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On the rationality of Fano–Enriques threefolds

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Abstract

A Fano–Enriques threefold is a 3-dimensional non-Gorenstein Fano variety of index 1 with at most canonical singularities. We study the birational geometry of Fano–Enriques threefolds with terminal cyclic quotient singularities. We investigate their rationality and also provide an example of a Fano–Enriques threefold whose pliability is 9, that is, a Fano–Enriques threefold birationally equivalent to exactly nine Mori fibre spaces in the Sarkisov category.

1. Introduction

Throughout this paper we work over the field of complex numbers $\mathbb{C}$, and all varieties are assumed to be projective unless stated otherwise.

Three-dimensional varieties whose hyperplane sections are Enriques surfaces were first introduced and studied by G. Fano in [Fan38]. He has attempted to classify such varieties; however, the provided proof contained inaccuracies due to the lack of a proper theory of 3-dimensional birational geometry at that time. Later, the ideas of G. Fano were summarised and presented in a modern mathematical language by A. Conte and J. P. Murre in [CM85]. The varieties described above are said to be Fano–Enriques threefolds, and their modern definition is the following.

Definition 1.1. A 3-dimensional variety X is called a *Fano–Enriques threefold* if it has at most canonical singularities, $-K_X$ is not a Cartier divisor, and $-K_X \sim_{\mathbb{Q}} H$ for some ample Cartier divisor H. The numbers $-K_X^3$ and $g(X) = -\frac{1}{2}K_X^3 + 1$ are called the *degree* and *genus* of X, respectively.

Remark 1.2. Actually, one has $-2K_X \sim 2H$ by the theorems below.

The connection between the threefolds studied by G. Fano in [Fan38] and Fano–Enriques threefolds is reflected in the following theorems.

Theorem 1.3 ([Pro95, Proposition 3.3]). *Let X be a Fano–Enriques threefold and $-K_X \sim_{\mathbb{Q}} H$, where H is an ample Cartier divisor. Then a general surface in the linear system $|H|$ is an Enriques surface with at most canonical singularities. It is smooth if the singularities of X are isolated and $-K_X^3 \neq 2$.*

Theorem 1.4 ([Che96]). *Let X be a normal threefold and $H \subset X$ be an Enriques surface with at most canonical singularities. Assume that H is an ample Cartier divisor on X. Then $-2K_X \sim 2H$, and X is either a Fano–Enriques threefold or a contraction of a section of* $\mathrm{Proj}(\mathcal{O}_H \oplus \mathcal{O}_H(H|_H))$.

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Since $-2K_X \sim 2H$ for all Fano–Enriques threefolds, then one can consider the so-called *canonical covering* of X, that is, the variety $V = \operatorname{Spec}(\mathcal{O}_X \oplus \mathcal{O}_X(K_X + H))$. It can be proved that V is a Fano variety with Gorenstein canonical singularities and the natural double covering $\pi\colon V \to X$ is ramified at finitely many points, which are exactly the non-Gorenstein points of X.

Although the classification of Fano–Enriques threefolds is as yet unknown, there are some partial results. For example, it has been proved by I. Cheltsov in [Che99] that the genus $g(X)$ is bounded by 47. However, most likely this bound is far from being sharp because if H is a smooth Enriques surface, then $g(X) \leqslant 17$, and this bound is sharp by [Pro07, Theorem 1.1] or [KLM11, Theorem 1.5]. Also, in [Bay94] L. Bayle has classified Fano–Enriques threefolds whose canonical covering V is smooth (see also [San95]). He has proved the following theorem.

Theorem 1.5 ([Bay94]). *Let X be a Fano–Enriques threefold, and let V be its canonical covering. Assume that V is smooth. Then X has eight singular points that are quotient singularities of type $\frac{1}{2}(1,1,1)$, and V is one of the following:*

(1) *the double covering of a quadric ramified in a divisor of degree 8, $g(X) = 2$;*
(2) *the complete intersection of three quadrics in $\mathbb{P}^6$, $g(X) = 3$;*
(3) *the blow-up of a smooth hypersurface of degree 4 in $\mathbb{P}(1^4, 2)$ along an elliptic curve cut out by two hypersurfaces of degree 1, $g(X) = 3$;*
(4) $\mathbb{P}^1 \times S_2$, $g(X) = 4$;
(5) *the double covering of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ramified in a divisor of degree $(2,2,2)$, $g(X) = 4$;*
(6) *the blow-up of a smooth complete intersection of two quadrics in $\mathbb{P}^5$ along an elliptic curve cut out by two hyperplane sections, $g(X) = 5$;*
(7) *the hypersurface of degree 4 in $\mathbb{P}(1^4, 2)$, $g(X) = 5$;*
(8) *the complete intersection of three divisors of degree $(1,1)$ in $\mathbb{P}^3 \times \mathbb{P}^3$, $g(X) = 6$;*
(9) $\mathbb{P}^1 \times S_4$, $g(X) = 7$;
(10) *the divisor of degree $(1,1,1,1)$ in $(\mathbb{P}^1)^4$, $g(X) = 7$;*
(11) *the blow-up of the cone over a smooth quadric surface $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ along the disjoint union of the vertex and a smooth elliptic curve on $\mathbb{P}^1 \times \mathbb{P}^1$, $g(X) = 8$;*
(12) *the compete intersection of two quadrics in $\mathbb{P}^5$, $g(X) = 9$;*
(13) $\mathbb{P}^1 \times S_6$, $g(X) = 10$;
(14) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $g(X) = 13$,

where S_d is a smooth del Pezzo surface of degree d.

Moreover, in each deformation family listed above, there exist a smooth Fano threefold V and an involution $\sigma \in \operatorname{Aut}(V)$ that fixes finitely many (actually eight) points on V, such that the quotient $V/\langle\sigma\rangle$ is a Fano–Enriques threefold.

Remark 1.6. By [Min99] all Fano–Enriques threefolds with at most terminal singularities admit a $\mathbb{Q}$-smoothing; that is, any such variety is a deformation of one from Theorem 1.5.

The rationality of Fano–Enriques threefolds is also an open problem that goes back to the works of G. Fano and F. Enriques. The following classic result about it is due to L. P. Botta and A. Verra.

Theorem 1.7 ([BV83]). *Let X be a general Fano–Enriques threefold whose canonical covering belongs to the deformation family (5) from Theorem 1.5. Then X is not rational.*

Remark 1.8. More precisely, L. P. Botta and A. Verra have considered a threefold $Y \subset \mathbb{P}^4$ defined by

$$x_1x_2x_3x_4\left(x_0^2 + x_0\sum_{i=1}^{4} a_ix_i + \sum_{i,j=1}^{4} b_{ij}x_ix_j\right) + c_1x_2^2x_3^2x_4^2 + c_2x_1^2x_3^2x_4^2 + c_3x_1^2x_2^2x_4^2 + c_4x_1^2x_2^2x_3^2 = 0\,,$$

where a_i, b_{ij}, c_i are sufficiently general complex numbers, and proved that Y is not rational. However, one can see that the normalisation of Y is the Fano–Enriques threefold corresponding to the deformation family (5) from L. Bayle's list.

Theorem 1.9 ([Che97]). *Let X be a Fano–Enriques threefold. If $g(X) \geqslant 6$, then X is rational.*

In the case when V is smooth, more can be said regarding the rationality of X.

Theorem 1.10 ([Che04, Corollary 18]). *Let X be a Fano–Enriques threefold. Assume that X is not rational and its canonical covering V is smooth. Then V belongs to one of the deformation families (1), (2) or (5) from Theorem 1.5.*

In the same paper I. Cheltsov stated the following conjecture.

Conjecture 1.11 ([Che04, Conjecture 19]). *Let X be a Fano–Enriques threefold such that its canonical covering V is smooth. Let V be one of the following:*

(1) *the double covering of a quadric ramified in a divisor of degree 8;*
(2) *the complete intersection of three quadrics in $\mathbb{P}^6$;*
(3) *the double covering of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ramified in a divisor of degree $(2,2,2)$.*

Then X is not rational.

We know from Theorem 1.7 that this conjecture holds for a general X if its canonical covering belongs to the deformation family (5) of Theorem 1.5. The goal of this paper is to study the rationality of the remaining deformation families and give a positive answer to Conjecture 1.11 for a general member. The main results obtained in this paper are the following.

Main Theorem 1. *Let X be a general Fano–Enriques threefold whose canonical covering V is the smooth intersection of three quadrics in $\mathbb{P}^6$. Then X is not rational.*

Recall from [CM04] that the *pliability* of a variety X is defined to be the set of all Mori fibre spaces birational to X up to a square equivalence (see Definition 3.9) in Sarkisov category. It is denoted by $\mathcal{P}(X)$.

Main Theorem 2. *Let X be a Fano–Enriques threefold whose canonical covering V is the double covering of a quadric ramified in a divisor of degree 8. Then there exist eight Sarkisov links of type I:*

$$\begin{tikzcd} & \widetilde{X}_i \arrow[dl, "\kappa_i"'] \arrow[r, dashed, "\psi_i"] & U_i \arrow[d, "f_i"] \\ X & & \mathbb{P}^1, \end{tikzcd}$$

where κ_i is the Kawamata blow-up of a non-Gorenstein point q_i, the map ψ_i is a flop and $f_i\colon U_i \to \mathbb{P}^1$ is a Mori fibre space whose general fibre is a del Pezzo surfaces of degree 1 for $i = 1, \dots, 8$.

Moreover, let $\Phi\colon X \dashrightarrow W$ be a birational map to a Mori fibre space W/B. Then one has the following possibilities:

- *$B = \mathrm{pt}$, and Φ is an isomorphism;*
- *$B \cong \mathbb{P}^1$, and for $i = 1, \ldots, 8$ there exists the following commutative diagram:*

$$\begin{tikzcd} X \arrow[dr, dashed, "\rho_i"'] \arrow[drr, dashed, bend left, "\Phi"] & & \\ & U_i \arrow[r, dashed, "\chi"] \arrow[d, "f_i"'] & W \arrow[d] \\ & \mathbb{P}^1 \arrow[r, "\omega"] & B, \end{tikzcd}$$

where $\chi\colon U_i \dashrightarrow W$ is a birational map, $\rho_i = \psi_i \circ \kappa_i^{-1}$ and $\omega\colon \mathbb{P}^1 \to B$ is an isomorphism.

As a direct corollary of Main Theorem 2, we have the following.

