Algebraic Geometry 11 (3) (2024) 346–385
doi:10.14231/AG-2024-012

RDP del Pezzo surfaces with global vector fields in odd characteristic

Gebhard Martin and Claudia Stadlmayr

Abstract

We classify RDP del Pezzo surfaces with global vector fields over arbitrary algebraically closed fields of characteristic $p \neq 2$. In characteristic 0, every RDP del Pezzo surface X is *equivariant*, that is, $\mathrm{Aut}_X = \mathrm{Aut}_{\widetilde{X}}$, where $\widetilde{X}$ is the minimal resolution of X; hence the classification of RDP del Pezzo surfaces with global vector fields is equivalent to the classification of weak del Pezzo surfaces with global vector fields.

In this article, we show that if $p \neq 2, 3, 5, 7$, then it is still true that every RDP del Pezzo surface is equivariant. We classify the non-equivariant RDP del Pezzo surfaces in characteristic $p = 3, 5, 7$, giving explicit equations for every such RDP del Pezzo surface in all possible degrees. As an application, we construct regular non-smooth RDP del Pezzo surfaces over imperfect fields of characteristic 7, thereby showing that the known bound $p \leqslant 7$ for the characteristics, where such a surface can exist, is sharp.

1. Introduction

We are working over an algebraically closed field k of characteristic $p \geqslant 0$. Let X be a del Pezzo surface with at worst rational double points as singularities, and let $\pi\colon \widetilde{X} \to X$ be its minimal resolution, so that, by definition, $\widetilde{X}$ is a weak del Pezzo surface. Since $-K_X$ is ample, Aut_X is an affine group scheme of finite type, so its group of automorphisms $\mathrm{Aut}(X) := \mathrm{Aut}_X(k)$ is infinite if and only if the automorphism scheme Aut_X is positive-dimensional. Moreover, by Blanchard's lemma [Bri17, Theorem 7.2.1], since X is the anti-canonical model of $\widetilde{X}$, there is a closed immersion of group schemes $\pi_*\colon \mathrm{Aut}_{\widetilde{X}} \hookrightarrow \mathrm{Aut}_X$. We call X *equivariant* if π_* is an isomorphism. Summarizing, for all characteristics, there is the following chain of implications:

$$|\mathrm{Aut}(X)| = \big|\mathrm{Aut}\big(\widetilde{X}\big)\big| = \infty \implies H^0\big(\widetilde{X}, T_{\widetilde{X}}\big) \neq 0 \implies H^0(X, T_X) \neq 0\,. \tag{1.1}$$

Over the complex numbers, every RDP del Pezzo surface X (see Definition 3.1) is equivariant, so in particular we have $H^0(X, T_X) = H^0\big(\widetilde{X}, T_{\widetilde{X}}\big)$. Moreover, by Cartier's theorem, Aut_X^0 is smooth, so it is positive-dimensional if and only if $H^0(X, T_X) \neq 0$. In other words, in characteristic 0, all implications in (1.1) are in fact equivalences.

Received 14 August 2022, accepted in final form 5 June 2023.
2020 Mathematics Subject Classification 14E07, 14J26, 14J50, 14L15, 14J17, 14B05, 14G17.
Keywords: del Pezzo surfaces, automorphism schemes, rational double points, positive characteristic

Research of the first-named author was supported by the DFG Research Grant MA 8510/1-1. The second-named author gratefully acknowledges funding by the DFG Sachbeihilfe LI 1906/5-1 "Geometrie von rationalen Doppelpunkten" and support by the doctoral program TopMath and the TUM Graduate School.

In our recent article [MS20], we obtained the classification of weak del Pezzo surfaces with global vector fields over algebraically closed fields of arbitrary characteristic (over the complex numbers, an independent proof was given by Cheltsov and Prokhorov in [CP21]). By (1.1), this includes the classification of all RDP del Pezzo surfaces with infinite automorphism group, but we note that if $p = 2, 3$, there are RDP del Pezzo surfaces with finite automorphism group whose minimal resolution has global vector fields, so the first implication in (1.1) is not an equivalence precisely if $p = 2, 3$. In other words, we have

$$|\mathrm{Aut}(X)| = \big|\mathrm{Aut}\big(\widetilde{X}\big)\big| = \infty \overset{p \neq 2,3}{\Longleftarrow\!\!\!\Longrightarrow} H^0\big(\widetilde{X}, T_{\widetilde{X}}\big) \neq 0 \implies H^0(X, T_X) \neq 0 \tag{1.2}$$

and a classification of $\widetilde{X}$ with $H^0\big(\widetilde{X}, T_{\widetilde{X}}\big) \neq 0$ in all characteristics.

Now, the missing piece is a classification of RDP del Pezzo surfaces X with $H^0(X, T_X) \neq 0$. What makes this subtle in positive characteristic is the existence of non-equivariant rational double points. Recall that the dual graph of the exceptional locus of the minimal resolution of a rational double point is a Dynkin diagram of type A_n, D_n, or E_n. In positive characteristic, the Dynkin diagram does not always determine the formal isomorphism class of the singularity, but for each fixed graph Γ_n, there are only finitely many isomorphism classes Γ_n^r of RDPs with resolution graph Γ_n. These isomorphism classes have been classified by Lipman [Lip69] and Artin [Art77]. As a first step toward the classification of RDP del Pezzo surfaces with global vector fields, we extend Hirokado's result [Hir19] on the liftability of vector fields to group scheme actions on RDP del Pezzo surfaces as follows.

Theorem 1.1 (= Theorem 6.1). *Let X be an RDP del Pezzo surface, and let $\pi\colon \widetilde{X} \to X$ be its minimal resolution. Assume that one of the following conditions holds:*

(1) $p \notin \{2, 3, 5, 7\}$.
(2) *$p = 7$ and X does not contain an RDP of type A_6.*
(3) *$p = 5$ and X does not contain an RDP of type A_4 or E_8^0.*
(4) *$p = 3$ and X does not contain an RDP of type A_2, A_5, A_8, E_6^0, E_6^1, E_7^0, E_8^0, or E_8^1.*
(5) *$p = 2$ and X does not contain an RDP of type A_1, A_3, A_5, A_7, D_n^r, E_6^0, E_7^0, E_7^1, E_7^2, E_7^3, E_8^0, E_8^1, E_8^2, or E_8^3, where $n \leqslant 8$.*

Then, $\mathrm{Aut}_X = \mathrm{Aut}_{\widetilde{X}}$, and thus, in particular, $H^0\big(\widetilde{X}, T_{\widetilde{X}}\big) = H^0(X, T_X)$. Therefore, $H^0(X, T_X) \neq 0$ if and only if X is the anti-canonical model of one of the surfaces in the classification tables of [MS20].

Thus, in order to classify RDP del Pezzo surfaces X with global vector fields, we may restrict our attention to RDP del Pezzo surfaces containing a configuration Γ of RDPs excluded in Theorem 6.1. In Theorem 7.1, we give a criterion for X to be the blow-up of an RDP del Pezzo surface of higher degree with the same configuration Γ. In the language of the minimal model program, this means that we give a sufficient criterion for the existence of a $K_{\widetilde{X}}$-negative extremal ray on $\widetilde{X}$ which lies in the orthogonal complement of the exceptional locus over Γ. Using Blanchard's lemma, this allows us to set up an inductive argument for the classification of non-equivariant RDP del Pezzo surfaces X with RDP configuration Γ. This strategy will be carried out in Sections 8.1, 8.2, and 8.3 for characteristic $p = 7$, $p = 5$, and $p = 3$, respectively. The following theorem is obtained by combining Theorems 8.3, 8.6, and 8.8.

Theorem 1.2. *Let X be an RDP del Pezzo surface, and let $\pi\colon \widetilde{X} \to X$ be its minimal resolution. Assume that $H^0(X, T_X) \neq 0$. Then, the following hold:*

(1) *If $p = 7$ and X contains an RDP of type A_6, then X is one of the two surfaces in Table 5.*

(2) *If $p = 5$ and X contains an RDP of type A_4 or E_8^0, then X is one of the nine surfaces in Table 6.*

(3) *If $p = 3$ and X contains an RDP of type A_2, A_5, A_8, E_6^0, E_6^1, E_7^0, E_8^0, or E_8^1, then X is a member of one of the 56 families of surfaces in Tables 7, 8, 9, and 10.*

For each of these surfaces, we have $h^0(X, T_X) > h^0(\widetilde{X}, T_{\widetilde{X}})$ and in particular $\mathrm{Aut}_X \neq \mathrm{Aut}_{\widetilde{X}}$.

Remark 1.3. The reason why we do not treat the case $p = 2$ is due to the sheer amount of RDP del Pezzo surfaces with global vector fields in characteristic 2. Indeed, by Theorem 6.1, it is unclear whether $\mathrm{Aut}_{\widetilde{X}} = \mathrm{Aut}_X$ as soon as X has a single node, and in fact, even for the quadratic cone $\{x_0^2 - x_1x_2 = 0\} \subseteq \mathbb{P}^3$ in characteristic 2, it is not true that every vector field lifts to its minimal resolution; consider for example $x_3\partial_{x_0}$. However, in principle, our approach would also work if $p = 2$.

Comparing Tables 5, 6, 7, 8, 9, and 10 with the classification in [MS20], we see that in characteristics $p = 3, 5$, and 7, there exists an RDP del Pezzo surface X with $H^0(X, T_X) \neq 0$ whose minimal resolution admits no non-trivial global vector fields. In other words, we have the following picture, where the implications from right to left hold *only* in the indicated characteristics:

$$|\mathrm{Aut}(X)| = \left|\mathrm{Aut}(\widetilde{X})\right| = \infty \quad \overset{p\neq 2,3}{\underset{\Longrightarrow}{\Longleftarrow}} \quad H^0(\widetilde{X}, T_{\widetilde{X}}) \neq 0 \quad \overset{p\neq 2,3,5,7}{\underset{\Longrightarrow}{\Longleftarrow}} \quad H^0(X, T_X) \neq 0\,. \tag{1.3}$$

2. An application: Regular inseparable twists of RDP del Pezzo surfaces

A *twisted form* of a k-scheme X over a field extension $L \supseteq k$ is a scheme Y over L such that $Y_{\bar{L}} \cong X_{\bar{L}}$, where $\bar{L}$ is an algebraic closure of L. If X is a proper scheme over k, then smoothness of Aut_X is intimately related to properties of twisted forms of X, as the following proposition shows. Even though this proposition should be well known, we include the proof for the convenience of the reader.

Proposition 2.1. *Let $k \subseteq L$ be a field extension. Let X be a proper scheme over k, and let Y be a twisted form of X over L. Assume that Aut_X is smooth. Then, the following hold:*

(1) *If L is separably closed, then $Y \cong X_L$.*

(2) *If $\mathcal{P}$ is a property of schemes that is stable under field extensions and local in the étale topology, then, if X satisfies $\mathcal{P}$, then Y, as well, satisfies $\mathcal{P}$.*

Proof. Let us first prove claim (1). Recall that an fppf (*"fidèlement plat de présentation finie"*) cover of a scheme T is a set of flat morphisms $\{f_i \colon T_i \to T\}_{i\in I}$ that are locally of finite presentation and such that $T = \bigcup_{i\in I} f_i(T_i)$; see [Sta18, Tag 021L]. In particular, every finite field extension of L (separable or not) determines an fppf cover of $\mathrm{Spec}\, L$. As explained for example in [Mil80, Section III.4, p. 134], an isomorphism $\bar{\varphi} \colon Y_{\bar{L}} \cong X_{\bar{L}}$ gives rise to a Čech cocycle on the fppf site of $\mathrm{Spec}\, L$, hence to an element in $\check{H}^1_{\mathrm{fppf}}(\mathrm{Spec}\, L, \mathrm{Aut}_{X_L})$. By [Mil80, Chapter III, Theorem 4.3(b), Corollary 4.7, Remark 4.8], the smoothness of Aut_X implies that

$$\check{H}^1_{\mathrm{fppf}}(\mathrm{Spec}\, L, \mathrm{Aut}_{X_L}) = \check{H}^1_{\text{ét}}(\mathrm{Spec}\, L, \mathrm{Aut}_{X_L}),$$

and since L is separably closed, the latter is trivial. Hence, Y and X_L are already isomorphic over L.

For claim (2), let L^{sep} be the separable closure of L in $\bar{L}$. By claim (1), we have $X_{L^{\mathrm{sep}}} \cong Y_{L^{\mathrm{sep}}}$. Since X is proper, this isomorphism is defined over a finite subextension $L \subseteq L' \subseteq L^{\mathrm{sep}}$, so that $X_{L'} \cong Y_{L'}$. The morphism $\operatorname{Spec} L' \to \operatorname{Spec} L$ is finite and étale. Hence, by our assumptions on $\mathcal{P}$, if X satisfies $\mathcal{P}$, then $X_{L'}$ satisfies $\mathcal{P}$, and thus Y, as well, satisfies $\mathcal{P}$. □

Choosing for $\mathcal{P}$ the property that the singular locus is non-empty and specializing to the case where X is an RDP del Pezzo surface, we obtain the following.

Corollary 2.2. *Let X be an RDP del Pezzo surface over k. Let $k \subseteq L$ be a field extension, and let Y be a twisted form of X over L. If X has at least one singular point and Y is regular, then Aut_X is non-smooth.*

At a first glance, these twists seem rather hard to get a grip on geometrically, but it turns out that they can be written down explicitly if one has explicit descriptions of X and Aut_X. For example, consider $\mu_p \subseteq \mathrm{PGL}_{n+1,k}$ embedded diagonally with weights $(0, a_1, \dots, a_{n-1}, 1)$, that is, given on the level of scheme-valued points as $\lambda \mapsto \big(1, \lambda^{a_1}, \dots, \lambda^{a_{n-1}}, \lambda\big)$. Alternatively, the μ_p-action is given by the p-closed derivation $D = \sum_{i=1}^n a_i x_i \partial_{x_i}$ with $a_n = 1$. We can write μ_p as the kernel of the surjective homomorphism

$$f\colon \mathbb{G}_m^n \to \mathbb{G}_m^n\,, \quad (1, u_1, \dots, u_n) \mapsto \big(1, u_1 u_n^{-a_1}, \dots, u_{n-1} u_n^{-a_{n-1}}, u_n^p\big)\,.$$

By [Mil80, Proposition 4.5] and Hilbert's Theorem 90, for every field extension $k \subseteq L$, this yields a short exact sequence of abelian groups

$$0 \to \mathbb{G}_m^n(L) \xrightarrow{f(L)} \mathbb{G}_m^n(L) \xrightarrow{d} \check{H}^1_{\mathrm{fppf}}(\operatorname{Spec} L, \mu_p) \to 0\,.$$

Here, for $g \in \mathbb{G}_m^n(L)$, the element $d(g)$ is defined by choosing $\bar{g} \in \mathbb{G}_m^n\big(\bar{L}\big)$ such that $f\big(\bar{L}\big)(\bar{g}) = g_{\bar{L}}$ and setting $d(g)$ to be the image of the cocycle $(\bar{g} \otimes 1)^{-1}(1 \otimes \bar{g}) \in \mu_p\big(\bar{L} \otimes_L \bar{L}\big)$. If $X \subseteq \mathbb{P}^n_k$ is a subvariety stabilized by μ_p (which, at the level of derivations, means that $D(I_X) \subseteq I_X$, where I_X is the ideal of X), then $\bar{g}^{-1}(X_{\bar{L}}) \subseteq \mathbb{P}^n_{\bar{L}}$ is defined over L and the cocycle one associates with this twisted form is in the same class as $d(g)$. In other words, we can realize every twist of X corresponding to an element of $\check{H}^1_{\mathrm{fppf}}(\operatorname{Spec} L, \mu_p)$ by choosing a $\bar{g} \in \mathbb{G}_m^n\big(\bar{L}\big)$ such that $f\big(\bar{L}\big)(\bar{g})$ is defined over L and translating X along $\bar{g}^{-1}$.

Example 2.3. Consider the quartic curve $Q = \big\{x^3y + y^3z + z^3x = 0\big\} \subseteq \mathbb{P}^2_k$ in characteristic 7. It is stable under the μ_7-action with weights $(0, 3, 1)$. Let $L = k(t)$, and consider $\bar{g} = \big(1, t^{3/7}, t^{1/7}\big) \in \mathbb{G}_m^2\big(\bar{L}\big) \subseteq \mathrm{PGL}_3\big(\bar{L}\big)$. Then,

$$\bar{g}^{-1}(Q) = \big\{t^{3/7}x^3y + t^{10/7}y^3z + t^{3/7}z^3x = 0\big\} = \big\{x^3y + ty^3z + z^3x = 0\big\}\,.$$

Spreading out over $\mathbb{A}^1_k - \{0\} = \operatorname{Spec} k\big[t, t^{-1}\big]$, we obtain a fibered surface S over $\mathbb{A}^1_k - \{0\}$ which is easily checked to be smooth using the Jacobian criterion. Its generic fiber S_η is a twisted form of Q over $k(t)$, and this twisted form is regular because it is the generic fiber of a flat morphism between smooth k-schemes. Indeed, for all points $s \in S_\eta \subseteq S$, we have an isomorphism of local rings $\mathcal{O}_{S_\eta,s} \cong \mathcal{O}_{S,s}$, where $\mathcal{O}_{S,s}$ is regular because S is smooth over k. Taking the double cover of $\big(\mathbb{A}^1 - \{0\}\big) \times \mathbb{P}^2$ branched over S, we obtain a smooth fibered threefold T over $\mathbb{A}^1 - \{0\}$ whose generic fiber is the regular del Pezzo surface Y of degree 2 given by the equation

$$\big\{w^2 = x^3y + ty^3z + z^3x\big\} \subseteq \mathbb{P}_{k(t)}(1, 1, 1, 2)\,.$$

As before, Y is regular, being the generic fiber of a flat morphism of smooth k-schemes. Observe, however, that Y is not smooth because $Y_{\overline{k(t)}} \cong X$, where X is the del Pezzo surface of degree 2 with a singularity of type A_6 given in our Table 5.

Remark 2.4. Example 2.3 shows that the bound $p \leqslant 7$ given in [BT22, Proposition 5.2] for the characteristics in which non-smooth regular RDP del Pezzo surfaces can exist is sharp. Using the approach explained at the beginning of Example 2.3, it is not hard to construct similar examples if $p = 2, 3, 5$, but since this is not the topic of this article, we leave these constructions to the interested reader. Finally, we note that it is no mere coincidence that the Klein quartic in characteristic 7 appears in this context and refer the reader to [Stö04] for a closer study of this curve and its regular twists.

3. Preliminaries on (RDP) del Pezzo surfaces

In this section, we recall the definition of RDP del Pezzo surfaces and weak del Pezzo surfaces, which occur as minimal resolutions of RDP del Pezzo surfaces, as well as their basic properties.

Definition 3.1. Let X and $\widetilde{X}$ be projective surfaces.

- X is a *del Pezzo surface* if it is smooth and $-K_X$ is ample.
- $\widetilde{X}$ is a *weak del Pezzo surface* if it is smooth and $-K_{\widetilde{X}}$ is big and nef.
- X is an *RDP del Pezzo surface* if all its singularities are rational double points and $-K_X$ is ample.

In all the above cases, the number $\deg(X) = K_X^2$ $\big($respectively, $\deg\big(\widetilde{X}\big) = K_{\widetilde{X}}^2\big)$ is called the *degree of* X $\big($respectively, $\widetilde{X}\big)$.

Recall (for example, from [Dol12, Theorem 8.1.15, Corollary 8.1.24]) that $1 \leqslant \deg(X) = \deg\big(\widetilde{X}\big) \leqslant 9$ and that every weak del Pezzo surface of degree d and different from $\mathbb{P}^1 \times \mathbb{P}^1$ and the second Hirzebruch surface $\mathbb{F}_2$ can be realized as a blow-up of $\mathbb{P}^2$ in $9 - d$ (possibly infinitely near) points in almost general position.

As mentioned at the beginning of this section, weak del Pezzo surfaces arise as the minimal resolutions of RDP del Pezzo surfaces, and, conversely, every RDP del Pezzo surface X is the anti-canonical model of a weak del Pezzo surface $\widetilde{X}$. The linear systems $|-nK_{\widetilde{X}}|$ are well studied (see, for example, [BT22, Proposition 2.14, Theorem 2.15] for proofs in positive characteristic). We denote the morphism induced by a linear system $|D|$ by $\varphi_{|D|}$. We recall that a curve singularity is called *simple* if its completion is isomorphic to one of the normal forms given in [GK90, Section 1]. For the convenience of the reader, we recall the classification of simple curve singularities in characteristic different from 2 in Table 1. Here, the first column indicates the type of the simple curve singularity. Up to formal local isomorphism, their normal forms $f \in k[[x, y]]$ are given in the second column. Note that the type of the curve singularity coincides with the type of the rational double point that occurs on the double cover of $\mathbb{A}^2$ branched over $\{f = 0\}$.

The following description of the geometric picture is well known [Dol12, Theorem 8.3.2].

Theorem 3.2. *Let $\widetilde{X}$ be a weak del Pezzo surface of degree d.*

(1) *If $d \geqslant 3$, then $\varphi_{|-K_{\widetilde{X}}|}$ factors as $\widetilde{X} \xrightarrow{\pi} X \xhookrightarrow{\varphi_{|-K_X|}} \mathbb{P}^d$, where $\varphi_{|-K_X|}$ is a closed immersion that realizes X as a surface of degree d.*

(2) *If $d = 2$, then $\varphi_{|-K_{\widetilde{X}}|}$ factors as $\widetilde{X} \xrightarrow{\pi} X \xrightarrow{\varphi_{|-K_X|}} \mathbb{P}^2$, where $\varphi_{|-K_X|}$ is finite flat of degree 2. If $p \neq 2$, then $\varphi_{|-K_X|}$ is branched over a quartic curve Q with simple singularities.*

(3) *If $d = 1$, then $\varphi_{|-2K_{\widetilde{X}}|}$ factors as $\widetilde{X} \xrightarrow{\pi} X \xrightarrow{\varphi_{|-2K_X|}} \mathbb{P}(1, 1, 2) \subseteq \mathbb{P}^3$, where $\mathbb{P}(1, 1, 2)$ is the quadratic cone and $\varphi_{|-2K_X|}$ is finite flat of degree 2.*
If $p \neq 2$, then $\varphi_{|-2K_X|}$ is branched over a sextic curve S with simple singularities.

