



On the boundedness of n -folds with $\kappa(X) = n - 1$

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ABSTRACT

In this note, we study certain sufficient conditions for a set of minimal klt pairs (X, Δ) with $\kappa(X, \Delta) = \dim(X) - 1$ to be bounded.

1. Introduction

Throughout this paper, the base field will be an algebraically closed field of characteristic 0.

One of the main goals of algebraic geometry is to classify varieties according to a few main features. One natural object attached to any normal variety X is its canonical sheaf ω_X . A fruitful perspective is to group varieties according to the behavior of the canonical sheaf. Under this perspective, the minimal model program (MMP for short) suggests the existence of three main classes of varieties: Fano varieties, Calabi–Yau varieties, and varieties of general type. Each of these classes corresponds to one of the following behaviors of the canonical sheaf: $\omega_X^\vee$ induces a birational polarization, ω_X is torsion, and ω_X induces a birational polarization, respectively. Notice that, for the purpose of this work, Calabi–Yau will refer to the torsion property of the (log) canonical divisor, while there will be no assumption on the fundamental group or on the smoothness of the variety. Then, these notions naturally extend to the setup of pairs.

The MMP conjectures that every variety decomposes birationally as iterated fibrations, where the base of the tower of fibrations and the general fiber of each fibration belong to one of the three above families. Thus, a natural point of view is to use the knowledge of the three fundamental classes of varieties to have a deeper understanding of other types of varieties.

In this work, we are concerned with varieties of intermediate Kodaira dimension. That is, we consider varieties X that satisfy $\Gamma(X, \omega_X^{\otimes n}) \neq 0$ for some $n \in \mathbb{N}$ but are not Calabi–Yau or of general type. This is equivalent to the following: the rate of growth of the sections of $\Gamma(X, \omega_X^{\otimes n})$ is asymptotically polynomial, with degree d satisfying $1 \leq d \leq \dim(X) - 1$. In this scenario, the MMP predicts that X admits a minimal model X' with the following property: a suitable positive power of $\omega_{X'}$ is basepoint-free and defines a morphism $f: X' \rightarrow Y$, called the Iitaka fibration. In this case, the fibers of f are Calabi–Yau varieties, and the base Y is naturally endowed with the structure of a pair of general type.

One of the main topics in the classification of algebraic varieties is boundedness. Loosely speaking, a set of varieties or, more generally, pairs is called bounded if it can be parametrized by a scheme of finite type. In particular, addressing the boundedness of a class of varieties is the first step towards considering a moduli space. Hacon, McKernan, and Xu proved that pairs (X, Δ)

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of general type form a bounded family when one imposes restrictions on $\dim(X)$ and on the coefficients of Δ and the divisor $K_X + \Delta$ is ample of fixed volume [HMX18a]. Similarly, Birkar settled an optimal boundedness statement for varieties of Fano type: varieties of Fano type whose singularities are bounded form a bounded family [Bir19, Bir21]. On the other hand, in general, Calabi–Yau varieties do not form bounded families. An example of this phenomenon is given by abelian n -folds and K3 surfaces: both classes are not bounded, but each of them decomposes as the countable union of bounded families, corresponding to polarizations of different degrees. Therefore, it is natural to introduce some geometrical or numerical conditions when studying families of Calabi–Yau varieties. In this direction, there are some results obtaining (weak versions of) boundedness of Calabi–Yau varieties. The boundedness of Calabi–Yau varieties admitting elliptic fibrations is considered in [Gro94, DS21, Bir18, BDS20, FHS21], while some recent works consider fairly singular Calabi–Yau pairs [Bir18, CDH⁺21, HJ22]. Furthermore, works of Jiao consider the problem of the boundedness of varieties admitting a log Calabi–Yau fibration of higher relative dimension [Jia23b, Jia23a].

In this paper, we investigate the following question: If the base Y and the general fiber F of the Iitaka fibration $f: X' \rightarrow Y$ belong to a bounded family of varieties, can we infer any boundedness statement about X' ? If not, what are some natural additional conditions required to achieve boundedness? In this direction, recent work of Li analyzes the case when the Iitaka fibration has fibers of Fano type [Li23]. Similarly, the results in [Bir18] apply to the study of fibrations of Fano type.

In this work, we are concerned with the case of varieties X of Kodaira dimension $\dim(X) - 1$. More generally, we are interested in minimal Kawamata log terminal (klt for short) pairs (X, Δ) of Kodaira dimension $\kappa(X, \Delta) = \dim(X) - 1$. In this case, the general fiber of the Iitaka fibration is either an elliptic curve or $\mathbb{P}^1$. While the case when the general fiber is $\mathbb{P}^1$ follows from [Bir18], substantial work is needed to analyze the case of elliptic fibrations. If the elliptic fibration admits a section, using techniques developed in [DS21], one can induce a polarization that bounds the total space of the fibration. This direction is successfully explored in [FS20]. On the other hand, an elliptic fibration does not necessarily admit a rational section. Furthermore, if a set of pairs is bounded by a family $\mathcal{X} \rightarrow T$, one can stratify T so that the relative Iitaka fibration of $\mathcal{X} \rightarrow T$ induces the Iitaka fibration fiberwise. In particular, a multisection of the relative Iitaka fibration induces, up to a stratification, a multisection of the Iitaka fibration of the fibers. Thus, if a set of elliptic n -folds with $\kappa(X) = n - 1$ is bounded, the minimal degree of a multisection is bounded. Similarly, the bases of the Iitaka fibration have to be bounded too.

Our main result shows that the above constraints, namely on the degree of the multisection and on the boundedness of the bases, are actually sufficient to achieve boundedness in codimension 1. We refer to § 2.3 for the definition of the set of coefficients $\Phi(\mathcal{R})$.

THEOREM 1.1. *Fix positive integers n and d , a positive real number v , and a finite set of rational numbers $\mathcal{R} \subset \mathbb{Q} \cap [0, 1]$. Let $\mathfrak{D}(n, v, \Phi(\mathcal{R}), d)$ be the set of minimal projective klt pairs (X, Δ) of dimension n with $\kappa(X, \Delta) = n - 1$, $\text{vol}_{n-1}(X, K_X + \Delta) = v$, $\text{coeff}(\Delta) \subset \Phi(\mathcal{R})$, and whose Iitaka fibration admits a multisection of degree d . Then, $\mathfrak{D}(n, v, \Phi(\mathcal{R}), d)$ is bounded in codimension 1.*

Remark 1.2. The notation $\text{vol}_{n-1}(X, K_X + \Delta) = v$ is defined in Definition 5.2. It is the volume of the pair of general type naturally induced on the base of the Iitaka fibration.

Remark 1.3. The varieties in the statement of Theorem 1.1 are bounded together with their Iitaka fibrations. This is made precise by the more technical statements Theorems 6.1 and 7.2.

Remark 1.4. If the general fiber of the Iitaka fibration is $\mathbb{P}^1$, the statement of Theorem 1.1 is stronger: boundedness (not just in codimension 1) holds, and the requirement on the degree of the multisection is unnecessary (this is consistent with the fact that a $\mathbb{P}^1$ -fibration always has a multisection of degree 2). See Theorem 6.1.

As mentioned above, the main case of Theorem 1.1 is when the Iitaka fibration $f: X \rightarrow Y$ is an elliptic fibration. In this case, we can consider an auxiliary fibration, the Jacobian fibration $j: J(X) \rightarrow Y$, whose generic fiber is the Jacobian of the generic fiber of f . In particular, j admits a rational section. Thus, one can argue as in [DS21, FS20] and conclude that the set of Jacobian fibrations is bounded in codimension 1. To retrieve the original fibration, we make use of tools developed by Dolgachev and Gross [DG94, Gro94]. More precisely, we consider the geometric Tate–Shafarevich group of the Jacobian fibration. This group parametrizes certain elliptic fibrations over Y whose Jacobian fibration is j . Gross showed that the torsion parts of this group behave well in families: Roughly speaking, for every d , the d -torsion elements of this group form a finite set, which is constant in the étale topology under small deformations. In particular, one could hope to retrieve the whole family of fibrations from the family of Jacobians. On the other hand, the fibrations parametrized by this group have very restrictive geometric properties, which our fibrations only satisfy over some open set of the base. The main technical statement of this work is a bound on the complement of this subset, which allows us to show that it deforms along with the family of Jacobian fibrations. This is the content of Proposition 7.4.

In a similar flavor as [HMX18a], Theorem 1.1 is concerned with minimal models. Thus, it is interesting to ask whether any klt pair (X, Δ) with $\kappa(X, \Delta) = n - 1$ has a minimal model to which Theorem 1.1 applies. The MMP predicts that it should be the case. More precisely, every klt pair of non-negative Kodaira dimension is expected to admit a good minimal model. In general, this statement is known just in some cases, for instance in the case of varieties of general type. In this work, we show the existence of good minimal models when the fibers of the Iitaka fibration have dimension up to 3.

THEOREM 1.5. *Let (X, Δ) be a projective klt pair of dimension n with rational coefficients with $\kappa(X, \Delta) \geq 0$. Assume $\kappa(X, \Delta) \geq n - 3$. Then, (X, Δ) has a good minimal model.*

Theorem 1.5 is a generalization of ideas of Lai [Lai11], who considered terminal varieties with no boundary. The specific statement of Theorem 1.5 has already been observed in the case of smooth varieties without boundary [HS21, Theorem 2.1]. Also, the strategy of the proof generalizes the approach in [GW22], where the case of elliptic fibrations is considered. More precisely, we first consider a resolution of indeterminacies of the Iitaka fibration. Then, by the canonical bundle formula, we can induce the structure of a pair of general type on the base of the Iitaka fibration. On the one hand, by [BCHM10], we can run an MMP on the base of the Iitaka fibration, which terminates with a good minimal model Y^m . On the other hand, since the abundance conjecture is known in dimension up to 3, we can apply the results of [HX13], which guarantee that our variety X has a relative good minimal model over the base. Combining these two facts, the proof is completed.

2. Preliminaries

2.1 Conventions

Throughout this paper, the base field will be an algebraically closed field of characteristic 0.

2.2 Contractions

In this paper, a *contraction* is a projective morphism of quasi-projective varieties $f: X \rightarrow Z$ with $f_*\mathcal{O}_X = \mathcal{O}_Z$. Notice that if X is normal, then so is Z . An *elliptic fibration* is a contraction whose general fiber is a smooth elliptic curve.

2.3 Hyperstandard sets

Let $\mathcal{R}$ be a subset of $[0, 1]$. Then, we define the *set of hyperstandard multiplicities* associated with $\mathcal{R}$ as

$$\Phi(\mathcal{R}) := \left\{ 1 - \frac{r}{m} \mid r \in \mathcal{R}, m \in \mathbb{N} \right\}.$$

When $\mathcal{R} = \{0, 1\}$, we call it the set of *standard multiplicities*. Usually, with no mention, we assume $0, 1 \in \mathcal{R}$, so that $\Phi(\{0, 1\}) \subset \Phi(\mathcal{R})$. Furthermore, if $1 - r \in \mathcal{R}$ for every $r \in \mathcal{R}$, we have that $\mathcal{R} \subset \Phi(\mathcal{R})$. Now, assume that $\mathcal{R} \subset [0, 1]$ is a finite set of rational numbers. Then, $\Phi(\mathcal{R})$ is a set of rational numbers satisfying the *descending chain condition* (DCC for short) whose only accumulation point is 1.

2.4 Divisors

Let X be a normal quasi-projective variety. We say that D is a divisor on X if it is a $\mathbb{Q}$ -Weil divisor, that is, D is a finite sum of prime divisors on X with coefficients in $\mathbb{Q}$. The *support* of a divisor $D = \sum_{i=1}^n d_i P_i$ is the union of the prime divisors appearing in the formal sum $\text{Supp}(D) = \sum_{i=1}^n P_i$. Let $f: X \rightarrow Z$ be a projective morphism of quasi-projective varieties. Given a divisor $D = \sum d_i P_i$ on X , we define

$$D^v := \sum_{f(D_i) \subsetneq Z} d_i P_i, \quad D^h := \sum_{f(D_i) = Z} d_i P_i.$$

We call D^v and D^h the *vertical part* and *horizontal part* of D , respectively. Let D_1 and D_2 be divisors on X . We write $D_1 \sim_Z D_2$ (respectively, $D_1 \sim_{\mathbb{Q}, Z} D_2$) if there is a Cartier (respectively, $\mathbb{Q}$ -Cartier) divisor L on Z such that $D_1 - D_2 \sim f^*L$ (respectively, $D_1 - D_2 \sim_{\mathbb{Q}} f^*L$). Equivalently, we may also write $D_1 \sim D_2$ over Z . The case of $\mathbb{Q}$ -linear equivalence is denoted similarly.

2.5 Pairs

A *sub-pair* (X, B) is the datum of a normal quasi-projective variety and a divisor B such that $K_X + B$ is $\mathbb{Q}$ -Cartier. If $B \leq \text{Supp}(B)$, we say that B is a *sub-boundary*, and if in addition $B \geq 0$, we call it a *boundary*. A sub-pair (X, B) is called a *pair* if $B \geq 0$. A sub-pair (X, B) is *simple normal crossing* (or *log smooth*) if X is smooth, every irreducible component of $\text{Supp}(B)$ is smooth, and étale locally $\text{Supp}(B) \subset X$ is isomorphic to the intersection of $r \leq n$ coordinate hyperplanes in $\mathbb{A}^n$. A *log resolution* of a sub-pair (X, B) is a birational contraction $\pi: X' \rightarrow X$ such that $\text{Ex}(\pi)$ is a divisor and $(X', \pi_*^{-1}(\text{Supp}(B)) + \text{Ex}(\pi))$ is log smooth. Here, $\text{Ex}(\pi) \subset X'$ is the *exceptional set* of π , that is, the reduced subscheme of X' consisting of the points where π is not an isomorphism. If (X, B) is a sub-pair and $f: X \rightarrow U$ is a morphism, we say that (X, B) is *log smooth over U* if (X, B) is simple normal crossing and every stratum of $(X, \text{Supp}(B))$, including X itself, is smooth over U .

Let (X_1, B_1) and (X_2, B_2) be two pairs. We say that they are *crepant* to each other if there exist a normal variety Y and birational morphisms $p: Y \rightarrow X_1$ and $q: Y \rightarrow X_2$ so that $p^*(K_{X_1} + B_1) = q^*(K_{X_2} + B_2)$.

Let (X, B) be a sub-pair, and let $\pi: X' \rightarrow X$ be a birational contraction from a normal variety X' . Then, we can define a sub-pair (X', B') on X' via the identity

$$K_{X'} + B' = \pi^*(K_X + B),$$

where we assume $\pi_*(K_{X'}) = K_X$. We call (X', B') the *log pull-back* or *trace* of (X, B) on X' . The *log discrepancy* of a prime divisor E on X' with respect to (X, B) is defined as $a_E(X, B) := 1 - \text{mult}_E(B')$. Let ϵ be a non-negative number. We say that a sub-pair (X, B) is *ϵ -sub-log canonical* (respectively, *ϵ -sub-klt*) if $a_E(X, B) \geq \epsilon$ (respectively, $a_E(X, B) > \epsilon$) for every π and every E as above. If $\epsilon = 0$, we drop it from the notation. When (X, B) is a pair, we say that (X, B) is *ϵ -log canonical* or *ϵ -klt*, respectively. Notice that if (X, B) is log canonical (respectively, klt), we have $\text{coeff}(B) \subset [0, 1]$ (respectively, $\text{coeff}(B) \subset [0, 1)$).

Given a sub-pair (X, B) and an effective $\mathbb{Q}$ -Cartier divisor D , we define the *log canonical threshold* of D with respect to (X, B) as

$$\text{lct}(X, B; D) := \sup\{t \geq 0 \mid (X, B + tD) \text{ is sub-log canonical}\}.$$

2.6 b-divisors

Let X be a normal variety, and consider the set of all proper birational morphisms $\pi: X_\pi \rightarrow X$, where X_π is normal. This is a partially ordered set, where $\pi' \geq \pi$ if π' factors through π . We define the space of *Weil b-divisors* as the inverse limit

$$\mathbf{Div}(X) := \varprojlim_{\pi} \text{Div}(X_\pi),$$

where $\text{Div}(X_\pi)$ denotes the space of Weil divisors on X_π . Then, we define the space of $\mathbb{Q}$ -Weil b-divisors $\mathbf{Div}_{\mathbb{Q}}(X) := \mathbf{Div}(X) \otimes \mathbb{Q}$. In the following, by b-divisor, we will mean a $\mathbb{Q}$ -Weil b-divisor. Equivalently, a b-divisor $\mathbf{D}$ can be described as a (possibly infinite) sum of geometric valuations V_i of $k(X)$ with coefficients in $\mathbb{Q}$,

$$\mathbf{D} = \sum_{i \in I} b_i V_i, \quad b_i \in \mathbb{Q},$$

such that for every normal variety X' birational to X , only a finite number of the V_i can be realized by divisors on X' . The *trace* $\mathbf{D}_{X'}$ of $\mathbf{D}$ on X' is defined as

$$\mathbf{D}_{X'} := \sum_{\{i \in I \mid c_{X'}(V_i) = D_i, \text{codim}_{X'}(D_i) = 1\}} b_i D_i,$$

where $c_{X'}(V_i)$ denotes the center of the valuation on X' .

