Azumaya algebras with involution and classical semisimple group schemes

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Abstract

Let S be a non-empty scheme with 2 invertible. In this paper, we present a functor $F\colon \mathrm{AZ}_*^{n'} \to \mathrm{GS}_*^n$, where $\mathrm{AZ}_*^{n'}$ and GS_*^n are fibered categories over Sch_S given, respectively, by degree n' Azumaya algebras with an involution of type $*$ and rank n adjoint group schemes of classical type $*$ with absolutely simple fibers. Here n' is a function of n. We show that this functor is an equivalence of fibered categories using étale descent, thus giving a classification of adjoint (as well as simply connected) group schemes over S, generalizing the well-known case when the base scheme is the spectrum of a field. In particular, this implies that every adjoint group scheme of classical type with absolutely simple fibers is isomorphic to the neutral component of the automorphism group scheme of a unique (up to isomorphism) Azumaya algebra with involution. We also show interesting applications of this classification such as a specialization theorem for isomorphism classes of Azumaya algebra with involution over Henselian local rings and the uniqueness of an integral model for groups with good reduction over discrete valued fields, and we discuss its implications for the Grothendieck–Serre conjecture over certain domains.

1. Introduction

It is well known that the category of rank n (with some n excepted) absolutely simple adjoint (or simply connected) algebraic groups of a given classical type over any field F (char $F \neq 2$) is equivalent to the category of degree n' central simple algebras with involution of analogous type over F. Here n' is a function of n. Moreover, the functor which gives this equivalence is obtained by taking a given central simple algebra with involution over F to the identity component of its automorphism group (or its simply connected cover for the simply connected case). This result is originally due to Weil [Wei60], and the proof of this equivalence can also be found in [KMRT98, § VI.26]. This gives neat classification results for groups of classical types in terms of central simple algebras with involution which can be translated to the well-understood language of sesquilinear forms over division algebras. This kind of classification is very useful for studying many properties of algebraic groups and the projective homogeneous varieties associated with them.

In this paper, we show a similar classification for adjoint (as well as simply connected) group schemes over an arbitrary scheme where is 2 invertible. Since we are in the general case of arbitrary base scheme, we use the language of stacks and gerbes to prove that the fibered category of degree n' Azumaya algebras with an involution of a given type is equivalent to the fibered category of rank n classical adjoint group schemes with absolutely simple fibers of the corresponding type via étale descent. As before, n' is a function of n with some n excepted. This implies that an adjoint group scheme of classical type (see Definition 3.2) with absolutely simple fibers is isomorphic to the neutral component of the automorphism group scheme of a unique (up to isomorphism) Azumaya algebra with involution. We also give a few applications of this classification such as a specialization theorem for Azumaya algebras with involution and the uniqueness of an integral model for group schemes with good reduction. Another interesting corollary is that the Grothendieck–Serre conjecture on principal G-bundles holds whenever G is an adjoint group scheme with absolutely simple fibers over R, where R is a regular local ring containing a field of characteristic different from 2 or R is a semilocal Bézout domain with 2 invertible.

2. Notation

Throughout this paper, S denotes a non-empty scheme with $1/2 \in \mathcal{O}_S(S)$. The category of schemes over S will be denoted by Sch_S. Given a scheme X over S and a point s in S, we denote by $k(s)$, X_s, $X_{\overline{s}}$, respectively, the residue field, fiber and geometric fiber at s. For a presheaf $\mathcal{F}$ over a scheme X, we denote by $\mathcal{F}_x$ its stalk at the point $x \in X$. The ring of $n \times n$ matrices is denoted by $\mathcal{M}_n$. The identity matrix of size n is denoted by I_n. For a sheaf of algebras $\mathcal{A}$, we denote by $\mathcal{A}^{\mathrm{op}}$ the sheaf of opposite algebras given by $U \to \mathcal{A}(U)^{\mathrm{op}}$.

3. Group schemes over an arbitrary scheme

In this section, we recall the necessary results from the literature on group schemes over an arbitrary scheme. The main sources of reference are [SGA3] and [Dem65].

A group scheme G over S is called *reductive* (respectively, semisimple, adjoint, simply connected) if G is affine, smooth over S and for every $s \in S$, the geometric fiber $G_{\overline{s}}$ is a connected reductive (respectively, semisimple, adjoint, simply connected) group (see [Dem65, Définition 2.1.2] and [SGA3, XXII, Définition 4.3.3]).

Let G be an adjoint (or simply connected) group scheme G over S. The *type* (respectively, *rank*) of G at $s \in S$ is the type (respectively, rank) of $G_{\overline{s}}$ (see [SGA3, XXII, Définition 2.7]). The type and rank of G are locally constant functions over S (see [SGA3, XXII, Proposition 2.8 and XIX, Corollaire 2.6]).

