



A generic global Torelli theorem for certain Horikawa surfaces

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ABSTRACT

Algebraic surfaces of general type with $q = 0$, $p_g = 2$ and $K^2 = 1$ were described by Enriques and then studied in more detail by Horikawa. In this paper, we consider a 16-dimensional family of special Horikawa surfaces which are certain bidouble covers of \mathbb{P}^2 . The construction is motivated by that of special Kunev surfaces which are counterexamples for the infinitesimal Torelli and generic global Torelli problems. The main result of the paper is a generic global Torelli theorem for special Horikawa surfaces. To prove the theorem, we relate the periods of special Horikawa surfaces to the periods of certain lattice polarized $K3$ surfaces using eigenperiod maps and then apply a Torelli type result proved by Laza.

1. Introduction

There are two particular situations where the period map plays an essential role in the study of moduli spaces, namely principally polarized abelian varieties and (lattice)polarized $K3$ surfaces. In these cases, the period domains are Hermitian symmetric domains, and the period maps are both injective and dominant. It is an interesting problem to find more examples where the period maps are injective and the images lie in certain Mumford–Tate subdomains which are locally Hermitian symmetric (but the Griffiths infinitesimal period relations may be nontrivial on the ambient period domains). We mention the examples previously studied by Allcock, Carlson and Toledo [ACT02, ACT11], Kondo [Kon00], Borcea [Bor97], Voisin [Voi93], Rohde [Roh09], Garbagnati and Van Geemen [GvG10].

The general problem of determining whether the period map is injective is called a Torelli problem. There are four types of Torelli problems (we follow the terminology of [Cat84a]): local Torelli (whether the differential of the period map is injective), infinitesimal Torelli (local Torelli for the semi-universal deformation), global Torelli (whether the period map is injective) and generic global Torelli (whether the period map is injective over an open dense subset).

Various Torelli theorems have been proved for a large class of varieties (cf. [Cat84a]). However, Kunev [Kyn77] constructed a counterexample for the infinitesimal Torelli and generic global Torelli problems (see also [Cat79, Cat80, Tod80]). Let us briefly recall the construction. Let C_1 and C_2 be two smooth plane cubic curves intersecting transversely and L be a general line. Let

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X be the $(\mathbb{Z}/2\mathbb{Z})^2$ -cover of \mathbb{P}^2 branched along $C_1 + C_2 + L$. Then X is a minimal algebraic surface with $q(X) = 0$, $p_g(X) = 1$ and $K_X^2 = 1$. Following [Cat79], such surfaces X are *special Kunev surfaces* whose bicanonical maps are Galois covers of \mathbb{P}^2 . The infinitesimal period map and period map for special Kunev surfaces both have 2-dimensional fibers (the rough reason is that the period map only sees the Hodge structures of the intermediate $K3$ surfaces obtained as the desingularizations of the double planes along $C_1 + C_2$).

One may ask whether is it possible to modify the construction or the period map for Kunev surfaces to get a Torelli theorem. Usui [Usu83] and Letizia [Let84] considered the complement of the canonical curve $\Lambda \subset X$ and the mixed Hodge structure $H^2(X - \Lambda)$, and proved the infinitesimal mixed Torelli theorem and the generic global mixed Torelli theorem for special Kunev surfaces X , respectively. Our idea is to modify the branch data $C_1 + C_2 + L$ (see also [Par91a, Definition 2.1]). Specifically, we consider the $(\mathbb{Z}/2\mathbb{Z})^2$ -cover S of \mathbb{P}^2 along a smooth quintic C together with two generic lines L_0 and L_1 . The surfaces S are minimal surfaces with $q(S) = 0$, $p_g(S) = 2$ and $K_S^2 = 1$ which have been studied by Horikawa [Hor76]. We shall call such bidouble covers S *special Horikawa surfaces*. We should remark that these surfaces S are also mentioned in the recent paper [Gar18] by Garbagnati. But her perspective (classify the possible branch loci of a smooth double cover of a $K3$ surface) is quite different from ours.

Let us explain why we want to modify the branch data in this way. On the one hand, there are two (lattice polarized) $K3$ surfaces hidden in the construction of a special Horikawa surface S (this is also observed in [Gar18]). Namely, one resolves the singularities of the double cover of \mathbb{P}^2 branched along $C + L_1$ (respectively, $C + L_0$) and gets X_0 (respectively, X_1), which is a $K3$ surface. On the other hand, it is also natural to study the action of the Galois group $(\mathbb{Z}/2\mathbb{Z})^2$ on the periods of the bidouble covers S . We shall show that the eigenperiods (cf. [DK07, §7]) of S are determined by the Hodge structures of the $K3$ surfaces X_0 and X_1 and apply the global Torelli theorem for (lattice polarized) $K3$ surfaces (a modified version is needed, see the next paragraph) to prove a generic global Torelli theorem for special Horikawa surfaces.

By the work of Catanese [Cat84b] and Pardini [Par91a], the isomorphism classes of the bidouble covers S are determined by the branch data $C + L_0 + L_1$. This can be used to construct the coarse moduli space \mathcal{M} for special Horikawa surfaces (the moduli of special Kunev surfaces has been constructed in a similar manner; see [Usu89]). It also follows that we need the global Torelli theorem [Laz09, Theorem 4.1] for degree 5 pairs (C, L) consisting of plane quintics C and lines L (up to projective equivalence). A key point is that one needs to choose a suitable arithmetic group, as explained in [Laz09, Proposition 4.22].

A typical way to prove a generic global Torelli theorem is to study the infinitesimal variation of Hodge structure and first prove a variational Torelli theorem (cf. [CDT87]). Our approach is different. One advantage is that we are able to describe the open dense subset over which the period map is injective explicitly. We suspect that variational Torelli fails for special Horikawa surfaces (otherwise, by op. cit., the global Torelli theorem holds for any discrete group of the automorphism group of the period domain, as long as the period map is well defined, which seems not true; see also [Hay14]).

After “labeling” the lines L_0 and L_1 , we obtain a period map (using the period maps for the degree 5 pairs (C, L_0) and (C, L_1)) from (a double cover of) the moduli \mathcal{M} of special Horikawa surfaces S to a product of two arithmetic quotients of type IV domains. The period map is generically injective. Therefore, special Horikawa surfaces are along the lines of the examples mentioned at the beginning of the paper. In an ongoing project with Gallardo and Laza, we use this period map as our guide to compactify the moduli space of special Horikawa surfaces.

Let us say a few words on the structure of the paper. The construction of special Horikawa surfaces is given in Section 2. As is well known (cf. [Hor76]), the canonical model of an algebraic surface with $q = 0$, $p_g = 2$ and $K^2 = 1$ is a degree 10 hypersurface in the weighted projective space $\mathbb{P}(1, 1, 2, 5)$. We shall give the equations for (the canonical models of) special Horikawa surfaces and use the Griffiths residue to study the decomposition of the Hodge structures. The infinitesimal Torelli problem will be discussed in Section 3. Usui [Usu78] has proved the infinitesimal Torelli theorem for nonsingular weighted complete intersections satisfying certain conditions, which will be checked for special Horikawa surfaces. A second proof will also be included which can be viewed as a boundary case of [Par91b, Theorem 3.1] or [Par98, Theorem 4.2] and might be of independent interest. In Section 4, we discuss the generic global Torelli problem for special Horikawa surfaces and prove the main result.

2. Special Horikawa surfaces

Let C be a smooth plane quintic curve. Let L_0 and L_1 be two distinct lines which intersect C transversely such that $C \cap L_0 \cap L_1 = \emptyset$. We are interested in the bidouble cover S of \mathbb{P}^2 branched along $C + L_0 + L_1$.

