\text{Br}(X)$ (see Proposition 11.7). This uses some arguments that are local over a resolution of singularities of A , which we deal with in this section.

We give a Hasse principle for Brauer groups of tubular neighborhoods.

PROPOSITION 10.1. Let $\hat{K}_\eta$ be a complete discrete valuation field with ring of integer $\hat{\mathcal{O}}_{K_\eta}$ and residue field $\kappa(\eta)$. Assume that $\hat{K}_\eta$ has characteristic zero, $\zeta_p \in \hat{K}_\eta$ and $\kappa(\eta)$ is the function field of a proper smooth geometrically connected curve Y_η over $F = \bar{F}$. Let η_0 be the set of closed points of Y_η . For each $x \in \eta_0$, let $\hat{\kappa}(\eta_x)$ be the complete local field of Y_η at x . Let $\hat{\mathcal{O}}_{K_{\eta_x}}$ be the Kato canonical lifting of the relatively perfect $\kappa(\eta)$ -algebra $\hat{\kappa}(\eta_x)$ to $\hat{\mathcal{O}}_{K_\eta}$. Let $\hat{K}_{\eta_x}$ be the fraction field of $\hat{\mathcal{O}}_{K_{\eta_x}}$.

Then the kernel of the natural map

$$\mathrm{Br}(\hat{K}_\eta)[p] \rightarrow \prod_{x \in \eta_0} \mathrm{Br}(\hat{K}_{\eta_x})[p] \quad (10.1)$$

is canonically isomorphic to $H^1(Y_\eta, \Lambda)$.

Proof. Let e_η be the absolute ramification index of $\hat{K}_\eta$, and set $f_\eta = pe_\eta/(p-1)$. We have the usual filtrations by symbols on $\mathrm{Br}(\hat{K}_\eta)[p] \cong K_2(\hat{K}_\eta)/pK_2(\hat{K}_\eta)$. Let $\mathrm{gr}^m \mathrm{Br}(\hat{K}_\eta)[p]$ be the m th graded piece. Define $\mathrm{gr}^m \mathrm{Br}(\hat{K}_{\eta_x})[p]$ similarly for each $x \in \eta_0$. We have

$$\mathrm{gr}^m \mathrm{Br}(\hat{K}_\eta)[p] \cong \begin{cases} \kappa(\eta)^\times / \kappa(\eta)^{\times p} & \text{if } m = 0, \\ \kappa(\eta) / \kappa(\eta)^p & \text{if } 0 < m < f_\eta, p \mid m, \\ \Omega_{\kappa(\eta)}^1 & \text{if } 0 < m < f_\eta, p \nmid m, \\ \kappa(\eta) / \wp(\kappa(\eta)) & \text{if } m = f_\eta, \\ 0 & \text{else.} \end{cases}$$

Similarly,

$$\mathrm{gr}^m \mathrm{Br}(\hat{K}_{\eta_x})[p] \cong \begin{cases} \hat{\kappa}(\eta_x)^\times / \hat{\kappa}(\eta_x)^{\times p} & \text{if } m = 0, \\ \hat{\kappa}(\eta_x) / \hat{\kappa}(\eta_x)^p & \text{if } 0 < m < f_\eta, p \mid m, \\ \Omega_{\hat{\kappa}(\eta_x)}^1 & \text{if } 0 < m < f_\eta, p \nmid m, \\ \hat{\kappa}(\eta_x) / \wp(\hat{\kappa}(\eta_x)) & \text{if } m = f_\eta, \\ 0 & \text{else.} \end{cases}$$

The map $\mathrm{Br}(\hat{K}_\eta)[p] \rightarrow \mathrm{Br}(\hat{K}_{\eta_x})[p]$ preserves the filtration. The induced map on the m th graded piece is injective except in the case $m = f_\eta$. In the case $m = f_\eta$, the map can be identified with the natural map $H^1(\kappa(\eta), \Lambda) \rightarrow H^1(\hat{\kappa}(\eta_x), \Lambda)$. Hence the intersection of the kernels over all x is isomorphic to $H^1(Y_\eta, \Lambda)$. $\square$

Next, we study trace (or transfer) maps. Assume $\zeta_p \in A$ and take $S \subset P$ to be the set of primes dividing p . Let $\mathfrak{X} \rightarrow \mathrm{Spec} A$, $W_\eta \subset \mathfrak{X}$ and $T \subset Y$ be as in Section 7. By [Suz24, (10.4.7) and (10.5.1)], we have a morphism of distinguished triangles

$$\begin{array}{ccccc} R\Gamma_c(\mathbf{U}_S, \Lambda) & \longrightarrow & R\Gamma(\mathbf{U}_S, \Lambda) & \longrightarrow & \bigoplus_{\mathfrak{p} \in S} R\Gamma(\hat{\mathbf{K}}_{\mathfrak{p}}, \Lambda) \\ \parallel & & \downarrow & & \downarrow \\ R\Gamma_c(\mathbf{U}_S, \Lambda) & \longrightarrow & \bigoplus_{x \in T} R\Gamma(\hat{\mathbf{R}}_{x,S}, \Lambda) & \longrightarrow & \bigoplus_{\substack{x \in T \\ \eta_x \in Y_1^x}} R\Gamma(\hat{\mathbf{K}}_{\eta_x}, \Lambda) \\ & & \oplus \bigoplus_{\eta \in Y_1} R\Gamma(\hat{\mathbf{R}}_{\eta,T}, \Lambda) & & \oplus \bigoplus_{\mathfrak{p} \in S} R\Gamma(\hat{\mathbf{K}}_{\mathfrak{p}}, \Lambda) \end{array} \quad (10.2)$$

in $D(F_{\mathrm{proet}}^{\mathrm{indrat}})$. In particular, we have canonical morphisms

$$\mathbf{H}^1(\hat{\mathbf{K}}_{\eta_x}, \Lambda) \rightarrow \mathbf{H}_c^2(\mathbf{U}_S, \Lambda) \rightarrow \mathbf{H}^2(\mathbf{X}, \mathfrak{T}(1)) \quad (10.3)$$

for each pair $x \in T$ and $\eta_x \in Y_1^x$ (where the second morphism comes from $\zeta_p: \Lambda \rightarrow \mathfrak{T}(1)$). Hence for any algebraically closed field $F' \in F^{\text{perar}}$, we obtain a homomorphism

$$H^1(\hat{\mathbf{K}}_{\eta_x}(F'), \Lambda) \rightarrow H^2(\mathbf{X}(F'), \mathfrak{T}(1)). \quad (10.4)$$

PROPOSITION 10.2. *In the above setting, let $x \in T$ and $\eta_x \in Y_1^x$. Then the image of (10.4) is contained in the subgroup $H^1(\mathbf{X}(F'), \mathbf{G}_m) \otimes \Lambda$ of the target.*

Proof. We may assume $F = F' = \bar{F}$. We want to show that the image of $H^1(\hat{K}_{\eta_x}, \Lambda) \rightarrow H^2(X, \mathfrak{T}(1))$ is contained in the subgroup $H^1(X, \mathbf{G}_m) \otimes \Lambda$ of the target.

Let $R_S = A[1/p]$. We first briefly recall the construction of the diagram (10.2) on F -points. Let $R_{S, \text{Et}}$ be the big étale site of R_S . View $\Gamma(R_S, \cdot)$, $\Gamma(\hat{K}_{\mathfrak{p}}, \cdot)$, $\Gamma(\hat{K}_{\eta_x}, \cdot)$, $\Gamma(\hat{R}_{x,S}, \cdot)$ and $\Gamma(\hat{R}_{\eta,T}, \cdot)$ as the section functors $\text{Ab}(R_{S, \text{Et}}) \rightarrow \text{Ab}$ at the objects R_S , $\hat{K}_{\mathfrak{p}}$, $\hat{K}_{\eta_x}$, $\hat{R}_{x,S}$ and $\hat{R}_{\eta,T}$, respectively. We have a commutative diagram

$$\begin{array}{ccc} \Gamma(R_S, \cdot) & \longrightarrow & \bigoplus_{\mathfrak{p} \in S} \Gamma(\hat{K}_{\mathfrak{p}}, \cdot) \\ \downarrow & & \downarrow \\ \bigoplus_{x \in T} \Gamma(\hat{R}_{x,S}, \cdot) & \longrightarrow & \bigoplus_{\substack{x \in T \\ \eta_x \in Y_1^x}} \Gamma(\hat{K}_{\eta_x}, \cdot) \\ \bigoplus_{\eta \in Y_1} \Gamma(\hat{R}_{\eta,T}, \cdot) & & \bigoplus_{\mathfrak{p} \in S} \Gamma(\hat{K}_{\mathfrak{p}}, \cdot) \end{array}$$

of functors $\text{Ab}(R_{S, \text{Et}}) \rightarrow \text{Ab}$, where the left vertical map is the diagonal

$$a \mapsto ((a|_{\hat{R}_{x,S}})_{x \in T}, (a|_{\hat{R}_{\eta,T}})_{\eta \in Y_1}),$$

the lower horizontal map is

$$((a_x)_{x \in T}, (a_{\eta})_{\eta \in Y_1}) \mapsto ((a_x|_{\hat{K}_{\eta_x}} - a_{\eta}|_{\hat{K}_{\eta_x}})_{x \in T, \eta_x \in Y_1^x}, (a_x|_{\hat{K}_{\mathfrak{p}}})_{\mathfrak{p} \in S}),$$