Remark 3.1. Any adjoint (respectively, simply connected) group scheme over S is isomorphic to the Weil restriction $R_{S'/S}(G')$ of an adjoint (respectively, simply connected) group scheme G' with absolutely simple fibers over S', where $S' \to S$ is a finite étale cover. Moreover, the pair (S', G') is uniquely determined up to a unique S-isomorphism (see [Con14a, Proposition 6.4.4 and Remark 6.4.5]). Therefore, classification of adjoint group schemes over S reduces to classification of adjoint group schemes with absolutely simple fibers.

DEFINITION 3.2. Let G be an adjoint group scheme over S with absolutely simple fibers. We say that G is of type A (respectively, B, C, D) and rank n if every fiber $G_{\overline{s}}$, $s \in S$, is of type A (respectively, B, C, D) and rank n. In general, we say that G is of classical type if it is of type

A, B, C or D.

PROPOSITION 3.3. *Let G and H be adjoint group schemes over S with absolutely simple fibers of a given type and rank. Then locally for the étale topology on S, the group schemes G and H are isomorphic. In fact, any reductive group scheme with root datum $\mathcal{R}$ is étale locally isomorphic to a unique Chevalley group over $\operatorname{Spec}\mathbb{Z}$ with root datum $\mathcal{R}$.*

Proof. See [Dem65, Corollaire 5.1.4.I and Proposition 5.1.6]. $\square$

Let G be any group scheme of locally finite type over S. In [SGA3, VI$_{\mathrm{B}}$, Définition 3.1], the notion of *the neutral component of G*, denoted by G^0, is defined by the functor

$$T \to G^0(T) = \left\{ u \in G(T) \mid \forall s \in S,\, u_s(T_s) \subset (G_s)^0 \right\}.$$

We refer the reader to [SGA3, VI$_{\mathrm{B}}$, §3] for more details. It can be shown that the neutral component of G is stable under base change. We restate it below.

PROPOSITION 3.4 ([SGA3, VI$_{\mathrm{B}}$, Proposition 3.3]). *Let G be group scheme over S. Then for any scheme $S' \to S$, we have*

$$\left(G \times_S S'\right)^0 = G^0 \times_S S' \,; \tag{3.1}$$

that is, the functor $G \to G^0$ commutes with base change.

We will be using the following result about G^0.

PROPOSITION 3.5 ([Con14a, Proposition 3.1.3]). *Let G be a smooth separated group scheme of finite presentation such that $G_{\bar{s}}^0$ is reductive for all $s \in S$. Then G^0 is a reductive group scheme over S that is open and closed in G.*

With a group scheme G over S, we associate the automorphism functor $\underline{\operatorname{Aut}}(G)$ on Sch_S defined by

$$\underline{\operatorname{Aut}}(G) \colon S' \mapsto \operatorname{Aut}_{S'\text{-grp}}(G_{S'}) \,.$$

PROPOSITION 3.6. *Let G a semisimple group scheme over S. Then the functor $\underline{\operatorname{Aut}}(G)$ is represented by a smooth, affine scheme over S.*

Proof. See [SGA3, XXIV, Théorème 1.3(i) and Corollaire 1.6]. $\square$

Remark 3.7. A representable functor on Sch_S is a sheaf for the fpqc topology [Vis05, Theorem 2.55]. Therefore, if G is semisimple, by Proposition 3.6, $\underline{\operatorname{Aut}}(G)$ is a sheaf for the fpqc topology and hence for the étale topology on Sch_S.

4. Azumaya algebras with involution over an arbitrary scheme

Recall that an *Azumaya algebra* $\mathcal{A}$ over S is an $\mathcal{O}_S$-algebra that is locally free and of finite type as an $\mathcal{O}_S$-module such that the canonical homomorphism

$$\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \to \operatorname{End}_{\mathcal{O}_S\text{-mod}}(\mathcal{A}) \,, \quad a \otimes b \mapsto (x \mapsto a \cdot x \cdot b)$$

is an isomorphism. This implies that there is an étale covering $\{U_i \to S\}$ such that $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_{U_i} \simeq \mathcal{M}_{n_i}(\mathcal{O}_{U_i})$ for some n_i. If $n_i = n$ for all i (this happens for example when S is connected), we call $\mathcal{A}$ an Azumaya algebra of *degree n* over S. Also recall that for an Azumaya algebra $\mathcal{A}$ over S, $\mathcal{A}_s \otimes k(s)$ is a central simple algebra over $k(s)$ for every $s \in S$ (see [Mil80, §IV.2, Proposition 2.1]).