Corollary 1.12. *In the notation and with the assumptions of Main Theorem 2, the following hold:*

(1) *One has $2 \leqslant |\mathcal{P}(X)| \leqslant 9$. For a general X one has $|\mathcal{P}(X)| = 9$.*
(2) *One has $\mathrm{Aut}(X) = \mathrm{Bir}(X)$.*
(3) *The threefold X is not birational to a conic bundle.*
(4) *The threefold X is not rational.*

Corollary 1.13. *Let X be a general Fano–Enriques threefold such that its canonical covering V is smooth. Then X is not rational if and only if V is one of the following:*

(1) *the double covering of a quadric ramified in a divisor of degree 8;*
(2) *the complete intersection of three quadrics in $\mathbb{P}^6$;*
(3) *the double covering of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ramified in a divisor of degree $(2, 2, 2)$.*

2. The smooth intersection of three quadrics

2.1. The rationality of conic bundles. For many years the rationality of conic bundles has been well studied. In this section we would like to recall some classic results on conic bundles, which will be used later. For more details on the rationality of conic bundles, we refer the reader to [Pro18].

Let $f\colon Y \to S$ be a conic bundle, where $\dim Y = 3$ and $\dim S = 2$. Recall that a conic bundle $f\colon Y \to S$ is said to be a *standard* conic bundle if both Y and S are smooth and the relative Picard group of Y over S has rank 1. By [Sar83, Theorem 1.11] for a given conic bundle $f\colon Y \to S$ there exists the following commutative diagram:

$$\begin{tikzcd} Y \arrow[r, dashed, "\phi"] \arrow[d, "f"'] & Y' \arrow[d, "f'"] \\ S \arrow[r, dashed, "\psi"] & S', \end{tikzcd}$$

where ϕ and ψ are birational maps and $f'\colon Y' \to S'$ is a standard conic bundle. Thus, without loss of generality, we may assume that $f\colon Y \to S$ is already a standard conic bundle.

Let

$$\Delta := \{ s \in S \mid f^{-1}(s) \text{ is a degenerate conic} \} .$$

Then Δ is a divisor on S; it is called the *degeneration curve* of f. By [Sar83, Proposition 1.8] the singular points of Δ correspond to the fibres of f that are double lines, and by [Sar83, Corollary 1.11] the curve Δ is nodal. Further, let $\widetilde{\Delta}$ be the Hilbert scheme of lines in the fibres of f over Δ. Then f induces a double covering $\tilde{f}\colon \widetilde{\Delta} \to \Delta$ ramified in the singular points of the curve Δ, and hence one can consider a principally polarised abelian variety $\mathrm{Prym}\,(\widetilde{\Delta}, \Delta)$, the *Prym variety* of the pair $(\widetilde{\Delta}, \Delta)$; see [Bea77, Sho84].

THEOREM 2.1 ([Bea77, Proposition 2.8]). *In the above notation, let* $\mathrm{J}(Y)$ *be the intermediate Jacobian of* Y*. Then* $\mathrm{J}(Y) \cong \mathrm{Prym}(\widetilde{\Delta}, \Delta)$ *as principally polarised abelian varieties.*

Recall that a curve Δ is said to be *quasi-trigonal* if Δ is obtained from a hyperelliptic curve by identifying two smooth points.

THEOREM 2.2 ([Sho84, Bea77]). *The Prym variety* $\mathrm{Prym}\,(\widetilde{\Delta}, \Delta)$ *is a sum of Jacobians of smooth curves if and only if* Δ *is one of the following:*

- *hyperelliptic;*
- *trigonal;*
- *quasi-trigonal;*
- *a plane quintic curve, and the corresponding double covering is given by an even theta-characteristic.*

2.2. Conic bundles birational X. In this section let X be a Fano–Enriques threefold whose canonical covering V is the smooth intersection of three quadrics in $\mathbb{P}^6$, and denote by $\sigma\colon V \to V$ the involution corresponding to the double covering $\pi\colon V \to X$.

By [Bay94, Section 6.1.5] there exist homogeneous coordinates $x_0, \ldots, x_6$ on $\mathbb{P}^6$ such that $V \subset \mathbb{P}^6$ is defined by the following system of equations:

$$\begin{cases} P_1(x_0, x_1, x_2, x_3) + Q_1(x_4, x_5, x_6) = 0\,, \\ P_2(x_0, x_1, x_2, x_3) + Q_2(x_4, x_5, x_6) = 0\,, \\ P_3(x_0, x_1, x_2, x_3) + Q_3(x_4, x_5, x_6) = 0\,, \end{cases}$$

where P_i and Q_i are homogeneous polynomials of degree 2 and σ is given by

$$\sigma(x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6) = (x_0 : x_1 : x_2 : x_3 : -x_4 : -x_5 : -x_6)\,.$$

The involution σ fixes exactly eight points p_i for $i = 1, \ldots, 8$ on V, which are given by

$$V \cap \{x_4 = x_5 = x_6 = 0\}\,.$$

The smoothness of V implies that the intersection $V \cap \{x_0 = x_1 = x_2 = x_3 = 0\}$ is empty; that is,

$$\bigcap_{i=1}^{3} \{Q_i(x_4, x_5, x_6) = 0\} = \varnothing\,.$$

Denote by q_i for $i = 1, \dots, 8$ the singular points of X; that is, $q_i = \pi(p_i)$. Notice that they are all of the type $\frac{1}{2}(1,1,1)$. Moreover, by changing the coordinates system, we can assume that

$$\begin{aligned} p_1 &= (1:0:0:0:0:0:0)\,, \\ p_2 &= (0:1:0:0:0:0:0)\,, \\ p_3 &= (0:0:1:0:0:0:0)\,, \\ p_4 &= (0:0:0:1:0:0:0)\,, \end{aligned}$$

and hence the $P_n(x_0, x_1, x_2, x_3)$ do not contain monomials of the form αx_k^2.

Let us consider the linear subsystem $\mathcal{L} = \langle x_0, x_1, x_2, x_3 \rangle|_V \subset |-K_V|$. Notice that one has $\mathcal{L} = \pi^*|H|$ in the notation of Definition 1.1. Then $\mathcal{L}_{p_1} = \langle x_1, x_2, x_3 \rangle|_V \subset \mathcal{L}$, the linear subsystem of sections of $\mathcal{L}$ vanishing at p_1, yields a rational map

$$\phi_{\mathcal{L}_{p_1}} : V \dashrightarrow \mathbb{P}^2\,,$$

undefined at p_1 only. Further, denote by $\mathcal{L}_{p_1,p_r} \subset \mathcal{L}_{p_1}$ the linear subsystem of sections of $\mathcal{L}_{p_1}$ that additionally vanish at p_r.

Proposition 2.3. *For a general V the base locus $\operatorname{Bs}\mathcal{L}_{p_1,p_r}$ consists of four smooth conics.*

Proof. Suppose that $f_1 = \sum_{i=1}^3 a_i x_i$ and $f_2 = \sum_{i=1}^3 b_i x_i$ are linearly independent sections of $\mathcal{L}_{p_1,p_r}$. Let $C_r = \operatorname{Bs}\mathcal{L}_{p_1,p_r}$. Notice that C_r is defined by

$$V \cap \{f_1 = f_2 = 0\}\,.$$

Using the equations $f_1 = f_2 = 0$, we can linearly eliminate two variables; that is, up to a linear change of coordinates, say $x_2 = \lambda x_1$ and $x_3 = \mu x_1$ for λ, $\mu \in \mathbb{C}$. Consequently, we can define the curve C_r in $\mathbb{P}^4$ with coordinates x_0, x_1, x_4, x_5, x_6 by the following system:

$$\begin{cases} P_1(x_0, x_1, \lambda x_1, \mu x_1) + Q_1(x_4, x_5, x_6) = 0\,, \\ P_2(x_0, x_1, \lambda x_1, \mu x_1) + Q_2(x_4, x_5, x_6) = 0\,, \\ P_3(x_0, x_1, \lambda x_1, \mu x_1) + Q_3(x_4, x_5, x_6) = 0\,. \end{cases}$$

The polynomials $P_n(x_0, x_1, \lambda x_1, \mu x_1)$ have two non-zero solutions, that correspond to the points p_1 and p_r. Therefore, they are pairwise linearly dependent. Hence, without loss of generality, we can assume that C_r is defined by

$$\begin{cases} P_1(x_0, x_1, \lambda x_1, \mu x_1) + Q_1(x_4, x_5, x_6) = 0\,, \\ \alpha Q_1(x_4, x_5, x_6) + Q_2(x_4, x_5, x_6) = 0\,, \\ \beta Q_1(x_4, x_5, x_6) + Q_3(x_4, x_5, x_6) = 0\,, \end{cases}$$

where $\alpha, \beta \in \mathbb{C}$. For a general choice of V the last two equations give four distinct roots for (x_4, x_5, x_6). Plugging these roots into the first equation, we obtain four smooth reduced conics. $\square$

Hence, we write $C_r = C_{r,1} \cup C_{r,2} \cup C_{r,3} \cup C_{r,4}$, where the $C_{r,i}$ are smooth conics.

Proposition 2.4. *The blow-up of the point p_1 resolves the indeterminacy of $\phi_{\mathcal{L}_{p_1}}$. More precisely, one has the following commutative diagram:*

$$\begin{tikzcd} & \widehat{V} \arrow[dl, "\tau"'] \arrow[dr, "\epsilon"] & \\ V \arrow[rr, dashed, "\phi_{\mathcal{L}_{p_1}}"] & & \mathbb{P}^2\,, \end{tikzcd}$$

where $\tau\colon \widehat{V} \to V$ is the blow-up of p_1 and $\epsilon\colon \widehat{V} \to \mathbb{P}^2$ is a fibration whose general fibre is an elliptic curve.

Proof. A general fibre of ϵ is an elliptic curve by construction. Notice that it in order to prove that ϵ is a morphism, it is enough to show that the strict transform of $\mathcal{L}_{p_1}$ on $\widehat{V}$ is base-point-free on the τ-exceptional divisor $E \cong \mathbb{P}^2$. Let us consider the affine patch $\{x_0 \neq 0\}$. Let $u_i = x_i/x_0$ be affine coordinates. Then $V \cap \{x_0 \neq 0\}$ is given by

$$\begin{cases} \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + R_1(u_1, u_2, u_3) + Q_1(u_4, u_5, u_6) = 0\,, \\ \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + R_2(u_1, u_2, u_3) + Q_2(u_4, u_5, u_6) = 0\,, \\ \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3 + R_3(u_1, u_2, u_3) + Q_3(u_4, u_5, u_6) = 0\,, \end{cases} \tag{2.5}$$

where α_i, β_i, γ_i are some complex numbers and the $R_i(u_1, u_2, u_3)$ are homogeneous polynomials of degree 2. Since V is smooth, the matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}$$

is non-degenerate. So, by a linear change of coordinates, we can bring the system (2.5) into the form

$$\begin{cases} u_1 = \widetilde{Q}_1(u_4, u_5, u_6) + \widetilde{R}_1(u_1, u_2, u_3)\,, \\ u_2 = \widetilde{Q}_2(u_4, u_5, u_6) + \widetilde{R}_2(u_1, u_2, u_3)\,, \\ u_3 = \widetilde{Q}_3(u_4, u_5, u_6) + \widetilde{R}_3(u_1, u_2, u_3) \end{cases}$$

for some homogeneous polynomials $\widetilde{Q}_i(u_4, u_5, u_6)$ and $\widetilde{R}_i(u_1, u_2, u_3)$ of degree 2, where the polynomials $\widetilde{Q}_i(u_4, u_5, u_6)$ are linear combinations of the $Q_i(u_4, u_5, u_6)$. Notice that u_4, u_5, u_6 are local coordinates in a neighbourhood of the point p_1. Consequently, considering the Taylor series at the origin of u_i for $i = 1, 2, 3$, we see that

$$u_i = \widetilde{Q}_i(u_4, u_5, u_6) + \text{higher-order terms}\,.$$

Notice that one has

$$\bigcap_{i=1}^{3} \{\widetilde{Q}_i(u_4, u_5, u_6) = 0\} = \varnothing\,.$$

Consequently, the restriction on E of the strict transform of $\mathcal{L}_{p_1}$ on $\widehat{V}$ is a base-point-free linear system of conics. $\square$

Remark 2.6. Let $\omega_i\colon \widehat{V}' \to \widehat{V}$ be the blow-up of the point $\tau^{-1}(p_i)$. Then the curves $(\omega_i)_*^{-1}(C_{i,j}) \cap (\omega_i)_*^{-1}(C_{i,k})$ do not intersect for all i, j, k; otherwise, we get a contradiction with the smoothness of V.