the upper horizontal map is the diagonal and the right vertical map is the inclusion into the second summand. This diagram induces a commutative diagram of functors $\text{Ch}(R_{S, \text{Et}}) \rightarrow \text{Ch}$ of additive categories with translation in the sense of [KS06, Definition 10.1.1] (where Ch is the category of complexes in Ab). When we take the right derived functors [KS06, Sections 13.3 and 14.3], it induces a morphism

$$\begin{aligned} R \left[\Gamma(R_S, \cdot) \rightarrow \bigoplus_{\mathfrak{p} \in S} \Gamma(\hat{K}_{\mathfrak{p}}, \cdot) \right] [-1] \\ \rightarrow R \left[\begin{array}{ccc} \bigoplus_{x \in T} \Gamma(\hat{R}_{x,S}, \cdot) & \rightarrow & \bigoplus_{\substack{x \in T \\ \eta_x \in Y_1^x}} \Gamma(\hat{K}_{\eta_x}, \cdot) \\ \bigoplus_{\eta \in Y_1} \Gamma(\hat{R}_{\eta,T}, \cdot) & & \bigoplus_{\mathfrak{p} \in S} \Gamma(\hat{K}_{\mathfrak{p}}, \cdot) \end{array} \right] [-1] \end{aligned}$$

of triangulated functors $D(R_{S, \text{Et}}) \rightarrow D(\text{Ab})$. It becomes an isomorphism when precomposed with the natural functor $D_{\text{tor}}^+(R_{S, \text{et}}) \rightarrow D(R_{S, \text{Et}})$ from the bounded below derived category of complexes with torsion cohomology, and the resulting isomorphic functors $D_{\text{tor}}^+(R_{S, \text{et}}) \rightrightarrows D(\text{Ab})$ are further isomorphic to $R\Gamma_c(R_S, \cdot)$. In particular, we have a morphism of distinguished triangles

$$\begin{array}{ccccc} R\Gamma_c(R_S, \Lambda) & \longrightarrow & R\Gamma(R_S, \Lambda) & \longrightarrow & \bigoplus_{\mathfrak{p} \in S} R\Gamma(\hat{K}_{\mathfrak{p}}, \Lambda) \\ \parallel & & \downarrow & & \downarrow \\ R\Gamma_c(R_S, \Lambda) & \longrightarrow & \bigoplus_{x \in T} R\Gamma(\hat{R}_{x,S}, \Lambda) & \longrightarrow & \bigoplus_{\substack{x \in T \\ \eta_x \in Y_1^x}} R\Gamma(\hat{K}_{\eta_x}, \Lambda) \\ & & \bigoplus_{\eta \in Y_1} R\Gamma(\hat{R}_{\eta,T}, \Lambda) & & \bigoplus_{\mathfrak{p} \in S} R\Gamma(\hat{K}_{\mathfrak{p}}, \Lambda) \end{array}$$

which is the F -points of the diagram (10.2).³

Let $P_T \subset P$ be the set of prime ideals $\mathfrak{p} \in P$ that specialize to an element of T in $\mathfrak{X}$. Let $S' \subset P_T$ be an arbitrary finite subset containing S . Let $R_{S'} = \mathcal{O}(U_S \setminus S')$. We have a commutative diagram

$$\begin{array}{ccc} \Gamma(R_S, \cdot) & \longrightarrow & \bigoplus_{\mathfrak{p} \in S} \Gamma(\hat{K}_{\mathfrak{p}}, \cdot) \\ \downarrow & & \downarrow \\ \Gamma(R_{S'}, \cdot) & \longrightarrow & \bigoplus_{\mathfrak{p} \in S} \Gamma(\hat{K}_{\mathfrak{p}}, \cdot) \\ & & \oplus \bigoplus_{\mathfrak{p} \in S' \setminus S} \Gamma_c(\hat{A}_{\mathfrak{p}}, \cdot)[1] \end{array} \quad (10.5)$$

of functors $\text{Ch}(U_{S, \text{Et}}) \rightarrow \text{Ch}$ of additive categories with translation, where $\Gamma_c(\hat{A}_{\mathfrak{p}}, \cdot)[1]$ is the mapping cone of $\Gamma(\hat{A}_{\mathfrak{p}}, \cdot) \rightarrow \Gamma(\hat{K}_{\mathfrak{p}}, \cdot)$. The induced morphism

$$\begin{aligned} & R \left[\Gamma(R_S, \cdot) \rightarrow \bigoplus_{\mathfrak{p} \in S} \Gamma(\hat{K}_{\mathfrak{p}}, \cdot) \right] [-1] \\ & \rightarrow R \left[\Gamma(R_{S'}, \cdot) \rightarrow \bigoplus_{\mathfrak{p} \in S} \Gamma(\hat{K}_{\mathfrak{p}}, \cdot) \right. \\ & \quad \left. \oplus \bigoplus_{\mathfrak{p} \in S' \setminus S} \Gamma_c(\hat{A}_{\mathfrak{p}}, \cdot)[1] \right] [-1] \end{aligned}$$

of triangulated functors $D(R_{S, \text{Et}}) \rightarrow D(\text{Ab})$ precomposed with $D_{\text{tor}}^+(R_{S, \text{et}}) \rightarrow D(R_{S, \text{Et}})$ is an isomorphism by excision.

Similarly, we have a commutative diagram

$$\begin{array}{ccc} \bigoplus_{x \in T} \Gamma(\hat{R}_{x, S}, \cdot) & \longrightarrow & \bigoplus_{\substack{x \in T \\ \eta_x \in Y_1^x}} \Gamma(\hat{K}_{\eta_x}, \cdot) \\ \oplus \bigoplus_{\eta \in Y_1} \Gamma(\hat{R}_{\eta, T}, \cdot) & & \oplus \bigoplus_{\mathfrak{p} \in S} \Gamma(\hat{K}_{\mathfrak{p}}, \cdot) \\ \downarrow & & \downarrow \\ \bigoplus_{x \in T} \Gamma(\hat{R}_{x, S'}, \cdot) & \longrightarrow & \bigoplus_{\substack{x \in T \\ \eta_x \in Y_1^x}} \Gamma(\hat{K}_{\eta_x}, \cdot) \\ \oplus \bigoplus_{\eta \in Y_1} \Gamma(\hat{R}_{\eta, T}, \cdot) & & \oplus \bigoplus_{\mathfrak{p} \in S} \Gamma(\hat{K}_{\mathfrak{p}}, \cdot) \\ & & \oplus \bigoplus_{\mathfrak{p} \in S' \setminus S} \Gamma_c(\hat{A}_{\mathfrak{p}}, \cdot)[1]. \end{array} \quad (10.6)$$

The induced morphism

$$\begin{aligned} & R \left[\bigoplus_{x \in T} \Gamma(\hat{R}_{x, S}, \cdot) \rightarrow \bigoplus_{\substack{x \in T \\ \eta_x \in Y_1^x}} \Gamma(\hat{K}_{\eta_x}, \cdot) \right. \\ & \quad \left. \oplus \bigoplus_{\eta \in Y_1} \Gamma(\hat{R}_{\eta, T}, \cdot) \rightarrow \bigoplus_{\mathfrak{p} \in S} \Gamma(\hat{K}_{\mathfrak{p}}, \cdot) \right] [-1] \\ & \rightarrow R \left[\bigoplus_{x \in T} \Gamma(\hat{R}_{x, S'}, \cdot) \rightarrow \bigoplus_{\substack{x \in T \\ \eta_x \in Y_1^x}} \Gamma(\hat{K}_{\eta_x}, \cdot) \right. \\ & \quad \left. \oplus \bigoplus_{\eta \in Y_1} \Gamma(\hat{R}_{\eta, T}, \cdot) \rightarrow \bigoplus_{\mathfrak{p} \in S} \Gamma(\hat{K}_{\mathfrak{p}}, \cdot) \right. \\ & \quad \left. \oplus \bigoplus_{\mathfrak{p} \in S' \setminus S} \Gamma_c(\hat{A}_{\mathfrak{p}}, \cdot)[1] \right] [-1] \end{aligned}$$

of triangulated functors $D(R_{S, \text{Et}}) \rightarrow D(\text{Ab})$ precomposed with $D_{\text{tor}}^+(R_{S, \text{et}}) \rightarrow D(R_{S, \text{Et}})$ is an isomorphism by excision.

³The original construction in [Suz24, Section 4.6] uses fibered sites. For the purpose of proving Proposition 10.2 here, it is enough to use big étale sites, which gives the same diagram on F -points.