Involutions on central simple algebras are well studied in the literature [KMRT98]. In a similar fashion, one can also define involution on Azumaya algebras over the scheme S. This is discussed in detail in [KPS90, PS92]; we briefly recall it now. A classification of involutions on Azumaya algebras in a more general setting can be found in [FW20, §5].

An *involution of first kind* σ on $\mathcal{A}$ is an isomorphism of $\mathcal{O}_S$-algebras

$$\sigma\colon \mathcal{A} \to \mathcal{A}^{\mathrm{op}}$$

such that $\sigma^{\mathrm{op}} \circ \sigma$ is the identity. The involution σ on A is said to be of *orthogonal type* (respectively, *symplectic type*) if étale locally on S, it corresponds to a non-degenerate symmetric (respectively, skew-symmetric) bilinear form on a locally free $\mathcal{O}_S$-module with values in a line bundle over S (see [PS92, §1.1]).

Now let $\pi\colon T \to S$ be an étale covering of degree 2. Let $\mathcal{A}$ be an Azumaya algebra over T. An involution σ of *second kind* (also known as *unitary type*) on $\mathcal{A}$ is an anti-automorphism on $\mathcal{A}$ of order 2 which on $\mathcal{O}_T$ restricts to the non-trivial element of the Galois group of the covering. Locally for the étale topology on S, unitary involutions correspond to hermitian forms on locally free $\mathcal{O}_T$-modules [PS92, §1.2].

The data of an Azumaya algebra $\mathcal{A}$ with a involution σ over S is denoted by $(\mathcal{A}, \sigma)$. A homomorphism between Azumaya algebras with involution is a homomorphism between the algebras which respects the involution structure.

DEFINITION 4.1. In the case above where $\mathcal{A}$ is an Azumaya algebra over a quadratic étale extension T of S and σ is unitary, we will make a slight abuse of notation and call $(\mathcal{A}, \sigma)$ an *Azumaya algebra with unitary involution over S* even though the center of $\mathcal{A}$ is not $\mathcal{O}_S$. This agrees with the corresponding notion of central simple algebras with unitary involution defined in [KMRT98, §I.2.B].

Remark 4.2. If (A, σ) is a degree n Azumaya algebra with involution over S where σ is of a given type, then for any $s \in S$, the pair $(A_s \otimes k(s), \sigma_s \otimes k(s))$ is a degree n central simple algebra with involution of the same type over $k(s)$.

We now give an étale local description of Azumaya algebras with involution.

PROPOSITION 4.3. *Let (A, σ) be a degree n Azumaya algebra with involution over S. Then locally for the étale topology on S, we have*

(1) $(\mathcal{A}, \sigma) \simeq (\mathcal{M}_n, \mathrm{tr})$ *when σ is of orthogonal type, where $\mathrm{tr}\colon A \to A^{\mathrm{tr}}$ is the transpose involution;*

(2) $(\mathcal{A}, \sigma) \simeq (\mathcal{M}_{2m}, \mathrm{sp})$ *when σ is of symplectic type, where sp is the involution on $\mathcal{M}_{2m}$ given by $A \to J_m A^{\mathrm{tr}} J_m^{-1}$ with $J_m = \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}$ the standard matrix associated with an alternating form;*

(3) $(\mathcal{A}, \sigma) \simeq (\mathcal{M}_n \times \mathcal{M}_n^{\mathrm{op}}, \epsilon)$ *when σ is of unitary type, where $\epsilon\colon (A, B^{\mathrm{op}}) \to (B, A^{\mathrm{op}})$ is the exchange involution.*

Proof. All of the above are well known if the base scheme S is the spectrum of a field (see [KMRT98]). For the general case, proofs of statements (1) and (2) can be easily derived and can also be found in the literature. See for example [PS92, §1.1] and [Knu91, §III.8.5]. We could not find a proof of statement (3) anywhere in the literature, so we give one here.