Given a b-divisor $\mathbf{D}$ over X , we say that $\mathbf{D}$ is a *b- $\mathbb{Q}$ -Cartier* b-divisor if there exists a birational model X' of X such that $\mathbf{D}_{X'}$ is $\mathbb{Q}$ -Cartier on X' and for any model $r: X'' \rightarrow X'$, we have $\mathbf{D}_{X''} = r^* \mathbf{D}_{X'}$. When this is the case, we will say that $\mathbf{D}$ descends to X' and write $\mathbf{D} = \overline{\mathbf{D}_{X'}}$. We say that $\mathbf{D}$ is *b-effective* if $\mathbf{D}_{X'}$ is effective for any model X' . We say that $\mathbf{D}$ is *b-nef* if it is b- $\mathbb{Q}$ -Cartier and, moreover, there exists a model X' of X such that $\mathbf{D} = \overline{\mathbf{D}_{X'}}$ and $\mathbf{D}_{X'}$ is nef on X' . The notion of b-nef b-divisor can be extended analogously to the relative case.

Example 2.1. Let (X, B) be a sub-pair. The *discrepancy b-divisor* $\mathbf{A}(X, B)$ is defined as follows: on a birational model $\pi: X' \rightarrow X$, its trace $\mathbf{A}(X, B)_{X'}$ is given by the identity $K_{X'} = \pi^*(K_X + B) + \mathbf{A}(X, B)_{X'}$. Then, the b-divisor $\mathbf{A}^*(X, B)$ is defined by taking its trace $\mathbf{A}^*(X, B)_{X'}$ on X' to be $\mathbf{A}(X, B)_{X'} := \sum_{a_i > -1} a_i D_i$, where $\mathbf{A}(X, B)_{X'} = \sum_i a_i D_i$.

2.7 Generalized pairs

A *generalized sub-pair* $(X, B, \mathbf{M})/Z$ over Z is the datum of

- a normal variety admitting a projective morphism $X \rightarrow Z$,
- a divisor B on X ,
- a $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor $\mathbf{M}$ over X which descends to a nef/ Z $\mathbb{Q}$ -Cartier divisor $\mathbf{M}_{X'}$ on some birational model $X' \rightarrow X$.

Moreover, we require that $K_X + B + \mathbf{M}_X$ is $\mathbb{Q}$ -Cartier. If B is effective, we say that $(X, B, \mathbf{M})/Z$ is a *generalized pair*. The divisor B is called the *boundary part* of $(X, B, \mathbf{M})/Z$, and $\mathbf{M}$ is called the *moduli part*. In the definition, we can replace X' with a higher birational model X'' and $\mathbf{M}_{X'}$ with $\mathbf{M}_{X''}$ without changing the generalized pair. Whenever $\mathbf{M}_{X''}$ descends on X'' , the datum of the rational map $X'' \dashrightarrow X$, B , and $\mathbf{M}_{X''}$ encodes all the information of the generalized pair.

Let $(X, B, \mathbf{M})/Z$ be a generalized sub-pair and $\rho: Y \rightarrow X$ a projective birational morphism. Then, we may write

$$K_Y + B_Y + \mathbf{M}_Y = \pi^*(K_X + B + \mathbf{M}_X).$$

Given a prime divisor E on Y , we define the *generalized log discrepancy* of E with respect to $(X, B, \mathbf{M})/Z$ to be $a_E(X, B, \mathbf{M}) := 1 - \text{mult}_E(B')$. If $a_E(X, B, \mathbf{M}) \geq 0$ for all divisors E over X , we say that $(X, B, \mathbf{M})/Z$ is *generalized sub-log canonical*. Similarly, if $a_E(X, B + M) > 0$ for all divisors E over X and $\lfloor B \rfloor \leq 0$, we say that $(X, B, \mathbf{M})/Z$ is *generalized sub-klt*. When $B \geq 0$, we say that $(X, B, \mathbf{M})/Z$ is *generalized log canonical* or *generalized klt*, respectively.

2.8 Canonical bundle formula

We recall the statement of the *canonical bundle formula*. We refer to [FG14] for the notation involved and a more detailed discussion about the topic. Let (X, B) be a sub-pair. A contraction $f: X \rightarrow T$ is an *lc-trivial fibration* if

- (i) (X, B) is a sub-pair that is sub-log canonical over the generic point of T ;
- (ii) $\text{rank} f_* \mathcal{O}_X(\lceil \mathbf{A}^*(X, B) \rceil) = 1$, where $\mathbf{A}^*(X, B)$ is the $\mathbb{Q}$ -divisor defined in Example 2.1; and
- (iii) there exists a $\mathbb{Q}$ -Cartier divisor L_T on T such that $K_X + B \sim_{\mathbb{Q}} f^* L_T$.

Condition (ii) above is automatically satisfied if B is effective over the generic point of T . Given a sub-pair (X, B) and an lc-trivial fibration $f: X \rightarrow T$, there exist $\mathbb{Q}$ -divisors $\mathbf{B}$ and $\mathbf{M}$ over T such that the following linear equivalence relation, known as the *canonical bundle formula*, holds:

$$K_X + B \sim_{\mathbb{Q}} f^*(K_T + \mathbf{B}_T + \mathbf{M}_T). \quad (2.1)$$

For a prime divisor $P \subset T$, its coefficient in $\mathbf{B}_T$ is given by the formula $1 - \text{lct}_{\eta_P}(X, B; f^*P)$, where the symbol lct_{η_P} denotes the log canonical threshold over the generic point η_P of P . Then, we set $\mathbf{M}_T := L_T - (K_T + \mathbf{B}_T)$. If $f': X' \rightarrow T'$ is a higher model of f with morphisms $\phi: X' \rightarrow X$ and $\psi: T' \rightarrow T$, one repeats this algorithm with (X', B') and $L_{T'}$, where $K_{X'} + B' = \phi^*(K_X + B)$ and $L_{T'} = \psi^* L_T$. We refer to [PS09, § 7] for more details.

The $\mathbb{Q}$ -divisor $\mathbf{B}$ is often called the *boundary part* in the canonical bundle formula; it is a canonically defined $\mathbb{Q}$ -divisor. Furthermore, if B is effective, then so is $\mathbf{B}_T$. The $\mathbb{Q}$ -divisor $\mathbf{M}$, in turn, is often called the *moduli part* in the canonical bundle formula, and it is in general defined only up to $\mathbb{Q}$ -linear equivalence. The linear equivalence in (2.1) holds at the level of $\mathbb{Q}$ -divisors: namely,

$$\overline{(K_X + B)} \sim_{\mathbb{Q}} f^*(\mathbf{K} + \mathbf{B} + \mathbf{M}),$$

where $\mathbf{K}$ denotes the canonical b -divisor of T . Let I be a positive integer such that $I(K_X + B) \sim 0$ along the generic fiber of f . Then, by [PS09, Construction 7.5], we may choose $\mathbf{M}$ in its $\mathbb{Q}$ -linear equivalence class so that

$$\overline{I(K_X + B)} \sim If^*(\mathbf{K} + \mathbf{B} + \mathbf{M}).$$

The moduli b -divisor $\mathbf{M}$ is expected to detect the variation of the fibers of the morphism f . In this direction, we have the following statement.

THEOREM 2.2 (cf. [FG14, Theorem 3.6]). *Let $f: (X, B) \rightarrow T$ be an lc-trivial fibration, and let $\pi: T \rightarrow S$ be a projective morphism. Let $\mathbf{B}$ and $\mathbf{M}$ be the b -divisors that give the boundary and the moduli part, respectively. Then, $\mathbf{K} + \mathbf{B}$ and $\mathbf{M}$ are b - $\mathbb{Q}$ -Cartier b -divisors. Furthermore, $\mathbf{M}$ is b -nef over S .*

Remark 2.3. In the setup of Theorem 2.2, let T' be a model where the nef part $\mathbf{M}$ descends in the sense of b -divisors. Then, $\mathbf{M}_{T'}$ is nef over S . In particular, $(T, \mathbf{B}_T, \mathbf{M})/S$ is a generalized sub-pair.

Remark 2.4. In the setup of Theorem 2.2, the b -divisor $\mathbf{M}$ is expected to be b -semi-ample. Furthermore, it is expected that $C\mathbf{M}$ is b -free, where $C \in \mathbb{Z}_{\geq 1}$ depends only on $\dim(X)$ and the coefficients of B ; see [PS09, Conjecture 7.13]. These facts are known if $\dim(T) = \dim(X) - 1$, and we will constantly make use of them in this work [PS09, Theorem 8.1].

2.9 Boundedness

Here we recall the notion of boundedness for a set of pairs, and we introduce a suitable notion of boundedness for fibrations.

DEFINITION 2.5. Let $\mathfrak{D}$ be a set of projective pairs. Then, we say that $\mathfrak{D}$ is bounded (respectively, birationally bounded) if there exist a pair $(\mathcal{X}, \mathcal{B})$, where $\mathcal{B}$ is reduced, and a projective morphism $\pi: \mathcal{X} \rightarrow T$, where T is of finite type, such that for every $(X, B) \in \mathfrak{D}$, there are a closed point $t \in T$ and a morphism (respectively, a birational map) $f_t: \mathcal{X}_t \rightarrow X$ inducing an isomorphism $(X, \text{Supp}(B)) \cong (\mathcal{X}_t, \mathcal{B}_t)$ (respectively, such that $\text{Supp}(\mathcal{B}_t)$ contains the strict transform of $\text{Supp}(B)$ and all the f_t exceptional divisors). If a set of pairs is birationally bounded and the maps f_t and f_t^{-1} are isomorphisms in codimension 1, we say that $\mathfrak{D}$ is bounded in codimension 1.

DEFINITION 2.6. Let $\mathfrak{F}$ be a set of fibrations between projective pairs $\phi: (X, B) \rightarrow (Y, D)$. We say that $\mathfrak{F}$ is bounded (respectively, birationally bounded) if there exist pairs $(\mathcal{X}, \mathcal{B})$, $(\mathcal{Y}, \mathcal{D})$, where $\mathcal{B}$ and $\mathcal{D}$ are reduced, a variety of finite type T , and projective morphisms

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\sigma} & \mathcal{Y} \\ \pi \searrow & & \swarrow \rho \\ & T & \end{array}$$

such that

- (i) the above diagram is commutative; that is, we have $\pi = \rho \circ \sigma$;
- (ii) for every $(X, B) \rightarrow (Y, D) \in \mathfrak{F}$, there exist a closed point $t \in T$ and morphisms (respectively, birational maps) $f_t: \mathcal{X}_t \rightarrow X$ and $g_t: \mathcal{Y}_t \rightarrow Y$ inducing isomorphisms $(X, \text{Supp}(B)) \cong (\mathcal{X}_t, \mathcal{B}_t)$ (respectively, such that $\text{Supp}(\mathcal{B}_t)$ contains the strict transform of $\text{Supp}(B)$ and all the f_t exceptional divisors) and $(Y, \text{Supp}(D)) \cong (\mathcal{Y}_t, \mathcal{D}_t)$ (respectively, such that $\text{Supp}(\mathcal{D}_t)$ contains the strict transform of $\text{Supp}(D)$ and all the g_t exceptional divisors); and

- (iii) for every $(X, B) \rightarrow (Y, D) \in \mathfrak{F}$ and t as in condition (ii), we have $\phi \circ f_t = g_t \circ \sigma_t$ (respectively, as rational maps), where σ_t denotes the restriction of σ to $\mathcal{X}_t$.

If a set of fibrations is birationally bounded and the maps f_t , f_t^{-1} , g_t , and g_t^{-1} are isomorphisms in codimension 1, we say that $\mathfrak{F}$ is bounded in codimension 1.

2.10 The geometric Tate–Shafarevich group

Here, we recall a few facts about the geometric Tate–Shafarevich group. We will limit ourselves to stating only those facts regarding the Tate–Shafarevich group that will be needed in the article. We refer to [DG94, Gro94] for a detailed development of the theory we need. An introduction to the topic over a 1-dimensional base can be found in [ŠAV⁺65]. Finally, all the needed facts about étale cohomology and group cohomology can be found in [Mil80, Mil06].

Let Y be a variety defined over a field of characteristic 0, and let $k(Y)$ be its field of fractions. Let $\pi: X \rightarrow Y$ be an elliptic fibration, that is, a contraction whose general fiber is an elliptic curve. Then, the generic fiber X_{η_Y} is a genus 1 curve over $k(Y)$ with possibly no $k(Y)$ -rational points. Let $J(X)_{\eta_Y}$ denote its Jacobian. Then, X_{η_Y} is a principal homogeneous space over $J(X)_{\eta_Y}$ defined over $k(Y)$. Furthermore, we can consider a projective model $\rho: J(X) \rightarrow Y$ realizing $J(X)_{\eta_Y}$ over Y .

For some finite Galois extension $K: k(Y)$, the variety X_{η_Y} acquires a K -rational point. In particular, X_{η_Y} and $J(X)_{\eta_Y}$ are non-canonically isomorphic over K . From a geometric point of view, it means that we can find a finite Galois morphism $Y' \rightarrow Y$ such that $X \times_Y Y'$ admits a rational section.

Now, let E be an elliptic curve defined over $k(Y)$, and let $L: k(Y)$ be a finite Galois extension. We want to consider all the elliptic fibrations $\pi: X \rightarrow Y$ so that X_{η_Y} is a principal homogeneous space over E admitting an L -rational point. That is, we are interested in elliptic fibrations $\pi: X \rightarrow Y$ that acquire a rational section after the base change $\text{Spec}(L) \rightarrow Y$ and whose associated Jacobian fibration has E as generic fiber. Said otherwise, π is an elliptic fibration whose generic fiber becomes isomorphic to $E_L := E \otimes_{k(Y)} L$ as an L -scheme. This set is parametrized by the cohomology group $H^1(\text{Gal}(L: k(Y)), E(L))$. More precisely, this group parametrizes the isomorphism classes of the generic fibers X_{η_Y} so that $\text{Pic}^0(X_{\eta_Y}) \simeq E$ and X_{η_Y} has an L -rational point. Equivalently, the group $H^1(\text{Gal}(L: k(Y)), E(L))$ parametrizes the birational classes of the elliptic fibrations with base Y and associated Jacobian fibration corresponding to E that acquire a rational section after the base change $\text{Spec}(L) \rightarrow Y$. Notice that, by standard properties of group cohomology, the order of the elements of $H^1(\text{Gal}(L: k(Y)), E(L))$ is finite and divides $[L: k(Y)]$.

Now, if we want to consider all the principal homogeneous spaces over E , we need to consider all possible finite Galois extensions $L: k(Y)$. This information is encoded in the Weil–Châtelet group $\text{WC}(E) := H^1(\text{Gal}(\overline{k(Y)}: k(Y)), E(\overline{k(Y)}))$, which is the direct limit of all possible groups $H^1(\text{Gal}(L: k(Y)), E(L))$ as above. In particular, we have the following geometric consequence.

LEMMA 2.7. *Let E be an elliptic curve defined over $k(Y)$, and let $\pi: X \rightarrow Y$ be an elliptic fibration with $X_{\eta_Y} \in \text{WC}(E)$. If π admits a multisection of degree d , then the order of X_{η_Y} in $\text{WC}(E)$ divides $d!$.*

Proof. Let $\pi: X \rightarrow Y$ be as in the statement. Then, the multisection induces a field extension L of degree d so that X_{η_Y} admits an L -rational point. Notice that $L: k(Y)$ may not be a Galois extension. Thus, there is a Galois extension $L': k(Y)$ so that X_{η_Y} admits an

L' -rational point and $[L' : k(Y)]$ divides $d!$. Then, the claim follows, as X_{η_Y} is an element of $H^1(\text{Gal}(L' : k(Y)), E(L'))$. $\square$

The Weil–Châtelet group $\text{WC}(E)$ is a very large group that parametrizes all birational classes of all elliptic fibrations $\pi : X \rightarrow Y$ whose Jacobian fibration is a compactification of E over Y . That is, $\text{WC}(E)$ parametrizes all birational classes of elliptic fibrations $\pi : X \rightarrow Y$ whose geometric generic fiber is isomorphic to $E_{\overline{k(Y)}}$ as a $\overline{k(Y)}$ -scheme. This group admits a natural subgroup, called the Tate–Shafarevich group, which is denoted by $\text{III}_Y(E)$. For a formal definition of $\text{III}_Y(E)$, we refer to [DG94]. Here, we limit ourselves to the following characterization as a set:

$\text{III}_Y(E) = \{C \in \text{WC}(E) \mid X_C \rightarrow Y \text{ has a rational section étale locally at } y \text{ for every point } y \in Y\}$, where $X_C \rightarrow Y$ is some proper model of the curve C defined over $k(Y)$; see [Gro94, §3]. Thus, $\text{III}_Y(E)$ imposes pretty restrictive conditions on the type of singular fibers that can occur. In particular, multiple fibers cannot occur over codimension 1 points of the base.

