Algebraic Geometry 11 (2) (2024) 296–317
doi:10.14231/AG-2024-010

On K-stability of Fano weighted hypersurfaces

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Abstract

Let $X \subset \mathbb{P}(a_0, \dots, a_n)$ be a quasi-smooth weighted Fano hypersurface of degree d and index I_X such that $a_i \mid d$ for all i. If $I_X = 1$, we show that, under a suitable condition, the α-invariant of X is greater than or equal to $\dim X/(\dim X + 1)$ and X is K-stable. This can be applied in particular to any X as above such that $\dim X \leqslant 3$. If X is general and $I_X < \dim X$, then we show that X is K-stable. We also give a sufficient condition for the finiteness of automorphism groups of quasi-smooth Fano weighted complete intersections.

1. Introduction

Let X be a Fano variety with log terminal singularities. The *α-invariant* of X (also known as *global log canonical threshold*) is defined as

$$\alpha(X) = \operatorname{glct}(X) := \sup\{c \in \mathbb{Q} \mid (X, cD) \text{ is log canonical for all } 0 \leqslant D \sim_{\mathbb{Q}} -K_X\}\,.$$

This was introduced by Tian [Tia87] in analytic terms to find a Kähler–Einstein metric on a Fano manifold (see also [CS08, Appendix]). This is a fundamental invariant of X from many points of view. It is known that if $\alpha(X) > \dim X/(\dim X + 1)$, then X is K-stable (see [OS12, Theorem 1.4]). If X is smooth, then equality is enough; see [Fuj19, Theorem 1.3] (cf. [LZ22]).

In [Puk98, Che01], it is shown that if $X \subset \mathbb{P}^{n+1}$ is a smooth Fano hypersurface of degree $n + 1$, then $\alpha(X) \geqslant n/(n + 1)$ and so X is K-stable. (See also [AZ22, AZ23] for recent progress in the higher-index cases.) A natural case to then consider is that of Fano weighted hypersurfaces $X \subset \mathbb{P}(a_0, \dots, a_n)$ of index 1. Del Pezzo surfaces $X \subset \mathbb{P}(a_0, \dots, a_3)$ of index 1 have been classified in [JK01b], and the existence of Kähler–Einstein metrics has been determined (cf. [JK01b, BGN02, Ara02, CPS10, LP22]). Fano threefolds $X \subset \mathbb{P}(a_0, \dots, a_4)$ of index 1 have been classified in [JK01a], and the K-stability of the terminal ones is well studied (cf. [Che08, Che09, KOW18, KOW20]). Very little is known in higher dimension except for [JK01a, Proposition 3.3], [Zhu20b, Theorem 1.2] and [Zhu20a, Theorem 1.3], to the authors' knowledge.

The idea of this paper is to generalize the methods introduced in [Puk98] and [Che01, CP02] to study the α-invariant of weighted Fano hypersurfaces in any dimension. Some consequences of our work are collected in the following result.

Received 21 March 2022, accepted in final form 9 May 2023.
2020 Mathematics Subject Classification 14J40, 14J45.
Keywords: Fano varieties, K-stability.

The first author was partially supported by JSPS KAKENHI Grant Numbers JP17H06127, JP19K14509. The second author is a member of the GNSAGA group of INdAM.

Theorem 1.1. *Let $X \subset \mathbb{P}(a_0, \dots, a_n)$ be a well-formed quasi-smooth weighted hypersurface of degree d. Assume $a_i \mid d$ for all i and that X is Fano of index 1. Also assume that (at least) one of the following conditions holds:*

(i) *The hypersurface X is general.*
(ii) *We have $a_2 = \dots = a_n = 1$.*
(iii) *We have $\dim X \leqslant 3$.*
(iv) *The hypersurface X is smooth, and $\dim X \leqslant 49$.*

Then we have

$$\alpha(X) \geqslant \begin{cases} \frac{d-2}{d} & \text{if } d = 2a,\ a \geqslant 3 \text{ and (up to permutation) } \mathbb{P} = \mathbb{P}(a_0, \dots, a_{n-2}, 2, a)\,, \\ \frac{d-1}{d} & \text{otherwise}\,. \end{cases}$$

Moreover, X is K-stable and admits a Kähler–Einstein metric.

If in addition $a_i \geqslant 2$ for any i and we are not in the first case above, then $\alpha(X) \geqslant 1$.

Remark 1.2. Quasi-smooth Fano fourfold hypersurfaces $X \subset \mathbb{P}(a_0, \dots, a_4)$ of index 1 are classified in [BK16]; see [BK02] for a complete list. In this list, there are 661 cases such that $a_i \mid d$ for all i. Using the same argument as in the proof of Theorem 1.1 when condition (iii) holds, one can check that all members of 653 such families satisfy the conclusion of Theorem 1.1. See Example 2.4 for more details.

An important point in this approach, which is interesting in itself, is given by the following question, whose answer is positive in the standard projective space due to [Puk98, Section 3] (see [Che01, Statement 3.3]).

Question 1.3. *Let $X \subset \mathbb{P}(a_0, \dots, a_n)$ be a quasi-smooth weighted hypersurface of degree d which is not a linear cone. Let D be an effective divisor on X such that $D \sim_{\mathbb{Q}} H$, where $H := \mathcal{O}_X(1)$, and let C be a curve in X. Is it true that*

$$\operatorname{omult}_C D \leqslant 1\,?$$

Here omult is the orbifold multiplicity as in [KOW20, Definition 2.1.9] (see Remark 2.7). A positive answer to Question 1.3 implies that (X, D) is log canonical outside a finite set. Section 2 is devoted to studying Question 1.3. In particular, in Lemma 2.1, we give an explicit condition on the equation of X to have a positive answer to such a question. In Section 4, we then develop a method to compute the log canonical threshold of weighted hypersurfaces. The final consequence is the following.

Theorem 1.4. *Let $X \subset \mathbb{P}(a_0, \dots, a_n)$ be a well-formed quasi-smooth weighted hypersurface of degree d. Assume $a_i \mid d$ for all i and that X is Fano of index 1. Assume that Question 1.3 has a positive answer for X. Then we have*

$$\alpha(X) \geqslant \begin{cases} \frac{d-2}{d} & \text{if } d = 2a,\ a \geqslant 3 \text{ and (up to permutation) } \mathbb{P} = \mathbb{P}(a_0, \dots, a_{n-2}, 2, a)\,, \\ \frac{d-1}{d} & \text{otherwise}\,. \end{cases}$$

Moreover, X is K-stable and admits a Kähler–Einstein metric.

If in addition $a_i \geqslant 2$ for any i and we are not in the first case above, then $\alpha(X) \geqslant 1$.

We expect that this approach can be applied to several other cases to compute the α-invariant of Fano weighted complete intersections besides those treated in Theorems 1.1 and 1.4; see for instance Example 4.9, where the α-invariant is computed for a hypersurface $X_{2a+1} \subset \mathbb{P}\big(1^{(a+2)}, a\big)$ (cf. [KOW20, Example 7.2.2]). Further applications would be toward the study of birational rigidity following [Puk98] and [dFe13, dFe16] (see also [SZ19]).

Finally, in Section 6, we study the K-stability of Fano weighted hypersurfaces of index greater than 1. In Theorem 6.1, we obtain a criterion of the K-polystability (respectively, K-semistability) of weighted hypersurfaces of Fermat type by using the argument in [Zhu21, Corollary 4.17]. In Theorem 6.5, we also give a sufficient condition for the finiteness of automorphism groups of quasi-smooth weighted complete intersections. This is a generalization of [PS19, Theorem 1.3]. As a consequence, we show the following.

Corollary 1.5. *Let $X = X_d \subset \mathbb{P}(a_0, \dots, a_n)$ be a well-formed quasi-smooth general Fano weighted hypersurface of degree d such that the Fano index $I_X := d - \sum_{i=0}^{n} a_i$ is less than $\dim X$ and $a_i \mid d$ for all i. Then X is K-stable. In particular, a general smooth Fano weighted hypersurface is K-stable if it is not isomorphic to the projective space or a quadric hypersurface.*

We also exhibit some K-unstable hypersurfaces of Fermat type (Remark 6.3).

The existence of a Kähler–Einstein metric on a Fano orbifold hypersurface is closely related to the existence of a Sasaki–Einstein metric on the link of the corresponding weighted homogeneous singularity (cf. [BGK05, BG06, GMSY07, CS19]). In fact, a variant of Theorem 6.1 is used in [LST22] to construct infinitely many families of Sasaki–Einstein metrics on spheres.

Notation 1.6. We work over the complex number field $\mathbb{C}$.

We define $\mathbb{P} := \mathbb{P}(a_0, \dots, a_n)$ to be the *weighted projective space* with weights $a_0, \dots, a_n$, that is, $\mathbb{P} = \operatorname{Proj} \mathbb{C}[z_0, \dots, z_n]$, where z_i has weight a_i. For simplicity, we assume that $\mathbb{P} = \mathbb{P}(a_0, \dots, a_n)$ is *well formed* unless otherwise stated; that is, the greatest common divisor of any n weights is 1 (although non-well-formed weighted projective spaces appear in the proof of Proposition 4.3).

A closed subvariety $X = X_{d_1,\dots,d_c} \subset \mathbb{P}$ is said to be a *weighted complete intersection* of multidegree $(d_1, \dots, d_c)$ if its weighted homogenous ideal in $\mathbb{C}[z_0, \dots, z_n]$ is generated by a regular sequence of homogenous polynomials $\{f_j\}_{j=1}^{c}$ such that $\deg f_j = d_j$ for $j = 1, \dots, c$. Let $\pi \colon \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}$ be the natural projection. Then X is *quasi-smooth* if $\pi^{-1}(X)$ is smooth. We say that X is *well formed* if $\mathbb{P}$ is well formed and $\operatorname{codim}_X(X \cap \operatorname{Sing}(\mathbb{P})) \geqslant 2$.

Finally, $X_{d_1,\dots,d_c} \subset \mathbb{P}$ is said to be a *linear cone* if $d_j = a_i$ for some i and j. We recall that if $\mathbb{P}$ is well formed and $X \subset \mathbb{P}$ is a weighted complete intersection of dimension at least 3, then X is well formed, or it is a linear cone (see [Ian00] for generalities on weighted complete intersections).

We write $\mathbb{P}\big(b_1^{(k_1)}, \dots, b_l^{(k_l)}\big)$ for $\mathbb{P}(\underbrace{b_1, \dots, b_1}_{k_1}, \dots, \underbrace{b_l, \dots, b_l}_{k_l})$.