Type		Normal form $f \in k[[x,y]]$	
A_n		$x^2 + y^{n+1}$	$n \geqslant 1$
D_n		$x^2y + y^{n-1}$	$n \geqslant 4$
E_6	E_6^0	$x^3 + y^4$	
	E_6^1	$x^3 + y^4 + x^2y^2$	additionally if $p = 3$
E_7	E_7^0	$x^3 + xy^3$	
	E_7^1	$x^3 + xy^3 + x^2y^2$	additionally if $p = 3$
E_8	E_8^0	$x^3 + y^5$	
	E_8^1	$x^3 + y^5 + xy^4$	additionally if $p = 5$
	E_8^1	$x^3 + y^5 + x^2y^3$	additionally if $p = 3$
	E_8^2	$x^3 + y^5 + x^2y^2$	additionally if $p = 3$

TABLE 1. Simple curve singularities in characteristic $p \neq 2$

Next, we recall the notion of a *marking* of a weak del Pezzo surface $\widetilde{X} \notin \{\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{F}_2\}$ (see [Dol12, Definition 8.1.21]) and explain how to describe the negative curves on $\widetilde{X}$ in terms of such a marking.

- A *marking* of $\widetilde{X}$ is an isomorphism $\phi\colon \mathrm{I}^{1,9-d} \xrightarrow{\sim} \mathrm{Pic}(\widetilde{X})$, where $\mathrm{I}^{1,9-d}$ is the lattice of rank $10-d$ with quadratic form given by the diagonal matrix $(1,-1,\ldots,-1)$ with respect to a basis $e_0,\ldots,e_{9-d}$.
- A realization $\pi\colon \widetilde{X} \to \mathbb{P}^2$ of $\widetilde{X}$ as an iterated blow-up of $\mathbb{P}^2$ induces a marking ϕ where $\phi(e_0) = \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$ and $\phi(e_i)$ is the class of the preimage in $\widetilde{X}$ of the ith point blown up by π. A marking that arises in this way is called *geometric.*
- If ϕ is a geometric marking of $\widetilde{X}$, then $\phi^{-1}(K_{\widetilde{X}}) = (-3,1,\ldots,1) =: k_{9-d}$.
- The lattice E_{9-d} is defined as $\langle k_{9-d}\rangle^{\perp} \subseteq \mathrm{I}^{1,9-d}$.
  - ∗ For $d = 1,2,3$, the lattices E_{9-d} are precisely the three exceptional irreducible root lattices.
  - ∗ For $d = 4,5,6$, there are identifications $E_5 = D_5$, $E_4 = A_4$, $E_3 = A_2 \oplus A_1$.
  - ∗ For $d = 7,8$, the lattices E_{9-d} are not root lattices. Every maximal root lattice contained in E_2 is isomorphic to A_1, and E_1 does not contain any (-2)-vectors.

Remark 3.3. In particular, the number of (-2)-curves in the minimal resolution of an RDP del Pezzo surface of degree d is bounded above by $9-d$.

Following [Dol12, Section 8.2], we let

$$\mathrm{Exc}_{9-d} := \left\{v \in \mathrm{I}^{1,9-d} \mid v^2 = -1,\, v.k_{9-d} = -1\right\} \subseteq \mathrm{I}^{1,9-d}$$

be the subset of *exceptional vectors.* Let $\mathcal{R}$ be a set of linearly independent (-2)-vectors in E_{9-d}, and define the cone

$$C_{\mathcal{R}} := \left\{v \in \mathrm{I}^{1,9-d} \otimes \mathbb{R} \mid v.w \geqslant 0 \text{ for all } w \in \mathcal{R}\right\}.$$

For a sublattice Λ of E_{9-d}, we denote the Weyl group of Λ by $W(\Lambda)$. That is, $W(\Lambda)$ is the subgroup of the orthogonal group $\mathrm{O}(\mathrm{I}^{1,9-d})$ generated by reflections along (-2)-vectors in Λ.

With this notation, $W(\Lambda)$ preserves Exc_{9-d}, and, for $\Lambda = \langle\mathcal{R}\rangle$, the cone $C_{\mathcal{R}}$ is a fundamental domain for the action of $W(\Lambda)$ on $\mathrm{I}^{1,9-d} \otimes \mathbb{R}$.

Lemma 3.4. *With the above notation, we have the following description of certain sets of (-1)-curves on $\widetilde{X}$:*

(1) *If $\mathcal{R}$ is the preimage of the set of* all *(-2)-curves on $\widetilde{X}$ under a geometric marking ϕ, then ϕ induces a bijection*

$$\left\{(-1)\text{-curves on } \widetilde{X}\right\} \longleftrightarrow C_{\mathcal{R}} \cap \mathrm{Exc}_{9-d} \cong \mathrm{Exc}_{9-d}/W(\Lambda)\,.$$

(2) *If $\mathcal{R}' \subseteq \mathcal{R}$ with $\Lambda' := \langle\mathcal{R}'\rangle$, then ϕ induces a bijection*

$$\left\{(-1)\text{-curves on } \widetilde{X} \text{ disjoint from } \phi(\mathcal{R}')\right\} \longleftrightarrow C_{\mathcal{R}} \cap \mathrm{Exc}_{9-d} \cap (\Lambda')^{\perp} = C_{\mathcal{R}} \cap \mathrm{Exc}_{9-d}^{W(\Lambda')}\,.$$

(3) *If, moreover, Λ' is a sum of connected components of Λ, then ϕ induces a bijection*

$$\left\{(-1)\text{-curves on } \widetilde{X} \text{ disjoint from } \phi(\mathcal{R}')\right\} \longleftrightarrow \left(\mathrm{Exc}_{9-d}^{W(\Lambda')}\right)/W(\Lambda)\,.$$

Proof. For claims (1) and (2), see [Dol12, Lemma 8.2.22 and Proposition 8.2.34]. To prove claim (3), we note that we have an orthogonal decomposition $\Lambda = \Lambda' \oplus \Lambda''$, where $\Lambda = \langle\mathcal{R}\rangle$ and $\Lambda'' = \langle\mathcal{R} \setminus \mathcal{R}'\rangle$. Therefore, the $W(\Lambda)$-action preserves $\mathrm{Exc}_{9-d} \cap (\Lambda')^{\perp} = \mathrm{Exc}_{9-d}^{W(\Lambda')}$ and, by claim (1), we can write

$$C_{\mathcal{R}} \cap \mathrm{Exc}_{9-d} \cap (\Lambda')^{\perp} = (C_{\mathcal{R}} \cap \mathrm{Exc}_{9-d})^{W(\Lambda')} \cong \left(\mathrm{Exc}_{9-d}^{W(\Lambda')}\right)/W(\Lambda)\,. \qquad \square$$

4. Group scheme actions on anti-canonical models

The purpose of this section is to recall some basic facts about group scheme actions on (blow-ups of) normal projective surfaces and to describe the automorphism scheme of an RDP del Pezzo surface in terms of the anti-(bi-)canonical morphisms recalled in Theorem 3.2.

Quite generally, the key tool to control the behavior of group scheme actions under birational morphisms is Blanchard's lemma [Bri17, Theorem 7.2.1].

Theorem 4.1 (Blanchard's lemma). *Let $\pi\colon \widetilde{X} \to X$ be a morphism of proper schemes with $\pi_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$. Then, π induces a homomorphism of group schemes $\pi_*\colon \mathrm{Aut}^0_{\widetilde{X}} \to \mathrm{Aut}^0_X$. If π is birational, then π_* is a closed immersion.*

Given an action of a group scheme G on a scheme X and a closed subscheme $Z \subseteq X$, we let $\mathrm{Stab}_G(Z) \subseteq \mathrm{Aut}_X$ be the *stabilizer subgroup scheme* of Z. The following proposition describes the image of π_* if π is a blow-up of a closed point on a normal surface with at worst rational double points.

Proposition 4.2. *Let X be a normal surface with at worst rational double points, and let $\pi\colon \widetilde{X} \to X$ be the blow-up of X in a closed point P. Then, $\pi_*\left(\mathrm{Aut}^0_{\widetilde{X}}\right) = (\mathrm{Stab}_{\mathrm{Aut}_X}(P))^0$.*

Proof. By [Mar22, Proposition 2.7 and Remark 2.8], it suffices to find an infinitesimally rigid subscheme $E \subseteq \widetilde{X}$ whose schematic image is P. Recall that E is called infinitesimally rigid if it admits no infinitesimal deformations as a closed subscheme in $\widetilde{X}$ or, equivalently, if its normal sheaf $\mathcal{N}_{E/\widetilde{X}}$ has no non-zero global sections. Thus, let E be the exceptional divisor of π, with scheme structure given by the inverse image ideal sheaf of P. In particular, E is a Cartier divisor on $\widetilde{X}$.

If P is smooth, then E is a (-1)-curve, so E is infinitesimally rigid. If P is not smooth, then P is a rational double point. In particular, π factors the minimal resolution $\pi'\colon \widetilde{X}' \to X$ and $\pi'^*\omega_X \cong \omega_{\widetilde{X}'}$. Since $\widetilde{X}$ has at worst rational double points as well and $\widetilde{X}' \to \widetilde{X}$ is its minimal resolution, we obtain $\pi^*\omega_X \cong \omega_{\widetilde{X}}$ using the projection formula. Now, ω_X is trivial in a neighborhood of P, so $\omega_{\widetilde{X}}$ is trivial in a neighborhood of E. Thus, $\mathcal{N}_{E/\widetilde{X}} \cong \omega_E$ by adjunction.

On the other hand, the rational double point $P \in X$ has multiplicity 2 and embedding dimension 3, so E is isomorphic to a (possibly non-reduced) conic in $\mathbb{P}^2$. As such, it satisfies $\omega_E \cong \mathcal{O}_{\mathbb{P}^2}(-1)|_E$. Hence,

$$h^0\big(E, \mathcal{N}_{E/\widetilde{X}}\big) = h^0\big(E, \mathcal{O}_{\mathbb{P}^2}(-1)|_E\big) = 0\,,$$

so, by [Ser06, Proposition 3.2.1(ii)], the exceptional divisor E is infinitesimally rigid. This finishes the proof. □

If X is an RDP del Pezzo surface, then we have the following description of Aut_X in terms of the anti-canonical morphisms $\varphi_{|-nK_X|}$ of X.

PROPOSITION 4.3. *Let X be an RDP del Pezzo surface of degree d. Then, $\varphi_{|-nK_X|}$ is Aut_X-equivariant for all $n \geqslant 0$, and the following hold:*

(1) *If $d \geqslant 3$, then $\mathrm{Aut}_X = \mathrm{Stab}_{\mathrm{PGL}_{d+1}}(X)$.*

(2) *If $d = 2$ and $p \neq 2$, then there is an exact sequence of group schemes*

$$1 \to \underline{\mathbb{Z}/2\mathbb{Z}} \to \mathrm{Aut}_X \to \mathrm{Stab}_{\mathrm{PGL}_3}(Q) \to 1\,,$$

where Q is the branch quartic of the anti-canonical morphism $X \to \mathbb{P}^2$.

(3) *If $d = 1$ and $p \neq 2$, then there is an exact sequence of group schemes*

$$1 \to \underline{\mathbb{Z}/2\mathbb{Z}} \to \mathrm{Aut}_X \to \mathrm{Stab}_{\mathrm{Aut}_{\mathbb{P}(1,1,2)}}(S) \to 1\,,$$

where S is the branch sextic of the anti-bi-canonical morphism $X \to \mathbb{P}(1,1,2) \subseteq \mathbb{P}^3$.

Proof. By [Bri18, Remark 2.15(iv)], the line bundles $\omega_X^{\otimes(-n)}$ admit natural Aut_X-linearizations for all $n \geqslant 0$. Hence, the natural action of Aut_X on the space of global sections of $\omega_X^{\otimes(-n)}$ induces a homomorphism $f_n\colon \mathrm{Aut}_X \to \mathrm{PGL}_{N+1}$ for $N = \dim\big(H^0\big(X, \omega_X^{\otimes(-n)}\big)\big) - 1$ making the rational map $\varphi_{|-nK_X|}\colon X \dashrightarrow \mathbb{P}^N$ Aut_X-equivariant.

If $d \geqslant 3$, then the anti-canonical map is an embedding by Theorem 3.2, so f_1 is a monomorphism. By [Mar22, Lemma 2.5], the homomorphism f_1 factors through $\mathrm{Stab}_{\mathrm{PGL}_{d+1}}(X)$. Conversely, restricting the G-action on $\mathbb{P}^d$ to X yields a left-inverse $g_1\colon \mathrm{Stab}_{\mathrm{PGL}_{d+1}}(X) \to \mathrm{Aut}_X$ to f_1. Since X is not contained in a proper linear subspace of $\mathbb{P}^d$ and the fixed locus of a subgroup scheme of PGL_{d+1} is a linear subspace, g_1 has to be a monomorphism. Hence, f_1 is an isomorphism.

If $d = 2$ and $p \neq 2$, then the anti-canonical map is a finite flat cover of $\mathbb{P}^2$ branched over a quartic curve $Q \subseteq \mathbb{P}^2$ by Theorem 3.2. Let K be the kernel of f_1. Restricting the action of K on X to the generic point of X yields a $k\big(\mathbb{P}^2\big)$-linear action of $K_{k(\mathbb{P}^2)}$ on the degree 2 field extension $k(X)$ of $k\big(\mathbb{P}^2\big)$. Since $p \neq 2$, the field extension $k\big(\mathbb{P}^2\big) \subseteq k(X)$ is Galois, which shows that $K = \underline{\mathbb{Z}/2\mathbb{Z}}$. Since K is normal in Aut_X, the action of Aut_X on X preserves the fixed locus X^K, so by [Mar22, Lemma 2.5], the induced action of Aut_X on $\mathbb{P}^2$ preserves the scheme-theoretic image of X^K under $\varphi_{|-K_X|}$, which is nothing but Q. Hence, f_1 factors through $\mathrm{Stab}_{\mathrm{PGL}_3}(Q)$. In order to show the faithful flatness of $f_1'\colon \mathrm{Aut}_X \to \mathrm{Stab}_{\mathrm{PGL}_3}(Q)$, we write X

as $\{w^2 = Q(x,y,z)\} \subseteq \mathbb{P}(1,1,1,2)$. For every k-algebra R and every automorphism σ of $\mathbb{P}^2_R$ preserving $Q_R = \{Q(x,y,z) = 0\} \subseteq \mathbb{P}^2_R$, that is, mapping $Q(x,y,z)$ to $\lambda Q(x,y,z)$ for some $\lambda \in R^\times$, we can pass to the faithfully flat ring extension $R' := R[\sqrt{\lambda}]$ of R and lift σ to an automorphism of $X_{R'}$ by mapping w to $\pm\sqrt{\lambda}w$. Hence, f_1' is faithfully flat and thus the sequence in assertion (2) is exact.

If $d = 1$ and $p \neq 2$, we can apply essentially the same argument as in the previous paragraph to the anti-bi-canonical morphism $\varphi_{|-2K_X|}\colon X \to \mathbb{P}^3$: Indeed, the argument for $K = \overline{\mathbb{Z}/2\mathbb{Z}}$ is exactly the same as in the previous paragraph. To prove the faithful flatness of $f_2'\colon \overline{\mathrm{Aut}_X} \to \mathrm{Stab}_{\mathrm{Aut}_{\mathbb{P}(1,1,2)}}(S)$, we can write X as a hypersurface in $\mathbb{P}(1,1,2,3)$ by identifying the quadratic cone with $\mathbb{P}(1,1,2)$ and then argue as above. □

5. On equivariant resolutions

It is an immediate consequence of the uniqueness of the minimal resolution $\widetilde{X}$ of a projective surface X that the action of the automorphism *group* $\mathrm{Aut}_X(k)$ on X lifts to $\widetilde{X}$. Over the complex numbers, this implies that the action of the automorphism *scheme* Aut_X lifts to $\widetilde{X}$. In general, this is no longer true in positive characteristic. In this section, we will study this phenomenon.

Definition 5.1. Let $\pi\colon \widetilde{X} \to X$ be a proper birational morphism of schemes. Assume that X is integral and normal.

(1) The morphism π is called *T_X-equivariant* if the natural map $\pi_* T_{\widetilde{X}} \to T_X$ is an isomorphism.
(2) Assume additionally that X is proper (in particular, this implies $\pi_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$). Then, π is called *Aut^0_X-equivariant* if the closed immersion $\pi_*\colon \mathrm{Aut}^0_{\widetilde{X}} \hookrightarrow \mathrm{Aut}^0_X$ induced by Blanchard's lemma is an isomorphism.

Remark 5.2. Note that T_X-equivariance is local on X and implies $H^0(\widetilde{X}, T_{\widetilde{X}}) \cong H^0(X, T_X)$. If π is T_X-equivariant, then $\pi_*\colon \mathrm{Aut}^0_{\widetilde{X}} \hookrightarrow \mathrm{Aut}^0_X$ is an isomorphism on tangent spaces.

The study of the T_X-equivariance of the minimal resolution of a rational double point has been initiated by Wahl [Wah75] and extended to all positive characteristics by Hirokado [Hir19]. There, T_X-equivariance is simply called "equivariance." For the convenience of the reader, we recall the classification of RDPs whose minimal resolution is not T_X-equivariant (see [Hir19, Theorem 1.1]).

Proposition 5.3. *Let $\pi\colon \widetilde{X} \to X$ be the minimal resolution of a rational double point (X, x). Then, π is not T_X equivariant if and only if (X, x) is of type*

(1) *A_n with $p \mid (n+1)$,*
(2) *E_8^0 if $p = 5$,*
(3) *E_6^0, E_6^1, E_7^0, E_8^0, E_8^1 if $p = 3$, or*
(4) *D_n^r, E_6^0, E_7^0, E_7^1, E_7^2, E_7^3, E_8^0, E_8^1, E_8^2, E_8^3 if $p = 2$.*

In the next sections, we would like to apply the notions of Aut^0_X-equivariance and T_X-equivariance to RDP del Pezzo surfaces and their partial resolutions.

Definition 5.4. Let X be a proper surface. A *partial resolution* of X is a proper birational morphism $\pi\colon \widetilde{X} \to X$ such that the minimal resolution of X factors through π.

We thank the referee for suggesting a simplified proof of the following result.

Proposition 5.5. *Let X be a normal proper surface, and let $\pi\colon \widetilde{X} \to X$ be a partial resolution of X. Assume that there exists an open subset $U \subseteq X$ such that $\pi^{-1}(U) \to U$ is an isomorphism and all singularities in $X \setminus U$ admit a T_X-equivariant minimal resolution. Then, π is T_X-equivariant.*

Proof. Let $\pi'\colon X' \to X$ be the minimal resolution of X. By the definition of a partial resolution, π' factors through π. We get an induced sequence $\pi'_* T_{X'} \xrightarrow{\sigma} \pi_* T_{\widetilde{X}} \xrightarrow{\tau} T_X$. By assumption, τ is an isomorphism on U, and $\tau \circ \sigma$ is an isomorphism in a neighborhood of every point in $X \setminus U$. Hence, τ is surjective and generically an isomorphism. Since $\pi_* T_{\widetilde{X}}$ is torsion-free, this implies that τ is an isomorphism, so π is equivariant, as claimed. □

In some situations, Aut^0_X-equivariance can be deduced immediately from the simpler notion of T_X-equivariance, which is the content of the following proposition.

Proposition 5.6. *Let $\pi\colon \widetilde{X} \to X$ be a birational morphism of proper k-schemes. If $\mathrm{Aut}^0_{\widetilde{X}}$ is smooth and π is T_X-equivariant, then π is Aut^0_X-equivariant.*

Proof. If π is T_X-equivariant, then

$$\dim \mathrm{Aut}^0_{\widetilde{X}} \leqslant \dim \mathrm{Aut}^0_X \leqslant \dim_k H^0(X, T_X) = \dim_k H^0\big(\widetilde{X}, T_{\widetilde{X}}\big)\,,$$

and since $\mathrm{Aut}^0_{\widetilde{X}}$ is smooth, all inequalities above are in fact equalities. Thus, Aut^0_X is smooth and of the same dimension as $\mathrm{Aut}^0_{\widetilde{X}}$. Hence, we must have $\mathrm{Aut}^0_{\widetilde{X}} = \mathrm{Aut}^0_X$; that is, π is Aut^0_X-equivariant. □

Remark 5.7. In particular, if, in the situation of Proposition 5.5, we assume in addition that $\mathrm{Aut}^0_{\widetilde{X}}$ is smooth, then π is Aut^0_X-equivariant.

To the best of our knowledge, the question of whether partial resolutions of a proper normal surface with rational double points are Aut^0_X-equivariant has not been studied. In the following, we prove Aut^0_X-equivariance for A_n-singularities with $n < p-1$ and bound the failure of Aut^0_X-equivariance for A_{p-1}-singularities. While this does not cover all rational double points and not even all A_n-singularities, it will come in handy for the calculation of the automorphism schemes of non-equivariant RDP del Pezzo surfaces in Section 8.

Proposition 5.8. *Let X be a proper surface. Let $\pi\colon \widetilde{X} \to X$ be a partial resolution of X, and assume that the only singularities over which π is not an isomorphism are A_n-singularities with $n \leqslant p-1$. Then,*

$$\mathrm{length}\big(\mathrm{Aut}^0_X \,/\, \mathrm{Aut}^0_{\widetilde{X}}\big) \leqslant p^m\,,$$

where m is the number of A_{p-1}-singularities on X over which π is not an isomorphism. In particular, if $m = 0$, then π is Aut_X-equivariant.

Proof. It suffices to prove the statement if $\pi\colon \widetilde{X} \to X$ is not an isomorphism only over a single singularity P of type A_n. By Proposition 4.2, it suffices to show that $G := \mathrm{Aut}^0_X$ fixes P if $n < p-1$ and that the stabilizer of P has index 1 or p in G if $n = p-1$. To see this, we equip the singular locus X_{sing} of X with a scheme structure using Fitting ideals (see [Sta18, Tag 07Z6]), and we let Y be the irreducible component of X_{sing} containing P. Since the scheme structure on X_{sing} is canonical and G is connected, G preserves Y (see [Sta18, Tag 07ZA]), so we get a homomorphism $\varphi\colon G \to \mathrm{Aut}_Y$. To prove the proposition, it suffices to show that the stabilizer of P in Aut_Y has index 1 or p, with the latter only occurring for $n = p-1$.