Specifically, the surface S can be constructed in the following way. Take the double cover \bar{X}_0 of \mathbb{P}^2 branched along the sextic curve $C + L_1$. The surface \bar{X}_0 is a singular $K3$ surface with five A_1 singularities. Let X_0 be the $K3$ surface obtained by blowing up the singularities (that is, take the canonical resolution of \bar{X}_0). Denote by E_1, \dots, E_5 the exceptional curves on X_0 with self intersection (-2) . Set D_0 to be the preimage of L_0 in X_0 , and let $D_1 \subset X_0$ be the strict transform of L_1 . Choose $\mathcal{L} := \mathcal{O}_{X_0}(D_1 + E_1 + \dots + E_5)$. By computing the pull-back of $\mathcal{O}_{\mathbb{P}^2}(1)$, one sees that $D_0 \sim 2D_1 + E_1 + \dots + E_5$ and hence $\mathcal{L}^{\otimes 2} = \mathcal{O}_{X_0}(D_0 + E_1 + \dots + E_5)$. Now, we take the branched double cover S_0 of X_0 along $D_0 + E_1 + \dots + E_5$. The exceptional curves E_1, \dots, E_5 become (-1) -curves on S_0 . Contract these (-1) -curves, and one obtains a surface S . For later use, let us denote by σ_0 the involution of S such that $\bar{X}_0 = S/\sigma_0$. To summarize, we have the following diagram (the left one):

$$\begin{array}{ccc}
 S_0 & \xrightarrow{f_0} & S \curvearrowright \sigma_0 \\
 \downarrow \varphi_0 & & \downarrow \psi_0 \\
 X_0 & \xrightarrow{g_0} & \bar{X}_0 \\
 \downarrow \tau_0 & & \downarrow \delta_0 \\
 \tilde{\mathbb{P}}^2 & \xrightarrow{h_0} & \mathbb{P}^2,
 \end{array}
 \qquad
 \begin{array}{ccc}
 \sigma_1 \curvearrowleft S & \xleftarrow{f_1} & S_1 \\
 \downarrow \psi_1 & & \downarrow \varphi_1 \\
 \bar{X}_1 & \xleftarrow{g_1} & X_1 \\
 \downarrow \delta_1 & & \downarrow \tau_1 \\
 \mathbb{P}^2 & \xleftarrow{h_1} & \tilde{\mathbb{P}}^2.
 \end{array}
 \tag{2.1}$$

LEMMA 2.1. *The surface $\psi_0 \circ \delta_0: S \rightarrow \mathbb{P}^2$ is a Galois cover with group $(\mathbb{Z}/2\mathbb{Z})^2$.*

Proof. By a result of Zariski, the fundamental group $\pi_1(\mathbb{P}^2 - (C + L_0 + L_1))$ is abelian (see for example [Ful80] or [Cat84b, Theorem 1.6]). Therefore, the covering map $\psi_0 \circ \delta_0$ is defined by a normal subgroup and is Galois. Clearly, the Galois group is an abelian group of order 4. But it cannot be $\mathbb{Z}/4\mathbb{Z}$ because otherwise the branch locus $C + L_0 + L_1$ is divisible by 4 in $\text{Pic}(\mathbb{P}^2)$ (or one can directly check that $\sigma_0^2 = \text{id}$ in the group of deck transformations $\text{Deck}(S/\mathbb{P}^2)$). \square

Since the Galois group of the bidouble cover S is $(\mathbb{Z}/2\mathbb{Z})^2$, there is a symmetric construction for S . Namely, one takes the double cover $\delta_1: \bar{X}_1 \rightarrow \mathbb{P}^2$ branched along $C + L_0$ and resolves the five A_1 singularities to obtain a $K3$ surface $g_1: X_1 \rightarrow \bar{X}_1$. Call the exceptional curves

$F_1, \dots, F_5 \subset X_1$. It can be shown that $(\delta_1 \circ g_1)^{-1}(L_1) + F_1 + \dots + F_5$ is divisible by 2 in $\text{Pic}(X_1)$. Let S_1 be the double cover of X_1 along $(\delta_1 \circ g_1)^{-1}(L_1) + F_1 + \dots + F_5$. The surface S is obtained by contracting the (-1) -curves on S_1 . Let us use σ_1 to denote the involution of S with $\bar{X}_1 = S/\sigma_1$. (Note that σ_0 and σ_1 generate the Galois group $(\mathbb{Z}/2\mathbb{Z})^2$.) See the right part of diagram (2.1).

PROPOSITION 2.2. *Let C be a smooth quintic curve, and let L_0 and L_1 be two transverse lines with $C \cap L_0 \cap L_1 = \emptyset$. Let S be the $(\mathbb{Z}/2\mathbb{Z})^2$ -cover of \mathbb{P}^2 branched along $C + L_0 + L_1$. Then the surface S is a minimal algebraic surface of general type with $p_g(S) = 2$ and $K_S^2 = 1$. Moreover, S is simply connected, and the canonical bundle K_S is ample.*

Proof. We use the notation as above. Denote by $\pi: S \rightarrow \mathbb{P}^2$ the covering map $\delta_0 \circ \psi_0 = \delta_1 \circ \psi_1$. It is clear from the construction that S is smooth. By [Mor88, Lemma 3.2], the canonical bundle K_S of the double cover S can be computed as $2K_S \sim \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$. It follows that K_S is ample and $K_S^2 = 1$. In particular, K_S is big and nef, and hence S is a minimal surface of general type. Now, let us compute $h^{2,0}(S)$, which clearly equals $h^0(S_0, \mathcal{O}_{S_0}(K_{S_0}))$. Let $\mathcal{L} = \mathcal{O}_{X_0}(D_1 + E_1 + \dots + E_5)$ be the line bundle associated to the double cover $\varphi_0: S_0 \rightarrow X_0$ of the K3 surface X_0 . Note that $H^0(S_0, \mathcal{O}_{S_0}(K_{S_0})) = H^0(X_0, \varphi_{0*}\mathcal{O}_{S_0}(K_{S_0})) = H^0(X_0, \varphi_{0*}\varphi_0^*(\mathcal{L}))$. Since $\varphi_{0*}\varphi_0^*(\mathcal{L}) = \mathcal{L} \otimes \varphi_{0*}(\mathcal{O}_{X_0}) = \mathcal{L} \otimes (\mathcal{O}_{X_0} \oplus \mathcal{L}^{-1}) = \mathcal{L} \oplus \mathcal{O}_{X_0}$, we have $h^0(S_0, \mathcal{O}_{S_0}(K_{S_0})) = h^0(X_0, \mathcal{O}_{X_0}) + h^0(X_0, \mathcal{L})$. Because $D_1 + E_1 + \dots + E_5$ is effective and $(D_1 + E_1 + \dots + E_5)^2 = -2$, the space $H^0(X_0, \mathcal{L})$ is 1-dimensional. Thus $h^{2,0}(S) = h^0(S_0, K_{S_0}) = 2$ (this is also mentioned in [Cat79, Remark 8]). By [Bom73, Theorems 11 and 14] or [Cat84b, Proposition 2.7], the surface S is simply connected. \square

Algebraic surfaces of general type with $p_g = 2$ and $K^2 = 1$ have been studied by Horikawa [Hor76]. We call the bidouble covers S constructed above *special Horikawa surfaces*.

PROPOSITION 2.3. *The canonical model of an algebraic surface Y of general type with $q(Y) = 0$, $p_g(Y) = 2$ and $K_Y^2 = 1$ is a hypersurface of degree 10 in $\mathbb{P}(1, 1, 2, 5)$. If K_Y is ample, then Y is isomorphic to a quasi-smooth hypersurface of degree 10 in $\mathbb{P}(1, 1, 2, 5)$.*

Proof. This has been proved in [Hor76, §2]. See also [KP03, §VII.7]. \square

Remark 2.4. A weighted hypersurface $Y \subset \mathbb{P}$ is *quasi-smooth* if the associated affine quasicone is smooth outside the vertex 0 (cf. [Dol82]). If, in addition, $\text{codim}_Y(Y \cap \mathbb{P}_{\text{sing}}) \geq 2$, then $Y_{\text{sing}} = Y \cap \mathbb{P}_{\text{sing}}$. (In our case, we have $\mathbb{P} = \mathbb{P}(1, 1, 2, 5)$, which has two singular points $[0, 0, 1, 0]$ and $[0, 0, 0, 1]$, and hence $S_{\text{sing}} = S \cap \mathbb{P}_{\text{sing}}$.) Moreover, the cohomology $H^k(Y, \mathbb{C})$ of a quasi-smooth hypersurface Y admits a pure Hodge structure and can be computed explicitly using a (slightly generalized version of) the Griffiths residue.

We have shown that special Horikawa surfaces S have ample canonical bundles. A natural question is which degree 10 quasi-smooth hypersurfaces in $\mathbb{P}(1, 1, 2, 5)$ do they correspond to.