2. A multiplicity lemma

Let $X \subset \mathbb{P}(a_0, \dots, a_n)$ be a weighted hypersurface of degree d defined by a polynomial $F = F(z_0, \dots, z_n) \in \mathbb{C}[z_0, \dots, z_n]$. For $i = 0, 1, \dots, n$, let

$$\begin{aligned} \pi_i \colon \mathbb{P}(a_0, \dots, 1, \dots a_n) =: \mathbb{P}_i &\to \mathbb{P}(a_0, \dots, a_n)\,; \\ [x_0 : \dots : x_i : \dots : x_n] &\mapsto [x_0 : \dots : x_i^{a_i} : \dots : x_n] \end{aligned}$$

be the finite cover branched along the hyperplane $(z_i = 0)$. Also, let $\pi \colon \mathbb{P}^n \to \mathbb{P}(a_0, \dots, a_n)$ defined by $[x_0 : \dots : x_n] \mapsto [x_0^{a_0} : \dots : x_n^{a_n}]$ be the finite cover which is a composition of the morphisms $\pi_0, \dots, \pi_n$.

Note that $Y_i := \pi_i^{-1}(X) \subset \mathbb{P}_i$ (respectively, $Y := \pi^{-1}(X) \subset \mathbb{P}^n$) is defined by the polynomial $G_i = G_i(x_0, \ldots, x_n) := F\big(x_0, \ldots, x_i^{a_i}, \ldots, x_n\big) \in \mathbb{C}[x_0, \ldots, x_n]$ (respectively, $G(x_0, \ldots, x_n) := F(x_0^{a_0}, \ldots, x_n^{a_n})$).

Lemma 2.1. *Let $X \subset \mathbb{P}(a_0, \ldots, a_n)$ be a quasi-smooth weighted hypersurface of degree d defined by a polynomial F.*

(i) *If X is general and $a_i \mid d$ for all i, then $Y = \pi^{-1}(X) \subset \mathbb{P}^n$ is smooth.*

(ii) *Fix an i such that $a_i > 1$ and assume that the following condition holds:*

– *Let $z_0^{k_0} \cdots z_n^{k_n}$ be a monomial appearing in F with non-zero coefficient such that $k_i = 1$. Then*

$$a_i = \sum_{j \neq i\colon k_j > 0} m_j a_j\,,$$

where the m_j are non-negative integers.

Then we can take an automorphism $\phi \in \operatorname{Aut} \mathbb{P}$ such that $(\phi \circ \pi_i)^{-1}(X) \subset \mathbb{P}_i$ is quasi-smooth.

(iii) *Assume that the following condition holds:*

⋆ *Let $z_0^{k_0} \cdots z_n^{k_n}$ be a monomial appearing in F with non-zero coefficient. If $k_i = 1$ for some i such that $a_i > 1$, then*

$$a_i = \sum_{j \neq i\colon k_j > 0} m_j a_j\,,$$

where the m_j are non-negative integers.

Then we can construct $\pi' := \pi'_1 \circ \cdots \circ \pi'_m\colon \mathbb{P}^n \to \mathbb{P}(a_0, \ldots, a_n)$ such that $\pi'_1, \ldots, \pi'_m$ are finite covers of the form $\phi_i \circ \pi_i$, where the ϕ_i are automorphisms and $Y = (\pi')^{-1}(X) \subset \mathbb{P}^n$ is smooth.

Proof. (i) This follows from a direct calculation by using that

$$\frac{\partial G}{\partial x_i} = a_i x_i^{a_i - 1} \frac{\partial F}{\partial z_i}\big(x_0^{a_0}, \ldots, x_n^{a_n}\big)$$

and that a general F defines a quasi-smooth hypersurface in $(z_i = 0 \mid i \in I) \subset \mathbb{P}(a_0, \ldots, a_n)$ for all $I \subset \{0, 1, \ldots, n\}$.

(ii) Let $Z_i := (F_{z_0} = \cdots = \breve{F}_{z_i} = \cdots = F_{z_n} = 0)$, where $F_{z_j} := \frac{\partial F}{\partial z_j}$ for $j = 0, \ldots, n$ and $\breve{F}_{z_i}$ means that we skip the term F_{z_i}. Then Z_i is zero-dimensional since X is quasi-smooth. If $Z_i \cap (z_i = 0) = \emptyset$, then we see that $\pi_i^{-1}(X) = (G_i = 0) \subset \mathbb{P}_i$ is quasi-smooth since we compute

$$\frac{\partial G_i}{\partial x_i}(Q) = a_i q_i^{a_i - 1} \frac{\partial F}{\partial z_i}\big(q_0, \ldots, q_i^{a_i}, \ldots, q_n\big) \neq 0$$

for $Q = [q_0 : \cdots : q_n] \in \pi_i^{-1}(Z_i)$.

Hence we may assume $Z_i \cap (z_i = 0) \neq \emptyset$. Take $P = [p_0 : \cdots : p_n] \in Z_i \cap (z_i = 0)$. Note that $F_{z_i}(P) \neq 0$ by the quasi-smoothness of X. Since

$$\frac{\partial\big(z_0^{k_0} \cdots z_n^{k_n}\big)}{\partial z_i}(P)$$

can be non-zero at P only if $k_i = 1$, we have at least one monomial $z_0^{k_0} \cdots z_n^{k_n}$ appearing in F such that $k_i = 1$ and $p_j \neq 0$ for any $j \neq i$ such that $k_j > 0$ (otherwise, we have $F_{z_i}(P) = 0$). Now fix such a monomial $z_0^{k_0} \cdots z_n^{k_n}$.

Then we see that

$$\left(z_i + \lambda \prod_{j\neq i\colon k_j>0} z_j^{m_j} = 0\right) \cap Z_i = \emptyset$$

for a general $\lambda \in \mathbb{C}^*$. Now consider the automorphism $\phi \in \operatorname{Aut}\mathbb{P}$ such that

$$\phi(z_\ell) = \begin{cases} z_\ell & (\ell \neq i)\,, \\ z_i + \lambda \prod_{j\neq i\colon k_j>0} z_j^{m_j} & (\ell = i)\,. \end{cases}$$

Then we see that $(\phi \circ \pi_i)^{-1}(X) \subset \mathbb{P}_i$ is quasi-smooth, as before.

(iii) Take an i such that a_i is a minimum among the weights bigger than 1.

If $Z_i \cap (z_i = 0) = \emptyset$, then, as in the proof of item (ii), we have that $\pi_i^{-1}(X) = (G = 0) \subset \mathbb{P}_i$ is quasi-smooth, and we can easily check condition ($\star$).

If $Z_i \cap (z_i = 0) \neq \emptyset$, then we can take $\phi_i \in \operatorname{Aut}\mathbb{P}$ such that $(\phi_i \circ \pi_i)^{-1}(X) \subset \mathbb{P}_i$ is quasi-smooth as in the proof of item (ii). Since a_i is a minimum, condition ($\star$) implies that $a_j = a_i$ (or $a_j = 1$) for some $j \neq i$ such that $k_j > 0$ in the monomial $z_0^{k_0} \cdots z_n^{k_n}$. The latter case ($a_j = 1$) is easier, so we consider the former case. Then we can take $\phi_i(z_i) = z_i + \lambda z_j$ for a general $\lambda \in \mathbb{C}^*$. We can check that the equation G_ϕ of $(\phi \circ \pi_i)^{-1}(X) \subset \mathbb{P}_i$ satisfies condition($\star$) as follows. Note that $G_\phi(x_0, \dots, x_n) = F_\phi\big(x_0, \dots, x_i^{a_i}, \dots, x_n\big)$, where $F_\phi(z_0, \dots, z_n) := F(z_0, \dots, \phi(z_i), \dots, z_n)$. Also note that

$$F_\phi(z_0, \dots, z_n) = \sum c_T z_0^{t_0} \cdots (z_i + \lambda z_j)^{t_i} \cdots z_n^{t_n}\,;$$

thus new monomials to consider are of the form $c \cdot z_0^{t_0} \cdots \big(z_i^{t_i - 1} z_j\big) \cdots z_n^{t_n}$ for j such that $a_j = a_i$. Hence F_ϕ satisfies condition ($\star$), and we can check that G_ϕ also satisfies condition ($\star$).

Repeating this argument a finite number of times, we obtain a smooth cover Y as in the statement. □

Lemma 2.2. *Let a_0, a_1, a_2 be positive integers such that $a_i \neq m_j a_j + m_k a_k$ for $\{i, j, k\} = \{0, 1, 2\}$ and non-negative integers m_j, m_k. If $\gcd(a_i, a_j) = 1$ for any $i \neq j$, then*

$$a_0 a_1 a_2 - a_0 - a_1 - a_2 \geqslant 48\,.$$

Proof. Write $1 < a_0 < a_1 < a_2$ (equalities are not possible by assumption). Note that $a_0 \geqslant 3$. Indeed, if $a_0 = 2$, then a_1 and a_2 are both odd, but then there exists a positive integer m_0 such that $a_2 = a_1 + m_0 a_0$. The smallest $a_0 a_1 a_2 - a_0 - a_1 - a_2$ is now given by $(a_0, a_1, a_2) = (3, 4, 5)$; it is 48. □

Lemma 2.3. *Let $X \subset \mathbb{P}(a_0, \dots, a_n)$ be a well-formed quasi-smooth weighted hypersurface of degree d. Assume $a_i \mid d$ for all i. Assume that one of the following holds:*

(i) *We have $a_2 = a_3 = \cdots = a_n = 1$.*

(ii) *We have $a_3 = a_4 = \cdots = a_n = 1$, $\gcd(a_i, a_j) = 1$ for any $i \neq j$ and*

$$d - \sum_{i\colon a_i > 1} a_i < 48\,.$$

(iii) *We have $\dim X \leqslant 49$, and X is a smooth Fano of index 1.*

(iv) *We have $X \subset \mathbb{P}(a_0, a_1, a_2, 1, 1)$ with $\gcd(a_0, a_1, a_2) = 1$ and $d - \sum_{i\colon a_i>1} a_i < 48$.*

(v) *We have $\dim X \leqslant 3$, and X is a Fano of index 1.*

Then we can construct $\pi' := \pi_1' \circ \cdots \circ \pi_m' \colon \mathbb{P}^n \to \mathbb{P}(a_0, \dots, a_n)$ such that $\pi_1', \dots, \pi_m'$ are finite covers of the form $\phi_i \circ \pi_i$, where the ϕ_i are automorphisms and $Y = (\pi')^{-1}(X) \subset \mathbb{P}^n$ is smooth.