We have a morphism of diagrams from (10.5) to (10.6). It induces a commutative diagram

$$\begin{array}{ccc}
 \frac{\bigoplus_{\mathfrak{p} \in S} H^1(\hat{K}_{\mathfrak{p}}, \Lambda)}{H^1(R_S, \Lambda)} & \longrightarrow & \frac{\bigoplus_{\mathfrak{p} \in S} H^1(\hat{K}_{\mathfrak{p}}, \Lambda) \oplus \bigoplus_{\mathfrak{p} \in S' \setminus S} H_c^2(\hat{A}_{\mathfrak{p}}, \Lambda)}{H^1(R_{S'}, \Lambda)} \\
 \downarrow & & \downarrow \\
 \frac{\bigoplus_{\substack{x \in T \\ \eta_x \in Y_1^x}} H^1(\hat{K}_{\eta_x}, \Lambda) \oplus \bigoplus_{\mathfrak{p} \in S} H^1(\hat{K}_{\mathfrak{p}}, \Lambda)}{\bigoplus_{x \in T} H^1(\hat{R}_{x,S}, \Lambda) \oplus \bigoplus_{\eta \in Y_1} H^1(\hat{R}_{\eta,T}, \Lambda)} & \longrightarrow & \frac{\bigoplus_{\substack{x \in T \\ \eta_x \in Y_1^x}} H^1(\hat{K}_{\eta_x}, \Lambda) \oplus \bigoplus_{\mathfrak{p} \in S} H^1(\hat{K}_{\mathfrak{p}}, \Lambda) \oplus \bigoplus_{\mathfrak{p} \in S' \setminus S} H_c^2(\hat{A}_{\mathfrak{p}}, \Lambda)}{\bigoplus_{x \in T} H^1(\hat{R}_{x,S'}, \Lambda) \oplus \bigoplus_{\eta \in Y_1} H^1(\hat{R}_{\eta,T}, \Lambda)}
 \end{array}$$

of injective maps, with all the terms mapping injectively to $H_c^2(R_S, \Lambda)$. For a ring Q , set $Q^{\bar{\times}} = Q^{\times}/Q^{\times p}$. Let R_T be the direct limit of $R_{S'}$ over increasing S' . Taking the direct limit of the above diagram in increasing S' , we obtain a commutative diagram

$$\begin{array}{ccc}
 \frac{\bigoplus_{\mathfrak{p} \in S} \hat{K}_{\mathfrak{p}}^{\bar{\times}}}{H^1(R_S, \Lambda)} & \longrightarrow & \frac{\bigoplus_{\mathfrak{p} \in S} \hat{K}_{\mathfrak{p}}^{\bar{\times}} \oplus \bigoplus_{\mathfrak{p} \in P_T \setminus S} \Lambda}{H^1(R_T, \Lambda)} \\
 \downarrow & & \downarrow \\
 \frac{\bigoplus_{\substack{x \in T \\ \eta_x \in Y_1^x}} \hat{K}_{\eta_x}^{\bar{\times}} \oplus \bigoplus_{\mathfrak{p} \in S} \hat{K}_{\mathfrak{p}}^{\bar{\times}}}{\bigoplus_{x \in T} \hat{R}_{x,S}^{\bar{\times}} \oplus \bigoplus_{\eta \in Y_1} H^1(\hat{R}_{\eta,T}, \Lambda)} & \longrightarrow & \frac{\bigoplus_{\substack{x \in T \\ \eta_x \in Y_1^x}} \hat{K}_{\eta_x}^{\bar{\times}} \oplus \bigoplus_{\mathfrak{p} \in S} \hat{K}_{\mathfrak{p}}^{\bar{\times}} \oplus \bigoplus_{\mathfrak{p} \in P_T \setminus S} \Lambda}{\bigoplus_{x \in T} \hat{R}_{x,S}^{\bar{\times}} \oplus \bigoplus_{\eta \in Y_1} H^1(\hat{R}_{\eta,T}, \Lambda)}
 \end{array}$$

of injective maps, with all the terms mapping injectively to $H_c^2(R_S, \Lambda)$. For any $x \in T$, the map $\hat{K}_x^{\bar{\times}} \rightarrow \bigoplus_{\eta_x \in Y_1^x} \hat{K}_{\eta_x}^{\bar{\times}}$ is surjective by the approximation lemma. Therefore, the right vertical map is an isomorphism. Hence for any $x \in T$ and $\eta_x \in Y_1^x$, the image of the map $\hat{K}_{\eta_x}^{\bar{\times}} \rightarrow H_c^2(R_S, \Lambda)$ is contained in the image of the map $\bigoplus_{\mathfrak{p} \in S} \hat{K}_{\mathfrak{p}}^{\bar{\times}} \oplus \bigoplus_{\mathfrak{p} \in P_T \setminus S} \Lambda \rightarrow H_c^2(R_S, \Lambda)$. Hence the image of the map $\hat{K}_{\eta_x}^{\bar{\times}} \rightarrow H^2(X, \mathfrak{T}(1))$ is contained in the image of the map $\bigoplus_{\mathfrak{p} \in P_T} \Lambda \rightarrow H^2(X, \mathfrak{T}(1))$. The latter map factors through $H^1(X, \mathbf{G}_m) \otimes \Lambda$ via the divisor map. This proves the proposition. $\square$

11. Picard–Brauer duality

In this section, we prove Theorem 1.6. We deduce it basically from Theorems 1.2 and 1.4. We also give a duality result for $H^3(X, \mathbf{G}_m)$ (see Proposition 11.9) and for π_0 of $\text{Pic}(X)$ (see Proposition 11.10).

PROPOSITION 11.1. *The group $\pi_{\mathbf{X},*} \mathbf{G}_m = \mathbf{A}^{\times}$ is $\mathbf{G}_m$ times a connected pro-algebraic group in $\mathcal{W}_F$. More specifically, the natural reduction morphism $\mathbf{A}^{\times} \rightarrow \mathbf{G}_m$ and the Teichmüller section give a direct summand $\mathbf{G}_m$ of $\mathbf{A}^{\times}$. The quotient $\mathbf{A}^{\times}/\mathbf{G}_m$ is in $\mathcal{W}_F$.*

Proof. For $F' \in F^{\text{perar}}$ and $n \geq 1$, let $\mathbf{A}_n(F') = \mathbf{A}(F') \otimes_A A/\mathfrak{m}^n$. Then $\mathbf{A}^{\times} = \varprojlim_n \mathbf{A}_n^{\times}$. The sheaf $\mathbf{A}_n^{\times}$ is represented by the perfection of the Greenberg transform (or realization) of $\mathbf{G}_m$ over $A/\mathfrak{m}^n$; see [BG18, Proposition-Definition 6.2]. The same proof as that of [BG18, Proposition 11.1] (see also [BS24, Proposition 5.6]) shows that the kernel of the reduction morphism $\mathbf{A}_{n+1}^{\times} \twoheadrightarrow \mathbf{A}_n^{\times}$ is the perfection of a vector group over F . Hence the kernel of $\mathbf{A}^{\times} \twoheadrightarrow \mathbf{G}_m$, or $\mathbf{A}^{\times}/\mathbf{G}_m$, is in $\mathcal{W}_F$. $\square$

Given a resolution of singularities $\mathfrak{X} \rightarrow \text{Spec } A$ such that the reduced part Y of $\mathfrak{X} \times_A F$ is supported on a strict normal crossing divisor, we have the intersection pairing

$$\bigoplus_{\eta \in Y_1} \mathbb{Z} Y_{\eta} \times \bigoplus_{\eta \in Y_1} \mathbb{Z} Y_{\eta} \rightarrow \mathbb{Z}, \quad (Y_{\eta}, Y_{\eta'}) \mapsto Y_{\eta} \cdot Y_{\eta'}, \quad (11.1)$$

where Y_1 is the set of generic points of Y and Y_η the closure of η . See [Lip69, Section 13] for the details. This pairing is negative-definite by [Lip69, Lemma 14.1]. Let

$$\delta_Y = \frac{\text{Hom}(\bigoplus_{\eta \in Y_1} \mathbb{Z}Y_\eta, \mathbb{Z})}{\bigoplus_{\eta \in Y_1} \mathbb{Z}Y_\eta}$$

be its discriminant group. Applying this construction to the resolution $\mathfrak{X} \times_A \mathbf{A}(\overline{F}) \rightarrow \text{Spec } \mathbf{A}(\overline{F})$, we obtain a finite Galois module $\delta_{Y \times_F \overline{F}}$ over F . Denote the resulting finite étale group scheme over F by δ_Y .

PROPOSITION 11.2. *We have $R^1\pi_{\mathbf{X},*}\mathbf{G}_m \in \text{Alg}/F$. For a resolution of singularities $\mathfrak{X} \rightarrow \text{Spec } A$ such that the reduced part Y of $\mathfrak{X} \times_A F$ is supported on a strict normal crossing divisor, we have $\pi_0 R^1\pi_{\mathbf{X},*}\mathbf{G}_m \cong \delta_Y$.*

Proof. Let $\mathfrak{X} \rightarrow \text{Spec } A$ be as in the statement. First, assume $F = \overline{F}$. We recall the constructions made in [Lip69, Section 14] and [Sai86, proof of Lemma 7.2]. Let $\text{Pic}(\mathfrak{X}) \rightarrow \text{Hom}(\bigoplus_{\eta \in Y_1} \mathbb{Z}Y_\eta, \mathbb{Z})$ be the map given by the intersection pairing. Let $\text{Pic}^0(\mathfrak{X})$ be its kernel. Then the resulting sequence

$$0 \rightarrow \text{Pic}^0(\mathfrak{X}) \rightarrow \text{Pic}(X) \rightarrow \delta_Y \rightarrow 0$$

is exact. For $n \geq 1$, let $\mathfrak{X}_n = \mathfrak{X} \times_A A/\mathfrak{m}^n$. Let $\text{Pic}^0(\mathfrak{X}_n)$ be the kernel of a similar map $\text{Pic}(\mathfrak{X}_n) \rightarrow \text{Hom}(\bigoplus_{\eta \in Y_1} \mathbb{Z}Y_\eta, \mathbb{Z})$. The group $\text{Pic}^0(Y) = \text{Pic}^0(\mathfrak{X}_1)$ is the group of F -valued points of the semi-abelian variety $\text{Pic}_{Y/F}^0$. The natural map $\text{Pic}^0(\mathfrak{X}_{n+1}) \rightarrow \text{Pic}^0(\mathfrak{X}_n)$ is surjective, and its kernel has a natural structure of a finite-dimensional F -vector space. There exists a positive integer n_0 such that the natural map $\text{Pic}^0(\mathfrak{X}) \rightarrow \text{Pic}^0(\mathfrak{X}_n)$ is an isomorphism for all $n \geq n_0$.