Since an A_n-singularity is given in a formal neighborhood by the equation $z^{n+1} + xy$, the irreducible component Y is isomorphic to

$$Y_n := \operatorname{Spec}\big(k[z]/(z^n)\big) \quad \text{if } n < p-1\,,$$
$$Y_p := \operatorname{Spec}\big(k[z]/(z^p)\big) \quad \text{if } n = p-1\,.$$

Now, we calculate $\operatorname{Aut}_{Y_{i+1}}$ by computing its R-valued points for an arbitrary local k-algebra R. An element of $\operatorname{Aut}_{Y_{i+1}}(R)$ is an R-linear automorphism φ of $R[z]/(z^{i+1})$, so it is determined by where it sends z. Let $a_0, \ldots, a_i \in R$ be such that $\varphi(z) = \sum_{j=0}^{i} a_j z^j$. Let $\mathfrak{m}$ be the maximal ideal of R, so that $(\mathfrak{m}, z)$ is the maximal ideal of $R[z]/(z^{i+1})$. Since φ is an automorphism, it maps $(\mathfrak{m}, z)$ to itself, so $a_0 \in \mathfrak{m}$. If $a_1 \in \mathfrak{m}$, then the coefficient of z in every $\varphi(z^j)$ is in $\mathfrak{m}$, so z would not lie in the image of φ, which is absurd. Hence, $a_1 \in R \setminus \mathfrak{m} = R^\times$. Next, we know that $\varphi(z^{i+1}) = 0$. Since the degree 0 term of $\varphi(z^{i+1})$ is a_0^{i+1}, we have $a_0^{i+1} = 0$. The degree j term of $\varphi(z^{i+1})$ is of the form $\binom{i+1}{j} a_0^{i+1-j} a_1^j + a_0^{i+2-j} b_j$ for some $b_j \in R$. If $i < p-1$, then $p \nmid \binom{i+1}{j}$ for all j, so solving the above equations inductively shows $a_0 = 0$. If $i = p-1$, then $p \mid \binom{i+1}{j}$ for all $j > 0$, so we only get $a_0^p = 0$.

Conversely, given a sequence $(a_0, \ldots, a_i)$ in R with $a_1 \in R^\times$ and $a_0 = 0$ if $i < p-1$ (respectively, $a_0^p = 0$ if $i = p$), the morphism induced by $z \mapsto \sum_{j=0}^{i} a_j z^j$ is an automorphism since it is well defined and its inverse is given by $z \mapsto \big(\sum_{j=1}^{i} a_j z^{j-1}\big)^{-1} z - a_0$.

Summarizing, we have natural identifications

$$\operatorname{Aut}_{Y_{i+1}}(R) = \big\{(0, a_1, \ldots, a_i) \mid a_j \in R,\, a_1 \in R^\times\big\} \quad \text{for } i < p-1\,,$$
$$\operatorname{Aut}_{Y_p}(R) = \big\{(a_0, a_1, \ldots, a_{p-1}) \mid a_j \in R,\, a_1 \in R^\times,\, a_0^p = 0\big\}\,.$$

In both cases, the corresponding automorphism of Y_{i+1} preserves $P \times \operatorname{Spec} R \subseteq Y_{i+1} \times \operatorname{Spec} R$ if and only if $a_0 = 0$ since the ideal of $P \times \operatorname{Spec} R$ is (z). Hence, the index of the stabilizer of P in $\operatorname{Aut}_{Y_{i+1}}$ is 1 if $i < p-1$ and p if $i = p-1$. This finishes the proof. $\square$

Remark 5.9. The strategy of proof of Proposition 5.8 would, in principle, also apply to other rational double points. However, there are two obstacles to overcome:

(1) The automorphism scheme of the singular locus is more complicated for more general RDPs since the singular locus has a more complicated scheme structure in general. For example, if $p = 5$ and X admits an RDP of type E_8^0, then, in a neighborhood of this singularity, the singular locus of X looks like $\operatorname{Spec} k[[x,y]]/(x^2, y^5)$. This also makes the calculation of the stabilizer of the closed point more complicated.

(2) If $\pi\colon \widetilde{X} \to X$ is the minimal resolution of an RDP surface, then $\operatorname{Aut}^0_{\widetilde{X}}$ is the intersection of all stabilizers of all singularities that occur in the blow-ups making up π. For example, if $p = 3$ and X admits a single RDP of type A_4, then the argument of Proposition 5.8 shows that $\operatorname{Aut}^0_{X'} = \operatorname{Aut}^0_X$, where X' is the blow-up of the closed point of P, but X' has a singularity of type A_2, so the approach of Proposition 5.8 only shows that $\operatorname{length}\big(\operatorname{Aut}^0_X / \operatorname{Aut}^0_{\widetilde{X}}\big) \leqslant 3$ even though we would expect the two group schemes to be equal by Proposition 5.3.

One case where the argument of Proposition 5.8 goes through essentially unchanged is if $p = 3$ and the morphism $\pi\colon \widetilde{X} \to X$ is the minimal resolution of an RDP of type A_5. In this case, π factors as a composition of three blow-ups $\widetilde{X} \to X'' \to X' \to X$, where X' has an A_3-singularity and X'' has an A_1-singularity. Then, the argument of Proposition 5.8 shows that $\operatorname{Aut}^0_{\widetilde{X}} = \operatorname{Aut}^0_{X''} = \operatorname{Aut}^0_{X'}$ and $\operatorname{length}\big(\operatorname{Aut}^0_X / \operatorname{Aut}^0_{X'}\big) \leqslant 3$, hence $\operatorname{length}\big(\operatorname{Aut}^0_X / \operatorname{Aut}^0_{\widetilde{X}}\big) \leqslant 3$.

6. Automorphism schemes of equivariant RDP del Pezzo surfaces

In [MS20], we classified all weak del Pezzo surfaces $\widetilde{X}$ with global vector fields and calculated the identity component $\mathrm{Aut}^0_{\widetilde{X}}$ of their automorphism schemes. In particular, if X is a projective surface whose minimal resolution $\pi\colon \widetilde{X} \to X$ is Aut^0_X-equivariant and such that $\widetilde{X}$ is a weak del Pezzo surface, then $\mathrm{Aut}^0_X = \mathrm{Aut}^0_{\widetilde{X}}$ and thus, if Aut^0_X is non-trivial, then $\widetilde{X}$ appears in the classification tables of [MS20].

In the following, we will observe that all RDP del Pezzo surfaces in characteristic $p \geqslant 11$ fall into the above category, and we will give a list of possible candidates for exceptions in small characteristics.

Theorem 6.1. *Let X be an RDP del Pezzo surface over an algebraically closed field of characteristic p, and let $\pi\colon \widetilde{X} \to X$ be its minimal resolution. Assume that one of the following conditions holds:*

(1) $p \notin \{2,3,5,7\}$.

(2) $p = 7$ *and X does not contain an RDP of type A_6.*

(3) $p = 5$ *and X does not contain an RDP of type A_4 or E_8^0.*

(4) $p = 3$ *and X does not contain an RDP of type A_2, A_5, A_8, E_6^0, E_6^1, E_7^0, E_8^0, or E_8^1.*

(5) $p = 2$ *and X does not contain an RDP of type A_1, A_3, A_5, A_7, D_n^r, E_6^0, E_7^0, E_7^1, E_7^2, E_7^3, E_8^0, E_8^1, E_8^2, or E_8^3, where $n \leqslant 8$.*

Then, $\mathrm{Aut}_X = \mathrm{Aut}_{\widetilde{X}}$, and thus, in particular, $H^0(\widetilde{X}, T_{\widetilde{X}}) = H^0(X, T_X)$. Therefore, $H^0(X,T_X) \neq 0$ if and only if X is the anti-canonical model of one of the surfaces in the classification tables of [MS20].

Proof. By Propositions 5.5 and 5.6, the theorem holds for those RDP del Pezzo surfaces that satisfy the following two conditions:

(a) All singularities of X admit a T_X-equivariant minimal resolution.

(b) The group scheme $\mathrm{Aut}^0_{\widetilde{X}}$ is smooth.

By Proposition 5.3 and Remark 3.3, condition (a) holds if we exclude the types of RDPs in the statement of the theorem. In particular, note that if $p \geqslant 11$, we do not have to exclude any RDPs.

Once we exclude those types of RDPs, then, by Tables 1–6 in [MS20], condition (b) is also satisfied unless we are in one of the following three cases, where Γ is the RDP configuration on X and $d = \deg(X)$:

(i) $p = 3$, $d = 2$, $\Gamma = A_6$.

(ii) $p = 3$, $d = 2$, $\Gamma = D_6$.

(iii) $p = 2$, $d = 3$, $\Gamma = A_4$.

In [MS20], these exceptions correspond to cases $2J$, $2K$, and $3N$, respectively, and there is a unique weak del Pezzo surface of each of these types. In all cases, we have $H^0(\widetilde{X}, T_{\widetilde{X}}) = 1$, and therefore $H^0(X,T_X) = 1$ by Proposition 5.3, so the only remaining statement we have to show in these three cases is that π is Aut^0_X-equivariant. We will check this via explicit calculation.

(i) Assume $p = 3$. Consider the surface

$$X := \{w^2 = x^2z^2 + xy^2z + y^4 + x^3y\} \subseteq \mathbb{P}(1,1,1,2),$$

and let Q be the branch quartic of the induced double cover $X \to \mathbb{P}^2$. An elementary calculation shows that X admits an A_6-singularity over $[0:0:1]$ and no other singularities. By Proposition 4.3, we have $\mathrm{Aut}_X^0 = \mathrm{Stab}_{\mathrm{PGL}_3}(Q)^0$. The diagonal μ_3-action with weights $(0,1,2)$ on $\mathbb{P}^2$ preserves Q, so $\mu_3 \subseteq \mathrm{Aut}_X^0$. Since π is T_X-equivariant by Proposition 5.3, the minimal resolution $\widetilde{X}$ must be the surface of type $2J$ of [MS20], so $\mathrm{Aut}_{\widetilde{X}}^0 = \mu_3$. Since $H^0(X, T_X) = 1$, we have $\mathrm{Aut}_X^0[F] = \mu_3$, where $\mathrm{Aut}_X^0[F]$ denotes the kernel of the Frobenius morphism on Aut_X^0. Hence, μ_3 is normal in $\mathrm{Aut}_X^0 = \mathrm{Stab}_{\mathrm{PGL}_3}(Q)^0$, and thus $\mathrm{Stab}_{\mathrm{PGL}_3}(Q)^0$ preserves the eigenspaces of the μ_3-action, so $\mathrm{Stab}_{\mathrm{PGL}_3}(Q)^0$ acts diagonally. With this restriction, it is easy to compute that $\mathrm{Aut}_X^0 = \mathrm{Stab}_{\mathrm{PGL}_3}(Q)^0 = \mu_3$. Therefore, π is Aut_X^0-equivariant, which is what we wanted to show.

(ii) Assume $p = 3$. Consider the surface

$$X := \left\{ w^2 = x\left(x^3 + y^3 + xyz\right) \right\} \subseteq \mathbb{P}(1,1,1,2),$$

and let Q be the branch quartic of the induced double cover $X \to \mathbb{P}^2$. Note that Q is the union of a nodal cubic and one of its nodal tangents, with the node located at $[0:0:1]$, so X has a D_6-singularity at $[0:0:1:0]$ and no other singularities. The diagonal μ_3-action with weights $(0,1,2)$ preserves Q, so $H^0(X, T_X) \neq 0$, and thus $\widetilde{X}$ is the surface of type $2K$ of [MS20]. The rest of the argument is the same as in case (i) and shows that π is Aut_X^0-equivariant.

(iii) Assume $p = 2$. Consider the surface

$$X := \left\{ x_0x_1x_3 + x_1^2x_2 + x_0x_2^2 + x_0^2x_2 \right\} \subseteq \mathbb{P}^3,$$

which is a cubic surface with a single singularity, which is of type A_4, at $[0:0:0:1]$ (see, for example, [Roc96, Case B]). It admits a diagonal μ_2-action with weights $(0,1,0,1)$, we have $H^0(X, T_X) \neq 0$, and, therefore, as π is T_X-equivariant by Proposition 5.3, the surface X is the anti-canonical model of the surface of type $3N$ in [MS20]. Using that μ_2 is normal in $\mathrm{Stab}_{\mathrm{PGL}_4}(X)^0$, we can calculate $\mathrm{Aut}_X^0 = \mathrm{Stab}_{\mathrm{PGL}_4}(X)^0$ explicitly. Indeed, Aut_X^0 preserves the eigenspaces $\{x_1 = x_3 = 0\}$ and $\{x_0 = x_2 = 0\}$ of the μ_2-action. In particular, Aut_X^0 preserves $X \cap \{x_1 = x_3 = 0\} = \{[0:0:1:0], [1:0:0:0], [1:0:1:0]\}$, so it acts on $\mathbb{P}^3$ via substitutions of the form $x_1 \mapsto ax_1 + bx_3$, $x_3 \mapsto cx_1 + dx_3$. Such a substitution preserves the equation of X if and only if $ac = bd = b^2 = 0$ and $a^2 = ad + bc = 1$, which holds if and only if $a^2 = 1$, $d = a$, and $b = c = 0$. Thus, $\mathrm{Aut}_X^0 = \mu_2$, and we conclude that π is Aut_X^0-equivariant, as desired. □

7. Finding (-1)-curves in the equivariant locus

In view of Theorem 6.1, in order to classify RDP del Pezzo surfaces with global vector fields in odd characteristic, it remains to study RDP del Pezzo surfaces X which are not T_X-equivariant. Let Γ' be the configuration of rational double points on X which are not T_X-equivariant, and let $\pi\colon \widetilde{X} \to X$ the minimal resolution of X.

In this section, we will describe a criterion for X to be the anti-canonical model of a blow-up of an RDP del Pezzo surface X' of higher degree containing Γ' such that $X' \dashrightarrow X$ is an isomorphism around Γ'. On the corresponding minimal resolutions, this will amount to finding (-1)-curves away from the configuration of (-2)-curves over Γ'. In other words, we are trying to find (-1)-curves on $\widetilde{X}$ that map to the T_X-equivariant locus of π. In Section 8, this criterion will allow us to give a complete classification of non-equivariant RDP del Pezzo surfaces with global vector fields by setting up an inductive argument depending on the degree of the surface.

Theorem 7.1. *Let X_d be an RDP del Pezzo surface of degree $d \leqslant 8$, and let Γ' be a configuration of rational double points on X_d. Assume that its minimal resolution $\widetilde{X}_d$ is a blow-up of $\mathbb{P}^2$, and* *let $\Lambda' \subseteq \operatorname{Pic}(\widetilde{X}_d)$ be the sublattice generated by the components of the exceptional locus over Γ'. Then the following are equivalent:*

(1) *There exists a (-1)-curve on $\widetilde{X}_d$ whose image in X_d does not pass through Γ'.*

(2) *The surface X_d is the anti-canonical model of a blow-up in a smooth point P of an RDP del Pezzo surface X_{d+1} of degree $d+1$ containing Γ' such that $X_d \dashrightarrow X_{d+1}$ is an isomorphism around Γ'.*

(3) *The map $\Lambda' \hookrightarrow \langle K_{\widetilde{X}_d}\rangle^{\perp} \cong E_{9-d}$ factors through an embedding $E_{8-d} \hookrightarrow E_{9-d}$.*

Proof. First, we show (1)⇒(2). Let $\widetilde{C}$ be the (-1)-curve whose existence is asserted in item (1). Contracting $\widetilde{C}$, we obtain a weak del Pezzo surface $\widetilde{X}_{d+1}$ of degree $d+1$ such that $\widetilde{X}_d$ is the blow-up of $\widetilde{X}_{d+1}$ in a smooth point $\widetilde{P}$. Let X_{d+1} be the anti-canonical model of $\widetilde{X}_{d+1}$, and let P be the image of $\widetilde{P}$ in X_{d+1}. By our choice of $\widetilde{C}$, all components of the exceptional locus over Γ' stay (-2)-curves in $\widetilde{X}_{d+1}$, so X_{d+1} contains Γ'.

Since $\widetilde{X}_d$ is a weak del Pezzo surface, $\widetilde{P}$ cannot lie on a (-2)-curve (otherwise, the strict transform of such a curve would have negative intersection with $-K_{\widetilde{X}_d}$, which is impossible as $-K_{\widetilde{X}_d}$ is nef), so P is a smooth point on X_{d+1}. Thus, blowing up $P \in X_{d+1}$, we obtain a surface Y_d with the same singularities as X_{d+1}. In particular, Y_d has only rational double points as singularities, and its minimal resolution is $\widetilde{X}_d$. Therefore, pullback of sections induces isomorphisms $H^0(Y_d, -nK_{Y_d}) \cong H^0(\widetilde{X}_d, -nK_{\widetilde{X}_d})$ for all $n \geqslant 0$, where the surjectivity follows from the fact that Y_d is normal. Thus, the anti-canonical model of Y_d coincides with X_d. The situation is summarized in Figure 1.

$$\begin{tikzcd}
\widetilde{C} \subseteq \widetilde{X}_d \arrow[rr, "\text{contract } \widetilde{C}", "\text{blow up } \widetilde{P}"'] \arrow[d] \arrow[dd, bend right=60, "\text{anti-can.}"'] & & \widetilde{X}_{d+1} \ni \widetilde{P} \arrow[d, "\text{anti-can.}"] \\
C \subseteq Y_d \arrow[rr, "\text{contract } C", "\text{blow up } P"'] \arrow[d, "\text{anti-can.}"] & & X_{d+1} \ni P \\
X_d & &
\end{tikzcd}$$

Figure 1. Contracting a (-1)-curve disjoint from the singular locus

Note that $X_d \dashrightarrow Y_d \to X_{d+1}$ is an isomorphism in a neighborhood of Γ' since $\widetilde{C}$ is disjoint from the exceptional locus over Γ'.

Next, we show (2)⇒(3). Assume that assertion (2) is true. We have $\langle K_{\widetilde{X}_d}\rangle^{\perp} \cong E_{9-d}$ and $\langle K_{\widetilde{X}_{d+1}}\rangle^{\perp} \cong E_{8-d}$. Since X_{d+1} contains Γ', the embedding $\Lambda' \hookrightarrow \operatorname{Pic}(\widetilde{X}_d)$ factors through the pullback map $\operatorname{Pic}(\widetilde{X}_{d+1}) \hookrightarrow \operatorname{Pic}(\widetilde{X}_d)$, which maps $\langle K_{\widetilde{X}_{d+1}}\rangle^{\perp}$ to $\langle K_{\widetilde{X}_d}\rangle^{\perp}$. Hence, assertion (3) follows.

Finally, to show (3)⇒(1), we identify $\operatorname{Pic}(\widetilde{X}_d)$ and $\mathrm{I}^{1,9-d}$ via a geometric marking. Assuming that assertion (3) is true, we have to show that there is a (-1)-curve $\widetilde{C}$ on $\widetilde{X}_d$ that does not meet the set $\mathcal{R}'$ of exceptional curves over Γ'. Let Λ be the sublattice of $\operatorname{Pic}(\widetilde{X}_d)$ spanned by the classes of all (-2)-curves. Since Λ' is a sum of connected components of Λ, Lemma 3.4 shows that it suffices to prove

$$\big(\mathrm{Exc}_{9-d}^{W(\Lambda')}\big)/W(\Lambda) \neq \emptyset\,.$$

Clearly, this is the case if and only if $\mathrm{Exc}_{9-d}^{W(\Lambda')} \neq \emptyset$. Since $\Lambda' \hookrightarrow E_{9-d}$ factors through an embedding $E_{8-d} \hookrightarrow E_{9-d}$, we have

$$\mathrm{Exc}_{9-d}^{W(E_{8-d})} \subseteq \mathrm{Exc}_{9-d}^{W(\Lambda')},$$

so it suffices to show that $\mathrm{Exc}_{9-d}^{W(E_{8-d})} \neq \emptyset$. Since the action of $W(E_{9-d})$ on $\mathrm{Pic}(\widetilde{X}_d)$ preserves Exc_{9-d}, the condition $\mathrm{Exc}_{9-d}^{W(E_{8-d})} \neq \emptyset$ depends on the embedding $E_{8-d} \hookrightarrow E_{9-d}$ only up to conjugation by elements of $W(E_{9-d})$ and up to automorphisms of E_{8-d}.

If $d \leqslant 5$, then E_{8-d} is a root lattice. By [Dyn52, Table 11] and [Mar03, Exercises 4.2.1 and 4.6.2], the embedding $\iota\colon E_{8-d} \hookrightarrow E_{9-d}$ is unique up to the action of $\mathrm{O}(E_{9-d})$. Since $\mathrm{O}(E_{9-d})$ is generated by $\{\pm\mathrm{id}\}$ and $W(E_{9-d})$ in every case (see, for example, [Mar03, Proposition 4.2.2 and Theorems 4.3.3, 4.5.2, and 4.5.3]), the embedding ι is unique up to the action of $W(E_{9-d})$ and up to automorphisms of E_{8-d}. Therefore, in order to show that $\mathrm{Exc}_{9-d}^{W(E_{8-d})} \neq \emptyset$, it suffices to show that there exists *some* $\widetilde{X}_d$ containing a configuration of (-2)-curves of type E_{8-d} and such that a (-1)-curve disjoint from this configuration exists. This is known and can be seen for example in [MS20, Figures 23, 48, 57, 63, and 61].

If $d \geqslant 7$, then E_{8-d} does not contain any (-2)-vectors, so $\Lambda' = 0$ and the implication (3)$\Rightarrow$(1) holds since $\widetilde{X}_d$ is a blow-up of $\mathbb{P}^2$ by assumption.

Finally, if $d = 6$, then the maximal root lattice contained in $E_{8-d} = E_2$ is A_1. Thus, we may assume $\Lambda' = A_1$, for otherwise we can argue as in the previous case where $d \geqslant 7$. Up to the action of $\mathrm{O}(E_3) = \{\pm\mathrm{id}\} \times W(E_3)$, there are two embeddings of A_1 into $E_3 = A_2 \oplus A_1$. It is easy to check that $\iota\colon A_1 \hookrightarrow E_3$ factors through E_2 if and only if ι factors through the A_2-summand of E_3, and then ι is unique up to the action of $W(E_3)$ and up to automorphisms of A_1. Hence, similarly to what we did in the case $d \leqslant 5$, it suffices to find some $\widetilde{X}_6$ containing a (-2)-curve and a disjoint (-1)-curve. Again, this is known; see [MS20, Figure 24]. □

Corollary 7.2. *Let X_d be an RDP del Pezzo surface of degree $d \leqslant 8$, let Γ' be a configuration of rational double points on X_d, and let Λ' be the root lattice associated with Γ'. Assume that Γ' occurs on an RDP del Pezzo surface of degree $d+1$ and satisfies one of the following conditions:*

(1) $d \neq 4, 2, 1$.

(2) $d = 4$ *and* $\Lambda' \neq A_3$.

(3) $d = 2$ *and* $\Lambda' \notin \{A_5 + A_1, A_5, A_3 + 2A_1, A_3 + A_1, 4A_1, 3A_1\}$.

(4) $d = 1$ *and* $\Lambda' \notin \{A_7, 2A_3, A_5 + A_1, A_3 + 2A_1, 4A_1\}$.

Then, X_d is the anti-canonical model of a blow-up in a smooth point P of an RDP del Pezzo surface X_{d+1} of degree $d+1$ containing Γ' such that $X_d \dashrightarrow X_{d+1}$ is an isomorphism around Γ'.