PROPOSITION 2.8. *Let $f : X \rightarrow Y$ be an elliptic fibration, and let $j : J(X) \rightarrow Y$ be its associated Jacobian fibration. Let $p \in Y$ be a closed point. Assume that Y is smooth, $\dim(Y) \geq 2$, f is smooth over $Y \setminus \{p\}$, and j is smooth with a regular section. Then, $X \in \text{III}_Y(J(X)_{\eta_Y})$.*

Proof. Let $\mathcal{O}_{Y,\bar{p}}$ denote the strict henselization of $\mathcal{O}_{Y,p}$, and let $\bar{p}$ denote the closed point of $\text{Spec}(\mathcal{O}_{Y,\bar{p}})$. Let $\bar{S}$ denote $\text{Spec}(\mathcal{O}_{Y,\bar{p}})$, and let $S := \bar{S} \setminus \{\bar{p}\}$. Let A denote the generic fiber of $J(X) \times_Y \bar{S}$. Then, as $\bar{S}$ is strictly local, we have $\text{III}_{\bar{S}}(A) = 0$. Indeed, $\mathcal{O}_{Y,\bar{p}}$ coincides with its own strict henselization, and thus the description of $\text{III}_{\bar{S}}(A)$ as an intersection of kernels necessarily returns 0; see [DG94, §1]. If $f : X \rightarrow Y$ does not correspond to an element of $\text{III}_Y(J(X)_{\eta_Y})$, then it induces a non-zero element of $\text{III}_S(A)$. Thus, to conclude, it is enough to show $\text{III}_S(A) = 0$. For this type of argument, see for instance [DG94, §3].

By assumption, $J(X) \times_Y \bar{S} \rightarrow \bar{S}$ satisfies the conditions of [DG94, Theorem 3.1]. Thus, following the notation of [DG94, Theorem 3.1], the group $\text{III}_S(A)$ is a subgroup of $H^2(S, \mathcal{E})$. Here, $\mathcal{E}$ is the étale sheaf defined in [DG94, Definition 1.8]. Since j is smooth, all the fibers of $J(X) \times_Y \bar{S}$ are geometrically integral. Thus, by [DG94, Proposition 1.12], we have $\mathcal{E} = 0$. Then, the claim follows. $\square$

2.11 Technical statements

Here we collect the technical statements that will be needed in the course of the main proofs.

PROPOSITION 2.9. *Let (X, Δ) be a klt pair, and let $f : X \rightarrow Y$ be a contraction with $K_X + \Delta \sim_{\mathbb{Q},f} 0$. Let $\pi : Y' \rightarrow Y$ be a small morphism. Then, there exist a $\mathbb{Q}$ -factorial pair (X', Δ') admitting a contraction $g : X' \rightarrow Y'$ so that X and X' are isomorphic in codimension 1 and (X', Δ') is crepant to (X, Δ) .*

Proof. Let $\phi : \hat{X} \rightarrow X$ be a log resolution of (X, Δ) admitting a morphism $\hat{X} \rightarrow Y'$. Since (X, Δ) is klt, it is ϵ -log canonical for some $\epsilon > 0$. Then, let E be the reduced ϕ -exceptional divisor, and define $\Gamma := (1 - c)E$, where c is a rational number satisfying $0 < c < \epsilon$. Let X^ν denote the normalization of the main component of $X \times_Y Y'$. First, we run a partial $(K_{\hat{X}} + \phi_*^{-1}\Delta + \Gamma)$ -MMP over X^ν . By [Fuj11, Theorem 2.3], after finitely many steps, this MMP contracts the prime components of Γ that are exceptional over X^ν . In particular, all the prime components of Γ that dominate Y' are contracted, and all the prime components of Γ that are not contracted have a center of codimension at least 2 in Y' . Thus, by [Lai11, Lemma 2.10], we can contract these

leftover components by running an MMP with scaling relative to Y' . The model thus obtained satisfies the properties in the statement. $\square$

LEMMA 2.10. *Let $(\mathcal{X}, \mathcal{B})$ be a sub-pair admitting a tower of contractions $\mathcal{X} \rightarrow \mathcal{Y} \rightarrow T$. Assume that the following conditions are satisfied:*

- *The fibration $(\mathcal{X}, \mathcal{B}) \rightarrow \mathcal{Y}$ is lc-trivial, inducing a generalized pair $(\mathcal{Y}, \mathcal{B}_{\mathcal{Y}}, \mathbf{M})$ on $\mathcal{Y}$.*
- *The varieties $\mathcal{X}$, $\mathcal{Y}$, and T are smooth.*
- *The b-divisor $\mathbf{M}$ descends on $\mathcal{Y}$, and $(\mathcal{Y}, \mathcal{B}_{\mathcal{Y}})$ is log smooth over T .*
- *The pair $(\mathcal{X}, \mathcal{B})$ is log smooth over T .*

Let $t \in T$ be a closed point, and assume that $(\mathcal{X}_t, \mathcal{B}_t) \rightarrow \mathcal{Y}_t$ is an lc-trivial fibration. Then, the boundary and moduli b-divisors induced on $\mathcal{Y}_t$ by $(\mathcal{X}_t, \mathcal{B}_t)$ agree with the restrictions of $\mathcal{B}_{\mathcal{Y}}$ and $\mathbf{M}$, respectively.

We recall that, given a sub-pair (W, Δ) and a morphism $\pi: W \rightarrow U$, we say that (W, Δ) is *log smooth over U* if W is smooth, $\text{Supp}(\Delta)$ is a divisor with simple normal crossings, and every stratum of $(W, \text{Supp}(\Delta))$ (including W) is smooth over U .

Remark 2.11. In the setup of Lemma 2.10, for some special point t , the induced fibration $(\mathcal{X}_t, \mathcal{B}_t) \rightarrow \mathcal{T}_t$ may not satisfy condition (ii) § 2.8. Nevertheless, one can apply the construction of boundary and moduli divisors to every fiber as described in § 2.8.

Proof of Lemma 2.10. Fix $t \in T$ as in the statement, and let $H \subset T$ be a general smooth divisor through t . Then, as H is general away from $t \in T$, the fibration $(\mathcal{X}, \mathcal{B}) \times_T H \rightarrow \mathcal{Y} \times_T H$ is lc-trivial. By induction on the dimension of T , it suffices to show that this fibration has the same properties as $(\mathcal{X}, \mathcal{B}) \rightarrow \mathcal{Y} \rightarrow T$ and that the boundary and moduli b-divisors induced on $\mathcal{Y} \times_T H$ are the restrictions of $\mathcal{B}_{\mathcal{Y}}$ and $\mathbf{M}$, respectively.

By base change, the properties in the statement are satisfied by $(\mathcal{X}, \mathcal{B}) \times_T H \rightarrow \mathcal{Y} \times_T H$. Thus, we are left with showing the statement about the b-divisors. Since $(\mathcal{Y}, \mathcal{B}_{\mathcal{Y}})$ is log smooth over T , every fiber of $\mathcal{Y} \rightarrow T$ intersects properly and transversally every stratum of $\text{Supp}(\mathcal{B}_{\mathcal{Y}})$. Furthermore, by [Har77, Proposition III.10.1], the fiber product $H \times_T \mathcal{Y}$ is smooth. Thus, $H \times_T \mathcal{Y}$ satisfies properties (1), (2), and (3) in the proof of [Flo14, Lemma 3.1]. Then, by the proof of [Flo14, Lemma 3.1], the claim about b-divisors on the model $\mathcal{Y} \times_T H$ follows. By construction, the standard normal crossing assumptions in [Kol07, Definition 8.3.6] are satisfied. So, by [Kol07, Theorem 8.3.7], the moduli b-divisor induced by $(\mathcal{X}, \mathcal{B}) \times_T H \rightarrow \mathcal{Y} \times_T H$ descends onto $\mathcal{Y} \times_T H$. Thus, the statement holds at the level of b-divisors and not just on a specific model. This concludes the proof. $\square$

COROLLARY 2.12. *Let $(\mathcal{X}, \mathcal{B})$ be a pair admitting a tower of contractions $\mathcal{X} \rightarrow \mathcal{Y} \rightarrow T$. Assume that $(\mathcal{X}, \mathcal{B}) \rightarrow \mathcal{Y}$ is an lc-trivial fibration, $(\mathcal{X}, \mathcal{B}) \rightarrow T$ is a family of pairs, and $\mathcal{Y} \rightarrow T$ is a family of normal varieties. Then, there exists a stratification of T such that the following holds: Let $(\mathcal{Y}, \mathcal{B}_{\mathcal{Y}}, \mathbf{M})$ be the generalized pair induced by the canonical bundle formula, and let $t \in T$ be a closed point. Then, the boundary and moduli b-divisors induced on $\mathcal{Y}_t$ by $(\mathcal{X}_t, \mathcal{B}_t)$ agree with the restrictions of $\mathcal{B}_{\mathcal{Y}}$ and $\mathbf{M}$, respectively.*

Remark 2.13. In the statement of Corollary 2.12, we are abusing notation in the following sense: we replace T with a stratification, and we replace $\mathcal{X}$ and $\mathcal{Y}$ with the corresponding stratifications induced by base change.

Proof. Up to a first stratification, we may assume that T is smooth. Now, fix an irreducible component T_i of T , and let $(\mathcal{X}_i, \mathcal{B}_i)$ and $\mathcal{Y}_i$ denote the corresponding irreducible components. Let $\mathcal{Y}'_i$ be a log resolution of $(\mathcal{Y}'_i, \mathcal{B}_{\mathcal{Y}'_i})$ where $\mathbf{M}_i$ descends. Then, let $(\mathcal{X}'_i, \mathcal{B}'_i)$ be a log resolution of $(\mathcal{X}_i, \mathcal{B}_i)$ that factors through $\mathcal{Y}'_i$. Then, by generic smoothness, the assumptions of Lemma 2.10 are satisfied over a non-empty open subset $U_i \subset T_i$. As the b-divisors corresponding to a fiber $t \in U_i$ can be constructed considering either $(\mathcal{X}'_{i,t}, \mathcal{B}'_{i,t}) \rightarrow \mathcal{Y}'_{i,t}$ or $(\mathcal{X}_{i,t}, \mathcal{B}_{i,t}) \rightarrow \mathcal{Y}_{i,t}$, the claim of the statement holds for $t \in U_i$. By Noetherian induction on $T_i \setminus U_i$, we conclude that the statement holds over T_i . As there are finitely many T_i , the claim follows. $\square$

LEMMA 2.14. *Let (X, Δ) be a klt n -fold admitting an elliptic fibration $f: X \rightarrow Y$ with $K_X + \Delta \sim_{\mathbb{Q},f} 0$. Let $(Y, B_Y, \mathbf{M})$ denote the generalized pair induced on Y . Let $J(X) \rightarrow Y$ be a relatively minimal terminal model for the Jacobian fibration, and let $\pi: Y' \rightarrow Y$ be the relative ample model of $J(X)$. Then, the trace of $(Y, B_Y, \mathbf{M})$ on Y' is a generalized pair; that is, the trace of the boundary b-divisor on Y' is effective.*

Proof. Up to taking a small $\mathbb{Q}$ -factorialization, we may assume that X is $\mathbb{Q}$ -factorial. Then, let $\hat{X}$ be a relative minimal model for K_X over Y . Then, $X \dashrightarrow \hat{X}$ is a birational contraction, and the relative ample model $\hat{Y} \rightarrow Y$ defines a birational morphism. Let $(\hat{X}, \hat{\Delta})$ denote the trace of (X, Δ) on $\hat{X}$, and let $(\hat{Y}, B_{\hat{Y}}, \mathbf{M})$ denote the trace of $(Y, B_Y, \mathbf{M})$ on $\hat{Y}$. Let $(\hat{Y}, \Omega_{\hat{Y}}, \mathbf{M})$ be the generalized pair induced by $(\hat{X}, 0)$ on $\hat{Y}$. Since we have $K_{\hat{X}} \sim_{\mathbb{Q},\hat{Y}} 0$ and $\hat{\Delta} \geq 0$, we have $0 \leq \Omega_{\hat{Y}} \leq B_{\hat{Y}}$. In particular, $B_{\hat{Y}}$ is effective. Hence, for the purposes of the proof, we may replace (X, Δ) with $(\hat{X}, \hat{\Delta})$. Indeed, the ample model of the Jacobian fibration relative to $\hat{Y}$ naturally admits a rational contraction to Y' , and the coefficients of the divisors appearing on Y' agree with the coefficients on $\hat{Y}$. This follows from the fact that a minimal model of the Jacobian fibration relative to $\hat{Y}$ admits a birational contraction to a minimal model relative to Y . Thus, in the rest of the proof, we may assume that X is $\mathbb{Q}$ -factorial and $K_X \sim_{\mathbb{Q},f} 0$.

By Proposition 2.9, we may assume that Y is $\mathbb{Q}$ -factorial. Let $U \subset Y$ be a big open subset such that

- $\text{Supp}(B_Y)$ is simple normal crossing on U ;
- $K_X \sim_{\mathbb{Q}} 0$ over U ; and
- $K_{J(X)} \sim_{\mathbb{Q}} 0$ over U .

This open set exists, as $J(X)$ is relatively minimal over Y , and its relative ample model is birational to Y . Let B_U denote the restriction of B_Y to U , and let Γ_U denote the boundary divisor of the canonical bundle formula for $J(X) \times_Y U \rightarrow U$. Then, by [Gro94, Lemma 1.6], we have $0 \leq \Gamma_U \leq B_U$. Now, let $(Y', \Gamma, \mathbf{M})$ denote the generalized pair induced by $J(X) \rightarrow Y'$. Notice that the moduli b-divisor is the same as the one corresponding to the morphism $(X, \Delta) \rightarrow Y$, as the corresponding j -maps agree. We have $\Gamma \geq 0$. Furthermore, as Y' is the relative ample model of $K_{J(X)}$ over Y , the divisor $K_{Y'} + \Gamma + \mathbf{M}_{Y'}$ is ample over Y . Since Y is $\mathbb{Q}$ -factorial, we can consider the generalized pair $(Y, \pi_*\Gamma, \mathbf{M})$. Let Γ' denote the trace of the corresponding boundary b-divisor on Y' . Thus, the divisor $K_{Y'} + \Gamma + \mathbf{M}_{Y'} - \pi^*(K_Y + \pi_*\Gamma + \mathbf{M}_Y)$ is π -ample, and its support is π -exceptional. Then, by the negativity lemma [KM98, Lemma 3.39], we have

$$K_{Y'} + \Gamma + \mathbf{M}_{Y'} \leq K_Y + \Gamma' + \mathbf{M}_{Y'}.$$

Since $(\pi_*\Gamma)|_U = \Gamma_U$, we have $\pi_*\Gamma \leq B_Y$. The claim follows. $\square$

LEMMA 2.15. *Let (X_1, Δ_1) and (X_2, Δ_2) be two quasi-projective $\mathbb{Q}$ -factorial klt pairs admitting contractions $f_1: X_1 \rightarrow Y$ and $f_2: X_2 \rightarrow Y$ to the same smooth quasi-projective variety Y with*

$\dim(Y) = \dim(X) - 1$. Further assume $K_{X_i} + \Delta_i \sim_{\mathbb{Q}, f_i} 0$ for $i = 1, 2$, that X_1 and X_2 are isomorphic in codimension 1, and that (X_1, Δ_1) and (X_2, Δ_2) are crepant to each other. Fix $y \in Y$. Then, f_1 is smooth over y if and only if so is f_2 .