In this case, $\mathcal{A}$ an Azumaya algebra with center $\mathcal{O}_T$, where $T \to S$ is a degree 2 étale cover of S with the non-trivial element in its Galois group denoted by τ. Consider the degree 2 étale

cover $p\colon T \times_S T \to T$ obtained via the étale base change $\pi\colon T \to S$. It follows from Galois theory of schemes that $(T \times_S T, \pi^*\tau) \simeq (T \coprod T, \mathrm{ex})$ (see the proof of [Len08, Theorem 5.10]), where $\mathrm{ex}\colon (x, y) \to (y, x)$. Now $p^*\mathcal{A}$ is an Azumaya algebra over $T \coprod T$. Therefore, the center $\mathcal{O}_T \times \mathcal{O}_T$ of $p^*\mathcal{A}$ contains the idempotent $i = (1, 0)$, where $\mathrm{ex}(i) = 1 - i$. Note that $B = i(p^*\mathcal{A})$ is an Azumaya algebra over T, and we have an isomorphism of Azumaya algebras with unitary involution given by

$$(p^*\mathcal{A}, p^*\sigma) \xrightarrow{\simeq} \left(B \times B^{\mathrm{op}}, \epsilon\right), \quad a \mapsto \left(i \cdot a, (i \cdot (p^*\sigma(a)))^{\mathrm{op}}\right).$$

By taking a suitable étale covering of T which splits B, we obtain statement (3). □

DEFINITION 4.4. With notation as in Proposition 4.3, we say that a degree n Azumaya algebra with unitary (respectively, orthogonal, symplectic) involution over S is split if it is isomorphic to $\left(\mathcal{M}_n \times \mathcal{M}_n^{\mathrm{op}}, \epsilon\right)$ (respectively, $(\mathcal{M}_n, \mathrm{tr})$, $(\mathcal{M}_n, \mathrm{sp})$).

5. The group scheme of automorphisms of Azumaya algebras with involution

Let $(\mathcal{A}, \sigma)$ be a degree n Azumaya algebra with involution of any type over S. Consider the functor

$$\underline{\mathrm{Aut}}(\mathcal{A}, \sigma)\colon \mathrm{Sch}_S \to \mathrm{Groups}, \quad \left(U \xrightarrow{i} S\right) \mapsto \mathrm{Aut}_{\mathcal{O}_U\text{-alg}}(i^*\mathcal{A}, i^*\sigma),$$

where $\mathrm{Aut}_{\mathcal{O}_U\text{-alg}}(i^*\mathcal{A}, i^*\sigma)$ is the group of $\mathcal{O}_U$-algebra automorphisms of $i^*\mathcal{A}$ compatible with the involution $i^*\sigma$.

NOTATION. If a functor $\underline{F}$ on Sch_S is representable, let us denote the representing scheme by F.

THEOREM 5.1. *The functor $\underline{\mathrm{Aut}}(\mathcal{A}, \sigma)$ is representable by a smooth, affine group scheme over S.*

Proof. Let $S' \to S$ be an fpqc morphism. Recall that the category of affine S-schemes is equivalent to the category of affine S'-schemes with descent data. To see this, use [BLR90, Chapter 6, Theorem 4] and the fact that for any scheme X, the category of affine X-schemes is anti-equivalent to the category of quasi-coherent sheaves of $\mathcal{O}_X$-algebras (see [Sta20, Tag 01S5, Lemma 29.11.5] or [Vis05, Theorem 4.33]). Moreover, the properties smooth and affine are fpqc local over the base (see [Sta20, Tag 02YJ] or [EGAIV$_2$, Propositions 2.7.1 and 6.8.3]). Therefore, by Proposition 4.3, it suffices to prove the theorem when $(\mathcal{A}, \sigma)$ is split. So assume that $(\mathcal{A}, \sigma)$ is split. Note that in this case, the functor

$$\underline{\mathrm{Aut}}(\mathcal{A})\colon \left(U \xrightarrow{i} S\right) \mapsto \mathrm{Aut}_{\mathcal{O}_U\text{-alg}}(i^*\mathcal{A})$$

is representable by a closed subscheme of the affine $\mathbb{Z}$-scheme $\mathrm{End}_{\mathcal{O}_S\text{-mod}}(\mathcal{A}) \simeq \mathcal{M}_r$, where $r = \dim_{\mathcal{O}_S}(\mathcal{A})$ (see [Mil80, §IV.2]). Module homomorphisms of $\mathcal{A}$ that respect the involution can be expressed as vanishing of polynomials, and hence the set of such homomorphisms is representable by a closed subscheme of $\mathcal{M}_{r^2}$. The intersection of these two subschemes represents $\underline{\mathrm{Aut}}(\mathcal{A}, \sigma)$ and hence is an affine scheme over S.