Proof. The embeddings of root lattices into E_6, E_7, and E_8 have been classified by Dynkin [Dyn52, Table 11], and for the embeddings of root lattices into $E_3 = A_2 \oplus A_1$, $E_4 = A_4$, and $E_5 = D_5$, we refer the reader to [Mar03, Exercises 4.2.1 and 4.6.2]. It follows from these classifications that if an embedding of Λ' into E_{9-d} exists, then this embedding is unique (up to the action of $\mathrm{O}(E_{9-d})$), except precisely in the cases excluded in conditions (2), (3), and (4). Hence, if one embedding $\Lambda' \hookrightarrow E_{9-d}$ factors through an embedding $E_{8-d} \hookrightarrow E_{9-d}$, then *every* embedding factors through an embedding $E_{8-d} \hookrightarrow E_{9-d}$. If Γ' occurs on some RDP del Pezzo surface of degree $d+1$, then an embedding of Λ' with such a factorization exists and the claim follows from Theorem 7.1. □

8. Automorphism schemes of non-equivariant RDP del Pezzo surfaces

Throughout this section, X denotes an RDP del Pezzo surface, and $\pi\colon \widetilde{X} \to X$ is its minimal resolution. In this section, we will prove Theorem 1.2, that is, we will classify all X with $H^0(X, T_X) \neq 0$ over a field of characteristic $p \in \{3, 5, 7\}$ and such that X contains one of the RDPs excluded in Theorem 6.1. We will treat the cases $p = 7$, $p = 5$, and $p = 3$ in Sections 8.1, 8.2, and 8.3, respectively. This will complete the classification of all RDP del Pezzo surfaces with global vector fields in odd characteristic.

The strategy of proof is as follows. First, for each degree $1 \leqslant d \leqslant 9$, we give the list of RDP configurations Γ that can occur on an RDP del Pezzo surface X of degree d and that contain at least one RDP whose minimal resolution is not T_X-equivariant. Then, starting with the highest possible degree and working our way down with Theorem 7.1, we classify those X containing Γ and satisfying $H^0(X, T_X) \neq 0$. In each step, we give explicit equations and calculate Aut_X^0 using Propositions 4.2, 4.3, and 5.8.

Notation 8.1. If $d \geqslant 3$, we use the notation $x_0, \ldots, x_d$ for the coordinates of $\mathbb{P}^d$. If $d = 2$, we use the notation x, y, z, w for the coordinates of $\mathbb{P}(1,1,1,2)$, where w has weight 2. Finally, if $d = 1$, we use the notation s, t, x, y for the coordinates of $\mathbb{P}(1,1,2,3)$, where x has weight 2 and y has weight 3. In Tables 5, 6, 7, 8, 9, and 10, we describe Aut_X^0 as follows:

- We only describe the R-valued points of Aut_X^0, where R is an arbitrary local k-algebra. By [MS20, Lemma 3.5], this suffices to describe the scheme structure of Aut_X^0 completely. We do this either by describing a general R-valued point as a matrix or by describing the image of $[x_0 : \ldots : x_n]$ (respectively, $[x : y : z : w]$, $[s : t : x : y]$) under a general R-valued automorphism of X.
- We often describe Aut_X^0 as the group scheme $\langle G_1, G_2 \rangle$ generated by subgroup schemes G_1 and G_2 of Aut_X^0. By this we mean that we describe Aut_X^0, using Proposition 4.3, as the smallest subgroup scheme of PGL_{d+1} (respectively, $\mathrm{Aut}_{\mathbb{P}(1,1,1,2)}$ if $d = 2$, $\mathrm{Aut}_{\mathbb{P}(1,1,2,3)}$ if $d = 1$) containing both G_1 and G_2.
- We use the variables λ or λ_i for R-valued points of $\mathbb{G}_m$ and μ_{p^n} (where $\lambda^{p^n} = 1$), and we use the variables ε or ε_i for R-valued points of $\mathbb{G}_a$ and α_{p^n} (where $\varepsilon^{p^n} = 0$).

8.1 In characteristic 7

By Theorem 6.1, we have to list all RDP configurations containing A_6 that can occur on an RDP del Pezzo surface in characteristic 7.

Lemma 8.2. *If $p = 7$, $\deg(X) = d$, and X contains an A_6-singularity, then d and the configuration Γ of RDPs on X is one of the cases in Table 2.*

d	Γ	$\subseteq \langle k_{9-d} \rangle^{\perp}$
2	A_6	$\subseteq E_7$
1	$A_6,\ A_6 + A_1$	$\subseteq E_8$

Table 2. Non-equivariant RDP configurations in characteristic 7

Proof. Since A_6 has rank 6 and discriminant 7, it does not embed into E_{9-d} with $d \geqslant 3$. By [Dyn52, Table 11], the only root lattice containing A_6 as a direct summand and embedding into E_7 is A_6 itself, and there are precisely two root lattices containing A_6 and embedding into E_8, namely A_6 and $A_6 + A_1$. $\square$

Theorem 8.3. *Assume $p = 7$ and that X contains an RDP of type A_6. Then, $H^0(X, T_X) \neq 0$ if and only if X is given by an equation as in Table 5. Moreover, Aut_X^0 is as in Table 5, so that $\mathrm{Aut}_{\widetilde{X}}^0 \subsetneq \mathrm{Aut}_X^0$ and even $h^0(X, T_X) > h^0(\widetilde{X}, T_{\widetilde{X}})$.*

Proof. By Lemma 8.2, we have $\deg(X) = d \leqslant 2$. Assume $H^0(X, T_X) \neq 0$.

Case $d = 2$. If $d = 2$, then by Theorem 3.2, the anti-canonical system of X realizes X as a double cover of $\mathbb{P}^2$ branched over a quartic curve Q with a simple singularity of type A_6. Over the complex numbers, there is a unique such Q (see [BG81, Proposition 1.3.II]: up to projective equivalence, one can choose $L = x$, $B = 2y^2$, and $\varphi = y^4 + x^3y$ in the notation of loc. cit.), and the argument carries over without change to characteristic 7. Now, an elementary calculation shows that the Klein quartic equation

$$x^3y + y^3z + z^3x = 0$$

defines such a Q with an A_6-singularity at $[1 : 2 : -3]$. Indeed, setting $x = 1$ and applying the substitution $y \mapsto y + 2$, $z \mapsto z + 4$, we obtain the affine equation

$$3(y + 2z)^2 + \left(4y^2 + z^3 - y^2z\right) + y^3z = 0\,,$$

which is irreducible and of multiplicity 2 at $(0, 0)$. So the Klein quartic is irreducible with a double point at $[1 : 2 : -3]$. Moreover, the μ_7-action described in Table 5 preserves the corresponding double cover X and does not fix the singular point. Hence, by Proposition 5.3, the singularity of Q must be of type A_n with $7 \mid (n + 1)$. Since $n \leqslant 7$, we must have $n = 6$, so the singularity of Q is indeed an A_6-singularity. Thus, X is given by the equation in Table 5. Clearly, the μ_7-action described in Table 5 preserves X. Since $\mathrm{Aut}_{\widetilde{X}}^0$ is trivial by [MS20], Proposition 5.8 implies that $\mathrm{Aut}_X^0 = \mu_7$.

Case $d = 1$. If $d = 1$, then by Theorem 3.2, the anti-bi-canonical system of X realizes X as a double cover of the quadratic cone in $\mathbb{P}^3$ branched over a sextic curve S. By Table 2, the RDP configuration on X is either A_6 or $A_6 + A_1$. By Corollary 7.2, the surface X is the anti-canonical model of a blow-up Y_1 in a smooth point P of the surface X_2 of Case $d = 2$. By Proposition 5.8, the morphism $Y_1 \to X$ is Aut_X^0-equivariant since it is an isomorphism around the A_6-singularity. Hence, by Proposition 4.2, we have $\mathrm{Aut}_X^0 = \mathrm{Aut}_{Y_1}^0 = \mathrm{Stab}_{\mathrm{Aut}_{X_2}^0}(P)^0 = \mathrm{Stab}_{\mu_7}(P)^0$. So, since μ_7 is simple, Y_1 is the blow-up of X_2 in a smooth fixed point P of the μ_7-action on X_2, and $\mathrm{Aut}_X^0 = \mu_7$.

Next, we prove the uniqueness of X. We may assume that X_2 is given by the equation in Table 5. The fixed points of the μ_7-action are the three points $[1 : 0 : 0 : 0]$, $[0 : 1 : 0 : 0]$, and $[0 : 0 : 1 : 0]$. The automorphism $x \mapsto y \mapsto z \mapsto x$ of X_2 permutes these fixed points. So, Y_1 is unique, and hence so is X.

Therefore, X is the unique RDP del Pezzo surface of degree 1 with an A_6-singularity and non-zero global vector fields. Now, the equation

$$y^2 = x^3 + ts^3x + t^5s$$

defines such an RDP del Pezzo surface of degree 1 in $\mathbb{P}(1, 1, 2, 3)$ with an A_6-singularity at $[1 : -3 : 1 : 0]$ and, additionally, an A_1-singularity at $[1 : 0 : 0 : 0]$. Clearly, the μ_7-action described in Table 5 preserves X. Hence, this is the surface we were looking for. $\square$

Remark 8.4. We remark that by [MS20], in both cases of Theorem 8.3, the minimal resolution $\widetilde{X}$ of X does not admit any non-trivial global vector fields. In particular, $\pi\colon \widetilde{X} \to X$ is not T_X-equivariant.

8.2 In characteristic 5

By Theorem 6.1, we have to list all RDP configurations containing A_4 or E_8^0 that can occur on an RDP del Pezzo surface in characteristic 5.

Lemma 8.5. *If $p = 5$, $\deg(X) = d$, and X contains a singularity of type A_4 or E_8^0, then d and the configuration Γ of RDPs on X is one of the cases in Table 3.*

d	Γ	$\subseteq \langle k_{9-d}\rangle^{\perp}$
5	A_4	$\subseteq A_4$
4	A_4	$\subseteq D_5$
3	$A_4,\ A_4+A_1$	$\subseteq E_6$
2	$A_4,\ A_4+A_1,\ A_4+A_2$	$\subseteq E_7$
1	$A_4,\ A_4+A_1,\ A_4+2A_1,\ A_4+A_2,\ A_4+A_2+A_1,\ A_4+A_3,\ 2A_4,\ E_8^0$	$\subseteq E_8$

Table 3. Non-equivariant RDP configurations in characteristic 5

Proof. Since A_4 has rank 4 and discriminant 5, it does not embed into E_{9-d} with $d \geqslant 6$, and the only root lattice containing A_4 as a direct summand and embedding into E_{9-d} with $d \in \{4, 5\}$ is A_4 itself. The other cases can be found in [Dyn52, Table 11]. $\square$

Theorem 8.6. *Assume $p = 5$ and that X contains an RDP of type A_4 or E_8^0. Then, $H^0(X, T_X) \neq 0$ if and only if X is given by an equation as in Table 6. Moreover, Aut_X^0 is as in Table 6, so that $\mathrm{Aut}_{\widetilde{X}}^0 \subsetneq \mathrm{Aut}_X^0$ and even $h^0(X, T_X) > h^0(\widetilde{X}, T_{\widetilde{X}})$.*

Proof. By Lemma 8.5, we have $d \leqslant 5$.

Case $d = 5$. If $d = 5$, then X is a quintic surface in $\mathbb{P}^5$. By Table 3, the RDP configuration on X is A_4. By the same argument as in characteristic 0 (going through the possible configurations of four (possibly infinitely) near points in $\mathbb{P}^2$), there is a unique quintic surface in $\mathbb{P}^5$ containing an A_4-singularity. It is given by the equations in Table 6 (see [Der14, Section 3.3, p. 657]) with singular point at $[0 : 0 : 0 : 0 : 0 : 1]$. The α_5-action given in Table 6 preserves X and does not preserve the singular point, so $\alpha_5 \cap \mathrm{Aut}_{\widetilde{X}}^0 = \{\mathrm{id}\}$. By Proposition 5.8, this implies that $\mathrm{Aut}_X^0 = \langle \alpha_5, \mathrm{Aut}_{\widetilde{X}}^0 \rangle$.

Case $d = 4$. If $d = 4$, then X is a quartic surface in $\mathbb{P}^4$. By Table 3, the RDP configuration on X is A_4. By Corollary 7.2, the surface X is the anti-canonical model of a blow-up Y_4 in a smooth point P of the surface X_5 of Case $d = 5$.

Such an X is in fact unique: By [MS20], we have $\dim \mathrm{Aut}_{X_5}^0 = \dim \mathrm{Aut}_{\widetilde{X}_5}^0 = 4$, and since the orbit of P is at most 2-dimensional, the stabilizer of P is positive-dimensional, so $H^0(\widetilde{X}, T_{\widetilde{X}}) \neq 0$. By [MS20], there is a unique quartic del Pezzo surface with an A_4-singularity and whose minimal resolution has global vector fields, so X is unique.

In Table 6, we give equations for such a surface (which we took from [Der14]), so this is our X. The singular point is located at $[0 : 0 : 0 : 0 : 1]$. The α_5-action given in Table 6 preserves X and does not preserve the singular point. Again, by Proposition 5.8, this implies that $\mathrm{Aut}_X^0 = \langle \alpha_5, \mathrm{Aut}_{\widetilde{X}}^0 \rangle$.

Case $d = 3$. If $d = 3$, then X is a cubic surface in $\mathbb{P}^3$. By Table 3, the RDP configuration on X is A_4 or $A_4 + A_1$. By Corollary 7.2, the surface X is the anti-canonical model of a blow-up Y_3 in a smooth point P of the surface X_4 of Case $d = 4$.

Next, we show that there are at most two non-isomorphic such X. By [MS20], we have $\dim \mathrm{Aut}_{X_4}^0 = \dim \mathrm{Aut}_{\widetilde{X}_4}^0 = 2$. There is at most one 2-dimensional orbit on X_4, so there is at most one X whose minimal resolution does not admit global vector fields. On the other hand, by [MS20], there is precisely one X whose minimal resolution does have global vector fields.

In Table 6, we give two (non-isomorphic) equations for cubic surfaces with an A_4-singularity, distinguished by their RDP configuration Γ, so these are the two possible surfaces X:

(1) If $\Gamma = A_4$, the singular point is located at $[-2 : -1 : 2 : 1]$. We describe a μ_5-action on X in Table 6. In this case, $\mathrm{Aut}_{\widetilde{X}}^0$ is trivial by [MS20], so $\mathrm{Aut}_X^0 = \mu_5$ by Proposition 5.8.

(2) If $\Gamma = A_4 + A_1$, the A_4-singularity is $[0 : 0 : 0 : 1]$ while the A_1-singularity is $[1 : 0 : 0 : 0]$. We describe an $\alpha_5 \rtimes \mathbb{G}_m$-action on X in Table 6. In this case, $\mathrm{Aut}_{\widetilde{X}}^0 = \mathbb{G}_m$ by [MS20], so $\mathrm{Aut}_X^0 = \alpha_5 \rtimes \mathbb{G}_m$ by Proposition 5.8.

Case $d = 2$. If $d = 2$, then X is a double cover of $\mathbb{P}^2$ branched over a quartic curve Q. By Table 3, the possible RDP configurations on X are A_4, $A_4 + A_1$, and $A_4 + A_2$. By Corollary 7.2, the surface X is the anti-canonical model of a blow-up Y_2 in a smooth point P of an RDP del Pezzo surface X_3 of degree 3 with an A_4-singularity. Since $Y_2 \to X_3$ and $Y_2 \to X$ are isomorphisms around the A_4-singularities, Proposition 5.8 yields $\mathrm{Aut}_X^0 = \mathrm{Aut}_{Y_2}^0 = \mathrm{Stab}_{\mathrm{Aut}_{X_3}^0}(P)^0$. Hence, for each of the surfaces X_3 in Case $d = 3$, we have to determine the points with non-trivial stabilizer.

- If the RDP configuration on X_3 is A_4, then the points with non-trivial stabilizer under the action of $\mathrm{Aut}_{X_3}^0 = \mu_5$ are $[1 : 0 : 0 : 0]$, $[0 : 1 : 0 : 0]$, $[0 : 0 : 1 : 0]$, and $[0 : 0 : 0 : 1]$, and these fixed points are permuted by the automorphism $x_0 \mapsto x_1 \mapsto x_2 \mapsto x_3 \mapsto x_0$. Hence, there is a unique choice for P up to isomorphism.
- If the RDP configuration on X_3 is $A_4 + A_1$, then there are four lines on X_3: the lines $\ell_1 = \{x_0 = x_1 = 0\}$, $\ell_2 = \{x_0 = x_2 = 0\}$, $\ell_3 = \{x_1 = x_2 = 0\}$ pass through the A_4-singularity at $[0 : 0 : 0 : 1]$, and the line $\ell_4 = \{x_2 = x_3 = 0\}$ passes through the A_1-singularity at $[1 : 0 : 0 : 0]$ but not through A_4. Moreover, ℓ_2 and ℓ_4 intersect in $[0 : 1 : 0 : 0]$. Straightforward calculation shows that the points with non-trivial stabilizer in $X_3 \setminus \{\ell_1, \ell_2, \ell_3\}$ are precisely those lying on the hyperplane $H = \{x_3 = 0\}$. The intersection $X_3 \cap H$ is the union of the conic $C = \{x_3 = x_0x_2 + x_1^2 = 0\}$ and the line ℓ_4. Hence, either $P \in C \setminus (\ell_1 \cup \ell_2 \cup \ell_3) = C \setminus \{[1 : 0 : 0 : 0], [0 : 0 : 1 : 0]\}$ or $P \in \ell_4 \setminus (\ell_1 \cup \ell_2 \cup \ell_3) = \ell_4 \setminus \{[1 : 0 : 0 : 0], [0 : 1 : 0 : 0]\}$. The group scheme $\mathrm{Aut}_{X_3}^0$ acts transitively on both loci, so there are only two choices for P up to isomorphism. In both cases, one checks that $\mathrm{Aut}_{Y_2}^0 = \mathrm{Stab}_{\mathrm{Aut}_{X_3}}(P)^0 = \mu_5$. One of the two choices of P can be reduced to a previous case as follows:
  - If $P \in C \setminus \{[1 : 0 : 0 : 0], [0 : 0 : 1 : 0]\}$, then the RDP configuration on X is $A_4 + A_1$. The strict transform C' of C on the blow-up X' of X in the A_1-singularity is a (-1)-curve that passes through the (-2)-curve over A_1 and is disjoint from A_4. If we contract C', we obtain an RDP del Pezzo surface X_3' of degree 3 which contains an A_4-singularity as its only singularity and is such that $H^0(X_3', T_{X_3'}) \neq 0$ (by Proposition 5.5 and Blanchard's

lemma). Hence, X_3' is isomorphic to the cubic surface with RDP configuration A_4 in Table 3, and thus X coincides with the surface we constructed in the previous bullet point.

Summarizing, there are at most two RDP del Pezzo surfaces of degree 2 which contain an A_4-singularity and which admit a non-trivial global vector field. Moreover, $\mathrm{Aut}_X^0 = \mathrm{Stab}_{\mathrm{Aut}_{X_3}^0}(P) = \mu_5$ in both cases. In Table 6, we give two equations of such surfaces, distinguished by their RDP configuration Γ:

(1) If $\Gamma = A_4 + A_1$, the A_4-singularity is $[-2 : 1 : 2 : 0]$, the A_1-singularity is $[0 : 1 : 0 : 0]$, and the corresponding $\mathrm{Aut}_X^0 = \mu_5$-action is as in Table 6

(2) If $\Gamma = A_4 + A_2$, the A_4-singularity is $[2 : 1 : -1 : 0]$, the A_2-singularity is $[1 : 0 : 0 : 0]$, and the corresponding $\mathrm{Aut}_X^0 = \mu_5$-action is as in Table 6.

Case $d = 1$. If $d = 1$, then X is a double cover of the quadratic cone in $\mathbb{P}^3$ branched over a sextic curve S. We consider separately the three cases where X contains a single A_4-singularity (and possibly equivariant RDPs of other types), two A_4-singularities, or an E_8^0-singularity, respectively:

(a) The surface X contains a single A_4-singularity. By Corollary 7.2, the surface X is the anti-canonical model of a blow-up Y_1 in a smooth point P of an RDP del Pezzo surface X_2 with an A_4-singularity. By Proposition 5.8, we have $\mathrm{Aut}_X^0 = \mathrm{Aut}_{Y_1}^0 = \mathrm{Stab}_{\mathrm{Aut}_{X_2}^0}(P)^0$. Since $\mathrm{Aut}_{X_2}^0 = \mu_5$, we thus have to determine the fixed points of the μ_5-action on X_2.

- If the RDP configuration on X_2 is $A_4 + A_2$, then the fixed points of the μ_5-action are $[1 : 0 : 0 : 0]$, $[0 : 1 : 0 : 0]$, and $[0 : 0 : 1 : 0]$, and we recall that $[1 : 0 : 0 : 0]$ is the A_2-singularity. Hence, there are two choices for P. In fact, one of them does not occur, as the following argument shows:
  - If $P = [0 : 0 : 1 : 0]$, let Q be the branch quartic of $X_2 \to \mathbb{P}^2$. The line $\ell = \{y = 0\}$ is tangent to Q at $[1 : 0 : 0]$ and $[0 : 0 : 1]$, so the preimage of ℓ in X_2 consists of two smooth rational curves C and C' meeting in P and the A_2-singularity. In the minimal resolution $\widetilde{Y}_1$ of Y_1, their strict transforms $\widetilde{C}$ and $\widetilde{C}'$ are (-2)-curves, which, together with the two exceptional curves over the A_2-singularity, form an A_4-configuration of (-2)-curves. Thus, X contains two A_4-singularities, contradicting our assumption.
- If the RDP configuration on X_2 is $A_4 + A_1$, then the fixed points of the μ_5-action are $[1 : 0 : 0 : 1]$, $[1 : 0 : 0 : -1]$, $[0 : 1 : 0 : 0]$, and $[0 : 0 : 1 : 0]$. The first two points are interchanged by $w \mapsto -w$, and the point $[0 : 1 : 0 : 0]$ is the A_1-singularity on X_2, so we have two choices for P up to isomorphism. Let Q be the branch quartic of $X_2 \to \mathbb{P}^2$, and recall that the A_4-singularity is located at $[-2 : 1 : 2 : 0]$. We will now show that both choices for P lead to the same surface as the one in the previous bullet point:
  - If $P = [1 : 0 : 0 : 1]$, then the image $[1 : 0 : 0]$ of P in $\mathbb{P}^2$ lies on the two bitangents $\ell_1 = \{y = 0\}$ and $\ell_2 = \{z = 0\}$ of Q. Let C_i and C_i' be the two irreducible components of the preimage of ℓ_i for $i = 1, 2$, and assume that P lies on C_1 and C_2. On $\widetilde{Y}_1$, the strict transforms $\widetilde{C}_1$ and $\widetilde{C}_2$ are (-2)-curves, while the strict transforms $\widetilde{C}_1'$ and $\widetilde{C}_2'$ remain (-1)-curves. Thus, the RDP configuration on X is $A_4 + A_2 + A_1$, where the A_2 is obtained from the A_1 of Y_1 by also contracting $\widetilde{C}_2$, and the A_1 arises from the contraction of $\widetilde{C}_1$. Therefore, if we contract the image of C_1' in the surface Y_1' obtained from Y_1 by contracting C_2, we obtain an RDP del Pezzo surface of degree 2 with global vector fields and containing an RDP configuration of type $A_4 + A_2$, so this case is reduced to the previous bullet point.