Proof. It suffices to show that X_1 and X_2 are connected by a sequence of $(K_{X_i} + \Delta_i)$ -flops over Y . By the assumption $\dim(Y) = \dim(X) - 1$, a smooth fiber is a curve in X that deforms over an open set of Y . Thus, a smooth fiber cannot be contained in the flopped locus. Now, let A_1 be an ample irreducible divisor on X_1 , and let A_2 be its strict transform on X_2 . Then, A_2 is big and movable on X_2 . For $0 < \epsilon \ll 1$, the pair $(X_2, \Delta_2 + \epsilon A_2)$ is klt. Then, we can run a relative $(K_{X_2} + \Delta_2 + \epsilon A_2)$ -MMP with scaling over Y . Since $K_{X_2} + \Delta_2 \sim_{\mathbb{Q}, f_2} 0$, it is an A_2 -MMP. Since A_2 is movable over Y , it has to be a sequence of A_2 -flips. By [BCHM10], this MMP terminates with a good relative minimal model. By construction, this model is $\mathbb{Q}$ -factorial, it is isomorphic in codimension 1 to X_1 , and it admits a morphism to X_1 since X_1 is the relative ample model of A_1 . Since X_1 is $\mathbb{Q}$ -factorial, this morphism cannot be a small morphism. Therefore, this minimal model has to be $(X_1, \Delta_1 + \epsilon A_1)$. This concludes the proof. $\square$

LEMMA 2.16. *Let (X, Δ) be a klt pair admitting a contraction $f: X \rightarrow Y$ of relative dimension 1, where Y is smooth. Let $\pi: X' \rightarrow X$ be a small morphism, and let (X', Δ') denote the trace of (X, Δ) on X' . Fix $y \in Y$. Then, $X \rightarrow Y$ is smooth over y if and only if so is $X' \rightarrow Y$.*

Proof. Since Y is smooth, if $X \rightarrow Y$ is smooth over $y \in Y$, then X is smooth over a neighborhood of $y \in Y$. In particular, X does not admit small modifications over y . Now, assume that $X' \rightarrow X$ is not an isomorphism over $y \in Y$. Then, the fiber X'_y is not an irreducible curve as $X'_y \rightarrow X_y$ is not an isomorphism. This concludes the proof. $\square$

3. Examples

In this work, we are interested in n -folds (X, Δ) with $\kappa(X, \Delta) = n - 1$. Among these, one may be particularly interested in smooth varieties without boundary and their minimal models. If $n = 2$, we get smooth elliptic surfaces with $\kappa(X) = 1$. If a set $\mathfrak{Q}$ of elliptic surfaces with Kodaira dimension 1 is bounded, then the minimal degree of a multisection of the Iitaka fibration for $X \in \mathfrak{Q}$ is bounded. Indeed, let $\pi: \mathcal{X} \rightarrow T$ be a bounding family. Up to stratification, we may assume that π is smooth. Thus, by deformation invariance of plurigena [Wil78], the relative Iitaka fibration of π induces the Iitaka fibration fiberwise. Thus, π factors as an elliptic fibration over T , say $\mathcal{X} \rightarrow \mathcal{C} \rightarrow T$. For simplicity, assume that T is irreducible (in general, it has finitely many irreducible components). Then, a multisection of degree k of $\mathcal{X} \rightarrow \mathcal{C}$ induces a multisection of degree k of $\mathcal{X}_t \rightarrow \mathcal{C}_t$ for $t \in T$ general. Thus, by Noetherian induction, there is a positive integer d such that every $X \in \mathfrak{Q}$ admits a multisection of the Iitaka fibration of degree at most d . More generally, if we have a family of elliptic n -folds that are bounded and such that the elliptic fibration deforms in the family, we have an upper bound on the minimal degree of a multisection.

Therefore, the hypotheses of Theorem 7.2 are necessary to achieve boundedness. On the other hand, one may wonder whether the boundedness of the base of the Iitaka fibration may put any constraints on the minimal degree of a multisection. We will show that this is not the case. We will address the isotrivial and the non-isotrivial cases separately.

Example 3.1. Let C be a Riemann surface of genus $g(C) \geq 2$. Denote by $\pi_n: C_n \rightarrow C$ a cyclic cover with group $\mathbb{Z}/n\mathbb{Z}$. By choosing an appropriate n -torsion element in $\text{Pic}^0(C)$, we may assume that $\pi_n: C_n \rightarrow C$ is étale. Let E be an elliptic curve, and let $p_n \in E$ be a distinguished point

of order n on E . If E is chosen to be very general, we have $\text{Hom}(E, J(C_n)) = 0$ for all n . In particular, $\text{Pic}(C_n \times E) \simeq \text{Pic}(C_n) \times \text{Pic}(E)$. Let $\mathbb{Z}/n\mathbb{Z}$ act on E by translation by p_n . Then, $\mathbb{Z}/n\mathbb{Z}$ acts diagonally on $C_n \times E$. Let $\sigma_n: C_n \times E \rightarrow S_n$ denote the quotient by this action. Thus, we get a smooth and isotrivial fibration $f_n: S_n \rightarrow C$ with fiber E . By construction, we have $K_{S_n} \sim_{\mathbb{Q}, f_n} 0$. Since f_n is smooth and isotrivial, it follows from Kodaira's canonical bundle formula that we have $K_{S_n} \sim_{\mathbb{Q}} f_n^* K_C$. In particular, f_n is the Iitaka fibration of the surface S_n .

Now, a k -section of f_n corresponds to a $\mathbb{Z}/n\mathbb{Z}$ -invariant k -section of σ_n . If a divisor $D \subset C_n \times E$ is $\mathbb{Z}/n\mathbb{Z}$ -invariant, then so is $\mathcal{O}_{C_n \times E}(D)$ in $\text{Pic}(C_n \times E)$. By our choice of E , we have $D \sim P \boxtimes Q$ for some divisors P and Q on C_n and E , respectively. Since the action preserves the two natural projections on $C_n \times E$, it follows that the line bundles $\mathcal{O}_{C_n}(P)$ and $\mathcal{O}_E(Q)$ have a $\mathbb{Z}/n\mathbb{Z}$ -action. Thus, $\mathcal{O}_E(Q)$ is the pull-back of a line bundle under the quotient morphism $E \rightarrow E/(\mathbb{Z}/n\mathbb{Z}) \cong E$. In particular, we have $n \mid \deg(Q)$. As D is a k -section, Q has degree $k > 0$. Then, we have that $k \geq n$. Thus, the set of surfaces $\{S_n\}_{n \in \mathbb{N}}$ is not bounded as the minimal degree of a multisection of f_n is not bounded.

Example 3.2. Let B be a curve, and let $J \rightarrow B$ be a Jacobian fibration that is not isotrivial. Further, assume that the Tate–Shafarevich $\text{III}_B(J_\eta)$ group is infinite. An example where this holds can be realized by [ŠAV⁺65, Theorem VII.11]. For instance, this can be achieved by an elliptic K3 surface, as observed in [Gro94]. For an example of surface of Kodaira dimension 1, one could consider a base curve B with genus at least 2 and the elliptic surface induced by base change from an elliptic K3 surface. Then, we claim that there is a sequence of surfaces $\pi_n: V_n \rightarrow B$ that are locally étale isomorphic to J (that is, they are accounted for in $\text{III}_B(J_\eta)$) and such that the minimal degree of a multisection of π_n diverges to ∞ .

Let $\pi: V \rightarrow B$ be a fibration that is locally isomorphic to $J \rightarrow B$, and let d be the minimal degree of a multisection of π . Then, there is a degree d extension $K: k(B)$ such that V_η has a K -rational point. This extension may not be Galois, but we can find a further extension $L: K$ such that $L: k(B)$ is Galois of degree at most $d!$. Then, V_η corresponds to a non-zero element σ of $H^1(G, E(L))$, where we set $G := \text{Gal}(L: k(B))$. Thus, the order of σ divides $d!$. Therefore, the order of the image of σ in the Weil–Châtelet group $\text{WC}(J_\eta)$ divides $d!$. Call this element τ . By assumption, τ is an element of $\text{III}_B(J_\eta)$, which is a subgroup of $\text{WC}(J_\eta)$. On the other hand, by the choice of $J \rightarrow B$, the Tate–Shafarevich group contains a non-trivial divisible group. Therefore, for every $n \in \mathbb{N}$, we can find an element ρ_n of order n in it. Let $\pi_n: V_n \rightarrow B$ the minimal elliptic fibration corresponding to ρ_n . Then, by the above discussion, $\limsup d_n = \infty$, where d_n denotes the minimal degree of a multisection of π_n .

Let S be a minimal surface with $\kappa(S) = 1$, and let $\pi: S \rightarrow C$ be the Iitaka fibration. By Kodaira's canonical bundle formula, we may write $K_S \sim_{\mathbb{Q}} \pi^*(K_C + B_C + M_C)$, where $B_C \geq 0$ and M_C can be represented by an effective divisor. Examples 3.1 and 3.2 show that it is not enough to fix $\deg(K_C + B_C + M_C)$ to achieve boundedness of surfaces of Kodaira dimension 1. On the other hand, the discussion above shows that the minimal degree of a multisection is bounded in a family of elliptic fibrations. Therefore, the assumptions of Theorem 7.2 are both necessary and sufficient to achieve boundedness (in codimension 1).

4. Good minimal models

In this section, we prove the existence of good minimal models under some conditions on the Kodaira dimension. The statement generalizes results of Grassi and Wen for n -folds with no

boundary of Kodaira dimension $n - 1$, see [GW22], as well as results of Hao and Schreieder for smooth n -folds of Kodaira dimension at least $n - 3$; see [HS21, Theorem 2.1].

Proof of Theorem 1.5. Let $X \dashrightarrow Y$ be the Iitaka fibration of the pair (X, Δ) , where we have

$$Y = \text{Proj}(\oplus_{l \geq 0} \Gamma(X, \mathcal{O}_X(l(K_X + \Delta)))) .$$

Notice that the ring $\oplus_{l \geq 0} \Gamma(X, \mathcal{O}_X(l(K_X + \Delta)))$ is finitely generated by [BCHM10], so the Proj construction is well defined and provides a normal variety. Let $\pi: X' \rightarrow X$ be a birational morphism so that $f: X' \rightarrow Y$ is a morphism. We may assume that π is a log resolution of (X, Δ) . Let E' denote the reduced π -exceptional divisor, and set $\Delta' := \pi_*^{-1}\Delta$. Then, as (X, Δ) is klt, we may find a $0 < c \ll 1$ so that $(X', \Delta' + (1 - c)E')$ is klt and has the same log canonical ring as (X, Δ) . Furthermore, we may assume that the minimal log discrepancy satisfies $\text{mld}(X, \Delta) > c$. This implies that every prime component of E' lies in the stable base locus of $K_{X'} + \Delta' + (1 - c)E'$.

As argued in the proof of [FM00, Theorem 5.2], the general fiber of $f: (X', \Delta' + (1 - c)E') \rightarrow Y$ is a klt pair of Kodaira dimension 0. Furthermore, by assumption, the dimension of the general fiber is at most 3. Hence, by [KMM94], the general fiber of f has a good minimal model. Therefore, by generic smoothness, [HMX18b, Theorem 1.9.1], and [HX13, Theorem 1.1], it follows that $(X', \Delta' + (1 - c)E')$ has a relative good minimal model over Y . Call it $(X'', \Delta'' + (1 - c)E'')$, and let Y'' be the corresponding ample model. By construction, $Y'' \rightarrow Y$ is a birational morphism.

Since $K_{X''} + \Delta'' + (1 - c)E'' \sim_{\mathbb{Q}, Y''} 0$, we may apply the canonical bundle formula. Thus, a generalized klt pair $(Y'', B'', \mathbf{M})$ is induced on Y'' . Since $K_{Y''} + B'' + \mathbf{M}_{Y''}$ is big, by [BZ16, proof of Lemma 4.4(2)], we may run a $(K_{Y''} + B'' + \mathbf{M}_{Y''})$ -MMP with scaling, which terminates with a good minimal model. Let $(Y^m, B^m, \mathbf{M})$ denote the model thus obtained. By [HX13, Corollary 2.13], we can follow this MMP step by step on X'' . In particular, there is a birational contraction $X'' \dashrightarrow X^m$ such that $K_{X^m} + \Delta^m + (1 - c)E^m \sim_{\mathbb{Q}} g^*(K_{Y^m} + B^m + \mathbf{M}_{Y^m})$, where Δ^m and E^m denote the push-forwards of Δ'' and E'' on X^m , respectively. Furthermore, this contraction preserves the log canonical ring of $(X'', \Delta'' + (1 - c)E'')$. Since $K_{Y^m} + B^m + \mathbf{M}_{Y^m}$ is semi-ample, it follows that $K_{X^m} + \Delta^m + (1 - c)E^m$ is semi-ample. On the other hand, since E' is in the stable base locus of $(X', \Delta' + (1 - c)E')$, it follows that $E^m = 0$. In particular, $X \dashrightarrow X^m$ is a birational contraction.

To conclude, we need to show that $X \dashrightarrow X^m$ is $(K_X + \Delta)$ -negative. First, we notice that since $K_{X^m} + \Delta^m$ is semi-ample and (X^m, Δ^m) has the same log canonical ring as (X, Δ) , it follows that $X \dashrightarrow X^m$ is (X, Δ) -non-positive. Now, let P be any prime divisor on X . Then, by construction, we have $a_P(X, \Delta) = a_P(X', \Delta' + (1 - c)E')$. Notice that, by construction, $X' \dashrightarrow X^m$ is $(X', \Delta' + (1 - c)E')$ -negative. Therefore, if P is contracted by $X' \dashrightarrow X^m$, it follows that $a_P(X^c, \Delta^m) > a_P(X', \Delta' + (1 - c)E')$. Thus, $X \dashrightarrow X^m$ is (X, Δ) -negative. This concludes the proof. $\square$

5. Minimal models with $\kappa(X, \Delta) = \dim(X) - 1$

In this section, we collect some definitions and facts about projective varieties of dimension n and Kodaira dimension $n - 1$.

Remark 5.1. Let (X, Δ) be a minimal projective klt pair of dimension n with $\kappa(X, \Delta) = n - 1$ and rational coefficients. In view of Theorem 1.5 and [Lai11, Proposition 2.4], the Iitaka fibration of (X, Δ) is a morphism $f: X \rightarrow Y$. Furthermore, we have that $K_X + \Delta \sim_{\mathbb{Q}} f^*L$, where L is an ample $\mathbb{Q}$ -Cartier divisor. Then, by the canonical bundle formula recalled in §2.8, the morphism

f induces a generalized pair $(Y, B_Y, \mathbf{M})$ such that $K_Y + B_Y + \mathbf{M}_Y \sim_{\mathbb{Q}} L$. Furthermore, by [FM00] (see also [PS09, Theorem 8.1] and [HMX14]), we may choose $0 \leq \Delta_Y \sim_{\mathbb{Q}} \mathbf{M}_Y$ so that $(Y, B_Y + \Delta_Y)$ is klt and the coefficients of $B_Y + \Delta_Y$ belong to a DCC set of rational numbers depending only on n and $\text{coeff}(\Delta)$. In this setup, we claim that the discrepancies of $(Y, B_Y + \Delta_Y)$ are a lower bound for the generalized discrepancies of $(Y, B_Y, \mathbf{M})$. Indeed, let $\pi: Y' \rightarrow Y$ be a log resolution of (Y, B_Y) where $\mathbf{M}$ descends. We write $K_Y + B_Y + \mathbf{M}_Y = \pi^*(K_Y + B_Y + \mathbf{M}_Y)$. As $\mathbf{M}$ descends on Y , the generalized discrepancies of $(Y, B_Y, \mathbf{M})$ coincide with the discrepancies of the sub-pair $(Y', B_{Y'})$. Then, to define Δ_Y , by [PS09, Theorem 8.1], we choose a suitable effective divisor $0 \leq \Delta_{Y'} \sim_{\mathbb{Q}} \mathbf{M}_{Y'}$ and set $\Delta_Y := \pi_* \Delta_{Y'}$. Thus, the discrepancies of $(Y, B_Y + \Delta_Y)$ coincide with the discrepancies of $(Y', B_{Y'} + \Delta_{Y'})$, and the claim follows as $B_{Y'} \leq B_{Y'} + \Delta_{Y'}$.

DEFINITION 5.2. Let (X, Δ) be a minimal projective klt pair of dimension n with $\kappa(X, \Delta) = n - 1$ and rational coefficients. Let Y and L be as in Remark 5.1. Then, we define $\text{vol}_{n-1}(X, K_X + \Delta) := \text{vol}(Y, L)$.

DEFINITION 5.3. Fix positive integers n and d , a positive real number v , and a finite set $\mathcal{R} \subset \mathbb{Q} \cap [0, 1]$. We define $\mathfrak{D}(n, v, \Phi(\mathcal{R}))$ to be the set of minimal projective klt pairs (X, Δ) of dimension n with $\kappa(X, \Delta) = n - 1$, $\text{vol}_{n-1}(X, K_X + \Delta) = v$, and $\text{coeff}(\Delta) \subset \Phi(\mathcal{R})$. Then, we define $\mathfrak{D}(n, v, \Phi(\mathcal{R}), d) \subset \mathfrak{D}(n, v, \Phi(\mathcal{R}))$ to be the subset consisting of those $(X, \Delta) \in \mathfrak{D}(n, v, \Phi(\mathcal{R}))$ whose Iitaka fibration admits a multisection of degree d . We define $\mathfrak{B}(n, v, \Phi(\mathcal{R}))$ to be the set of pairs $(Y, B_Y + \Delta_Y)$ arising as bases of the Iitaka fibration of the elements of $\mathfrak{D}(n, v, \Phi(\mathcal{R}))$ (that is, $Y = \text{Proj}(\oplus_{l \geq 0} \Gamma(X, \mathcal{O}_X(l(K_X + \Delta))))$). Here, we choose Δ_Y as in Remark 5.1.