Thus, by Proposition 2.4 the linear subsystem $\mathcal{H}_{q_1} \subset |H|$ of sections of $|H|$ vanishing at $q_1 = \pi(p_1)$ determines a rational map

$$\phi_{\mathcal{H}_{q_1}} : X \dashrightarrow \mathbb{P}^2\,,$$

which, after the Kawamata blow-up $\theta\colon \widehat{X} \to X$ of the point q_1, gives a conic bundle $c : \widehat{X} \to \mathbb{P}^2$. Namely, there exists the following commutative diagram:

$$
\begin{tikzcd}
\widehat{V} \arrow[rrrr, "\widehat{\pi}"] \arrow[dr, "\tau"'] \arrow[ddrr, bend right, "\epsilon"'] & & & & \widehat{X}\,, \arrow[dl, "\theta"] \arrow[ddll, bend left, "c"] \\
& V \arrow[rr, "\pi"] \arrow[dr, dashed, "\phi_{\mathcal{L}_{p_1}}"'] & & X \arrow[dl, dashed, "\phi_{\mathcal{H}_{q_1}}"] & \\
& & \mathbb{P}^2 & &
\end{tikzcd}
$$

where $\widehat{\pi}\colon \widehat{V} \to \widehat{X}$ is the induced double covering ramified in the τ-exceptional divisor and in $\tau^{-1}(p_i)$ for $i \neq 1$. However, this conic bundle is non-standard because $\widehat{X}$ is singular.

2.3. From a non-standard to a standard conic bundle. In the previous section we arrived at the non-standard conic bundle $c\colon \widehat{X} \to \mathbb{P}^2$. In order to make it standard, firstly we need to resolve the singularities of $\widehat{X}$. More precisely, since there are only eight singular points q_i on X and they are all of type $\frac{1}{2}(1,1,1)$, it is enough to consider the Kawamata blow-up of them $\kappa\colon \widetilde{X} \to X$ to obtain a smooth threefold $\widetilde{X}$; see [Kaw96]. Notice that the morphism κ resolves indeterminacies of $\phi_{\mathcal{H}_{q_1}}$, and therefore we get a morphism $f\colon \widetilde{X} \to \mathbb{P}^2$ whose general fibre is a conic. However, f is not a standard conic bundle. Indeed, let $s_i = \phi_{\mathcal{H}_{q_1}}(q_i) \in \mathbb{P}^2$, where $i \neq 1$, and let E_{q_i} be the κ-exceptional divisor over the point q_i. Then we see that $f^{-1}(s_i)$ consists of four disjoint smooth reduced rational curves $Z_i = Z_{i,1} \sqcup Z_{i,2} \sqcup Z_{i,3} \sqcup Z_{i,4}$ (by Proposition 2.3 and Remark 2.6) and the divisor E_{q_i}.

We now introduce a method that will resolve this problem and provide a standard conic bundle. The method works locally; hence it is enough to treat only one κ-exceptional divisor E_{q_i}.

LEMMA 2.7. *One has* $\mathcal{N}_{Z_{i,j}/\widetilde{X}} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

Proof. Let $\mathcal{H}_{q_1,q_i} \subset \mathcal{H}_{q_1}$ be the linear subsystem of sections of $\mathcal{H}_{q_1}$ vanishing additionally at q_i. Assume that H_{q_1,q_i} is a general divisors from $\mathcal{H}_{q_1,q_i}$. Let $S \in |\kappa^* H_{q_1,q_i} - E_{q_1} - E_{q_i}|$ be a general surface. Then S contains the curve Z_i by construction, and $Z_i = S|_S$. Further, suppose that $D \in |H|$ is a surface that does not pass through q_1 and q_i. Then

$$(\kappa^* D - E_{q_1} - E_{q_i})|_S \sim S|_S\,.$$

Observe that

$$Z_{i,j} \cdot (\kappa^* D - E_{q_1} - E_{q_i})|_S = 1 - 1 - 1 = -1$$

for all j. Hence

$$-1 = Z_{i,j} \cdot (\kappa^* D - E_{q_1} - E_{q_i})|_S = Z_{i,j} \cdot Z_i = Z_{i,j}^2\,.$$

Since $Z_{i,j} \cong \mathbb{P}^1$, we have $\mathcal{N}_{Z_{i,j}/S} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$. Notice that locally $Z_{i,j}$ is a complete intersection of two general elements S_1, $S_2 \in |\kappa^* H_{q_1,q_i} - E_{q_1} - E_{q_i}|$. Consequently,

$$\mathcal{N}_{Z_{i,j}/\widetilde{X}} \cong \mathcal{N}_{Z_{i,j}/S_1} \oplus \mathcal{N}_{Z_{i,j}/S_2} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\,. \qquad \square$$

So, by Lemma 2.7 flops of the curves $Z_{i,j}$ are the Atiyah flops, that is, the blow-up of each curve $Z_{i,j}$ and the contraction of the exceptional divisor isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ along another ruling class.

Denote by $\psi\colon \widetilde{X} \dashrightarrow U$ the composition of four Atiyah flops of $Z_{i,j}$ and by $g\colon Y \to \mathbb{P}^2$ the blow-up of s_i with the exceptional divisor $B \cong \mathbb{P}^1$. And finally, let $\xi = g^{-1} \circ f \circ \psi^{-1}$ be a rational map $U \dashrightarrow Y$. Therefore, we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[d, dashed, "\phi_{\mathcal{H}_{q_1}}"'] & \widetilde{X} \arrow[l, "\kappa"'] \arrow[dl, "f"] \arrow[r, dashed, "\psi"] & U \arrow[dl, dashed, "\xi"] \\ \mathbb{P}^2 & Y \arrow[l, "g"] & \end{tikzcd}$$

Proposition 2.8. *The map ξ is a morphism, and $\xi(\psi(E_{q_i})) = B$. Moreover, the restriction $\xi|_{\psi(E_{q_i})}\colon \psi(E_{q_i}) \to B$ is a conic bundle with three singular fibres.*

Proof. Denote by ω_1 the blow-up of $Z_{i,1}$, $Z_{i,2}$, $Z_{i,3}$ and $Z_{i,4}$. Then there is the following commutative diagram for the flop ψ:

$$\begin{tikzcd} & W \arrow[dl, "\omega_1"'] \arrow[dr, "\omega_2"] & \\ \widetilde{X} \arrow[rr, dashed, "\psi"] & & U \end{tikzcd}$$

Let $G_{i,j} \cong \mathbb{P}^1 \times \mathbb{P}^1$ be the ω_1-exceptional divisor over $Z_{i,j}$, and let $\ell_1^{i,j}$, $\ell_2^{i,j}$ be the ruling classes of $G_{i,j}$ contracted by ω_2, ω_1, respectively. Then

$$G_{i,j}|_{G_{i,j}} = -\ell_1^{i,j} - \ell_2^{i,j}\,.$$

Let $\mathcal{D}$ be the pencil of curves on Y that are proper transforms of lines on $\mathbb{P}^2$ passing through the point s_i. Note that the class $D + R$, where $D \in \mathcal{D}$ and $R \in |g^*\mathcal{O}_{\mathbb{P}^2}(1)|$, is very ample on Y and the proper transform of $|g^*\mathcal{O}_{\mathbb{P}^2}(1)|$ on U is base-point-free. Therefore, in order to prove that $\xi\colon U \dashrightarrow Y$ is a morphism, it is enough to show that the proper transform $\mathcal{D}_U$ of $\mathcal{D}$ on U is base-point-free.

Let us first consider the proper transform $\mathcal{D}_W$ of $\mathcal{D}$ on W. By construction, the base locus of $\mathcal{D}_W$ is contained in

$$G_{i,1} \cup G_{i,2} \cup G_{i,3} \cup G_{i,4} \cup E_{q_i}^W\,,$$

where $E_{q_i}^W$ is the strict transform of E_{q_i} on W. One has

$$\mathcal{D}_W \sim (f \circ \omega_1)^*\mathcal{O}_{\mathbb{P}^2}(1) - \sum_{j=1}^{4} G_{i,j} - E_{q_i}^W\,.$$

Since $E_{q_i}^W|_{G_{i,j}} = \ell_2^{i,j}$, we have $\mathcal{D}_W|_{G_{i,j}} = \ell_1^{i,j}$. Consequently, either $\operatorname{Bs}\mathcal{D}_W \cap G_{i,j}$ is empty, or all elements of $\mathcal{D}_W$ intersect by the same ruling class $\ell_1^{i,j}$. However, the latter case is impossible since the proper transforms of elements of $\mathcal{H}_{q_1}$ on $\widetilde{X}$ intersect transversely at a general point of $Z_{i,j}$.

So, it remains to consider the base locus of $\mathcal{D}_W$ on $E_{q_i}^W$. However, Proposition 2.4 implies that the restriction of the linear system $|\kappa^*H_{q_1} - E_{q_1} - E_{q_i}|$ on E_{q_i} is a linear system of conics on E_{q_i} with four base points that are exactly $E_{q_i} \cap Z_{i,j}$. These points are in general position because they are determined by the intersection of two conics from the linear system $\langle Q_1(x_4,x_5,x_6), Q_2(x_4,x_5,x_6), Q_3(x_4,x_5,x_6)\rangle$. But we have already proved that there is no base locus of $\mathcal{D}_W$ on $G_{i,j}$. Thereby, we have $\operatorname{Bs}\mathcal{D}_W = \varnothing$.