Proof. *Case* 1: Condition (i) holds. Let us show that condition ($\star$) of Lemma 2.1(iii) holds. Assume toward a contradiction that there exists a monomial $M = z_0^{i_0} \cdots z_n^{i_n}$ appearing in F with $i_j = 1$ for some j such that $a_j > 1$ and that for any other k such that $i_k \geqslant 1$, we have $a_k \nmid a_j$. In particular, $a_k > 1$ for any k such that $i_k > 0$. The monomial M is thus of the form $z_j z_k^{i_k}$ with $i_k > 0$. Then $d = a_j + a_k i_k$, which implies $a_k \mid a_j$, giving a contradiction.

Case 2: Condition (ii) holds. We can assume $a_0, a_1, a_2 > 1$ by Case 1. By Lemma 2.2 we get an i such that $a_i = m_j a_j + m_k a_k$ for $\{i, j, k\} = \{0, 1, 2\}$ and non-negative integers m_j, m_k. Hence the conditions of Lemma 2.1(ii) are satisfied, and we can take a cover $(\phi \circ \pi_i)^{-1}(X) \subset \mathbb{P}_i$. We can now apply Case 1 to conclude.

Case 3: Condition (iii) holds. Since X is smooth, the weights are pairwise coprime. If there are at most two weights bigger than 1, then we can apply Case 1. If there are at least three weights bigger than 1, then (using that $\dim X \leqslant 49$) it is easy to check that the only possible cases are $X_{30} \subset \mathbb{P}\big(2, 3, 5, 1^{(21)}\big)$ and $X_{42} \subset \mathbb{P}\big(2, 3, 7, 1^{(31)}\big)$, which are covered by Case 2 (see Example 2.5 for the case $X_{60} \subset \mathbb{P}\big(3, 4, 5, 1^{(49)}\big)$ of dimension 50).

Case 4: Condition (iv) holds. If $\gcd(a_i, a_j) = 1$ for any $i \neq j$, then we can apply Case 2, so assume that (up to reordering) $\gcd(a_0, a_1) > 1$. If $a_2 = 1$, then we are done by Case 1, so assume $a_2 > 1$. We claim that we can now apply Lemma 2.1(ii) for $i = 2$. In fact, consider a monomial of the form $z_0^{k_0} \cdots z_4^{k_4}$ such that $k_2 = 1$. Since $a_3 = a_4 = 1$, the only case to check is $k_3 = k_4 = 0$. But then $d = a_2 + k_0 a_0 + k_1 a_1$ and so $\gcd(a_0, a_1) \mid a_2$, giving a contradiction. Hence, by Lemma 2.1(ii), we can take a cover $\phi \circ \pi_2$ and then apply Case 1 to conclude.

Case 5: Condition (v) holds. We will show that there is always a smooth cover as in Lemma 2.1. To make the proof short, we are going to use the available classification results.

First assume $\dim X = 2$, that is, $X \subset \mathbb{P}(a_0, a_1, a_2, a_3)$. By [JK01b, Theorem 8], there are only four possible cases satisfying $a_i \mid d$: $X_3 \subset \mathbb{P}^3$, $X_4 \subset \mathbb{P}(1, 1, 1, 2)$, $X_6 \subset \mathbb{P}(1, 1, 2, 3)$ and $X_{15} \subset \mathbb{P}(3, 3, 5, 5)$. In all cases, it is immediate to see that we can apply Lemma 2.1(iii).

Now assume $\dim X = 3$, that is, $X \subset \mathbb{P}(a_0, a_1, a_2, a_3, a_4)$ (see Table 1 for a list obtained using the classification given in [JK01a, Theorem 2.2]). If $a_3 = a_4 = 1$, then $d = a_0 + a_1 + a_2 + 1$, which implies $\gcd(a_0, a_1, a_2) = 1$ since $a_i \mid d$ for any i. Then the result follows from Case 4. If $a_0, a_1, a_2, a_3 > 1$ and $a_4 = 1$, then we see from Table 1 that there are only two possible cases: $X_{12} \subset \mathbb{P}(2, 3, 3, 4, 1)$ and $X_{30} \subset \mathbb{P}(2, 3, 10, 15, 1)$. In both cases, we can apply Lemma 2.1(ii) to take a cover $\phi \circ \pi_3$ and then conclude by Case 4. We are left with the case $1 < a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$. One can check from Table 1 that it is possible to apply Lemma 2.1(ii) first with $i = 4$, then after the cover with $i = 3$ and finally with $i = 2$. Then to conclude, it is enough to use Case 1. □

Example 2.4. In $\mathbb{P}(3, 4, 5, 4, 15, 30)$, consider X given by

$$z_0^{17} z_1 z_2 + z_0 z_1^{13} z_2 + \big(z_0^4 + z_1^3\big)^5 + \big(z_0^5 + z_2^3\big)^4 + \big(z_1^5 + z_2^4\big)^3 - z_0^{20} - z_1^{15} - z_2^{12} + G(z_3, z_4, z_5) = 0\,,$$

where G is general of degree 60. Then X is a quasi-smooth Fano fourfold of index 1. (The quasi-smoothness was checked by computer.) Moreover, $X \cap (z_i = 0)$ is not quasi-smooth for $i = 0, 1, 2$, and it is not possible to perform a procedure as in Lemma 2.1 to get a smooth cover Y.

Similar examples can be constructed for $X_{105} \subset \mathbb{P}(40, 40, 30, 5, 3, 3)$, $X_{140} \subset \mathbb{P}(70, 35, 20, 7, 5, 4)$, $X_{210} \subset \mathbb{P}(105, 42, 35, 14, 10, 5)$, $X_{420} \subset \mathbb{P}(210, 140, 35, 28, 5, 3)$, $X_{714} \subset \mathbb{P}(357, 238, 51, 34, 21, 14)$, $X_{1386} \subset \mathbb{P}(693, 462, 198, 14, 11, 9)$ and $X_{1890} \subset \mathbb{P}(945, 630, 270, 27, 14, 5)$. Using the classification given in [BK16], we could check that for any other Fano fourfold quasi-smooth hypersurface $X_d \subset \mathbb{P}(a_0, \dots, a_4)$ of index 1 such that $a_i \mid d$, there exists a smooth cover.

Table 1. Weights for Fano 3-folds of index 1 with $a_i \mid d$ ($\forall i$).

a_0	a_1	a_2	a_3	a_4	d	a_0	a_1	a_2	a_3	a_4	d
1	1	1	1	1	4	2	2	3	3	9	18
1	1	1	1	3	6	2	3	3	14	21	42
1	1	1	2	2	6	2	3	5	6	15	30
1	1	1	2	4	8	2	4	5	5	5	20
1	1	1	4	6	12	2	5	9	30	45	90
1	1	2	2	5	10	2	6	7	7	21	42
1	1	2	3	6	12	3	3	3	8	8	24
1	1	2	6	9	18	3	3	5	5	15	30
1	1	3	4	4	12	3	3	5	10	10	30
1	1	3	8	12	24	3	3	5	20	30	60
1	1	4	5	10	20	3	3	15	20	20	60
1	1	6	14	21	42	4	4	7	7	7	28
1	2	3	3	4	12	5	5	18	18	45	90
1	2	3	10	15	30	5	7	10	14	35	70
2	2	3	3	3	12	6	6	11	11	33	66

Example 2.5. In $\mathbb{P}\big(1^{(49)}, 3, 4, 5\big)$, consider X given by

$$z_0^{60} + \cdots + z_{n-3}^{60} + z_{n-2}z_{n-1}^{13}z_n + z_{n-2}^2 z_{n-1} z_n^{10} + \big(z_{n-2}^4 + z_{n-1}^3\big)^5 + \big(z_{n-2}^5 + z_n^3\big)^4 + \big(z_{n-1}^5 + z_n^4\big)^3 - z_{n-2}^{20} - z_{n-1}^{15} - z_n^{12} = 0\,.$$

Then X is a smooth Fano of index 1 and dimension 50. Moreover, $X \cap (z_i = 0)$ is not quasi-smooth for $i = n-2, n-1, n$, and it is not possible to perform a procedure as in Lemma 2.1 to get a smooth cover Y.

Proposition 2.6. *Let $X \subset \mathbb{P}(a_0, \dots, a_n)$ be a weighted hypersurface of degree d. With the above notation, assume that we have a finite cover $\pi' \colon \mathbb{P}^n \to \mathbb{P}(a_0, \dots, a_n)$ with the ramification formula*

$$K_{\mathbb{P}^n} = (\pi')^*\Big(K_{\mathbb{P}} + \sum_{i=0}^{n} \frac{a_i - 1}{a_i} H_i'\Big) \tag{2.1}$$

for some hyperplanes $H_i' \in |\mathcal{O}_{\mathbb{P}}(a_i)|$ for $i = 0, \dots, n$ such that $Y := (\pi')^{-1}(X) \subset \mathbb{P}^n$ is smooth and $(\pi')^*\mathcal{O}_{\mathbb{P}}(1) = \mathcal{O}_{\mathbb{P}^n}(1)$. (Such a cover exists for X as in Lemmas 2.1 and 2.3.) Let D be an effective $\mathbb{Q}$-divisor on X such that

$$D \sim_{\mathbb{Q}} H\,,$$

where $H := \mathcal{O}_X(1)$ is the hyperplane section.

Then (X, D) is log canonical outside a finite set $Z \subset X$.

Proof. We can write $D = (1/r)D_r$ for some $r \in \mathbb{Z}_{>0}$ and $D_r \in |\mathcal{O}_X(r)|$. Since we have the ramification formula (2.1), we obtain

$$K_Y = \pi^*\Big(K_X + \sum_{i=0}^{n} \frac{a_i - 1}{a_i} H_i\Big),$$

where $H_i := H_i' \cap X$ for $i = 0, \dots, n$. Let $\tilde{D} := \pi^*(D)$ so that $\tilde{D} \sim_{\mathbb{Q}} \mathcal{O}_Y(1)$. Take an irreducible curve $\tilde{C} \subset Y$. By [Che01, Statement 3.3], we see that

$$\operatorname{mult}_{\tilde{C}}\big(\tilde{D}\big) \leqslant 1 \,.$$

This implies that $\big(Y, \tilde{D}\big)$ is log canonical on $Y \setminus \tilde{Z}$ for some finite set $\tilde{Z} \subset Y$.