PROPOSITION 5.4. *The set of pairs $\mathfrak{B}(n, v, \Phi(\mathcal{R}))$ is bounded.*

Proof. Fix $(Y, B_Y + \Delta_Y) \in \mathfrak{B}(n, v, \Phi(\mathcal{R}))$. Then, by the definition of $\mathfrak{B}(n, v, \Phi(\mathcal{R}))$, we have that $K_Y + B_Y + \Delta_Y$ is ample and $\text{vol}(Y, K_Y + B_Y + \Delta_Y) = v$. Since the coefficients of $B_Y + \Delta_Y$ belong to a DCC set that depends only on n and $\mathcal{R}$, the claim follows from [HMX18a, Theorem 1.1]. $\square$

COROLLARY 5.5. *Fix positive integers n and d , a positive real number v , and a finite set $\mathcal{R} \subset \mathbb{Q} \cap [0, 1]$. Then, there exists an $\epsilon > 0$ depending only on n, d, v , and $\mathcal{R}$ such that if $(X, \Delta) \in \mathfrak{D}(n, v, \Phi(\mathcal{R}))$, it is ϵ -log canonical.*

Proof. Fix $(X, \Delta) \in \mathfrak{D}(n, v, \Phi(\mathcal{R}))$, and let $f: X \rightarrow Y$ be its Iitaka fibration. Let $(Y, B_Y, \mathbf{M})$ and $(Y, B_Y + \Delta_Y)$ be the generalized pair and pair induced on Y , respectively. Since $\mathfrak{B}(n, v, \Phi(\mathcal{R}))$ is a bounded set of klt pairs with coefficients in a finite set, and since the generalized discrepancies of $(Y, B_Y, \mathbf{M})$ are bounded below by the discrepancies of $(Y, B_Y + \Delta_Y)$, there exists an $\epsilon > 0$ depending only on n, d, v , and $\mathcal{R}$ such that $(Y, B_Y, \mathbf{M})$ is generalized ϵ -log canonical. By assumption, the generic fiber is a smooth log Calabi–Yau curve whose boundary has coefficients in $\Phi(\mathcal{R})$. Thus, by the global ascending chain condition (global ACC for short), see [HMX14], the horizontal part of Δ can only attain finitely many values. Let $1 - \delta$ denote the maximum of such values. Notice that we have $0 < \delta \leq 1$ as (X, Δ) is klt. Then, we may assume $\epsilon < \delta$. Thus, (X, Δ) is ϵ -log canonical over the generic point of Y . Indeed, as f has relative dimension 1, the generic fiber of f is smooth, and in a neighborhood of it, the singularities of (X, Δ) are controlled by the coefficients of Δ . In particular, over a non-empty open subset of Y , the pair (X, Δ) is δ -log canonical. Therefore, in order to bound the singularities of (X, Δ) , it suffices to bound the discrepancies of those geometric valuations over X whose centers on X are vertical over Y . By [Amb99, Proposition 3.4], the pair (X, Δ) is ϵ -log canonical. Notice that we can apply [Amb99, Proposition 3.4] to the model Y even though $\mathbf{M}$ does not descend on Y since we are considering the generalized log discrepancies of $(Y, B_Y, \mathbf{M})$. $\square$

DEFINITION 5.6. Fix positive integers n and d , a positive real number v , and a finite set $\mathcal{R} \subset \mathbb{Q} \cap [0, 1]$. We define $\mathfrak{C}(n, v, \Phi(\mathcal{R}), d)$ to be the set of fibrations $(X, \Delta) \rightarrow (Y, B_Y + \Delta_Y)$, where $(X, \Delta) \in \mathfrak{D}(n, v, \Phi(\mathcal{R}), d)$ and $(Y, B_Y + \Delta_Y)$ is the corresponding element in $\mathfrak{B}(n, v, \Phi(\mathcal{R}))$.

6. Iitaka fibration of Fano type

In this section, we prove a stronger version of Theorem 1.1 under the additional assumption that the general fiber of the Iitaka fibration is $\mathbb{P}^1$.

THEOREM 6.1. *Let $\mathfrak{F}(n, v, \Phi(\mathcal{R}))$ be the subset of $\bigcup_{d \in \mathbb{N}} \mathfrak{C}(n, v, \Phi(\mathcal{R}), d)$ consisting of those fibrations whose general fiber is $\mathbb{P}^1$. Then, $\mathfrak{F}(n, v, \Phi(\mathcal{R}))$ is bounded.*

Proof. By Proposition 5.4, the set of pairs $(Y, B_Y + \Delta_Y)$ is bounded. By Corollary 5.5, every (X, Δ) is ϵ -log canonical for a fixed $\epsilon > 0$. Therefore, by [Bir18, Theorem 1.3], the set of pairs (X, Δ) is bounded. Let $(\mathcal{X}, \mathcal{B}) \rightarrow T$ be a bounding family. We recall that the notions of boundedness introduced in § 2.9 only control the support of the boundary divisor Δ and that the divisor $\mathcal{B}$ is reduced. Yet, if we want to retrieve the Iitaka fibration of the pairs (X, Δ) , we need to put the appropriate coefficients to $\mathcal{X}$ to retrieve each (X, Δ) , rather than $(X, \text{Supp}(\Delta))$. A priori, the coefficients of Δ belong to the countable set $\Phi(\mathcal{R})$. If we argue that they actually belong to a finite subset of $\Phi(\mathcal{R})$, then, by considering finitely many copies of $(\mathcal{X}, \mathcal{B}) \rightarrow T$, we can assign to the components of $\mathcal{B}$ all the possible coefficients. But this is indeed the case. By global ACC [HMX14], the coefficients of the horizontal part of Δ belong to a finite subset of $\Phi(\mathcal{R})$. Then, by Proposition 5.4 (note that the possible coefficients appearing therein are finitely many by [HMX18a, Theorem 1.1]) and the algorithm to compute the boundary divisor in the canonical bundle formula, it readily follows that the vertical components of Δ , as well, can attain finitely many coefficients. Thus, up to replacing $(\mathcal{X}, \mathcal{B}) \rightarrow T$ with finitely many copies of itself and assigning coefficients to $\mathcal{B}$, we may assume that for every (X, Δ) , there is a closed point $t \in T$ such that $(\mathcal{X}_t, \mathcal{B}_t) \cong (X, \Delta)$. Furthermore, we may stratify T so that T is smooth and there exists a log resolution of $(\mathcal{X}, \mathcal{B})$ that induces a log resolution fiberwise. In particular, this guarantees that deformation invariance of plurigena applies [HMX18b, Theorem 1.9.2]. Thus, the relative ample model of $(\mathcal{X}, \mathcal{B}) \rightarrow T$ induces the ample model of each fiber; see, for example, [FS22, Theorem 4.5]. Call the relative ample model $\mathcal{Y}$. Thus, the claim of the theorem follows, except for the boundedness of the boundary divisors. The latter is recovered by Corollary 2.12. $\square$

7. Elliptic Iitaka fibration

Recall that $\mathfrak{D}(n, v, \Phi(\mathcal{R}))$ is the set of minimal projective klt pairs (X, Δ) of dimension n with coefficients in $\Phi(\mathcal{R})$ and $\text{vol}_{n-1}(X, \Delta) = v$. In this section, we will only consider elements of $\mathfrak{D}(n, v, \Phi(\mathcal{R}))$ whose Iitaka fibration is an elliptic fibration. Now, consider $(X, \Delta) \in \mathfrak{D}(n, v, \Phi(\mathcal{R}))$, and let $f: X \rightarrow Y$ be its Iitaka fibration. Then, we may consider the corresponding Jacobian fibration $J(X) \rightarrow Y$. This fibration is defined up to birational equivalence over Y , and the generic fiber is $\text{Pic}^0(X_\eta)$, where X_η denotes the generic fiber of f . Indeed, in general, $J(X)$ is a projective variety over Y having $\text{Pic}^0(X_\eta)$ as generic fiber of the morphism $J(X) \rightarrow Y$. Then, depending on the context, some additional assumptions, such as smoothness or minimality over Y , may be required. Notice that neither smoothness nor relative minimality lead to a unique choice of representative of this birational class. For our purposes, we will proceed as follows. Assume that $J(X)$ is smooth and admits a contraction to Y . Then, $J(X)$ admits a relative good minimal model over Y by [HX13, Theorem 1.1]. Replace $J(X)$ with this model, and

let $Y' \rightarrow Y$ be the relative ample model over Y . Then, we may assume that $J(X)$ is terminal, $\mathbb{Q}$ -factorial, and that $K_{J(X)} \sim_{\mathbb{Q}, Y'} 0$. Let $(Y, B_Y, \mathbf{M})$ be the generalized pair induced by X on Y , and let $(Y', B_{Y'}, \mathbf{M})$ denote its trace on Y' . Then, $B_{Y'}$ is effective by Lemma 2.14. Notice that, as $(Y, B_Y + \Delta_Y)$ is bounded and Δ_Y can be chosen generically with bounded denominator [PS09, Theorem 8.1], it follows that the coefficients of $B_{Y'}$ belong to a finite set depending only on $\mathfrak{D}(n, v, \Phi(\mathcal{R}))$. Thus, it follows by [MST20] that $(Y', B_{Y'} + \Delta_{Y'})$ is bounded with coefficients in a finite set. Then, by Lemma 2.14, we have that $J(X)$ induces a generalized pair $(Y', C_{Y'}, \mathbf{M})$, where $0 \leq C_{Y'} \leq B_{Y'}$. By [FM00], the coefficients of $C_{Y'}$ belong to a DCC set whose only accumulation point is 1. As they are bounded away from 1 by the coefficients of $B_{Y'}$, it follows that $\text{coeff}(C_{Y'})$ varies in a finite set depending only on $\mathfrak{D}(n, v, \Phi(\mathcal{R}))$.

DEFINITION 7.1. Fix positive integers n and d , a positive real number v , and a finite set of rational numbers $\mathcal{R} \subset \mathbb{Q} \cap [0, 1]$. We define $\mathfrak{J}(n, v, \Phi(\mathcal{R}))$ to be the set of Jacobian fibrations constructed above. That is, $J(X) \in \mathfrak{J}(n, v, \Phi(\mathcal{R}))$ is a terminal, $\mathbb{Q}$ -factorial model of the Jacobian fibration of $(X, \Delta) \in \mathfrak{D}(n, v, \Phi(\mathcal{R}))$. Furthermore, $J(X)$ is relatively minimal over Y , the base of the Iitaka fibration of X , with relative ample model $Y' \rightarrow Y$. Then, we define $\mathfrak{B}(n, v, \Phi(\mathcal{R}))$ to be the set of fibrations $J(X) \rightarrow (Y', B_{Y'} + \Delta_{Y'})$, where $J(X) \in \mathfrak{J}(n, v, \Phi(\mathcal{R}))$ and $(Y', B_{Y'} + \Delta_{Y'})$ is the trace on Y' of the corresponding element in $\mathfrak{B}(n, v, \Phi(\mathcal{R}))$.

THEOREM 7.2. Fix positive integers n and d , a positive real number v , and a finite set of rational numbers $\mathcal{R} \subset \mathbb{Q} \cap [0, 1]$. The subset of $\mathfrak{C}(n, v, \Phi(\mathcal{R}), d)$ consisting of elliptic fibrations is bounded in codimension 1. Let $(\mathcal{X}, \mathcal{D})$, $(\mathcal{Y}, \mathcal{D})$, and T be varieties bounding these fibrations in codimension 1 as in Definition 2.6. Then, $(\mathcal{Y}, \mathcal{D}) \rightarrow T$ bounds (that is, not just birationally in codimension 1) the set of bases of the fibrations of interest.

Remark 7.3. Notice that if $n = 2$, it follows that the family of elliptic fibrations in Theorem 7.2 is actually bounded. Indeed, two normal projective surfaces that are isomorphic in codimension 1 are actually isomorphic.

PROPOSITION 7.4. Fix positive integers n and d , a positive real number v , and a finite set of rational numbers $\mathcal{R} \subset \mathbb{Q} \cap [0, 1]$. Assume $n \geq 3$. Assume that Theorem 7.2 holds for $\mathfrak{C}(m, w, \Phi(\mathcal{S}), e)$, where $m \leq n - 1$, w , e and the finite set $\mathcal{S} \subset \mathbb{Q} \cap [0, 1]$ are arbitrary. Then, there exists a positive integer C , depending only on $\mathfrak{C}(n, v, \Phi(\mathcal{R}), d)$, such that for every $(X, \Delta) \rightarrow (Y, B_Y + \Delta_Y) \in \mathfrak{C}(n, v, \Phi(\mathcal{R}), d)$, the linear series $|C(K_Y + B_Y + \Delta_Y)|$ is very ample and there exists an element $D \in |C(K_Y + B_Y + \Delta_Y)|$ such that $Y_{\text{sing}} \subset \text{Supp}(D)$ and $X \rightarrow Y$ is smooth over $Y \setminus \text{Supp}(D)$, except possibly at finitely many isolated points.

Proof. By Proposition 5.4, the set $\mathfrak{B}(n, v, \Phi(\mathcal{R}))$ is bounded. Fix

$$f: (X, \Delta) \rightarrow (Y, B_Y + \Delta_Y) \in \mathfrak{C}(n, d, \Phi(\mathcal{R}), v).$$

By Proposition 5.4, there is a positive integer C' , depending only on n and v , such that $C'(K_Y + B_Y + \Delta_Y)$ is a very ample Cartier divisor. Pick a general $H \in |C'(K_Y + B_Y + \Delta_Y)|$, define $X_H := f^*H$, and let (X_H, Δ_{X_H}) be the pair obtained by adjunction. As H is general, it follows that $\text{coeff}(\Delta_{X_H}) \subset \Phi(\mathcal{R})$. By adjunction, we define $(H, B_H + \Delta_H)$ and $(H, B_H, \mathbf{N})$, where $\mathbf{N} := \mathbf{M}|_H$. Notice that $\Delta_H \sim_{\mathbb{Q}} \mathbf{N}_H$. Then, arguing as in [Flo14, Lemma 3.1] (notice that the smoothness assumptions in the reference are not needed and it suffices that H is chosen generically in a free linear series as the computations can be done on a log resolution of Y and then pushed forward to Y and H), it follows that $(H, B_H, \mathbf{N})$ is the generalized pair induced by $X_H \rightarrow H$ via the canonical bundle formula. Furthermore, $(X_H, \Delta_{X_H}) \rightarrow (H, B_H + \Delta_H) \in \mathfrak{C}(n - 1, w, \Phi(\mathcal{R}), d)$, where w

depends only on $\mathfrak{C}(n, v, \Phi(\mathcal{R}), d)$. Notice that in this argument, we are using the assumptions on the singularities of X and H to control the behavior of the different [Kol13, Proposition 4.5].

Let Y_{sing} and H_{sing} denote the singular loci of Y and H , respectively. Then, by Bertini's theorem, we have $H_{\text{sing}} = Y_{\text{sing}} \cap H$. Define $Y_{\text{sm}} := Y \setminus Y_{\text{sing}}$ and $H_{\text{sm}} := H \setminus H_{\text{sing}}$. Let $\Sigma_{Y_{\text{sm}}}$ denote the minimal closed subset of Y_{sm} such that $X \rightarrow Y$ is a smooth fibration with smooth base over $Y_{\text{sm}} \setminus \Sigma_{Y_{\text{sm}}}$. Let Σ_Y denote the closure of $\Sigma_{Y_{\text{sm}}}$ in Y . Since H is general and Σ_Y is independent of the choice of H , we may assume that H meets properly every irreducible component of Σ_Y . Then, we let $\Sigma_{H_{\text{sm}}}$ denote the minimal closed subset of H_{sm} such that $X_H \rightarrow H$ is a smooth fibration with smooth base over $H_{\text{sm}} \setminus \Sigma_{H_{\text{sm}}}$, and we let Σ_H denote its closure in H . By construction, we have $\Sigma_{H_{\text{sm}}} = \Sigma_{Y_{\text{sm}}} \cap H_{\text{sm}}$. This is clear over the locus where the fibers are not 1-dimensional. Then, over the locus in Y_{sm} where the fibers are 1-dimensional, the morphism is flat by [Sta20, Tag 00R4]. Thus, over this locus, by [Sta20, Tag 01V9], the morphism is smooth if and only if so is the restriction to H_{sm} , as the fibers over closed points agree scheme-theoretically. Hence, we have $\Sigma_H = \Sigma_Y \cap H$. Notice that, by Lemmas 2.15 and 2.16, these loci are well defined, even though we may need to replace X and X_H with models that are isomorphic in codimension 1.