Next, let F be general. The above constructions are functorial in residue field extensions. That is, for any algebraically closed field $F' \in F^{\text{perar}}$, we can apply the above constructions for $\mathfrak{X}(F') = \mathfrak{X} \times_A \mathbf{A}(F') \rightarrow \text{Spec } \mathbf{A}(F')$. This gives an exact sequence

$$0 \rightarrow \text{Pic}^0(\mathfrak{X}(F')) \rightarrow \text{Pic}(\mathbf{X}(F')) \rightarrow \delta_{\mathbf{Y}(F')} \rightarrow 0$$

and quotients $\text{Pic}^0(\mathfrak{X}_n(F'))$ of $\text{Pic}^0(\mathfrak{X}(F'))$ functorial in F' (where $\mathbf{Y}(F') = Y' \times_F F'$ and $\mathfrak{X}_n(F') = \mathfrak{X}_n \times_A \mathbf{A}(F')$).

The sheaf $R^1\pi_{\mathbf{X},*}\mathbf{G}_m$ sends a field $F' \in F^{\text{perar}}$ to the $\text{Gal}(\overline{F'}/F')$ -invariant part of $\text{Pic}(\mathbf{X}(\overline{F'}))$. The above shows that this sheaf has a finite filtration whose graded pieces are sheaves that send a field $F' \in F^{\text{perar}}$ to the $\text{Gal}(\overline{F'}/F')$ -invariant parts of

- (i) $\delta_{\mathbf{Y}(\overline{F'})}$,
- (ii) $\text{Pic}^0(\mathbf{Y}(\overline{F'}))$ or
- (iii) $\text{Ker}(\text{Pic}^0(\mathfrak{X}_{n+1}(\overline{F'})) \rightarrow \text{Pic}^0(\mathfrak{X}_n(\overline{F'})))$ for $n < n_0$.

In this list, the first sheaf is a finite étale group (which is δ_Y), the second a semi-abelian variety and the third a vector group. Being an extension of such groups, the sheaf $R^1\pi_{\mathbf{X},*}\mathbf{G}_m$ is in Alg/F with component group δ_Y . $\square$

PROPOSITION 11.3. *Let $n \geq 1$. Then $R^0\pi_{\mathbf{X},*}\mathfrak{T}_n(1)$ is a finite étale group and $R^1\pi_{\mathbf{X},*}\mathfrak{T}_n(1)$ is pro-algebraic in $\mathcal{W}_F$.*

Proof. The sheaf $R^0\pi_{\mathbf{X},*}\mathfrak{T}_n(1)$ is the finite étale group of p^n th roots of unity in $\mathbf{A}(\overline{F})$. We have an exact sequence

$$0 \rightarrow \mathbf{A}^\times / \mathbf{A}^{\times p^n} \rightarrow R^1\pi_{\mathbf{X},*}\mathfrak{T}_n(1) \rightarrow (R^1\pi_{\mathbf{X},*}\mathbf{G}_m)[p^n] \rightarrow 0$$

in $\text{Ab}(F_{\text{et}}^{\text{perar}})$. The fourth term $(R^1\pi_{\mathbf{X},*}\mathbf{G}_m)[p^n]$ is in Alg_u/F by Proposition 11.2. We have an exact sequence

$$0 \rightarrow R^0\pi_{\mathbf{X},*}\mathfrak{T}_n(1) \rightarrow \mathbf{A}^\times/\mathbf{G}_m \xrightarrow{p} \mathbf{A}^\times/\mathbf{G}_m \rightarrow \mathbf{A}^\times/\mathbf{A}^{\times p^n} \rightarrow 0$$

in $\text{Ab}(F_{\text{et}}^{\text{perar}})$. The sheaf $\mathbf{A}^\times/\mathbf{G}_m$ is pro-algebraic in $\mathcal{W}_F$ by Proposition 11.1. Therefore, $\mathbf{A}^\times/\mathbf{A}^{\times p^n}$ is pro-algebraic in $\mathcal{W}_F$. $\square$

PROPOSITION 11.4. (i) *The object $R\pi_{\mathbf{X},*}\mathbf{G}_m$ is h -acyclic.*

(ii) *Let $q \geq 2$. Then the sheaf $R^q\pi_{\mathbf{X},*}\mathbf{G}_m$ is torsion. We have*

$$R^q\pi_{\mathbf{X},*}\mathfrak{T}_\infty(1) \simeq (R^q\pi_{\mathbf{X},*}\mathbf{G}_m)[p^\infty],$$

where $\mathfrak{T}_\infty(1) = \varinjlim_n \mathfrak{T}_n(1)$. For any prime $l \neq p$, the sheaf $(R^q\pi_{\mathbf{X},*}\mathbf{G}_m)[l^\infty]$ is the étale group over F with group of $\bar{F}$ -points given by $H^q(\mathbf{X}(\bar{F}), \mathbb{Q}_l/\mathbb{Z}_l(1))$.

Proof. The truncation $\tau_{\leq 1}R\pi_{\mathbf{X},*}\mathbf{G}_m$ is h -acyclic by Propositions 11.1 and 11.2 and [Suz21, Proposition 7.2]. The sheaf $R^q\pi_{\mathbf{X},*}\mathbf{G}_m$ is torsion for $q \geq 2$ by [Gro95, Proposition 1.4]. Since $(R^1\pi_{\mathbf{X},*}\mathbf{G}_m) \otimes \mathbb{Q}/\mathbb{Z} = 0$ by Proposition 11.2, we have

$$R^q\pi_{\mathbf{X},*}\mathfrak{T}_\infty(1) \oplus \bigoplus_{l \neq p} R^q\pi_{\mathbf{X},*}\mathbb{Q}_l/\mathbb{Z}_l(1) \simeq R^q\pi_{\mathbf{X},*}\mathbf{G}_m$$

for $q \geq 2$. For any field $F' \in F^{\text{perar}}$, the map $A \rightarrow \mathbf{A}(F')$ is regular and hence ind-smooth by Popescu's theorem. With [ILO14, Introduction, Theorem 1], we know that $R^q\pi_{\mathbf{X},*}\mathbb{Z}/l^n\mathbb{Z}(1)$ is finite étale for all q and all primes $l \neq p$. This implies that $R^q\pi_{\mathbf{X},*}\mathbb{Q}_l/\mathbb{Z}_l(1)$ is h -acyclic by [Suz21, Proposition 3.11]. For any $n \geq 1$, the object $\tau_{\geq 2}R\pi_{\mathbf{X},*}\mathfrak{T}_n(1)$ is in $\langle \mathcal{W}_F \rangle_{F^{\text{et}}}$ and h -acyclic by [Suz24, Propositions 10.1.5 and 3.1.4] and Proposition 11.3. Since $R\Gamma(\mathbf{X}, \mathfrak{T}_n(1)) \in D^b(\text{IPAlg}_u/F)$ is concentrated in non-negative degrees for all n by Proposition 8.3, we know that $Lh^*R\pi_{\mathbf{X},*}\mathfrak{T}_n(1) \in D^b(\text{IPAlg}_u/F)$ is concentrated in non-negative degrees by [Suz21, Proposition 7.6]. In particular, $\{Lh^*R\pi_{\mathbf{X},*}\mathfrak{T}_n(1)\}_{n \geq 1}$ as an object of the derived category of the ind-category of $\text{Ab}(F_{\text{profppf}}^{\text{perf}})$ is bounded below. On the other hand, $\{Lh^*\tau_{\leq 1}R\pi_{\mathbf{X},*}\mathfrak{T}_n(1)\}_{n \geq 1}$ is concentrated in degrees 0 and 1 by Proposition 11.3. Hence $\{Lh^*\tau_{\geq 2}R\pi_{\mathbf{X},*}\mathfrak{T}_n(1)\}_{n \geq 1}$ is bounded below. The functor Rh_* commutes with $\varinjlim_n$ on the bounded below derived categories by [Suz20b, Proposition 2.2.4]. The functor Lh^* commutes with $\varinjlim_n$ by [Suz22, Lemma 3.7.2] and [KS06, Corollary 14.4.6(ii)]. Therefore,

$$Rh_*Lh^*\tau_{\geq 2}R\pi_{\mathbf{X},*}\mathfrak{T}_\infty(1) \cong \varinjlim_n Rh_*Lh^*\tau_{\geq 2}R\pi_{\mathbf{X},*}\mathfrak{T}_n(1) \cong \tau_{\geq 2}R\pi_{\mathbf{X},*}\mathfrak{T}_\infty(1),$$

showing the h -acyclicity of $\tau_{\geq 2}R\pi_{\mathbf{X},*}\mathfrak{T}_\infty(1)$. Thus $\tau_{\geq 2}R\pi_{\mathbf{X},*}\mathbf{G}_m$ is h -acyclic. $\square$

By Proposition 11.4(i), for any field $F' \in F^{\text{perar}}$ with algebraic closure $\bar{F}'$ and any $q \in \mathbb{Z}$, we have

$$\mathbf{H}^q(\mathbf{X}, \mathbf{G}_m)(F') \cong H^q(\mathbf{X}(\bar{F}'), \mathbf{G}_m)^{\text{Gal}(\bar{F}'/F')}$$

functorially in F' .