Now we prove that this affine scheme is smooth. For the unitary case, consider the functor

$$\underline{\mathrm{Aut}}_{\mathcal{O}_S \times \mathcal{O}_S}\left(\mathcal{M}_n \times \mathcal{M}_n^{\mathrm{op}}, \epsilon\right)\colon \left(U \xrightarrow{i} S\right) \mapsto \mathrm{Aut}_{\mathcal{O}_U \times \mathcal{O}_U\text{-alg}}\left(\mathcal{M}_n(\mathcal{O}_U) \times \mathcal{M}_n(\mathcal{O}_U)^{\mathrm{op}}, \epsilon\right)$$
$$\simeq \mathrm{Aut}_{\mathcal{O}_U\text{-alg}}(\mathcal{M}_n(\mathcal{O}_U)).$$

This functor is representable by a smooth affine $\mathbb{Z}$-scheme GL_n (see [Mil80, §IV.2] for affineness and [DG70, §II.5, Proposition 2.7] for smoothness). The functor $\underline{\mathrm{Aut}}_{\mathcal{O}_S}(\mathcal{O}_S \times \mathcal{O}_S)$ is representable

by the finite group scheme $\mathbb{Z}/2\mathbb{Z}$. Since representable functors on Sch_s are sheaves for the fpqc topology [Vis05, Theorem 2.55], we have the following exact sequence of sheaves over S:

$$1 \to \underline{\mathrm{Aut}}_{\mathcal{O}_S \times \mathcal{O}_S}\big(\mathcal{M}_n \times \mathcal{M}_n^{\mathrm{op}}, \epsilon\big) \to \underline{\mathrm{Aut}}\big(\mathcal{M}_n \times \mathcal{M}_n^{\mathrm{op}}, \epsilon\big) \to \underline{\mathrm{Aut}}_{\mathcal{O}_S}(\mathcal{O}_S \times \mathcal{O}_S) \to 1\,. \qquad (5.1)$$

The smoothness of $\mathrm{Aut}(\mathcal{M}_n \times \mathcal{M}_n^{\mathrm{op}}, \epsilon)$ now follows from [SGA3, VI$_\mathrm{B}$, Proposition 9.2].

For the other cases, note that $\mathrm{Aut}(\mathcal{M}_n, \mathrm{sp}) \simeq \mathrm{Sp}_n/Z$ and $\mathrm{Aut}(\mathcal{M}_n, \mathrm{tr}) \simeq \mathrm{O}_n/Z$ (as fpqc quotients), where Sp_n, O_n denote, respectively, the symplectic group and orthogonal group and Z denotes their respective centers. The required smoothness results for these group schemes follow from [DG70, §II.5, 2.7] and [SGA3, VI$_\mathrm{B}$, Proposition 9.2]. $\qquad \square$

Remark 5.2. By [Vis05, Theorem 2.55] and Theorem 5.1, the functor $\underline{\mathrm{Aut}}(\mathcal{A}, \sigma)$ is a sheaf for the fpqc topology.

DEFINITION 5.3. Let $\mathrm{Aut}^0(\mathcal{A}, \sigma) := \mathrm{Aut}(\mathcal{A}, \sigma)^0$ denote the neutral component of $\mathrm{Aut}(\mathcal{A}, \sigma)$ defined in Section 3.

THEOREM 5.4. *Let $(\mathcal{A}, \sigma)$ be a degree n Azumaya algebra with involution over S, where $n \neq 2, 4$ whenever σ is orthogonal. Then $\mathrm{Aut}^0(\mathcal{A}, \sigma)$ is an adjoint group scheme over S with absolutely simple fibers. Moreover, $\mathrm{Aut}^0(\mathcal{A}, \sigma)$ is of type A if σ is unitary, type B if σ is orthogonal and n is odd, type C if σ is symplectic and type D if σ is orthogonal and n is even.*

Proof. By Lemma 7.1 below, for every $s \in S$, we have

$$\mathrm{Aut}^0(\mathcal{A}, \sigma)_s = \big(\mathrm{Aut}(\mathcal{A}, \sigma)_s\big)^0 \simeq \mathrm{Aut}\big(\mathcal{A}_{k(s)}, \sigma_{k(s)}\big)^0 = \mathrm{Aut}^0\big(\mathcal{A}_{k(s)}, \sigma_{k(s)}\big)\,. \qquad (5.2)$$

Now $\mathrm{Aut}^0(\mathcal{A}_{k(s)}, \sigma_{k(s)})$ is an absolutely simple adjoint algebraic group over $k(s)$ (see [KMRT98, §VI.26]). This together with Theorem 5.1 and Proposition 3.5 shows that $\mathrm{Aut}^0(\mathcal{A}, \sigma)$ is a reductive group scheme over S. The rest of the claim follows from (5.2), unraveling the definitions in Section 3 and [KMRT98, §VI.26]. $\qquad \square$

Remark 5.5. In the above theorem, when $n = 2$, $\mathrm{Aut}^0(\mathcal{A}, \sigma)$ is not adjoint, and when $n = 4$, the fibers of $\mathrm{Aut}^0(\mathcal{A}, \sigma)$ are not absolutely simple.