Finally, we notice that the restriction of $\mathcal{D}_W$ on $G_{i,j}$ is contained in the fibres of ω_2; consequently, $\mathcal{D}_U$ is also base-point-free, so ξ is indeed a morphism. To prove the last assertion, it is enough to see that $\psi\big(E_{q_i}^W\big)$ is a del Pezzo surface of degree 5. $\square$

Remark 2.9. Notice that a construction similar to that in Proposition 2.8 can be applied to V. So, there is a diagram

$$\begin{tikzcd}
 & & W' \arrow[dl, "\omega_1'"'] \arrow[dr, "\omega_2'"] & \\
V \arrow[d, "\pi"'] & \widetilde{V} \arrow[l, "\widetilde{\kappa}"'] \arrow[d, "\widetilde{\pi}"'] & W \arrow[dl, "\omega_1"'] \arrow[dr, "\omega_2"] & U' \arrow[dd, bend left, "e"] \\
X \arrow[dr, dashed, "\phi_{\mathcal{H}_{q_1}}"'] & \widetilde{X} \arrow[l, "\kappa"'] \arrow[rr, dashed, "\psi"] \arrow[d, "f"'] & & U \arrow[d, "\xi"'] \\
 & \mathbb{P}^2 & & Y \arrow[ll, "g"'] ,
\end{tikzcd}$$

where

- $\widetilde{\kappa}$ is the blow-up of $p_1, \dots, p_8$;
- ω_1' is the blow-up of $\widetilde{\kappa}_*^{-1}(C_{i,1}), \dots, \widetilde{\kappa}_*^{-1}(C_{i,4})$ (notice, that the ω_1-exceptional divisor over $\widetilde{\kappa}_*^{-1}(C_{i,j})$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ since one has
$$\mathcal{N}_{\widetilde{\kappa}_*^{-1}(C_{i,j})} \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$$
by Lemma 2.7);
- ω_2' is the contraction of the exceptional divisors $\mathbb{P}^1 \times \mathbb{P}^1$ along another ruling class to a singular variety U';
- e is a morphism whose fibre is an elliptic curve.

Thus, applying Proposition 2.8 to seven κ-exceptional divisors E_{q_i} for $i \neq 1$, we obtain a standard conic bundle $\chi \colon \mathfrak{X} \to B$, where B is the blow-up of $\mathbb{P}^2$ in seven points s_i. More precisely, one has the following diagram:

$$\begin{tikzcd}
X \arrow[d, dashed, "\phi_{\mathcal{H}_{q_1}}"'] & \widetilde{X} \arrow[l, "\kappa"'] \arrow[dl, "f"'] \arrow[r, dashed, "\Psi"] & \mathfrak{X} , \arrow[dl, "\chi"] \\
\mathbb{P}^2 & B \arrow[l, "h"'] &
\end{tikzcd}$$

where Ψ is a composition of Atiyah flops in curves $Z_{i,j}$ and h is the blow-up of $s_2, \dots, s_8$.

Let C be the degeneration curve of $f \colon \widetilde{X} \to \mathbb{P}^2$. We want to compute $\deg C$. We write

$$\widetilde{H}_{q_1} \sim_{\mathbb{Q}} \kappa^* H - E_{q_1}, \quad K_{\widetilde{X}} = \kappa^* K_X + \tfrac{1}{2} \sum E_i .$$

So,

$$K^2_{\widetilde{H}_{q_1}} = \big(K_{\widetilde{X}} + \widetilde{H}_{q_1}\big)^2 \cdot \widetilde{H}_{q_1} = -\tfrac{1}{4} E^3_{q_1} = -1 .$$

Hence, by Noether's formula $\deg C = 9$.

Now, let us summarise our generality assumptions on V. Considering general V, we can assume that:

(1) the points $s_2, \dots, s_8$ are in general position; that is, no three of them lie on a line, and no six of them lie on a conic;

(2) the base locus $\operatorname{Bs} \mathcal{L}_{p_1, p_i}$ consists of four smooth conics for all $i \neq 1$.

Hence, the curve C has ordinary triple points at $s_2, \dots, s_8$, and the flops ψ are exactly Atiyah flops.

Consequently, under the generality conditions, one has that B is the del Pezzo surface of degree 2, the degeneration curve of χ is $\Delta = h_*^{-1}(C)$ and $\Delta \sim -3K_B$.

Now we are ready to prove the main result of this section.

Theorem 2.10. *Under our generality assumptions X is not rational.*

Proof. By Theorem 2.2 it is enough to show that the curve $\Delta \subset B$ is not hyperelliptic, trigonal or quasi-trigonal.

Notice that $-2K_B$ is very ample. Since B is smooth and Δ is a Cohen–Macaulay scheme, we can use the adjunction formula for the dualising sheaf of Δ, that we will denote by K_Δ. Therefore, $K_\Delta = (K_B + \Delta)|_\Delta = -2K_B|_\Delta$, and hence K_Δ is very ample. Thus, Δ is not hyperelliptic; otherwise, K_Δ would give a finite morphism of degree 2.

Let us prove that Δ is not trigonal. Assume the contrary. Then the image $\phi_{|K_\Delta|}(\Delta) \subset \mathbb{P}^6$ would have a 3-section, that is, a line ℓ in $\mathbb{P}^6$ such that $|\ell \cap \phi_{|K_\Delta|}(\Delta)| = 3$. Since one has $H^0(B, \mathcal{O}_B(K_B)) = H^1(B, \mathcal{O}_B(K_B)) = 0$, the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_B(K_B) \longrightarrow \mathcal{O}_B(-2K_B) \longrightarrow \mathcal{O}_\Delta(-2K_B) \longrightarrow 0$$

implies that the restriction morphism of global sections

$$H^0(B, \mathcal{O}_B(-2K_B)) \longrightarrow H^0(\Delta, \mathcal{O}_\Delta(K_\Delta))$$

is an isomorphism. Consequently, $\phi_{|-2K_B|}(\Delta) = \phi_{|K_\Delta|}(\Delta)$. Now one can show that the image of $\phi_{|-2K_B|}\colon B \hookrightarrow \mathbb{P}^6$ is contained in an intersection of quadrics. Since, as we have noticed, $\phi_{|-2K_B|}(\Delta) = \phi_{|K_\Delta|}(\Delta)$, then $\ell \subset \phi_{|-2K_B|}(B)$. However, $\phi_{|-2K_B|}(B)$ obviously does not contain any lines. So we have a contradiction.

Finally, we claim that Δ is not quasi-trigonal. Indeed, if Δ was quasi-trigonal, then the image $\phi_{|K_\Delta|}(\Delta)$ would have a 3-section as well; see, for example, [Pro18, Remark 7.5.1]. So we have a contradiction as above. $\square$

2.4. An explicit example. Notice that all required generality assumptions on V are Zariski open. Thus, it is enough to provide an example of such a variety V satisfying our assumptions to finish the proof of the non-rationality of a general X from this family.

Consider $V \subset \mathbb{P}^6$ given by

$$\begin{cases} 40x_0x_1 + 55x_1x_2 + 12x_1x_3 + 3x_2x_3 + x_4^2 + x_5x_6 = 0\,, \\ 12x_0x_2 + 7x_1x_2 + 4x_1x_3 - 9x_2x_3 + x_5^2 + x_4x_6 = 0\,, \\ 5x_1x_2 + 8x_0x_3 + 4x_1x_3 + 9x_2x_3 + x_6^2 + x_4x_5 = 0 \end{cases}$$

and the involution σ as in Section 2.2. Then, the σ-invariant points are the following:

$$\begin{aligned} p_1 &= (1,0,0,0,0,0,0)\,, & p_2 &= (0,1,0,0,0,0,0)\,, \\ p_3 &= (0,0,1,0,0,0,0)\,, & p_4 &= (0,0,0,1,0,0,0)\,, \\ p_5 &= (-1,1,1,-1,0,0,0)\,, & p_6 &= (2,3,-2,3,0,0,0)\,, \\ p_7 &= (-3,3,1,5,0,0,0)\,, & p_8 &= (21/2,9,-10,15,0,0,0)\,. \end{aligned}$$

We keep the notation from Section 2.3. Let $\pi(p_1) = q_1$. Now we will explicitly describe the degeneration curve Δ.

By construction $\mathcal{L}_{p_1} = \langle x_1, x_2, x_3\rangle|_V$. So, in an appropriate affine chart, the section of $\mathcal{L}_{p_1}$ can given by $x_2 = \lambda x_1$ and $x_3 = \mu x_1$, where λ, $\mu \in \mathbb{C}$. Thereby, in the affine chart where $x_1 \neq 0$, a general fibre of $\phi_{\mathcal{L}_{p_1}}$ is determined by the following equations:

$$\begin{cases} 40u_0 + 55\lambda + 12\mu + 3\lambda\mu + u_4^2 + u_5u_6 = 0\,, \\ 12u_0\lambda + 7\lambda + 4\mu - 9\lambda\mu + u_5^2 + u_4u_6 = 0\,, \\ 5\lambda + 8u_0\mu + 4\mu + 9\lambda\mu + u_6^2 + u_4u_5 = 0\,, \end{cases}$$

where (u_0, u_4, u_5, u_6) are affine coordinates. Eliminating u_0 from the first equation, we have

$$\begin{cases} 7\lambda - 33/2\lambda^2 + 4\mu - 63/5\lambda\mu - 9/10\lambda^2\mu - 3/10\lambda\big(u_4^2 + u_5u_6\big) + u_5^2 + u_4u_6 = 0\,, \\ 5\lambda + 4\mu - 2\lambda\mu - 12/5\mu^2 - 3/5\lambda\mu^2 - 1/5\mu\big(u_4^2 + u_5u_6\big) + u_6^2 + u_4u_5 = 0\,. \end{cases}$$

Let us homogenise the coordinates u_i; that is, let $u_i = y_i/y_1$, where the y_i are the homogeneous coordinates. Then,

$$\begin{cases} \big(7\lambda - 33/2\lambda^2 + 4\mu - 63/5\lambda\mu - 9/10\lambda^2\mu\big)y_1^2 - 3/10\lambda\big(y_4^2 + y_5y_6\big) + y_5^2 + y_4y_6 = 0\,, \\ \big(5\lambda + 4\mu - 2\lambda\mu - 12/5\mu^2 - 3/5\lambda\mu^2\big)y_1^2 - 1/5\mu\big(y_4^2 + y_5y_6\big) + y_6^2 + y_4y_5 = 0\,. \end{cases}$$