PROPOSITION 11.5. *The group $\Gamma(\mathbf{X}, \mathbf{G}_m) \cong \mathbf{A}^\times$ is $\mathbf{G}_m$ times a connected pro-algebraic group in $\mathcal{W}_F$, $\mathbf{H}^1(\mathbf{X}, \mathbf{G}_m)$ is in Alg/F , $\mathbf{H}^2(\mathbf{X}, \mathbf{G}_m)$ and $\mathbf{H}^3(\mathbf{X}, \mathbf{G}_m)$ are torsion with p -primary part in IAlg_u/F and étale l -primary part for $l \neq p$, and $\mathbf{H}^q(\mathbf{X}, \mathbf{G}_m) = 0$ for all $q \neq 0, 1, 2, 3$. We have*

$$\mathbf{H}^q(\mathbf{X}, \mathfrak{T}_\infty(1)) \simeq \mathbf{H}^q(\mathbf{X}, \mathbf{G}_m)[p^\infty]$$

for all $q \geq 2$.

Proof. This follows from Propositions 2.1, 8.3, 11.1, 11.2 and 11.4. $\square$

Recall from [Suz20a, Section 2.1] that an object $G \in \text{IAlg}_u/F$ is said to be of cofinite type if $G[p^n] \in \text{Alg}_u/F$ for all $n \geq 1$. For a cofinite-type G , let G_{div}^0 be the maximal divisible and connected part of G (see [Suz20a, Proposition 2.2.2]; the notation there was $G_{0\text{div}}$).

PROPOSITION 11.6. *The object $\mathbf{H}^2(\mathbf{X}, \mathbf{G}_m)[p^\infty] \in \text{IAlg}_u/F$ is of cofinite type.*

Proof. For any $n \geq 1$, we have an exact sequence

$$0 \rightarrow \mathbf{H}^1(\mathbf{X}, \mathbf{G}_m) \otimes \Lambda_n \rightarrow \mathbf{H}^2(\mathbf{X}, \mathfrak{T}_n(1)) \rightarrow \mathbf{H}^2(\mathbf{X}, \mathbf{G}_m)[p^n] \rightarrow 0.$$

The direct limit of the first term in n is zero by Proposition 11.5. As $\mathbf{H}^2(\mathbf{X}, \mathfrak{T}_n(1)) \in \text{Alg}_u/F$ by Proposition 8.3, we get the result. $\square$

PROPOSITION 11.7. *We have $\mathbf{H}^2(\mathbf{X}, \mathbf{G}_m)^0 \in \text{Alg}_u/F$.*

Proof. It is enough to show that $\mathbf{H}^2(\mathbf{X}, \mathbf{G}_m)_{\text{div}}^0 = 0$ by [Suz20a, Proposition 2.2.3]. We may assume $\zeta_p \in A$ and $F = \overline{F}$. We have

$$\mathbf{H}^2(\mathbf{X}, \mathbf{G}_m)^0[p^\infty] \cong \varinjlim_n \text{Ext}_{F_{\text{proet}}^{\text{indrat}}}^1(\mathbf{H}^2(\mathbf{X}, \mathfrak{T}_n(1)), \Lambda_\infty)$$

by duality. We have a natural morphism from the right-hand side to

$$\varinjlim_n \text{Ext}_{F_{\text{proet}}^{\text{indrat}}}^1(\mathbf{H}^1(\mathbf{X}, \mathbf{G}_m) \otimes \Lambda_n, \Lambda_\infty) \cong \text{Ext}_{F_{\text{proet}}^{\text{indrat}}}^1(\mathbf{H}^1(\mathbf{X}, \mathbf{G}_m), \Lambda_\infty)^0.$$

This final group is in Alg_u/F by Proposition 11.2 and hence killed by a power of p . Hence the composite of the above two morphisms maps $\mathbf{H}^2(\mathbf{X}, \mathbf{G}_m)_{\text{div}}^0$ to zero. Let G be the inverse image of the subgroup $\mathbf{H}^2(\mathbf{X}, \mathbf{G}_m)_{\text{div}}^0[p]^0$ under the natural morphism

$$\mathbf{H}^2(\mathbf{X}, \mathfrak{T}(1))^0 \rightarrow \mathbf{H}^2(\mathbf{X}, \mathbf{G}_m)^0.$$

The image of G^0 in $\mathbf{H}^2(\mathbf{X}, \mathbf{G}_m)$ is $\mathbf{H}^2(\mathbf{X}, \mathbf{G}_m)_{\text{div}}^0[p]^0$. We have a natural isomorphism and morphism

$$\mathbf{H}^2(\mathbf{X}, \mathfrak{T}(1))^0 \cong \text{Ext}_{F_{\text{proet}}^{\text{indrat}}}^1(\mathbf{H}^2(\mathbf{X}, \mathfrak{T}(1)), \Lambda_\infty) \rightarrow \text{Ext}_{F_{\text{proet}}^{\text{indrat}}}^1(\mathbf{H}^1(\mathbf{X}, \mathbf{G}_m) \otimes \Lambda, \Lambda_\infty).$$

We have a commutative diagram

$$\begin{array}{ccc} \mathbf{H}^2(\mathbf{X}, \mathfrak{T}(1))^0 & \longrightarrow & \text{Ext}_{F_{\text{proet}}^{\text{indrat}}}^1(\mathbf{H}^1(\mathbf{X}, \mathbf{G}_m) \otimes \Lambda, \Lambda_\infty) \\ \downarrow & & \downarrow \\ \mathbf{H}^2(\mathbf{X}, \mathbf{G}_m)^0[p^\infty] & \longrightarrow & \text{Ext}_{F_{\text{proet}}^{\text{indrat}}}^1(\mathbf{H}^1(\mathbf{X}, \mathbf{G}_m), \Lambda_\infty). \end{array}$$

The image of G in the right lower term is zero. The right vertical morphism has finite kernel. Hence the image of G^0 in the right upper term is zero.

Let $S \subset P$ be the set of primes dividing p . Let $\mathfrak{X} \rightarrow \text{Spec } A$, $W_\eta \subset \mathfrak{X}$ and $T \subset Y$ be as in Section 7. Let $x \in Y_0$ and $\eta_x \in Y_1^x$. Take T large enough so that $x \in T$. We have $\mathbf{H}^1(\hat{\mathbf{K}}_{\eta_x}, \Lambda) \in \mathcal{W}_F$ and $\mathbf{H}^2(\mathbf{X}, \mathfrak{T}(1)), \mathbf{H}^1(\mathbf{X}, \mathbf{G}_m) \otimes \Lambda \in \text{Alg}_u/F$. Hence the image of $\mathbf{H}^1(\hat{\mathbf{K}}_{\eta_x}, \Lambda) \rightarrow \mathbf{H}^2(\mathbf{X}, \mathfrak{T}(1))$ is in Alg_u/F by Propositions 2.4 and 2.2. Therefore, by Proposition 10.2, the image of the composite

$$\mathbf{H}^1(\hat{\mathbf{K}}_{\eta_x}, \Lambda(1)) \rightarrow \mathbf{H}^2(\mathbf{X}, \mathfrak{T}(1))$$

of (10.3) is contained in the subgroup $\mathbf{H}^1(\mathbf{X}, \mathbf{G}_m) \otimes \Lambda$ of the target. Hence the dual morphism

$$\text{Ext}_{F_{\text{proet}}^{\text{indrat}}}^1(\mathbf{H}^2(\mathbf{X}, \mathfrak{T}(1)), \Lambda_\infty) \rightarrow \text{Ext}_{F_{\text{proet}}^{\text{indrat}}}^1(\mathbf{H}^1(\hat{\mathbf{K}}_{\eta_x}, \Lambda(1)), \Lambda_\infty)$$

factors through $\mathbf{Ext}_{F_{\text{proet}}}^1(\mathbf{H}^1(\mathbf{X}, \mathbf{G}_m) \otimes \Lambda, \Lambda_\infty)$. Therefore, the image of G^0 in $\mathbf{Ext}_{F_{\text{proet}}}^1(\mathbf{H}^1(\hat{\mathbf{K}}_{\eta_x}, \Lambda(1)), \Lambda_\infty)$ is zero. By the diagram in the proof of [Suz24, Proposition 10.1.5] (which is given in [Suz24, Section 10.6]), we have a commutative diagram

$$\begin{array}{ccc} \mathbf{H}^2(\mathbf{X}, \mathfrak{T}(1))^0 & \xrightarrow{\sim} & \mathbf{Ext}_{F_{\text{proet}}}^1(\mathbf{H}^2(\mathbf{X}, \mathfrak{T}(1)), \Lambda_\infty) \\ \downarrow & & \downarrow \\ \mathbf{H}^2(\mathbf{U}_S, \Lambda(1))^0 & \xrightarrow{\sim} & \mathbf{Ext}_{F_{\text{proet}}}^1(\mathbf{H}_c^2(\mathbf{U}_S, \Lambda(1)), \Lambda_\infty). \end{array}$$