PROPOSITION 2.5. *Let S be a $(\mathbb{Z}/2\mathbb{Z})^2$ -cover of \mathbb{P}^2 branched over a smooth quintic C and two general lines L_0 and L_1 (that is, S is a special Horikawa surface). Then S is isomorphic to a quasi-smooth hypersurface $z^2 = F(x_0^2, x_1^2, y)$ in $\mathbb{P}(1, 1, 2, 5) = \text{Proj}(\mathbb{C}[x_0, x_1, y, z])$, where F is a quintic polynomial.*

Proof. Denote by $\pi: S \rightarrow \mathbb{P}^2$ the covering map $\delta_0 \circ \psi_0 = \delta_1 \circ \psi_1$. For $i = 0, 1$, we let Λ_i be the reduced inverse image $\pi^{-1}(L_i)$ of L_i in S . By the proof of Proposition 2.2, we have $\Lambda_i \in |K_S|$. Choose a section $x_i \in H^0(S, \mathcal{O}_S(K_S))$ which cuts out Λ_i . Clearly, $\{x_0, x_1\}$ forms a basis of $H^0(S, \mathcal{O}_S(K_S))$. Since $2K_S \sim \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$, the covering map π is defined by a subspace of $|2K_S|$. Choose $y \in H^0(S, \mathcal{O}_S(2K_S))$ such that the subspace of $|2K_S|$ is generated by x_0^2, x_1^2 and y . Note

that $y \neq x_0x_1$. By choosing a suitable $z \in H^0(S, \mathcal{O}_S(5K_S))$, we assume that the equation for S is $z^2 = F'(x_0, x_1, y)$, where F' is a weighted homogeneous polynomial of degree 10 in $\mathbb{P}(1, 1, 2, 5)$. (The defining equation must contain z^2 ; otherwise, S is not quasi-smooth. In the latter case, we complete the square for z , which does not affect the other coordinates.) The ramification locus π consists of three components ($x_0 = 0$), ($x_1 = 0$) and ($z = 0$), which are mapped to L_0 , L_1 and C , respectively. The proposition then follows.

Alternatively, one considers the action of σ_i (see diagram (2.1)) on the canonical ring

$$\bigoplus_{m \geq 0} H^0(S, \mathcal{O}_S(mK_S))$$

and hence on $\mathbb{P}(1, 1, 2, 5)$. Note that the involution σ_0 fixes Λ_0 pointwise and $\sigma_0(\Lambda_1) = \Lambda_1$, and similarly for the involution σ_1 . As in the proof of [Cat79, Theorem 3], one can choose y and z such that σ_0 acts on $\mathbb{P}(1, 1, 2, 5)$ by $[x_0, x_1, y, z] \mapsto [-x_0, x_1, y, z]$. Since σ_0 acts on S , we assume that the defining equation for S is an eigenvector for σ_0 . It is not difficult to see that only even powers of x_0 can appear in the defining equation. Next, one normalizes the equation and gets $z^2 = F'(x_0, x_1, y)$. Since F' is a weighted homogeneous polynomial of degree 10, the variable x_1 also occurs only in even powers. \square

Remark 2.6. (i) Because S is quasi-smooth, the quintic F must contain y^5 (more generally, see [FPR17, Theorem 3.3]).

(ii) We have shown that a special Horikawa surface S is isomorphic to a quasi-smooth hypersurface in $\mathbb{P}(1, 1, 2, 5)$ with the equation $z^2 = F(x_0^2, x_1^2, y)$. The covering map $S \rightarrow \mathbb{P}^2$ is given by $[x_0, x_1, y, z] \mapsto [x_0^2, x_1^2, y]$. The Galois group is generated by $\sigma_{x_0}: [x_0, x_1, y, z] \mapsto [-x_0, x_1, y, z]$ and $\sigma_{x_1}: [x_0, x_1, y, z] \mapsto [x_0, -x_1, y, z]$, which correspond to σ_0 and σ_1 , respectively.

(iii) Special Horikawa surfaces have moduli dimension $\binom{2+5}{2} - 1 + 2 + 2 - 8 = 16$ (dimension of moduli for plane quintics together with two lines minus dimension of $\mathrm{PGL}(3)$). The dimension of the moduli for degree 10 quasi-smooth hypersurfaces in $\mathbb{P}(1, 1, 2, 5) = \mathrm{Proj}(\mathbb{C}[x_0, x_1, y, z])$ cut out by $z^2 = F(x_0^2, x_1^2, y)$ is $\binom{2+5}{2} - 5 = 16$ (dimension of the quintic polynomials F minus dimension of the subgroup of the automorphism group of $\mathbb{P}(1, 1, 2, 5)$ acting on $z^2 = F(x_0^2, x_1^2, y)$, a semidirect product of the group consisting of the elements $[x_0, x_1, y, z] \mapsto [ax_0, bx_1, cy + dx_0^2 + ex_1^2, z]$ with the group generated by $[x_0, x_1, y, z] \mapsto [x_1, x_0, y, z]$).

Now, let us study how the Galois group $(\mathbb{Z}/2\mathbb{Z})^2$ acts on the Hodge structures of special Horikawa surfaces S . We view S as a degree 10 quasi-smooth hypersurface in $\mathbb{P}(1, 1, 2, 5)$ cut out by the equation $z^2 = F(x_0^2, x_1^2, y)$. We choose the Kähler form corresponding to one of the canonical curves ($x_j = 0$) (which is the reduced inverse image of L_j) and describe the primitive cohomology using the Griffiths residue method.

PROPOSITION 2.7. *Let Y be a quasi-smooth hypersurface of degree d in a weighted projective space $\mathbb{P}(a_0, a_1, \dots, a_n)$. That is, Y is given by a weighted homogeneous polynomial $G(z_0, z_1, \dots, z_n)$ of degree d whose partial derivatives have no common zero other than the origin. Let*

$$E = \sum_{i=0}^n z_i \frac{\partial}{\partial z_i}$$

be the Euler vector field. Let $dV = dz_0 \wedge \dots \wedge dz_n$ be the Euclidean volume form, and let $\Omega = i(E)dV$ (where i denotes interior multiplication) be the projective volume form (which has

degree $a_0 + \dots + a_n$). Consider expressions of the form

$$\Omega(A) = \frac{A \cdot \Omega}{G^q},$$

where A is a homogeneous polynomial whose degree is such that $\Omega(A)$ is homogeneous of degree 0. Then the Poincaré residues of $\Omega(A)$ span $F^{n-q} H_{\text{prim}}^{n-1}(Y, \mathbb{C})$, where F^\bullet denotes the Hodge filtration. Moreover, the residue lies in F^{n-q+1} if and only if A lies in the Jacobian ideal J_G of G (the ideal generated by the first partial derivatives of G).

Proof. This is [ACT11, Proposition 1.2]. See also [Dol82]. □

PROPOSITION 2.8. *Let $S \in \mathbb{P}(1, 1, 2, 5)$ be a quasi-smooth hypersurface of degree 10 given by $z^2 = F(x_0^2, x_1^2, y)$. Set σ_0 and σ_1 to be the automorphisms defined by $[x_0, x_1, y, z] \mapsto [-x_0, x_1, y, z]$ and $[x_0, x_1, y, z] \mapsto [x_0, -x_1, y, z]$, respectively. Let χ_0 and χ_1 be the corresponding characters of the Galois group defined by $\chi_0(\sigma_0) = 1$ and $\chi_0(\sigma_1) = -1$, and similarly for χ_1 . Then we have the following decomposition of Hodge structures:*

$$H_{\text{prim}}^2(S, \mathbb{Q}) = H_{\chi_0}^2(S, \mathbb{Q}) \oplus H_{\chi_1}^2(S, \mathbb{Q}),$$

where $H_{\chi_l}^2(S, \mathbb{Q})$ is the eigenspace corresponding to χ_l for $l = 0, 1$. Moreover, $H_{\chi_l}^2(S, \mathbb{Q})$ ($l = 0, 1$) has Hodge numbers $[1, 14, 1]$.