By the inductive hypothesis, Theorem 7.2 applies to $\mathfrak{C}(n - 1, w, \Phi(\mathcal{R}), d)$. Let $(\mathcal{X}_{\mathcal{H}}, \mathcal{B}_{\mathcal{H}}) \rightarrow (\mathcal{H}, \mathcal{D}) \rightarrow T$ be the corresponding family of fibrations. Up to a stratification, we may assume that T is smooth. Then, we may assume that the singularities of the fibers of $\mathcal{H} \rightarrow T$ are induced by the singularities of $\mathcal{H}$ itself. That is, we may assume $\mathcal{H}_{t,\text{sing}} = (\mathcal{H}_{\text{sing}})_t$. Let $\Sigma_{\mathcal{H}_{\text{sm}}}$ be the minimal closed subset of $\mathcal{H}_{\text{sm}}$ such that $\mathcal{X}_{\mathcal{H}} \rightarrow \mathcal{H}$ is a smooth fibration over $\mathcal{H}_{\text{sm}} \setminus \Sigma_{\mathcal{H}_{\text{sm}}}$. By a stratification, we may assume $(\Sigma_{\mathcal{H}_{\text{sm}}})_t = \Sigma_{\mathcal{H}_{t,\text{sm}}}$. Similarly, we have $(\Sigma_{\mathcal{H}})_t = \Sigma_{\mathcal{H}_t}$, where $\Sigma_{\mathcal{H}}$ and $\Sigma_{\mathcal{H}_t}$ denote the closures of $\Sigma_{\mathcal{H}_{\text{sm}}}$ in $\mathcal{H}$ and of $\Sigma_{\mathcal{H}_{t,\text{sm}}}$ in $\mathcal{H}_t$, respectively.

By [Kol13, Claim 4.38.1], up to a stratification and a base change, we may assume that the restrictions of the irreducible components of $\mathcal{B}_{\mathcal{H}}$ and $\mathcal{D}$ to the fibers of the corresponding morphisms to T are irreducible. By Corollary 5.5, the coefficients of Δ_{X_H} can attain finitely many values. Thus, as the coefficients of Δ_{X_H} and $B_H + \Delta_H$ take finitely many values, we may assume that $\mathcal{B}_{\mathcal{H}}$ restricts to Δ_{X_H} fiberwise. Similarly, we may assume that $\mathcal{D}$ restricts to $B_H + \Delta_H$. Then, by [HX15, Proposition 2.4] and Corollary 5.5, up to a stratification, we may assume that $K_{\mathcal{X}_{\mathcal{H}}} + \mathcal{B}_{\mathcal{H}}$ and $K_{\mathcal{H}} + \mathcal{D}$ are $\mathbb{Q}$ -Cartier.

By generic flatness, up to a stratification of T , we may assume that $\mathcal{D}$ and every irreducible component of $\Sigma_{\mathcal{H}}$ are flat over T . Now, write $T = \sqcup T_i$, where each T_i is irreducible, and let $\mathcal{X}_{\mathcal{H},i}$ and $\mathcal{H}_i$ be the corresponding irreducible components of $\mathcal{X}_{\mathcal{H}}$ and $\mathcal{H}$, respectively. Fix an irreducible component Ξ_i of $\Sigma_{\mathcal{H}_i}$ of dimension $\dim(T_i) + k$. Recall that $\dim(X) = n$. Therefore, we have $\dim(H) = n - 2$. Then, for every $t \in T_i$, we have

$$(K_{\mathcal{H},t} + \mathcal{D}_t)^k \cdot \Xi_{i,t} = (K_{\mathcal{H}} + \mathcal{D})^k \cdot \Xi_i \cdot \mathcal{H}_t, \quad (7.1)$$

and this expression is independent of $t \in T_i$. Thus, the intersection products between $K_{\mathcal{H},t} + \mathcal{D}_t$ and the irreducible components of $\Sigma_{\mathcal{H},t}$ are locally constant. In particular, they are bounded.

Now, recall that for every pair $(H, B_H + \Delta_H)$ appearing as a fiber, we have

$$K_{\mathcal{H},t} + \mathcal{D}_t = K_H + B_H + \Delta_H.$$

By construction, there is a $C' \in \mathbb{N}$ depending only on n, v , and $\mathcal{R}$ so that $H \sim C'(K_Y + B_Y + \Delta_Y)$. For brevity, set $A_Y := C'(K_Y + B_Y + \Delta_Y)$. Notice that A_Y is a very ample polarization. Now, fix an irreducible component Ω of Σ_Y of dimension $l \geq 1$, and set $\Omega_H := \Omega \cap H$. Then, by the

construction of Σ_Y and Σ_H , we have

$$\Omega \cdot A^l = \Omega \cdot H \cdot A^{l-1} = \Omega_H \cdot A_H^{l-1} = (C')^{l-1} \Omega_H \cdot (K_H + B_H + \Delta_H)^{l-1},$$

and the right-hand side is bounded, by (7.1) and the fact that $K_H + B_H + \Delta_H \sim_{\mathbb{Q}} K_{\mathcal{H},t} + \mathcal{D}_t$ for some $t \in T$. Thus, the positive-dimensional irreducible components of Σ_Y have bounded degree, and they are bounded in number. Let $M \in \mathbb{N}$ be a positive integer that bounds both of these quantities.

Now, recall that the degree with respect to A of a subvariety of X is the same as the degree of the same variety with respect to the hyperplane class of $\mathbb{P}(H^0(X, \mathcal{O}_X(A)))$. Thus, every irreducible component Ω of Σ_Y of positive dimension is a subvariety of $\mathbb{P}(H^0(X, \mathcal{O}_X(A)))$ of degree at most M . So, set-theoretically, Ω is the intersection of hypersurfaces of degree at most M . Choosing one of these hypersurfaces generically so that it does not contain X , it follows that there is a $D_\Omega \in |MA|$ such that $\Omega \subset \text{Supp}(D_\Omega)$. Thus, we may find an element $D \in |M^2 A|$ whose support contains the positive-dimensional part of Σ_Y . Notice that the degree of the singular locus of Y is bounded as Y comes in a bounded family. Therefore, we may also assume that $\text{Supp}(D)$ contains the singular locus of Y . The claim follows. $\square$

Remark 7.5. In the proof of Proposition 7.4, we may also assume $\text{Supp}(B_Y + \Delta_Y) \subset \text{Supp}(D)$. Indeed, as these divisors are bounded as the cycle Σ_Y is, we may find a bounded multiple of $K_Y + B_Y + \Delta_Y$ such that some element of the corresponding linear series vanishes along these divisors.

Remark 7.6. Fix $X \in \mathfrak{C}(n, v, \Phi(\mathcal{R}), d)$ and the associated $(Y, B_Y + \Delta_Y + D)$, where D is as in Proposition 7.4. Also, fix $J(X)$ as discussed above. Consider $(Y', B_{Y'} + \Delta_{Y'})$ as in Definition 7.1, and let D' be the pull-back of D to D' . Since $(Y', B_{Y'} + \Delta_{Y'})$ is bounded and $0 \leq D' \sim_{\mathbb{Q}} C(K_{Y'} + B_{Y'} + \Delta_{Y'})$, where C depends only on n, v, d , and $\mathcal{R}$, it follows that $(Y', B_{Y'} + \Delta_{Y'} + D')$ is bounded. Then, we can regard $J(X) \rightarrow (Y', B_{Y'} + \Delta_{Y'} + D')$ as an element of $\mathfrak{P}(n, v, \Phi(\mathcal{R}))$ together with the datum of D' . Notice that D' depends on $\mathfrak{C}(n, v, \Phi(\mathcal{R}), d)$ and not just on $\mathfrak{C}(n, v, \Phi(\mathcal{R}))$.

DEFINITION 7.7. We define $\mathfrak{P}(n, v, \Phi(\mathcal{R}), d)$ to be the set of fibrations $J(X) \rightarrow (Y', B_{Y'} + \Delta_{Y'} + D')$, where $J(X) \rightarrow (Y', B_{Y'} + \Delta_{Y'}) \in \mathfrak{P}(n, d, \Phi(\mathcal{R}))$ and D' is the pull-back of D , which is as in Proposition 7.4.

THEOREM 7.8. Fix positive integers n and d , a positive real number v , and a finite set $\mathcal{R} \subset \mathbb{Q} \cap [0, 1]$. Assume that Theorem 7.2 holds true in dimension strictly less than n . Then, the set of fibrations $\mathfrak{P}(n, v, \Phi(\mathcal{R}), d)$ is bounded in codimension 1. Furthermore, we may choose the birationally bounding family so that the bases of the fibrations are actually bounded.

Remark 7.9. Notice that the inductive assumption in Theorem 7.2 is only needed for the existence of the divisor D' .

Proof. Fix a fibration $j: J(X) \rightarrow (Y', B_{Y'} + \Delta_{Y'} + D') \in \mathfrak{P}(n, v, \Phi(\mathcal{R}), d)$. Then, by construction, we have $K_{J(X)} \sim_{\mathbb{Q}} j^*(K_{Y'} + C_{Y'} + \Delta_{Y'})$, where $0 \leq C_{Y'} \leq B_{Y'}$. By Remark 7.6, the pairs of the form $(Y', B_{Y'} + \Delta_{Y'} + D')$ are bounded. Since $(Y', B_{Y'} + \Delta_{Y'})$ is bounded and klt with coefficients in a finite set, we may find a $0 < c \ll 1$ depending only on $\mathfrak{C}(n, v, \Phi(\mathcal{R}), d)$ so that the pairs of the form $(Y', B_{Y'} + \Delta_{Y'} + cD')$ are klt. This can be achieved by taking a bounding family for the pairs $(Y', B_{Y'} + \Delta_{Y'} + D')$, where we prescribe the coefficients of the bounding divisors to restrict to $B_{Y'} + \Delta_{Y'}$. Notice that, by the construction of D in Proposition 7.4, the divisor $K_{Y'} + B_{Y'} + \Delta_{Y'} + cD'$ is semi-ample and big with fixed volume. Now, we divide the proof into several steps. We follow the proof of [DS21, Theorem 1.1].

Step 1: In this step, we reduce to the case when $J(X)$ maps to a $\mathbb{Q}$ -factorialization $Y^\mathbb{Q}$ of Y' .

Since $(Y', B_{Y'} + \Delta_{Y'} + cD')$ is klt, Y' admits a small $\mathbb{Q}$ -factorialization $(Y^\mathbb{Q}, B_Y^\mathbb{Q} + \Delta_Y^\mathbb{Q} + cD^\mathbb{Q}) \rightarrow (Y', B_{Y'} + \Delta_{Y'} + cD')$. By [MST20], the pair $(Y^\mathbb{Q}, B_Y^\mathbb{Q} + \Delta_Y^\mathbb{Q} + cD^\mathbb{Q})$ also belongs to a bounded family which depends only on $\mathfrak{C}(n, v, \Phi(\mathcal{R}), d)$. Up to replacing $J(X)$ with a model that is isomorphic in codimension 1, by Proposition 2.9, we may assume that $J(X) \rightarrow Y'$ factors through $Y^\mathbb{Q}$.

Step 2: In this step, we reduce to the case when the rational section satisfies certain positivity assumptions.

Now, denote by $\hat{Y}$ the closure of the rational section of $j: J(X) \rightarrow Y'$. Then, $\hat{Y}$ is relatively big over $Y^\mathbb{Q}$. Also, for $0 < \gamma \ll 1$, the pair $(J(X), \gamma\hat{Y})$ is klt. Thus, by [BCHM10], any $(K_{J(X)} + \gamma\hat{Y})$ -MMP over $Y^\mathbb{Q}$ with scaling of an ample divisor terminates. Let $(\tilde{J}(X), \gamma\tilde{Y})$ be the resulting model. Denote by $\tilde{j}: \tilde{J}(X) \rightarrow Y^\mathbb{Q}$ the resulting morphism. Notice that $K_{J(X)} + \gamma\hat{Y} \sim_{\mathbb{Q}, Y^\mathbb{Q}} \gamma\hat{Y}$. Hence, this MMP is independent of γ , and $\tilde{Y}$ is relatively big and semi-ample over $Y^\mathbb{Q}$. Furthermore, since $\hat{Y}$ is irreducible and dominates $Y^\mathbb{Q}$, every step of the above MMP has to be a $(K_{J(X)} + \gamma\hat{Y})$ -flip. Thus, $\tilde{J}(X)$ is isomorphic to $J(X)$ in codimension 1. Moreover, as $K_{J(X)} \sim_{\mathbb{Q}, Y^\mathbb{Q}} 0$, the terminality of $J(X)$ implies that of $\tilde{J}(X)$. Hence, up to relabeling, we may assume $(J(X), \hat{Y}) = (\tilde{J}(X), \tilde{Y})$.

Step 3: In this step, we show that $(J(X), \hat{Y})$ is a purely log terminal (plt for short) pair. This implies that $\hat{Y}$ is a normal variety admitting a structure of klt pair.

The normality of $\hat{Y}$ and the existence of a structure of klt pair on $\hat{Y}$ will follow from the plt-ness of $(J(X), \hat{Y})$ by [KM98, Proposition 5.51] and inversion of adjunction; see [Kaw07]. To show that $(J(X), \hat{Y})$ is plt, it suffices to show that $(J(X), \hat{Y})$ is log canonical and that $\hat{Y}$ is its only log canonical center. Let $\phi: \hat{Y}^\nu \rightarrow \hat{Y}$ be the normalization of $\hat{Y}$, and let $\text{Diff}(0)$ be the different defined by

$$K_{\hat{Y}^\nu} + \text{Diff}(0) := \phi^*((K_{J(X)} + \hat{Y})|_{\hat{Y}}).$$

By construction, $K_{\hat{Y}^\nu} + \text{Diff}(0)$ is nef and big over $Y^\mathbb{Q}$. By [DS21, Lemma 5.1], the different $\text{Diff}(0)$ is exceptional over $Y^\mathbb{Q}$. Thus, we have $(\hat{j} \circ \phi)_*(K_{\hat{Y}^\nu} + \text{Diff}(0)) = K_{Y^\mathbb{Q}}$, where we set $\hat{j}: J(X) \rightarrow Y^\mathbb{Q}$. Since $Y^\mathbb{Q}$ is $\mathbb{Q}$ -factorial, the negativity lemma [KM98, Lemma 3.39] implies that

$$K_{\hat{Y}^\nu} + \text{Diff}(0) = (\hat{j} \circ \phi)^* K_{Y^\mathbb{Q}} - F, \quad (7.2)$$

where $F \geq 0$ is $(\hat{j} \circ \phi)$ -exceptional. As $(Y^\mathbb{Q}, B_Y^\mathbb{Q} + \Delta_Y^\mathbb{Q} + cD^\mathbb{Q})$ is klt, so is $(Y^\mathbb{Q}, 0)$. Therefore, it follows from (7.2) that $(\hat{Y}^\nu, \text{Diff}(0))$ is klt. Inversion of adjunction implies that $\hat{Y}$ is the only log canonical center of $(J(X), \hat{Y})$. In particular, $(J(X), \hat{Y})$ is plt, and the other conclusions follow as indicated above.

Step 4: In this step, we show that there exists an effective divisor $\hat{G}$ on $J(X)$ such that the pair $(J(X), \frac{1}{2}\hat{Y} + \frac{1}{2}\hat{G})$ is $\frac{1}{2}$ -log canonical and $K_{J(X)} + \frac{1}{2}\hat{Y} + \frac{1}{2}\hat{G}$ is big.