By the diagram in the proof of [Suz24, Proposition 10.5.5], we have a commutative diagram

$$\begin{array}{ccc} \mathbf{H}^2(\mathbf{U}_S, \Lambda(1))^0 & \xrightarrow{\sim} & \mathbf{Ext}_{F_{\text{proet}}}^1(\mathbf{H}_c^2(\mathbf{U}_S, \Lambda(1)), \Lambda_\infty) \\ \downarrow & & \downarrow \\ \mathbf{H}^2(\hat{\mathbf{K}}_{\eta_x}, \Lambda(1))^0 & \xrightarrow{\sim} & \mathbf{Ext}_{F_{\text{proet}}}^1(\mathbf{H}^1(\hat{\mathbf{K}}_{\eta_x}, \Lambda(1)), \Lambda_\infty). \end{array}$$

Combining the above two diagrams, we obtain a commutative diagram

$$\begin{array}{ccc} \mathbf{H}^2(\mathbf{X}, \mathfrak{T}(1))^0 & \xrightarrow{\sim} & \mathbf{Ext}_{F_{\text{proet}}}^1(\mathbf{H}^2(\mathbf{X}, \mathfrak{T}(1)), \Lambda_\infty) \\ \downarrow & & \downarrow \\ \mathbf{H}^2(\hat{\mathbf{K}}_{\eta_x}, \Lambda(1))^0 & \xrightarrow{\sim} & \mathbf{Ext}_{F_{\text{proet}}}^1(\mathbf{H}^1(\hat{\mathbf{K}}_{\eta_x}, \Lambda(1)), \Lambda_\infty). \end{array}$$

Hence the image of G^0 in $\mathbf{H}^2(\hat{\mathbf{K}}_{\eta_x}, \Lambda(1))^0$ is zero. On F -valued points, the left vertical morphism factors as

$$H^2(X, \mathfrak{T}(1)) \rightarrow \text{Br}(K)[p] \rightarrow \text{Br}(\hat{K}_\eta)[p] \rightarrow \text{Br}(\hat{K}_{\eta_x})[p].$$

By Proposition 10.1, we know that the image of $G^0(F)$ in $\text{Br}(\hat{K}_\eta)[p]$ is contained in $H^1(Y_\eta, \Lambda)$. Thus we have a homomorphism $G^0(F) \rightarrow H^1(Y_\eta, \Lambda)$. This is compatible with base field extensions. Hence we have a homomorphism $G^0(F') \rightarrow H^1(Y_\eta(F'), \Lambda)$ for any algebraically closed field $F' \in F^{\text{perar}}$ functorial in F' . Hence we have a morphism $G^0 \rightarrow \mathbf{H}^1(Y_\eta, \Lambda)$. It is a morphism from a connected group to a finite group, hence zero. Therefore, the image of $G^0(F)$ in $\text{Br}(\hat{K}_\eta)[p]$ is zero. By the purity for Brauer groups, we know that the image of $G^0(F)$ in $\text{Br}(K)[p]$ is contained in $\text{Br}(\mathfrak{X})[p]$. As in the final paragraph of [Sai86, Section 3], we have $\text{Br}(\mathfrak{X}) \cong \text{Br}(Y)$ by the same argument as in the proof of [Gro68, Theorem 3.1]. As $F = \bar{F}$, we have $\text{Br}(Y) = 0$. Thus the image of $G^0(F)$ in $\text{Br}(X)$ is zero. The result then follows. $\square$

PROPOSITION 11.8. *There exists a canonical isomorphism*

$$\mathbf{H}^2(\mathbf{X}, \mathbf{G}_m)[p^\infty] \cong \mathbf{Ext}_{F_{\text{proet}}}^1(\mathbf{H}^1(\mathbf{X}, \mathbf{G}_m), \Lambda_\infty).$$

Proof. Let $n \geq 1$. We have

$$\mathbf{H}^2(\mathbf{X}, \mathfrak{T}_n(1)) \simeq \mathbf{Ext}_{F_{\text{proet}}}^{-1}(R\Gamma(\mathbf{X}, \mathfrak{T}_n(1)), \Lambda_\infty)$$

by duality. The right-hand side is isomorphic to $\mathbf{Ext}_{F_{\text{proet}}}^{-1}(\tau_{\geq 1}\tau_{\leq 2}R\Gamma(\mathbf{X}, \mathfrak{T}_n(1)), \Lambda_\infty)$. Let G_n be the canonical mapping cone of the natural morphism

$$\mathbf{A}^\times / \mathbf{A}^{\times p^n}[-1] \rightarrow \tau_{\geq 1}\tau_{\leq 2}R\Gamma(\mathbf{X}, \mathfrak{T}_n(1)).$$

Since $\mathbf{A}^\times / \mathbf{A}^{\times p^n}$ is connected, we have

$$\mathbf{Ext}_{F_{\text{proet}}}^{-1}(\tau_{\geq 1}\tau_{\leq 2}R\Gamma(\mathbf{X}, \mathfrak{T}_n(1)), \Lambda_\infty) \cong \mathbf{Ext}_{F_{\text{proet}}}^{-1}(G_n, \Lambda_\infty).$$

We have a canonical distinguished triangle

$$\mathbf{H}^1(\mathbf{X}, \mathbf{G}_m) \otimes^L \Lambda_n[-2] \rightarrow G_n \rightarrow \mathbf{H}^2(\mathbf{X}, \mathbf{G}_m)[p^n](-2).$$

Applying $\mathbf{Ext}_{F_{\text{proet}}^{\text{indrat}}}(\cdot, \Lambda_\infty)$, we obtain an exact sequence

$$\begin{aligned} \mathbf{Ext}_{F_{\text{proet}}^{\text{indrat}}}^1(\mathbf{H}^2(\mathbf{X}, \mathbf{G}_m)[p^n], \Lambda_\infty) &\rightarrow \mathbf{H}^2(\mathbf{X}, \mathfrak{T}_n(1)) \\ &\rightarrow \mathbf{Ext}_{F_{\text{proet}}^{\text{indrat}}}^1(\mathbf{H}^1(\mathbf{X}, \mathbf{G}_m) \otimes^L \Lambda_n, \Lambda_\infty) \rightarrow 0. \end{aligned} \quad (11.2)$$

All the terms are in Alg_u/F . By Proposition 11.7, we have

$$\varinjlim_n \mathbf{Ext}_{F_{\text{proet}}^{\text{indrat}}}^1(\mathbf{H}^2(\mathbf{X}, \mathbf{G}_m)[p^n], \Lambda_\infty) = 0.$$

Also,

$$\varinjlim_n \mathbf{Ext}_{F_{\text{proet}}^{\text{indrat}}}^1(\mathbf{H}^1(\mathbf{X}, \mathbf{G}_m) \otimes^L \Lambda_n, \Lambda_\infty) \cong \mathbf{Ext}_{F_{\text{proet}}^{\text{indrat}}}^1(\mathbf{H}^1(\mathbf{X}, \mathbf{G}_m), \Lambda_\infty)$$

by [Suz20b, Proposition 2.4.1(d)]. Thus we get the result by taking the direct limit in n . $\square$

Recall from [Suz20b, Proposition 2.4.1] that for $G \in \text{Alg}/F$, the group $\pi_0 \mathbf{Ext}_{F_{\text{proet}}^{\text{indrat}}}^1(G, \Lambda_\infty)$ is the Pontryagin dual of the p -adic Tate module of the semi-abelian part $G_{\text{sAb}} \subset G$ and the group $\mathbf{Ext}_{F_{\text{proet}}^{\text{indrat}}}^1(G, \Lambda_\infty)^0$ is the Serre dual of the unipotent group G^0/G_{sAb} .

Now we define

$$\mathbf{Pic}_X = \mathbf{H}^1(\mathbf{X}, \mathbf{G}_m), \quad \mathbf{Br}_X = \mathbf{H}^2(\mathbf{X}, \mathbf{G}_m).$$

Theorem 1.6 follows from the above results.

PROPOSITION 11.9. *The group $\mathbf{H}^3(\mathbf{X}, \mathbf{G}_m)[p^\infty]$ is connected. It is isomorphic to the Serre dual of $\mathbf{A}^\times/\mathbf{G}_m$. In particular, we have $\mathbf{H}^3(\mathbf{X}, \mathbf{G}_m)[p^\infty] \in \mathcal{W}_F$.*

Proof. The component group of $\mathbf{H}^3(\mathbf{X}, \mathbf{G}_m)[p^\infty]$ is Pontryagin dual to the inverse limit of $\mathbf{H}^0(\mathbf{X}, \mathfrak{T}_n(1)) \cong \mathbf{A}^\times[p^n]$. This inverse limit is zero since A contains only finitely many roots of unity of order a power of p . For any $n \geq 1$, we have an exact sequence

$$0 \rightarrow \mathbf{A}^\times/\mathbf{A}^{\times p^n} \rightarrow \mathbf{H}^1(\mathbf{X}, \mathfrak{T}_n(1)) \rightarrow \mathbf{H}^1(\mathbf{X}, \mathbf{G}_m)[p^n] \rightarrow 0.$$