Proof. We use the notation of Proposition 2.7. The decomposition is obtained by a Griffiths residue calculus. Specifically, let $G = F(x_0^2, x_1^2, y) - z^2$. Take a basis for $H_{\text{prim}}^2(S, \mathbb{C})$ consisting of the forms $\text{Res}((A \cdot \Omega)/G^q)$ with A being certain monomials $x_0^i x_1^j y^k$. The cohomology group $H^{2,0}(S)$ (respectively, $H_{\text{prim}}^{1,1}(S)$, $H^{0,2}(S)$) corresponds to $q = 1$ (respectively, $q = 2$, $q = 3$), and hence $i + j + 2k = 1$ (respectively, $i + j + 2k = 11$, $i + j + 2k = 21$). In particular, $i + j$ must be odd. It follows that only the characters χ_0 and χ_1 appear in the decomposition of $H_{\text{prim}}^2(S, \mathbb{C})$ (and hence $H_{\text{prim}}^2(S, \mathbb{Q})$). The eigenspace $H_{\chi_0}^2(S, \mathbb{Q})$ is a sub-Hodge structure because $H_{\chi_0}^2(S, \mathbb{Q}) = \ker(\sigma_0^* - \text{id})$ ($= \ker(\sigma_1^* + \text{id})$), and similarly for $H_{\chi_1}^2(S, \mathbb{Q})$. The claim on Hodge numbers can be checked for a special surface $z^2 = x_0^{10} + x_1^{10} + y^5$ (consider the induced action of σ_0 or σ_1 on the polarized variation of rational Hodge structure with fibers $H_{\text{prim}}^2(S, \mathbb{Q})$). □

Remark 2.9. Let X_0 and X_1 be the $K3$ surfaces associated to a special Horikawa surface S in diagram (2.1). One can show that $T_{\mathbb{Q}}(S) \cong T_{\mathbb{Q}}(X_0) \oplus T_{\mathbb{Q}}(X_1)$, where $T_{\mathbb{Q}} = \text{Tr} \otimes \mathbb{Q}$ denotes the rational transcendental lattice. Specifically, we consider the induced actions σ_0^* and σ_1^* on $T_{\mathbb{Q}}(S)$. The space $T_{\mathbb{Q}}(S)$ decomposes as a direct sum of the eigenspaces of σ_0^* with eigenvalues 1 and -1 . By [Shi86, Proposition 5], we have $T_{\mathbb{Q}}(X_0) \cong T_{\mathbb{Q}}(S)^{\sigma_0^*}$ and $T_{\mathbb{Q}}(X_1) \cong T_{\mathbb{Q}}(S)^{\sigma_1^*}$ (where $T_{\mathbb{Q}}(S)^{\sigma_i^*}$ denotes the invariant part). Proposition 2.8 allows us to identify $T_{\mathbb{Q}}(S)^{\sigma_1^*}$ with the (-1) -eigenspace of σ_0^* , which completes the proof. Generically, $T_{\mathbb{Q}}(S)$ has Hodge numbers $[2, 28, 2]$ (one applies [Moo93, Proposition 4] to $z^2 = x_0^{10} + x_1^{10} + y^5$ to see that the generic Picard number is 1) and $T_{\mathbb{Q}}(X_l)$ has Hodge numbers $[1, 14, 1]$ (cf. [Laz09, Corollary 4.15]) for $l = 0, 1$.

3. The infinitesimal Torelli theorem

We shall show in this section that (unlike for special Kunev surfaces [Kyn77, Cat79, Tod80]) the infinitesimal Torelli holds for special Horikawa surfaces S .

THEOREM 3.1. *Let S be a bidouble cover of \mathbb{P}^2 branched along a smooth quintic C and two transverse lines L_0 and L_1 with $C \cap L_0 \cap L_1 = \emptyset$. The natural map*

$$p: H^1(S, \mathcal{T}_S) \rightarrow \text{Hom}(H^0(S, \omega_S), H^1(S, \Omega_S^1)), \quad (3.1)$$

given by the cup product, is injective.

Proof. Usui [Usu78] has proved the infinitesimal Torelli theorem for the periods of holomorphic d -forms on certain d -dimensional complete intersections ($d \geq 2$) in certain weighted projective spaces. One can check that the conditions in [Usu78, Theorem 2.1] are satisfied for the special Horikawa surface S and then apply the theorem. Specifically, by Proposition 2.5, the surface $S \subset \mathbb{P}(1, 1, 2, 5)$ is defined by $z^2 = F(x_0^2, x_1^2, y)$, and it is not difficult to verify that $z^2 - F(x_0^2, x_1^2, y)$, x_0, x_1, y forms a regular sequence for $\mathbb{C}[x_0, x_1, y, z]$. (One can apply [Usu78, Proposition 3.1] to check the conditions and prove the infinitesimal Torelli theorem for the periods of holomorphic 2-forms on any smooth hypersurface with ample canonical bundle in $\mathbb{P}(1, 1, 2, 5)$.) \square

Pardini [Par91b, Theorem 3.1], [Par98, Theorem 4.2] has considered the infinitesimal Torelli problem for certain abelian covers (including bidouble covers). The conditions of her theorems do not hold for special Horikawa surfaces S , but the strategy still works. (In particular, we need the notion of the prolongation bundle discussed in [Par98, § 2].) Let us sketch the proof, which might be useful for finding more boundary cases of Pardini's theorems.

Proof. The idea is to decompose the infinitesimal Torelli map

$$p: H^1(S, \mathcal{T}_S) \rightarrow \text{Hom}(H^0(S, \omega_S), H^1(S, \Omega_S^1))$$

using the Galois group action. The first step is to figure out the building data (see [Cat84b, § 2] and [Par91a, Definition 2.1]) of the bidouble cover $\pi: S \rightarrow \mathbb{P}^2$. The (reduced) branch locus of π consists of three irreducible components:

$$D_0 := L_0, \quad D_1 := L_1 \quad \text{and} \quad D_z := C.$$

Let σ_0 and σ_1 be the involutions in diagram (2.1). Let $\sigma_z := \sigma_0 \circ \sigma_1$. The Galois group G of the abelian cover $S \rightarrow \mathbb{P}^2$ consists of $\text{id}, \sigma_0, \sigma_1, \sigma_z$. Let χ_0, χ_1, χ_z be the corresponding nontrivial characters of G (and we shall denote the character group by G^*). Write

$$\pi_* \mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{L}_{\chi_0}^{-1} \oplus \mathcal{L}_{\chi_1}^{-1} \oplus \mathcal{L}_{\chi_z}^{-1},$$

where \mathcal{L}_{χ}^{-1} denotes the eigensheaf on which G acts via the character χ . One easily verifies that $\mathcal{L}_{\chi_0} = \mathcal{L}_{\chi_1} = \mathcal{O}_{\mathbb{P}^2}(3)$ and $\mathcal{L}_{\chi_z} = \mathcal{O}_{\mathbb{P}^2}(1)$.

Next, we compute the direct images of various sheaves. Let $D = D_0 + D_1 + D_z$. For the trivial character $\chi = 1$, define $\Delta_1 = D$. Set

$$\Delta_{\chi_0} := D_0, \quad \Delta_{\chi_1} := D_1, \quad \Delta_{\chi_z} := D_z.$$

We also let

$$D_{1,1^{-1}} := \emptyset, \quad D_{\chi_0, \chi_0^{-1}} := D_1 + D_z, \quad D_{\chi_1, \chi_1^{-1}} := D_0 + D_z, \quad D_{\chi_z, \chi_z^{-1}} := D_0 + D_1.$$

(For every pair of characters $\chi, \phi \in G^*$, the divisor $D_{\chi, \phi}$ is defined in [Par91b] and [Par98]. The fundamental relations of the building data are $\mathcal{L}_{\chi} + \mathcal{L}_{\phi} \equiv \mathcal{L}_{\chi\phi} + D_{\chi, \phi}$; see [Par91a, Theorem 2.1].) For any character χ , we have (see [Par91a, Proposition 4.1])

- $(\pi_* \mathcal{T}_S)^{(\chi)} = \mathcal{T}_{\mathbb{P}^2}(-\log \Delta_{\chi}) \otimes \mathcal{L}_{\chi}^{-1}$ (in particular, $(\pi_* \mathcal{T}_S)^{(\text{inv})} = \mathcal{T}_{\mathbb{P}^2}(-\log D)$);
- $(\pi_* \Omega_S^1)^{(\chi)} = \Omega_{\mathbb{P}^2}^1(\log D_{\chi, \chi^{-1}}) \otimes \mathcal{L}_{\chi}^{-1}$ (in particular, $(\pi_* \Omega_S^1)^{(\text{inv})} = \Omega_{\mathbb{P}^2}^1$);

– $(\pi_*\omega_S)^{(\chi)} = \omega_{\mathbb{P}^2} \otimes \mathcal{L}_{\chi^{-1}}$ (in particular, $(\pi_*\omega_S)^{(\text{inv})} = \omega_{\mathbb{P}^2}$).