Let $R := \sum_{i=0}^{n} (a_i - 1) a_i^{-1} H_i$. Then we have the ramification formula

$$K_Y + \tilde{D} = \pi^*(K_X + D + R) \,.$$

From this, we conclude that $(X, D + R)$ is log canonical on $X \setminus Z$ for $Z := \pi\big(\tilde{Z}\big)$ (cf. [KM98, Proposition 5.20]); thus (X, D) is log canonical on $X \setminus Z$. □

Remark 2.7. In Proposition 2.6, we can show that $\operatorname{omult}_C(D) \leqslant 1$ for any irreducible curve $C \subset X$. We recall the definition of orbifold multiplicity (see [KOW20, Definition 2.1.9]): if $p \in X$ is a cyclic quotient singularity and D is an effective divisor on X, then $\operatorname{omult}_p D = \operatorname{mult}_{p'} \pi^* D$, where $\pi \colon X' \to X$ is the quotient map and p' is a preimage of p.

Let $p \in C$ be a general point and $U_p \subset X$ be a small neighborhood of p. Let $V_p := \pi^{-1}(U_p)$, and let $\nu_p \colon \tilde{U}_p \to U_p$ be a finite cover from some smooth variety $\tilde{U}_p$ such that ν_p is étale in codimension 1 and $U_p \simeq \tilde{U}_p / \mathbb{Z}_m$ for some m. Let $\tilde{V}_p$ be the normalization of the fiber product $V_p \times_{U_p} \tilde{U}_p$. Then we have the following diagram:

$$\begin{array}{ccc} \tilde{V}_p & \xrightarrow{\ \tilde{\nu}_p\ } & V_p \\ \Big\downarrow{\scriptstyle \tilde{\pi}_p} & & \Big\downarrow{\scriptstyle \pi_p} \\ \tilde{U}_p & \xrightarrow{\ \nu_p\ } & U_p \,. \end{array}$$

Note that $\tilde{V}_p$ is smooth and $\tilde{\nu}_p$ is étale by the purity of the branch locus. For an irreducible curve $\tilde{C} \subset Y$ such that $\pi\big(\tilde{C}\big) = C$, we see that $\operatorname{mult}_{\tilde{C}} \tilde{D} \leqslant 1$, as above. This implies that $\operatorname{mult}_q\big(\tilde{D}\big) \leqslant 1$ for $q \in \pi_p^{-1}(p)$; thus we see that $\operatorname{mult}_{\tilde{q}} \tilde{\nu}_p^{-1}\big(\tilde{D} \cap V_p\big) \leqslant 1$ for $\tilde{q} \in \tilde{\nu}_p^{-1}(q)$ since $\tilde{\nu}_p$ is étale. Then we see that $\operatorname{mult}_{\tilde{p}} \nu_p^{-1}(U_p \cap D) \leqslant 1$ for $\tilde{p} \in \nu_p^{-1}(p)$ by considering the local homomorphism $\tilde{\pi}_p^{\sharp} \colon \mathcal{O}_{\tilde{U}_p, \tilde{p}} \to \mathcal{O}_{\tilde{V}_p, \tilde{q}}$ on the stalks. This implies that $\operatorname{omult}_C D \leqslant 1$.

3. A Nadel vanishing–type theorem

The following is a version of Nadel vanishing for $\mathbb{Q}$-Cartier integral Weil divisors (not necessary Cartier) that we are going to use to compute the α-invariant.

Lemma 3.1. *Let (X, B) be a log canonical pair and D a $\mathbb{Q}$-Cartier integral Weil divisor on X such that $A = D - K_X - B$ is nef and big. Let $\mathcal{J} = \mathcal{J}((X, B); -D)$ be the multiplier ideal sheaf associated with $-D$ with respect to (X, B).*

(i) *There is an inclusion $\mathcal{J} \hookrightarrow \mathcal{O}_X(D)$.*
(ii) *We have $H^i(X, \mathcal{J}) = 0$ for any $i > 0$.*
(iii) *Let $x \in X$ be such that $\mathcal{O}_X(D)_x \cong \mathcal{O}_{X,x}$; that is, D is Cartier at x. Then $\mathcal{J}((X, B); -D)_x = \mathcal{J}(X, B)_x \otimes \mathcal{O}_X(D)_x$, where $\mathcal{J}(X, B) := \mathcal{J}(X, B; 0)$.*

Proof. Let $\mu \colon W \to X$ be a log resolution of $(X, B + D)$, and define a $\mathbb{Q}$-divisor B_W by

$$K_W + B_W = \mu^*(K_X + B) \,.$$

We can write $\mu^* D = \tilde{D} + \sum_{k=1}^{m} b_k E_k$, where $\tilde{D} \subset W$ is the strict transform of D and $E_1, \ldots, E_m$ are exceptional divisors of μ. Since D is integral, $\tilde{D}$ is a Cartier divisor on W.

Set

$$\mathcal{L} := \mathcal{O}_W\left(\tilde{D} + \left\lceil \sum b_k E_k - B_W \right\rceil\right) = \mathcal{O}_W\left(\tilde{D} - \left(\left\lfloor -\sum b_k E_k + B_W \right\rfloor\right)\right).$$

Then by the definition of the multiplier ideal sheaf (see [Laz04, Definition 9.3.56]),

$$\mathcal{J} = \mathcal{J}((X, B); -D) = \mu_* \mathcal{L}.$$

If E is the exceptional locus of μ and $Z := \mu(E)$, we have

$$\mathcal{L}_{|(W \setminus E)} \cong \mathcal{O}_{X\setminus Z}(D + \lceil -B \rceil) \hookrightarrow \mathcal{O}_{X \setminus Z}(D).$$

Hence we have $\mu_*\mathcal{L} \hookrightarrow (\mu_*\mathcal{L})^{\vee\vee} \simeq \mathcal{O}_X(D + \lceil -B \rceil) \hookrightarrow \mathcal{O}_X(D)$ and obtain an injection $\mathcal{J} \hookrightarrow \mathcal{O}_X(D)$ as a composition.

We now prove item (ii). Since

$$\tilde{D} + \sum b_k E_k - B_W \equiv \mu^*(D) - B_W \equiv \mu^*(K_X + B + A) - B_W = K_W + \mu^*(A),$$

the relative Kawamata–Viehweg vanishing theorem [KMM87, Theorem 1-2-3] implies

$$R^i \mu_*(\mathcal{L}) = 0$$

for $i > 0$, and so by Leray spectral sequence, we get that $H^i(X, \mathcal{J}) = H^i(W, \mathcal{L})$. By the Kawamata–Viehweg vanishing theorem, we also get $H^i(W, \mathcal{L}) = 0$ for $i > 0$, and so item (ii) is proven.

Now assume that D is Cartier at $x \in X$. Then

$$\mathcal{J}_x = \mu_*\mathcal{O}_W\left(\tilde{D} + \left\lceil \sum b_k E_k - B_W \right\rceil\right)_x = \mu_*\mathcal{O}_W(\mu^*D + \lceil -B_W \rceil)_x = \mathcal{J}(X, B)_x \otimes \mathcal{O}_X(D)_x$$

by the projection formula. □

The following example shows that in Lemma 3.1(iii), one cannot simply drop the condition that D is Cartier at the point x.

Example 3.2. Let $X \subset \mathbb{A}^3$ be the affine cone over a smooth cubic curve $C \subset \mathbb{P}^2$, and let D be a line through the vertex of X that passes through a flex of C, so that D is a $\mathbb{Q}$-Cartier divisor ($3D$ is Cartier). Denote by $\mu\colon W \to X$ the minimal resolution of X with exceptional curve E. Since $E^2 = -3$, the following hold:

$$\mu^*D = \tilde{D} + \frac{1}{3}E \quad \text{and} \quad K_W = \mu^*K_X - E.$$

Since $\mu_*\mathcal{O}_W(\lfloor \mu^*D \rfloor) = \mathcal{O}_X(\lfloor D \rfloor) = \mathcal{O}_X(D)$ (use [Nak04, Lemma 2.11] and the fact that D is integral), we get

$$\mathcal{J}(X; -D) = \mu_*\mathcal{O}_W\left(\tilde{D} + \left\lceil \frac{1}{3}E - E \right\rceil\right) = \mu_*\mathcal{O}_W(\tilde{D}) = \mu_*\mathcal{O}_W(\lfloor \mu^*D \rfloor) = \mathcal{O}_X(D).$$

On the other hand, $\mathcal{J}(X)$ is the ideal sheaf of the vertex of X, which is not trivial. We also note that the inclusion $\mathcal{J}(X; -D) \hookrightarrow \mathcal{O}_X(D)$ is not strict in this case, even if the vertex is a log canonical center (lc center for short) of X. (This example reflects the necessity of H and H_j being Cartier at the isolated lc center Q in the proof of Proposition 4.3.)

4. Log canonical threshold computation

We start off with the following numerical lemma.

Lemma 4.1. *Let $a_0, \dots, a_n$ and d be positive integers such that*

(i) $\gcd(a_0, \dots, \breve{a}_i, \dots, a_n) = 1$ *for all* i,

(ii) $a_i \mid d$ *for all* i,

(iii) $d = \sum_{i=0}^n a_i - 1$.

We use the notation $(*)$ *for the condition*

$(*)$ $\quad d = 2a$ *and (up to permutation)* $(a_0, \dots, a_n) = (a_0, \dots, a_{n-2}, 2, a)$ *for some* $a \geqslant 3$.

Set

$$c := \begin{cases} \frac{d-2}{d} = \frac{a-1}{a} & \text{if } a_0, \dots, a_n \text{ and } d \text{ satisfy } (*)\,, \\ \frac{d-1}{d} & \text{otherwise}\,. \end{cases}$$

Then, for $i, j \in \{0, 1, \dots, n\}$ *such that* $i \neq j$, *we have*

$$-d - 1 + a_i + cd \leqslant -1 \qquad \text{if } a_i = 1\,, \tag{4.1}$$

$$-d - 1 + a_i + c\frac{d}{a_i} \leqslant -a_j \quad \text{if } (*) \text{ holds and } a_i = 2,\ a_j = a\,, \tag{4.2}$$

$$-d - 1 + a_i + \frac{d}{a_i} \leqslant -a_j \quad \text{for all } j \text{ otherwise } (a_i > 1)\,. \tag{4.3}$$

We also have $c \geqslant (n-1)/n$ *in either case. Equality holds only if* $(d, a_0, \dots, a_n) = (n, 1, \dots, 1)$ *or* $(2a, 1, \dots, 1, 2, a)$ *for some* $a \geqslant 3$.

Proof. If $a_i = 1$, then

$$-d - 1 + a_i + c\frac{d}{a_i} \leqslant -d + \frac{d-1}{d}d = -1\,,$$

where we used that if $(*)$ holds, then $(d-2)/d \leqslant (d-1)/d$.