Thus, we have defined a general fibre of $\phi_{\mathcal{L}_{p_1}}$ as an elliptic curve embedded into $\mathbb{P}^3$ with coordinates y_1, y_4, y_5, y_6 depending on a point $(\lambda, \mu) \in \mathbb{C}^2$. Finally, eliminating y_1^2, we find the equation of the quotient of a general fibre of $\phi_{\mathcal{L}_{p_1}}$ by σ, that is, the equation of a general fibre of $f\colon \widetilde{X} \to \mathbb{P}^2_{(x_1:x_2:x_3)}$. Homogenising λ and μ back, we have the following equation of a general fibre of f:

$$\begin{aligned} &(3x_2 + 2x_3)(5x_1x_2 - 4x_1x_3 + 9x_2x_3)\big(y_4^2 + y_5y_6\big) \\ &\quad + 2(3x_3 - 5x_1)(5x_1x_2 + 4x_1x_3 + x_2x_3)\big(y_5^2 + y_4y_6\big) \\ &\quad + \big(-9x_2^2x_3 + 10x_1^2(7x_2 + 4x_3) - 3x_1x_2(55x_2 + 42x_3)\big)\big(y_6^2 + y_4y_5\big) = 0\,. \end{aligned}$$

Direct computations from the above equation show that the degeneration curve $C \subset \mathbb{P}^2$ of f is given by

$$\begin{aligned} &-109000x_1^6x_2^3 + 1081500x_1^5x_2^4 - 2549250x_1^4x_2^5 + 2244375x_1^3x_2^6 - 144000x_1^6x_2^2x_3 \\ &\quad + 2074600x_1^5x_2^3x_3 - 5674500x_1^4x_2^4x_3 + 5284350x_1^3x_2^5x_3 + 358425x_1^2x_2^6x_3 - 48000x_1^6x_2x_3^2 \\ &\quad + 1278400x_1^5x_2^2x_3^2 - 4514680x_1^4x_2^3x_3^2 + 4652580x_1^3x_2^4x_3^2 + 312930x_1^2x_2^5x_3^2 + 3645x_1x_2^6x_3^2 \\ &\quad + 214400x_1^5x_2x_3^3 - 1482560x_1^4x_2^2x_3^3 + 2009592x_1^3x_2^3x_3^3 - 322956x_1^2x_2^4x_3^3 - 85374x_1x_2^5x_3^3 \\ &\quad - 9477x_2^6x_3^3 - 25600x_1^5x_3^4 - 140800x_1^4x_2x_3^4 + 551328x_1^3x_2^2x_3^4 - 411984x_1^2x_2^3x_3^4 - 113616x_1x_2^4x_3^4 \\ &\quad - 23328x_2^5x_3^4 + 15360x_1^4x_3^5 + 99072x_1^3x_2x_3^5 - 115776x_1^2x_2^2x_3^5 - 31104x_1x_2^3x_3^5 - 15552x_2^4x_3^5 \\ &\quad - 6656x_1^3x_3^6 - 6912x_1^2x_2x_3^6 + 2592x_1x_2^2x_3^6 - 3024x_2^3x_3^6 = 0\,. \end{aligned}$$

Indeed, the curve obtained above is irreducible, has the same degree as C and coincides with C on the affine patch by the construction. Hence it coincides with C everywhere.

Now direct computations show that:

- the points s_i have coordinates $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,-1)$, $(3,-2,3)$, $(3,1,5)$, $(9,-10,15)$, and they are in general position;
- the curve C is singular only at the points s_i, and these points are ordinary triple points of C.

Consequently, after the blow-up $h\colon U \to \mathbb{P}^2$ of the points $s_2, \ldots, s_8$, we get a smooth curve $\Delta = h_*^{-1}(C) \in |-3K_U|$ on the del Pezzo surface U of degree 2, which is the degeneration curve of a standard conic bundle $\chi\colon \mathfrak{X} \to U$. Thereby, X is not rational by Theorem 2.10. Hence, a general member of this deformation family is also not rational, which completes the proof of Main Theorem 1.

3. The double covering of a quadric ramified in a divisor of degree 8

In this section let X be a Fano–Enriques threefold whose canonical covering V is the double covering of a quadric ramified in a divisor of degree 8 and $\sigma\colon V \to V$ be the involution corresponding to the double covering $\pi\colon V \to X$ as before.

By [Bay94, Section 6.1.6] there are homogeneous coordinates $x_0, \dots, x_4, x_5$ on $\mathbb{P}(1^5, 2)$, where x_5 is the coordinate of weight 2, such that $V \subset \mathbb{P}(1^5, 2)$ is defined by the following system of equations:

$$\begin{cases} P_2(x_0, x_1, x_2) + R_2(x_3, x_4) = 0\,, \\ x_5^2 + A_2(x_0, x_1, x_2)x_3^2 + B_2(x_0, x_1, x_2)x_3x_4 + C_2(x_0, x_1, x_2)x_4^2 + F_4(x_0, x_1, x_2) + G_4(x_3, x_4) = 0\,, \end{cases}$$

where P_2, R_2, A_2, B_2, C_2, are homogeneous polynomials of degree 2 and F_4, G_4 are homogeneous polynomials of degree 4. The involution σ is given by

$$\sigma(x_0 : x_1 : x_2 : x_3 : x_4 : x_5) = (x_0 : x_1 : x_2 : -x_3 : -x_4 : -x_5)\,.$$

It fixes exactly eight points p_i for $i = 1, \dots, 8$ on V, which are given by

$$V \cap \{x_3 = x_4 = x_5 = 0\}\,.$$

Remark 3.1. Notice that $R_2(x_3, x_4)$ is reduced. Assume the contrary. Then, without loss of generality, we can suppose that $R_2(x_3, x_4) = \mu(x_3 - \lambda x_4)^2$ for some $\mu, \lambda \in \mathbb{C}$. Let $\beta \in \mathbb{C}$ be a complex number such that

$$\beta^2 + G_4(\lambda, 1) = 0\,.$$

Then the point $(0 : 0 : 0 : \lambda : 1 : \beta) \in V$ is singular. This contradicts the smoothness of V.

Thus, linearly changing the coordinate system, we can assume $R_2(x_3, x_4) = x_3x_4$.

Remark 3.2. The polynomial $G_4(x_3, x_4)$ is co-prime with x_3x_4. Indeed, assume that $G_4(x_3, x_4)$ is divisible by x_3. Then the point $(0 : 0 : 0 : 0 : 1 : 0) \in V$ is singular. This gives a contradiction. If $G_4(x_3, x_4)$ is divisible by x_4, then we get the same contradiction with the point $(0 : 0 : 0 : 1 : 0 : 0) \in V$.

Denote by q_i for $i = 1, \dots, 8$ the singular points of X; that is, $q_i = \pi(p_i)$. They are all of type $\frac{1}{2}(1, 1, 1)$.

3.1. Fibrations on del Pezzo surfaces birational to X. Consider the linear subsystem $\mathcal{L} = \pi^*|H| = \langle x_0, x_1, x_2\rangle|_V \subset |-K_V|$. Let us fix a σ-invariant point p. Then the linear subsystem $\mathcal{L}_p \subset \mathcal{L}$ of sections of $\mathcal{L}$ vanishing at p yields a rational map

$$\phi_{\mathcal{L}_p}\colon V \dashrightarrow \mathbb{P}^1\,,$$

undefined in the singular curve $Z = \operatorname{Bs} \mathcal{L}_p$.

Changing the coordinate system, we can also assume $p = (1 : 0 : 0 : 0 : 0 : 0)$. Let $\tau\colon \widehat{V} \to V$ be the blow-up of the point p with exceptional divisor G and $\widehat{\sigma}\colon \widehat{V} \to \widehat{V}$ be the lift of the involution σ.

Proposition 3.3. *The restriction of the strict transform of $\tau_*^{-1}\mathcal{L}_p$ on G is a mobile linear system of conics.*

Proof. Consider the affine patch $\{x_0 \neq 0\}$ with affine coordinates

$$u_1 = x_1/x_0\,,\ \dots\,,\ u_4 = x_4/x_0\,,\ u_5 = x_5/x_0^2\,.$$

Then one can see that u_3, u_4 and u_5 are the local coordinates of $V \cap \{x_0 \neq 0\}$ at the origin. Hence, we can consider them as homogeneous coordinates on $G \cong \mathbb{P}^2$. Thereby, the restriction of $\tau_*^{-1}\mathcal{L}_p$ on G is generated by two conics:

$$\begin{aligned}\mathcal{Q}_1 &= \left\{u_5^2 + au_3^2 + cu_4^2 = 0\right\},\\ \mathcal{Q}_2 &= \{u_3u_4 = 0\}\,,\end{aligned}$$

where $a = A_2(1,0,0)$ and $c = C_2(1,0,0)$. Now we see that this linear system of conics is mobile on G. Moreover, from the above equations one can see that

$$\operatorname{Bs}\left(\tau_*^{-1}\mathcal{L}_p|_G\right) = \begin{cases} 4 \text{ points} & \text{if both } a \text{ and } c \text{ are not } 0\,,\\ 3 \text{ points} & \text{if only one of } a \text{ or } b \text{ is not } 0\,,\\ 2 \text{ points} & \text{if both } a \text{ and } b \text{ are } 0\,.\end{cases} \qquad \square$$

Proposition 3.4. *The curve Z is a union of two curves Z_1 and Z_2 of arithmetic genus 1, intersecting in the singular point p. Moreover, let $\widehat{Z}_i$ be the strict transform of Z_i on $\widehat{V}$. Then $C_i = \widehat{Z}_i/\langle\widehat{\sigma}\rangle$ is a smooth rational curve.*

Proof. The curve Z is given in $\mathbb{P}\left(1^3,2\right)$ with coordinates x_0, x_3, x_4, x_5 by the following system of equations:

$$\begin{cases} x_3x_4 = 0\,,\\ x_5^2 + ax_0^2x_3^2 + bx_0^2x_3x_4 + cx_0^2x_4^2 + G_4(x_3,x_4) = 0\,.\end{cases}$$

Notice that the first equation restricted to the plane $\{x_5 = 0\}$ defines a conic that is singular at the point p. So, Z is a double covering of the conic by Remark 3.2. Hence, Z consists of two components, namely $Z_1 = Z \cap \{x_3 = 0\}$ and $Z_2 = Z \cap \{x_4 = 0\}$. Up to a change of coordinates, it is enough to study Z_1 only.

If $c \neq 0$, then, using the second equation, we see that Z_1 is a nodal curve of arithmetic genus 1 with a node at p. So, $\widehat{Z}_1$ is a smooth rational curve, intersecting G in two points. Hence, C_1 is a smooth rational curve.