Since the map π is finite, for every coherent sheaf \mathcal{F} on S one has $H^k(S, \mathcal{F}) = H^k(\mathbb{P}^2, \pi_*\mathcal{F})$ ($k = 0, 1, 2$). In particular, we have

$$H^1(S, \mathcal{T}_S) = H^1(\mathbb{P}^2, \pi_*\mathcal{T}_S), \quad H^1(S, \Omega_S) = H^1(\mathbb{P}^2, \pi_*\Omega_S), \quad H^0(S, \omega_S) = H^0(\mathbb{P}^2, \pi_*\omega_S).$$

Combining this with the splittings of $\pi_*\mathcal{T}_S$, $\pi_*\Omega_S$ and $\pi_*\omega_S$, we obtain the following decompositions:

$$H^1(S, \mathcal{T}_S) = H^1(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(-\log D)) \oplus \left(\bigoplus_{\chi \in G^* \setminus \{1\}} H^1(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(-\log \Delta_\chi) \otimes \mathcal{L}_\chi^{-1}) \right), \quad (3.2)$$

$$H^1(S, \Omega_S^1) = H^1(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^1) \oplus \left(\bigoplus_{\chi \in G^* \setminus \{1\}} H^1(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^1(\log D_{\chi, \chi^{-1}}) \otimes \mathcal{L}_\chi^{-1}) \right), \quad (3.3)$$

$$H^0(S, \omega_S) = H^0(\mathbb{P}^2, \omega_{\mathbb{P}^2}) \oplus \left(\bigoplus_{\chi \in G^* \setminus \{1\}} H^0(\mathbb{P}^2, \omega_{\mathbb{P}^2} \otimes \mathcal{L}_{\chi^{-1}}) \right). \quad (3.4)$$

Since the cup product (and hence the infinitesimal Torelli map p) is compatible with the group action, for characters $\chi, \phi \in G^*$, we consider

$$\begin{aligned} p_{\chi, \phi}: H^1(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(-\log \Delta_\chi) \otimes \mathcal{L}_\chi^{-1}) \\ \rightarrow \text{Hom} \left(H^0(\mathbb{P}^2, \omega_{\mathbb{P}^2} \otimes \mathcal{L}_{\phi^{-1}}), H^1(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^1(\log D_{\chi\phi, (\chi\phi)^{-1}}) \otimes \mathcal{L}_{\chi\phi}^{-1}) \right). \end{aligned} \quad (3.5)$$

Clearly, one has

$$p = \bigoplus_{\chi, \phi \in G^*} p_{\chi, \phi}. \quad (3.6)$$

LEMMA 3.2. *The infinitesimal Torelli theorem holds for S (that is, the map p is injective) if and only if $\bigcap_{\phi \in G^*} \ker p_{\chi, \phi} = \{0\}$ for any character χ .*

Let us take a closer look at the maps $p_{\chi, \phi}$. For every pairs of characters $\chi, \phi \in G^*$ and every section $\xi \in H^0(\mathbb{P}^2, \omega_{\mathbb{P}^2} \otimes \mathcal{L}_{\phi^{-1}})$, consider the diagram

$$\begin{array}{ccc} \mathcal{T}_{\mathbb{P}^2}(-\log \Delta_\chi) \otimes \mathcal{L}_\chi^{-1} & \longrightarrow & \Omega_{\mathbb{P}^2}^1(\log D_{\chi\phi, (\chi\phi)^{-1}}) \otimes \omega_{\mathbb{P}^2}^{-1} \otimes (\mathcal{L}_{\chi\phi} \otimes \mathcal{L}_{\phi^{-1}})^{-1} \\ \downarrow & & \downarrow \\ \mathcal{T}_{\mathbb{P}^2}(-\log \Delta_\chi) \otimes \mathcal{L}_\chi^{-1} \otimes \omega_{\mathbb{P}^2} \otimes \mathcal{L}_{\phi^{-1}} & \longrightarrow & \Omega_{\mathbb{P}^2}^1(\log D_{\chi\phi, (\chi\phi)^{-1}}) \otimes \mathcal{L}_{\chi\phi}^{-1}, \end{array}$$

where the vertical maps are given by multiplication by ξ and the horizontal maps are defined by contraction of tensors and by the fundamental relations [Par91a, Theorem 2.1] of the building data (n. b.: there is a canonical isomorphism between $\mathcal{T}_{\mathbb{P}^2}(-\log D_{\chi\phi, (\chi\phi)^{-1}})$ and $\Omega_{\mathbb{P}^2}^1(\log D_{\chi\phi, (\chi\phi)^{-1}}) \otimes (\omega_{\mathbb{P}^2}(D_{\chi\phi, (\chi\phi)^{-1}}))^{-1}$). Consider

$$\begin{aligned} q_{\chi, \phi}: H^1(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(-\log \Delta_\chi) \otimes \mathcal{L}_\chi^{-1}) \\ \rightarrow H^1(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^1(\log D_{\chi\phi, (\chi\phi)^{-1}}) \otimes \omega_{\mathbb{P}^2}^{-1} \otimes (\mathcal{L}_{\chi\phi} \otimes \mathcal{L}_{\phi^{-1}})^{-1}) \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} r_{\chi, \phi}: H^1(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^1(\log D_{\chi, \chi^{-1}}) \otimes (\omega_{\mathbb{P}^2} \otimes \mathcal{L}_\chi \otimes \mathcal{L}_{\phi^{-1}})^{-1}) \\ \rightarrow \text{Hom} \left(H^0(\mathbb{P}^2, \omega_{\mathbb{P}^2} \otimes \mathcal{L}_{\phi^{-1}}), H^1(\mathbb{P}^2, \Omega_{\mathbb{P}^2}^1(\log D_{\chi, \chi^{-1}}) \otimes \mathcal{L}_\chi^{-1}) \right). \end{aligned} \quad (3.8)$$

Obviously, we have $p_{\chi, \phi} = r_{\chi\phi, \phi} \circ q_{\chi, \phi}$.

Let us analyze the maps $r_{\chi, \phi}$.

- The maps $r_{1,1}, r_{1,\chi_0}, r_{1,\chi_1}, r_{1,\chi_z}$ are injective; this follows by an explicit computation and Bott’s vanishing theorem.
- The maps $r_{\chi_0,1}, r_{\chi_1,1}, r_{\chi_z,1}, r_{\chi_0,\chi_z}, r_{\chi_1,\chi_z}$ are zero since $H^0(\mathbb{P}^2, \omega_{\mathbb{P}^2} \otimes \mathcal{L}_{\phi^{-1}}) = 0$.
- The maps $r_{\chi_0,\chi_0}, r_{\chi_0,\chi_1}, r_{\chi_1,\chi_0}, r_{\chi_1,\chi_1}$ are injective; we use the prolongation bundle of the irreducible components of $D_{\chi,\chi^{-1}}$ defined in [Par98, § 2] and show that if the multiplication map

$$\begin{aligned} & H^0(\mathbb{P}^2, \omega_{\mathbb{P}^2} \otimes \mathcal{L}_{\phi^{-1}}) \otimes \left(\bigoplus_{\substack{B \text{ irreducible} \\ \text{components of } D_{\chi,\chi^{-1}}} } H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(B) \otimes \omega_{\mathbb{P}^2} \otimes \mathcal{L}_{\chi}) \right) \\ & \rightarrow \bigoplus_{\substack{B \text{ irreducible} \\ \text{components of } D_{\chi,\chi^{-1}}} } H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(B) \otimes \omega_{\mathbb{P}^2}^{\otimes 2} \otimes \mathcal{L}_{\chi} \otimes \mathcal{L}_{\phi^{-1}}) \end{aligned}$$

is surjective, then the map $r_{\chi,\phi}$ is injective (cf. [Par98, § 3]). As argued in [Par98, Proposition 3.5], the surjectivity of the multiplication map follows from a special case of [EL93, Theorem 2.1].

- The maps $r_{\chi_z,\chi_0}, r_{\chi_z,\chi_1}, r_{\chi_z,\chi_z}$ are injective; we also consider the prolongation bundle and also note that $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(B) \otimes \omega_{\mathbb{P}^2}^{\otimes 2} \otimes \mathcal{L}_{\chi} \otimes \mathcal{L}_{\phi^{-1}}) = 0$ (with B any irreducible component of $D_{\chi,\chi^{-1}}$) in these cases.