– If $P = [0:0:1:0]$, then the image $[0:0:1]$ of P in $\mathbb{P}^2$ lies on the bitangent $\ell = \{y = 0\}$ and P is the non-transversal intersection point of the two irreducible components C and C' of the preimage of ℓ in X_2. Thus, the strict transforms $\widetilde{C}$ and $\widetilde{C}'$ of C and C' on $\widetilde{Y}_1$ are (-2)-curves, so the RDP configuration on X is $A_4 + A_2 + A_1$, where the A_2 is obtained by contracting $\widetilde{C}$ and $\widetilde{C}'$. The preimage D of the line $\{x = 0\}$ in X_2 is a cuspidal cubic with cusp at $[0:1:0:0]$. On $\widetilde{X}_2$, the strict transform of D is a smooth rational curve of self-intersection 0 by adjunction. Hence, the strict transform $\widetilde{D}$ of D on $\widetilde{Y}_1$ is a (-1)-curve which passes through the exceptional curve over the A_1-singularity. Contracting the image of $\widetilde{D}$ on the surface Y_1' obtained from Y_1 by contracting C and C' and resolving the A_1-singularity, we obtain an RDP del Pezzo surface of degree 2 with global vector fields containing an RDP configuration of type $A_4 + A_2$, so this case is also reduced to the previous bullet point.

Summarizing, there is at most one RDP del Pezzo surface of degree 1 with global vector fields and containing a single A_4-singularity. In Table 6, we give an equation of such a surface, so this is our X. The singularities of X are as follows: A_4 at $[1:-2:2:0]$, A_2 at $[0:1:0:0]$, and A_1 at $[1:0:0:0]$. The μ_5-action we describe in Table 6 preserves the equation, so $\mathrm{Aut}_X^0 = \mu_5$.

(b) The surface X contains two A_4-singularities. By [Lan94, Theorem 4.1], there is a unique RDP del Pezzo surface of degree 1 with RDP configuration $A_4 + A_4$, namely the Weierstrass model of the rational elliptic surface with singular fibers of type I_5, I_5, II. Its equation is given in Table 6. The $\alpha_5 \rtimes \mu_5$-action we describe in Table 6 preserves the equation. By [MS20], the group scheme $\mathrm{Aut}_{\widetilde{X}}^0$ is trivial, so $\mathrm{Aut}_X^0 = \alpha_5 \rtimes \mu_5$ follows from Proposition 5.8.

(c) The surface X contains an E_8^0-singularity. By [Lan94], there are two RDP del Pezzo surfaces of degree 1 with an RDP of type E_8. The one whose equation we give in Table 6 has an E_8^0-singularity, while the other one has an E_8^1-singularity (see [Sta21, Table 1]). The $\alpha_5 \rtimes \mathbb{G}_m$-action we describe in Table 6 preserves the equation. We leave it to the reader to check that $\mathrm{Aut}_X^0 = \mathrm{Stab}_{\mathrm{Aut}_{\mathbb{P}(1,1,2)}}(S)^0 = \alpha_5 \rtimes \mathbb{G}_m$. □

8.3 In characteristic 3

By Theorem 6.1, we have to list all RDP configurations containing A_2, A_5, A_8, E_6^0, E_6^1, E_7^0, E_8^0, or E_8^1 that can occur on an RDP del Pezzo surface.

Lemma 8.7. *If $p = 3$, $\deg(X) = d$, and X contains a singularity of type A_2, A_5, A_8, E_6^0, E_6^1, E_7^0, E_8^0, or E_8^1, then d and the configuration Γ of RDPs on X is one of the cases in Table 4.*

Proof. The maximal root lattice contained in E_2 is isomorphic to A_1, so none of the root lattices in the statement of the lemma embed into E_{9-d} with $d \geqslant 7$. The list for $4 \leqslant d \leqslant 6$ follows from [Mar03, Exercises 4.2.1 and 4.6.2], and the one for $7 \leqslant d \leqslant 9$ follows from [Dyn52, Table 11] (note that the lattice $A_6 + A_2$ in Dynkin's table of root lattices in E_8 should be $E_6 + A_2$), where we marked those RDP configurations whose associated root lattice embeds in two non-conjugate ways into E_{9-d} by a prime $'$. □

Theorem 8.8. *Assume $p = 3$ and that X contains an RDP of type A_2, A_5, A_8, E_6^0, E_6^1, E_7^0, E_8^0, or E_8^1. Then, $H^0(X, T_X) \neq 0$ if and only if X is given by an equation as in Tables 7, 8, 9 and 10. Moreover, Aut_X^0 is as given in these tables, so that $\mathrm{Aut}_{\widetilde{X}}^0 \subsetneq \mathrm{Aut}_X^0$ and even $h^0(X, T_X) > h^0(\widetilde{X}, T_{\widetilde{X}})$.*

Proof. By Lemma 8.7, we have $d \leqslant 6$.

d	Γ	$\subseteq \langle k_{9-d}\rangle^{\perp}$
6	$A_2,\ A_2+A_1$	$\subseteq A_2+A_1$
5	$A_2,\ A_2+A_1$	$\subseteq A_4$
4	$A_2,\ A_2+A_1,\ A_2+2A_1$	$\subseteq D_5$
3	$A_2,\ A_2+A_1,\ A_2+2A_1,\ 2A_2,\ 2A_2+A_1,\ A_5,\ 3A_2,\ A_5+A_1,\ E_6^0,\ E_6^1,$	$\subseteq E_6$
2	$A_2,\ A_2+A_1,\ A_2+2A_1,\ 2A_2,\ A_2+3A_1,\ 2A_2+A_1,\ A_3+A_2,\ (A_5)',\ 3A_2,$ $A_3+A_2+A_1,\ A_4+A_2,\ (A_5+A_1)',\ E_6^0,\ E_6^1,\ A_5+A_2,\ E_7^0$	$\subseteq E_7$
1	$A_2,\ A_2+A_1,\ A_2+2A_1,\ 2A_2,\ A_2+3A_1,\ 2A_2+A_1,\ A_3+A_2,\ A_5,$ $A_2+4A_1,\ 2A_2+2A_1,\ 3A_2,\ A_3+A_2+A_1,\ A_4+A_2,\ D_4+A_2,\ (A_5+A_1)',$ $E_6^0,\ E_6^1,\ 3A_2+A_1,\ A_3+A_2+2A_1,\ A_4+A_2+A_1,\ A_5+2A_1,\ A_5+A_2,$ $D_5+A_2,\ E_6^0+A_1,\ E_6^1+A_1,\ E_7^0,\ 4A_2,\ A_5+A_2+A_1,$ $E_6^0+A_2,\ E_6^1+A_2,\ A_8,\ E_8^0,\ E_8^1$	$\subseteq E_8$

TABLE 4. Non-equivariant RDP configurations in characteristic 3

Case $d=6$. If $d=6$, then X is a sextic surface in $\mathbb{P}^6$. The RDP configuration Γ on X is either A_2 or A_2+A_1 by Table 4. In both cases, X is uniquely determined by its singularities, by the same argument as in characteristic 0, that is, by checking the possible configurations of infinitely near points in $\mathbb{P}^2$ (see, for example, [Dol12, Section 8.4.2]).

(1) If $\Gamma = A_2$, then, by [Der14, Section 3.2], the surface X is given by the equations in Table 7. The A_2-singularity is $[1:0:0:0:0:0:0]$, and we give an α_3-action on X which does not preserve the singularity in Table 7. By Proposition 5.8, this implies $\mathrm{Aut}_X^0 = \langle \alpha_3, \mathrm{Aut}_{\tilde{X}}^0 \rangle$.

(2) If $\Gamma = A_2+A_1$, then, by [KN09, Appendix, p. 3], the surface X is given by the equation in Table 7. The A_2-singularity is $[0:1:0:0:0:0:0]$, and the A_1-singularity is $[0:0:0:0:0:0:1]$. We give an α_3-action on X which does not fix the A_2-singularity in Table 7. By Proposition 5.8, this implies $\mathrm{Aut}_X^0 = \langle \alpha_3, \mathrm{Aut}_{\tilde{X}}^0 \rangle$.

Case $d=5$. If $d=5$, then X is a quintic surface in $\mathbb{P}^5$. By Table 4, the RDP configuration Γ on X is either A_2 or A_2+A_1. As in Case $d=6$, the surface X is uniquely determined by Γ, as can be seen by checking the possible configurations of four infinitely near points in $\mathbb{P}^2$ (see, for example [Dol12, Section 8.5.1, p. 430]).

(1) If $\Gamma = A_2$, then, by [Der14, Section 3.3], the surface X is given by the equation in Table 7. The A_2-singularity is $[0:0:1:0:0:0]$, and we give an α_3-action on X which does not preserve the singularity in Table 7. By Proposition 5.8, this implies $\mathrm{Aut}_X^0 = \langle \alpha_3, \mathrm{Aut}_{\tilde{X}}^0 \rangle$.

(2) If $\Gamma = A_2+A_1$, then by [KN09, Appendix, p. 5], the surface X is given by the equation in Table 7. The A_2-singularity is $[0:1:0:0:0:0]$, and the A_1-singularity is $[0:0:0:0:0:1]$. We give an α_3-action on X which does not fix the A_2-singularity in Table 7. By Proposition 5.8, this implies $\mathrm{Aut}_X^0 = \langle \alpha_3, \mathrm{Aut}_{\tilde{X}}^0 \rangle$.

Case $d=4$. If $d=4$, then X is a quartic surface in $\mathbb{P}^4$. By Table 4, the RDP configuration on X is A_2, A_2+A_1, or A_2+2A_1. By Corollary 7.2, the surface X is the anti-canonical model of a blow-up Y_4 in a smooth point P of an RDP del Pezzo surface X_5 of degree 5 with an

A_2-singularity.

Next, we show that there are at most three non-isomorphic such X. By [MS20], there are at most two X whose minimal resolution has global vector fields. So, we claim that there is at most one X whose minimal resolution $\widetilde{X}$ does not have global vector fields. By Propositions 4.2 and 5.8, the stabilizer of P must be 0-dimensional in this case. Since the orbit of P under $\mathrm{Aut}^0_{X_5}$ is at most 2-dimensional (being contained in X_5) and isomorphic to $\mathrm{Aut}^0_{X_5}/\mathrm{Stab}_{\mathrm{Aut}^0_{X_5}}(P)$, we conclude that $\dim \mathrm{Aut}^0_{X_5} \leqslant 2$. Hence, by [MS20], the surface X_5 is the unique RDP del Pezzo surface of degree 5 whose RDP configuration is A_2. For this surface, $\dim \mathrm{Aut}^0_{\widetilde{X}_5} = \dim \mathrm{Aut}^0_{X_5} = 2$, so there is at most one 2-dimensional orbit on X_5. By the orbit-stabilizer theorem, P must lie in this maximal orbit. Blowing up a different point in this orbit yields an isomorphic surface, so we conclude that there is at most one X whose minimal resolution does not have global vector fields. In Table 7, we give equations for three such surfaces, distinguished by their RDP configuration Γ:

(1) If $\Gamma = A_2$, the A_2-singularity is at $[1:1:1:1:1]$. We give a μ_3-action on X in Table 7, while the group scheme $\mathrm{Aut}^0_{\widetilde{X}}$ is trivial by [MS20]. By Proposition 5.8, this implies $\mathrm{Aut}^0_X = \mu_3$.

(2) If $\Gamma = A_2 + A_1$, the A_2-singularity is $[1:0:0:0:0]$, while the A_1-singularity is $[0:0:0:0:1]$. We give an $\alpha_3 \rtimes \mathbb{G}_m$-action on X in Table 7, while $\mathrm{Aut}^0_{\widetilde{X}} = \mathbb{G}_m$ by [MS20], so $\mathrm{Aut}^0_X = \alpha_3 \rtimes \mathbb{G}_m$ by Proposition 5.8.

(3) If $\Gamma = A_2 + 2A_1$, the A_2-singularity is $[0:0:0:0:1]$ and the two A_1-singularities are $[0:1:0:0:0]$ and $[0:0:1:0:0]$. We give an $\alpha_3 \rtimes \mathbb{G}_m^2$-action on X in Table 7, while $\mathrm{Aut}^0_{\widetilde{X}} = \mathbb{G}_m^2$ by [MS20], so $\mathrm{Aut}^0_X = \alpha_3 \rtimes \mathbb{G}_m^2$ by Proposition 5.8.

Case $d = 3$. If $d = 3$, then X is a cubic surface in $\mathbb{P}^3$. By Table 4, the surface X contains either a single A_2-singularity, two A_2-singularities, three A_2-singularities, one A_5-singularity, one E_6^0-singularity, or one E_6^1-singularity. In the following, we consider these six cases separately.

(a) The surface X contains a single A_2-singularity. By Corollary 7.2, the surface X is the anti-canonical model of a blow-up Y_3 in a smooth point P of an RDP del Pezzo surface X_4 with an A_2-singularity. More precisely, since $X \dashrightarrow X_4$ is an isomorphism around the only non-equivariant RDP on X and all other singularities of X are A_1-singularities by Table 4, Propositions 5.8 and 4.2 imply that $\mathrm{Aut}^0_X = \mathrm{Aut}^0_{Y_3} = \mathrm{Stab}_{\mathrm{Aut}^0_{X_4}}(P)^0$. In particular, P is a point with non-trivial stabilizer on one of the surfaces X_4 in Table 7. Before we go on, note that by [MS20], the group scheme $\mathrm{Aut}^0_{\widetilde{X}}$ is trivial, so the stabilizer of P is 0-dimensional.

- Assume that the RDP configuration on X_4 is $A_2 + 2A_1$. In this case, $\mathrm{Aut}^0_{X_4}$ is 2-dimensional, so there is at most one 2-dimensional orbit for this action. Hence, there is at most one choice for P up to isomorphism.
- Assume that the RDP configuration on X_4 is $A_2 + A_1$. The surface X_4 contains the six lines $\ell_1 = \{x_0 = x_2 = x_3 = 0\}$, $\ell_2 = \{x_0 = x_2 = x_4 = 0\}$, $\ell_3 = \{x_0 = x_3 = x_1 + x_4 = 0\}$, $\ell_4 = \{x_1 = x_2 = x_3 = 0\}$, $\ell_5 = \{x_1 = x_2 = x_4 = 0\}$, $\ell_6 = \{x_1 = x_3 = x_4 = 0\}$. The lines ℓ_4, ℓ_5, and ℓ_6 pass through the A_2-singularity at $[1:0:0:0:0]$, and the others do not. The lines ℓ_1 and ℓ_4 pass through the A_1-singularity at $[0:0:0:0:1]$, and the others do not. Using the description of the $\mathrm{Aut}^0_{X_4}$-action in Table 7, one can calculate the smooth points $P = [x_0 : x_1 : x_2 : x_3 : x_4]$ with non-trivial and 0-dimensional stabilizer on X_4 that do not lie on the lines ℓ_4, ℓ_5, ℓ_6 through the A_2-singularity: If $x_4 \neq 0$, then we have $x_2 = 0$ or $x_0 = 0$ from the description of the $\mathrm{Aut}^0_{X_4}$-action. Hence, P lies in the intersection of X_4 with the coordinate hyperplanes. From the equations of X, we see that this intersection is exactly the

union of the six lines on X_4. Since the line ℓ_1 is fixed pointwise by $\mathbb{G}_m$, we obtain that

$$\begin{aligned} P &\in (\ell_2 \cup \ell_3) \setminus (\ell_1 \cup \ell_4 \cup \ell_5 \cup \ell_6) \\ &= (\ell_2 \cup \ell_3) \setminus \{[0:1:0:0:0], [0:0:1:0:0], [0:0:0:1:0], [0:1:0:0:-1]\}. \end{aligned}$$

Since the automorphism $x_2 \leftrightarrow x_3$, $x_4 \leftrightarrow -x_4 - x_1$ interchanges the two lines ℓ_2 and ℓ_3 and $\mathbb{G}_m \subseteq \mathrm{Aut}_{X_4}$ acts transitively on the locus of points with 0-dimensional stabilizer on each of ℓ_2 and ℓ_3, there is a unique choice for P up to isomorphism. We will now reduce this case to the previous bullet point:

- Without loss of generality, assume $P \in \ell_2 \setminus \{[0:1:0:0:0], [0:0:0:1:0]\}$. Then, the strict transform of ℓ_2 on Y_3 is a (-2)-curve, and the RDP configuration on X is $A_2 + 2A_1$. Since ℓ_3 is disjoint from ℓ_2, it remains a (-1)-curve on X, so we can contract it to obtain a realization of X as a blow-up of an RDP del Pezzo surface X_4' with RDPs of type $A_2 + 2A_1$. Hence, this case is reduced to the previous bullet point.

- Assume that the RDP configuration on X_4 is A_2. Using the description of the $\mathrm{Aut}_{X_4}^0 = \mu_3$-action given in Table 7, we see that the points P with non-trivial stabilizer on X_4 are $[1:0:0:0:0]$, $[0:1:0:0:0]$, $[0:0:1:0:0]$, $[0:0:0:1:0]$, and $[0:0:0:0:1]$. The surface X_4 admits the two involutions $x_0 \leftrightarrow x_1$ and $(x_0, x_1) \leftrightarrow (x_2, x_3)$. Hence, blowing up any of the first four points leads to the same surface. In fact, we already treated the resulting surface, as the following argument shows:
  - Without loss of generality, assume $P = [1:0:0:0:0]$. There are two lines on X_4 passing through P, namely $\ell_1 = \{x_1 = x_2 = x_4 = 0\}$ and $\ell_2 = \{x_1 = x_3 = x_4 = 0\}$, and both of these lines do not pass through the A_2-singularity at $[1:1:1:1:1]$. Their strict transforms on Y_3 are disjoint (-2)-curves. Thus, the RDP configuration on X is $A_2 + 2A_1$. Now, the conic $C = \{x_1 = x_2 + x_3 = x_0 x_4 - x_3^2 = 0\}$ meets ℓ_1 and ℓ_2 transversally at P and does not pass through $[1:1:1:1:1]$, so we can contract the image of the strict transform of C in X and obtain an RDP del Pezzo surface X_4' with RDP configuration $A_2 + 2A_1$ and with global vector fields, so X_4' is the surface in the first bullet point.

Summarizing, there are at most two RDP del Pezzo surfaces of degree 3 with global vector fields and containing a single A_2-singularity. Moreover, $\mathrm{Aut}_X^0 = \mathrm{Stab}_{\mathrm{Aut}_{X_4}^0}(P)^0 = \mu_3$. In Table 8, we give equations for two such surfaces, distinguished by their RDP configuration Γ:

(1) If $\Gamma = A_2$, the A_2-singularity is at $[1:1:-1:-1]$. Clearly, the μ_3-action we give preserves the equation.

(2) If $\Gamma = A_2 + 2A_1$, the A_2-singularity is at $[1:-1:-1:1]$ and the two A_1-singularities are at $[0:1:0:0]$ and $[0:0:1:0]$. Again, the μ_3-action we give preserves the equation.

(b) The surface X contains two A_2-singularities. By Table 4, the RDP configuration Γ on X is $2A_2$ or $2A_2 + A_1$. Simplifying the normal form of Roczen given in [Roc96], we obtain the equations given in Table 8. Indeed, by [Roc96, Case B] and after permuting the coordinates, a cubic surface with an A_2-singularity admits an equation of the form

$$x_0^3 + x_1 x_2 x_3 + x_2\big(a_0 x_2^2 + a_1 x_0 x_2 + a_2 x_0^2\big) + x_1\big(a_3 x_1^2 + a_4 x_0 x_1 + a_5 x_0^2\big) = 0$$

with $a_i \in k$. Moreover, again by [Roc96, Case B], the existence of a second A_2-singularity implies that we can choose an equation as above with $a_0 = a_1 = a_2 = 0$. Now, using a substitution of the form $x_0 \mapsto x_0 + \lambda x_1$ for a suitable $\lambda \in k$ allows us to set $a_3 = 0$. If $a_4 \neq 0$, we can rescale coordinates to assume $a_4 = 1$. Then, by [Roc96, Case B], the two A_2-singularities on X are the only singularities of the cubic surface if and only if $x_0^3 + a_5 x_0^2 x_1 + x_0 x_1^2$ has only simple roots, so if and only if $a_5^2 \neq 1$. This is our normal form in Table 8 for the cubic surfaces with two

A_2-singularities. If $a_5^2 = 1$, then we can change coordinates to assume $a_4 = 0$ in the original equation. Then, we can rescale coordinates to get our normal form for the cubic surface with singularities $2A_2 + A_1$.

In particular, there is a 1-dimensional family of X with $\Gamma = 2A_2$ and a unique X with $\Gamma = 2A_2 + A_1$.

(1) If $\Gamma = 2A_2$, the two A_2-singularities are at $[0:0:1:0]$ and $[0:0:0:1]$. The two α_3-actions and the $\mathrm{Aut}^0_{\widetilde{X}} = \mathbb{G}_m$-action we give in Table 8 preserve the equation. Each of the α_3-actions fixes one of the A_2-singularities and does not preserve the other one, so Proposition 5.8 shows that $\mathrm{Aut}^0_X = \langle \alpha_3, \alpha_3, \mathbb{G}_m \rangle$.

(2) If $\Gamma = 2A_2 + A_1$, the two A_2-singularities are at $[0:0:1:0]$ and $[0:0:0:1]$ and the A_1-singularity is at $[0:1:0:0]$. The two α_3-actions and the $\mathrm{Aut}^0_{\widetilde{X}} = \mathbb{G}_m$-action we describe in Table 8 preserve the equation. By the same argument as in item (1), we have $\mathrm{Aut}^0_X = \langle \alpha_3, \alpha_3, \mathbb{G}_m \rangle$.