Assume $a_i > 1$. Fix $j \in \{0, \dots, n\}$.

Case 1: $d < a_i a_j$. Set $a_M := \max\{a_i, a_j\}$. If $d \geqslant 3a_M$, then

$$d + 1 - \frac{d}{a_i} - a_i - a_j \geqslant d + 1 - 3a_M \geqslant 1\,.$$

If $d < 3a_M$, then we must have $d = 2a_M$ by the assumption that $a_i \mid d$ for all i.

If $a_i = a_j$, then the condition $d = 2a_i$ together with assumption (iii) would imply that $(a_0, \dots, a_n) = (1, a_i, a_i)$ up to permutation, which contradicts assumption (i).

If $a_i > a_j$, then $d = 2a_i$ and

$$d + 1 - \frac{d}{a_i} - a_i - a_j \geqslant 2a_i + 1 - 2 - a_i - (a_i - 1) = 0\,.$$

If $a_i < a_j$, then $d = 2a_j$. We must have $a_i \geqslant 3$ because we are in the case $d < a_i a_j$. Note that $k := 2a_j/a_i$ satisfies $k \geqslant 3$ since it is an integer and $a_i < a_j$. Then

$$d + 1 - \frac{d}{a_i} - a_i - a_j = 2a_j + 1 - k - a_i - a_j = \frac{ka_i}{2} + 1 - k - a_i = (k-2)\left(\frac{a_i}{2} - 1\right) - 1\,,$$

which is non-negative unless $k = a_i = 3$, which is not possible since it would give $9 = 2a_j$.

Case 2: $d \geqslant a_i a_j$. If $a_j = 1$, then

$$d + 1 - \frac{d}{a_i} - a_i - a_j = d - \frac{d}{a_i} - a_i = \left(\frac{d}{a_i} - 1\right)(a_i - 1) - 1 \geqslant 0\,,$$

where we used $d \geqslant 2a_i$. So we may assume $a_j > 1$.

Assume $a_i \geqslant 3$.

Since we have $d \geqslant a_i a_j$, we get the required inequality from

$$d+1-\left(a_i+a_j+\frac{d}{a_i}\right)=\left(\frac{d}{a_i}a_i-a_i-\frac{d}{a_i}+1\right)-a_j=\left(\frac{d}{a_i}-1\right)(a_i-1)-a_j$$
$$\geqslant 2(a_j-1)-a_j=a_j-2\geqslant 0\,.$$

Now consider the case $a_i=2$ and recall that $d \geqslant 2a_j$ and $a_j \mid d$. If $d \geqslant 4a_j$, then the inequality follows as

$$d-1-d/2-a_j=d(1-1/2)-a_j-1\geqslant 2a_j-a_j-1=a_j-1>0\,.$$

If $d=3a_j$, then

$$d+1-\frac{d}{a_i}-a_i-a_j=3a_j+1-\frac{3a_j}{2}-2-a_j=\frac{a_j}{2}-1\geqslant 0\,.$$

If $d=2a_j$, then condition $(*)$ holds, that is, $a=a_j\geqslant 3$, $(a_i=2)$ and $c=(a-1)/a$ since $a_j=2$ implies $(a_0,\dots,a_n)=(1,2,2)$ up to permutation, as before. Then we obtain the required inequality as

$$d+1-c\frac{d}{a_i}-a_i-a_j=2a+1-ca-2-a=a-1-ca=a-1-a\cdot\frac{a-1}{a}=0\,.$$

Finally, we check the last statement. If condition $(*)$ holds, then we have $2+a+\sum_{i=0}^{n-2}a_i=2a+1$. Hence we have

$$a-1=\sum_{i=0}^{n-2}a_i\geqslant n-1\,;$$

thus we see that $c=(a-1)/a\geqslant(n-1)/n$ if condition $(*)$ holds, and equality holds only when $a_0=\cdots=a_{n-2}=1$. Otherwise, we have

$$d=\sum_{i=0}^{n}a_i-1\geqslant n\,;$$

thus see that $c=(d-1)/d\geqslant(n-1)/n$. Equality holds only if $a_0=\cdots=a_n=1$. □

Lemma 4.2. *Let $X=X_d\subset\mathbb{P}(a_0,\dots,a_n)=:\mathbb{P}$ be a quasi-smooth weighted hypersurface of degree d that is not a linear cone. Assume $a_i\mid d$ for any $i=0,\dots,n$. Then, up to a linear automorphism of $\mathbb{P}$, we can assume $P_i\notin X_d$ for any $i=0,\dots,n$, where P_i is the ith coordinate point of $\mathbb{P}$.*

Proof. Assume that there exists an i such that $P_i\in X$. Since X is quasi-smooth, there exists a j such that $\frac{\partial F}{\partial z_j}(P_i)\neq 0$. This implies that there exists a monomial in F of the form $z_j z_i^{c_i}$, that is, $d=a_j+c_ia_i$, which tells us that $a_i\mid a_j$. We can then consider an automorphism of the form $z_j\mapsto z_j+\lambda z_i^{a_j/a_i}$ with $\lambda\in\mathbb{C}^*$ general. Since λ is general, we can apply the argument for any $P_i\in X$ to obtain the statement of the lemma. □

Proposition 4.3. *Let $X\subset\mathbb{P}(a_0,\dots,a_n)=:\mathbb{P}$ be a well-formed quasi-smooth weighted hypersurface of degree d that is not a linear cone. Let $D\sim_{\mathbb{Q}}H$ be a $\mathbb{Q}$-divisor on X, where $H:=\mathcal{O}_X(1)$. Assume that*

(i) *X is Fano of index 1,*

(ii) *$a_i | d$ for any $i = 0, \dots, n$,*

(iii) *the non-klt (non–Kawamata log terminal) locus of (X, D) is at most zero-dimensional.*

We use the notation $(*)$ for the condition

$$(*) \qquad d = 2a \text{ and (up to permutation) } (a_0, \dots, a_n) = (a_0, \dots, a_{n-2}, 2, a) \text{ for some } a \geqslant 3.$$

Then the log canonical threshold of (X, D) satisfies

$$\operatorname{lct}(X, D) \geqslant \begin{cases} \frac{d-2}{d} = \frac{a-1}{a} & \text{if } (*) \text{ holds}, \\ 1 & \text{if } (*) \text{ does not hold and } a_i \geqslant 2 \text{ for any } i, \\ \frac{d-1}{d} & \text{otherwise}. \end{cases}$$

Remark 4.4. The second case of the inequality really occurs. For example, let $X_{6(m+l+1)} \subset \mathbb{P}\big(2^{(2+3m)}, 3^{(1+2l)}\big)$ be a general hypersurface of degree $6(m + l + 1)$ for some $m, l \in \mathbb{Z}_{>0}$. Then this satisfies the condition.

Proof. The $\mathbb{Q}$-divisor D is of the form $D = D_m/m$ for some $m > 0$ and $D_m \in |\mathcal{O}_X(m)|$. Let c be the log canonical threshold of (X, D). Assume toward a contradiction that $c < (a - 1)/a$ if $(*)$ holds, that $c < 1$ if $(*)$ does not hold and $a_i \geqslant 2$ for any i and that $c < (d - 1)/d$ otherwise.

By assumption (iii), the log canonical locus $\operatorname{LCS}(X, cD)$ of (X, cD) consists of a finite number of points. By the Shokurov connectedness theorem (see for example [Che01, Theorem 2.8]), we get that $\operatorname{LCS}(X, cD)$ consists of one single point Q.

By Lemma 4.2, we can assume $P_i \notin X_d$ for any $i = 0, \dots, n$, where P_i is the ith coordinate point of $\mathbb{P}$. Then we have a well-defined finite morphism of degree d/a_i,

$$p_i \colon X_d \to \mathbb{P}_i := \mathbb{P}(a_0, \dots, \check{a}_i, \dots, a_n)$$

induced by the ith projection $\mathbb{P} \dashrightarrow \mathbb{P}_i$ on $\mathbb{P}$. Note that the finiteness of the projection follows from $P_i \notin X_d$ and $\mathbb{P}_i$ may not be well formed. Let $c_{ij} := \gcd(a_0, \dots, \check{a}_i, \dots, \check{a}_j, \dots, a_n)$ for $j \neq i$ and $c_i := \prod_{j \neq i} c_{ij}$. Then, by the operation as in [Ian00, Lemma 5.7] (cf. [Dol82, Section 1.3.1]), we see that

$$\mathbb{P}_i \simeq \mathbb{P}\left(\frac{c_{i0} a_0}{c_i}, \dots, \check{\imath}, \dots, \frac{c_{in} a_n}{c_i}\right) = \mathbb{P}\big(\bar{a}_0, \dots, \check{\imath}, \dots, \bar{a}_n\big) =: \bar{\mathbb{P}}_i,$$

where $\bar{a}_j := \frac{c_{ij} a_j}{c_i}$ for $j \neq i$ ($\check{\imath}$ means that we skip the ith term). The isomorphism follows from $\mathbb{P}_i \simeq \operatorname{Proj} \mathbb{C}\big[z_0^{c_{i0}}, \dots, \check{\imath}, \dots, z_n^{c_{in}}\big]$ and by dividing all weights by c_i.

Set

$$B_{\mathbb{P}_i} := c \cdot \frac{p_i(D_m)}{m}.$$

Claim 4.5. (i) *The morphism p_i is étale on $X \setminus \big(\frac{\partial F}{\partial z_i} = 0\big) \cup p_i^{-1}(\operatorname{Sing} \mathbb{P}_i)$.*

(ii) *There exists an $i \in \{0, \dots, n\}$ such that $\frac{\partial F}{\partial z_i}(Q) \neq 0$ and this implies that $Q_i := p_i(Q) \in \mathbb{P}_i$ is an isolated lc center of the pair $(\mathbb{P}_i, B_{\mathbb{P}_i})$.*

Proof of Claim 4.5. (i) Let

$$Q' := [q'_0 : \cdots : q'_n] \in X \setminus \left(\frac{\partial F}{\partial z_i} = 0\right) \cup p_i^{-1}(\operatorname{Sing} \mathbb{P}_i).$$

The fiber of p_i over $Q'_i := p_i(Q') = [q'_0 : \cdots : \check{q}'_i : \cdots : q'_n]$ is given by the zeros of the univariate polynomial $F(q'_0, \dots, q'_{i-1}, x, q'_{i+1}, \dots, q'_n)$. The condition $\frac{\partial F}{\partial z_i}(Q') \neq 0$ implies that Q' is not a multiple root of $F(q'_0, \dots, q'_{i-1}, x, q'_{i+1}, \dots, q'_n)$, and so p_i is unramified over Q'_i since $p_i^{-1}(Q'_i)$ consists of $\deg p_i$ points.