We use the method of [Par91b, § 3] to study the maps $q_{\chi,\phi}$ (especially diagram (3.4) and Lemma 3.1). The results are as follows:

- For $\chi = 1$, we have $\ker q_{1,\chi_0} \cap \ker q_{1,\chi_1} = \{0\}$ (and hence $\ker p_{1,\chi_0} \cap \ker p_{1,\chi_1} = \{0\}$).
- For $\chi = \chi_0$, both q_{χ_0,χ_1} and q_{χ_0,χ_z} are injective (and hence $p_{\chi_0,\chi_1} = r_{\chi_z,\chi_1} \circ q_{\chi_0,\chi_1}$ is injective).
- For $\chi = \chi_1$, both q_{χ_1,χ_0} and q_{χ_1,χ_z} are injective (and hence $p_{\chi_1,\chi_0} = r_{\chi_z,\chi_0} \circ q_{\chi_1,\chi_0}$ is injective).
- For $\chi = \chi_z$, both q_{χ_z,χ_0} and q_{χ_z,χ_1} are injective (and hence $p_{\chi_z,\chi_0} = r_{\chi_1,\chi_0} \circ q_{\chi_z,\chi_0}$ is injective, and $p_{\chi_z,\chi_1} = r_{\chi_0,\chi_1} \circ q_{\chi_z,\chi_1}$ is also injective).

The theorem clearly follows from these observations. □

4. Degree 5 pairs and a generic global Torelli theorem

Let us review the period map for degree 5 pairs, which will be used later in this section to prove a generic global Torelli theorem for special Horikawa surfaces S .

Following [Laz09, Definition 2.1], we call a pair (C, L) consisting of a plane quintic curve C and a line $L \subset \mathbb{P}^2$ a *degree 5 pair*. Two such pairs are equivalent if they are projectively equivalent. We are interested in the degree 5 pairs (C, L) with $C + L$ defining a sextic curve admitting at worst *ADE* singularities. The coarse moduli space \mathcal{M}_{ADE} is contained in the geometric invariant theory (GIT) quotient $(\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(5)) \times \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))) // \mathrm{SL}_3(\mathbb{C})$ (with respect to the linearization $\pi_1^* \mathcal{O}_{\mathbb{P}^2}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^2}(1)$).

To every degree 5 pair (C, L) such that $C + L$ has at worst *ADE* singularities, we associate a *K3* surface $X_{(C,L)}$ obtained by taking the canonical resolution of the double cover $\bar{X}_{(C,L)}$ of \mathbb{P}^2 along the sextic $C + L$. The period map for degree 5 pairs (C, L) is defined via the periods of $X_{(C,L)}$. Specifically, one first considers a generic pair (C, L) , where C is smooth and L is a transversal. The *K3* surface $X_{(C,L)}$ contains five exceptional curves e_1, \dots, e_5 corresponding to

the five intersection points of $C \cap L$ and the strict transform l' of L . By [Laz09, Proposition 4.12, Definition 4.17], the lattice generated by $\{l', e_1, \dots, e_5\}$ is a primitive sublattice of $\text{Pic}(X_{(C,L)})$, and hence $X_{(C,L)}$ is a lattice polarized $K3$ surface (note that the lattice polarization depends on the labeling of the points of intersection $C \cap L$).

Notation 4.1. Let Λ be an even lattice. We define

- Λ^* : the dual lattice;
- $A_\Lambda = \Lambda^*/\Lambda$: the discriminant group endowed with the induced quadratic form q_Λ ;
- $O(\Lambda)$: the group of isometries of Λ ;
- $O(q_\Lambda)$: the automorphisms of A_Λ that preserve the quadratic form q_Λ ;
- $O_-(\Lambda)$: the subgroup of isometries of Λ of spinor norm 1 (see also [Sca87, § 3.6]);
- $\tilde{O}(\Lambda) = \ker(O(\Lambda) \rightarrow O(A_\Lambda))$: the group of isometries of Λ that induce the identity on A_Λ ;
- $O^*(\Lambda) := O_-(\Lambda) \cap \tilde{O}(\Lambda)$.

We also introduce

- Λ_{K3} : the $K3$ lattice $U^{\oplus 3} \oplus E_8^{\oplus 2}$ (we denote the bilinear form by $(\cdot, \cdot)_{K3}$);
- M : the abstract lattice generated by $\{l', e_1, \dots, e_5\}$ which admits a unique primitive embedding into Λ_{K3} (one can also show that M is the generic Picard group of $K3$ surfaces $X_{(C,L)}$ and $M \cong U(2) \oplus D_4$; see [Laz09, Corollary 4.15, Lemma 4.18]);
- $T = M_{\Lambda_{K3}}^\perp$: the orthogonal complement of M in Λ_{K3} , which is isomorphic to $U \oplus U(2) \oplus D_4 \oplus E_8$;
- $\mathcal{D}_M := \{\omega \in \mathbb{P}(T \otimes \mathbb{C}) \mid (\omega, \omega)_{K3} = 0, (\omega, \bar{\omega})_{K3} > 0\}$, which is the period domain for M -polarized $K3$ surfaces;
- \mathcal{D}_M^0 : a connected component of \mathcal{D}_M which is a type IV Hermitian symmetric domain.

Let \mathcal{U} be an open subset of \mathcal{M}_{ADE} parameterizing the generic degree 5 pairs (C, L) with C smooth and with transverse intersections $C \cap L$. Let $\tilde{\mathcal{U}}$ be the \mathfrak{S}_5 -cover of \mathcal{U} that consists of triples (C, L, σ) with $\sigma: \{1, \dots, 5\} \rightarrow C \cap L$ labelings of $C \cap L$. By [Dol96], there is a period map $\tilde{\mathcal{U}} \rightarrow \mathcal{D}_M^0/O^*(T)$ sending (C, L, σ) to the periods of $X_{(C,L)}$ with the M -polarization determined by σ . By the global Torelli theorem and surjectivity of the period map for $K3$ surfaces and [Laz09, Proposition 4.14], the period map is birational. Note that there is a natural \mathfrak{S}_5 -action on $\tilde{\mathcal{U}}$. Moreover, the group $O^*(T)$ is a normal subgroup of $O_-(T)$ with $O_-(T)/O^*(T) \cong \mathfrak{S}_5$, and the residual \mathfrak{S}_5 -action on M is the permutation of the five points of intersection $C \cap L$ [Laz09, Proposition 4.22]. (In fact, $O_-(T)$ is the monodromy group for the degree 5 pairs.) Thus, the period map is \mathfrak{S}_5 -equivariant and descends to a birational map $\mathcal{U} \rightarrow \mathcal{D}_M^0/O_-(T)$.

The birational map can be extended to a morphism $\mathcal{M}_{ADE} \rightarrow \mathcal{D}_M^0/O_-(T)$ using normalized M -polarizations. In particular, one needs to construct an M -polarization in the case of non-transversal intersections $C \cap L$. We briefly summarize the construction and refer the readers to [Laz09, § 4.2.3] for the details. The construction is a modification of the canonical resolution of singularities of double covers (see [KP03, Theorem III.7.2]). The role of modification is to keep track of the points of intersection $C \cap L$. More precisely, one chooses a labeling of the intersection $\sigma: \{1, 2, 3, 4, 5\} \rightarrow C \cap L$ such that for any $p \in C \cap L$, we have $|\sigma^{-1}(p)| = \text{mult}_p(C \cap L)$. Set $Y_0 = \mathbb{P}^2$ and $B_0 = C + L$. We blow up one singularity at a time (instead of doing simultaneous blow-ups) and do the first five blow-ups in points belonging to L . The new branched divisor B_i is the strict transform of B_{i-1} together with the exceptional divisor of the blow-up reduced mod 2.

The process is repeated until the resulting divisor B_N is smooth. Denote the blow-up sequence by $Y_N \rightarrow \cdots \rightarrow Y_i \rightarrow Y_{i-1} \rightarrow \cdots \rightarrow Y_0 = \mathbb{P}^2$. The double cover $X_{(C,L)}$ of Y_N along B_N is a minimal resolution of $\bar{X}_{(C,L)}$. Let $p_i \in Y_{i-1}$ ($1 \leq i \leq 5$) be the centers of the blow-ups which lie on the corresponding strict transforms of L . Construct a primitive embedding of M into $\text{Pic}(X_{(C,L)})$ by sending l to the class of the reduced preimage of L and sending e_i ($1 \leq i \leq 5$) to the fundamental cycle associated to the simple singularity of B_{i-1} in the point p_i . The embedding is normalized in the sense of [Laz09, Definition 4.24], and the construction fits well in families.