Let H' be a very ample polarization of bounded degree on Y' . Notice that its existence is guaranteed by the boundedness of the pairs $(Y', B_{Y'} + \Delta_{Y'} + cD')$. We may assume that $H' \pm (K_{Y'} + B_{Y'} + \Delta_{Y'})$ and $H' \pm (K_{Y'} + C_{Y'} + \Delta_{Y'})$ are ample. Since D' is $\mathbb{Q}$ -linearly equivalent to $K_Y + C_Y + \Delta_Y$ up to a bounded multiple, we may also assume that $H' \pm D'$ is ample. Furthermore, we may assume $H^0(Y', \mathcal{O}_{Y'}(H' - \text{Supp}(B_{Y'} + \Delta_{Y'}))) \neq 0$. Let $\hat{G}$ be a general member of $|(2n+3)j^*H'|$. Then, $(J(X), \hat{Y} + \hat{G})$ is log canonical. On the other hand, $J(X)$ is terminal, and the discrepancies of valuations are linear functions of the boundary divisor of a pair. Hence, it follows that $(J(X), \frac{1}{2}\hat{Y} + \frac{1}{2}\hat{G})$ is $\frac{1}{2}$ -log canonical. Since $K_{J(X)} + j^*H'$ is the

pull-back of an ample divisor on Y' and $\hat{Y}$ is effective and relatively big over Y' , it follows that $K_{J(X)} + \frac{1}{2}\hat{Y} + \frac{1}{2}\hat{G}$ is big. Since we have

$$K_{J(X)} + \frac{1}{2}\hat{Y} + \frac{1}{2}\hat{G} \sim_{\mathbb{Q}} \frac{1}{2}(K_{J(X)} + j^*H') + \frac{1}{2}(K_{J(X)} + \hat{Y} + (2n+2)j^*H')$$

and $K_{J(X)} + j^*H'$ is nef, it suffices to show that $K_{J(X)} + \hat{Y} + (2n+2)j^*H'$ is nef to conclude that the same holds for $K_{J(X)} + \frac{1}{2}\hat{Y} + \frac{1}{2}\hat{G}$. The nefness of $K_{J(X)} + \hat{Y} + (2n+2)j^*H'$ follows by the boundedness of the negative extremal rays [Fuj14, Theorem 1.19]. Indeed, let R be a $(K_{J(X)} + \hat{Y})$ -negative extremal ray. There exists a rational curve C spanning R such that we have $-2n \leq (K_{J(X)} + \hat{Y}) \cdot C < 0$. Since $K_{J(X)} + \hat{Y}$ is nef relatively to Y' , we then have that $j(C)$ is a curve. In particular, we have $(2n+2)j^*H' \cdot C \geq (2n+2)H' \cdot j(C) \geq 2n+2$. So, it follows that $K_{J(X)} + \hat{Y} + (2n+2)j^*H'$ is non-negative on every $(K_{J(X)} + \hat{Y})$ -negative extremal ray. Thus, $K_{J(X)} + \hat{Y} + (2n+2)j^*H'$ is nef. In particular, $K_{J(X)} + \frac{1}{2}\hat{Y} + \frac{1}{2}\hat{G}$ is nef and big.

Step 5: In this step, we show that there exist positive constants C_1 and C_2 , depending only on $\mathfrak{C}(n, v, \Phi(\mathcal{R}), d)$, such that $C_1 \leq (K_{J(X)} + \frac{1}{2}\hat{Y} + \frac{1}{2}\hat{G})^n \leq C_2$.

The existence of C_1 follows from [HMX14, Theorem 1.3]. Thus, we are left to show the existence of C_2 .

Up to a rescaling factor depending only on n , we need to consider the following quantity:

$$2K_{J(X)} \cdot (2K_{J(X)} + \hat{Y} + \hat{G})^{n-1} + \hat{Y} \cdot (2K_{J(X)} + \hat{Y} + \hat{G})^{n-1} + \hat{G} \cdot (2K_{J(X)} + \hat{Y} + \hat{G})^{n-1}. \quad (7.3)$$

By the choice of H' in Step 4, the divisor $\hat{G} - 2K_{J(X)}$ is semi-ample. Thus, as $2K_{J(X)} + \hat{Y} + \hat{G}$ is nef, in order to bound $2K_{J(X)} \cdot (2K_{J(X)} + \hat{Y} + \hat{G})^{n-1}$, it suffices to bound $\hat{G} \cdot (2K_{J(X)} + \hat{Y} + \hat{G})^{n-1}$. Hence, we are left with finding an upper bound for the second and third summands in (7.3).

By (7.2) and adjunction, we have $(K_{J(X)} + \hat{Y})|_{\hat{Y}} = (\hat{j}|_{\hat{Y}})^* K_{Y^{\mathbb{Q}}} - F$, where $F \geq 0$ is $\hat{j}|_{\hat{Y}}$ -exceptional. Thus, we get

$$\begin{aligned} \hat{Y} \cdot (2K_{J(X)} + \hat{Y} + \hat{G})^{n-1} &= ((K_{J(X)} + \hat{Y})|_{\hat{Y}} + K_{J(X)}|_{\hat{Y}} + \hat{G}|_{\hat{Y}})^{n-1} \\ &= \text{vol}(\hat{Y}, (K_{J(X)} + \hat{Y})|_{\hat{Y}} + K_{J(X)}|_{\hat{Y}} + \hat{G}|_{\hat{Y}}) \\ &\leq \text{vol}(\hat{Y}, (\hat{j}|_{\hat{Y}})^* K_{Y^{\mathbb{Q}}} + K_{J(X)}|_{\hat{Y}} + \hat{G}|_{\hat{Y}}) \\ &\leq \text{vol}(Y', (2n+5)H') \\ &= (2n+5)^{n-1}(H')^{n-1}, \end{aligned} \quad (7.4)$$

where the last quantity is bounded as n is fixed and H' is the fixed very ample polarization of bounded degree Y' . In (7.4), the second inequality follows from (7.2), while the third inequality follows from the projection formula and the fact that $H' - (K_{Y'} + B_{Y'} + \Delta_{Y'})$ is ample.

Thus, we are left with bounding the summand $\hat{G} \cdot (2K_{J(X)} + \hat{Y} + \hat{G})^{n-1}$. For an integer $1 \leq k \leq n-1$, we have

$$\begin{aligned} \hat{G}^k \cdot (2K_{J(X)} + \hat{Y} + \hat{G})^{n-k} &= \hat{G}^k \cdot (2K_{J(X)} + \hat{Y} + \hat{G}) \cdot (2K_{J(X)} + \hat{Y} + \hat{G})^{n-k-1} \\ &\leq \hat{G}^k \cdot (\hat{Y} + 2\hat{G}) \cdot (2K_{J(X)} + \hat{Y} + \hat{G})^{n-k-1} \\ &= 2\hat{G}^{k+1} \cdot (2K_{J(X)} + \hat{Y} + \hat{G})^{n-k-1} \\ &\quad + \hat{G}^k \cdot \hat{Y} \cdot (2K_{J(X)} + \hat{Y} + \hat{G})^{n-k-1}, \end{aligned} \quad (7.5)$$

where the inequality follows from the fact that, by construction, $\hat{G} - 2K_{J(X)}$ is semi-ample. Since $\hat{G}^n = 0$, by iteratively applying (7.5) $n-1$ times, bounding the intersection number $\hat{G} \cdot (2K_{J(X)} + \hat{Y} + \hat{G})^{n-1}$ reduces to bounding $\hat{G}^k \cdot \hat{Y} \cdot (2K_{J(X)} + \hat{Y} + \hat{G})^{n-k-1}$ for $1 \leq k \leq n-1$.

This can be achieved by (7.2) and the fact that $H' - (K_{Y'} + B_{Y'} + \Delta_{Y'})$ is ample as in the treatment of (7.4). This concludes the step.

Step 6: In this step, we show that the birational representative of the Jacobian fibration $J(X)$ chosen at the end of Step 2 is bounded.

As showed in the previous steps, $(J(X), \frac{1}{2}\hat{Y} + \frac{1}{2}\hat{G})$ is $\frac{1}{2}$ -log canonical, and its coefficients belong to the set $\{\frac{1}{2}\}$. Thus, by [Fil20, Theorem 1.3], the volume $\text{vol}(K_{J(X)} + \frac{1}{2}\hat{Y} + \frac{1}{2}\hat{G})$ belongs to a discrete set depending only on $\mathfrak{C}(n, v, \Phi(\mathcal{R}), d)$. By Step 5, this volume is also bounded from above and below. Thus, we conclude that $\text{vol}(K_{J(X)} + \frac{1}{2}\hat{Y} + \frac{1}{2}\hat{G})$ attains only finitely many values, depending only on $\mathfrak{C}(n, v, \Phi(\mathcal{R}), d)$. Then, by [MST20, Theorem 6], the set of pairs $(J(X), \frac{1}{2}\hat{Y} + \frac{1}{2}\hat{G})$ is bounded. In particular, the varieties $J(X)$ are bounded. Notice that the boundedness of this specific model guarantees the boundedness in codimension 1 of any other model with the properties required at the beginning of §7 since the distinguished model chosen at the end of Step 2 is isomorphic in codimension 1 to any arbitrary model taken as input at the beginning of this proof.

Step 7: In this step, we conclude the proof.

Since the boundary divisor $\hat{G}$ is bounded, the intersection between a fixed very ample polarization and $\hat{G}$ is bounded. Now, we consider j^*D' . By the choice of H' in Step 4, the divisor $H' - D'$ is ample. Thus, we have that the intersection between any polarization on $J(X)$ and $\text{Supp}(j^*D')$ is bounded above by the intersection with $\hat{G}$. Similarly, as $H^0(Y', \mathcal{O}_{Y'}(H' - \text{Supp}(B_{Y'} + \Delta_{Y'}))) \neq 0$, the divisor $H^{\mathbb{Q}} - \text{Supp}(B_{Y^{\mathbb{Q}}} + \Delta_{Y^{\mathbb{Q}}})$ is linearly equivalent to an effective divisor. Thus, we can bound $\text{Supp}(\hat{j}^*(B_{Y^{\mathbb{Q}}} + \Delta_{Y^{\mathbb{Q}}}))$. Hence, the pairs $(J(X), \hat{Y} + \hat{G} + \hat{j}^*(B_{Y^{\mathbb{Q}}} + \Delta_{Y^{\mathbb{Q}}} + D^{\mathbb{Q}}))$ are bounded, where we choose the birational representative of $J(X)$ according to the previous step. Choose a bounding family $(\mathcal{J}, \mathcal{E}) \rightarrow U$, where the reduced divisor $\mathcal{E}$ bounds the support of all the above divisors. That is, if the closed point $u \in U$ corresponds to $(J(X), \hat{Y} + \hat{G} + \hat{j}^*(B_{Y^{\mathbb{Q}}} + \Delta_{Y^{\mathbb{Q}}} + D^{\mathbb{Q}}))$, we have $(J(X), \text{Supp}(\hat{Y} + \hat{G} + \hat{j}^*(B_{Y^{\mathbb{Q}}} + \Delta_{Y^{\mathbb{Q}}} + D^{\mathbb{Q}}))) \cong (\mathcal{J}_u, \mathcal{E}_u)$.

By Step 4, $K_{J(X)} + \frac{1}{2}\hat{G}$ is $\mathbb{Q}$ -linearly equivalent to the pull-back of an ample divisor on Y' . The pairs of the form $(J(X), \frac{1}{2}\hat{G})$ are bounded and klt. Thus, up to stratifying U so that a log resolution of $(\mathcal{J}, \mathcal{E})$ induces a log resolution of the fibers of $\mathcal{J} \rightarrow U$, we may apply the conclusions of [HMX18a, Corollary 1.4]. Furthermore, up to a base change, we may assume that the restrictions of the irreducible components of $\mathcal{E}$ to the fibers are irreducible [Kol13, Claim 4.38.1]. Thus, we can define the divisor $\mathcal{G}$, which is supported on $\mathcal{E}$ and restricts to the divisor $\hat{G}$ on the fibers of the family. Up to a stratification of the family [HX15, Proposition 2.4], the divisor $K_{\mathcal{J}} + \frac{1}{2}\mathcal{G}$ is $\mathbb{Q}$ -Cartier. Then, the relative ample model of $K_{\mathcal{J}} + \frac{1}{2}\mathcal{G}$ induces the ample model of each fiber. Thus, the family $\mathcal{J} \rightarrow U$ factors as $\mathcal{J} \rightarrow \mathcal{Y}' \rightarrow U$, where the family $\mathcal{Y}' \rightarrow U$ bounds the surfaces Y' .

Let $\mathcal{F}$ denote the reduced divisorial part of the image of $\mathcal{E}$ in $\mathcal{Y}$. By construction, the components of $\mathcal{E}$ corresponding to $\hat{j}^*(B_{Y^{\mathbb{Q}}} + \Delta_{Y^{\mathbb{Q}}} + D^{\mathbb{Q}})$ induce components of $\mathcal{E}$ corresponding to $B_{Y'} + \Delta_{Y'} + D'$. The claim follows by considering the family $\mathcal{J} \rightarrow (\mathcal{Y}, \mathcal{F}) \rightarrow U$. $\square$

Remark 7.10. By Step 7 in the proof of Theorem 7.8, the rational sections $\hat{Y}$ are bounded together with the fibrations $J(X) \rightarrow (Y, B_Y + \Delta_Y + D)$. Thus, for every irreducible component $U_i \subset U$, the corresponding fibration $\mathcal{J}_i \rightarrow \mathcal{Y}_i$ is an elliptic fibration with a rational section. Furthermore, the rational section restricts to the rational section of $J(X) \rightarrow Y$ for every element of $\mathfrak{P}(n, v, \Phi(\mathcal{R}), d)$.

LEMMA 7.11. *Let X be an element of $\mathfrak{C}(n, v, \Phi(\mathcal{R}), d)$, and let $(Y, B_Y + \Delta_Y)$ be the associated pair. Furthermore, let D be as in Proposition 7.4 and $(Y', B_{Y'} + \Delta_{Y'})$ be as in Definition 7.1. Then, the induced morphism $Y' \setminus \text{Supp}(B_{Y'} + D') \rightarrow Y$ is an isomorphism with its image, and*

its image is included in $Y \setminus \text{Supp}(D)$. Here, D' denotes the pull-back of D to D' .

Proof. By construction, $\text{Supp}(D)$ contains the singular locus of Y . Thus, $\text{Supp}(D')$ contains the preimage of the singular locus of Y . Hence, up to shrinking, we may assume that Y is smooth and, in particular, $\mathbb{Q}$ -factorial. This guarantees that the exceptional locus is purely divisorial.

As argued at the beginning of § 7 and in Lemma 2.14, there is a divisor $C_{Y'}$ so that

- $0 \leq C_{Y'} \leq B_{Y'}$;
- $K_{Y'} + C_{Y'} + \Delta_{Y'}$ is relatively ample over Y ;
- $K_{Y'} + C_{Y'} + \Delta_{Y'} - \pi^*(\pi_*(K_{Y'} + C_{Y'} + \Delta_{Y'}))$ is π -exceptional (notice that here we are using the reduction to the case when Y is $\mathbb{Q}$ -factorial).

By the negativity lemma [KM98, Lemma 3.39], it follows that

$$K_{Y'} + C_{Y'} + \Delta_{Y'} = \pi^*(\pi_*(K_{Y'} + C_{Y'} + \Delta_{Y'})) - F,$$

where $F \geq 0$ is fully supported on the π -exceptional locus. Since $B_Y \geq \pi_*(C_{Y'})$, it follows that $0 \leq C_{Y'} \leq C_{Y'} + F \leq B_{Y'}$. In particular, $\text{Supp}(B_{Y'})$ contains the π -exceptional locus. This concludes the proof. $\square$

PROPOSITION 7.12. *Let (X, Δ) be a minimal klt pair with $\dim(X) = 2$ and $\kappa(X, \Delta) = 1$. Let $f: X \rightarrow Y$ be the Iitaka fibration, and assume that it is an elliptic fibration. Let B_Y be the boundary divisor of the canonical bundle formula for the pair (X, Δ) and the morphism f . Then, for every closed point $P \in Y \setminus \text{Supp}(B_Y)$, the fibration f admits a section étale locally at P .*

Proof. Let X' be the minimal resolution of X . Notice that if $X' \rightarrow Y$ admits a section étale locally, then so does $X \rightarrow Y$ by composition with the morphism $X' \rightarrow X$. Thus, we may assume that X is smooth. Since there is an inclusion $f(\text{Supp}(\Delta)) \subset \text{Supp}(B_Y)$, the fibration is a minimal elliptic fibration in the sense of Kodaira over $Y \setminus \text{Supp}(B_Y)$. Therefore, all the fibers are reduced over $Y \setminus \text{Supp}(B_Y)$. The claim follows. $\square$

Proof of Theorem 7.2. We proceed by induction on n . Proposition 7.12 implies that an analog of Proposition 7.4 holds true if $n = 2$. This will guarantee that the base case $n = 2$ can be proved following the steps below. Since the strategy below applies with no changes to the base case $n = 2$ and to the inductive step, we do not make mention of the induction in the rest of the proof. Our strategy follows the one in the proof of [Gro94, Theorem 4.3]. We proceed in several steps.

Step 1: In this step, we consider the family of Jacobian fibration guaranteed by Theorem 7.8 and construct some closed subsets on this family.

Let $\mathcal{J} \rightarrow (\mathcal{Y}, \mathcal{F}) \rightarrow U$ be the family of fibrations constructed in Theorem 7.8. Up to a stratification of U , we may assume that U is smooth. Similarly, we may assume that the closed subset of $\mathcal{Y}$ where the morphism $\mathcal{J} \rightarrow \mathcal{Y}$ is not smooth does not contain any fiber of $\mathcal{Y} \rightarrow U$. Let $\mathcal{Z} \subset \mathcal{Y}$ denote a closed subset so that $\mathcal{J} \rightarrow \mathcal{Y}$ is a smooth fibration over $\mathcal{Y} \setminus \mathcal{Z}$. Up to enlarging $\mathcal{Z}$, by Remark 7.10, we may assume that $\mathcal{J} \rightarrow \mathcal{Y}$ has a regular section over $\mathcal{Y} \setminus \mathcal{Z}$. Up to a further stratification and Noetherian induction, we may assume that $\mathcal{Z}$ does not contain any fiber of $\mathcal{Y} \rightarrow U$.