If $c = 0$, then Z_1 can be given in $\mathbb{A}^2$ with coordinates x, y by $y^2 = \alpha x^4$, where $\alpha \neq 0$ by Remark 3.2. The involution σ acts on $\mathbb{A}^2$ as $(x,y) \longmapsto (-x,-y)$. So, by taking the quotient we obtain a curve $\widetilde{Z}_1 \in \mathbb{A}^3$ with coordinates u, v, w, where $u = x^2$, $v = y^2$, $w = xy$ are given by the following system:

$$\begin{cases} uv = w^2\,,\\ v = \alpha u^2\,.\end{cases}$$

Hence, we have $\alpha u^3 = w^2$. So, $\widetilde{Z}_1$ is a cuspidal curve. Thus, blowing up the singular point, we see that C_1 is a smooth rational curve. $\square$

Therefore, by Proposition 3.4 the linear subsystem $\mathcal{H}_q \subset |H|$ of sections of $|H|$ vanishing at q, where $q = \pi(p)$, yields a rational map

$$\phi_{\mathcal{H}_q}\colon X \dashrightarrow \mathbb{P}^1\,,$$

undefined in a curve that splits into a union of two rational curves C_1 and C_2 after the Kawamata blow-up $\kappa\colon \widetilde{X} \to X$ of the point q. Let E be the κ-exceptional divisor over q.

Lemma 3.5. *One has* $\mathcal{N}_{C_j/\widetilde{X}} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

Proof. This is similar to Lemma 2.7. Let $S \in |\kappa^* H_q - E|$ be a general surface. We observe as in the proof of Lemma 2.7 that $C_j^2 = -1$, where C_j is regarded as a divisor on S. Since locally C_j is a complete intersection of two elements of $|\kappa^* H_q - E|$,

$$\mathcal{N}_{C_j/\widetilde{X}} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\,. \qquad \square$$

Consequently, by Lemma 3.5 the flops of the curves C_1 and C_2 are the Atiyah flops. Denote by $\psi\colon \widetilde{X} \dashrightarrow U$ the composition of two Atiyah flops. As in the proof of Proposition 2.8, the linear system $\psi_*(\kappa^* H_q - E)$ is base-point-free and hence gives a morphism $f\colon U \to \mathbb{P}^1$. Since $\mathrm{Pic}(U) \cong \mathbb{Z}^2$, then $f\colon U \to \mathbb{P}^1$ is a Mori fibre space; that is, a general fibre F of f is a del Pezzo surface.

Proposition 3.6. *Let F be a general fibre of $f\colon U \to \mathbb{P}^1$. Then F is a del Pezzo surface of degree* 1.

Proof. Let $S \in |\kappa^* H_q - E|$ be a general surface. First, let us compute K_S^2. We write $K_{\widetilde{X}} = \kappa^* K_X + \frac{1}{2}E$; hence $K_S^2 = (K_{\widetilde{X}} + S)^2 \cdot S = -\frac{1}{4}E^3 = -1$. Notice that $\psi|_S$ contracts C_1 and C_2 on S and makes S a fibre of f; hence $K_F^2 = 1$. $\square$

3.2. The birational geometry of singular del Pezzo fibrations of degree 1. In the previous section we constructed the Mori fibre space $f\colon U \to \mathbb{P}^1$ that is birationally equivalent to X, whose fibre is a del Pezzo surface of degree 1. The aim of this section is to study the birational geometry of U.

The birational geometry of singular del Pezzo fibrations of degree 1 was studied in [Oka20]; see also [Puk98b, Gri00, Gri03, Gri06] for the smooth case. Let us recall some definitions and results. Throughout this section let $g\colon Y \to \mathbb{P}^1$ be a del Pezzo fibration, where Y is not assumed to be smooth, and let F be the fibre class of g. Also, we will use the shorthand W/B to denote a Mori fibre space $W \to B$.

Definition 3.7.

(1) In the notation above, we define the *nef threshold* of $Y/\mathbb{P}^1$ as

$$\mathrm{nef}\left(Y/\mathbb{P}^1\right) = \inf\{r \mid -K_Y + rF \text{ is nef}\}\,.$$

(2) For a given number $\delta \in \mathbb{R}$ we say that $Y/\mathbb{P}^1$ satisfies the *K_δ^3-condition* if

$$\left(-K_Y\right)^3 + \mathrm{nef}\left(Y/\mathbb{P}^1\right) \leqslant \delta\,.$$

Remark 3.8 ([Oka20, Remark 3.5]). Let $R \subset \overline{\mathrm{NE}}(X)$ be the extremal ray which is not generated by a curve contracted by g. Let $\xi \in R$ be a class. Then

$$\left(-K_Y + \mathrm{nef}\left(Y/\mathbb{P}^1\right)F\right) \cdot \xi = 0\,,$$

and $F \cdot \xi > 0$. Thereby, we have

$$\mathrm{nef}\left(Y/\mathbb{P}^1\right) = -\frac{-K_Y \cdot \xi}{F \cdot \xi}\,.$$

Let us recall a classic definition.

Definition 3.9. A birational map $\Phi\colon W \dashrightarrow W'$ between Mori fibre spaces W/B and W'/B' is a *square equivalence* if there is a birational map $h\colon B \dashrightarrow B'$ such that the diagram

$$\begin{tikzcd} W \arrow[r, dashed, "\Phi"] \arrow[d] & W' \arrow[d] \\ B \arrow[r, dashed, "h"] & B' \end{tikzcd}$$

commutes and the induced birational map between generic fibres of W/B and W'/B' is an isomorphism.

Proposition 3.10 ([Oka20, Proposition 3.6]). *Let $Y/\mathbb{P}^1$ be a fibration on del Pezzo surfaces of degree 1 such that Y has only terminal quotient singular points of type $\frac{1}{2}(1,1,1)$. Suppose that there is a birational map $\Phi\colon Y \dashrightarrow W$ to a Mori fibre space W/B, and let a linear system $\mathcal{M}_Y \subset |-nK_Y + aF|$ be the strict transform of a very ample complete linear system on W. If $a \geqslant 0$ and $Y/\mathbb{P}^1$ satisfies the $K^3_{3/2}$-condition, then Φ is a square equivalence.*

3.3. Birational models of X. In this subsection we will completely describe the birational geometry of X and finish the proof of Main Theorem 2.

Recall that in Section 3.1 for each singular point $q_i \in X$ we have constructed a birational map $\rho_i\colon X \dashrightarrow U_i$, where $f_i\colon U_i \to \mathbb{P}^1$ is a fibration on del Pezzo surfaces of degree 1 over $\mathbb{P}^1$.

Let us fix notation. Let $\Phi\colon X \dashrightarrow W$ be a birational map to a Mori fibre space W/B, and let $\mathcal{M}_X$ be the strict transform of a very ample complete linear system on W. Denote by $\lambda \in \mathbb{Q}_{>0}$ the positive rational number such that

$$K_X + \lambda\mathcal{M}_X \sim_{\mathbb{Q}} 0\,.$$

By the Noether–Fano inequality the pair $(X, \lambda\mathcal{M}_X)$ is not canonical if Φ is not an isomorphism; see [Cor95, Theorem 4.2].

Further, let $\Phi_i = \Phi \circ \rho_i^{-1}$ and $\mathcal{M}_{U_i} = (\rho_i)_*\mathcal{M}_X$. Also let $\mu_i \in \mathbb{Q}_{>0}$ be the positive rational number such that

$$K_{U_i} + \mu_i\mathcal{M}_{U_i} \sim_{\mathbb{Q}} a_iF_i$$

for some $a_i \in \mathbb{Q}$, where F_i is a class of the fibre of f_i.

The next lemma is known as Pukhlikov's inequality.

Lemma 3.11 ([Puk98a]). *Let p be a smooth point of X. Assume that p is a centre of non-canonical singularities of the pair $(X, \lambda\mathcal{M}_X)$. Then*

$$\mathrm{mult}_p(M_1 \cdot M_2) > 4/\lambda^2$$

for any two general elements $M_1, M_2 \in \mathcal{M}_X$.

Lemma 3.12. *Let p be a smooth point of X. Then p is not a centre of non-canonical singularities.*

Proof. Notice that

$$\mathscr{G} := \pi^*|-2K_X| = \left\langle x_0^2, x_1^2, x_2^2, x_3^2, x_4^2, x_0x_1, x_0x_2, x_1x_2, x_3x_4\right\rangle|_V\,.$$

Let $o = (a_0 : \cdots : a_5) \in V$ be a point such that $\pi(o) = p$. Since p is assumed to be a smooth point of X, then o is not σ-invariant. Denote by $\mathscr{G}_o \subset \mathscr{G}$ the linear subsystem of section of $\mathscr{G}$ vanishing at o. Hence, up to a permutation of the coordinates, we have two cases.

Case 1: $a_3 \neq 0$. We may assume $a_3 = 1$. Then direct computations show that

$$\mathscr{G}_o = \langle x_0^2 - a_0^2x_3^2, x_1^2 - a_1^2x_3^2, x_2^2 - a_2^2x_3^2, x_4^2 - a_4^2x_3^2,\\ x_0x_1 - a_0a_1x_3^2, x_0x_2 - a_0a_2x_3^2, x_1x_2 - a_1a_2x_3^2, x_3x_4 - a_4x_3^2\rangle|_V .$$

Therefore,

$$\mathrm{Bs}\,\mathscr{G}_o = \{o, \sigma(o), \nu(o), (\nu\circ\sigma)(o)\},$$

where $\nu\colon V \to V$ is the involution induced by the double covering $g\colon V \to Q$ of a quadric $Q \subset \mathbb{P}^4$ ramified in a divisor of degree 8.

Case 2: $a_5 \neq 0$. If $a_3 \neq 0$ or $a_4 \neq 0$, then, by the previous case, $\mathrm{Bs}\,\mathscr{G}_o$ does not contain any curves. Thus, it remains to consider the case when $o = (a_0 : a_1 : a_2 : 0 : 0 : a_5)$. Notice that one of a_i for $i = 0, 1, 2$ must be non-zero. Without loss of generality, we can assume $a_0 \neq 0$, hence $a_0 = 1$. Then

$$\mathscr{G}_o = \langle x_1^2 - a_1^2x_0^2, x_2^2 - a_2^2x_0^2, x_3^2, x_4^2, x_0x_1 - a_1x_0^2, x_0x_2 - a_2x_0^2, x_1x_2 - a_1a_2x_0^2, x_3x_4\rangle|_V .$$

Consequently, $\mathrm{Bs}\,\mathscr{G}_o = \{o, \sigma(o)\}$.

Thus, we just have shown that $\mathrm{Bs}\,\mathscr{G}_o$ contains no curves. Denote by $\mathscr{L}_p \subset |-2K_X|$ the linear subsystem of section of $|-2K_X|$ vanishing at p. Then $\mathrm{Bs}\,\mathscr{L}_p$ contains no curves as well.

Now assume the contrary; that is, p is a centre of non-canonical singularities of $(X, \lambda\mathcal{M}_X)$. Let $M_1, M_2 \in \mathcal{M}_X$ and $S \in \mathscr{L}_p$ be general elements. Then

$$4/\lambda^2 = S \cdot M_1 \cdot M_2 > 4/\lambda^2 .$$

This contradicts Lemma 3.11. □

Remark 3.13. Alternatively, one can pull back the linear system $\mathcal{M}_X$ on V via π and prove similarly that if P is a smooth point of V, then the pair $(V, \lambda\pi^*(\mathcal{M}_X))$ is canonical at P; see [Isk80, Theorem 3.8] or [Che05, Theorem 2.2.1] for a modern proof.