(c) The surface X contains three A_2-singularities. Again, we can simplify the normal form of Roczen given in [Roc96] and obtain the equation in Table 8, which admits A_2-singularities at $[0:1:0:0]$, $[0:0:1:0]$, and $[0:0:0:1]$. In Table 8, we give an action of $\alpha_3^3 \rtimes \mathbb{G}_m^2$ on X. By [MS20], we have $\mathrm{Aut}^0_{\widetilde{X}} = \mathbb{G}_m^2$. Each of the three factors of the α_3^3-action preserves two of the A_2-singularities and moves the other one, so Proposition 5.8 yields $\mathrm{Aut}^0_X = \alpha_3^3 \rtimes \mathbb{G}_m^2$.

(d) The surface X contains an A_5-singularity. As above, we simplify Roczen's normal form [Roc96] to the two equations in Table 8. Let Γ be the RDP configuration on X.

(1) If $\Gamma = A_5$, then the A_5-singularity is at $[0:0:0:1]$. The α_3-action we describe preserves the equation and does not fix the A_5-singularity. By [MS20], we have $\mathrm{Aut}^0_{\widetilde{X}} = \mathbb{G}_a \rtimes \mu_3$, and we describe this action in terms of the equation in Table 8. By Remark 5.9, we have $\mathrm{Aut}^0_X = \langle \alpha_3, \mathbb{G}_a \rtimes \mu_3 \rangle$.

(2) If $\Gamma = A_5 + A_1$, then the A_5-singularity is at $[0:0:0:1]$ and the A_1-singularity is at $[0:0:1:0]$. The α_3-action we describe preserves the equation. By [MS20], we have $\mathrm{Aut}^0_{\widetilde{X}} = \mathbb{G}_a \rtimes \mathbb{G}_m$, and we describe this action in terms of the equation in Table 8. By Remark 5.9, we have $\mathrm{Aut}^0_X = \langle \alpha_3, \mathbb{G}_a \rtimes \mathbb{G}_m \rangle$.

(e) The surface X contains an E_6^0-singularity. By [Roc96, Case C3], the surface X is given by the equation in Table 8, and the singularity is at $[0:0:0:1]$. In Table 8, we give an action of a group scheme G of order 27 such that no subgroup scheme of G lifts to $\widetilde{X}$, as well as the action of $\mathrm{Aut}^0_{\widetilde{X}} = \mathbb{G}_a^2 \rtimes \mathbb{G}_m$ in terms of the equation. We claim that these actions generate $\mathrm{Aut}^0_X = \mathrm{Stab}_{\mathrm{PGL}_4}(X)^0$. Indeed, let $\sigma \in \mathrm{Aut}^0_X(R)$ be an R-valued automorphism deforming the identity, where R is an Artinian local k-algebra with residue field k. After composing with sections of $\mathrm{Aut}^0_{\widetilde{X}}(R) = R^2 \rtimes R^\times$, we may assume that σ is given by a substitution acting on the x_i via a matrix

$$A = \begin{pmatrix} 1 & a & 0 & b \\ c & 1 & 0 & d \\ e & f & g & h \\ i & j & k & l \end{pmatrix}$$

with coefficients in R and such that $A = \mathrm{id}$ modulo the maximal ideal of R. Now, first, in order to understand $\mathrm{Aut}^0_X[F]$, assume $R = k[\varepsilon]/(\varepsilon^2)$. Then, any two entries off the diagonal in A multiply to 0. Applying A to the equation of X and comparing coefficients, we obtain that $c = e = i = 0$, so σ fixes $[1:0:0:0]$. Since the schematic fixed locus of the action of the Frobenius kernel $\mathrm{Aut}^0_{\widetilde{X}}[F] \subseteq \mathrm{Aut}^0_X[F]$ on X is $\{x_0 = x_2 = 0\} \cup \{[1:0:0:0]\}$, we deduce that the schematic fixed locus of all of $\mathrm{Aut}^0_X[F]$ contains the reduced point $[1:0:0:0]$ as a component. As $\mathrm{Aut}^0_X[F]$ is normal in Aut^0_X, this implies that Aut^0_X fixes $[1:0:0:0]$.

This means that, for general σ and R, we have $c = e = i = 0$ in the matrix A. Now, applying A to the equation of X and comparing coefficients, we obtain

$$a^3 + f^2 j + f = 0\,,$$
$$g^2 k = 0\,, \tag{8.1}$$
$$b^3 + d^2 h + h^2 l = 0\,,$$
$$f^2 k + 2fgj + g = 1\,, \tag{8.2}$$
$$2df + f^2 l + 2fhj + h = 0\,,$$
$$2fgk + g^2 j = 0\,, \tag{8.3}$$
$$g^2 l + 2ghk = 1\,, \tag{8.4}$$
$$d^2 f + 2dh + 2fhl + h^2 j = 0\,,$$
$$d^2 g + 2ghl + h^2 k = 0\,,$$
$$2dg + 2fgl + 2fhk + 2ghj = 0\,.$$

Since g and l are units in R, we deduce that $k = j = 0$ from (8.1) and (8.3), and then $g = l = 1$ from (8.2) and (8.4). Then, the remaining equations yield $f = -a^3$, $d = a^3$, $h = a^6$, $a^9 = 0$, and $b^3 = 0$. Hence, $\sigma \in G(R)$, as desired.

(f) The surface X contains an E_6^1-singularity. By [Roc96], the surface X is given by the equation in Table 8, and the singularity is at $[0:0:0:1]$. In Table 8, we give a μ_3-action that does not preserve the singular point, as well as the action of $\mathrm{Aut}^0_{\tilde{X}} = \mathbb{G}_a^2$ in terms of the equation. We leave it to the reader to check that these two actions generate $\mathrm{Aut}^0_X = \mathrm{Stab}_{\mathrm{PGL}_4}(X)^0$.

Case $d = 2$. If $d = 2$, then X is a double cover of $\mathbb{P}^2$ branched over a quartic curve Q. In this case, we will take a slightly different approach which will turn out to be more economical than using Corollary 7.2. Namely, we classify all possible Q with global vector fields; by Proposition 4.3, this is equivalent to the classification of RDP del Pezzo surfaces of degree 2 with global vector fields. If Q admits a global vector field, then it also admits an additive or multiplicative global vector field. This vector field is induced by a vector field on $\mathbb{P}^2$. Up to conjugation, there are two non-zero vector fields D on $\mathbb{P}^2$ with $D^3 = D$ and two non-zero vector fields with $D^3 = 0$. They correspond to the following four matrices in Jordan normal form in the Lie algebra of PGL_3:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Integrating the corresponding vector fields (see, for example, [Tzi17, Proposition 3.1]), we obtain the following four μ_3- and α_3-actions on $\mathbb{P}^2$:

(a) $\mu_3\colon [x:y:z] \mapsto \big[x : \lambda y : \lambda^2 z\big]$,
(b) $\mu_3\colon [x:y:z] \mapsto [x : y : \lambda z]$,
(c) $\alpha_3\colon [x:y:z] \mapsto [x + \varepsilon y : y : z]$,
(d) $\alpha_3\colon [x:y:z] \mapsto \big[x + \varepsilon y - \varepsilon^2 z : y + \varepsilon z : z\big]$.

For each of these actions, we will now classify the quartics that are invariant under it:

(a) There are three types of quartics which are invariant under this μ_3-action, namely

$$ax^4 + bx^2yz + cxy^3 + dxz^3 + ey^2z^2\,,$$
$$ax^3y + bx^2z^2 + cxy^2z + dy^4 + eyz^3\,,$$
$$ax^3z + bx^2y^2 + cxyz^2 + dy^3z + ez^4\,.$$

These families are identified by permuting x, y, and z, so it suffices to study the first one. Now, we simplify this equation and identify the singularities. We sort the classification according to the number of coefficients that are 0. Since Q is reduced, at least two coefficients are non-zero. Except in the case where $c = d = 0$, we can scale three of the non-zero coefficients to 1.

- If three of the coefficients are 0, the other two can be scaled to 1. The fact that Q is reduced leaves us with the following three cases after using the symmetry $y \leftrightarrow z$:

$$x^4 + y^2z^2\,, \tag{8.5}$$
$$x^2yz + xy^3\,, \tag{8.6}$$
$$x^2yz + y^2z^2\,. \tag{8.7}$$

  In Case (8.5), the surface X has two A_3-singularities, one at $[0 : 1 : 0 : 0]$ and one at $[0 : 0 : 1 : 0]$. In particular, X contains only equivariant RDPs, so it does not appear in Table 9.
  In Case (8.6), the surface X has a D_6-singularity at $[0 : 0 : 1 : 0]$ and an A_1-singularity at $[1 : 0 : 0 : 0]$. As before, X contains only equivariant RDPs, so it does not appear in Table 9.
  In Case (8.7), the surface X has an A_1-singularity at $[1 : 0 : 0 : 0]$ and two A_3-singularities at $[0 : 1 : 0 : 0]$ and $[0 : 0 : 1 : 0]$. Again, X contains only equivariant RDPs, so it does not appear in Table 9.
- If two of the coefficients are 0, we get the following cases, again using scaling and the symmetry $y \leftrightarrow z$:

$$x^4 + ax^2yz + y^2z^2\,, \tag{8.8}$$
$$x^4 + x^2yz + xy^3\,, \tag{8.9}$$
$$x^4 + y^2z^2 + xy^3\,, \tag{8.10}$$
$$x^2yz + y^2z^2 + xy^3\,, \tag{8.11}$$
$$x^2yz + xy^3 + xz^3\,, \tag{8.12}$$
$$xy^3 + xz^3 + y^2z^2\,. \tag{8.13}$$

  In Case (8.8), we must have $a^2 \neq 1$, otherwise Q is a double conic. Then, Q is the union of two conics, tangent at the points $[0 : 1 : 0]$ and $[0 : 0 : 1]$. The singularities of X over these two points are two A_3-singularities, so X contains only equivariant RDPs and does not appear in Table 9.
  In Case (8.9), the surface X has a D_6-singularity at $[0 : 0 : 1 : 0]$, so X contains only equivariant RDPs and does not appear in Table 9.
  In Case (8.10), the surface X has an A_3-singularity at $[0 : 0 : 1 : 0]$ and an A_2-singularity at $[1 : -1 : 0 : 0]$. By [MS20], $\mathrm{Aut}^0_{\tilde{X}}$ is trivial, so Proposition 5.8 implies $\mathrm{Aut}^0_X = \mu_3$.
  In Case (8.11), the surface X has an A_3-singularity at $[0 : 0 : 1 : 0]$, an A_2-singularity at $[1 : 1 : 1 : 0]$, and an A_1-singularity at $[1 : 0 : 0 : 0]$. By [MS20], the group scheme $\mathrm{Aut}^0_{\tilde{X}}$ is trivial, so Proposition 5.8 implies $\mathrm{Aut}^0_X = \mu_3$.
  In Case (8.12), the surface X has an A_5-singularity at $[0 : 1 : -1 : 0]$ and an A_1-singularity

at $[1:0:0:0]$. By [MS20], the group scheme $\mathrm{Aut}^0_{\widetilde{X}}$ is trivial, so Proposition 5.8 implies $\mathrm{Aut}^0_X = \mu_3$.

In Case (8.13), the surface X has an E_6^1-singularity at $[1:0:0:0]$. It is elementary to check that $\mathrm{Aut}^0_X = \mathrm{Stab}_{\mathrm{PGL}_3}(Q) = \mu_3$ in this case. Alternatively, observe that this case does not occur in the classification of invariant quartics for the actions (b), (c), and (d), so $\mathrm{Aut}^0_X[F] = \mu_3$, where $\mathrm{Aut}^0_X[F]$ is the Frobenius kernel of Aut^0_X. In particular, μ_3 is normal in Aut^0_X, so the Aut^0_X-action preserves the eigenspaces of the μ_3-action. Hence, the induced action on $\mathbb{P}^2$ is diagonal. This simplifies the calculation of the stabilizer of Q considerably.

- If only one coefficient is 0, then we get the following cases, again using scaling and the symmetry $y \leftrightarrow z$:

$$x^4 + ax^2yz + xy^3 + xz^3 \,, \tag{8.14}$$
$$x^4 + a^3x^2yz + xy^3 + y^2z^2 \,, \tag{8.15}$$
$$x^4 + xy^3 + xz^3 + ay^2z^2 \,, \tag{8.16}$$
$$ax^2yz + xy^3 + xz^3 + y^2z^2 \,. \tag{8.17}$$

In Case (8.14), the surface X has an A_5-singularity at $[0:1:-1:0]$. By [MS20], the group scheme $\mathrm{Aut}^0_{\widetilde{X}}$ is trivial, so $\mathrm{Aut}^0_X = \mu_3$ by Remark 5.9.

Consider Case (8.15). If $a^2 = 1$, then X has an A_6-singularity at $[0:0:1:0]$, so X contains only equivariant RDPs and does not occur in Table 9. If $a^2 \neq 1$, then X has an A_3-singularity at $[0:0:1:0]$ and an A_2-singularity at $\left[a^2-1 : a^4+a^2+1 : a^3 : 0\right]$. In this case, $\mathrm{Aut}^0_{\widetilde{X}}$ is trivial by [MS20], so $\mathrm{Aut}^0_X = \mu_3$ by Proposition 5.8.

In Case (8.16), the surface X has two A_2-singularities, one at $[1:0:-1:0]$ and one at $[1:-1:0:0]$. We leave it to the reader to check that $\mathrm{Aut}^0_X = \mu_3$ in this case.

In Case (8.17), the surface X has an A_1-singularity at $[1:0:0:0]$ and additional singularities at $\left[u : au^2 : 1\right]$, where u is a solution of $a^3u^6 - au^3 + 1 = 0$. If $a = 1$, then $u = -1$ is unique and the resulting singularity of X is of type A_5. If $a \neq 1$, then X has two singularities of type A_2. In both cases, $\mathrm{Aut}^0_{\widetilde{X}}$ is trivial by [MS20], so $\mathrm{Aut}^0_X = \mu_3$ if $a = 1$ by Remark 5.9. If $a \neq 1$, one can check $\mathrm{Aut}^0_X = \mu_3$ directly.

- If no coefficient is 0, we get the case

$$x^4 + xy^3 + xz^3 + ax^2yz + by^2z^2 \,. \tag{8.18}$$

Here, X has singularities at the points $\left[bu : au^2 : b\right]$, where u is a solution of the equation $a^3u^6 + \left(b^3 - a^2b^2\right)u^3 + b^3 = 0$. If $\left(b^3 - a^2b^2\right)^2 = a^3b^3$, then x is unique and the resulting singularity of X is of type A_5. If $\left(b^3 - a^2b^2\right)^2 \neq a^3b^3$, then X has two singularities of type A_2. In both cases, $\mathrm{Aut}^0_{\widetilde{X}}$ is trivial by [MS20], so $\mathrm{Aut}^0_X = \mu_3$ by Proposition 5.8 and Remark 5.9.

(b) There are three types of quartics which are invariant under this μ_3-action, namely

$$f_2(x,y)z^2 \,, \quad f_3(x,y)z + f_0z^4 \,, \quad f_4(x,y) + f_1(x,y)z^3 \,,$$

where the f_i are homogeneous polynomials of degree i in x and y. All quartics in the first family contain a double line, so they lead to non-normal X. For the same reason, we have $f_3, f_4 \neq 0$ in the latter two families. In the third family, we have $f_1 \neq 0$, for otherwise Q has a quadruple point and the corresponding singularity on X is not an RDP. The GL_2-action on x, y normalizes the μ_3-action, so acts on the space of invariant quartics. Similarly, substitutions of the form $z \mapsto z + \beta x + \gamma y$ with $\beta, \gamma \in k$ act on the space of invariant quartics. Using these substitutions, and keeping in mind that Q must be reduced, we obtain the following simplified normal forms:

$$xy(x+y)z\,, \tag{8.19}$$

$$xy(x+y)z + z^4\,, \tag{8.20}$$

$$x^2yz + z^4\,, \tag{8.21}$$

$$y^4 + xz^3\,, \tag{8.22}$$

$$y^4 + x^2y^2 + xz^3\,, \tag{8.23}$$

$$x^2y^2 + xz^3\,, \tag{8.24}$$

$$x^3y + xz^3\,. \tag{8.25}$$

In Case (8.19), the surface X has a D_4-singularity at $[0:0:1:0]$ and three A_1-singularities, at $[1:0:0:0]$, $[0:1:0:0]$, and $[1:-1:0:0]$. In particular, X contains only equivariant RDPs, so it does not occur in Table 9.

In Case (8.20), the surface X has an A_2-singularity at $[1:1:1:0]$ and three A_1-singularities, at $[1:0:0:0]$, $[0:1:0:0]$, and $[1:-1:0:0]$. By [MS20], the group scheme $\mathrm{Aut}^0_{\widetilde{X}}$ is trivial, so $\mathrm{Aut}^0_X = \mu_3$ by Proposition 5.8.

In Case (8.21), the surface X has a D_5-singularity at $[0:1:0:0]$ and an A_1-singularity at $[1:0:0:0]$. In particular, X contains only equivariant RDPs, so it does not occur in Table 9.

In Case (8.22), the surface X has a singularity of type E_6^0 at $[1:0:0:0]$. In Table 9, we give an action of a group scheme G of length 27 which preserves X and is such that no subgroup scheme of G lifts to $\widetilde{X}$. Moreover, we give an action of $\mathrm{Aut}^0_{\widetilde{X}} = \mathbb{G}_m$ on X. We claim that these two actions generate $\mathrm{Aut}^0_X \cong \mathrm{Stab}_{\mathrm{PGL}_3}(Q)^0$. To see this, first note that the line $\{x=0\}$ is the image of a (-1)-curve E in the smooth locus of X. Hence, E is preserved by Aut^0_X and, when we contract E, the action of Aut^0_X on X descends to an action on the cubic del Pezzo surface X_3 with an E_6^0-singularity, so X_3 is given by the equation in Table 8. By Proposition 4.2, we have $\mathrm{Aut}^0_X \cong \mathrm{Stab}_{\mathrm{Aut}^0_{X_3}}(P)^0$, where P is the image of E in X_3. Note that P does not lie on the unique line $\ell = \{x_0 = x_2 = 0\} \subseteq X_3$ since this line passes through the singular point of X_3. By Table 8, the group scheme $\mathrm{Aut}^0_{X_3}$ acts transitively on $X_3 \setminus \ell$, so we may assume $P = [0:0:1:0]$. Then, from the description of the $\mathrm{Aut}^0_{X_3}$-action in Table 8, it is clear that $\mathrm{Stab}_{\mathrm{Aut}^0_{X_3}}(P)^0 \cong \langle G, \mathbb{G}_m \rangle$. Hence, $\mathrm{Aut}^0_X \cong \langle G, \mathbb{G}_m \rangle$, as claimed.

In Case (8.23), the surface X has three A_2-singularities, at $[1:0:0:0]$, $[1:-1:1:0]$, and $[1:1:1:0]$. In Table 9, we describe an action of $\alpha_3^2 \rtimes \mu_3$ on X. By [MS20], the group scheme $\mathrm{Aut}^0_{\widetilde{X}}$ is trivial, so $\mathrm{Aut}^0_X = \alpha_3^2 \rtimes \mu_3$ by Proposition 5.8.

In Case (8.24), the surface X has an A_5-singularity at $[0:1:0:0]$ and an A_2-singularity at $[1:0:0:0]$. In Table 9, we give an action of $\alpha_3^2 \rtimes \mathbb{G}_m$ on X. By [MS20], we have $\mathrm{Aut}^0_{\widetilde{X}} = \mathbb{G}_m$, so Proposition 5.8 and Remark 5.9 show that $\mathrm{Aut}^0_X = \alpha_3^2 \rtimes \mathbb{G}_m$.

In Case (8.25), the surface X admits a singularity of type E_7^0 at $[0:1:0:0]$. In Table 9, we give an action of α_3 on X that does not fix the E_7^0-singularity as well as the action of $\mathrm{Aut}^0_{\widetilde{X}} = \mathbb{G}_a \rtimes \mathbb{G}_m$. We leave it to the reader to check that these two actions generate Aut^0_X.

(c) We can write the equation of Q as

$$f_0x^4 + x^3f_1(y,z) + x^2f_2(y,z) + xf_3(y,z) + f_4(y,z)\,,$$

where the f_i are homogeneous of degree i in y and z. Applying the α_3-action, we obtain

$$f_0x^4 + \varepsilon f_0x^3y + x^3f_1(y,z) + \left(x^2 - \varepsilon xy + \varepsilon^2y^2\right)f_2(y,z) + (x+\varepsilon y)f_3(y,z) + f_4(y,z)\,.$$

By considering the non-zero term of highest degree in x, we see that this is a multiple of the original equation if and only if it is equal to it. Comparing the coefficients of ε^2 and ε, we see that this happens if and only if $f_0 = f_2 = f_3 = 0$. Hence, Q is of the form

$$x^3 f_1(y,z) + f_4(y,z)\,.$$

But then Q is invariant under the μ_3-action $[x:y:z] \mapsto [\lambda x : y : z]$ and hence, after interchanging x and z, the curve Q is as in Cases (8.22), (8.23), (8.24), and (8.25).

(d) We write the equation of Q as $\sum a_{ijk} x^i y^j z^k$. As in Case (c), one checks that Q is preserved by the α_3-action if and only if its equation is preserved by the α_3-action. Applying the α_3-action and comparing the coefficients of ε and ε^2, respectively, we see that Q is α_3-invariant if and only if the following two conditions are satisfied:

$$\begin{aligned} &a_{400}x^3y + a_{310}x^3z - a_{220}x^2yz + a_{211}x^2z^2 - a_{220}xy^3 - a_{211}xy^2z - (a_{202}+a_{121})xyz^2 \\ &\quad + a_{112}xz^3 + a_{130}y^4 + (a_{121}+a_{040})y^3z + a_{112}y^2z^2 + (a_{103}-a_{022})yz^3 + a_{013}z^4 = 0\,, \\ &-a_{400}x^3z + a_{220}x^2z^2 - a_{220}xy^2z + (a_{121}+a_{202})xz^3 + a_{220}y^4 + (a_{211}-a_{130})y^3z \\ &\quad + (a_{121}+a_{202})y^2z^2 + (a_{022}-a_{103})z^4 = 0\,. \end{aligned}$$

In other words, Q is α_3-invariant if and only if it is given by an equation of the form

$$a\big(xz+y^2\big)^2 + bz^2\big(xz+y^2\big) + cx^3z + dy^3z + ez^4\,.$$

The substitutions of the form $x \mapsto \beta^2 x + \beta\gamma y + \delta z$, $y \mapsto \beta y + \gamma z$ with $\beta \in k^\times$ and $\gamma, \delta \in k$ normalize the α_3-action, so they preserve the space of α_3-invariant quartics. If $a \neq 0$, we can scale it to $a = 1$ and use δ to kill b. If $a = 0$, then $b \neq 0$; otherwise, Q contains a triple line. Then, we can assume $b = 1$. Using the other substitutions, we arrive at one of the following five simplified normal forms:

$$\big(xz+y^2\big)^2 + z^4\,, \tag{8.26}$$
$$\big(xz+y^2\big)^2 + y^3z\,, \tag{8.27}$$
$$\big(xz+y^2\big)^2 + x^3z + a^6z^4\,, \tag{8.28}$$
$$z^2\big(xz+y^2\big) + y^3z\,, \tag{8.29}$$
$$z^2\big(xz+y^2\big) + x^3z\,. \tag{8.30}$$

In Case (8.26), the surface X has an A_7-singularity at $[1:0:0:0]$, so X contains only equivariant RDPs and does not occur in Table 9.