DEFINITION 5.6. Based on Theorem 5.4, we say that the pair $(\mathcal{A}, \sigma)$ is of type $*$, where $*$ is

- A if σ is unitary,
- B if σ is orthogonal and the degree of $\mathcal{A}$ is odd,
- C if σ is symplectic,
- D if σ is orthogonal and the degree of $\mathcal{A}$ is even.

6. The fibered categories $\mathbf{Az}_*^n$ and $\mathbf{GS}_*^n$

We will now construct the fibered categories of degree n Azumaya algebras with involution of a given type and the fibered category of rank n adjoint group schemes with absolutely simple fibers of a given type over Sch_S and show that they are stacks (in fact gerbes) with respect to the étale topology. The classical reference for stacks and gerbes is [Gir71]. Other references are [Vis05, Moe02] and the appendix in [DM18].

DEFINITION 6.1. Given a scheme U in the category Sch_S, let $\mathrm{Az}_*^n(U)$ denote the groupoid of degree n Azumaya algebras with involution of type $*$ (where $*$ is A, B, C or D) over U (morphisms are isomorphisms of Azumaya algebras over U that respect the involution structures). For any morphism $f\colon V \to U$ in Sch_S, we have a pullback functor

$$\mathrm{Az}_*^n(U) \to \mathrm{Az}_*^n(V)\,, \quad (\mathcal{A}, \sigma) \mapsto f^*(\mathcal{A}, \sigma) := \left(f^*\mathcal{A}, f^*\sigma\right),$$

which makes the assignment $U \to \mathrm{Az}_*^n(U)$ a pseudo-functor on Sch_S (see [Vis05, Chapter 3]).

DEFINITION 6.2. The fibered category of degree n Azumaya algebras with involution of type $*$ associated with the above pseudo-functor is denoted by $\mathrm{Az}_*^n \to \mathrm{Sch}_S$.

DEFINITION 6.3. The fibered category $\mathrm{GS}_*^n \to \mathrm{Sch}_S$ of rank n adjoint group schemes with absolutely simple fibers of type $*$ (where $*$ is A, B, C or D) is defined in a similar way, where morphisms in every fiber $\mathrm{GS}_*^n(U)$ are isomorphisms.

PROPOSITION 6.4. *The fibered categories $\mathrm{Az}_*^n \to \mathrm{Sch}_S$ and $\mathrm{GS}_*^n \to \mathrm{Sch}_S$ are stacks for the étale topology. In fact, they are gerbes.*

Proof. By standard arguments from decent theory, it is easy to see that $\mathrm{GS}_*^n \to \mathrm{Sch}_S$ is a stack for the étale topology. It is a gerbe by Proposition 3.3. For the case of Az_*^n, we note from descent theory that quasi-coherent sheaves (as well as the morphisms) satisfy descent for the fpqc topology [BLR90, Chapter 6, Theorem 4] and hence also for the étale topology. Quasi-coherent sheaves together with additional structure such as an algebra structure and involutions also descend (see for example [Vis05, § 4.2.2 and § 4.2.3] or [KO74, Theorem II.3.4]). This shows that $\mathrm{Az}_*^n \to \mathrm{Sch}_S$ is a stack for the étale topology. It is a gerbe by Proposition 4.3. $\square$

7. Equivalence of $\mathrm{Az}_*^{n'}$ and GS_*^n

Let $\mathrm{Az}_*^n \to \mathrm{Sch}_S$ and $\mathrm{GS}_*^n \to \mathrm{Sch}_S$ be the fibered categories defined in the previous section. In this section, we describe a morphism between $\mathrm{Az}_*^{n'}$ and GS_*^n (where n' is determined by n) that will yield the required equivalence of fibered categories. We will need the following lemma.

LEMMA 7.1. *Let $(\mathcal{A}, \sigma)$ be an Azumaya algebra with involution over a scheme X. The assignment $(\mathcal{A}, \sigma) \to \mathrm{Aut}^0(\mathcal{A}, \sigma)$ respects pullbacks; that is, for $f\colon Y \to X$, we have a canonical isomorphism*

$$f^*\left(\mathrm{Aut}^0(\mathcal{A}, \sigma)\right) \simeq \mathrm{Aut}^0(f^*(\mathcal{A}, \sigma))\,.$$

Proof. We note that $f^*(\underline{\mathrm{Aut}}(\mathcal{A}, \sigma)) = \underline{\mathrm{Aut}}(f^*(\mathcal{A}, \sigma))$. Hence by the Yoneda lemma, there is a canonical isomorphism $f^*(\mathrm{Aut}(\mathcal{A}, \sigma)) \simeq \mathrm{Aut}(f^*(\mathcal{A}, \sigma))$. This together with Proposition 3.4 proves the claim. $\square$