(ii) Since $X_d = (F = 0) \subset \mathbb{P}$ is quasi-smooth, there exists an i such that

$$\frac{\partial F}{\partial z_i}(Q) \neq 0\,.$$

Since X and $\mathbb{P}_i$ are smooth in codimension 1, we conclude that p_i is étale in codimension 1 around Q by assertion (i), and so $K_X + cD = p_i^*(K_{\mathbb{P}_i} + B_{\mathbb{P}_i})$ locally around Q. Then, by a standard lemma about discrepancies (cf. [KM98, Proposition 5.20]), we get that the pair $(\mathbb{P}_i, B_{\mathbb{P}_i})$ is log canonical but not Kawamata log terminal, and $p_i(Q)$ is an isolated lc center. □

Remark 4.6. If $Q \in X$ is smooth and $p_i(Q)$ is a smooth point of $\mathbb{P}_i$, then by the implicit function theorem, the projection p_i induces a local analytic isomorphism of a neighborhood of Q and one of $p_i(Q)$. Hence $B_{\mathbb{P}_i}$ and cD are also locally isomorphic, and we obtain Claim 4.5 in a more direct way.

We have

$$[D_m : p_i(D_m)]p_i(D_m) = (p_i)_*(D_m) \sim \mathcal{O}_{\bar{\mathbb{P}}_i}(md/a_ic_i)\,,$$

where $\mathcal{O}_{\bar{\mathbb{P}}_i}(1) \in \operatorname{Cl}\bar{\mathbb{P}}_i$ is the ample generator and $[D_m : p_i(D_m)]$ is the degree of the map p_i restricted to D_m. Note that we can calculate $(p_i)_*(D_m) \sim \mathcal{O}_{\bar{\mathbb{P}}_i}(md/a_ic_i)$ by taking some explicit hyperplane and the fact that the push-forward preserves linear equivalence (cf. [Nak04, Section 2.e]).

This implies that

$$K_{\mathbb{P}_i} + B_{\mathbb{P}_i} \equiv \mathcal{O}_{\bar{\mathbb{P}}_i}\left(\frac{1}{c_i}\left(-\sum_{j\neq i} c_{ij}a_j + cd_i\right)\right), \tag{4.4}$$

where $d_i := d/(a_i[D_m : p_i(D_m)])$.

We now distinguish two cases, depending on whether $Q_i = p_i(Q) \in \mathbb{P}_i$ is a smooth or a singular point.

Case 1. Assume that $Q_i \in \mathbb{P}_i$ is a smooth point. Take a Weil divisor H on $\mathbb{P}_i$ whose class is $\mathcal{O}_{\bar{\mathbb{P}}_i}(1)$, and consider the multiplier ideal sheaf $\mathcal{J} = \mathcal{J}((\mathbb{P}_i, B_{\mathbb{P}_i}); H)$. Set $\mathcal{Q} := \mathcal{O}_{\bar{\mathbb{P}}_i}(-1)/\mathcal{J}$. By Lemma 3.1(i), we have an inclusion $\mathcal{J} \hookrightarrow \mathcal{O}_{\bar{\mathbb{P}}_i}(-1)$. Since H is Cartier at the smooth point Q_i, by Lemma 3.1(iii), we see that such an inclusion is strict at Q_i. Thus the support of $\mathcal{Q}$ contains Q_i as a connected component and $H^0(\mathcal{Q}) \neq 0$.

Claim 4.7. *We have*

$$-H - (K_{\mathbb{P}_i} + B_{\mathbb{P}_i}) = \mathcal{O}_{\bar{\mathbb{P}}_i}\left(-1 + \frac{1}{c_i}\left(\sum_{j\neq i} c_{ij}a_j - cd_i\right)\right),$$

and it is ample as a $\mathbb{Q}$-line bundle.

Proof of Claim 4.7. The equality follows from (4.4).

If $\mathbb{P}_i$ is well formed, the ampleness follows from Lemma 4.1; thus assume $c_i > 1$, that is, $c_{ij} > 1$ for some $j \neq i$. Then we have

$$\frac{1}{c_i}\left(\sum_{j\neq i} c_{ij}a_j - cd_i\right) > \sum_{j\neq i}\frac{c_{ij}a_j}{c_i} - \frac{d}{c_ia_i} = \sum_{j\neq i}\frac{c_{ij}a_j}{c_i} - \sum_{j\neq i}\frac{a_j}{c_ia_i} \geqslant 1$$

since $c_{ij}a_j/c_i$, $d/(c_ia_i) \in \mathbb{Z}$. This implies the required ampleness. □

Claim 4.7 and Lemma 3.1 (ii) give a surjection

$$H^0\big(\bar{\mathbb{P}}_i, \mathcal{O}_{\bar{\mathbb{P}}_i}(-1)\big) \twoheadrightarrow H^0(\mathcal{Q}) \neq 0\,,$$

which gives a contradiction.

Case 2. Now assume that $Q_i \in \mathbb{P}_i$ is a singular point of $\mathbb{P}_i$.

We first deal with the case $a_i = 1$. Write $Q = [q_0 : \cdots : q_n]$. Since $Q_i \in \mathbb{P}_i \simeq \bar{\mathbb{P}}_i$ is singular, we have $\gcd\{\bar{a}_j \colon j \neq i \text{ and } q_j \neq 0\} > 1$; thus

$$\gcd\{a_j \colon j \neq i \text{ and } q_j \neq 0\} > 1\,. \tag{4.5}$$

The fact that $\frac{\partial F}{\partial z_i}(Q) \neq 0$ implies that there exists a monomial $G = z_0^{b_0} \cdots z_n^{b_n}$ of degree d that appears in F with non-zero coefficient and satisfies $\frac{\partial G}{\partial z_i}(Q) \neq 0$. This means that if $b_j > 0$ for $j \neq i$, then $q_j \neq 0$. By (4.5), we get that $g := \gcd\{a_j \colon j \neq i \text{ and } b_j > 0\}$ satisfies $g > 1$ and $g \mid d$ because $a_\ell \mid d$ for any ℓ. Hence G must be divisible by z_i^g since $a_i = 1$. This gives $q_i \neq 0$ since $\frac{\partial G}{\partial z_i}(Q) \neq 0$.

By (4.5), we have that $q_j = 0$ for any $j \neq i$ such that $a_j = 1$, and so from the Euler identity

$$0 = dF(Q) = \sum_{\ell=0}^{n} a_\ell q_\ell \frac{\partial F}{\partial z_\ell}(Q)\,,$$

we deduce that there exists a k such that $a_k > 1$ and $\frac{\partial F}{\partial z_k}(Q) \neq 0$. We can therefore consider $p_k \colon X \to \mathbb{P}_k$. Since $q_i \neq 0$ and $a_i = 1$, we see that $Q_k = p_k(Q)$ is a smooth point of $\mathbb{P}_k$, and we are reduced to Case 1.

So we can assume $a_i > 1$. Let $j \neq i$ be such that the jth coordinate of Q is non-zero, and note that $\mathcal{O}_{\bar{\mathbb{P}}_i}(-c_{ij}a_j/c_i) = \mathcal{O}_{\bar{\mathbb{P}}_i}(-\bar{a}_j)$ is invertible at Q_i. Take a Weil divisor H_j on $\mathbb{P}_i$ whose class is $\mathcal{O}_{\bar{\mathbb{P}}_i}(\bar{a}_j)$, and consider the multiplier ideal sheaf $\mathcal{J} = \mathcal{J}((\mathbb{P}_i, B_{\mathbb{P}_i}); H_j)$. Set $\mathcal{Q} := \mathcal{O}_{\bar{\mathbb{P}}_i}(-\bar{a}_j)/\mathcal{J}$.

Claim 4.8. *We have that*

$$-H_j - (K_{\mathbb{P}_i} + B_{\mathbb{P}_i}) = \mathcal{O}_{\bar{\mathbb{P}}_i}\bigg(-\bar{a}_j + \frac{1}{c_i}\bigg(\sum_{k\neq i} c_{ik}a_k - cd_i\bigg)\bigg),$$

and this is ample as a $\mathbb{Q}$*-line bundle.*

Proof of Claim 4.8. The equality follows from (4.4).

If $c_i = 1$, then the required inequality is

$$-a_j + \sum_{k\neq i} a_k - cd_i = d + 1 - a_i - a_j - cd_i > 0\,,$$

and it follows from (4.2) and (4.3) in Lemma 4.1. Thus assume $c_i > 1$. Note that $(*)$ does not occur in this case. Then we have

$$\begin{aligned}\sum_{k\neq i} c_{ik}a_k - cd_i > \sum_{k\neq i} c_{ik}a_k - d_i \geqslant (c_{ij}-1)a_j + \sum_{k\neq i} a_k - \frac{d}{a_i} \\ = (c_{ij}-1)a_j + d + 1 - a_i - \frac{d}{a_i} \overset{(4.3)}{\geqslant} (c_{ij}-1)a_j + a_j = c_{ij}a_j\,.\end{aligned}$$

This implies the required ampleness. □

As in Case 1, we reach a contradiction using Claim 4.8 and Lemma 3.1(ii), (iii). □

It is sometimes possible to use the above argument for the computation of the alpha invariants without assumption (ii) of Proposition 4.3, as follows.

Example 4.9 ([KOW20, Example 7.2.2]). Consider a hypersurface $X = X_{2a+1} \subset \mathbb{P}\big(1^{(a+2)}, a\big)$ of degree $2a+1$ with $a \geqslant 2$ given by

$$X = \big(y^2 x_1 + f(x_1, \dots, x_{a+2}) = 0\big)\,,$$

where f is general. Then the coordinate point $P_y = [0 : \cdots : 0 : 1] \in X$ is a singular point of X, and there is no automorphism to move it outside X. Also note that

$$\alpha(X) \leqslant \operatorname{lct}_{P_y}(X, H_1) = \frac{a+1}{2a+1}\,,$$

where $H_1 := (x_1 = 0) \subset X$ and $\operatorname{lct}_{P_y}(X, H_1)$ denotes the log canonical threshold of the pair (X, H_1) locally around the point P_y.