In Case (8.27), the surface X has an A_4-singularity at $[1:0:0:0]$ and an A_2-singularity at $[0:0:1:0]$. By [MS20], the group scheme $\mathrm{Aut}^0_{\widetilde{X}}$ is trivial, so $\mathrm{Aut}^0_X = \alpha_3$ by Proposition 5.8.

In Case (8.28), the surface X is singular precisely at $\big[-a^2 : \pm a : 1\big]$. If $a = 0$, then this singularity is of type A_5 and, by [MS20] and Remark 5.9, we have $\mathrm{Aut}^0_X = \alpha_3$. If $a \neq 0$, then both singularities are of type A_2. Direct calculation shows that $\mathrm{Aut}^0_X = \alpha_3$.

In Case (8.29), the surface X has an RDP of type E_7^1 at $[1:0:0:0]$. In particular, X contains only equivariant RDPs and does not occur in Table 9.

Finally, in Case (8.30), the surface X has an RDP of type A_5 at $[0:1:0:0]$. By [MS20], the group scheme $\mathrm{Aut}^0_{\widetilde{X}}$ is trivial, so, by Remark 5.9, we have $\mathrm{Aut}^0_X = \alpha_3$. Note that this surface is not isomorphic to the one in Case (8.28) (with $a = 0$) since here, Q contains a line, while in the other case, Q is irreducible.

Case $d = 1$. If $d = 1$, then X is a double cover of the quadratic cone in $\mathbb{P}^3$ branched over a sextic curve S. In particular, X is given in $\mathbb{P}(1,1,2,3)$ by an equation in Weierstrass form

$$y^2 = x^3 + a_2(s,t)x^2 + a_4(s,t)x + a_6(s,t)\,,$$

where a_i is a homogeneous polynomial of degree i in s and t, and S is given by the equation

$$x^3 + a_2(s,t)x^2 + a_4(s,t)x + a_6(s,t) = 0 \tag{8.31}$$

in $\mathbb{P}(1,1,2)$. By Proposition 4.3, we have $\mathrm{Aut}_X^0 = \mathrm{Stab}_{\mathrm{Aut}_{\mathbb{P}(1,1,2)}}(S)^0$. Since $\mathbb{P}(1,1,2)$ is an Aut_X-equivariant RDP del Pezzo surface of degree 8, Theorem 6.1 and [MS20] show that $\mathrm{Aut}_{\mathbb{P}(1,1,2)} = (\mathbb{G}_a^3 \rtimes \mathrm{GL}_2)/(\mathbb{Z}/2\mathbb{Z})$, which acts via substitutions of the form

$$x \mapsto x + f_2(s,t)\,, \quad s \mapsto \alpha s + \beta t\,, \quad t \mapsto \gamma s + \delta t\,,$$

where f_2 is a homogeneous polynomial of degree 2 in s and t, and α, β, γ, δ are scalars such that $\alpha\delta - \beta\gamma$ is invertible.

Now, assume that Aut_X^0 is non-trivial. Then, it contains $G \in \{\alpha_3, \mu_3\}$. If $G = \mu_3$, we claim that there are only three embeddings of G into $\mathrm{Aut}_{\mathbb{P}(1,1,2)}^0 = \mathbb{G}_a^3 \rtimes \mathrm{GL}_2$. First, by counting the possible Jordan normal forms, observe that there are only three conjugacy classes of embeddings of G into GL_2. Then, applying [Ray64, Théorème 5.1.1(ii)(b)], we see that every such embedding lifts uniquely, up to conjugation by $\mathbb{G}_a^3$, to an embedding of μ_3 into $\mathbb{G}_a^3 \rtimes \mathrm{GL}_2$. Hence, if $G = \mu_3$, we may assume that it acts in one of the following three ways on $\mathbb{P}(1,1,2)$:

(a) $\mu_3\colon [s:t:x] \mapsto [s:\lambda t:x]$,

(b) $\mu_3\colon [s:t:x] \mapsto [\lambda s:\lambda t:x]$,

(c) $\mu_3\colon [s:t:x] \mapsto [\lambda s:\lambda^{-1} t:x]$.

Next, assume $G = \alpha_3$ and that the image of G in GL_2 is trivial. Then, the embedding of G into $\mathbb{G}_a^3 \rtimes \mathrm{GL}_2$ is given by a homomorphism $\alpha_3 \to \mathbb{G}_a^3$, which in turn corresponds to a choice of homogeneous polynomial f_2 of degree 2 in s and t. According to whether this polynomial has one double zero or two simple zeroes, we can conjugate the embedding of α_3 using the GL_2-action to get one of the following two α_3-actions:

(d) $\alpha_3\colon [s:t:x] \mapsto \left[s:t:x+\varepsilon s^2\right]$,

(e) $\alpha_3\colon [s:t:x] \mapsto [s:t:x+\varepsilon st]$.

Finally, assume $G = \alpha_3$ and that the image of G in GL_2 is non-trivial. After conjugating by elements of GL_2, we can assume that the image of G in GL_2 acts as $(s,t) \mapsto (s+\varepsilon t, t)$. An action of α_3 on $\mathbb{P}(1,1,2)$ that lifts this embedding of α_3 into GL_2 acts on x via

$$x \mapsto x + p(\varepsilon)s^2 + q(\varepsilon)st + r(\varepsilon)t^2\,,$$

where p, q, r are polynomials of degree 2 in ε satisfying the following conditions:

$$\begin{aligned}
&p(0) = q(0) = r(0) = 0\,,\\
&p(\varepsilon+\varepsilon') = p(\varepsilon) + p(\varepsilon')\,,\\
&q(\varepsilon+\varepsilon') = q(\varepsilon) + q(\varepsilon') - p(\varepsilon)\varepsilon'\,,\\
&r(\varepsilon+\varepsilon') = r(\varepsilon) + r(\varepsilon') + q(\varepsilon)\varepsilon' + p(\varepsilon)\varepsilon'^2\,.
\end{aligned}$$

Solving this system of equations, we obtain $p = 0$, $q(\varepsilon) = \alpha\varepsilon$, and $r(\varepsilon) = \beta\varepsilon - \alpha\varepsilon^2$ for scalars $\alpha, \beta \in k$. Conjugating with the substitution $x \mapsto x - \alpha s^2 + \beta st$, we obtain that our α_3-action is conjugate to the following:

(f) $\alpha_3\colon [s:t:x] \mapsto [s+\varepsilon t : t : x]$.

We will now classify the sextics as in equation (8.31) which are reduced with only simple singularities and invariant under one of the above actions. In particular, note that if all the a_i are scalar multiples of the ith power of the same linear polynomial in s and t, then S has a non-simple singularity, so it will not appear in our list. Calculating Aut_X^0 is straightforward here, using our description of $\mathrm{Aut}_{\mathbb{P}(1,1,2)}$ above, so it will be left to the reader without further mention. The results can be found in Table 10.

(a) The sextic S is invariant if and only if the t-degree of every monomial that occurs in the equation of S is divisible by 3. Note that the substitutions

$$x \mapsto x + \alpha s^2\,, \quad s \mapsto \beta s\,, \quad t \mapsto \gamma t + \delta s$$

act on the space of sextics satisfying the condition on the t-degree, so we can apply them to arrive at the following normal forms for S:

$$x^3 + s^4x + t^6\,, \tag{8.32}$$
$$x^3 + s^4x + s^3t^3\,, \tag{8.33}$$
$$x^3 + st^3x\,, \tag{8.34}$$
$$x^3 + st^3x + as^3t^3 + t^6\,, \tag{8.35}$$
$$x^3 + st^3x + s^3t^3\,, \tag{8.36}$$
$$x^3 + s^2x^2 + s^3t^3\,, \tag{8.37}$$
$$x^3 + s^2x^2 + a^3s^3t^3 + t^6\,, \tag{8.38}$$
$$x^3 + s^2x^2 + st^3x + a^3s^3t^3 + b^3t^6\,. \tag{8.39}$$

In Case (8.32), the surface X has an E_6^0-singularity at $[0:1:-1:0]$.

In Case (8.33), the surface X has an E_8^1-singularity at $[0:1:0:0]$.

In Case (8.34), the surface X has an E_7^0-singularity at $[1:0:0:0]$ and an A_1-singularity at $[0:1:0:0]$.

In Case (8.35), if $a \neq 0$, then X has an E_6^0-singularity at $[1:0:0:0]$. If $a = 0$, then X has an E_7^0-singularity at $[1:0:0:0]$.

In Case (8.36), the surface X has an E_6^0-singularity at $[1:0:0:0]$ and an A_1-singularity at $[0:1:0:0]$.

In Case (8.37), the surface X has an E_6^1-singularity at $[0:1:0:0]$ and an A_2-singularity at $[1:0:0:0]$.

In Case (8.38), the surface X is singular at $[1:0:0:0]$, $[1:-a:0:0]$, and $[0:1:-1:0]$. If $a \neq 0$, then all singular points are A_2-singularities. If $a = 0$, then the first two combine to an A_5-singularity, while the latter stays an A_2-singularity.

Consider Case (8.39). Here, X is singular at $[1:0:0:0]$ and $\left[u:1:u^{-1}:0\right]$, where u is a solution of $au^2+(b-1)u+1=0$. If $b=0$, then X has an additional A_1-singularity at $[0:1:0:0]$. If the first three singular points are distinct, which happens if and only if $a \neq 0, (b-1)^2$, then they are A_2-singularities. Now, consider the case where $a=0$ or $a=(b-1)^2$ but not both: then $[1:0:0:0]$ is an A_5-singularity and $\left[u:1:u^{-1}:0\right]$ an A_2-singularity if $0 = a \neq (b-1)^2$, and the other way round if $0 \neq a = (b-1)^2$. Note that the substitution $t \mapsto t+(b-1)s$, $x \mapsto x+(b-1)^3s^2$ maps the family with $a=(b-1)^2$ to the one with $a=0$, which is why only the latter occurs in Table 10. Finally, if $a=0$ and $b=1$, then X has an A_8-singularity at $[1:0:0:0]$.

(b) The sextic S is invariant if and only if the degrees of the a_i are divisible by 3. This happens if and only if $a_2 = a_4 = 0$. Using a substitution from $\mathrm{Aut}_{\mathbb{P}(1,1,2)}$, we obtain the following normal forms:

$$x^3 + s^5 t\,, \tag{8.40}$$
$$x^3 + s^4 t^2\,, \tag{8.41}$$
$$x^3 + s^4 t^2 + s^2 t^4\,. \tag{8.42}$$

In Case (8.40), the surface X has an E_8^0-singularity at $[0:1:0:0]$.

In Case (8.41), the surface X has an E_6^0-singularity at $[0:1:0:0]$ and an A_2-singularity at $[1:0:0:0]$.

In Case (8.42), the surface X has four A_2-singularities, at $[1:0:0:0]$, $[0:1:0:0]$, $[1:-1:0:0]$, and $[1:1:0:0]$.

(c) The sextic S is invariant if and only if for every monomial in the equation of S, the difference between the s- and t-degree is divisible by 3. We may assume that not both a_2 and a_4 are zero; otherwise, we get Cases (8.40), (8.41), and (8.42). Note that the substitutions

$$x \mapsto x + \alpha st\,, \quad s \mapsto \beta s\,, \quad t \mapsto \gamma t$$

normalize our μ_3-action, so we can apply them to arrive at the following normal forms for S:

$$x^3 + s^2 t^2 x\,, \tag{8.43}$$
$$x^3 + s^2 t^2 x + t^6 + a^3 s^6\,, \tag{8.44}$$
$$x^3 + stx^2 + s^6\,, \tag{8.45}$$
$$x^3 + stx^2 + a^3 s^6 + s^3 t^3\,, \tag{8.46}$$
$$x^3 + stx^2 + a^3 s^6 + b^3 s^3 t^3 + t^6\,. \tag{8.47}$$

In Case (8.43), the surface X has two D_4-singularities, one at $[1:0:0:0]$ and one at $[0:1:0:0]$, so X contains only equivariant RDPs and does not occur in Table 10.

Consider Case (8.44). If $a = 0$, then X has a D_4-singularity at $[1:0:0:0]$ and an A_2-singularity at $[0:1:-1:0]$. If $a \neq 0$, then X has two A_2-singularities, one at $[1:0:-a:0]$ and one at $[0:1:-1:0]$.

In Case (8.45), the surface X has an RDP of type D_7 at $[0:1:0:0]$. In particular, X contains only equivariant RDPs and does not occur in Table 10.

Consider Case (8.46). If $a = 0$, then X has RDPs of type D_4 at $[1:0:0:0]$ and $[0:1:0:0]$, so X contains only equivariant RDPs and does not occur in Table 10. If $a \neq 0$, then X has an RDP of type D_4 at $[0:1:0:0]$ and an A_2-singularity at $[1:-a:0:0]$.

Consider Case (8.47). If $a = 0$, we can interchange s and t to reduce to one of the previous two cases. Here, X has singularities at $[1:u:0:0]$, where u is a solution of $u^2 + bu + a = 0$. If $b^2 = a$, then the unique singularity of X is an A_5-singularity. If $b^2 \neq a$, then X has two A_2-singularities.

(d) If $a_2 \neq 0$ or $a_4 \neq 0$, then S cannot be α_3-invariant. Hence, $a_2 = a_4 = 0$. But then S is given by one of the equations in Case (b), so we are done.

(e) By the same argument as in Case (d), we can reduce this to Case (b).

(f) The sextic S is α_3-invariant if and only if each a_i is invariant under the α_3-action. This happens if and only if the s-degree of each monomial in the equation of S is divisible by 3. Interchanging the roles of s and t, we can therefore reduce this case to Case (a). In fact, this explains why each of the surfaces in Case (a) admits an α_3-action of this form. □

Appendix: Classification tables

In this appendix, we give equations for all RDP del Pezzo surfaces X in odd characteristic which contain at least one of the RDPs excluded in Theorem 6.1 and satisfy $H^0(X, T_X) \neq 0$. Moreover, we describe the identity component Aut^0_X of their automorphism schemes. The completeness of the following classification tables is proved in Theorems 8.3, 8.6, and 8.8.

d	RDPs	Equation of X	Aut^0_X
2	A_6	$w^2 = x^3y + y^3z + z^3x$	$\mu_7\colon \left[\lambda x : \lambda^4 y : \lambda^2 z : w\right]$
1	$A_6 + A_1$	$y^2 = x^3 + ts^3x + t^5 s$	$\mu_7\colon \left[\lambda s : \lambda^4 t : x : y\right]$

TABLE 5. Non-equivariant RDP del Pezzo surfaces with vector fields in characteristic 7

d	RDPs	Equation(s) of X	Aut^0_X
5	A_4	$x_0x_2 - x_1^2 = 0$ $x_0x_3 - x_1x_4 = 0$ $x_2x_4 - x_1x_3 = 0$ $x_1x_2 + x_4^2 + x_0x_5 = 0$ $x_2^2 + x_3x_4 + x_1x_5 = 0$	$\langle \alpha_5, \mathrm{Aut}^0_{\widetilde{X}} \rangle$ with $\alpha_5\colon \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2\varepsilon^2 & 2\varepsilon^3 & \varepsilon & 2\varepsilon^4 \\ 0 & 0 & 1 & 2\varepsilon & 0 & -\varepsilon^2 \\ 0 & 0 & 0 & 1 & 0 & -\varepsilon \\ 0 & 0 & \varepsilon & \varepsilon^2 & 1 & -2\varepsilon^3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
4	A_4	$x_0x_1 - x_2x_3 = 0$ $x_0x_4 + x_1x_2 + x_3^2 = 0$	$\langle \alpha_5, \mathrm{Aut}^0_{\widetilde{X}} \rangle$ with $\alpha_5\colon \begin{pmatrix} 1 & -\varepsilon^3 & -2\varepsilon & 2\varepsilon^2 & 2\varepsilon^4 \\ 0 & 1 & 0 & 0 & 2\varepsilon \\ 0 & -\varepsilon^2 & 1 & -2\varepsilon & \varepsilon^3 \\ 0 & \varepsilon & 0 & 1 & \varepsilon^2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
3	A_4	$x_0^2x_1 + x_1^2x_2 + x_2^2x_3 + x_3^2x_0 = 0$	$\mu_5\colon \left[x_0 : \lambda x_1 : \lambda^4 x_2 : \lambda^3 x_3\right]$
	$A_4 + A_1$	$x_0x_1x_3 + x_0x_2^2 + x_1^2x_2 = 0$	$\alpha_5 \rtimes \mathbb{G}_m$ with $\alpha_5\colon \begin{pmatrix} 1 & \varepsilon & \varepsilon^2 & -2\varepsilon^3 \\ 0 & 1 & 2\varepsilon & -\varepsilon^2 \\ 0 & 0 & 1 & -\varepsilon \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\mathbb{G}_m\colon \left[x_0 : \lambda x_1 : \lambda^2 x_2 : \lambda^3 x_3\right]$
2	$A_4 + A_1$	$w^2 = x^4 + xy^2z + yz^3$	$\mu_5\colon \left[x : \lambda y : \lambda^3 z : w\right]$
	$A_4 + A_2$	$w^2 = xy^3 + yz^3 + x^2z^2$	$\mu_5\colon \left[\lambda^2 x : \lambda y : \lambda^3 z : w\right]$
1	$A_4 + A_2 + A_1$	$y^2 = x^3 + s^3tx + s^2t^4$	$\mu_5\colon \left[s : \lambda t : \lambda^3 x : \lambda^2 y\right]$
	$2A_4$	$y^2 = x^3 + t^4x + s^5t$	$\alpha_5 \rtimes \mu_5\colon [\lambda s + \varepsilon t : t : x : y]$
	E_8^0	$y^2 = x^3 + s^5t$	$\alpha_5 \rtimes \mathbb{G}_m\colon \left[\lambda s + \varepsilon t : \lambda^{-5} t : x : y\right]$

TABLE 6. Non-equivariant RDP del Pezzo surfaces with vector fields in characteristic 5

d	RDPs	Equation(s) of X	Aut_X^0
6	A_2	$\begin{gathered} x_0x_5 - x_3x_4 = 0 \\ x_0x_6 - x_1x_4 = 0 \\ x_0x_6 - x_2x_3 = 0 \\ x_3x_6 - x_1x_5 = 0 \\ x_4x_6 - x_2x_5 = 0 \\ x_1x_6 + x_3^2 + x_3x_4 = 0 \\ x_2x_6 + x_3x_4 + x_4^2 = 0 \\ x_6^2 + x_3x_5 + x_4x_5 = 0 \\ x_1x_2 + x_0x_3 + x_0x_4 = 0 \end{gathered}$	$\langle \alpha_3, \mathrm{Aut}^0_{\widetilde{X}} \rangle$ with $\alpha_3\colon \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\varepsilon & 1 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon & 0 & 1 & 0 & 0 & 0 & 0 \\ -\varepsilon^2 & -\varepsilon & 0 & 1 & 0 & 0 & 0 \\ -\varepsilon^2 & 0 & \varepsilon & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\varepsilon^2 & -\varepsilon^2 & -\varepsilon & \varepsilon & 0 & 1 \end{pmatrix}$
	$A_2 + A_1$	$\begin{gathered} x_0^2 - x_1x_5 = 0 \\ x_0x_2 - x_1x_4 = 0 \\ x_0x_3 - x_2x_4 = 0 \\ x_0x_4 - x_2x_5 = 0 \\ x_0x_5 - x_2x_6 = 0 \\ x_1x_3 - x_2^2 = 0 \\ x_3x_5 - x_4^2 = 0 \\ x_3x_6 - x_4x_5 = 0 \\ x_4x_6 - x_5^2 = 0 \end{gathered}$	$\langle \alpha_3, \mathrm{Aut}^0_{\widetilde{X}} \rangle$ with $\alpha_3\colon \begin{pmatrix} 1 & -\varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon & 0 & 1 & 0 & 0 \\ \varepsilon & \varepsilon^2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
5	A_2	$\begin{gathered} x_0x_2 - x_1x_5 = 0 \\ x_0x_2 - x_3x_4 = 0 \\ x_0x_3 + x_1^2 + x_1x_4 = 0 \\ x_0x_5 + x_1x_4 + x_4^2 = 0 \\ x_3x_5 + x_1x_2 + x_2x_4 = 0 \end{gathered}$	$\langle \alpha_3, \mathrm{Aut}^0_{\widetilde{X}} \rangle$ with $\alpha_3\colon \begin{pmatrix} 1 & \varepsilon & 0 & -\varepsilon^2 & -\varepsilon & -\varepsilon^2 \\ 0 & 1 & -\varepsilon^2 & \varepsilon & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 1 & 0 & 0 \\ 0 & 0 & -\varepsilon^2 & 0 & 1 & -\varepsilon \\ 0 & 0 & -\varepsilon & 0 & 0 & 1 \end{pmatrix}$
	$A_2 + A_1$	$\begin{gathered} x_0^2 - x_1x_4 = 0 \\ x_0x_2 - x_1x_3 = 0 \\ x_0x_3 - x_2x_4 = 0 \\ x_0x_4 - x_2x_5 = 0 \\ x_3x_5 - x_4^2 = 0 \end{gathered}$	$\langle \alpha_3, \mathrm{Aut}^0_{\widetilde{X}} \rangle$ with $\alpha_3\colon \begin{pmatrix} 1 & -\varepsilon & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon & 1 & 0 & 0 \\ \varepsilon & \varepsilon^2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
4	A_2	$\begin{gathered} x_0x_1 + x_2x_4 + x_3x_4 = 0 \\ x_0x_4 + x_1x_4 + x_2x_3 = 0 \end{gathered}$	$\mu_3\colon [x_0 : x_1 : \lambda x_2 : \lambda x_3 : \lambda^2 x_4]$
	$A_2 + A_1$	$\begin{gathered} x_0x_1 - x_2x_3 = 0 \\ x_1x_2 + x_2x_4 + x_3x_4 = 0 \end{gathered}$	$\alpha_3 \rtimes \mathbb{G}_m$ with $\alpha_3\colon \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\varepsilon^2 & 1 & \varepsilon & -\varepsilon & 0 \\ -\varepsilon & 0 & 1 & 0 & 0 \\ \varepsilon & 0 & 0 & 1 & 0 \\ -\varepsilon^2 & 0 & -\varepsilon & 0 & 1 \end{pmatrix}$ $\mathbb{G}_m\colon [\lambda^2 x_0 : x_1 : \lambda x_2 : \lambda x_3 : x_4]$
	$A_2 + 2A_1$	$\begin{gathered} x_0^2 - x_3x_4 = 0 \\ x_0x_3 - x_1x_2 = 0 \end{gathered}$	$\alpha_3 \rtimes \mathbb{G}_m^2$ with $\alpha_3\colon \begin{pmatrix} 1 & 0 & 0 & 0 & -\varepsilon \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \varepsilon & 0 & 0 & 1 & \varepsilon^2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ $\mathbb{G}_m^2\colon [x_0 : \lambda_1 x_1 : \lambda_2 x_2 : \lambda_1\lambda_2 x_3 : (\lambda_1\lambda_2)^{-1} x_4]$