By the global Torelli theorem for $K3$, the surface $X_{(C,L)}$ is unique up to isomorphism. Moreover, one can recover the degree 5 pair (C, L) because the classes $2l' + e_1 + \cdots + e_5$ (which corresponds to the pull-back of $\mathcal{O}_{\mathbb{P}^2}(1)$ and determines the covering map and the branched curve) and l' (which determines the line L and hence the residue quintic C) are fixed by the monodromy group $O_-(T)$. It follows that the period map

$$\mathcal{M}_{ADE} \hookrightarrow \mathcal{D}_M^0/O_-(T)$$

for degree 5 pairs (C, L) with $C + L$ admitting at worst ADE singularities is injective. This is the part we shall need later. For completeness, let us mention that one can verify that the period map is surjective (see [Laz09, § 4.3.1], especially Proposition 4.31). By Zariski's main theorem, the bijective birational morphism between two normal varieties $\mathcal{M}_{ADE} \rightarrow \mathcal{D}_M^0/O_-(T)$ is an isomorphism [Laz09, Theorem 4.1].

Let us focus on the generic global Torelli problem for special Horikawa surfaces. By [Cat84b] or [Par91a, Theorem 2.1], we construct the coarse moduli space \mathcal{M} for special Horikawa surfaces as the open subset of the quotient¹

$$(\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(5)) \times \text{Sym}^2(\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)))) // \text{SL}_3(\mathbb{C})$$

corresponding to triples (C, L, L') which consist of smooth quintics C and transversals L and L' with $C \cap L \cap L' = \emptyset$. It is more convenient to work with a double cover \mathcal{M}' of \mathcal{M} . Specifically, \mathcal{M}' is the open subset of the GIT quotient²

$$(\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(5)) \times \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \times \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))) // \text{SL}_3(\mathbb{C}),$$

which parameterizes (up to projective equivalence) triples (C, L_0, L_1) with C smooth quintics and L_0 and L_1 “labeled” lines which intersect C transversely and satisfy $C \cap L_0 \cap L_1 = \emptyset$.

Choose a sufficiently general reference point $b \in \mathcal{M}'$ (in particular, we label the two lines), and let S_b be the corresponding bidouble cover. Let $V = H_{\text{prim}}^2(S_b, \mathbb{R})$ (with respect to the class of a canonical curve or equivalently a hyperplane section in $\mathbb{P}(1, 1, 2, 5)$). Let Q be the polarization on V defined using cup product. We also write $V_{\mathbb{Q}} = H_{\text{prim}}^2(S_b, \mathbb{Q})$. Consider the action of the Galois group $(\mathbb{Z}/2\mathbb{Z})^2$ on S_b , and define $\rho: (\mathbb{Z}/2\mathbb{Z})^2 \rightarrow \text{Aut}(V, Q)$ to be the corresponding representation. We use the notation of diagram (2.1). The Galois group $(\mathbb{Z}/2\mathbb{Z})^2$ is generated by σ_0 and σ_1 . Denote the corresponding characters by χ_0 and χ_1 .

Notation 4.2. We shall use the following notation:

- $\mathcal{D} = \mathcal{D}(V, Q)$: the period domain parameterizing Q -polarized Hodge structures of weight 2 on V with Hodge numbers $[2, 28, 2]$;
- $\mathcal{D}^\rho = \{x \in \mathcal{D} \mid \rho(a)(x) = x, \forall a \in (\mathbb{Z}/2\mathbb{Z})^2\}$;

¹This is the quotient for the linearization induced by $\pi_1^* \mathcal{O}_{\mathbb{P}^2}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^2}(1) \otimes \pi_3^* \mathcal{O}_{\mathbb{P}^2}(1)$.

²This is the quotient with respect to the linearization $\pi_1^* \mathcal{O}_{\mathbb{P}^2}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^2}(1) \otimes \pi_3^* \mathcal{O}_{\mathbb{P}^2}(1)$.

- $V(\chi)$: the eigenspace of V corresponding to the character χ (it is not difficult to see that the eigenspaces $V(\chi)$ and $V(\chi')$ are orthogonal with respect to Q if $\chi \neq \chi'$);
- $V(\chi)_{\mathbb{Q}} := V(\chi) \cap V_{\mathbb{Q}}$;
- $\mathcal{D}(\chi)$: the period domain $\mathcal{D}(V(\chi), Q|_{V(\chi)})$ of type [1, 14, 1].

LEMMA 4.3. *There is a natural map $\mathcal{D}^{\rho} \rightarrow \mathcal{D}(\chi_0) \times \mathcal{D}(\chi_1)$ which is injective.*

Proof. The lemma follows from Proposition 2.8 and [DK07, § 7]. Specifically, only the characters χ_0 and χ_1 appear in the decomposition of the vector space V . Let $V \otimes \mathbb{C} = V^{2,0} \oplus V^{1,1} \oplus V^{0,2}$ be a Q -polarized Hodge structure on V . The map is defined by sending the Hodge structure to the induced Q -polarized Hodge structures on $V(\chi_0)$ and $V(\chi_1)$, which is clearly injective. \square

Now, we show that the period spaces $\mathcal{D}(\chi_0)$ and $\mathcal{D}(\chi_1)$ are both isomorphic to the period space \mathcal{D}_M for M -polarized $K3$ surfaces.

LEMMA 4.4. *There exists an isomorphism $(V(\chi_l)_{\mathbb{Q}}, \frac{1}{2}Q) \cong (T \otimes \mathbb{Q}, (\cdot, \cdot)_{K3} \otimes \mathbb{Q})$ for $l = 0$ or 1 .*

Proof. We take χ_0 as an example. Let $S = S_b$ be the bidouble cover corresponding to the reference point $b \in \mathcal{M}'$. We use the notation of diagram (2.1). In particular, by abuse of notation, σ_0 also denotes the involution relative to $\varphi_0: S_0 \rightarrow X_0$. Label the points of intersection $C \cap L_1$ (the isomorphism we shall describe does not depend on the labeling), and there is a primitive embedding of M (and hence T) into $H^2(X_0, \mathbb{Z})$. Since b is sufficiently general, M is the Picard lattice of the $K3$ surface X_0 [Laz09, Corollary 4.15]. Consider the composition of linear maps

$$T \otimes \mathbb{Q} \hookrightarrow H^2(X_0, \mathbb{Q}) \xrightarrow{\varphi_0^*} H^2(S_0, \mathbb{Q})^{\sigma_0^*}.$$

The map φ_0^* is an isomorphism of vector spaces. Let $D_1 \subset X_0$ be the strict transform of L_1 , and set E_1, \dots, E_5 to be the exceptional curves. Clearly, $T \otimes \mathbb{Q}$ is the orthogonal complement of $\mathbb{Q}[D_1] \oplus \mathbb{Q}[E_1] \oplus \dots \oplus \mathbb{Q}[E_5]$ in $H^2(X_0, \mathbb{Q})$. Thus, $T \otimes \mathbb{Q}$ is mapped onto $V(\chi_0)_{\mathbb{Q}} = H_{\text{prim}}^2(S, \mathbb{Q})^{\sigma_0^*}$ by φ_0^* . The claim on the bilinear forms is clear. \square

COROLLARY 4.5. *The period domain $\mathcal{D}(\chi_l)$ has two connected components which are both isomorphic to the 14-dimensional type IV Hermitian symmetric domain $\text{SO}(2, 14)/\text{S}(\text{O}(2) \times \text{O}(14))$.*

To formulate the theorem, we also need to choose a discrete group. Let Γ_0 and Γ_1 be the discrete subgroups of $\text{Aut}(V(\chi_0)_{\mathbb{Q}}, Q)$ and $\text{Aut}(V(\chi_1)_{\mathbb{Q}}, Q)$, respectively, corresponding to $O_-(T)$ (using Lemma 4.4). Set Γ to be the discrete subgroup in $\text{Aut}(V_{\mathbb{Q}}, Q)$ which projects onto Γ_0 and Γ_1 under the isomorphism $V_{\mathbb{Q}} = V(\chi_0)_{\mathbb{Q}} \oplus V(\chi_1)_{\mathbb{Q}}$. Let

$$\mathcal{P}: \mathcal{M} \rightarrow \mathcal{D}^{\rho}/\Gamma$$

be the period map for special Horikawa surfaces (which are canonically polarized). Because the monodromy group (for the very general base point) is contained in the generic Mumford–Tate group and σ_0^* and σ_1^* are Hodge tensors for every member of the family, the monodromy representation commutes with the representation ρ (see also [GGK12, III.C, Discussion and Example, pp. 67–68] and [DK07, § 7]). By [Laz09, Proposition 4.22], the discrete subgroup Γ contains the image of the monodromy representation.