THEOREM 7.2. *The functor*

$$\mathrm{Aut}^{\mathbf{0}}\colon \mathrm{Az}_*^{n'} \to \mathrm{GS}_*^n\,, \quad (\mathcal{A}, \sigma) \mapsto \mathrm{Aut}^0(\mathcal{A}, \sigma)\,, \ \left((\mathcal{A}, \sigma) \xrightarrow{i} (\mathcal{B}, \tau)\right) \mapsto \left(\phi \to i \circ \phi \circ i^{-1}\right)$$

is an equivalence of fibered categories, where

- $n > 1$ *and* $n' = n + 1$ *if* $*$ *is of type A;*
- $n' = 2n + 1$ *if* $*$ *is of type B;*
- $n' = 2n$ *if* $*$ *is of type C;*
- $n > 2$, $n \neq 4$ *and* $n' = 2n$ *if* $*$ *is of type D.*

Proof. The functor $\mathrm{Aut}^{\mathbf{0}}$ defines a morphism of fibered categories by Lemma 7.1. Since $\mathrm{Az}^{n'}_* \to$ Sch_S and $\mathrm{GS}^n_* \to \mathrm{Sch}_S$ are gerbes by Proposition 6.4, to show that $\mathrm{Aut}^{\mathbf{0}}$ is an equivalence, it suffices to show that for any object $(\mathcal{A}, \sigma)$ in $\mathrm{Az}^{n'}_*(S)$ (say the split object), $\mathrm{Aut}^{\mathbf{0}}$ induces an isomorphism between the sheaves $\underline{\mathrm{Aut}}(\mathcal{A}, \sigma)$ and $\underline{\mathrm{Aut}}\left(\mathrm{Aut}^0(\mathcal{A}, \sigma)\right)$ (see [Gir71, § IV.3.1] or [Moe02]). Now by Theorem 5.1 and Proposition 3.6, both $\underline{\mathrm{Aut}}(\mathcal{A}, \sigma)$ and $\underline{\mathrm{Aut}}\left(\mathrm{Aut}^0(\mathcal{A}, \sigma)\right)$ are represented by smooth affine group schemes over S denoted by $\mathrm{Aut}(\mathcal{A}, \sigma)$ and $\mathrm{Aut}\left(\mathrm{Aut}^0(\mathcal{A}, \sigma)\right)$, respectively. So it suffices to check that $\mathrm{Aut}^{\mathbf{0}}$ induces an isomorphism at every fiber [Sta20, Tag 039E], [Sta20, Tag 025G]. Again by the proof of Lemma 7.1, we see that for every $s \in S$, the functor $\mathrm{Aut}^{\mathbf{0}}$ induces morphism of schemes over $k(s)$:

$$\mathrm{Aut}^{\mathbf{0}}{}_{k(s)} \colon \mathrm{Aut}\left(\mathcal{A}_{k(s)}, \sigma_{k(s)}\right) \to \mathrm{Aut}\left(\mathrm{Aut}^0\left(\mathcal{A}_{k(s)}, \sigma_{k(s)}\right)\right).$$

Now we are in the case of fields, and the fact that $\mathrm{Aut}^{\mathbf{0}}{}_{k(s)}$ is an isomorphism follows from [KMRT98, § VI.26]. □

Remark 7.3. In the case $n = 1$ for type A, the above equivalence holds if Az^2_A is defined to be the fibered category of Azumaya algebras of degree 2 (see [KMRT98, § 26.A, p. 366]). Similarly, in the case $n = 2$ for type D, since the corresponding root system is not irreducible, one has to remove the phrase "with absolutely simple fibers" in the definition of GS^D_2 for the above equivalence to hold. The proofs are similar.

Remark 7.4. The classification of simply connected group schemes with absolutely simple fibers over S is similar to the adjoint case. The functor in this case is given by

$$\overline{\mathrm{Aut}^{\mathbf{0}}} \colon (\mathcal{A}, \sigma) \to \overline{\mathrm{Aut}^0(\mathcal{A}, \sigma)},$$

where for a semisimple group scheme G over S, we denote by $\overline{G}$ its unique simply connected cover (see [Con14a, Exercise 6.5.2]).