Lemma 3.14. *Let $C \subset X$ be an irreducible curve that does not pass through the singular points of X. Then C is not a centre of non-canonical singularities.*

Proof. Firstly, recall that by definition if a curve is a centre of non-canonical singularities, then it is irreducible. Now suppose that an irreducible curve C is a centre of non-canonical singularities of $(X, \lambda\mathcal{M}_X)$. Then one has $\mathrm{mult}_C\,\mathcal{M}_X > 1/\lambda$, for example by [KSC04, Exercise 6.18]. This implies that $H \cdot C = 1$. Notice that the morphism $\pi\colon V \to X$ is étale outside of the points p_i; hence the pair $(V, \lambda\pi^*(\mathcal{M}_X))$ is non-canonical at the curve Z, where Z is the preimage of the curve C via π. Therefore, $-K_V \cdot Z = 2$ by the projection formula. On the other hand, the proofs of [Isk80, Theorems 3.8 and 3.10] show that the only centres of non-canonical singularities on V are curves $\mathscr{C} \subset V$ such that $-K_V \cdot \mathscr{C} = 1$, which immediately leads to a contradiction. However, the author has omitted several important technical details. So, for convenience of the reader, we will give an independent proof. In order to finish the proof of the lemma, we need to show that:

- if Z is an irreducible curve such that $-K_V \cdot Z = 2$, then Z cannot be a centre of non-canonical singularities;
- if $Z = Z_1 \cup Z_2$, where $Z_2 = \sigma(Z_1)$ and $-K_V \cdot Z_i = 1$, then Z_1 and Z_2 cannot simultaneously be centres of non-canonical singularities.

Assume that Z is irreducible. In this case the proof of [Che05, Theorem 2.3.1] implies that if Z is a centre of non-canonical singularities, then $-K_V \cdot Z = 1$. So we have a contradiction.

Now assume $Z = Z_1 \cup Z_2$, where $Z_2 = \sigma(Z_1)$ and $-K_V \cdot Z_i = 1$. Then Z_i is a smooth rational curve. Notice that $Z_1 \cap Z_2 = \varnothing$ because the curve C does not pass through the singular points of X by assumption. Recall that we denote by $\nu\colon V \to V$ the involution induced by the double covering $g\colon V \to Q$ of a quadric $Q \subset \mathbb{P}^4$. Notice that the involutions σ and ν do commute; hence either both Z_1 and Z_2 are contained in the ramification divisor or neither.

Firstly, we prove that $\nu(Z_i) \neq Z_i$; that is, the curves Z_1 and Z_2 are not contained in the ramification divisor of g. Assume the contrary. Let $Y \subset Q$ be a hyperplane section containing $g(Z_1)$ and $g(Z_2)$. Then $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$, and both curves $g(Z_1)$ and $g(Z_2)$ lie in the same ruling class of Y. Let L be a general curve on V such that $g(L) \subset Y$ is a general curve from another ruling class. Then $-K_V \cdot L = 2$, and L intersects Z_1 and Z_2 each in a single point. Let S be a general surface of $\pi^*(\mathcal{M}_X)$. Then one has $L \not\subset S$ because the linear system $\pi^*(\mathcal{M}_X)$ has no fixed components. So,

$$2/\lambda = S \cdot L \geqslant \sum_{O \in L \cap Z} \operatorname{mult}_O S \operatorname{mult}_O L \geqslant \sum_{O \in L \cap Z} \operatorname{mult}_Z S > 2/\lambda\,.$$

So we have a contradiction. Thus, the curves Z_i are not ν-invariant.

Let $\beta\colon W \to V$ be the blow-up of Z_1 and Z_2, and let E_1 and E_2 be the β-exceptional divisors over Z_1 and Z_2, respectively. Let us prove that the divisor $\beta^*(-2K_V) - E_1 - E_2$ is nef. Firstly, denote by $\mathcal{L} \subset |-2K_V|$ the linear subsystem of sections of $|-2K_V|$ containing both Z_1 and Z_2. We want to show that

$$\operatorname{Bs}\mathcal{L} \subseteq \{Z_1, \nu(Z_1), Z_2, \nu(Z_2)\}\,.$$

Indeed, let us consider the linear subsystem $\mathcal{Q} \subset |\mathcal{O}_Q(2)|$ of sections of $|\mathcal{O}_Q(2)|$ containing both $g(Z_1)$ and $g(Z_2)$. Further, let $\mathcal{H}_1, \mathcal{H}_2 \subset |\mathcal{O}_Q(1)|$ be the linear subsystems of sections of $|\mathcal{O}_Q(1)|$ containing $g(Z_1)$ and $g(Z_2)$, respectively. Then $U_1 + U_2 \in \mathcal{Q}$, where $U_i \in \mathcal{H}_i$, and $\operatorname{Bs}|U_1 + U_2| = \{g(Z_1), g(Z_2)\}$. Since one has $|U_1 + U_2| \subset \mathcal{Q}$ and $g^*(\mathcal{Q}) \subset \mathcal{L}$, then $\operatorname{Bs}\mathcal{L} \subseteq \{Z_1, \nu(Z_1), Z_2, \nu(Z_2)\}$.

Therefore, in order to prove that $\beta^*(-2K_V) - E_1 - E_2$ is nef, it is enough to show that it intersects positively with the strict transforms of the curves $\nu(Z_i)$ via β and curves contained in the β-exceptional divisors. By construction, $-2K_V \cdot \nu(Z_i) = 2$. Consequently,

$$(\beta^*(-2K_V) - E_1 - E_2) \cdot \beta_*^{-1}(\nu(Z_i)) \geqslant 2 - 1 - 1 \geqslant 0\,.$$

Now let us consider curves contained in the β-exceptional divisors. Let $\mathcal{N}_{Z_i/V}$ be the normal sheaf of the curve Z_i on V. Hence,

$$\mathcal{N}_{Z_i/V} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$$

for some integers a, b with $a \geqslant b$. Using the exact sequence

$$0 \longrightarrow \mathcal{T}_{Z_i} \longrightarrow \mathcal{T}_V|_{Z_i} \longrightarrow \mathcal{N}_{Z_i/V} \longrightarrow 0\,,$$

one considers the top exterior powers to get

$$0 \longrightarrow \mathcal{O}_{Z_i}(-K_{Z_i}) \longrightarrow \mathcal{O}_V(-K_V)|_{Z_i} \longrightarrow \det\mathcal{N}_{Z_i/V} \longrightarrow 0\,.$$

Since

$$\det\mathcal{N}_{Z_i/V} \cong \det(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)) \cong \mathcal{O}_{\mathbb{P}^1}(a) \otimes \mathcal{O}_{\mathbb{P}^1}(b) \cong \mathcal{O}_{\mathbb{P}^1}(a+b)\,,$$

by taking degrees of the previous exact sequence, one has

$$a + b = -K_V \cdot Z_i + 2g(Z_i) - 2 = -1\,.$$

Let S be a general hyperplane section of V containing the curve Z_i. Then S is a smooth $K3$ surface because Z_i is not contained in the ramification divisor of g. Therefore, $\mathcal{N}_{Z_i/S} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$.

Using the exact sequence

$$0 \longrightarrow \mathcal{N}_{Z_i/S} \longrightarrow \mathcal{N}_{Z_i/V} \longrightarrow \mathcal{N}_{S/V}|_{Z_i} \longrightarrow 0\,,$$

one has $b \geqslant -2$. In particular, $a - b \leqslant 3$. Let $\mathfrak{s}_i$ be the exceptional section of the Hirzebruch surface $\beta|_{E_i}\colon E_i \to Z_i$. Then

$$(\beta^*(-2K_V) - E_1 - E_2) \cdot \mathfrak{s}_i = \frac{5+b-a}{2} > 0\,.$$

Thus, $\beta^*(-2K_V) - E_1 - E_2$ is nef.

Now let $\mathcal{M}_W = \beta_*^{-1}(\pi^*\mathcal{M}_X)$. Set $m_i = \operatorname{mult}_{Z_i}(\pi^*\mathcal{M}_X)$. Then, on the one hand

$$\begin{aligned}&\big(\beta^*\big(-\tfrac{1}{\lambda}K_V\big) - m_1E_1 - m_2E_2\big)^2 \cdot (\beta^*(-2K_V) - E_1 - E_2)\\ &= -3m_1^2 - 3m_2^2 - \frac{2m_1}{\lambda} - \frac{2m_2}{\lambda} + \frac{8}{\lambda^2} < 0\,,\end{aligned}$$

but on the other hand, for general surfaces S_1 and S_2 from $\mathcal{M}_W$,

$$\begin{aligned}&\big(\beta^*\big(-\tfrac{1}{\lambda}K_V\big) - m_1E_1 - m_2E_2\big)^2 \cdot (\beta^*(-2K_V) - E_1 - E_2)\\ &= S_1 \cdot S_2 \cdot (\beta^*(-2K_V) - E_1 - E_2) \geqslant 0\,.\end{aligned}$$

So we have a contradiction. □

Lemma 3.15. *Assume $a_i \geqslant 0$ for some i. Then Φ_i is a square equivalence.*

Proof. Let $\kappa_i\colon \widetilde{X}_i \to X$ be the Kawamata blow-up of q_i and E_i be the κ_i-exceptional divisor. Then

$$(-K_{U_i})^3 = (-K_{\widetilde{X}_i})^3 = \big(\kappa_i^*(-K_X) - \tfrac{1}{2}E_i\big)^3 = \kappa_i^*(-K_X)^3 - \tfrac{1}{8}E_i^3 = 3/2\,,$$

and $\operatorname{nef}\big(U_i/\mathbb{P}^1\big) = 0$ by Remark 3.8 because in our case ξ_i is the class of the flopping curves, so $-K_{U_i} \cdot \xi_i = 0$. Thereby, $U_i/\mathbb{P}^1$ satisfies the $K^3_{3/2}$-condition, and we are done by Proposition 3.10. □

Lemma 3.16. *Assume $a_i < 0$ for all i. Then $\Phi\colon X \dashrightarrow W$ is an isomorphism.*

Proof. Assume the contrary. Then we have the following commutative diagram:

$$\begin{tikzcd} & W & \\ X \arrow[ur, dashed, "\Phi"] \arrow[rr, dashed, "\rho_i"] & & U_i \arrow[ul, dashed, "\Phi_i"'] \\ & \widetilde{X}_i\,. \arrow[ul, "\kappa_i"] \arrow[ur, dashed, "\psi_i"'] & \end{tikzcd}$$