Claim 4.10. *We have $\alpha(X) = (a+1)/(2a+1)$.*

Proof of Claim 4.10. Let $D = D_m/m \sim_{\mathbb{Q}} \mathcal{O}_X(1)$ be an effective $\mathbb{Q}$-divisor as in the proof of Proposition 4.3. Note that we have a smooth cover $Y := \big(z^{2a}x_1 + f(x_1, \dots, x_{a+2}) = 0\big) \subset \mathbb{P}^n$ and can apply Proposition 2.6. Let $c := \operatorname{lct}(X, D)$ be the log canonical threshold of X with respect to D.

Suppose $c < (a+1)/(2a+1)$. We will obtain a contradiction as in the proof of Proposition 4.3. By Proposition 2.6, the pair (X, cD) has an isolated lc center $Q = [q_1 : \cdots : q_{a+2} : r]$. Let $F := y^2 x_1 + f(x_1, \dots, x_{a+2})$ be the defining equation of X. Note that for $i = 1, \dots, a+2$, the projection $p_i \colon X \to \mathbb{P}_i \simeq \mathbb{P}\big(1^{(a+1)}, a\big)$ is well defined since the ith coordinate point P_i satisfies $P_i \notin X$. On the other hand, since $P_y \in X$, the projection $p_y \colon X \dashrightarrow \mathbb{P}_y \simeq \mathbb{P}^{a+1}$ is not defined at P_y.

Case 1. First consider the case where $Q = P_y = [0 : \cdots : 0 : 1]$. Then the first projection $p_1 \colon X \to \mathbb{P}_1 \simeq \mathbb{P}\big(1^{(a+1)}, a\big)$ is étale at Q because $\frac{\partial F}{\partial x_1}(Q) \neq 0$. Let $B_{\mathbb{P}_1} := c \cdot p_1(D_m)/m$ and $e_1 := [D_m : p_1(D_m)]$ be the degree of $p_1|_{D_m}$. Then we see that

$$B_{\mathbb{P}_1} \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}_1}\bigg(c \cdot \frac{2a+1}{e_1}\bigg).$$

Since $p_1(Q) = [0 : \cdots : 0 : 1] \in \mathbb{P}_1$ is a singularity of index a, we see that $H = \mathcal{O}_{\mathbb{P}_1}(a)$ is Cartier. Since we have

$$-H - (K_{\mathbb{P}_1} + B_{\mathbb{P}_1}) \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}_1}\bigg(-a + (2a+1)\bigg(1 - \frac{c}{e_1}\bigg)\bigg)$$

and $-a + (2a+1)\big(1 - \frac{c}{e_1}\big) \geqslant -a + (2a+1)(1-c) > -a + (a) = 0$, we see that $-H - (K_{\mathbb{P}_1} + B_{\mathbb{P}_1})$ is ample. Then we can argue as in the proof of Proposition 4.3 to obtain a contradiction.

Case 2. Consider the case where $Q \neq P_y$. If we have $\frac{\partial F}{\partial x_i}(Q) \neq 0$ for some $1 \leqslant i \leqslant a+2$, then we may assume that the coordinate point P_{x_i} for x_i satisfies $P_{x_i} \notin X$ as in Lemma 4.2. We see that the ith projection $p_i \colon X \to \mathbb{P}_i \simeq \mathbb{P}\big(1^{(a+1)}, a\big)$ is étale at Q; thus we can argue as in Case 1 to obtain a contradiction.

Hence we may assume $\frac{\partial F}{\partial x_i}(Q) = 0$ for $i = 1, \dots, a+2$. Note that

$$\frac{\partial F}{\partial x_1} = y^2 + \frac{\partial f}{\partial x_1}\,, \quad \frac{\partial F}{\partial x_2} = \frac{\partial f}{\partial x_2}\,, \quad \dots\,, \quad \frac{\partial F}{\partial x_{a+2}} = \frac{\partial f}{\partial x_{a+2}}\,.$$

The point $Q = [q_1 : \cdots : q_{a+2} : r]$ satisfies $r \neq 0$. Indeed, if $r = 0$, then $\frac{\partial f}{\partial x_i}(q_1, \dots, q_{a+2}) = 0$ for all i, and this gives a contradiction since f is general. We also see that $q_i \neq 0$ for some i since

$Q \neq P_y$. By considering an automorphism φ of $\mathbb{P}\big(1^{(a+2)}, a\big)$ such that $\varphi(y) = y + \lambda x_i^a$ for some $\lambda \in \mathbb{C}^*$ and $\varphi(x_j) = x_j$ for $j = 1, \ldots, a+2$, it is enough to consider the case where $r \neq 0$ and $\frac{\partial F}{\partial x_1}(Q) \neq 0$. Thus, by considering the projection p_1, we obtain the same contradiction as Case 1.

From these, we obtain $c \geqslant (a+1)/(2a+1)$, thus the claim. □

5. Proofs of Theorems 1.4 and 1.1

Proof of Theorem 1.4. Let $X \subset \mathbb{P}(a_0, \ldots, a_n)$ be as in the statement, and let $D \sim_{\mathbb{Q}} H$ be a $\mathbb{Q}$-divisor on X, where $H = \mathcal{O}_X(1)$.

Since $a_i \mid d$ for all i and Question 1.3 has a positive answer for X, we can apply Proposition 4.3 to conclude that

$$\operatorname{lct}(X, D) \geqslant \begin{cases} \frac{a-1}{a} & \text{if } d = 2a,\ a \geqslant 3 \text{ and (up to permutation) } \mathbb{P} = \mathbb{P}(a_0, \ldots, a_{n-2}, 2, a)\,, \\ \frac{d-1}{d} & \text{otherwise}\,. \end{cases}$$

We also get that $\alpha(X) \geqslant 1$ if $a_i \geqslant 2$ for any i and we are not in case $(*)$ (note that X is not smooth in this case by [PS20, Lemma 3.3] or [PST17, Theorem 1.2]).

Now assume that $(*)$ holds and $a_0 = \ldots = a_{n-2} = 1$. Then we have $\dim X_{2a} = a - 1$ since the Fano index is 1; thus we only see the K-semistability of X_{2a} from the criterion [OS12, Theorem 1.4]. Nevertheless, we see the K-stability of X_{2a} as follows. If a is odd, then we see that X_{2a} is smooth; thus X_{2a} is K-stable by [Fuj19, Theorem 1.3]. If a is even, then X_{2a} has only $\frac{1}{2}(1, \ldots, 1)$-singularities. We see that the singularities are not weakly exceptional by [CS11, Corollary 3.20] since a cyclic quotient singularity is defined by a reducible representation of a cyclic group. From this and [LZ22, Theorem 3.1], we see that X_{2a} is K-stable when a is even.

If $(*)$ holds but we do not have $a_0 = \cdots = a_{n-2} = 1$, then $d = \sum_{i=0}^n a_i - 1 > n+1 = \dim X + 2$. If $(*)$ does not hold, then we have $d = \sum_{i=0}^n a_i - 1 > n = \dim X + 1$ since we may assume $\mathbb{P}(a_0, \ldots, a_n) \not\simeq \mathbb{P}^n$. In all these cases, we obtain the K-stability of X_d from [OS12, Theorem 1.4].

Since X is K-stable, we see that X admits a Kähler–Einstein metric (cf. [DK01, Section 6], [LTW22, Li22]). □

Proof of Theorem 1.1. Assume that condition (i) holds. Theorem 1.4 can be applied to any Fermat type hypersurface $X = \big\{z_0^{d/a_0} + \cdots + z_n^{d/a_n} = 0\big\} \subset \mathbb{P}(a_0, \ldots, a_n)$ since it has a smooth cover as in Lemma 2.1. Then, by the openness of (uniform) K-stability [BL22, BLX22, LXZ22], we obtain the K-stability of general X_d. Another proof can be given following the proof of Theorem 1.4, replacing Lemma 2.1(iii) by Lemma 2.1(i).

Now assume that one of conditions (ii), (iii) and (iv) holds. In all these cases, Lemma 2.3 assures that we have a smooth cover so that we can apply Proposition 2.6. The conclusion then follows from Proposition 4.3. □

6. Automorphism groups and Fano weighted hypersurfaces of Fermat type

Adapting the argument in [Zhu21, Corollary 4.17] using the criterion [Fuj21, Corollary 1.6], we have the following criterion for the K-polystability of Fano weighted hypersurfaces of Fermat type.

Theorem 6.1. *Let $X_d := \big(z_0^{d_0} + \cdots + z_n^{d_n} = 0\big) \subset \mathbb{P}(a_0, \ldots, a_n)$ be a quasi-smooth Fano hypersurface of degree d such that $a_i \mid d$ for all i and $d_i := d/a_i$ satisfies $d_i \geqslant 2$. Let $I_X := \sum_{i=0}^n a_i - d$*

be the Fano index of X_d. Assume $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_n$.

Then X_d is K-polystable (respectively, K-semistable) if and only if $I_X < na_0$ (respectively, $I_X \leqslant na_0$). In particular, X_d is K-polystable when X_d is smooth.

Remark 6.2. It is known that the condition $I_X \leqslant na_0$ is a necessary condition for the existence of a Kähler–Einstein metric on a well-formed quasi-smooth hypersurface $X \subset \mathbb{P}(a_0, \dots, a_n)$ (see [GMSY07, (3.23)] and [CPS10, Example 1.8]).