As $\mathbf{H}^1(\mathbf{X}, \mathbf{G}_m) \in \text{Alg}/F$, the inverse limit of $\mathbf{H}^1(\mathbf{X}, \mathbf{G}_m)[p^n]$ in n is profinite. Hence the Serre dual of $\mathbf{H}^3(\mathbf{X}, \mathbf{G}_m)[p^\infty]$ is the inverse limit of $\mathbf{A}^\times/\mathbf{A}^{\times p^n}$, which is $\mathbf{A}^\times/\mathbf{G}_m$. $\square$

Applying $\varprojlim_n$, π_0 and $(\cdot)_{\text{tor}}$ to (11.2), we obtain an isomorphism

$$\pi_0 \mathbf{H}^1(\mathbf{X}, \mathbf{G}_m)[p^\infty] \xrightarrow{\sim} \mathbf{Hom}(\pi_0 \mathbf{H}^1(\mathbf{X}, \mathbf{G}_m)[p^\infty], \Lambda_\infty)$$

or a Pontryagin duality

$$\pi_0 \mathbf{H}^1(\mathbf{X}, \mathbf{G}_m)[p^\infty] \leftrightarrow \pi_0 \mathbf{H}^1(\mathbf{X}, \mathbf{G}_m)[p^\infty]. \quad (11.3)$$

On the other hand, if $L \times M \rightarrow \mathbb{Z}$ is a non-degenerate pairing of finitely generated free abelian groups, then it induces a Pontryagin duality $\mathbf{Hom}(M, \mathbb{Z})/L \leftrightarrow \mathbf{Hom}(L, \mathbb{Z})/M$ on the discriminant groups. Hence the non-degeneracy of the intersection pairing (11.1) on $\mathfrak{X}$ induces a Pontryagin duality

$$\delta_Y[p^\infty] \leftrightarrow \delta_Y[p^\infty]. \quad (11.4)$$

The following states that our duality is compatible with the intersection pairing.

PROPOSITION 11.10. *The pairings (11.3) and (11.4) are compatible under the isomorphism $\pi_0 \mathbf{H}^1(\mathbf{X}, \mathbf{G}_m) \cong \delta_Y$ in Proposition 11.2.*

Proof. We may assume $F = \overline{F}$, and it is enough to compare the two pairings on F -valued points. By construction, (11.3) is given as follows. Let $n \geq 1$. The cup product and the trace map give a pairing

$$H^1(X, \mathfrak{T}_n(1)) \times H^2(X, \mathfrak{T}_n(1)) \rightarrow H^3(X, \mathfrak{T}_n(2)) \rightarrow \Lambda_n.$$

With the exact sequences

$$0 \rightarrow A^\times \otimes \Lambda_n \rightarrow H^1(X, \mathfrak{T}_n(1)) \rightarrow \text{Pic}(X)[p^n] \rightarrow 0,$$

$$0 \rightarrow \text{Pic}(X) \otimes \Lambda_n \rightarrow H^2(X, \mathfrak{T}_n(1)) \rightarrow \text{Br}(X)[p^n] \rightarrow 0,$$

since $A^\times \otimes \Lambda_n$ is connected, this pairing induces a pairing

$$\text{Pic}(X)[p^n] \times \text{Pic}(X) \otimes \Lambda_n \rightarrow \Lambda_n.$$

Taking suitable direct and inverse limits, this factors as a pairing between component groups, which gives (11.3). We calculate it explicitly. Let $\mathfrak{p} \in P$ be arbitrary. Consider the natural maps

$$H^1(X, \mathfrak{T}_n(1)) \rightarrow H^1(K, \Lambda_n(1)),$$

$$H^1(\hat{K}_{\mathfrak{p}}, \Lambda_n(1)) \rightarrow H^2(\hat{A}_{\mathfrak{p}}, \mathfrak{T}_n(1)) \rightarrow H^2(X, \mathfrak{T}_n(1)).$$

We have a commutative diagram of pairings

$$\begin{array}{ccccccc} H^1(X, \mathfrak{T}_n(1)) & \times & H^2(X, \mathfrak{T}_n(1)) & \longrightarrow & H^3(X, \mathfrak{T}_n(2)) & \longrightarrow & \Lambda_n \\ \downarrow & & \uparrow & & \uparrow & & \parallel \\ H^1(K, \Lambda_n(1)) & \times & H^1(\hat{K}_{\mathfrak{p}}, \Lambda_n(1)) & \longrightarrow & H^2(\hat{K}_{\mathfrak{p}}, \Lambda_n(2)) & \longrightarrow & \Lambda_n, \end{array}$$

where the lower pairings are the cup-product pairing and the trace map. We also have a commutative diagram

$$\begin{array}{ccc} H^1(X, \mathfrak{T}_n(1)) & \longrightarrow & H^1(K, \Lambda_n(1)) \\ \downarrow & & \downarrow \\ \text{Pic}(X)[p^n] & \longrightarrow & (K^\times/A^\times) \otimes \Lambda_n, \end{array}$$

where the lower horizontal map comes from the exact sequence

$$0 \rightarrow A^\times \rightarrow K^\times \rightarrow \bigoplus_{\mathfrak{q} \in P} \mathbb{Z} \rightarrow \text{Pic}(X) \rightarrow 0,$$

and a commutative diagram

$$\begin{array}{ccc} \hat{K}_{\mathfrak{p}}^\times \otimes \Lambda_n & \longrightarrow & \text{Pic}(X) \otimes \Lambda_n \\ \parallel & & \downarrow \\ H^1(\hat{K}_{\mathfrak{p}}, \Lambda_n(1)) & \longrightarrow & H^2(X, \mathfrak{T}_n(1)), \end{array}$$

where the upper horizontal map factors as the valuation map $\hat{K}_{\mathfrak{p}}^\times \otimes \Lambda_n \rightarrow \Lambda_n$ followed by the map $1 \mapsto [\mathfrak{p}]$. Hence we have a commutative diagram of pairings

$$\begin{array}{ccccccc} \text{Pic}(X)[p^n] & \times & \text{Pic}(X) \otimes \Lambda_n & \longrightarrow & \Lambda_n \\ \downarrow & & \uparrow & & \parallel \\ (K^\times/A^\times) \otimes \Lambda_n & \times & \hat{K}_{\mathfrak{p}}^\times \otimes \Lambda_n & \longrightarrow & \Lambda_n, \end{array}$$

where the lower pairing is given by the tame symbol map $\hat{K}_{\mathfrak{p}}^{\times} \times \hat{K}_{\mathfrak{p}}^{\times} \rightarrow \kappa(\mathfrak{p})^{\times}$ followed by the valuation map on $\kappa(\mathfrak{p})^{\times}$. Hence the value of the upper pairing at an element $(D, [\mathfrak{p}])$ is given as follows. Represent $p^n D$ by a rational function $f \in K^{\times}$. Take a prime element π of the local ring of A at $\mathfrak{p}$. Consider the tame symbol $\{f, \pi\}_{\mathfrak{p}} \in \kappa(\mathfrak{p})^{\times}$. Then $v_{\kappa(\mathfrak{p})}(\{f, \pi\}_{\mathfrak{p}})/p^n \in \Lambda_{\infty}$ is the value of the pairing (11.3) at $(D, [\mathfrak{p}])$.

The second pairing (11.4) is given as follows. Let $D, \mathfrak{p}$ and f be as above. The divisor on $\mathfrak{X}$ defined by f is the sum of $p^n D$ and another divisor E supported on Y . Consider the intersection number $E \cdot [\mathfrak{p}]$. Then $(E \cdot [\mathfrak{p}])/p^n \in \Lambda_{\infty}$ is the value of the second pairing at $(D, [\mathfrak{p}])$.

Therefore, it is enough to show that

$$v_{\kappa(\mathfrak{p})}(\{f, \pi\}_{\mathfrak{p}}) = E \cdot [\mathfrak{p}].$$

Suitably replacing the resolution $\mathfrak{X}$, we may assume that the divisor on $\mathfrak{X}$ defined by $f, \mathfrak{p}$ and Y is supported on a strict normal crossing divisor. Let η be the generic point of the irreducible component of Y containing the specialization of $\mathfrak{p}$ in Y . Then we can see that these two numbers are both equal to $v_{\hat{K}_{\eta}}(f)$. $\square$

12. The case of finite residue field

In this section, we prove Theorem 1.7. We deduce it from Theorems 1.2 and 1.4 by applying $R\Gamma(F, \cdot)$.

Assume $F = \mathbb{F}_q$. Let $*_{\text{proet}}$ be the pro-étale site of a point [BS15, Example 4.1.10]. Identify a profinite set with the affine F -scheme having the same set of points. This defines a morphism of sites

$$\text{Spec } F_{\text{proet}}^{\text{indrat}} \rightarrow *_{\text{proet}}.$$

Its pushforward functor is simply denoted by $\Gamma(F, \cdot)$ by abuse of notation. Let Fin_p be the category of finite abelian p -groups. Let IPFin_p be the ind-category of the pro-category of Fin_p .

DEFINITION 12.1. Define $\mathcal{W}_0 \subset \text{IPFin}_p$ to be the full subcategory consisting of objects G that fit in an exact sequence

$$0 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 0,$$

where $G' = \varprojlim_{n \geq 1} G'_n$ is profinite (with $G'_n \in \text{Fin}_p$) and $G'' = \varinjlim_{n \geq 1} G''_n$ is ind-finite (with $G''_n \in \text{Fin}_p$), both indexed by $\mathbb{N}$.

All objects of $\mathcal{W}_0$ are locally compact and, in particular, Hausdorff.