Step 2: In this step, we show that, after shrinking the family according to the closed subsets constructed in Step 1, the fibrations of interest are parametrized by the Tate–Shafarevich group.

Define $\mathring{\mathcal{Y}} := \mathcal{Y} \setminus (\mathcal{F} \cup \mathcal{Z})$ and $\mathring{\mathcal{J}} := \mathcal{J} \times_{\mathcal{Y}} \mathring{\mathcal{Y}}$. For an element $J(X) \rightarrow (Y', B_{Y'} + \Delta_{Y'} + D') \in \mathfrak{P}(n, v, \Phi(\mathcal{R}), d)$ corresponding to the fiber over $u \in U$, we denote by $\mathring{Y} \subset Y'$ the open subset

induced by $\mathring{\mathcal{Y}}$. By Lemma 7.11, we may identify $\mathring{Y}$ with an open subset of Y as well. Thus, we may write $J(\mathring{X}) := J(X) \times_Y \mathring{Y}$ and $\mathring{X} := X \times_Y \mathring{Y}$. Then, by construction and Proposition 7.4, $X \rightarrow Y$ is smooth over $\mathring{Y}$, except possibly at finitely many isolated points. Call these points $P_1, \dots, P_k$. Furthermore, the fibration $J(X) \rightarrow Y'$ is smooth over $\mathring{Y}$, and it admits a section. By construction, we know that $X \rightarrow Y$ corresponds to an element of $\text{III}_{\mathring{Y} \setminus \{P_1, \dots, P_k\}}(J(X)_{\eta_Y})$. Then, by Proposition 2.8, it actually belongs to $\text{III}_{\mathring{Y}}(J(X)_{\eta_Y})$. Notice that, by Lemma 2.7, this element has order dividing $d!$ in $\text{III}_{\mathring{Y}}(J(X)_{\eta_Y})$.

Step 3: In this step, we introduce a log resolution of $\mathcal{J} \rightarrow (\mathcal{Y}, \mathcal{F} \cup \mathcal{Z}) \rightarrow U$.

Up to a further stratification of U , we may assume that $\mathcal{Y}$ and $\mathcal{J}$ admit log resolutions $\mathcal{Y}'$ and $\mathcal{J}'$ so that

- $\mathcal{J}' \rightarrow \mathcal{J}$ and $\mathcal{Y}' \rightarrow \mathcal{Y}$ are isomorphisms over $\mathring{\mathcal{J}}$ and $\mathring{\mathcal{Y}}$, respectively;
- the induced rational map $\mathcal{J}' \dashrightarrow \mathcal{Y}'$ is an actual morphism which admits a section;
- the complement of $\mathring{\mathcal{Y}}$ in $\mathcal{Y}'$ is simple normal crossing over U ;
- $\mathcal{J}' \rightarrow U$ and $\mathcal{Y}' \rightarrow U$ are smooth.

Notice that, by construction, the ramification locus of $\mathcal{J}' \rightarrow \mathcal{Y}'$ is contained in the complement of $\mathring{Y}$.

Step 4: In this step, we show that the set of fibrations (without the boundary divisors) we are considering is birationally bounded. For this purpose, we will refer to several statements in [Gro94, § 4]. While the results in [Gro94, § 4] are stated for families of elliptic threefolds, we will make use only of the ones that hold without any restriction on the dimension. One of the needed statements, namely [Gro94, Lemma 4.7], holds in full generality only in dimension 3. Yet, for this specific statement, we only need the conclusion of the first half of its proof, which holds true with no restriction on the dimension (specifically, we only need the statement when the ramification divisor is removed, that is, $(\mathcal{X}', \mathcal{S}', \mathcal{T})$ in the notation of [Gro94, proof of Lemma 4.7]).

By abusing notation, we proceed as if U were irreducible: to be precise, the following steps should be performed on each irreducible component of the family of fibrations. Now, let $\eta_{\mathcal{Y}'}$ denote the generic point of $\mathcal{Y}'$, and let $\iota: \eta_{\mathcal{Y}'} \rightarrow \mathcal{Y}'$ be the natural inclusion. We regard $\iota_* \mathcal{J}'_{\eta_{\mathcal{Y}'}}$ as a sheaf in the étale topology of $\mathcal{Y}'$. Similarly, its (derived) push-forwards and pull-backs will be understood in the étale site. We write $\pi: \mathcal{Y}' \rightarrow U$ and $\mathring{\pi}: \mathring{\mathcal{Y}} \rightarrow U$. Then, by [Gro94, Lemma 4.4], the $d!$ -torsion subsheaf ${}_d R^1 \mathring{\pi}_*(\iota_* \mathcal{J}'_{\eta_{\mathcal{Y}'}})$ of $R^1 \mathring{\pi}_*(\iota_* \mathcal{J}'_{\eta_{\mathcal{Y}'}})$, is a constructible sheaf on U . By [Mil80, Proposition V.1.8], there exists an open subset $V \subset U$ such that ${}_d R^1 \mathring{\pi}_*(\iota_* \mathcal{J}'_{\eta_{\mathcal{Y}'}})|_V$ is locally constant with finite stalks. Thus, there is an étale cover $V' \rightarrow V$ such that ${}_d R^1 \mathring{\pi}_*(\iota_* \mathcal{J}'_{\eta_{\mathcal{Y}'}})|_{V'}$ is a constant sheaf with stalks isomorphic to a finite group G . For every $v \in V'$ and $g \in G$, there exists an étale neighborhood $V_{v,g} \rightarrow V'$ of v such that the element g of the stalk $({}_d R^1 \mathring{\pi}_*(\iota_* \mathcal{J}'_{\eta_{\mathcal{Y}'}})|_{V'})_{\bar{v}} = G$ is represented by an element of $E_{v,g} \in H^1(V_{v,g} \times_U \mathring{\mathcal{Y}}, \iota_* \mathcal{J}'_{\eta_{\mathcal{Y}'}})$. Fix $g \in G$. Then, by quasi-compactness, we may find a finite subcover of $\{V_{v,g} | v \in V'\}$ that covers V' . Thus, we produce a collection of connected étale schemes $V_1, \dots, V_k$ over V' and elements $E_j \in H^1(V_j \times_U \mathring{\mathcal{Y}}, \iota_* \mathcal{J}'_{\eta_{\mathcal{Y}'}})$ such that for all closed points $v \in V$, restricting each E_j to $\mathring{\mathcal{Y}}_v$ gives all elements of ${}_d \text{III}_{\mathring{\mathcal{Y}}_v}((\mathcal{J}'_v)_{\eta_{\mathring{\mathcal{Y}}_v}})$, by [Gro94, Lemma 4.7]. By Noetherian induction, we can repeat the above argument over $U \setminus V$. Thus, arguing as in the proof of [Gro94, Theorem 4.3], we obtain a family (over a possibly disconnected base) $\tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{Y}} \rightarrow T$ such that for every $u \in U$ and every element of ${}_d \text{III}_{\mathring{\mathcal{Y}}_u}((\mathcal{J}_u)_{\eta_{\mathring{\mathcal{Y}}_u}})$, there is a closed point $t \in T$ such that the fibration $\tilde{\mathcal{X}}_t \rightarrow \tilde{\mathcal{Y}}_t$ corresponds to such an element. Furthermore, we have that $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{X}} \rightarrow T$ are smooth. We conclude that the varieties underlying the pairs in $\mathfrak{C}(n, v, \Phi(\mathcal{R}), d)$ are birationally bounded.

Step 5: In this step, we retrieve the birational boundedness of the divisors corresponding to the elements of $\mathfrak{C}(n, v, \Phi(\mathcal{R}), d)$.

In Step 4, the variety $\tilde{\mathcal{Y}}$ together with the morphism $\tilde{\mathcal{Y}} \rightarrow T$ were constructed by iterated base changes and stratifications of the original base U . In particular, the simple normal crossing divisor $\mathcal{Y}' \setminus \check{Y}$ pulls back to $\tilde{\mathcal{Y}}$. Call this divisor Σ , and let Ξ be its pull-back to $\tilde{\mathcal{X}}$. Now, recall that $\mathcal{Y}' \rightarrow \mathcal{Y}$ is an isomorphism over $\check{Y}$. Let $\tilde{\mathcal{X}}_t$ be a closed fiber corresponding to the fibration $(X, \Delta) \rightarrow (Y, B_Y + \Delta_Y)$, and let $K_{\tilde{\mathcal{X}}_t} + \tilde{\Delta}_t$ denote the crepant pull-back of $K_X + \Delta$ to $\tilde{\mathcal{X}}_t$. Notice that, while $\tilde{\mathcal{X}}_t \dashrightarrow X$ may not be a morphism, the pull-back is crepant since $K_X + \Delta \sim_{\mathbb{Q}, Y} 0$. Let Σ_t denote the restriction of Σ to $\tilde{\mathcal{Y}}_t$. Then, every prime component of $\tilde{\Delta}_t$ or of the exceptional locus of $\tilde{\mathcal{X}}_t \dashrightarrow X$ that maps to a divisor on $\tilde{\mathcal{Y}}_t$ maps to Σ_t . Thus, every other component of $\tilde{\Delta}_t$ or of the exceptional locus of $\tilde{\mathcal{X}}_t \dashrightarrow X$ is exceptional for $\tilde{\mathcal{X}}_t \rightarrow \tilde{\mathcal{Y}}_t$ in the sense of [Lai11, Definition 2.9]. In order to conclude this step, it suffices to show the boundedness of these divisors.

Now, define the closed subset $\mathcal{V} \subset \tilde{\mathcal{Y}}$ as the set of points y so that $\dim(\tilde{\mathcal{X}}_y) > 1$. Let $\mathcal{W}$ be the inverse image of $\mathcal{V}$ in $\tilde{\mathcal{X}}$ with the reduced structure. Now, we may stratify T so that $\mathcal{W}$ is flat over T . Thus, the divisors that are exceptional for the morphism $\tilde{\mathcal{X}}_t \rightarrow \tilde{\mathcal{Y}}_t$ for some $t \in T$ are bounded since they deform in the divisorial part of $\mathcal{W}$. Denote by Ω the reduced divisor obtained as the union of Ξ and the divisorial part of $\mathcal{W}$. This divisor bounds the strict transform of the support of Δ and the exceptional divisor of $\tilde{\mathcal{X}}_t \dashrightarrow X$, as required.

Step 6: In this step, we perform some suitable modifications to the birationally bounding family in order to prepare it for the run of a suitable MMP.

In Step 5, we achieved the birational boundedness of the fibrations of interest. In order to retrieve (up to codimension 1 indeterminacy) the fibrations $(X, \Delta) \rightarrow Y$ in $\mathfrak{C}(n, v, \Phi(R), d)$, we will use that the pairs (X, Δ) are good minimal models. Thus, the ultimate goal of the proof is to replace the family $(\tilde{\mathcal{X}}, \Omega) \rightarrow T$ constructed in Step 5 with a new family where each fiber is a good minimal model of a fiber of $(\tilde{\mathcal{X}}, \Omega) \rightarrow T$. This final replacement will be performed in Step 7. In this step, we will improve the family constructed in Step 5 to have the additional properties needed to run the MMP in families. In particular, two key properties are needed. First, we need that if $(\tilde{\mathcal{X}}_t, \Omega_t)$ corresponds to (X, Δ) , the rational map $\tilde{\mathcal{X}}_t \dashrightarrow X$ is a rational contraction. To achieve this, we will use the fact that (X, Δ) is ϵ -log canonical for a uniform $\epsilon > 0$, and thus this property is guaranteed up to replacing $(\tilde{\mathcal{X}}_t, \Omega_t)$ by the terminalization of $(\tilde{\mathcal{X}}_t, (1 - \frac{1}{2}\epsilon)\Omega_t)$. Second, in order to apply [HMX18b, Theorem 1.9.1] in Step 7, we need for this replacement to occur on the whole family and for the new birational model of $(\tilde{\mathcal{X}}, \Omega) \rightarrow T$ to be log smooth over T . To this end, we will consider log resolutions and terminalizations of the total space $\tilde{\mathcal{X}}$, and we will recur to stratifications of T and Noetherian induction to guarantee that resolutions and terminalizations of the total space restrict to resolutions and terminalizations fiberwise.

Up to a further resolution, we may assume that Ω is simple normal crossing. In doing so, we include in Ω the divisors extracted in this process. By Corollary 5.5, each (X, Δ) is ϵ -log canonical for a fixed $\epsilon > 0$. Thus, we may replace $(\tilde{\mathcal{X}}, \Omega)$ with a log resolution of the terminalization of $(\tilde{\mathcal{X}}, (1 - \frac{1}{2}\epsilon)\Omega)$. Furthermore, up to a stratification of T , we may assume that this process induces a terminalization fiberwise. By abusing notation, we call the outcome of this process (X, Ω) . This process guarantees that no divisor is contracted by the birational map $X \dashrightarrow \tilde{\mathcal{X}}_t$. Then, up to a stratification, we may assume that every stratum of Ω is smooth over T . Finally, by [Kol13, Claim 4.38.1], we may perform an étale base change of a stratification of T so that every prime component of Ω restrict to a prime divisor fiberwise. This reduction is needed in order to be able to assign coefficients to the divisors Ω_t with a choice of coefficients for Ω .

Step 7: In this step, we conclude the proof.

Fix $(X, \Delta) \rightarrow Y$ in $\mathfrak{C}(n, v, \Phi(\mathcal{R}), d)$, and let $\tilde{\mathcal{X}}_t$ be the corresponding closed fiber of $\tilde{\mathcal{X}} \rightarrow T$. Then, the strict transform of Δ is supported on Ω_t . Let Γ_t be the divisor supported on Ω_t obtained as follows: it is the sum of the strict transform of Δ and the divisors that are exceptional for $\tilde{\mathcal{X}}_t \dashrightarrow X$, the latter having coefficients $1 - \frac{1}{2}\epsilon$. Then, by construction, (X, Δ) is a good minimal model for $(\tilde{\mathcal{X}}_t, \Gamma_t)$. Now, let $(\tilde{\mathcal{X}}, \Gamma)$ be the pair supported on $\tilde{\mathcal{X}}$ whose restriction to $\tilde{\mathcal{X}}_t$ is $(\tilde{\mathcal{X}}_t, \Gamma_t)$ (notice that this process may actually involve just one of the finitely many connected components of $\tilde{\mathcal{X}}$). Then, by [HMX18b, Theorem 1.9.1], the pair $(\tilde{\mathcal{X}}, \Gamma)$ admits a relative good minimal model over T that induces a good minimal model fiberwise. Furthermore, the relative ample model induces the ample model fiberwise. Call the minimal model $(\hat{\mathcal{X}}, \hat{\Gamma})$, and let $\hat{\mathcal{Y}}$ be the relative ample model. Then, by [HX13, Lemma 2.4], the pair $(\hat{\mathcal{X}}, \hat{\Gamma}_t)$ is isomorphic in codimension 1 to (X, Δ) , and $\hat{\mathcal{Y}}_t$ is isomorphic to Y .

Now, notice that $\tilde{\mathcal{X}} \rightarrow T$ is a morphism of schemes of finite type. In particular, they have finitely many irreducible components. Furthermore, Ω has finitely many irreducible components. Finally, the possible coefficients involved in the construction of Γ_t as above are finite. Indeed, $\Phi(\mathcal{R}) \cap [0, 1 - \epsilon]$ is a finite set. Thus, up to performing all the possible combinations of choices involving irreducible components of $\tilde{\mathcal{X}}$ and all the possible coefficients to assign to Ω , the above process retrieves every (X, Δ) up to isomorphism in codimension 1. This implies that there exist schemes of finite type $\overline{\mathcal{X}}$, $\overline{\mathcal{Y}}$, and S , a reduced divisor $\overline{\mathcal{B}}$ on $\overline{\mathcal{X}}$, and projective morphisms $(\overline{\mathcal{X}}, \overline{\mathcal{B}}) \rightarrow \overline{\mathcal{Y}} \rightarrow S$ that bound in codimension 1 the pairs (X, Δ) and bound the varieties Y . In order to conclude, we need to show that this family also bounds the divisors arising from the canonical bundle formula applied to any $(X, \Delta) \rightarrow Y$. Up to a stratification of S , this can be achieved by Corollary 2.12. This concludes the proof. $\square$

Proof of Theorem 1.1. It is immediate from Theorems 6.1 and 7.2. $\square$

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