Proof. We will show this by adapting [Zhu21, Corollary 4.17]. Consider $\mathbb{P}^n$ with coordinates $[w_0 : \cdots : w_n]$. Then X_d admits a Galois covering $\pi \colon X_d \to H \subset \mathbb{P}^n$ defined by $[z_0 : \cdots : z_n] \mapsto [z_0^{d_0} : \cdots : z_n^{d_n}]$, where $H := (w_0 + \cdots + w_n = 0) \subset \mathbb{P}^n$. Indeed, we see that the Galois group of π is $\bigoplus_{i=0}^n \mathbb{Z}/(d_i)$ since it is deduced from the injection $\mathbb{C}[w_0, \dots, w_n] \hookrightarrow \mathbb{C}[z_0, \dots, z_n]$ determined by $w_i \mapsto z_i^{d_i}$ for $i = 0, \dots, n$. Let $H_i := (w_i = 0) \subset H \simeq \mathbb{P}^{n-1}$ for $i = 0, \dots, n$. Then we see that $\bigcup_{i=0}^n H_i \subset H$ is a simple normal crossings divisor and

$$K_{X_d} = \pi^* \left(K_H + \sum_{i=0}^n \left(1 - \frac{1}{d_i}\right) H_i \right).$$

By [Zhu21, Corollary 4.13], in order to check the K-polystability of X_d, it is enough to check the K-polystability of the log Fano hyperplane arrangement $\left(H, \sum_{i=0}^n (1 - 1/d_i) H_i\right)$. By the isomorphism $H \simeq \mathbb{P}^{n-1}$ and the criterion [Fuj21, Corollary 1.6], the above pair is K-semistable (respectively, uniformly K-stable) if and only if

$$k \sum_{i=0}^n \left(1 - \frac{1}{d_i}\right) \geqslant n \sum_{j=1}^k \left(1 - \frac{1}{d_{i_j}}\right) \quad \left(\text{respectively, } k \sum_{i=0}^n \left(1 - \frac{1}{d_i}\right) > n \sum_{j=1}^k \left(1 - \frac{1}{d_{i_j}}\right)\right)$$

for any $k = 1, \dots, n-1$ and $0 \leqslant i_1 < \cdots < i_k \leqslant n$. The difference (LHS − RHS) is equal to

$$\begin{aligned} k\left(n + 1 - \sum_{i=0}^n \frac{1}{d_i}\right) - n\left(k - \sum_{j=1}^k \frac{1}{d_{i_j}}\right) &= k - k \sum_{i=0}^n \frac{1}{d_i} + n \sum_{j=1}^k \frac{1}{d_{i_j}} \\ &= \frac{k}{d}\left(d - \sum_{i=0}^n a_i + \frac{n}{k} \sum_{j=1}^k a_{i_j}\right) = \frac{k}{d}\left(-I_X + \frac{n}{k} \sum_{j=1}^k a_{i_j}\right). \end{aligned}$$

Then the K-semistability (respectively, uniform K-stability) of the arrangement is equivalent to the non-negativity (respectively, the positivity) of the term $-I_X + (n/k) \sum_{j=1}^k a_{i_j}$. Note that we have

$$\min \left\{ -I_X + \frac{n}{k} \sum_{j=1}^k a_{i_j} \mid k = 1, \dots, n-1, 0 \leqslant i_1 < \cdots < i_k \leqslant n \right\} = -I_X + na_0$$

since $a_0 = \min\{a_0, \dots, a_n\}$ and $(1/k) \sum_{j=1}^k a_{i_j} \geqslant a_0$. Hence the positivity is equivalent to the positivity of $-I_X + na_0$.

When X_d is smooth, we always have the K-polystability since $X \simeq \mathbb{P}^{n-1}$ if $I_X \geqslant n$. □

Remark 6.3. It is easy to find a quasi-smooth K-unstable hypersurface of Fermat type $X_d \subset \mathbb{P}(a_0, \dots, a_n)$ as follows. For example, let

$$X_6 := \left(y_1^3 + \cdots + y_m^3 + z_1^2 + \cdots + z_l^2 = 0\right) \subset \mathbb{P}\left(2^{(m)}, 3^{(l)}\right)$$

for some $m \geqslant 1, l \geqslant 5$. Then we see that $I_X = 2m + 3l - 6$ and $n = m + l - 1$, $a_0 = 2$; thus we have

$$-I_X + na_0 = -(2m + 3l - 6) + 2(m + l - 1) = -l + 4 < 0.$$

Hence we see that X_6 is not K-semistable by Theorem 6.1.

One can check that $H^0(X_6, \mathcal{T}_{X_6}) \neq 0$ and $|\operatorname{Aut}(X_6)| = \infty$. Indeed, since we have $z_1^2 + z_2^2 = (z_1 + \sqrt{-1}z_2)(z_1 - \sqrt{-1}z_2)$, we obtain automorphisms as in [PS21, Example 5.1].

Let $X_{12} \subset \mathbb{P}(3^{(m)}, 4^{(l)})$ for $m \geqslant 1$, $l \geqslant 10$. Then we see that X_{12} is also K-unstable by Theorem 6.1 and the relation

$$-I_X + na_0 = -(3m + 4l - 12) + 3(m + l - 1) = -l + 9\,.$$

We also see that $\operatorname{Aut}(X_{12})$ is finite by Theorem 6.5(i) and the inequality $12 > 4 + 4$.

The following proposition is useful to study the infinitesimal automorphisms of weighted complete intersections.

Proposition 6.4. *Let $n \geqslant 3$ and $X = X_{d_1,\dots,d_c} \subset \mathbb{P}(a_0, \dots, a_n)$ be a quasi-smooth weighted complete intersection that is not a linear cone. Let $C_X \subset \mathbb{A}^{n+1}$ be the affine cone of X and $\pi\colon C'_X \to X$ be the quotient morphism from $C'_X := C_X \setminus \{0\}$. Let $R := H^0(C_X, \mathcal{O}_{C_X})$ be the coordinate ring of C_X.*

(i) *We have a surjection $H^0(C_X, \mathcal{T}_{C_X})^{\mathbb{C}^*} \twoheadrightarrow H^0(X, \mathcal{T}_X)$, where $\mathcal{T}_{C_X}$ and $\mathcal{T}_X$ are tangent sheaves and $M^{\mathbb{C}^*} \subset M$ is the $\mathbb{C}^*$-invariant part of an R-module M with a $\mathbb{C}^*$-action.*

(ii) *We have a surjection*

$$\left\{\wedge^{n-c}\left(\bigoplus_{i=0}^{n} R(a_i)\right)\right\}_{I_X} \twoheadrightarrow H^0(C_X, \mathcal{T}_{C_X})^{\mathbb{C}^*}\,,$$

where $R(a_i) = R$ is a free R-module with a $\mathbb{C}^*$-action such that $\lambda \cdot s_j = \lambda^{a_i + j} s_j$ for a homogeneous $s_j \in R_j$ of degree j and, for an R-module M with a $\mathbb{C}^*$-action, we set $M_{I_X} := \{x \in M \mid \lambda \cdot x = \lambda^{I_X} x\}$.

Proof. (i) Note that $C'_X \simeq \operatorname{Spec}_X \bigoplus_{i\in\mathbb{Z}} \mathcal{O}_X(i)$ and that we have an exact sequence

$$0 \to (\mathcal{T}_{C'_X/X})^{\vee\vee} \to \mathcal{T}_{C'_X} \to (\pi^*\mathcal{T}_X)^{\vee\vee} \to 0\,.$$

Also note that $(\mathcal{T}_{C'_X/X})^{\vee\vee} \simeq (\pi^*\mathcal{O}_X(1))^{\vee\vee}$ by the above description of C'_X. By taking π_* and its $\mathbb{C}^*$-invariant part, we obtain an exact sequence

$$H^0(C'_X, \mathcal{T}_{C'_X})^{\mathbb{C}^*} \to H^0(X, \mathcal{T}_X) \to H^1(X, \mathcal{O}_X(1))\,.$$

Since $\mathcal{T}_{C_X}$ is reflexive and $H^0(C_X, \mathcal{T}_{C_X}) \simeq H^0(C'_X, \mathcal{T}_{C'_X})$, the required surjectivity follows from $H^1(X, \mathcal{O}_X(1)) = 0$ because $\dim X = n - 1 \geqslant 2$ (cf. [Dol82] and [Ian00, Lemma 7.1]).

(ii) Note that $\dim C_X = n - c + 1$ and that we have $\mathcal{T}_{C_X} \simeq \mathcal{T}_{C_X} \otimes \omega_{C_X} \otimes \omega_{C_X}^{-1} \simeq \Omega^{n-c}_{C_X} \otimes \omega_{C_X}^{-1}$. Since the generator $s \in H^0(\omega_{C_X}^{-1})$ satisfies $\lambda \cdot s = \lambda^{-I_X} s$, we see that

$$H^0(C_X, \mathcal{T}_{C_X})^{\mathbb{C}^*} \simeq H^0(C_X, \Omega^{n-c}_{C_X} \otimes \omega_{C_X}^{-1})^{\mathbb{C}^*} \simeq H^0(C_X, \Omega^{n-c}_{C_X})_{I_X}\,.$$

The surjection $\Omega^{n-c}_{\mathbb{A}^{n+1}}|_{C_X} \twoheadrightarrow \Omega^{n-c}_{C_X}$ induces a surjection

$$H^0(C_X, \Omega^{n-c}_{\mathbb{A}^{n+1}}|_{C_X})_{I_X} \to H^0(C_X, \Omega^{n-c}_{C_X})_{I_X}\,.$$

Since we have $H^0(C_X, \Omega^{n-c}_{\mathbb{A}^{n+1}}|_{C_X}) \simeq \wedge^{n-c}(\bigoplus_{i=0}^{n} R(a_i))$, we obtain the required surjection. □

We now give a sufficient condition for the finiteness of the automorphism groups of quasi-smooth weighted complete intersections as follows. This is a generalization of [PS19, Theorem 1.3], which is based on the calculations in [Fle81].

Theorem 6.5. *Let $X = X_{d_1,\dots,d_c} \subset \mathbb{P}(a_0,\dots,a_n)$ be a quasi-smooth Fano weighted complete intersection that is not a linear cone with $a_0 \leqslant \cdots \leqslant a_n$. Then we have the following:*

(i) *The automorphism group $\mathrm{Aut}(X)$ is finite if $\sum_{j=1}^{c} d_j > a_{n-c} + \cdots + a_n$.*

(ii) *In particular, $\mathrm{Aut}(X)$ is finite if $I_X < \dim X$.*

Proof. (i) To prove the first statement, it is enough to show that $H^0(X, \mathcal{T}_X) = 0$ when $a_0 + \cdots + a_{n-c-1} > I_X$. Let $L := \wedge^{n-c}\big(\bigoplus_{i=0}^{n} R(a_i)\big)$ and $L = \bigoplus_{i\in\mathbb{Z}} L_i$ be the eigen-decomposition with respect to the $\mathbb{C}^*$-action on L. We see that $L_i = 0$ if $i < a_0 + \cdots + a_{n-c-1}$ by the construction of L. In particular, we obtain $L_{I_X} = 0$; thus the first statement follows by Proposition 6.4.

(ii) This follows from the inequality $a_0 + \cdots + a_{n-c-1} \geqslant n - c = \dim X$. □

Proof of Corollary 1.5. If X_d is a Fermat hypersurface, then X_d is K-stable since X_d is K-polystable by Theorem 6.1 and its automorphism group is finite by Theorem 6.5. This implies that general hypersurfaces are K-stable by the openness of uniform K-stability (see [BL22, Theorem A] and [BLX22]) and the equivalence of uniform K-stability and K-stability [LXZ22, Theorem 1.5]. □

Acknowledgements

We thank Giulio Codogni, Kento Fujita, Yuchen Liu, Yuji Odaka, Takuzo Okada and Andrea Petracci for valuable comments. We are grateful to the referee for helpful comments. Part of this project has been realized when the first author visited Università degli Studi di Milano. He would like to thank the university for its support.

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