Remark 7.5. Recall that when the base scheme is the spectrum of a field of characteristic 2, the groups $\mathrm{Aut}^0(A, \sigma)$ are not smooth when σ is orthogonal [KMRT98, § 23, p. 347]. Hence when 2 is not invertible in S, things look a little different for types B and D. One could possibly work fppf locally instead of étale locally for these types to get a similar classification as in the case of fields [Con14b, Lemma C.2.1]. We did not want to deal with these subtle technicalities in this paper that will take away the nice picture when 2 is invertible.

8. Applications

In this section, we show some interesting consequences of Theorem 7.2.

For a ring R, let $P_{n'}(R, *)$ denote the isomorphism classes of degree n' Azumaya algebras with involution of type $*$ over R.

COROLLARY 8.1. *Let R be a Henselian local ring with residue field k. Assume char $k \neq 2$. Then the restriction map $P_{n'}(R, *) \to P_{n'}(k, *)$ is bijective for every $(n', *)$ listed in Theorem 7.2.*

Proof. This follows from Lemma 7.1, Theorem 7.2 and [SGA3, XXIV, Proposition 1.21]. □

Remark 8.2. The above corollary is a generalization of a similar statement about isomorphism classes of Azumaya algebras; see [Gro95, Theorem 6.1].

We will recall the concept of *good reduction* for algebraic groups defined in [CRR19]. Let k be a discrete valued field with valuation v. Let k_v, $\mathcal{O}_v$ and $k^{(v)}$ denote, respectively, the completion

of k, the valuation ring of k_v and the residue field. An absolutely almost simple linear algebraic group G over k is said to have *good reduction at* v if there exists a reductive group scheme $\mathcal{G}$ over $\mathcal{O}_v$ such that $\mathcal{G} \otimes_{\mathcal{O}_v} k_v \simeq G \otimes_k k_v$. In this case, let us call $\mathcal{G}$ an $\mathcal{O}_v$-*model for* G. Studying good reduction of G has many applications such as computing the genus of algebraic group and proving Hasse principles. We refer the reader to [CRR19] for more details.

COROLLARY 8.3. *Let G be a rank n absolutely simple adjoint (or simply connected) algebraic group of classical type ($n \neq 4$ if G is of type D) over a discrete valued field k with valuation v. Assume that the characteristic of the residue field $k^{(v)}$ is different from 2. Suppose that G has good reduction at v. Then any two $\mathcal{O}_v$-models of G are isomorphic. In other words, an $\mathcal{O}_v$-model of G if it exists is unique up to isomorphism.*

Proof. This follows from Lemma 7.1, Theorem 7.2 and [BVG14, Theorem 3.7]. □

Remark 8.4. The classification in Theorem 7.2 can be used to translate the notion of good reduction of absolutely simple adjoint (or simply connected) classical algebraic groups to the notion of good reduction of the underlying sesquilinear forms over division algebras. See for example [Sri20, Remark 4.3].

Let R be a regular local ring, and let G be a reductive groups scheme over R. A conjecture of Grothendieck and Serre ([Gro58, § 5, Remarque 3, pp. 26–27], [Gro68, Remarque 1.11(a)] and [Ser58, § 5.5, Remarque, p. 31]) states that rationally trivial principal G-homogeneous spaces are trivial; that is, the kernel of the canonical map

$$H^1(R, G) \to H^1(K, G), \tag{8.1}$$

where K is the fraction field of R, is trivial. While the conjecture is still open to be proved in complete generality, the proofs for various cases of R and G have been established since Ye. A. Nisnevich's thesis (Harvard, 1982). If R is a regular local ring containing a field of characteristic different from 2 or if R is a semilocal Bézout domain with 2 invertible, Panin [Pan03, Theorem 1.1] and Beke [BVG14, Theorem 3.7], respectively, have proved that any two Azumaya algebras with involutions over R that are rationally isomorphic (that is, isomorphic over the fraction field of R) are already isomorphic. This implies that for the above cases of R, the kernel of the map in (8.1) is trivial when $G \simeq \text{Aut}(A, \sigma)$, the automorphism scheme of an Azumaya algebra with involution over R. Since any rank n adjoint group scheme of classical type ($n \neq 4$ if the group is of type D) over R with absolutely simple fibers is isomorphic to $\text{Aut}^0(A, \sigma)$ for some (A, σ) by Theorem 7.2, we conclude the following.

COROLLARY 8.5. *Let R be a integral domain with 2 invertible that satisfies the property that any Azumaya algebra with involution over R that is split over the fraction field of R is already split. Then the Grothendieck–Serre conjecture is true for any rank n adjoint group scheme of classical type ($n \neq 4$ if the group is of type D) with absolutely simple fibers over R. In particular, this happens when R is a regular local ring containing a field of characteristic different from 2 or when R is a semilocal Bézout domain with 2 invertible.*

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