To prove the generic global Torelli theorem for $\mathcal{P}: \mathcal{M} \rightarrow \mathcal{D}^{\rho}/\Gamma$, we consider the map

$$\mathcal{P}_0 \times \mathcal{P}_1: \mathcal{M}' \rightarrow \mathcal{D}_M^0/O_-(T) \times \mathcal{D}_M^0/O_-(T),$$

which is defined using the period maps for the degree 5 pairs (C, L_0) and (C, L_1) .

PROPOSITION 4.6. *The map $\mathcal{P}_0 \times \mathcal{P}_1: \mathcal{M}' \rightarrow \mathcal{D}_M^0/O_-(T) \times \mathcal{D}_M^0/O_-(T)$ is generically injective.*

Proof. By [Laz09, § 4.2.3 or Theorem 4.1], one can recover a degree 5 pair (C, L) (up to projective equivalence) from the periods of the $K3$ surface $X_{(C,L)}$. As a result, the isomorphism class of a triple (C, L_0, L_1) is determined by the periods of the $K3$ surfaces $X_{(C,L_0)}$ and $X_{(C,L_1)}$ provided that the quintic C has no nontrivial automorphism. More specifically, we assume $(\mathcal{P}_0 \times \mathcal{P}_1)(C, L_0, L_1) = (\mathcal{P}_0 \times \mathcal{P}_1)(C', L'_0, L'_1)$. Then there exist $f, g \in \mathrm{PGL}_3(\mathbb{C})$ such that $f(C) = C'$, $f(L_0) = L'_0$, $g(C) = C'$, and $g(L_1) = L'_1$. In particular, one has $(g^{-1} \circ f)(C) = C$. Because $\mathrm{Aut}(C) = \{\mathrm{id}\}$, we get $f = g$. Thus, the map $\mathcal{P}_0 \times \mathcal{P}_1: \mathcal{M}' \rightarrow \mathcal{D}_M^0/O_-(T) \times \mathcal{D}_M^0/O_-(T)$ is generically injective. \square

THEOREM 4.7. *The period map $\mathcal{P}: \mathcal{M} \rightarrow \mathcal{D}^\rho/\Gamma$ is generically injective.*

Proof. Let $\mathcal{P}': \mathcal{M}' \rightarrow \mathcal{D}^{\rho,0}/\Gamma \hookrightarrow \mathcal{D}^0(\chi_0)/\Gamma_0 \times \mathcal{D}^0(\chi_1)/\Gamma_1$ (where the superscript 0 denotes the choice of a connected component) be the map sending a labeled triple (C, L_0, L_1) to the periods of the corresponding special Horikawa surface S (and to the eigenperiods on the underlying eigenspaces $V(\chi_0)$ and $V(\chi_1)$). We would like to compare \mathcal{P}' and $\mathcal{P}_0 \times \mathcal{P}_1$. Note that the transcendental lattice T has been identified with the invariant parts of the underlying vector space $V_{\mathbb{Q}} = H_{\mathrm{prim}}^2(S_b, \mathbb{Q})$ for the involutions σ_l^* ($l = 0, 1$) via the natural pull-backs (see Lemma 4.4). We claim that under these identifications, \mathcal{P}' coincides with $\mathcal{P}_0 \times \mathcal{P}_1$ up to the order of the periods (after relabeling the two lines, one needs to switch the periods of the two $K3$ surfaces, but the period of the special Horikawa surface and the eigenperiods remain the same). The reason is that the eigenperiods are obtained by pulling back the holomorphic 2-forms of the two $K3$ surfaces $X_0 = X_{(C,L_1)}$ and $X_1 = X_{(C,L_0)}$ (see diagram (2.1) and Proposition 2.8). Specifically, we choose the orderings for the intersection points of $C \cap L_1$ and $C \cap L_0$ and define respective primitive embeddings of M and T into $H^2(X_0, \mathbb{Z})$ and $H^2(X_1, \mathbb{Z})$ accordingly. Let $\phi_l: \Lambda_{K3} \xrightarrow{\cong} H^2(X_l, \mathbb{Z})$ ($l = 0, 1$) be the markings compatible with the primitive embeddings of M . The map $\mathcal{P}_0 \times \mathcal{P}_1$ is defined by considering (the $O_-(T)$ -orbits of) the $K3$ periods $\phi_{0,\mathbb{C}}^{-1}(H^{2,0}(X_0))$ and $\phi_{1,\mathbb{C}}^{-1}(H^{2,0}(X_1))$.

Now, let us discuss the (eigen)period map \mathcal{P}' . We denote σ_0 and σ_1 in diagram (2.1) by σ and σ' , respectively, to reflect the fact that they do not depend on how one labels the lines. Let χ and χ' be the corresponding characters. We now construct a natural ρ -marking (cf. [DK07, § 7]) on $H_{\mathrm{prim}}^2(S, \mathbb{Q})$ using the $K3$ markings ϕ_0 and ϕ_1 . On one hand, Lemma 4.4 gives us $V(\chi)_{\mathbb{Q}} \cong T \otimes \mathbb{Q}$ and $V(\chi')_{\mathbb{Q}} \cong T \otimes \mathbb{Q}$. On the other hand, one has $T \otimes \mathbb{Q} \xrightarrow{\phi_0} H^2(X_0, \mathbb{Q}) \xrightarrow{\cong} H^2(S_0, \mathbb{Q})^{\sigma^*}$, where the second homomorphism is induced by the natural pull-back map. As in the proof of Lemma 4.4, one can show that the image of the composition (which is clearly injective) is $H_{\mathrm{prim}}^2(S, \mathbb{Q})^{\sigma^*}$. In other words, we get $T \otimes \mathbb{Q} \xrightarrow{\cong} H_{\mathrm{prim}}^2(S, \mathbb{Q})^{\sigma^*}$. Similarly, the $K3$ marking ϕ_1 and the natural pull-back map allow us to identify $T \otimes \mathbb{Q}$ and $H_{\mathrm{prim}}^2(S, \mathbb{Q})^{\sigma'^*}$. Combining these observations, we get the markings $V(\chi)_{\mathbb{Q}} \xrightarrow{\cong} H^2(S_0, \mathbb{Q})^{\sigma^*}$ and $V(\chi')_{\mathbb{Q}} \xrightarrow{\cong} H^2(S_0, \mathbb{Q})^{\sigma'^*}$. Taking the preimage of the holomorphic 2-form invariant for σ^* (respectively, σ'^*) in $V(\chi)_{\mathbb{Q}} \otimes \mathbb{C}$ (respectively, $V(\chi')_{\mathbb{Q}} \otimes \mathbb{C}$), one obtains the eigenperiod map \mathcal{P}' . Now, our claim is clear. By Proposition 4.6, generically, \mathcal{P}' has degree 2 (depending on the labelings of the lines). As a result, the period map $\mathcal{P}: \mathcal{M} \rightarrow \mathcal{D}^\rho/\Gamma$ is generically injective. \square

Remark 4.8. Assume that the quintic curve C admits a nontrivial automorphism σ satisfying $\sigma(L_0) \neq L_0$ and $\sigma(L_1) \neq L_1$. Then the triples (C, L_0, L_1) and $(C, \sigma(L_0), L_1)$ are mapped by the period map \mathcal{P} to the same point.

Remark 4.9. Let \mathcal{W} be the subset of $|\mathcal{O}_{\mathbb{P}^2}(5)| \times |\mathcal{O}_{\mathbb{P}^2}(1)| \times |\mathcal{O}_{\mathbb{P}^2}(1)|$ corresponding to triples (C, L_0, L_1) with $C + L_0$ and $C + L_1$ admitting at worst ADE singularities and $C \cap L_0 \cap L_1 = \emptyset$.

By taking bidouble covers, we obtain a family of surfaces $\mathcal{S} \rightarrow \mathcal{W}$ with only du Val singularities. By applying a simultaneous resolution to the family \mathcal{S} , we obtain a family $\tilde{\mathcal{S}} \rightarrow \mathcal{W}$ (after a finite base change of \mathcal{W}) of Horikawa surfaces (which are surfaces of general type with $p_g = 2$ and $K^2 = 1$). Consider the period map $\mathcal{P}_0 \times \mathcal{P}_1: \mathcal{W} \rightarrow \mathcal{D}_M^0/O_-(T) \times \mathcal{D}_M^0/O_-(T)$. The generic global Torelli theorem holds for this family. Namely, if two generic points in \mathcal{W} have the same image in $\mathcal{D}_M^0/O_-(T) \times \mathcal{D}_M^0/O_-(T)$, then the corresponding triples are projectively equivalent.

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