Notice that $\rho_i\colon X \dashrightarrow U_i$ is the composition of the inverse map to the Kawamata blow-up $\kappa_i\colon \widetilde{X}_i \to X$ and two flops $\psi_i\colon \widetilde{X}_i \dashrightarrow U_i$. Consequently, without loss of generality, we can identify linear systems on U_i with linear systems on $\widetilde{X}_i$ via $(\psi_i)_*$. We write $\mathcal{M}_X \sim_{\mathbb{Q}} dH$, and hence we have

$$\begin{aligned}\kappa_i^*H &= H_{\widetilde{X}_i} + E_i\,,\\ K_{\widetilde{X}_i} &= -H_{\widetilde{X}_i} - \tfrac{1}{2}E_i\,,\\ \mathcal{M}_{\widetilde{X}_i} &= \kappa_i^*\mathcal{M}_X - m_iE = dH_{\widetilde{X}_i} + (d-m_i)E_i\,.\end{aligned}$$

Therefore,

$$dH_{\widetilde{X}_i} + (d - m_i)E_i = -n_i K_{\widetilde{X}_i} + a_i F = (n_i + a_i)H_{\widetilde{X}_i} + \tfrac{n_i}{2}E_i\,.$$

Thus,

$$d - 2m_i = -a_i\,. \tag{3.17}$$

By [Kaw96, Lemma 7] and Lemmas 3.12 and 3.14, a point q_j must be a centre of non-canonical singularities of the pair $\big(X, \frac{1}{d}\mathcal{M}_X\big)$ for some j. However, all $-a_i > 0$, so the pair $\big(X, \frac{1}{d}\mathcal{M}_X\big)$ is canonical, and thus Φ is an isomorphism by the Noether–Fano inequality. □

LEMMA 3.18. *Let q_i be a centre of non-canonical singularities of the pair $(X, \lambda\mathcal{M}_X)$. Then $\rho_i\colon U_i \dashrightarrow W$ a square equivalence.*

Proof. This follows directly from (3.17) and Proposition 3.10. □

Finally, Lemmas 3.12, 3.14, 3.15, 3.16, 3.18 and the Noether–Fano inequality imply the following theorem.

THEOREM 3.19. *Let $\Phi\colon X \dashrightarrow W$ be a birational map to a Mori fibre space W/B. Then one of the following holds:*

(1) *$B = \mathrm{pt}$ and Φ is an isomorphism;*

(2) *$B \cong \mathbb{P}^1$ and for $i = 1, \ldots, 8$ there exists the following commutative diagram:*

$$\begin{tikzcd} X \arrow[dr, dashed, "\rho_i"'] \arrow[drr, dashed, bend left, "\Phi"] & & \\ & U_i \arrow[r, dashed, "\chi"] \arrow[d, "f_i"'] & W \arrow[d] \\ & \mathbb{P}^1 \arrow[r, "\omega"] & B\,, \end{tikzcd}$$

where $\chi\colon U_i \dashrightarrow W$ is a birational map and $\omega\colon \mathbb{P}^1 \to B$ is an isomorphism.

Now, Main Theorem 2 follows from the construction provided in Section 3.1 and Theorem 3.19.

COROLLARY 3.20. *In the notation and with the assumptions above, the following hold:*

(1) *One has $2 \leqslant |\mathcal{P}(X)| \leqslant 9$. For a general X one has $|\mathcal{P}(X)| = 9$.*

(2) *One has $\mathrm{Aut}(X) = \mathrm{Bir}(X)$.*

(3) *The threefold X is not birational to a conic bundle.*

(4) *The threefold X is not rational.*

Proof. Assertions (2), (3), (4) are obvious. Let us prove assertion (1).

Clearly $2 \leqslant |\mathcal{P}(X)| \leqslant 9$, so we need to prove that $|\mathcal{P}(X)| = 9$ for a general X. Assume that there is a birational map $g\colon U_1 \dashrightarrow U_2$. Then g induces a birational automorphism $h\colon X \dashrightarrow X$, where $h = \rho_1 \circ g \circ \rho_2^{-1}$. However, by assertion (2), the morphism h is biregular. Consequently, h must permute the points q_1 and q_2. Now let us consider the map determined by the linear system $|H|$:

$$\phi_{|H|}\colon X \dashrightarrow \mathbb{P}^2 = \mathbb{P}\big(H^0(X, \mathcal{O}_X(H))\big)\,.$$

Hence, h induces a biregular action on $\mathbb{P}^2 = \mathbb{P}\big(H^0(X, \mathcal{O}_X(H))\big)$. Now we notice that for a general X the images of the q_i on $\mathbb{P}^2$ cannot be non-trivially permuted by any element of $\mathrm{PGL}(3, \mathbb{C})$. Hence, $|\mathcal{P}(X)| = 9$ for a general X. Example 3.21 finishes the proof. $\square$

Example 3.21. Let $x_0, \dots, x_4, x_5$ be homogeneous coordinates on $\mathbb{P}(1^5, 2)$, where x_5 is the coordinate of weight 2. Let $V \subset \mathbb{P}(1^5, 2)$ be defined by the following equations:

$$\begin{cases} x_0x_1 - x_2^2 + x_3x_4 = 0\,, \\ x_5^2 + x_0x_1x_3^2 + x_2^2x_4^2 + 2x_0^3x_1 - 4x_0x_1^3 + 30x_1^4 - 42x_0^3x_2 + 5x_0^2x_1x_2 - 2x_0x_1^2x_2 + 79x_1^3x_2 \\ +363x_0^2x_2^2 + x_0x_1x_2^2 - 803x_1^2x_2^2 - 890x_0x_2^3 + 850x_1x_2^3 + 411x_2^4 + x_3^4 + x_4^4 = 0\,. \end{cases}$$

Let the involution $\sigma\colon V \to V$ be given by

$$\sigma(x_0 : x_1 : x_2 : x_3 : x_4 : x_5) = (x_0 : x_1 : x_2 : -x_3 : -x_4 : -x_5)\,.$$

Then one can see that the σ-invariant points are the following:

$$\begin{aligned} p_1 &= (1:0:0:0:0:0)\,, & p_2 &= (1:1:1:0:0:0)\,, \\ p_3 &= (1:1:-1:0:0:0)\,, & p_4 &= (4:1:2:0:0:0)\,, \\ p_5 &= (1:9:3:0:0:0)\,, & p_6 &= (9:4:6:0:0:0)\,, \\ p_7 &= (25:1:5:0:0:0)\,, & p_8 &= (1:49:-7:0:0:0)\,. \end{aligned}$$

Let $r_i = (\phi_{|H|} \circ \pi)(p_i)$. Then $\Sigma = \{r_1, \dots, r_8\} = Y_1 \cap Y_2$, where

$$\begin{aligned} Y_1 &= \big\{x_0x_1 - x_2^2 = 0\big\}\,, \\ Y_2 &= \big\{2x_0^3x_1 - 4x_0x_1^3 + 30x_1^4 - 42x_0^3x_2 + 5x_0^2x_1x_2 - 2x_0x_1^2x_2 + 79x_1^3x_2 \\ &\quad + 363x_0^2x_2^2 + x_0x_1x_2^2 - 803x_1^2x_2^2 - 890x_0x_2^3 + 850x_1x_2^3 + 411x_2^4 = 0\big\}\,. \end{aligned}$$

Let $M\colon \mathbb{P}^2 \to \mathbb{P}^2$ be an automorphism such that $M(\Sigma) = \Sigma$. Obviously $M(Y_1) = Y_1$, so one can consider $r_1, \dots, r_8$ as points on $\mathbb{P}^1$. Now direct computations show that there are no automorphisms of $\mathbb{P}^1$ respecting this set of points.

In fact, for a special X we may have $|\mathcal{P}(X)| = 2$. Consider the following example.

Example 3.22. Let V be defined by the following equations:

$$\begin{cases} -2ix_0x_1 + x_2^2 + x_3x_4 = 0\,, \\ x_5^2 + x_0x_1x_3^2 + x_2^2x_4^2 + x_0^4 + x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0\,, \end{cases}$$

and let involution σ be the same.

Then one can see that the σ-invariant points are the following:

$$\begin{aligned} p_1 &= \big((-1-i)\big(2+\sqrt{3}\big)^{1/4} : (1+i)\big(2-\sqrt{3}\big)^{1/4} : 2:0:0:0\big)\,, \\ p_2 &= \big((1-i)\big(2+\sqrt{3}\big)^{1/4} : (1-i)\big(2-\sqrt{3}\big)^{1/4} : 2:0:0:0\big)\,, \\ p_3 &= \big((1+i)\big(2+\sqrt{3}\big)^{1/4} : (-1-i)\big(2-\sqrt{3}\big)^{1/4} : 2:0:0:0\big)\,, \\ p_4 &= \big((-1+i)\big(2+\sqrt{3}\big)^{1/4}, (-1+i)\big(2-\sqrt{3}\big)^{1/4}, 2:0:0:0\big)\,, \\ p_5 &= \big((-1-i)\big(2-\sqrt{3}\big)^{1/4} : (1+i)\big(2+\sqrt{3}\big)^{1/4} : 2:0:0:0\big)\,, \\ p_6 &= \big((1-i)\big(2-\sqrt{3}\big)^{1/4} : (1-i)\big(2+\sqrt{3}\big)^{1/4} : 2:0:0:0\big)\,, \end{aligned}$$

$$p_7 = \big((-1+i)\big(2-\sqrt{3}\big)^{1/4} : (-1+i)\big(2+\sqrt{3}\big)^{1/4} : 2 : 0 : 0 : 0\big)\,,$$
$$p_8 = \big((1+i)\big(2-\sqrt{3}\big)^{1/4} : (-1-i)\big(2+\sqrt{3}\big)^{1/4} : 2 : 0 : 0 : 0\big)\,.$$

Consider the morphisms $\alpha\colon \mathbb{P}\big(1^5,2\big) \to \mathbb{P}\big(1^5,2\big)$ given by

$$\alpha(x_0 : x_1 : x_2 : x_3 : x_4 : x_5) = (ix_0 : -ix_1 : x_2 : x_3 : x_4 : x_5)\,,$$

and $\beta\colon \mathbb{P}\big(1^5,2\big) \to \mathbb{P}\big(1^5,2\big)$ given by

$$\beta(x_0 : x_1 : x_2 : x_3 : x_4 : x_5) = (x_1 : x_0 : x_2 : x_3 : x_4 : x_5)\,.$$

Then α and β induce automorphisms of V, which we will still denote by α and β. Set $\Omega = \{p_1, \ldots, p_8\}$. Representing the point p_i by the number i, we notice that α acts on the set of eight elements Ω as $(1234)(5678) \in \mathfrak{S}_8$ and β acts on Ω as $(18)(26)(35)(47) \in \mathfrak{S}_8$. Therefore, the subgroup $\langle \alpha, \beta \rangle \subset \mathrm{Aut}(V)$ permutes transitively the set Ω. Thus, all $U_i/\mathbb{P}^1$ are square equivalent, and hence $|\mathcal{P}(X)| = 2$.

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