PROPOSITION 12.2. *Let $G \in \mathcal{W}_0$. Then $R\mathbf{Hom}_{*_{\text{proet}}}(G, \Lambda_{\infty})$ is concentrated in degree zero with cohomology given by the usual Pontryagin dual of G . In particular, $\mathbf{Hom}_{*_{\text{proet}}}(G, \Lambda_{\infty}) \in \mathcal{W}_0$.*

Proof. This is obvious if G is finite. Let $G = \varprojlim_{n \geq 1} G_n$ be profinite (with $G_n \in \text{Fin}_p$). Then

$$R\mathbf{Hom}_{*_{\text{proet}}}(G, \Lambda_{\infty}) \cong \varinjlim_n R\mathbf{Hom}_{*_{\text{proet}}}(G_n, \Lambda_{\infty})$$

by the same argument as in the proof of [Suz20b, Theorem 2.3.1(c)] (see also the proof of [BS24, Proposition 8.10]). Let $G = \varinjlim_{n \geq 1} G_n$ be ind-finite (with $G_n \in \text{Fin}_p$). Then

$$R\mathbf{Hom}_{*_{\text{proet}}}(G, \Lambda_{\infty}) \cong R\varprojlim_n R\mathbf{Hom}_{*_{\text{proet}}}(G_n, \Lambda_{\infty})$$

by [Suz20b, Proposition 2.2.3]. Combining all these, we get the result. $\square$

Note that non-Hausdorff groups may have non-vanishing $\mathbf{Ext}^1$: we have

$$R\mathbf{Hom}_{*\text{proet}}(\Lambda^{\mathbb{N}}/\Lambda^{\oplus \mathbb{N}}, \Lambda_{\infty}) \cong \Lambda^{\mathbb{N}}/\Lambda^{\oplus \mathbb{N}}[-1].$$

PROPOSITION 12.3. *Let $G \in D^b(F_{\text{proet}}^{\text{indrat}})$ be such that $H^q G \in \mathcal{W}_F$ for all q . Then $R\Gamma(F, G) \in D^b(*_{\text{proet}})$ and $H^q(F, G) \in \mathcal{W}_0$ for all q . The group $H^q(F, G)$ is profinite if G is pro-algebraic and ind-finite if G is ind-algebraic. We have an exact sequence*

$$0 \rightarrow H^1(F, \pi_0(H^{q-1}G)) \rightarrow H^q(F, G) \rightarrow \Gamma(F, H^q G) \rightarrow 0$$

for all q .

Proof. Let $G \in \text{Alg}_u/F$ be finite étale. Then $H^q(F, G)$ is finite for $q = 0, 1$ and is zero otherwise. Let $G \in \text{Alg}_u/F$ be connected. Then $H^q(F, G)$ is the finite group $G(F)$ for $q = 0$ and is zero otherwise. Let $G = \varprojlim_n G_n \in \mathcal{W}_F$ be connected pro-algebraic (with $G_n \in \text{Alg}_u/F$ connected such that $G_{n+1} \rightarrow G_n$ is surjective with connected kernel). Then

$$R\Gamma(F, G) \cong R\varprojlim_n R\Gamma(F, G_n) \cong R\varprojlim_n (G_n(F)) \cong G(F)$$

by [Suz20b, Proposition 2.2.4(b)], and $G(F)$ is profinite. Let $G = \varinjlim_n G_n \in \mathcal{W}_F$ be connected ind-algebraic (with $G_n \in \text{Alg}_u/F$ connected unipotent such that $G_n \hookrightarrow G_{n+1}$ is injective). Then

$$R\Gamma(F, G) \cong \varinjlim_n R\Gamma(F, G_n) \cong \varinjlim_n (G_n(F)) \cong G(F)$$

by [Suz20b, Proposition 2.2.4(b)], and $G(F)$ is ind-finite. Combining all these, we get the result. $\square$

The composite functor

$$D(\mathbf{U}_{S,\text{et}}) \xrightarrow{R\Gamma(\mathbf{U}_S, \cdot)} D(F_{\text{proet}}^{\text{indrat}}) \xrightarrow{R\Gamma(F, \cdot)} D(*_{\text{proet}})$$

is denoted by $R\Gamma(\mathbf{U}_S, \cdot)$ by abuse of notation, with cohomologies $H^q(\mathbf{U}_S, \cdot)$. The composite functor

$$D(\mathbf{U}_{S,\text{et}}) \xrightarrow{R\Gamma_c(\mathbf{U}_S, \cdot)} D(F_{\text{proet}}^{\text{indrat}}) \xrightarrow{R\Gamma(F, \cdot)} D(*_{\text{proet}})$$

is denoted by $R\Gamma_c(\mathbf{U}_S, \cdot)$ by abuse of notation, with cohomologies $H_c^q(\mathbf{U}_S, \cdot)$.

PROPOSITION 12.4. *Let $n \geq 1$ and $q, r \in \mathbb{Z}$. Then the objects $H^q(\mathbf{U}_S, \mathfrak{T}_n(r))$ and $H_c^q(\mathbf{U}_S, \mathfrak{T}_n(r))$ are in $\mathcal{W}_0$ for all q and are zero for all but finitely many q . We have a perfect pairing*

$$R\Gamma(\mathbf{U}_S, \mathfrak{T}_n(r)) \otimes^L R\Gamma_c(\mathbf{U}_S, \mathfrak{T}_n(2-r)) \rightarrow \Lambda_{\infty}[-4]$$

in $D(*_{\text{proet}})$ and a Pontryagin duality

$$H^q(\mathbf{U}_S, \mathfrak{T}_n(r)) \leftrightarrow H_c^{4-q}(\mathbf{U}_S, \mathfrak{T}_n(2-r)).$$

Proof. This follows from the perfect pairing

$$R\Gamma(\mathbf{U}_S, \mathfrak{T}_n(r)) \otimes^L R\Gamma_c(\mathbf{U}_S, \mathfrak{T}_n(2-r)) \rightarrow \Lambda_{\infty}[-3],$$

[Suz20a, Proposition 4.2.1] and Propositions 12.2 and 12.3. $\square$

This proves Theorem 1.7.

Finally, we give a local version of this theorem. Let K and k be as in Section 5. The composite functor

$$D(\mathbf{K}_{\text{et}}) \xrightarrow{R\Gamma(\mathbf{K}, \cdot)} D(F_{\text{proet}}^{\text{indrat}}) \xrightarrow{R\Gamma(F, \cdot)} D(*_{\text{proet}})$$

is denoted by $R\Gamma(K, \cdot)$ by abuse of notation, with cohomologies $H^q(K, \cdot)$. The composite functor

$$D(\mathbf{O}_{K,\text{et}}) \xrightarrow{R\Gamma(\mathbf{O}_K, \cdot)} D(F_{\text{proet}}^{\text{indrat}}) \xrightarrow{R\Gamma(F, \cdot)} D(*_{\text{proet}})$$

is denoted by $R\Gamma(\mathcal{O}_K, \cdot)$ by abuse of notation, with cohomologies $H^q(\mathcal{O}_K, \cdot)$. The composite functor

$$D(\mathbf{O}_{K,\text{et}}) \xrightarrow{R\Gamma_c(\mathbf{O}_K, \cdot)} D(F_{\text{proet}}^{\text{indrat}}) \xrightarrow{R\Gamma(F, \cdot)} D(*_{\text{proet}})$$

is denoted by $R\Gamma_c(\mathcal{O}_K, \cdot)$ by abuse of notation, with cohomologies $H_c^q(\mathcal{O}_K, \cdot)$.

PROPOSITION 12.5. *Let $n \geq 1$ and $q, r \in \mathbb{Z}$. Then we have*

$$H^q(K, \Lambda_n(r)), H^q(\mathcal{O}_K, \mathfrak{T}_n(r)), H_c^q(\mathcal{O}_K, \mathfrak{T}_n(r)) \in \mathcal{W}_0.$$

We have perfect pairings

$$\begin{aligned} R\Gamma(K, \Lambda_n(r)) \otimes^L R\Gamma(K, \Lambda_n(2-r)) &\rightarrow \Lambda_\infty[-3], \\ R\Gamma(\mathcal{O}_K, \mathfrak{T}_n(r)) \otimes^L R\Gamma_c(\mathcal{O}_K, \mathfrak{T}_n(2-r)) &\rightarrow \Lambda_\infty[-4] \end{aligned}$$

in $D(*_{\text{proet}})$ and Pontryagin dualities

$$\begin{aligned} H^q(K, \Lambda_n(r)) &\leftrightarrow H^{3-q}(K, \Lambda_n(2-r)), \\ H^q(\mathcal{O}_K, \mathfrak{T}_n(r)) &\leftrightarrow H_c^{4-q}(\mathcal{O}_K, \mathfrak{T}_n(2-r)). \end{aligned}$$

Proof. This follows from the perfect pairings

$$\begin{aligned} R\Gamma(\mathbf{K}, \Lambda_n(r)) \otimes^L R\Gamma(\mathbf{K}, \Lambda_n(2-r)) &\rightarrow \Lambda_\infty[-2], \\ R\Gamma(\mathbf{O}_K, \mathfrak{T}_n(r)) \otimes^L R\Gamma_c(\mathbf{O}_K, \mathfrak{T}_n(2-r)) &\rightarrow \Lambda_\infty[-3], \end{aligned}$$

[Suz20a, Proposition 4.2.1] and Propositions 12.2 and 12.3. □

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