TABLE 7. Non-equivariant RDP del Pezzo surfaces of degree at least 4 with vector fields in characteristic 3.

d	RDPs	Equation(s) of X	Aut_X^0
3	A_2	$x_0^2x_1 + x_0x_1^2 + x_2^2x_3 + x_2x_3^2 = 0$	$\mu_3\colon [x_0 : x_1 : \lambda x_2 : \lambda x_3]$
	$A_2 + 2A_1$	$x_0^2x_1 + x_0^2x_2 + x_0x_3^2 + x_1x_2x_3 = 0$	$\mu_3\colon [x_0 : \lambda x_1 : \lambda x_2 : \lambda^2 x_3]$
	$2A_2$	$x_0^3 + x_1x_2x_3 + x_0x_1^2 + ax_0^2x_1 = 0$ with $a^2 \neq 1$	$\langle \alpha_3, \alpha_3, \mathbb{G}_m \rangle$ with $\alpha_3\colon [x_0 + \varepsilon x_2 : x_1 : x_2 : a\varepsilon x_0 - \varepsilon x_1 - a\varepsilon^2 x_2 + x_3]$ $\alpha_3\colon [x_0 + \varepsilon x_3 : x_1 : a\varepsilon x_0 - \varepsilon x_1 + x_2 - a\varepsilon^2 x_3 : x_3]$ $\mathbb{G}_m\colon [x_0 : x_1 : \lambda x_2 : \lambda^{-1} x_3]$
	$2A_2 + A_1$	$x_0^3 + x_1x_2x_3 + x_0^2x_1 = 0$	$\langle \alpha_3, \alpha_3, \mathbb{G}_m \rangle$ with $\alpha_3\colon [x_0 + \varepsilon x_2 : x_1 : x_2 : \varepsilon x_0 - \varepsilon^2 x_2 + x_3]$ $\alpha_3\colon [x_0 + \varepsilon x_3 : x_1 : a\varepsilon x_0 + x_2 - \varepsilon^2 x_3 : x_3]$ $\mathbb{G}_m\colon [x_0 : x_1 : \lambda x_2 : \lambda^{-1} x_3]$
	$3A_2$	$x_0^3 + x_1x_2x_3 = 0$	$\alpha_3^3 \rtimes \mathbb{G}_m^2$ with $\alpha_3^3\colon [x_0 + \varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 : x_1 : x_2 : x_3]$ $\mathbb{G}_m^2\colon [x_0 : \lambda_1 x_1 : \lambda_2 x_2 : (\lambda_1\lambda_2)^{-1} x_3]$
	A_5	$x_0^3 + x_0x_2x_3 + x_1^2x_2 + x_2^3 = 0$	$\langle \alpha_3, \mathbb{G}_a \rtimes \mu_3 \rangle$ with $\alpha_3\colon [x_0 + \varepsilon x_1 - \varepsilon^2 x_3 : x_1 + \varepsilon x_3 : x_2 : x_3]$ $\mathbb{G}_a\colon [x_0 : \varepsilon x_0 + x_1 : x_2 : -\varepsilon^2 x_0 + \varepsilon x_1 + x_3]$ $\mu_3\colon [x_0 : \lambda x_1 : \lambda x_2 : \lambda^2 x_3]$
	$A_5 + A_1$	$x_0^3 + x_0x_2x_3 + x_1^2x_2 = 0$	$\langle \alpha_3, \mathbb{G}_a \rtimes \mathbb{G}_m \rangle$ with $\alpha_3\colon [x_0 + \varepsilon x_1 - \varepsilon^2 x_3 : x_1 + \varepsilon x_3 : x_2 : x_3]$ $\mathbb{G}_a\colon [x_0 : \varepsilon x_0 + x_1 : x_2 : -\varepsilon^2 x_0 + \varepsilon x_1 + x_3]$ $\mathbb{G}_m\colon [x_0 : \lambda x_1 : \lambda x_2 : \lambda^2 x_3]$
	E_6^0	$x_0^3 + x_1^2x_2 + x_2^2x_3 = 0$	$\langle G, \mathbb{G}_a^2 \rtimes \mathbb{G}_m \rangle$ with $\mathbb{G}_a\colon [x_0 + \varepsilon x_2 : x_1 : x_2 : -\varepsilon^3 x_2 + x_3]$ $\mathbb{G}_a\colon [x_0 : x_1 + \varepsilon x_2 : x_2 : \varepsilon^3 x_1 - \varepsilon^2 x_2 + x_3]$ $\mathbb{G}_m\colon [x_0 : \lambda x_1 : \lambda^{-2} x_2 : \lambda^4 x_3]$ and G non-commutative, $\lvert G \rvert = 27$, acting as $[x_0 + \varepsilon_1 x_1 + \varepsilon_2 x_3 : x_1 + \varepsilon_1^3 x_3 :$ $-\varepsilon_1^3 x_1 + x_2 + \varepsilon_1^6 x_3 : x_3]$, where $\varepsilon_1^9 = \varepsilon_2^3 = 0$
	E_6^1	$x_0^3 + x_1^3 + x_0x_1x_2 + x_2^2x_3 = 0$	$\langle \mu_3, \mathbb{G}_a^2 \rangle$ with $\mu_3\colon [\lambda x_0 : \lambda^2 x_1 : x_2 : x_3]$ $\mathbb{G}_a\colon [x_0 - \varepsilon x_2 : x_1 : x_2 : \varepsilon x_1 + \varepsilon^3 x_2 + x_3]$ $\mathbb{G}_a\colon [x_0 : x_1 - \varepsilon x_2 : x_2 : \varepsilon x_0 + \varepsilon^3 x_2 + x_3]$

Table 8. Non-equivariant RDP del Pezzo surfaces of degree 3 with vector fields in characteristic 3.

d	RDPs	Equation(s) of X	Aut_X^0
2	A_2+3A_1	$w^2 = z\big(xy(x+y)+z^3\big)$	$\mu_3\colon [x:y:\lambda z:\lambda^{-1}w]$
	A_2+A_3	$w^2 = x^4+a^3x^2yz+xy^3+y^2z^2$ with $a^2 \neq 1$	$\mu_3\colon [x:\lambda y:\lambda^{-1}z:w]$
	$A_2+A_3+A_1$	$w^2 = x^2yz+xy^3+y^2z^2$	$\mu_3\colon [x:\lambda y:\lambda^{-1}z:w]$
	A_2+A_4	$w^2 = \big(xz+y^2\big)^2+y^3z$	$\alpha_3\colon [x+\varepsilon y-\varepsilon^2 z:y+\varepsilon z:z:w]$
	$2A_2$	$w^2 = x^4+xy^3+xz^3+ax^2yz+by^2z^2$ with $\big(b^3-a^2b^2\big)^2 \neq a^3b^3,\ b\neq 0$	$\mu_3\colon [x:\lambda y:\lambda^{-1}z:w]$
	$2A_2$	$w^2 = \big(xz+y^2\big)^2+x^3z+a^6z^4$ with $a\neq 0$	$\alpha_3\colon [x+\varepsilon y-\varepsilon^2 z:y+\varepsilon z:z:w]$
	$2A_2+A_1$	$w^2 = ax^2yz+xy^3+xz^3+y^2z^2$ with $a\neq 0,1$	$\mu_3\colon [x:\lambda y:\lambda^{-1}z:w]$
	$3A_2$	$w^2 = y^4+x^2y^2+xz^3$	$\alpha_3^2 \rtimes \mu_3\colon [x:y:\varepsilon_1 x+\varepsilon_2 y+\lambda z:w]$
	A_5	$w^2 = x^4+xy^3+xz^3+ax^2yz+by^2z^2$ with $\big(b^3-a^2b^2\big)^2 = a^3b^3,\ b\neq 0$	$\mu_3\colon [x:\lambda y:\lambda^{-1}z:w]$
	A_5	$w^2 = x^4+ax^2yz+xy^3+xz^3$ with $a\neq 0$	$\mu_3\colon [x:\lambda y:\lambda^{-1}z:w]$
	A_5	$w^2 = (xz+y^2)^2+x^3z$	$\alpha_3\colon [x+\varepsilon y-\varepsilon^2 z:y+\varepsilon z:z:w]$
	A_5	$w^2 = z(z(xz+y^2)+x^3)$	$\alpha_3\colon [x+\varepsilon y-\varepsilon^2 z:y+\varepsilon z:z:w]$
	A_5+A_1	$w^2 = x^2yz+xy^3+xz^3$	$\mu_3\colon [x:\lambda y:\lambda^{-1}z:w]$
	A_5+A_1	$w^2 = x^2yz+xy^3+xz^3+y^2z^2$	$\mu_3\colon [x:\lambda y:\lambda^{-1}z:w]$
	A_5+A_2	$w^2 = x^2y^2+xz^3$	$\alpha_3^2 \rtimes \mathbb{G}_m\colon [x:\lambda^3 y:\varepsilon_1 x+\varepsilon_2 y+\lambda^2 z:\lambda^3 w]$
	E_6^0	$w^2 = y^4+xz^3$	$\langle G,\mathbb{G}_m\rangle$ with $\mathbb{G}_m\colon [x:\lambda^3 y:\lambda^4 z:\lambda^6 w]$ and G non-commutative, $\lvert G\rvert = 27$, acting as $[x:y-\varepsilon_1^3 x:\varepsilon_2 x+\varepsilon_1 y+z:w]$ where $\varepsilon_1^9=\varepsilon_2^3=0$
	E_6^1	$w^2 = \big(y^3+z^3\big)x+y^2z^2$	$\mu_3\colon [x:\lambda y:\lambda^{-1}z:w]$
	E_7^0	$w^2 = x^3y+xz^3$	$\langle \alpha_3,\mathbb{G}_a\rtimes\mathbb{G}_m\rangle$ with $\alpha_3\colon [x:y:z+\varepsilon y:w]$ $\mathbb{G}_a\colon [x:y+\varepsilon^3 x:z-\varepsilon x:w]$ $\mathbb{G}_m\colon [x:\lambda^6 y:\lambda^2 z:\lambda^3 w]$

Table 9. Non-equivariant RDP del Pezzo surfaces of degree 2 with vector fields in characteristic 3.

d	RDPs	Equation(s) of X	Aut^0_X
1	$A_2 + D_4$	$y^2 = x^3 + stx^2 + a^3s^6 + s^3t^3$ with $a \neq 0$	$\mu_3\colon \big[\lambda s : \lambda^{-1}t : x : y\big]$
	$A_2 + D_4$	$y^2 = x^3 + s^2t^2x + t^6$	$\mu_3\colon \big[\lambda s : \lambda^{-1}t : x : y\big]$
	$2A_2$	$y^2 = x^3 + stx^2 + a^3s^6 + b^3s^3t^3 + t^6$ with $a \neq 0,\ b^2 \neq a$	$\mu_3 : \big[\lambda s : \lambda^{-1}t : x : y\big]$
	$2A_2$	$y^2 = x^3 + s^2t^2x + a^3s^6 + t^6$ with $a \neq 0$	$\mu_3\colon \big[\lambda s : \lambda^{-1}t : x : y\big]$
	$3A_2$	$y^2 = x^3 + s^2x^2 + st^3x + a^3s^3t^3 + b^3t^6$ with $a \notin \{0, (b-1)^2\},\ b \neq 0$	$\alpha_3 \rtimes \mu_3\colon [s : \varepsilon s + \lambda t : x : y]$
	$3A_2$	$y^2 = x^3 + s^2x^2 + a^3s^3t^3 + t^6$ with $a \neq 0$	$\alpha_3 \rtimes \mu_3\colon [s : \varepsilon s + \lambda t : x : y]$
	$3A_2 + A_1$	$y^2 = x^3 + s^2x^2 + st^3x + a^3s^3t^3$ with $a \notin \{0, 1\}$	$\alpha_3 \rtimes \mu_3\colon [s : \varepsilon s + \lambda t : x : y]$
	$4A_2$	$y^2 = x^3 + s^4t^2 + s^2t^4$	$\alpha_3^3 \rtimes \mu_3\colon \big[\lambda s : \lambda t : x + \varepsilon_1 s^2 + \varepsilon_2 st + \varepsilon_3 t^2 : y\big]$
	A_5	$y^2 = x^3 + stx^2 + b^6s^6 + b^3s^3t^3 + t^6$ with $b \neq 0$	$\mu_3\colon \big[\lambda s : \lambda^{-1}t : x : y\big]$
	$A_5 + A_2$	$y^2 = x^3 + s^2x^2 + st^3x + b^3t^6$ with $b \neq 0, 1$	$\alpha_3 \rtimes \mu_3\colon [s : \varepsilon s + \lambda t : x : y]$
	$A_5 + A_2$	$y^2 = x^3 + s^2x^2 + t^6$	$\alpha_3 \rtimes \mu_3\colon [s : \varepsilon s + \lambda t : x : y]$
	$A_5 + A_2 + A_1$	$y^2 = x^3 + s^2x^2 + st^3x$	$\alpha_3 \rtimes \mu_3\colon [s : \varepsilon s + \lambda t : x : y]$
	E_6^0	$y^2 = x^3 + st^3x + as^3t^3 + t^6$ with $a \neq 0$	$\alpha_3 \rtimes \mu_3\colon [s : \varepsilon s + \lambda t : x : y]$
	E_6^0	$y^2 = x^3 + s^4x + t^6$	$\mu_3\colon [s : \lambda t : x : y]$
	$E_6^0 + A_1$	$y^2 = x^3 + st^3x + s^3t^3$	$\alpha_3 \rtimes \mu_9\colon \big[\lambda^6 s : \varepsilon s + \lambda t : x + \big(1 - \lambda^3\big)s^2 : y\big]$
	$E_6^0 + A_2$	$y^2 = x^3 + s^4t^2$	$\langle G, \mathbb{G}_m\rangle$ with $\mathbb{G}_m\colon \big[\lambda s : \lambda^{-2}t : x : y\big]$ and G non-commutative, $\lvert G\rvert = 81$, acting as $\mathbb{G}_m\colon \big[s : t - \varepsilon_2^3 s : x + \varepsilon_1 s^2 + \varepsilon_2 st + \varepsilon_3 t^2 : y\big]$ with $\varepsilon_1^3 = \varepsilon_2^9 = \varepsilon_3^3 = 0$
	$E_6^1 + A_2$	$y^2 = x^3 + s^2x^2 + s^3t^3$	$\alpha_3 \rtimes \mu_3\colon [s : \varepsilon s + \lambda t : x : y]$
	E_7^0	$y^2 = x^3 + st^3x + t^6$	$\alpha_3 \rtimes \mu_3\colon [s : \varepsilon s + \lambda t : x : y]$
	$E_7^0 + A_1$	$y^2 = x^3 + st^3x$	$\alpha_3 \rtimes \mathbb{G}_m\colon \big[\lambda^{-3}s : \varepsilon s + \lambda t : x : y\big]$
	A_8	$y^2 = x^3 + s^2x^2 + st^3x + t^6$	$\alpha_9 \rtimes \mu_3\colon \big[s : \varepsilon s + \lambda t : x + \varepsilon^3 s^2 : y\big]$
	E_8^0	$y^2 = x^3 + s^5t$	$\langle \alpha_3^2, \mathbb{G}_a \rtimes \mathbb{G}_m\rangle$ with $\mathbb{G}_a\colon \big[s : t - a^3 s : x + as^2 : y\big]$ $\mathbb{G}_m\colon \big[\lambda s : \lambda^{-5}t : x : y\big]$ $\alpha_3^2\colon \big[s : t : x + \varepsilon_1 st + \varepsilon_2 t^2 : y\big]$
	E_8^1	$y^2 = x^3 + s^4x + s^3t^3$	$\mathbb{G}_a \rtimes \mu_3$ with $\mathbb{G}_a\colon \big[s : t - \big(a^3 + a\big)s : x + as^2 : y\big]$ $\mu_3\colon [s : \lambda t : x : y]$

Table 10. Non-equivariant RDP del Pezzo surfaces of degree 1 with vector fields in characteristic 3.

References

Art77 M. Artin, *Coverings of the Rational Double Points in Characteristic p*, in *Complex Analysis and Algebraic Geometry* (Iwanami Shoten Publishers, Tokyo; Cambridge Univ. Press, Cambridge, 1977), 11–22; doi:10.1017/CBO9780511569197.003.

BT22 F. Bernasconi and H. Tanaka, *On del Pezzo fibrations in positive characteristic*, J. Inst. Math. Jussieu **21** (2022), no. 1, 197–239; doi:10.1017/S1474748020000067.

Bri17 M. Brion, *Some structure theorems for algebraic groups*, in *ALgebraic Groups: Structure and Actions*, Proc. Sympos. Pure Math., vol. 94 (Amer. Math. Soc., Providence, RI, 2017), 53–126; doi:10.1090/pspum/094/04.

Bri18 ———, *Notes on automorphism groups of projective varieties*, Summer school and Workshop on Varieties and Group Actions (Warsaw, 2018). Available at https://www-fourier.univ-grenoble-alpes.fr/~mbrion/autos_final.pdf.

BG81 J. W. Bruce and P. J. Giblin, *A stratification of the space of plane quartic curves*, Proc. Lond. Math. Soc. (3) **42** (1981), no. 2, 270–298; doi:10.1112/plms/s3-42.2.270.

CP21 I. Cheltsov and Yu. Prokhorov, *Del Pezzo surfaces with infinite automorphism groups*, Algebr. Geom. **8** (2021), no. 3, 319–357; doi:10.14231/ag-2021-008.

Der14 U. Derenthal, *Singular del Pezzo surfaces whose universal torsors are hypersurfaces*, Proc. Lond. Math. Soc. (3) **108** (2014), no. 3, 638–681; doi:10.1112/plms/pdt041.

Dol12 I. V. Dolgachev, *Classical algebraic geometry* (Cambridge Univ. Press, Cambridge, 2012); doi:10.1017/CBO9781139084437.

Dyn52 E. B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Mat. Sb. N.S. **30/72** (1952), 349–462.

GK90 G.-M. Greuel and H. Kröning, *Simple singularities in positive characteristic*, Math. Z. **203** (1990), no. 2, 339–354; doi:10.1007/BF02570742.

Hir19 M. Hirokado, *Further Evaluation of Wahl Vanishing Theorems for Surface Singularities in Characteristic p*, Michigan Math. J. **68** (2019), no. 3, 621–636; doi:10.1307/mmj/1560391418.

KN09 T. Kikuchi and T. Nakano, *On the projective embeddings of Gorenstein toric del Pezzo surfaces*, Illinois J. Math. **53** (2009), no. 4, 1051–1059; doi:10.1215/ijm/1290435338.

Lan94 W. E. Lang, *Extremal rational elliptic surfaces in characteristic p. II. Surfaces with three or fewer singular fibres*, Ark. Mat. **32** (1994), no. 2, 423–448; doi:10.1007/BF02559579.

Lip69 J. Lipman, *Rational singularities, with applications to algebraic surfaces and unique factorization*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 195–279; doi:10.1007/BF02684604.

Mar22 G. Martin, *Infinitesimal automorphisms of algebraic varieties and vector fields on elliptic surfaces*, Algebra Number Theory **16** (2022), no. 7, 1655–1704; doi:10.2140/ant.2022.16.1655

MS20 G. Martin and C. Stadlmayr, *Weak del Pezzo surfaces with global vector fields*, Geom. Topol., to appear, arXiv:2007.03665.

Mar03 J. Martinet, *Perfect lattices in Euclidean spaces*, Grundlehren math. Wiss., vol. 327 (Springer, Berlin, 2003); doi:10.1007/978-3-662-05167-2.

Mil80 J. S. Milne, *Étale cohomology*, Princeton Math. Series, vol. 33 (Princeton Univ. Press, Princeton, NJ, 1980); doi:10.1515/9781400883981.

Ray64 M. Raynaud, *Groupes algébriques unipotents. Extensions entre groupes unipotents et groupes de type multiplicatif*, Schémas en Groupes (Sém. Géométrie Algébrique, Inst. Hautes Études Sci., 1963/64), Fasc. 5b, Exposé XVII (Inst. Hautes Études Sci., Paris, 1964).

Roc96 M. Roczen, *Cubic surfaces with double points in positive characteristic*, Algebraic Geometry and Singularities (La Rábida, 1991), Progr. Math., vol. 134, (Birkhäuser, Basel, 1996), 375–382.

Ser06 E. Sernesi, *Deformations of algebraic schemes*, Grundlehren Math. Wiss., vol. 334 (Springer, Berlin, 2006); doi:10.1007/978-3-540-30615-3.

Sta18 The Stacks Project Authors, *Stacks Project*, 2018, https://stacks.math.columbia.edu.

Sta21 C. Stadlmayr, *Which rational double points occur on del Pezzo surfaces?*, Épijournal Géom. Algébrique **5** (2021), Paper No. 17; doi:10.46298/epiga.2021.7041.

Stö04 K.-O. Stöhr, *On Bertini's theorem in characteristic p for families of canonical curves in* $\mathbb{P}^{(p-3)/2}$, Proc. Lond. Math. Soc. (3) **89** (2004), no. 2, 291–316; doi:10.1112/S0024611504014753.

Tzi17 N. Tziolas, *Quotients of schemes by* α_p *or* μ_p *actions in characteristic* $p > 0$, Manuscripta Math. **152** (2017), no. 1–2, 247–279; doi:10.1007/s00229-016-0854-y.

Wah75 J. M. Wahl, *Vanishing theorems for resolutions of surface singularities*, Invent. Math. **31** (1975), no. 1, 17–41; doi:10.1007/BF01389864.

Gebhard Martin gmartin@math.uni-bonn.de
Mathematisches Institut, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

Claudia Stadlmayr claudia.stadlmayr@ma.tum.de c.stadlmayr.math@gmx.net
Zentrum Mathematik – M11, Technische Universität München, Boltzmannstraße 3, 85748 Garching bei München, Germany